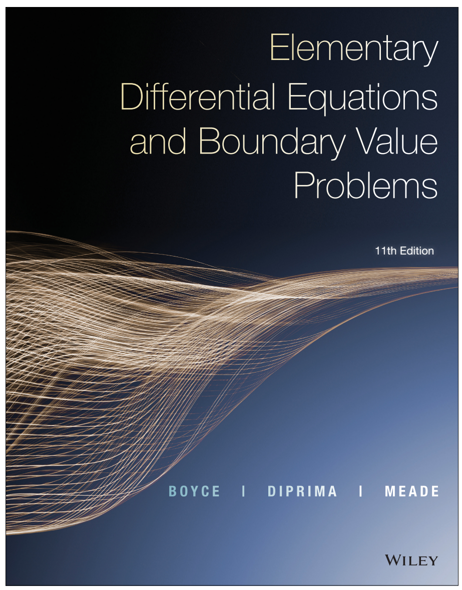
A Solution Manual For

Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade



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1 Chapter 4.1, Higher order linear differential equations. General theory. page 173

1.1 problem 1

Internal problem ID [812]
Internal file name [`OUTPUT/812_Sunday_June_05_2022_01_50_23_AM_98049129/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 4.1, Higher order linear differential equations. General theory. page 173
Problem number: 1.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _with_linear_symmetries]]
```

$$\boxed{y'''' + 4y''' + 3y = t}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 4y''' + 3y = 0$$

The characteristic equation is

$$\lambda^4 + 4\lambda^3 + 3 = 0$$

The roots of the above equation are

$$
\begin{aligned}
\lambda_1 &= -1 \\
\lambda_2 &= -\left(4+2\sqrt{2}\right)^{\frac{1}{3}} - \frac{2}{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}} - 1 \\
\lambda_3 &= \frac{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}}{2} + \frac{1}{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}} - 1 + \frac{i\sqrt{3}\left(-\left(4+2\sqrt{2}\right)^{\frac{1}{3}} + \frac{2}{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}}\right)}{2} \\
\lambda_4 &= \frac{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}}{2} + \frac{1}{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}} - 1 - \frac{i\sqrt{3}\left(-\left(4+2\sqrt{2}\right)^{\frac{1}{3}} + \frac{2}{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}}\right)}{2}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
y_h(t) = c_1 \mathrm{e}^{-t} + \mathrm{e}^{\left(\frac{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}}{2} + \frac{1}{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}} - 1 - \frac{i\sqrt{3}\left(-\left(4+2\sqrt{2}\right)^{\frac{1}{3}} + \frac{2}{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}}\right)}{2}\right)t} c_2 + \mathrm{e}^{\left(-\left(4+2\sqrt{2}\right)^{\frac{1}{3}} - \frac{2}{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}} - 1\right)t} c_3 + \mathrm{e}^{\left(\frac{(4+2\sqrt{2}}{2}\right.}
$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-t} \\
y_2 &= \mathrm{e}^{\left(\frac{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}}{2} + \frac{1}{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}} - 1 - \frac{i\sqrt{3}\left(-\left(4+2\sqrt{2}\right)^{\frac{1}{3}} + \frac{2}{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}}\right)}{2}\right)t} \\
y_3 &= \mathrm{e}^{\left(-\left(4+2\sqrt{2}\right)^{\frac{1}{3}} - \frac{2}{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}} - 1\right)t} \\
y_4 &= \mathrm{e}^{\left(\frac{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}}{2} + \frac{1}{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}} - 1 + \frac{i\sqrt{3}\left(-\left(4+2\sqrt{2}\right)^{\frac{1}{3}} + \frac{2}{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}}\right)}{2}\right)t}
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
y'''' + 4y''' + 3y = t
$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$t$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, t\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{\left(-\left(4+2\sqrt{2}\right)^{\frac{1}{3}}-\frac{2}{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}}-1\right)t}, e^{\left(\frac{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}}{2}+\frac{1}{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}}-1-\frac{i\sqrt{3}\left(-\left(4+2\sqrt{2}\right)^{\frac{1}{3}}+\frac{2}{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}}\right)}{2}\right)t}, e^{\left(\frac{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}}{2}+\frac{1}{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}}-1+\frac{i\sqrt{3}}{\cdots}\right.}\right.$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 t + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_2 t + 3A_1 = t$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{t}{3}$$

Therefore the general solution is

$$y = y_h + y_p$$

$$= \left(c_1 e^{-t} + e^{\left(\frac{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}}{2}+\frac{1}{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}}-1-\frac{i\sqrt{3}\left(-\left(4+2\sqrt{2}\right)^{\frac{1}{3}}+\frac{2}{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}}\right)}{2}\right)t} c_2 + e^{\left(-\left(4+2\sqrt{2}\right)^{\frac{1}{3}}-\frac{2}{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}}-1\right)t} c_3 + e^{\left(\frac{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}}{2}+\frac{1}{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}}-1+\frac{i\sqrt{3}\left(-\left(4+2\sqrt{2}\right)^{\frac{1}{3}}+\frac{2}{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}}\right)}{2}\right)t} c_4\right) + \left(\frac{t}{3}\right)$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-t} + e^{\left(\frac{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}}{2}+\frac{1}{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}}-1-\frac{i\sqrt{3}\left(-\left(4+2\sqrt{2}\right)^{\frac{1}{3}}+\frac{2}{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}}\right)}{2}\right)t} c_2 + e^{\left(-\left(4+2\sqrt{2}\right)^{\frac{1}{3}}-\frac{2}{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}}-1\right)t} c_3 + e^{\left(\frac{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}}{2}+\frac{1}{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}}-1+\frac{i\sqrt{3}\left(-\left(4+2\sqrt{2}\right)^{\frac{1}{3}}+\frac{2}{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}}\right)}{2}\right)t} c_4 + \frac{t}{3} \quad (1)$$

Verification of solutions

$$y = c_1 e^{-t} + e^{\left(\frac{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}}{2} + \frac{1}{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}} - 1 - \frac{i\sqrt{3}\left(-\left(4+2\sqrt{2}\right)^{\frac{1}{3}} + \frac{2}{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}}\right)}{2}\right)t} c_2 + e^{\left(-\left(4+2\sqrt{2}\right)^{\frac{1}{3}} - \frac{2}{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}} - 1\right)t} c_3 + e^{\left(\frac{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}}{2} + \frac{1}{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}} - 1 + \frac{i\sqrt{3}\left(-\left(4+2\sqrt{2}\right)^{\frac{1}{3}} + \frac{2}{\left(4+2\sqrt{2}\right)^{\frac{1}{3}}}\right)}{2}\right)t} c_4 + \frac{t}{3}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 182

```
dsolve(diff(y(t),t$4)+4*diff(y(t),t$3)+3*y(t)=t,y(t), singsol=all)
```

$$y(t) = \frac{t}{3} + e^{-t}c_1 + c_2 e^{\frac{t\left(\left(\sqrt{2}-2\right)\left(4+2\sqrt{2}\right)^{\frac{2}{3}}-2\left(4+2\sqrt{2}\right)^{\frac{1}{3}}-2\right)}{2}} + c_3 e^{-\frac{t\left(\left(\sqrt{2}-2\right)\left(4+2\sqrt{2}\right)^{\frac{2}{3}}-2\left(4+2\sqrt{2}\right)^{\frac{1}{3}}+4\right)}{4}} \cos\left(\frac{t\left(4+2\sqrt{2}\right)^{\frac{1}{3}}\left(2+\left(\sqrt{2}-2\right)\left(4+2\sqrt{2}\right)^{\frac{1}{3}}\right)\sqrt{3}}{4}\right) + c_4 e^{-\frac{t\left(\left(\sqrt{2}-2\right)\left(4+2\sqrt{2}\right)^{\frac{2}{3}}-2\left(4+2\sqrt{2}\right)^{\frac{1}{3}}+4\right)}{4}} \sin\left(\frac{t\left(4+2\sqrt{2}\right)^{\frac{1}{3}}\left(2+\left(\sqrt{2}-2\right)\left(4+2\sqrt{2}\right)^{\frac{1}{3}}\right)\sqrt{3}}{4}\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 100

```
DSolve[y''''[t]+4*y'''[t]+3*y[t]==t,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to c_2 \exp\left(t\text{Root}\left[\#1^3+3\#1^2-3\#1+3\&,2\right]\right) + c_3 \exp\left(t\text{Root}\left[\#1^3+3\#1^2-3\#1+3\&,3\right]\right) + c_1 \exp\left(t\text{Root}\left[\#1^3+3\#1^2-3\#1+3\&,1\right]\right) + \frac{t}{3} + c_4 e^{-t}$$

1.2 problem 2

Internal problem ID [813]
Internal file name [OUTPUT/813_Sunday_June_05_2022_01_50_25_AM_31574293/index.tex]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 4.1, Higher order linear differential equations. General theory. page 173
Problem number: 2.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_high_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{t(-1+t)\,y'''' + \mathrm{e}^{t} y'' + 4y t^{2} = 0}$$

Unable to solve this ODE.

1.2.1 Maple step by step solution

Let's solve

$t(-1+t)\,y'''' + \mathrm{e}^{t} y'' + 4y t^{2} = 0$

- Highest derivative means the order of the ODE is 4

  y''''

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying high order exact linear fully integrable
trying to convert to a linear ODE with constant coefficients
trying differential order: 4; missing the dependent variable
trying a solution in terms of MeijerG functions
trying differential order: 4; missing the dependent variable
trying a solution in terms of MeijerG functions
-> Try computing a Rational Normal Form for the given ODE...
<- unable to resolve the Equivalence to a Rational Normal Form
trying reduction of order using simple exponentials
trying differential order: 4; exact nonlinear
--- Trying Lie symmetry methods, high order ---
`, `-> Computing symmetries using: way = 3`[0, y]
```

✗ Solution by Maple

```
dsolve(t*(t-1)*diff(y(t),t$4)+exp(t)*diff(y(t),t$2)+4*t^2*y(t)=0,y(t), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[t*(t-1)*y''''[t]+Exp[t]*y''[t]+4*t^2*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

Not solved

1.3 problem 8

Internal problem ID [814]
Internal file name [OUTPUT/814_Sunday_June_05_2022_01_50_26_AM_60538498/index.tex]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 4.1, Higher order linear differential equations. General theory. page 173
Problem number: 8.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + y'' = 0}$$

The characteristic equation is

$$\lambda^4 + \lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$
$$\lambda_3 = i$$
$$\lambda_4 = -i$$

Therefore the homogeneous solution is

$$y_h(t) = c_2 t + c_1 + \mathrm{e}^{-it} c_3 + \mathrm{e}^{it} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = t$$
$$y_3 = \mathrm{e}^{-it}$$
$$y_4 = \mathrm{e}^{it}$$

Summary
The solution(s) found are the following

$$y = c_2 t + c_1 + e^{-it} c_3 + e^{it} c_4 \tag{1}$$

Verification of solutions

$$y = c_2 t + c_1 + e^{-it} c_3 + e^{it} c_4$$

Verified OK.

1.3.1 Maple step by step solution

Let's solve

$y'''' + y'' = 0$

- Highest derivative means the order of the ODE is 4

  y''''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(t)$

    $y_1(t) = y$

  - Define new variable $y_2(t)$

    $y_2(t) = y'$

  - Define new variable $y_3(t)$

    $y_3(t) = y''$

  - Define new variable $y_4(t)$

    $y_4(t) = y'''$

  - Isolate for $y_4'(t)$ using original ODE

    $y_4'(t) = -y_3(t)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(t) = y_1'(t), y_3(t) = y_2'(t), y_4(t) = y_3'(t), y_4'(t) = -y_3(t)]$

- Define vector

$$\vec{y}(t)=\left[\begin{array}{c} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{array}\right]$$

- System to solve

$$\vec{y}'(t)=\left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right]\cdot\vec{y}(t)$$

- Define the coefficient matrix

$$A=\left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

- Rewrite the system as

$\vec{y}'(t)=A\cdot\vec{y}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0,\left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array}\right]\right],\left[0,\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right]\right],\left[-\mathrm{I},\left[\begin{array}{c} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{array}\right]\right],\left[\mathrm{I},\left[\begin{array}{c} \mathrm{I} \\ -1 \\ -\mathrm{I} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[0,\left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I}t} \cdot \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(t) - \mathrm{I}\sin(t)) \cdot \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\mathrm{I}(\cos(t) - \mathrm{I}\sin(t)) \\ -\cos(t) + \mathrm{I}\sin(t) \\ \mathrm{I}(\cos(t) - \mathrm{I}\sin(t)) \\ \cos(t) - \mathrm{I}\sin(t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(t) = \begin{bmatrix} -\sin(t) \\ -\cos(t) \\ \sin(t) \\ \cos(t) \end{bmatrix}, \vec{y}_4(t) = \begin{bmatrix} -\cos(t) \\ \sin(t) \\ \cos(t) \\ -\sin(t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(t) + c_4\vec{y}_4(t)$

- Substitute solutions into the general solution

$$\vec{y} = \begin{bmatrix} -c_4\cos(t) - c_3\sin(t) + c_1 \\ c_4\sin(t) - c_3\cos(t) \\ c_4\cos(t) + c_3\sin(t) \\ -c_4\sin(t) + c_3\cos(t) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$y = -c_4\cos(t) - c_3\sin(t) + c_1$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(t),t$4)+diff(y(t),t$2)=0,y(t), singsol=all)
```

$$y(t) = c_1 + c_2 t + c_3 \sin(t) + c_4 \cos(t)$$

✓ Solution by Mathematica

Time used: 0.098 (sec). Leaf size: 24

```
DSolve[y''''[t]+y''[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to c_4 t - c_1 \cos(t) - c_2 \sin(t) + c_3$$

1.4 problem 9

Internal problem ID [815]
Internal file name [`OUTPUT/815_Sunday_June_05_2022_01_50_27_AM_43069891/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 4.1, Higher order linear differential equations. General theory. page 173
Problem number: 9.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + 2y'' - y' - 2y = 0}$$

The characteristic equation is

$$\lambda^3 + 2\lambda^2 - \lambda - 2 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -2$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(t) = c_1 e^{-t} + e^{-2t} c_2 + c_3 e^{t}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-t}$$
$$y_2 = e^{-2t}$$
$$y_3 = e^{t}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-t} + e^{-2t} c_2 + c_3 e^{t} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-t} + e^{-2t} c_2 + c_3 e^{t}$$

Verified OK.

1.4.1 Maple step by step solution

Let's solve

$y''' + 2y'' - y' - 2y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- ☐ Convert linear ODE into a system of first order ODEs
  - ◦ Define new variable $y_1(t)$

    $y_1(t) = y$
  - ◦ Define new variable $y_2(t)$

    $y_2(t) = y'$
  - ◦ Define new variable $y_3(t)$

    $y_3(t) = y''$
  - ◦ Isolate for $y_3'(t)$ using original ODE

    $y_3'(t) = -2y_3(t) + y_2(t) + 2y_1(t)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(t) = y_1'(t), y_3(t) = y_2'(t), y_3'(t) = -2y_3(t) + y_2(t) + 2y_1(t)]$

- Define vector

  $$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- System to solve

$$\vec{y}'(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix} \cdot \vec{y}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(t) = A \cdot \vec{y}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-2t} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{-t} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-2t} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 e^{-t} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(4c_3 e^{3t} + 4c_2 e^{t} + c_1) e^{-2t}}{4}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(t),t$3)+2*diff(y(t),t$2)-diff(y(t),t)-2*y(t)=0,y(t), singsol=all)
```

$$y(t) = \left(c_1 \mathrm{e}^{3t} + c_3 \mathrm{e}^{t} + c_2\right) \mathrm{e}^{-2t}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 28

```
DSolve[y'''[t]+2*y''[t]-y'[t]-2*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-2t}\left(c_2 e^{t} + c_3 e^{3t} + c_1\right)$$

1.5 problem 10

Internal problem ID [816]
Internal file name [OUTPUT/816_Sunday_June_05_2022_01_50_28_AM_7133267/index.tex]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 4.1, Higher order linear differential equations. General theory. page 173
Problem number: 10.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_missing_y"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{xy''' - y'' = 0}$$

Since y is missing from the ode then we can use the substitution $y' = v(x)$ to reduce the order by one. The ODE becomes

$$xv''(x) - v'(x) = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (xv''(x) - v'(x))\,dx = 0$$
$$v'(x)\,x - 2v(x) = c_1$$

Which is now solved for $v(x)$. In canonical form the ODE is

$$\begin{aligned} v' &= F(x, v) \\ &= f(x)g(v) \\ &= \frac{2v + c_1}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(v) = 2v + c_1$. Integrating both sides gives

$$\frac{1}{2v + c_1}\, dv = \frac{1}{x}\, dx$$
$$\int \frac{1}{2v + c_1}\, dv = \int \frac{1}{x}\, dx$$
$$\frac{\ln (2v + c_1)}{2} = \ln (x) + c_2$$

Raising both side to exponential gives

$$\sqrt{2v + c_1} = \mathrm{e}^{\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{2v + c_1} = c_3 x$$

But since $y' = v(x)$ then we now need to solve the ode $y' = \frac{c_3^2 \mathrm{e}^{2c_2} x^2}{2} - \frac{c_1}{2}$. Integrating both sides gives

$$y = \int \frac{c_3^2 \mathrm{e}^{2c_2} x^2}{2} - \frac{c_1}{2}\, \mathrm{d}x$$
$$= \frac{c_3^2 \mathrm{e}^{2c_2} x^3}{6} - \frac{c_1 x}{2} + c_4$$

Summary

The solution(s) found are the following

$$y = \frac{c_3^2 \mathrm{e}^{2c_2} x^3}{6} - \frac{c_1 x}{2} + c_4 \tag{1}$$

Verification of solutions

$$y = \frac{c_3^2 \mathrm{e}^{2c_2} x^3}{6} - \frac{c_1 x}{2} + c_4$$

Verified OK.

1.5.1 Maple step by step solution

Let's solve

$xy''' - y'' = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = \frac{y''}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - \frac{y''}{x} = 0$

- Multiply by denominators of the ODE

  $xy''' - y'' = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  ◦ Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

  ◦ Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  ◦ Calculate the 3rd derivative of y with respect to x , using the chain rule

    $y''' = \left(\frac{d^3}{dt^3}y(t)\right) t'(x)^3 + 3t'(x)\, t''(x) \left(\frac{d^2}{dt^2}y(t)\right) + t'''(x) \left(\frac{d}{dt}y(t)\right)$

  ◦ Compute derivative

    $y''' = \frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}$

  Substitute the change of variables back into the ODE

$$x\left(\frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}\right) - \frac{\frac{d^2}{dt^2}y(t)}{x^2} + \frac{\frac{d}{dt}y(t)}{x^2} = 0$$

- Simplify

$$\frac{\frac{d^3}{dt^3}y(t) - 4\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t)}{x^2} = 0$$

- Isolate 3rd derivative

$$\frac{d^3}{dt^3}y(t) = 4\frac{d^2}{dt^2}y(t) - 3\frac{d}{dt}y(t)$$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$$\frac{d^3}{dt^3}y(t) - 4\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) = 0$$

☐ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(t)$

    $$y_1(t) = y(t)$$

  - Define new variable $y_2(t)$

    $$y_2(t) = \frac{d}{dt}y(t)$$

  - Define new variable $y_3(t)$

    $$y_3(t) = \frac{d^2}{dt^2}y(t)$$

  - Isolate for $\frac{d}{dt}y_3(t)$ using original ODE

    $$\frac{d}{dt}y_3(t) = 4y_3(t) - 3y_2(t)$$

  Convert linear ODE into a system of first order ODEs

  $$\left[y_2(t) = \frac{d}{dt}y_1(t), y_3(t) = \frac{d}{dt}y_2(t), \frac{d}{dt}y_3(t) = 4y_3(t) - 3y_2(t)\right]$$

- Define vector

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- System to solve

$$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & 4 \end{bmatrix} \cdot \vec{y}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & 4 \end{bmatrix}$$

- Rewrite the system as

  $\frac{d}{dt}\overrightarrow{y}(t) = A \cdot \overrightarrow{y}(t)$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$
- Consider eigenpair

  $$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$
- Solution to homogeneous system from eigenpair

  $$\overrightarrow{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
- Consider eigenpair

  $$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$
- Solution to homogeneous system from eigenpair

  $$\overrightarrow{y}_2 = e^t \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{3t} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_2 e^{t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{3t} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix} + \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y(t) = c_2 e^{t} + \frac{c_3 e^{3t}}{9} + c_1$$

- Change variables back using $t = \ln(x)$

$$y = c_2 x + \frac{1}{9} c_3 x^3 + c_1$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(x*diff(y(x),x$3)-diff(y(x),x$2)=0,y(x), singsol=all)
```

$$y(x) = c_3x^3 + c_2x + c_1$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 21

```
DSolve[x*y'''[x]-y''[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1x^3}{6} + c_3x + c_2$$

1.6 problem 11

Internal problem ID [817]
Internal file name [`OUTPUT/817_Sunday_June_05_2022_01_50_29_AM_64841157/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 4.1, Higher order linear differential equations. General theory. page 173
Problem number: 11.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x^3y''' + x^2y'' - 2y'x + 2y = 0}$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned} y' &= \lambda\, x^{\lambda-1} \\ y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\ y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \end{aligned}$$

Substituting these back into

$$x^3y''' + x^2y'' - 2y'x + 2y = 0$$

gives

$$-2x\lambda\, x^{\lambda-1} + x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} + 2x^\lambda = 0$$

Which simplifies to

$$-2\lambda\, x^\lambda + \lambda(\lambda-1)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\, x^\lambda + 2x^\lambda = 0$$

And since $x^\lambda \neq 0$ then dividing through by x^λ, the above becomes

$$-2\lambda + \lambda(\lambda - 1) + \lambda(\lambda - 1)(\lambda - 2) + 2 = 0$$

Simplifying gives the characteristic equation as

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

Solving the above gives the following roots

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = -1$$

This table summarises the result

root	multiplicity	type of root
-1	1	real root
1	1	real root
2	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a c_1x^λ basis solution. Each real root of multiplicty two, generates c_1x^λ and $c_2x^\lambda \ln(x)$ basis solutions. Each real root of multiplicty three, generates c_1x^λ and $c_2x^\lambda \ln(x)$ and $c_3x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = \frac{c_1}{x} + c_2x + c_3x^2$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \frac{1}{x}$$
$$y_2 = x$$
$$y_3 = x^2$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x} + c_2x + c_3x^2 \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + c_2x + c_3x^2$$

Verified OK.

1.6.1 Maple step by step solution

Let's solve

$x^3y''' + x^2y'' - 2y'x + 2y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = -\frac{2y}{x^3} - \frac{y''x-2y'}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' + \frac{y''}{x} - \frac{2y'}{x^2} + \frac{2y}{x^3} = 0$

- Multiply by denominators of the ODE

  $x^3y''' + x^2y'' - 2y'x + 2y = 0$

- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  - Calculate the 3rd derivative of y with respect to x , using the chain rule

    $y''' = \left(\frac{d^3}{dt^3}y(t)\right) t'(x)^3 + 3t'(x) t''(x) \left(\frac{d^2}{dt^2}y(t)\right) + t'''(x) \left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $$y''' = \frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}$$

  Substitute the change of variables back into the ODE

  $$x^3\left(\frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}\right) + x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 2\frac{d}{dt}y(t) + 2y(t) = 0$$

- Simplify

  $$\frac{d^3}{dt^3}y(t) - 2\frac{d^2}{dt^2}y(t) - \frac{d}{dt}y(t) + 2y(t) = 0$$

- ☐ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(t)$

    $$y_1(t) = y(t)$$

  - Define new variable $y_2(t)$

    $$y_2(t) = \frac{d}{dt}y(t)$$

  - Define new variable $y_3(t)$

    $$y_3(t) = \frac{d^2}{dt^2}y(t)$$

  - Isolate for $\frac{d}{dt}y_3(t)$ using original ODE

    $$\frac{d}{dt}y_3(t) = 2y_3(t) + y_2(t) - 2y_1(t)$$

    Convert linear ODE into a system of first order ODEs

    $$\left[y_2(t) = \frac{d}{dt}y_1(t), y_3(t) = \frac{d}{dt}y_2(t), \frac{d}{dt}y_3(t) = 2y_3(t) + y_2(t) - 2y_1(t)\right]$$

- Define vector

  $$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- System to solve

  $$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix} \cdot \vec{y}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix}$$

- Rewrite the system as

  $\frac{d}{dt}\vec{y}(t) = A \cdot \vec{y}(t)$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-t} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \mathrm{e}^{t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_3 = \mathrm{e}^{2t} \cdot \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

$$\overrightarrow{y} = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2 + c_3 \overrightarrow{y}_3$$

- Substitute solutions into the general solution

$$\overrightarrow{y} = c_1 \mathrm{e}^{-t} \cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right] + c_2 \mathrm{e}^{t} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right] + c_3 \mathrm{e}^{2t} \cdot \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y(t) = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{t} + \frac{c_3 \mathrm{e}^{2t}}{4}$$

- Change variables back using $t = \ln(x)$

$$y = \frac{c_1}{x} + c_2 x + \frac{c_3 x^2}{4}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve(x^3*diff(y(x),x$3)+x^2*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_2 x^3 + c_1 x^2 + c_3}{x}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 22

```
DSolve[x^3*y'''[x]+x^2*y''[x]-2*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_3 x^2 + c_2 x + \frac{c_1}{x}$$

1.7 problem 16

Internal problem ID [818]
Internal file name [`OUTPUT/818_Sunday_June_05_2022_01_50_30_AM_97126028/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 4.1, Higher order linear differential equations. General theory. page 173
Problem number: 16.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + 2y'' - y' - 3y = 0}$$

The characteristic equation is

$$\lambda^3 + 2\lambda^2 - \lambda - 3 = 0$$

The roots of the above equation are

$$\lambda_1 = \frac{\left(188 + 12\sqrt{93}\right)^{\frac{1}{3}}}{6} + \frac{14}{3\left(188 + 12\sqrt{93}\right)^{\frac{1}{3}}} - \frac{2}{3}$$

$$\lambda_2 = -\frac{\left(188 + 12\sqrt{93}\right)^{\frac{1}{3}}}{12} - \frac{7}{3\left(188 + 12\sqrt{93}\right)^{\frac{1}{3}}} - \frac{2}{3} + \frac{i\sqrt{3}\left(\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{6} - \frac{14}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}\right)}{2}$$

$$\lambda_3 = -\frac{\left(188 + 12\sqrt{93}\right)^{\frac{1}{3}}}{12} - \frac{7}{3\left(188 + 12\sqrt{93}\right)^{\frac{1}{3}}} - \frac{2}{3} - \frac{i\sqrt{3}\left(\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{6} - \frac{14}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}\right)}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{\left(-\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{12}-\frac{7}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}-\frac{2}{3}+\frac{i\sqrt{3}\left(\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{6}-\frac{14}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}\right)}{2}\right)x} c_1 + \mathrm{e}^{\left(-\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{12}-\frac{7}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}-\right.}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{\left(-\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{12}-\frac{7}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}-\frac{2}{3}+\frac{i\sqrt{3}\left(\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{6}-\frac{14}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}\right)}{2}\right)x}$$

$$y_2 = \mathrm{e}^{\left(-\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{12}-\frac{7}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}-\frac{2}{3}-\frac{i\sqrt{3}\left(\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{6}-\frac{14}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}\right)}{2}\right)x}$$

$$y_3 = \mathrm{e}^{\left(\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{6}+\frac{14}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}-\frac{2}{3}\right)x}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\left(-\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{12}-\frac{7}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}-\frac{2}{3}+\frac{i\sqrt{3}\left(\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{6}-\frac{14}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}\right)}{2}\right)x} c_1 + \mathrm{e}^{\left(-\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{12}-\frac{7}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}-\frac{2}{3}-\frac{i\sqrt{3}\left(\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{6}-\frac{14}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}\right)}{2}\right)x} c_2 + \mathrm{e}^{\left(\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{6}+\frac{14}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}-\frac{2}{3}\right)x} c_3 \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{\left(-\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{12}-\frac{7}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}-\frac{2}{3}+\frac{i\sqrt{3}\left(\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{6}-\frac{14}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}\right)}{2}\right)x} c_1 + \mathrm{e}^{\left(-\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{12}-\frac{7}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}-\frac{2}{3}-\frac{i\sqrt{3}\left(\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{6}-\frac{14}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}\right)}{2}\right)x} c_2 + \mathrm{e}^{\left(\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{6}+\frac{14}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}-\frac{2}{3}\right)x} c_3$$

Verified OK.

1.7.1 Maple step by step solution

Let's solve

$y''' + 2y'' - y' - 3y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -2y_3(x) + y_2(x) + 3y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -2y_3(x) + y_2(x) + 3y_1(x)]$

- Define vector

$$\overrightarrow{y}(x) = \left[\begin{array}{c} y_1(x) \\ y_2(x) \\ y_3(x) \end{array}\right]$$

- System to solve

$$\overrightarrow{y}'(x) = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 1 & -2 \end{array}\right] \cdot \overrightarrow{y}(x)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 1 & -2 \end{array}\right]$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{6} + \frac{14}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}} - \frac{2}{3}, \left[\begin{array}{c} \frac{1}{\left(\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{6} + \frac{14}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}} - \frac{2}{3}\right)^{2}} \\ \frac{1}{\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{6} + \frac{14}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}} - \frac{2}{3}} \\ 1 \end{array}\right]\right], \left[-\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{12} - \frac{\ldots}{3\ldots}\right.\right.$$

- Consider eigenpair

$$\left[\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{6}+\frac{14}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}-\frac{2}{3}, \left[\begin{array}{c} \frac{1}{\left(\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{6}+\frac{14}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}-\frac{2}{3}\right)^{2}} \\ \frac{1}{\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{6}+\frac{14}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}-\frac{2}{3}} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_{1} = \mathrm{e}^{\left(\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{6}+\frac{14}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}-\frac{2}{3}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{6}+\frac{14}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}-\frac{2}{3}\right)^{2}} \\ \frac{1}{\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{6}+\frac{14}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}-\frac{2}{3}} \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{12}-\frac{7}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}-\frac{2}{3}-\frac{\mathrm{I}\sqrt{3}\left(\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{6}-\frac{14}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}\right)}{2}, \left[\begin{array}{c} \frac{\cdots}{\left(-\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{12}-\frac{7}{3\left(188+12\cdots\right.} \cdots\right.} \\ \frac{\cdots}{-\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{12}-\frac{7}{3\left(188+12\cdots\right.} \cdots} \end{array}\right.\right.$$

- Solution from eigenpair

$$e^{\left(-\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{12}-\frac{7}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}-\frac{2}{3}-\frac{I\sqrt{3}\left(\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{6}-\frac{14}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}\right)}{2}\right)x} \cdot \left[\frac{\left(-\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{12}-\frac{7}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}-\right.}{-\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{12}-\frac{7}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}-}\right.$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{\left(-\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{12}-\frac{7}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}-\frac{2}{3}\right)x} \cdot \left(\cos\left(\frac{\sqrt{3}\left(\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{6}-\frac{14}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}\right)x}{2}\right)-I\sin\left(\frac{\sqrt{3}\left(\frac{\left(188+12\sqrt{9\right.}}{6}\right.}{}\right.\right.$$

- Simplify expression

$$
e^{\left(-\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{12}-\frac{7}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}-\frac{2}{3}\right)x}\cdot\left[\begin{array}{c}
\frac{\cos\left(\frac{\sqrt{3}\left(\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{6}-\frac{14}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}\right)x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\left(\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{6}\cdots\right.}{\cdots}\right.}{\left(-\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{12}-\frac{7}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}-\frac{2}{3}-\frac{\mathrm{I}\sqrt{3}\left(\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{6}-\frac{\cdots}{3\left(18\cdots\right.}\right.}{2}\cdots\right.} \\
\frac{\cos\left(\frac{\sqrt{3}\left(\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{6}-\frac{14}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}\right)x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\left(\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{6}\cdots\right.}{\cdots}\right.}{-\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{12}-\frac{7}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}-\frac{2}{3}-\frac{\mathrm{I}\sqrt{3}\left(\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{6}-\frac{\cdots}{3\left(1\cdots\right.}\right.}{2}\cdots} \\
\cos\left(\frac{\sqrt{3}\left(\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{6}-\frac{14}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}\right)x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\left(\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}\cdots}{6}\right.}{\cdots}\right.
\end{array}\right.
$$

- Both real and imaginary parts are solutions to the homogeneous system

$$
\vec{y}_2(x)=e^{\left(-\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{12}-\frac{7}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}-\frac{2}{3}\right)x}\cdot\left[\begin{array}{c}
-\frac{18\left(188+12\sqrt{93}\right)^{\frac{2}{3}}\left(\left(188+12\sqrt{93}\right)^{\frac{4}{3}}\sqrt{3}\sin\left(\frac{\sqrt{3}\left(\left(188+12\sqrt{93}\right)\cdots\right.}{12\left(188+12\sqrt{93}\cdots\right.}\right.\right.}{\cdots} \\
\\
\\
\end{array}\right.
$$

- General solution to the system of ODEs

$$\vec{y}=c_1\vec{y}_1+c_2\vec{y}_2(x)+c_3\vec{y}_3(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 \mathrm{e}^{\left(\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{6}+\frac{14}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}-\frac{2}{3}\right)x} \cdot \begin{bmatrix} \frac{1}{\left(\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{6}+\frac{14}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}-\frac{2}{3}\right)^{2}} \\ \frac{1}{\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{6}+\frac{14}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}-\frac{2}{3}} \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{\left(-\frac{\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}{12}-\cdots\right.}$$

- First component of the vector is the solution to the ODE

$$y = \frac{13\left(c_1\left(\frac{40}{3}+\frac{7\left(\sqrt{3}\,\sqrt{31}+11\right)\left(188+12\sqrt{3}\,\sqrt{31}\right)^{\frac{2}{3}}}{78}+\left(\frac{47}{3}+\sqrt{3}\,\sqrt{31}\right)\left(188+12\sqrt{3}\,\sqrt{31}\right)^{\frac{1}{3}}+\frac{4\sqrt{3}\,\sqrt{31}}{39}\right)\mathrm{e}^{-\frac{2x\left(-\frac{\left(188+12\sqrt{93}\right)^{\frac{2}{3}}}{4}+\left(188+12\cdots\right.\right.}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}}\cdots\right.}{\cdots}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 183

```
dsolve(diff(y(x),x$3)+2*diff(y(x),x$2)-diff(y(x),x)-3*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-\frac{2x\left(-\frac{\left(188+12\sqrt{93}\right)^{\frac{2}{3}}}{4}+\left(188+12\sqrt{93}\right)^{\frac{1}{3}}-7\right)}{3\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}} - c_2 e^{-\frac{\left(28+\left(188+12\sqrt{93}\right)^{\frac{2}{3}}+8\left(188+12\sqrt{93}\right)^{\frac{1}{3}}\right)x}{12\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}} \sin\left(\frac{\sqrt{3}\left(\left(188+12\sqrt{3}\sqrt{31}\right)^{\frac{2}{3}}-28\right)x}{12\left(188+12\sqrt{3}\sqrt{31}\right)^{\frac{1}{3}}}\right) + c_3 e^{-\frac{\left(28+\left(188+12\sqrt{93}\right)^{\frac{2}{3}}+8\left(188+12\sqrt{93}\right)^{\frac{1}{3}}\right)x}{12\left(188+12\sqrt{93}\right)^{\frac{1}{3}}}} \cos\left(\frac{\sqrt{3}\left(\left(188+12\sqrt{3}\sqrt{31}\right)^{\frac{2}{3}}-28\right)x}{12\left(188+12\sqrt{3}\sqrt{31}\right)^{\frac{1}{3}}}\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 87

```
DSolve[y'''[x]+2*y''[x]-y'[x]-3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 \exp\left(x\text{Root}\left[\#1^3+2\#1^2-\#1-3\&,2\right]\right) + c_3 \exp\left(x\text{Root}\left[\#1^3+2\#1^2-\#1-3\&,3\right]\right) + c_1 \exp\left(x\text{Root}\left[\#1^3+2\#1^2-\#1-3\&,1\right]\right)$$

1.8 problem 17

Internal problem ID [819]
Internal file name [OUTPUT/819_Sunday_June_05_2022_01_50_31_AM_27651476/index.tex]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 4.1, Higher order linear differential equations. General theory. page 173
Problem number: 17.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{ty''' + 2y'' - y' + yt = 0}$$

Unable to solve this ODE.

1.8.1 Maple step by step solution

Let's solve

$ty''' + 2y'' - y' + yt = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Check to see if $t_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(t) = \frac{2}{t}, P_3(t) = -\frac{1}{t}, P_4(t) = 1\right]$

  ◦ $t \cdot P_2(t)$ is analytic at $t = 0$

    $(t \cdot P_2(t))\Big|_{t=0} = 2$

- $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $$\left(t^2 \cdot P_3(t)\right)\bigg|_{t=0} = 0$$

- $t^3 \cdot P_4(t)$ is analytic at $t = 0$

  $$\left(t^3 \cdot P_4(t)\right)\bigg|_{t=0} = 0$$

- $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k t^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $t \cdot y$ to series expansion

  $$t \cdot y = \sum_{k=0}^{\infty} a_k t^{k+r+1}$$

- Shift index using $k- >k - 1$

  $$t \cdot y = \sum_{k=1}^{\infty} a_{k-1} t^{k+r}$$

- Convert y' to series expansion

  $$y' = \sum_{k=0}^{\infty} a_k (k+r) \, t^{k+r-1}$$

- Shift index using $k- >k + 1$

  $$y' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1) \, t^{k+r}$$

- Convert y'' to series expansion

  $$y'' = \sum_{k=0}^{\infty} a_k (k+r) \, (k+r-1) \, t^{k+r-2}$$

- Shift index using $k- >k + 2$

  $$y'' = \sum_{k=-2}^{\infty} a_{k+2} (k+2+r) \, (k+r+1) \, t^{k+r}$$

- Convert $t \cdot y'''$ to series expansion

$$t \cdot y''' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)(k+r-2)\, t^{k+r-2}$$

- Shift index using $k->k+2$

$$t \cdot y''' = \sum_{k=-2}^{\infty} a_{k+2}(k+2+r)(k+r+1)(k+r)\, t^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2(-1+r)\, t^{-2+r} + \left(a_1(1+r)^2 r - a_0 r\right) t^{-1+r} + \left(a_2(2+r)^2 (1+r) - a_1(1+r)\right) t^r + \left(\sum_{k=1}^{\infty} (a_{k-}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r^2(-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 1\}$$

- The coefficients of each power of t must be 0

$$\left[a_1(1+r)^2 r - a_0 r = 0, a_2(2+r)^2 (1+r) - a_1(1+r) = 0\right]$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+2}(k+2+r)^2 (k+r+1) + (-k-r-1)\, a_{k+1} + a_{k-1} = 0$$

- Shift index using $k->k+1$

$$a_{k+3}(k+3+r)^2 (k+2+r) + (-k-2-r)\, a_{k+2} + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+3} = \frac{k a_{k+2} + r a_{k+2} - a_k + 2a_{k+2}}{(k+3+r)^2 (k+2+r)}$$

- Recursion relation for $r = 0$

$$a_{k+3} = \frac{k a_{k+2} - a_k + 2a_{k+2}}{(k+3)^2 (k+2)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k t^k, a_{k+3} = \frac{k a_{k+2} - a_k + 2a_{k+2}}{(k+3)^2 (k+2)}, 0 = 0, 4a_2 - a_1 = 0\right]$$

- Recursion relation for $r = 1$

$$a_{k+3} = \frac{k a_{k+2} - a_k + 3a_{k+2}}{(k+4)^2 (k+3)}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+3} = \frac{k a_{k+2} - a_k + 3a_{k+2}}{(k+4)^2 (k+3)}, 4a_1 - a_0 = 0, 18a_2 - 2a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k t^k\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+1}\right), a_{k+3} = \frac{k a_{k+2} - a_k + 2a_{k+2}}{(k+3)^2 (k+2)}, 0 = 0, 4a_2 - a_1 = 0, b_{k+3} = \frac{k b_{k+2} - b_k + 3b_{k\cdot}}{(k+4)^2 (k+3)}\right.$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying high order exact linear fully integrable
trying to convert to a linear ODE with constant coefficients
trying differential order: 3; missing the dependent variable
trying Louvillian solutions for 3rd order ODEs, imprimitive case
-> pFq: Equivalence to the 3F2 or one of its 3 confluent cases under a power @ Moebius
trying a solution in terms of MeijerG functions
-> pFq: Equivalence to the 3F2 or one of its 3 confluent cases under a power @ Moebius
trying a solution in terms of MeijerG functions
   checking if the LODE is of Euler type
   Calling dsolve with: (t-1)/t*y(t)-(t-1)/t*diff(y(t),t)+diff(diff(y(t),t),t) = 0
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Kummer successful
   <- special function solution successful
<- differential factorization successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 159

```
dsolve(t*diff(y(t),t$3)+2*diff(y(t),t$2)-diff(y(t),t)+t*y(t)=0,y(t), singsol=all)
```

$$y(t) = \mathrm{e}^{-\frac{t(i\sqrt{3}-1)}{2}} \left(\mathrm{KummerM}\left(\frac{1}{2} - \frac{i\sqrt{3}}{6}, 1, i\sqrt{3}\,t\right) \left(\int \mathrm{KummerU}\left(\frac{1}{2} - \frac{i\sqrt{3}}{6}, 1, i\sqrt{3}\,t\right) \mathrm{e}^{-\frac{t(i\sqrt{3}+3)}{2}} dt\right) c_3 - \mathrm{KummerU}\left(\frac{1}{2} - \frac{i\sqrt{3}}{6}, 1, i\sqrt{3}\,t\right) \left(\int \mathrm{KummerM}\left(\frac{1}{2} - \frac{i\sqrt{3}}{6}, 1, i\sqrt{3}\,t\right) \mathrm{e}^{-\frac{t(i\sqrt{3}+3)}{2}} dt\right) c_3 + c_1 \mathrm{KummerM}\left(\frac{1}{2} - \frac{i\sqrt{3}}{6}, 1, i\sqrt{3}\,t\right) + c_2 \mathrm{KummerU}\left(\frac{1}{2} - \frac{i\sqrt{3}}{6}, 1, i\sqrt{3}\,t\right)\right)$$

✓ Solution by Mathematica

Time used: 0.639 (sec). Leaf size: 520

```
DSolve[t*y'''[t]+2*y''[t]-y'[t]+t*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{\frac{1}{2}\left(t-i\sqrt{3}t\right)} \left(c_3 \,\text{HypergeometricU}\left(\frac{1}{6}\left(3-i\sqrt{3}\right), 1, i\sqrt{3}t\right) \int_1^t \frac{}{\left(-1-i\sqrt{3}\right) K[1] \left(\text{Hypergeometric1F1}\left(\frac{1}{6}\right.\right.} + c_3 \,\text{LaguerreL}\left(\frac{1}{6}i\left(3i+\sqrt{3}\right), i\sqrt{3}t\right) \int_1^t -\frac{2ie^{\frac{1}{2}i\left(3i+\sqrt{3}\right)K[2]} \text{Hypergeometric}}{\left(-i+\sqrt{3}\right) K[2] \left(\text{Hypergeometric1F1}\left(\frac{1}{6}\left(9-i\sqrt{3}\right), 2, i\sqrt{3}K[2]\right) \text{HypergeometricU}\left(\frac{1}{6}\left(3-i\sqrt{3}\right), 1,\right.\right.} + c_1 \,\text{HypergeometricU}\left(\frac{1}{6}\left(3-i\sqrt{3}\right), 1, i\sqrt{3}t\right) + c_2 \,\text{LaguerreL}\left(\frac{1}{6}i\left(3i+\sqrt{3}\right), i\sqrt{3}t\right)\right)$$

1.9 problem 20

Internal problem ID [820]
Internal file name [`OUTPUT/820_Sunday_June_05_2022_01_50_33_AM_70836635/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 4.1, Higher order linear differential equations. General theory. page 173
Problem number: 20.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{(-t+2)\,y''' + (-3+2t)\,y'' - ty' + y = 0}$$

Unable to solve this ODE.

1.9.1 Maple step by step solution

Let's solve

$(-t+2)\,y''' + (-3+2t)\,y'' - ty' + y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Check to see if $t_0 = 2$ is a regular singular point

  ◦ Define functions

    $\left[P_2(t) = -\frac{-3+2t}{t-2}, P_3(t) = \frac{t}{t-2}, P_4(t) = -\frac{1}{t-2}\right]$

  ◦ $(t-2)\cdot P_2(t)$ is analytic at $t = 2$

    $\left.\left((t-2)\cdot P_2(t)\right)\right|_{t=2} = -1$

- $(t-2)^2 \cdot P_3(t)$ is analytic at $t=2$

  $$\left((t-2)^2 \cdot P_3(t)\right)\bigg|_{t=2} = 0$$

- $(t-2)^3 \cdot P_4(t)$ is analytic at $t=2$

  $$\left((t-2)^3 \cdot P_4(t)\right)\bigg|_{t=2} = 0$$

- $t=2$is a regular singular point

  Check to see if $t_0 = 2$ is a regular singular point

  $t_0 = 2$

- Multiply by denominators

  $$(t-2)\, y''' + (-2t+3)\, y'' + ty' - y = 0$$

- Change variables using $t = u+2$ so that the regular singular point is at $u = 0$

  $$u\left(\frac{d^3}{du^3}y(u)\right) + (-2u-1)\left(\frac{d^2}{du^2}y(u)\right) + (u+2)\left(\frac{d}{du}y(u)\right) - y(u) = 0$$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- > k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 0..1$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- > k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

- Convert $u \cdot \left(\frac{d^3}{du^3}y(u)\right)$ to series expansion

  $$u \cdot \left(\frac{d^3}{du^3}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,(k+r-2)\, u^{k+r-2}$$

- Shift index using $k->k+2$

  $u\cdot\left(\frac{d^3}{du^3}y(u)\right)=\sum_{k=-2}^{\infty}a_{k+2}(k+2+r)(k+1+r)(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-1+r)(-3+r)\,u^{-2+r}+(a_1(1+r)\,r(-2+r)-2a_0r(-2+r))\,u^{-1+r}+\left(\sum_{k=0}^{\infty}(a_{k+2}(k+2+\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+r)(-3+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{0,1,3\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)(a_{k+2}(k+2+r)(k+1+r)-2a_{k+1}k-2a_{k+1}r+a_k-2a_{k+1})=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{2a_{k+1}k+2a_{k+1}r-a_k+2a_{k+1}}{(k+2+r)(k+1+r)}$

- Recursion relation for $r=0$

  $a_{k+2}=\frac{2a_{k+1}k-a_k+2a_{k+1}}{(k+2)(k+1)}$

- Solution for $r=0$

  $\left[y(u)=\sum_{k=0}^{\infty}a_ku^k,a_{k+2}=\frac{2a_{k+1}k-a_k+2a_{k+1}}{(k+2)(k+1)},0=0\right]$

- Revert the change of variables $u=t-2$

  $\left[y=\sum_{k=0}^{\infty}a_k(t-2)^k,a_{k+2}=\frac{2a_{k+1}k-a_k+2a_{k+1}}{(k+2)(k+1)},0=0\right]$

- Recursion relation for $r=1$

  $a_{k+2}=\frac{2a_{k+1}k-a_k+4a_{k+1}}{(k+3)(k+2)}$

- Solution for $r=1$

  $\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k+1},a_{k+2}=\frac{2a_{k+1}k-a_k+4a_{k+1}}{(k+3)(k+2)},-2a_1+2a_0=0\right]$

- Revert the change of variables $u=t-2$

  $\left[y=\sum_{k=0}^{\infty}a_k(t-2)^{k+1},a_{k+2}=\frac{2a_{k+1}k-a_k+4a_{k+1}}{(k+3)(k+2)},-2a_1+2a_0=0\right]$

- Recursion relation for $r=3$

$a_{k+2} = \frac{2a_{k+1}k - a_k + 8a_{k+1}}{(k+5)(k+4)}$

- Solution for $r = 3$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+3}, a_{k+2} = \frac{2a_{k+1}k - a_k + 8a_{k+1}}{(k+5)(k+4)}, 12a_1 - 6a_0 = 0\right]$$

- Revert the change of variables $u = t - 2$

$$\left[y = \sum_{k=0}^{\infty} a_k (t-2)^{k+3}, a_{k+2} = \frac{2a_{k+1}k - a_k + 8a_{k+1}}{(k+5)(k+4)}, 12a_1 - 6a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (t-2)^k\right) + \left(\sum_{k=0}^{\infty} b_k (t-2)^{k+1}\right) + \left(\sum_{k=0}^{\infty} c_k (t-2)^{k+3}\right), a_{k+2} = \frac{2ka_{k+1} - a_k + 2a_{k+1}}{(k+2)(k+1)}, 0 = 0\right.$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying high order exact linear fully integrable
trying to convert to a linear ODE with constant coefficients
trying differential order: 3; missing the dependent variable
Equation is the LCLM of -1/t*y(t)+diff(y(t),t), -y(t)+diff(y(t),t), (-1-1/t)*y(t)+diff(y(t),
trying differential order: 1; missing the dependent variable
checking if the LODE is of Euler type
<- LODE of Euler type successful
Euler equation successful
trying differential order: 1; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful
trying differential order: 1; missing the dependent variable
checking if the LODE is of Euler type
   checking if the LODE is of Euler type
   exponential solutions successful
<- differential factorization successful
<- solving the LCLM ode successful `
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve([(2-t)*diff(y(t),t$3)+(2*t-3)*diff(y(t),t$2)-t*diff(y(t),t)+y(t)=0,exp(t)],singsol=al
```

$$y(t) = e^{t}(c_3 t + c_2) + c_1 t$$

✓ Solution by Mathematica

Time used: 0.079 (sec). Leaf size: 28

```
DSolve[(2-t)*y'''[t]+(2*t-3)*y''[t]-t*y'[t]+y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to t(c_2 e^t + c_1) + (c_3 - 4c_2)e^t$$

1.10 problem 21

Internal problem ID [821]
Internal file name [OUTPUT/821_Sunday_June_05_2022_01_50_34_AM_48811924/index.tex]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 4.1, Higher order linear differential equations. General theory. page 173
Problem number: 21.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_3rd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{t^2(t+3)\,y''' - 3t(2+t)\,y'' + 6(t+1)\,y' - 6y = 0}$$

Unable to solve this ODE.

1.10.1 Maple step by step solution

Let's solve

$t^2(t+3)\,y''' - 3t(2+t)\,y'' + 6(t+1)\,y' - 6y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Check to see if t_0 is a regular singular point
  - Define functions

    $\left[P_2(t) = -\frac{3(2+t)}{t(t+3)}, P_3(t) = \frac{6(t+1)}{t^2(t+3)}, P_4(t) = -\frac{6}{t^2(t+3)}\right]$
  - $(t+3)\cdot P_2(t)$ is analytic at $t=-3$

$\left((t+3)\cdot P_2(t)\right)\Big|_{t=-3} = -1$

- $(t+3)^2 \cdot P_3(t)$ is analytic at $t=-3$

  $\left((t+3)^2\cdot P_3(t)\right)\Big|_{t=-3} = 0$

- $(t+3)^3 \cdot P_4(t)$ is analytic at $t=-3$

  $\left((t+3)^3\cdot P_4(t)\right)\Big|_{t=-3} = 0$

- $t=-3$is a regular singular point

  Check to see if t_0 is a regular singular point

  $t_0 = -3$

- Multiply by denominators

  $-6y + (6t+6)\,y' + t^2(t+3)\,y''' - 3t(2+t)\,y'' = 0$

- Change variables using $t = u-3$ so that the regular singular point is at $u=0$

  $(u^3-6u^2+9u)\left(\frac{d^3}{du^3}y(u)\right) + (-3u^2+12u-9)\left(\frac{d^2}{du^2}y(u)\right) + (6u-12)\left(\frac{d}{du}y(u)\right) - 6y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 0..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^3}{du^3}y(u)\right)$ to series expansion for $m = 1..3$

  $u^m \cdot \left(\frac{d^3}{du^3}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)(k+r-2)\,u^{k+r-3+m}$

- Shift index using $k- >k+3-m$

  $u^m \cdot \left(\frac{d^3}{du^3}y(u)\right) = \sum_{k=-3+m}^{\infty} a_{k+3-m}(k+3-m+r)(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $9a_0r(-1+r)(-3+r)\,u^{-2+r} + (9a_1(1+r)\,r(-2+r) - 6a_0r(-2+r)(-3+r))\,u^{-1+r} + \left(\sum_{k=0}^{\infty} (9\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $9r(-1+r)(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 1, 3\}$

- Each term in the series must be 0, giving the recursion relation

  $((a_k - 6a_{k+1} + 9a_{k+2})\,k^2 + (2(a_k - 6a_{k+1} + 9a_{k+2})\,r - 5a_k + 6a_{k+1} + 27a_{k+2})\,k + (a_k - 6a_{k+1} +$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{k^2a_k - 6k^2a_{k+1} + 2kra_k - 12kra_{k+1} + r^2a_k - 6r^2a_{k+1} - 5ka_k + 6ka_{k+1} - 5ra_k + 6ra_{k+1} + 6a_k + 12a_{k+1}}{9(k^2+2kr+r^2+3k+3r+2)}$

- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+2} = -\frac{k^2a_k - 6k^2a_{k+1} - 5ka_k + 6ka_{k+1} + 6a_k + 12a_{k+1}}{9(k^2+3k+2)}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2a_k - 6k^2a_{k+1} - 5ka_k + 6ka_{k+1} + 6a_k + 12a_{k+1}}{9(k^2+3k+2)}, 0 = 0\right]$

- Revert the change of variables $u = t + 3$

  $\left[y = \sum_{k=0}^{\infty} a_k(t+3)^k, a_{k+2} = -\frac{k^2a_k - 6k^2a_{k+1} - 5ka_k + 6ka_{k+1} + 6a_k + 12a_{k+1}}{9(k^2+3k+2)}, 0 = 0\right]$

- Recursion relation for $r = 1$; series terminates at $k = 1$

  $a_{k+2} = -\frac{k^2a_k - 6k^2a_{k+1} - 3ka_k - 6ka_{k+1} + 2a_k + 12a_{k+1}}{9(k^2+5k+6)}$

- Solution for $r = 1$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+1}, a_{k+2} = -\frac{k^2a_k - 6k^2a_{k+1} - 3ka_k - 6ka_{k+1} + 2a_k + 12a_{k+1}}{9(k^2+5k+6)}, -18a_1 - 12a_0 = 0\right]$

- Revert the change of variables $u = t + 3$

$$\left[y=\sum_{k=0}^{\infty} a_k (t+3)^{k+1}, a_{k+2}=-\frac{k^2 a_k-6k^2 a_{k+1}-3k a_k-6k a_{k+1}+2a_k+12a_{k+1}}{9(k^2+5k+6)}, -18a_1-12a_0=0\right]$$

- Recursion relation for $r=3$

$$a_{k+2}=-\frac{k^2 a_k-6k^2 a_{k+1}+k a_k-30k a_{k+1}-24a_{k+1}}{9(k^2+9k+20)}$$

- Solution for $r=3$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+3}, a_{k+2}=-\frac{k^2 a_k-6k^2 a_{k+1}+k a_k-30k a_{k+1}-24a_{k+1}}{9(k^2+9k+20)}, 108a_1=0\right]$$

- Revert the change of variables $u=t+3$

$$\left[y=\sum_{k=0}^{\infty} a_k (t+3)^{k+3}, a_{k+2}=-\frac{k^2 a_k-6k^2 a_{k+1}+k a_k-30k a_{k+1}-24a_{k+1}}{9(k^2+9k+20)}, 108a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k (t+3)^{k}\right)+\left(\sum_{k=0}^{\infty} b_k (t+3)^{k+1}\right)+\left(\sum_{k=0}^{\infty} c_k (t+3)^{k+3}\right), a_{k+2}=-\frac{k^2 a_k-6k^2 a_{k+1}-5k a_k+6k}{9(k^2+3k}\right.$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying high order exact linear fully integrable
trying to convert to a linear ODE with constant coefficients
trying differential order: 3; missing the dependent variable
Equation is the LCLM of -1/(t+1)*y(t)+diff(y(t),t), -2/t*y(t)+diff(y(t),t), -3/t*y(t)+diff(y
trying differential order: 1; missing the dependent variable
checking if the LODE is of Euler type
<- LODE of Euler type successful
Euler equation successful
trying differential order: 1; missing the dependent variable
checking if the LODE is of Euler type
<- LODE of Euler type successful
Euler equation successful
trying differential order: 1; missing the dependent variable
checking if the LODE is of Euler type
<- LODE of Euler type successful
Euler equation successful
<- solving the LCLM ode successful `
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 19

```
dsolve([t^2*(t+3)*diff(y(t),t$3)-3*t*(t+2)*diff(y(t),t$2)+6*(1+t)*diff(y(t),t)-6*y(t)=0,[t^2
```

$$y(t) = c_2t^3 + c_1t^2 + c_3t + c_3$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 58

```
DSolve[t^2*(t+3)*y'''[t]-3*t*(t+2)*y''[t]+6*(1+t)*y'[t]-6*y[t]==0,y[t],t,IncludeSingularSolu
```

$$y(t) \to \frac{1}{8}\left(2c_1\left(t^3 - 3t^2 + 3t + 3\right) - (t-1)\left(4c_2\left(t^2 - 2t - 1\right) + c_3\left(-3t^2 + 2t + 1\right)\right)\right)$$

2 Chapter 4.2, Higher order linear differential equations. Constant coefficients. page 180

2.1 problem 8

Internal problem ID [822]
Internal file name [OUTPUT/822_Sunday_June_05_2022_01_50_35_AM_13639714/index.tex]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 4.2, Higher order linear differential equations. Constant coefficients. page 180
Problem number: 8.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - y'' - y' + y = 0}$$

The characteristic equation is

$$\lambda^3 - \lambda^2 - \lambda + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + x\,\mathrm{e}^{x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = \mathrm{e}^{x}$$
$$y_3 = x\,\mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{x} + x e^{x} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{x} + x e^{x} c_3$$

Verified OK.

2.1.1 Maple step by step solution

Let's solve

$y''' - y'' - y' + y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = y_3(x) + y_2(x) - y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = y_3(x) + y_2(x) - y_1(x)]$

- Define vector

  $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \cdot \overrightarrow{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 1

$$\vec{y}_2(x) = \mathrm{e}^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 1$ is the eigenvalue, an

  $$\vec{y}_3(x) = \mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right)$$

- Note that the x multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\vec{y}_3(x)$ into the homogeneous system

  $$\lambda\,\mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda x}\vec{v} = \left(\mathrm{e}^{\lambda x}A\right) \cdot \left(x\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

  $$\lambda\,\mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda x}\vec{v} = \mathrm{e}^{\lambda x}\left(\lambda x\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

  $$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

  $$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{y}_3(x)$ to be a solution to the homogeneous system

  $$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 1

  $$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

  $$\vec{p} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 1

$$\vec{y}_3(x) = e^x \cdot \left(x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}\right)$$

- General solution to the system of ODEs

  $\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2(x) + c_3 \vec{y}_3(x)$
- Substitute solutions into the general solution

  $$\vec{y} = c_1 e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 e^x \cdot \left(x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}\right)$$
- First component of the vector is the solution to the ODE

  $y = c_1 e^{-x} + ((x-1) c_3 + c_2) e^x$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$3)-diff(y(x),x$2)-diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-x} c_1 + (c_3 x + c_2) e^x$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 25

```
DSolve[y'''[x]-y''[x]-y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-x} + e^x (c_3 x + c_2)$$

2.2 problem 9

Internal problem ID [823]
Internal file name [OUTPUT/823_Sunday_June_05_2022_01_50_35_AM_63213515/index.tex]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 4.2, Higher order linear differential equations. Constant coefficients. page 180
Problem number: 9.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 3y'' + 3y' + y = 0}$$

The characteristic equation is

$$\lambda^3 - 3\lambda^2 + 3\lambda + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = -2^{\frac{1}{3}} + 1$$
$$\lambda_2 = \frac{2^{\frac{1}{3}}}{2} - \frac{i\sqrt{3}\,2^{\frac{1}{3}}}{2} + 1$$
$$\lambda_3 = \frac{2^{\frac{1}{3}}}{2} + \frac{i\sqrt{3}\,2^{\frac{1}{3}}}{2} + 1$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{\left(\frac{2^{\frac{1}{3}}}{2} + \frac{i\sqrt{3}\,2^{\frac{1}{3}}}{2} + 1\right)x} c_1 + \mathrm{e}^{\left(\frac{2^{\frac{1}{3}}}{2} - \frac{i\sqrt{3}\,2^{\frac{1}{3}}}{2} + 1\right)x} c_2 + \mathrm{e}^{\left(-2^{\frac{1}{3}}+1\right)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{\left(\frac{2^{\frac{1}{3}}}{2}+\frac{i\sqrt{3}\,2^{\frac{1}{3}}}{2}+1\right)x} \\
y_2 &= \mathrm{e}^{\left(\frac{2^{\frac{1}{3}}}{2}-\frac{i\sqrt{3}\,2^{\frac{1}{3}}}{2}+1\right)x} \\
y_3 &= \mathrm{e}^{\left(-2^{\frac{1}{3}}+1\right)x}
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \mathrm{e}^{\left(\frac{2^{\frac{1}{3}}}{2}+\frac{i\sqrt{3}\,2^{\frac{1}{3}}}{2}+1\right)x} c_1 + \mathrm{e}^{\left(\frac{2^{\frac{1}{3}}}{2}-\frac{i\sqrt{3}\,2^{\frac{1}{3}}}{2}+1\right)x} c_2 + \mathrm{e}^{\left(-2^{\frac{1}{3}}+1\right)x} c_3 \tag{1}
$$

<u>Verification of solutions</u>

$$
y = \mathrm{e}^{\left(\frac{2^{\frac{1}{3}}}{2}+\frac{i\sqrt{3}\,2^{\frac{1}{3}}}{2}+1\right)x} c_1 + \mathrm{e}^{\left(\frac{2^{\frac{1}{3}}}{2}-\frac{i\sqrt{3}\,2^{\frac{1}{3}}}{2}+1\right)x} c_2 + \mathrm{e}^{\left(-2^{\frac{1}{3}}+1\right)x} c_3
$$

Verified OK.

2.2.1 Maple step by step solution

Let's solve

$y''' - 3y'' + 3y' + y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 3y_3(x) - 3y_2(x) - y_1(x)$

  Convert linear ODE into a system of first order ODEs

$$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 3y_3(x) - 3y_2(x) - y_1(x)]$$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & 3 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & 3 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2^{\frac{1}{3}}+1, \begin{bmatrix} \frac{1}{\left(-2^{\frac{1}{3}}+1\right)^2} \\ \frac{1}{-2^{\frac{1}{3}}+1} \\ 1 \end{bmatrix}\right], \left[\frac{2^{\frac{1}{3}}}{2} - \frac{I\sqrt{3}\,2^{\frac{1}{3}}}{2} + 1, \begin{bmatrix} \frac{1}{\left(\frac{2^{\frac{1}{3}}}{2} - \frac{I\sqrt{3}\,2^{\frac{1}{3}}}{2} + 1\right)^2} \\ \frac{1}{\frac{2^{\frac{1}{3}}}{2} - \frac{I\sqrt{3}\,2^{\frac{1}{3}}}{2} + 1} \\ 1 \end{bmatrix}\right], \left[\frac{2^{\frac{1}{3}}}{2} + \frac{I\sqrt{3}\,2^{\frac{1}{3}}}{2} + 1, \begin{bmatrix} \frac{1}{\left(\frac{2^{\frac{1}{3}}}{2} \right.} \\ \frac{1}{\frac{2^{\frac{1}{3}}}{2}} \\ \end{bmatrix}\right.\right.$$

- Consider eigenpair

$$\left[-2^{\frac{1}{3}}+1, \begin{bmatrix} \frac{1}{\left(-2^{\frac{1}{3}}+1\right)^2} \\ \frac{1}{-2^{\frac{1}{3}}+1} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{\left(-2^{\frac{1}{3}}+1\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(-2^{\frac{1}{3}}+1\right)^2} \\ \frac{1}{-2^{\frac{1}{3}}+1} \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[\frac{2^{\frac{1}{3}}}{2} - \frac{\mathrm{I}\sqrt{3}\,2^{\frac{1}{3}}}{2} + 1, \left[\begin{array}{c} \frac{1}{\left(\frac{2^{\frac{1}{3}}}{2} - \frac{\mathrm{I}\sqrt{3}\,2^{\frac{1}{3}}}{2}+1\right)^2} \\ \frac{1}{\frac{2^{\frac{1}{3}}}{2} - \frac{\mathrm{I}\sqrt{3}\,2^{\frac{1}{3}}}{2}+1} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{\left(\frac{2^{\frac{1}{3}}}{2} - \frac{\mathrm{I}\sqrt{3}\,2^{\frac{1}{3}}}{2}+1\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(\frac{2^{\frac{1}{3}}}{2} - \frac{\mathrm{I}\sqrt{3}\,2^{\frac{1}{3}}}{2}+1\right)^2} \\ \frac{1}{\frac{2^{\frac{1}{3}}}{2} - \frac{\mathrm{I}\sqrt{3}\,2^{\frac{1}{3}}}{2}+1} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{\left(\frac{2^{\frac{1}{3}}}{2}+1\right)x} \cdot \left(\cos\left(\frac{2^{\frac{1}{3}}\sqrt{3}\,x}{2}\right) - \mathrm{I}\sin\left(\frac{2^{\frac{1}{3}}\sqrt{3}\,x}{2}\right)\right) \cdot \left[\begin{array}{c} \frac{1}{\left(\frac{2^{\frac{1}{3}}}{2} - \frac{\mathrm{I}\sqrt{3}\,2^{\frac{1}{3}}}{2}+1\right)^2} \\ \frac{1}{\frac{2^{\frac{1}{3}}}{2} - \frac{\mathrm{I}\sqrt{3}\,2^{\frac{1}{3}}}{2}+1} \\ 1 \end{array}\right]$$

- Simplify expression

$$\mathrm{e}^{\left(\frac{2^{\frac{1}{3}}}{2}+1\right)x} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{2^{\frac{1}{3}}\sqrt{3}\,x}{2}\right) - \mathrm{I}\sin\left(\frac{2^{\frac{1}{3}}\sqrt{3}\,x}{2}\right)}{\left(\frac{2^{\frac{1}{3}}}{2} - \frac{\mathrm{I}\sqrt{3}\,2^{\frac{1}{3}}}{2}+1\right)^2} \\ \frac{\cos\left(\frac{2^{\frac{1}{3}}\sqrt{3}\,x}{2}\right) - \mathrm{I}\sin\left(\frac{2^{\frac{1}{3}}\sqrt{3}\,x}{2}\right)}{\frac{2^{\frac{1}{3}}}{2} - \frac{\mathrm{I}\sqrt{3}\,2^{\frac{1}{3}}}{2}+1} \\ \cos\left(\frac{2^{\frac{1}{3}}\sqrt{3}\,x}{2}\right) - \mathrm{I}\sin\left(\frac{2^{\frac{1}{3}}\sqrt{3}\,x}{2}\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\vec{y}_2(x) = e^{\left(\frac{2^{\frac{1}{3}}}{2}+1\right)x} \cdot \left[\begin{array}{c} -\frac{-\sin\left(\frac{2^{\frac{1}{3}}\sqrt{3}\,x}{2}\right)2^{\frac{2}{3}}\sqrt{3}+2^{\frac{2}{3}}\cos\left(\frac{2^{\frac{1}{3}}\sqrt{3}\,x}{2}\right)-2\sin\left(\frac{2^{\frac{1}{3}}\sqrt{3}\,x}{2}\right)2^{\frac{1}{3}}\sqrt{3}-2\,2^{\frac{1}{3}}\cos\left(\frac{2^{\frac{1}{3}}\sqrt{3}\,x}{2}\right)-2\cos\left(\cdots\right.}{2\left(2^{\frac{2}{3}}+2^{\frac{1}{3}}+1\right)^2} \\ \frac{\sin\left(\frac{2^{\frac{1}{3}}\sqrt{3}\,x}{2}\right)2^{\frac{1}{3}}\sqrt{3}+2^{\frac{1}{3}}\cos\left(\frac{2^{\frac{1}{3}}\sqrt{3}\,x}{2}\right)+2\cos\left(\frac{2^{\frac{1}{3}}\sqrt{3}\,x}{2}\right)}{2\left(2^{\frac{2}{3}}+2^{\frac{1}{3}}+1\right)} \\ \cos\left(\frac{2^{\frac{1}{3}}\sqrt{3}\,x}{2}\right) \end{array}\right.$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{\left(-2^{\frac{1}{3}}+1\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(-2^{\frac{1}{3}}+1\right)^2} \\ \frac{1}{-2^{\frac{1}{3}}+1} \\ 1 \end{array}\right] + c_2 e^{\left(\frac{2^{\frac{1}{3}}}{2}+1\right)x} \cdot \left[\begin{array}{c} -\frac{-\sin\left(\frac{2^{\frac{1}{3}}\sqrt{3}\,x}{2}\right)2^{\frac{2}{3}}\sqrt{3}+2^{\frac{2}{3}}\cos\left(\frac{2^{\frac{1}{3}}\sqrt{3}\,x}{2}\right)-2\sin\left(\frac{2^{\frac{1}{3}}\sqrt{3}\,x}{2}\right)\cdots}{2\left(2^{\frac{2}{3}}+2^{\frac{1}{3}}+1\right)\cdots} \\ \frac{\sin\left(\frac{2^{\frac{1}{3}}\sqrt{3}\,x}{2}\right)2^{\frac{1}{3}}\sqrt{3}+2^{\frac{1}{3}}\cos\left(\frac{2^{\frac{1}{3}}\sqrt{3}\,x}{2}\cdots\right.}{2\left(2^{\frac{2}{3}}+2^{\frac{1}{3}}+1\right)} \\ \cos\left(\frac{2^{\frac{1}{3}}\sqrt{3}\,x}{2}\right) \end{array}\right.$$

- First component of the vector is the solution to the ODE

$$y = -2\left(\left(-\sqrt{3}\,c_3 + c_2\right)2^{\frac{1}{3}} + \frac{3\left(\sqrt{3}\,c_3+c_2\right)2^{\frac{2}{3}}}{4} - \frac{5c_2}{2}\right)e^{\frac{\left(2+2^{\frac{1}{3}}\right)x}{2}}\cos\left(\frac{2^{\frac{1}{3}}\sqrt{3}\,x}{2}\right) + 2\left(\left(\sqrt{3}\,c_2 + c_3\right)2^{\frac{1}{3}} + \cdots\right.$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 58

```
dsolve(diff(y(x),x$3)-3*diff(y(x),x$2)+3*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-\left(2^{\frac{1}{3}}-1\right)x} + c_2 \mathrm{e}^{\frac{\left(2^{\frac{1}{3}}+2\right)x}{2}} \sin\left(\frac{2^{\frac{1}{3}}\sqrt{3}\,x}{2}\right) + c_3 \mathrm{e}^{\frac{\left(2^{\frac{1}{3}}+2\right)x}{2}} \cos\left(\frac{2^{\frac{1}{3}}\sqrt{3}\,x}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 87

```
DSolve[y'''[x]-3*y''[x]+3*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\begin{aligned} y(x) \to\ & c_1 \exp\left(x\text{Root}\left[\#1^3 - 3\#1^2 + 3\#1 + 1\&, 1\right]\right) \\ & + c_2 \exp\left(x\text{Root}\left[\#1^3 - 3\#1^2 + 3\#1 + 1\&, 2\right]\right) \\ & + c_3 \exp\left(x\text{Root}\left[\#1^3 - 3\#1^2 + 3\#1 + 1\&, 3\right]\right) \end{aligned}$$

2.3 problem 10

Internal problem ID [824]
Internal file name [`OUTPUT/824_Sunday_June_05_2022_01_50_36_AM_67205334/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 4.2, Higher order linear differential equations. Constant coefficients. page 180
Problem number: 10.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - 4y''' + 4y'' = 0}$$

The characteristic equation is

$$\lambda^4 - 4\lambda^3 + 4\lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$
$$\lambda_3 = 2$$
$$\lambda_4 = 2$$

Therefore the homogeneous solution is

$$y_h(x) = c_2 x + c_1 + c_3 e^{2x} + x\, e^{2x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= x \\
y_3 &= e^{2x} \\
y_4 &= x\,e^{2x}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_2x + c_1 + c_3e^{2x} + x\,e^{2x}c_4 \tag{1}$$

Verification of solutions

$$y = c_2x + c_1 + c_3e^{2x} + x\,e^{2x}c_4$$

Verified OK.

2.3.1 Maple step by step solution

Let's solve

$y'''' - 4y''' + 4y'' = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = 4y_4(x) - 4y_3(x)$

  Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 4y_4(x) - 4y_3(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & 4 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & 4 \end{bmatrix}$$

- Rewrite the system as

$\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\begin{bmatrix} 0, & \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 0, & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\begin{bmatrix} 2, & \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \end{bmatrix}$$

- First solution from eigenvalue 2

$$\vec{y}_3(x) = \mathrm{e}^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 2$ is the eigenvalue, an

  $\vec{y}_4(x) = \mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right)$
- Note that the x multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\vec{y}_4(x)$ into the homogeneous system

  $\lambda\,\mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda x}\vec{v} = \left(\mathrm{e}^{\lambda x}A\right) \cdot \left(x\vec{v} + \vec{p}\right)$
- Use the fact that $\vec{v}$ is an eigenvector of A

  $\lambda\,\mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda x}\vec{v} = \mathrm{e}^{\lambda x}\left(\lambda x\vec{v} + A \cdot \vec{p}\right)$
- Simplify equation

  $\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$
- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$
- Condition $\vec{p}$ must meet for $\vec{y}_4(x)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$
- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 2

$$\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & 4 \end{bmatrix} - 2 \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} -\frac{1}{16} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 2

$$\vec{y}_4(x) = e^{2x} \cdot \left(x \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{16} \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)$$

- General solution to the system of ODEs

$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x)$

- Substitute solutions into the general solution

$$\vec{y} = c_3 e^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + e^{2x} c_4 \cdot \left(x \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{16} \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) + \begin{bmatrix} c_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$y = \frac{((2x-1)c_4+2c_3)e^{2x}}{16} + c_1$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$4)-4*diff(y(x),x$3)+4*diff(y(x),x$2)=0,y(x), singsol=all)
```

$$y(x) = (c_4x + c_3)\, e^{2x} + c_2x + c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 22

```
DSolve[y''''[x]-4*y'''[x]+4*y'''[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(x(c_4x + c_3) + c_2) + c_1$$

2.4 problem 11

Internal problem ID [825]
Internal file name [`OUTPUT/825_Sunday_June_05_2022_01_50_37_AM_32825979/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 4.2, Higher order linear differential equations. Constant coefficients. page 180
Problem number: 11.
ODE order: 6.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y^{(6)}+y=0}$$

The characteristic equation is

$$\lambda^6+1=0$$

The roots of the above equation are

$$\begin{aligned}
\lambda_1 &= i\\
\lambda_2 &= -i\\
\lambda_3 &= \frac{\sqrt{-2i\sqrt{3}+2}}{2}\\
\lambda_4 &= -\frac{\sqrt{-2i\sqrt{3}+2}}{2}\\
\lambda_5 &= \frac{\sqrt{2+2i\sqrt{3}}}{2}\\
\lambda_6 &= -\frac{\sqrt{2+2i\sqrt{3}}}{2}
\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{-\frac{\sqrt{2+2i\sqrt{3}}\,x}{2}} c_1 + c_2 \mathrm{e}^{-ix} + \mathrm{e}^{\frac{\sqrt{-2i\sqrt{3}+2}\,x}{2}} c_3 + \mathrm{e}^{ix} c_4 + \mathrm{e}^{-\frac{\sqrt{-2i\sqrt{3}+2}\,x}{2}} c_5 + \mathrm{e}^{\frac{\sqrt{2+2i\sqrt{3}}\,x}{2}} c_6$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-\frac{\sqrt{2+2i\sqrt{3}}\,x}{2}}$$
$$y_2 = \mathrm{e}^{-ix}$$
$$y_3 = \mathrm{e}^{\frac{\sqrt{-2i\sqrt{3}+2}\,x}{2}}$$
$$y_4 = \mathrm{e}^{ix}$$
$$y_5 = \mathrm{e}^{-\frac{\sqrt{-2i\sqrt{3}+2}\,x}{2}}$$
$$y_6 = \mathrm{e}^{\frac{\sqrt{2+2i\sqrt{3}}\,x}{2}}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{\sqrt{2+2i\sqrt{3}}\,x}{2}} c_1 + c_2 \mathrm{e}^{-ix} + \mathrm{e}^{\frac{\sqrt{-2i\sqrt{3}+2}\,x}{2}} c_3 + \mathrm{e}^{ix} c_4 + \mathrm{e}^{-\frac{\sqrt{-2i\sqrt{3}+2}\,x}{2}} c_5 + \mathrm{e}^{\frac{\sqrt{2+2i\sqrt{3}}\,x}{2}} c_6 \quad (1)$$

Verification of solutions

$$y = \mathrm{e}^{-\frac{\sqrt{2+2i\sqrt{3}}\,x}{2}} c_1 + c_2 \mathrm{e}^{-ix} + \mathrm{e}^{\frac{\sqrt{-2i\sqrt{3}+2}\,x}{2}} c_3 + \mathrm{e}^{ix} c_4 + \mathrm{e}^{-\frac{\sqrt{-2i\sqrt{3}+2}\,x}{2}} c_5 + \mathrm{e}^{\frac{\sqrt{2+2i\sqrt{3}}\,x}{2}} c_6$$

Verified OK.

2.4.1 Maple step by step solution

Let's solve

$y^{(6)} + y = 0$

- Highest derivative means the order of the ODE is 6

  $y^{(6)}$

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

$y_3(x) = y''$

- Define new variable $y_4(x)$

  $y_4(x) = y'''$

- Define new variable $y_5(x)$

  $y_5(x) = y''''$

- Define new variable $y_6(x)$

  $y_6(x) = y^{(5)}$

- Isolate for $y_6'(x)$ using original ODE

  $y_6'(x) = -y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_5(x) = y_4'(x), y_6(x) = y_5'(x), y_6'(x) = -y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \\ y_5(x) \\ y_6(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\begin{bmatrix} -I, & \begin{bmatrix} I \\ 1 \\ -I \\ -1 \\ I \\ 1 \end{bmatrix} \end{bmatrix}, \begin{bmatrix} I, & \begin{bmatrix} -I \\ 1 \\ I \\ -1 \\ -I \\ 1 \end{bmatrix} \end{bmatrix}, \begin{bmatrix} -\frac{I}{2} - \frac{\sqrt{3}}{2}, & \begin{bmatrix} \frac{1}{\left(-\frac{I}{2}-\frac{\sqrt{3}}{2}\right)^5} \\ \frac{1}{\left(-\frac{I}{2}-\frac{\sqrt{3}}{2}\right)^4} \\ \frac{1}{\left(-\frac{I}{2}-\frac{\sqrt{3}}{2}\right)^3} \\ \frac{1}{\left(-\frac{I}{2}-\frac{\sqrt{3}}{2}\right)^2} \\ \frac{1}{-\frac{I}{2}-\frac{\sqrt{3}}{2}} \\ 1 \end{bmatrix} \end{bmatrix}, \begin{bmatrix} -\frac{I}{2} + \frac{\sqrt{3}}{2}, & \begin{bmatrix} \frac{1}{\left(-\frac{I}{2}+\frac{\sqrt{3}}{2}\right)^5} \\ \frac{1}{\left(-\frac{I}{2}+\frac{\sqrt{3}}{2}\right)^4} \\ \frac{1}{\left(-\frac{I}{2}+\frac{\sqrt{3}}{2}\right)^3} \\ \frac{1}{\left(-\frac{I}{2}+\frac{\sqrt{3}}{2}\right)^2} \\ \frac{1}{-\frac{I}{2}+\frac{\sqrt{3}}{2}} \\ 1 \end{bmatrix} \end{bmatrix}, \Bigg[\frac{I}{2} -$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\begin{bmatrix} -I, & \begin{bmatrix} I \\ 1 \\ -I \\ -1 \\ I \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution from eigenpair

$$e^{-\mathrm{I}x} \cdot \begin{bmatrix} \mathrm{I} \\ 1 \\ -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(x) - \mathrm{I}\sin(x)) \cdot \begin{bmatrix} \mathrm{I} \\ 1 \\ -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} \mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \\ -\mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ -\cos(x) + \mathrm{I}\sin(x) \\ \mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_1(x) = \begin{bmatrix} \sin(x) \\ \cos(x) \\ -\sin(x) \\ -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_2(x) = \begin{bmatrix} \cos(x) \\ -\sin(x) \\ -\cos(x) \\ \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{I}{2}-\frac{\sqrt{3}}{2}, \left[\begin{array}{c} \frac{1}{\left(-\frac{I}{2}-\frac{\sqrt{3}}{2}\right)^{5}} \\ \frac{1}{\left(-\frac{I}{2}-\frac{\sqrt{3}}{2}\right)^{4}} \\ \frac{1}{\left(-\frac{I}{2}-\frac{\sqrt{3}}{2}\right)^{3}} \\ \frac{1}{\left(-\frac{I}{2}-\frac{\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{I}{2}-\frac{\sqrt{3}}{2}} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{\left(-\frac{I}{2}-\frac{\sqrt{3}}{2}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(-\frac{I}{2}-\frac{\sqrt{3}}{2}\right)^{5}} \\ \frac{1}{\left(-\frac{I}{2}-\frac{\sqrt{3}}{2}\right)^{4}} \\ \frac{1}{\left(-\frac{I}{2}-\frac{\sqrt{3}}{2}\right)^{3}} \\ \frac{1}{\left(-\frac{I}{2}-\frac{\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{I}{2}-\frac{\sqrt{3}}{2}} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{-\frac{\sqrt{3}\,x}{2}} \cdot \left(\cos\left(\frac{x}{2}\right) - I\sin\left(\frac{x}{2}\right)\right) \cdot \left[\begin{array}{c} \frac{1}{\left(-\frac{I}{2}-\frac{\sqrt{3}}{2}\right)^{5}} \\ \frac{1}{\left(-\frac{I}{2}-\frac{\sqrt{3}}{2}\right)^{4}} \\ \frac{1}{\left(-\frac{I}{2}-\frac{\sqrt{3}}{2}\right)^{3}} \\ \frac{1}{\left(-\frac{I}{2}-\frac{\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{I}{2}-\frac{\sqrt{3}}{2}} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{-\frac{\sqrt{3}\,x}{2}} \cdot \begin{bmatrix} \frac{\cos\left(\frac{x}{2}\right)-\mathrm{I}\sin\left(\frac{x}{2}\right)}{\left(-\frac{\mathrm{I}}{2}-\frac{\sqrt{3}}{2}\right)^5} \\ \frac{\cos\left(\frac{x}{2}\right)-\mathrm{I}\sin\left(\frac{x}{2}\right)}{\left(-\frac{\mathrm{I}}{2}-\frac{\sqrt{3}}{2}\right)^4} \\ \frac{\cos\left(\frac{x}{2}\right)-\mathrm{I}\sin\left(\frac{x}{2}\right)}{\left(-\frac{\mathrm{I}}{2}-\frac{\sqrt{3}}{2}\right)^3} \\ \frac{\cos\left(\frac{x}{2}\right)-\mathrm{I}\sin\left(\frac{x}{2}\right)}{\left(-\frac{\mathrm{I}}{2}-\frac{\sqrt{3}}{2}\right)^2} \\ \frac{\cos\left(\frac{x}{2}\right)-\mathrm{I}\sin\left(\frac{x}{2}\right)}{-\frac{\mathrm{I}}{2}-\frac{\sqrt{3}}{2}} \\ \cos\left(\frac{x}{2}\right)-\mathrm{I}\sin\left(\frac{x}{2}\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = e^{-\frac{\sqrt{3}\,x}{2}} \cdot \begin{bmatrix} \Re\left(\frac{\cos\left(\frac{x}{2}\right)-\mathrm{I}\sin\left(\frac{x}{2}\right)}{\left(-\frac{\mathrm{I}}{2}-\frac{\sqrt{3}}{2}\right)^5}\right) \\ \Re\left(\frac{\cos\left(\frac{x}{2}\right)-\mathrm{I}\sin\left(\frac{x}{2}\right)}{\left(-\frac{\mathrm{I}}{2}-\frac{\sqrt{3}}{2}\right)^4}\right) \\ \sin\left(\frac{x}{2}\right) \\ \frac{\cos\left(\frac{x}{2}\right)}{2}-\frac{\sqrt{3}\sin\left(\frac{x}{2}\right)}{2} \\ -\frac{\cos\left(\frac{x}{2}\right)\sqrt{3}}{2}+\frac{\sin\left(\frac{x}{2}\right)}{2} \\ \cos\left(\frac{x}{2}\right) \end{bmatrix}, \vec{y}_4(x) = e^{-\frac{\sqrt{3}\,x}{2}} \cdot \begin{bmatrix} \Im\left(\frac{\cos\left(\frac{x}{2}\right)-\mathrm{I}\sin\left(\frac{x}{2}\right)}{\left(-\frac{\mathrm{I}}{2}-\frac{\sqrt{3}}{2}\right)^5}\right) \\ \Im\left(\frac{\cos\left(\frac{x}{2}\right)-\mathrm{I}\sin\left(\frac{x}{2}\right)}{\left(-\frac{\mathrm{I}}{2}-\frac{\sqrt{3}}{2}\right)^4}\right) \\ \cos\left(\frac{x}{2}\right) \\ -\frac{\cos\left(\frac{x}{2}\right)\sqrt{3}}{2}-\frac{\sin\left(\frac{x}{2}\right)}{2} \\ \frac{\cos\left(\frac{x}{2}\right)}{2}+\frac{\sqrt{3}\sin\left(\frac{x}{2}\right)}{2} \\ -\sin\left(\frac{x}{2}\right) \end{bmatrix}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{\mathrm{I}}{2}+\frac{\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{\left(-\frac{\mathrm{I}}{2}+\frac{\sqrt{3}}{2}\right)^5} \\ \frac{1}{\left(-\frac{\mathrm{I}}{2}+\frac{\sqrt{3}}{2}\right)^4} \\ \frac{1}{\left(-\frac{\mathrm{I}}{2}+\frac{\sqrt{3}}{2}\right)^3} \\ \frac{1}{\left(-\frac{\mathrm{I}}{2}+\frac{\sqrt{3}}{2}\right)^2} \\ \frac{1}{-\frac{\mathrm{I}}{2}+\frac{\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{\left(-\frac{I}{2}+\frac{\sqrt{3}}{2}\right)x} \cdot \begin{bmatrix} \frac{1}{\left(-\frac{I}{2}+\frac{\sqrt{3}}{2}\right)^5} \\ \frac{1}{\left(-\frac{I}{2}+\frac{\sqrt{3}}{2}\right)^4} \\ \frac{1}{\left(-\frac{I}{2}+\frac{\sqrt{3}}{2}\right)^3} \\ \frac{1}{\left(-\frac{I}{2}+\frac{\sqrt{3}}{2}\right)^2} \\ \frac{1}{-\frac{I}{2}+\frac{\sqrt{3}}{2}} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{\frac{\sqrt{3}\,x}{2}} \cdot \left(\cos\left(\frac{x}{2}\right) - I\sin\left(\frac{x}{2}\right)\right) \cdot \begin{bmatrix} \frac{1}{\left(-\frac{I}{2}+\frac{\sqrt{3}}{2}\right)^5} \\ \frac{1}{\left(-\frac{I}{2}+\frac{\sqrt{3}}{2}\right)^4} \\ \frac{1}{\left(-\frac{I}{2}+\frac{\sqrt{3}}{2}\right)^3} \\ \frac{1}{\left(-\frac{I}{2}+\frac{\sqrt{3}}{2}\right)^2} \\ \frac{1}{-\frac{I}{2}+\frac{\sqrt{3}}{2}} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{\frac{\sqrt{3}\,x}{2}} \cdot \begin{bmatrix} \frac{\cos\left(\frac{x}{2}\right)-I\sin\left(\frac{x}{2}\right)}{\left(-\frac{I}{2}+\frac{\sqrt{3}}{2}\right)^5} \\ \frac{\cos\left(\frac{x}{2}\right)-I\sin\left(\frac{x}{2}\right)}{\left(-\frac{I}{2}+\frac{\sqrt{3}}{2}\right)^4} \\ \frac{\cos\left(\frac{x}{2}\right)-I\sin\left(\frac{x}{2}\right)}{\left(-\frac{I}{2}+\frac{\sqrt{3}}{2}\right)^3} \\ \frac{\cos\left(\frac{x}{2}\right)-I\sin\left(\frac{x}{2}\right)}{\left(-\frac{I}{2}+\frac{\sqrt{3}}{2}\right)^2} \\ \frac{\cos\left(\frac{x}{2}\right)-I\sin\left(\frac{x}{2}\right)}{-\frac{I}{2}+\frac{\sqrt{3}}{2}} \\ \cos\left(\frac{x}{2}\right) - I\sin\left(\frac{x}{2}\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_5(x) = \mathrm{e}^{\frac{\sqrt{3}\,x}{2}} \cdot \left[\begin{array}{c} \Re\left(\frac{\cos\left(\frac{x}{2}\right)-\mathrm{I}\sin\left(\frac{x}{2}\right)}{\left(-\frac{\mathrm{I}}{2}+\frac{\sqrt{3}}{2}\right)^{5}}\right) \\ \Re\left(\frac{\cos\left(\frac{x}{2}\right)-\mathrm{I}\sin\left(\frac{x}{2}\right)}{\left(-\frac{\mathrm{I}}{2}+\frac{\sqrt{3}}{2}\right)^{4}}\right) \\ \sin\left(\frac{x}{2}\right) \\ \frac{\cos\left(\frac{x}{2}\right)}{2}+\frac{\sqrt{3}\,\sin\left(\frac{x}{2}\right)}{2} \\ \frac{\cos\left(\frac{x}{2}\right)\sqrt{3}}{2}+\frac{\sin\left(\frac{x}{2}\right)}{2} \\ \cos\left(\frac{x}{2}\right) \end{array}\right], \vec{y}_6(x) = \mathrm{e}^{\frac{\sqrt{3}\,x}{2}} \cdot \left[\begin{array}{c} \Im\left(\frac{\cos\left(\frac{x}{2}\right)-\mathrm{I}\sin\left(\frac{x}{2}\right)}{\left(-\frac{\mathrm{I}}{2}+\frac{\sqrt{3}}{2}\right)^{5}}\right) \\ \Im\left(\frac{\cos\left(\frac{x}{2}\right)-\mathrm{I}\sin\left(\frac{x}{2}\right)}{\left(-\frac{\mathrm{I}}{2}+\frac{\sqrt{3}}{2}\right)^{4}}\right) \\ \cos\left(\frac{x}{2}\right) \\ \frac{\cos\left(\frac{x}{2}\right)\sqrt{3}}{2}-\frac{\sin\left(\frac{x}{2}\right)}{2} \\ \frac{\cos\left(\frac{x}{2}\right)}{2}-\frac{\sqrt{3}\,\sin\left(\frac{x}{2}\right)}{2} \\ -\sin\left(\frac{x}{2}\right) \end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + c_5\vec{y}_5(x) + c_6\vec{y}_6(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_3\mathrm{e}^{-\frac{\sqrt{3}\,x}{2}} \cdot \left[\begin{array}{c} \Re\left(\frac{\cos\left(\frac{x}{2}\right)-\mathrm{I}\sin\left(\frac{x}{2}\right)}{\left(-\frac{\mathrm{I}}{2}-\frac{\sqrt{3}}{2}\right)^{5}}\right) \\ \Re\left(\frac{\cos\left(\frac{x}{2}\right)-\mathrm{I}\sin\left(\frac{x}{2}\right)}{\left(-\frac{\mathrm{I}}{2}-\frac{\sqrt{3}}{2}\right)^{4}}\right) \\ \sin\left(\frac{x}{2}\right) \\ \frac{\cos\left(\frac{x}{2}\right)}{2}-\frac{\sqrt{3}\,\sin\left(\frac{x}{2}\right)}{2} \\ -\frac{\cos\left(\frac{x}{2}\right)\sqrt{3}}{2}+\frac{\sin\left(\frac{x}{2}\right)}{2} \\ \cos\left(\frac{x}{2}\right) \end{array}\right] + c_4\mathrm{e}^{-\frac{\sqrt{3}\,x}{2}} \cdot \left[\begin{array}{c} \Im\left(\frac{\cos\left(\frac{x}{2}\right)-\mathrm{I}\sin\left(\frac{x}{2}\right)}{\left(-\frac{\mathrm{I}}{2}-\frac{\sqrt{3}}{2}\right)^{5}}\right) \\ \Im\left(\frac{\cos\left(\frac{x}{2}\right)-\mathrm{I}\sin\left(\frac{x}{2}\right)}{\left(-\frac{\mathrm{I}}{2}-\frac{\sqrt{3}}{2}\right)^{4}}\right) \\ \cos\left(\frac{x}{2}\right) \\ -\frac{\cos\left(\frac{x}{2}\right)\sqrt{3}}{2}-\frac{\sin\left(\frac{x}{2}\right)}{2} \\ \frac{\cos\left(\frac{x}{2}\right)}{2}+\frac{\sqrt{3}\,\sin\left(\frac{x}{2}\right)}{2} \\ -\sin\left(\frac{x}{2}\right) \end{array}\right] + c_5\mathrm{e}^{\frac{\sqrt{3}\,x}{2}} \cdot \left[\begin{array}{c} \Re\left(\frac{\cos\left(\frac{x}{2}\right)}{\left(-\right.}\right. \\ \Re\left(\frac{\cos\left(\frac{x}{2}\right)}{\left(-\right.}\right. \\ \sin \\ \frac{\cos\left(\frac{x}{2}\right)}{2}+ \\ \frac{\cos\left(\frac{x}{2}\right)\sqrt{3}}{2} \\ \cos \end{array}\right.$$

- First component of the vector is the solution to the ODE

$$y = -32c_3\mathrm{e}^{-\frac{\sqrt{3}\,x}{2}}\Im\left(\frac{\sin\left(\frac{x}{2}\right)+\mathrm{I}\cos\left(\frac{x}{2}\right)}{\left(\sqrt{3}+\mathrm{I}\right)^{5}}\right) + 32c_4\mathrm{e}^{-\frac{\sqrt{3}\,x}{2}}\Re\left(\frac{\sin\left(\frac{x}{2}\right)+\mathrm{I}\cos\left(\frac{x}{2}\right)}{\left(\sqrt{3}+\mathrm{I}\right)^{5}}\right) - 32c_5\mathrm{e}^{\frac{\sqrt{3}\,x}{2}}\Im\left(\frac{\sin\left(\frac{x}{2}\right)+\mathrm{I}\cos\left(\frac{x}{2}\right)}{\left(\mathrm{I}-\sqrt{3}\right)^{5}}\right)$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 56

```
dsolve(diff(y(x),x$6)+y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(-\sin\left(\frac{x}{2}\right)c_4 + c_6\cos\left(\frac{x}{2}\right)\right)e^{-\frac{\sqrt{3}\,x}{2}} + \left(\sin\left(\frac{x}{2}\right)c_3 + \cos\left(\frac{x}{2}\right)c_5\right)e^{\frac{\sqrt{3}\,x}{2}} + c_1\sin(x) + c_2\cos(x)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 92

```
DSolve[y''''''[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-\frac{\sqrt{3}x}{2}}\left(c_1e^{\sqrt{3}x} + c_3\right)\cos\left(\frac{x}{2}\right) + c_2\cos(x) + c_4e^{-\frac{\sqrt{3}x}{2}}\sin\left(\frac{x}{2}\right) + c_6e^{\frac{\sqrt{3}x}{2}}\sin\left(\frac{x}{2}\right) + c_5\sin(x)$$

2.5 problem 12

Internal problem ID [826]
Internal file name [`OUTPUT/826_Sunday_June_05_2022_01_50_38_AM_47084674/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 4.2, Higher order linear differential equations. Constant coefficients. page 180
Problem number: 12.
ODE order: 6.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y^{(6)} - 3y'''' + 3y'' - y = 0}$$

The characteristic equation is

$$\lambda^6 - 3\lambda^4 + 3\lambda^2 - 1 = 0$$

The roots of the above equation are

$$\begin{aligned}
\lambda_1 &= 1 \\
\lambda_2 &= -1 \\
\lambda_3 &= 1 \\
\lambda_4 &= -1 \\
\lambda_5 &= 1 \\
\lambda_6 &= -1
\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + x^2 \mathrm{e}^{-x} c_3 + \mathrm{e}^{x} c_4 + x\,\mathrm{e}^{x} c_5 + x^2 \mathrm{e}^{x} c_6$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-x} \\
y_2 &= x\,\mathrm{e}^{-x} \\
y_3 &= x^2\mathrm{e}^{-x} \\
y_4 &= \mathrm{e}^{x} \\
y_5 &= x\,\mathrm{e}^{x} \\
y_6 &= x^2\mathrm{e}^{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\mathrm{e}^{-x} + x\,\mathrm{e}^{-x}c_2 + x^2\mathrm{e}^{-x}c_3 + \mathrm{e}^{x}c_4 + x\,\mathrm{e}^{x}c_5 + x^2\mathrm{e}^{x}c_6 \tag{1}
$$

Verification of solutions

$$
y = c_1\mathrm{e}^{-x} + x\,\mathrm{e}^{-x}c_2 + x^2\mathrm{e}^{-x}c_3 + \mathrm{e}^{x}c_4 + x\,\mathrm{e}^{x}c_5 + x^2\mathrm{e}^{x}c_6
$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(y(x),x$6)-3*diff(y(x),x$4)+3*diff(y(x),x$2)-y(x)=0,y(x), singsol=all)
```

$$
y(x) = (c_6x^2 + c_5x + c_4)\,\mathrm{e}^{-x} + \mathrm{e}^{x}(c_3x^2 + c_2x + c_1)
$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 50

```
DSolve[y''''''[x]-3*y''''[x]+3*y''[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
y(x) \to e^{-x}\left(x^2\left(c_6e^{2x} + c_3\right) + x\left(c_5e^{2x} + c_2\right) + c_4e^{2x} + c_1\right)
$$

2.6 problem 13

Internal problem ID [827]
Internal file name [`OUTPUT/827_Sunday_June_05_2022_01_50_39_AM_36738676/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 4.2, Higher order linear differential equations. Constant coefficients. page 180
Problem number: 13.
ODE order: 6.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y^{(6)} - y'' = 0}$$

The characteristic equation is

$$\lambda^6 - \lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$
$$\lambda_3 = 1$$
$$\lambda_4 = -1$$
$$\lambda_5 = i$$
$$\lambda_6 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + c_2 + c_3 x + \mathrm{e}^{x} c_4 + \mathrm{e}^{-ix} c_5 + \mathrm{e}^{ix} c_6$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-x} \\
y_2 &= 1 \\
y_3 &= x \\
y_4 &= \mathrm{e}^{x} \\
y_5 &= \mathrm{e}^{-ix} \\
y_6 &= \mathrm{e}^{ix}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-x} + c_2 + c_3 x + \mathrm{e}^{x} c_4 + \mathrm{e}^{-ix} c_5 + \mathrm{e}^{ix} c_6 \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{-x} + c_2 + c_3 x + \mathrm{e}^{x} c_4 + \mathrm{e}^{-ix} c_5 + \mathrm{e}^{ix} c_6
$$

Verified OK.

2.6.1 Maple step by step solution

Let's solve

$y^{(6)} - y'' = 0$

- Highest derivative means the order of the ODE is 6

  $y^{(6)}$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Define new variable $y_4(x)$

    $y_4(x) = y'''$

  ◦ Define new variable $y_5(x)$

$y_5(x) = y''''$

- Define new variable $y_6(x)$

  $y_6(x) = y^{(5)}$

- Isolate for $y_6'(x)$ using original ODE

  $y_6'(x) = y_3(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_5(x) = y_4'(x), y_6(x) = y_5'(x), y_6'(x) = y_3(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \\ y_5(x) \\ y_6(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{array}\right]\right], \left[0, \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]\right], \left[0, \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]\right], \left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array}\right]\right], \left[-\mathrm{I}, \left[\begin{array}{c} \mathrm{I} \\ 1 \\ -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{array}\right]\right], \left[\mathrm{I}, \left[\begin{array}{c} -\mathrm{I} \\ 1 \\ \mathrm{I} \\ -1 \\ -\mathrm{I} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix}0\\0\\0\\0\\0\\0\end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \begin{bmatrix}0\\0\\0\\0\\0\\0\end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_4 = e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-I, \begin{bmatrix} I \\ 1 \\ -I \\ -1 \\ I \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-Ix} \cdot \begin{bmatrix} I \\ 1 \\ -I \\ -1 \\ I \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(x) - \mathrm{I}\sin(x)) \cdot \begin{bmatrix} \mathrm{I} \\ 1 \\ -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} \mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \\ -\mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ -\cos(x) + \mathrm{I}\sin(x) \\ \mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_5(x) = \begin{bmatrix} \sin(x) \\ \cos(x) \\ -\sin(x) \\ -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_6(x) = \begin{bmatrix} \cos(x) \\ -\sin(x) \\ -\cos(x) \\ \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + c_4\vec{y}_4 + c_5\vec{y}_5(x) + c_6\vec{y}_6(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 \mathrm{e}^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \mathrm{e}^{x} c_4 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} c_2 + c_5\sin(x) + c_6\cos(x) \\ c_5\cos(x) - c_6\sin(x) \\ -c_5\sin(x) - c_6\cos(x) \\ -c_5\cos(x) + c_6\sin(x) \\ c_5\sin(x) + c_6\cos(x) \\ c_5\cos(x) - c_6\sin(x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y = -c_1 \mathrm{e}^{-x} + \mathrm{e}^{x} c_4 + c_6 \cos(x) + c_5 \sin(x) + c_2$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$6)-diff(y(x),x$2)=0,y(x), singsol=all)
```

$$y(x) = c_1 + c_2 x + c_3 \mathrm{e}^{x} + c_4 \mathrm{e}^{-x} + c_5 \sin(x) + c_6 \cos(x)$$

✓ Solution by Mathematica

Time used: 0.112 (sec). Leaf size: 38

```
DSolve[y''''''[x]-y''[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{x} + c_3 e^{-x} + c_6 x - c_2 \cos(x) - c_4 \sin(x) + c_5$$

2.7 problem 14

Internal problem ID [828]
Internal file name [`OUTPUT/828_Sunday_June_05_2022_01_50_40_AM_14549149/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 4.2, Higher order linear differential equations. Constant coefficients. page 180
Problem number: 14.
ODE order: 5.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y^{(5)} - 3y'''' + 3y''' - 3y'' + 2y' = 0}$$

The characteristic equation is

$$\lambda^5 - 3\lambda^4 + 3\lambda^3 - 3\lambda^2 + 2\lambda = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= 0\\ \lambda_2 &= 1\\ \lambda_3 &= 2\\ \lambda_4 &= i\\ \lambda_5 &= -i\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + c_2\mathrm{e}^x + c_3\mathrm{e}^{2x} + \mathrm{e}^{-ix}c_4 + \mathrm{e}^{ix}c_5$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= \mathrm{e}^{x} \\
y_3 &= \mathrm{e}^{2x} \\
y_4 &= \mathrm{e}^{-ix} \\
y_5 &= \mathrm{e}^{ix}
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 + c_2\mathrm{e}^{x} + c_3\mathrm{e}^{2x} + \mathrm{e}^{-ix}c_4 + \mathrm{e}^{ix}c_5 \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1 + c_2\mathrm{e}^{x} + c_3\mathrm{e}^{2x} + \mathrm{e}^{-ix}c_4 + \mathrm{e}^{ix}c_5$$

Verified OK.

2.7.1 Maple step by step solution

Let's solve

$y^{(5)} - 3y'''' + 3y''' - 3y'' + 2y' = 0$

- Highest derivative means the order of the ODE is 5

  $y^{(5)}$

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Define new variable $y_4(x)$

    $y_4(x) = y'''$

  - Define new variable $y_5(x)$

    $y_5(x) = y''''$

- Isolate for $y_5'(x)$ using original ODE

  $y_5'(x) = 3y_5(x) - 3y_4(x) + 3y_3(x) - 2y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_5(x) = y_4'(x), y_5'(x) = 3y_5(x) - 3y_4(x) + 3y_3(x) -$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \\ y_5(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & 3 & -3 & 3 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & 3 & -3 & 3 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}\right], \left[1, \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}\right], \left[2, \begin{bmatrix}\frac{1}{16}\\\frac{1}{8}\\\frac{1}{4}\\\frac{1}{2}\\1\end{bmatrix}\right], \left[-I, \begin{bmatrix}1\\-I\\-1\\I\\1\end{bmatrix}\right], \left[I, \begin{bmatrix}1\\I\\-1\\-I\\1\end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{16} \\ \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{2x} \cdot \begin{bmatrix} \frac{1}{16} \\ \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} 1 \\ -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix} \right]$$

- Solution from eigenpair

$$e^{-Ix} \cdot \begin{bmatrix} 1 \\ -I \\ -1 \\ I \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(x) - I\sin(x)) \cdot \begin{bmatrix} 1 \\ -I \\ -1 \\ I \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} \cos(x) - I\sin(x) \\ -I(\cos(x) - I\sin(x)) \\ -\cos(x) + I\sin(x) \\ I(\cos(x) - I\sin(x)) \\ \cos(x) - I\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_4(x) = \begin{bmatrix} \cos(x) \\ -\sin(x) \\ -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_5(x) = \begin{bmatrix} -\sin(x) \\ -\cos(x) \\ \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + c_4\vec{y}_4(x) + c_5\vec{y}_5(x)$

- Substitute solutions into the general solution

$$\vec{y} = c_2 e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{2x} \cdot \begin{bmatrix} \frac{1}{16} \\ \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} c_1 + c_4 \cos(x) - c_5 \sin(x) \\ -c_4 \sin(x) - c_5 \cos(x) \\ -c_4 \cos(x) + c_5 \sin(x) \\ c_4 \sin(x) + c_5 \cos(x) \\ c_4 \cos(x) - c_5 \sin(x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y = c_2 e^x + \frac{c_3 e^{2x}}{16} - c_5 \sin(x) + c_4 \cos(x) + c_1$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$5)-3*diff(y(x),x$4)+3*diff(y(x),x$3)-3*diff(y(x),x$2)+2*diff(y(x),x)=0,y(
```

$$y(x) = c_1 + e^x c_2 + c_3 e^{2x} + c_4 \sin(x) + c_5 \cos(x)$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 36

```
DSolve[y'''''[x]-3*y''''[x]+3*y'''[x]-3*y''[x]+2*y'[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \rightarrow c_3 e^x + \frac{1}{2} c_4 e^{2x} - c_2 \cos(x) + c_1 \sin(x) + c_5$$

2.8 problem 15

Internal problem ID [829]
Internal file name [`OUTPUT/829_Sunday_June_05_2022_01_50_41_AM_94805634/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 4.2, Higher order linear differential equations. Constant coefficients. page 180
Problem number: 15.
ODE order: 8.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y^{(8)} + 8y'''' + 16y = 0}$$

The characteristic equation is

$$\lambda^8 + 8\lambda^4 + 16 = 0$$

The roots of the above equation are

$$\begin{aligned}
\lambda_1 &= 1 - i \\
\lambda_2 &= 1 + i \\
\lambda_3 &= -1 - i \\
\lambda_4 &= -1 + i \\
\lambda_5 &= 1 - i \\
\lambda_6 &= 1 + i \\
\lambda_7 &= -1 - i \\
\lambda_8 &= -1 + i
\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = e^{(-1-i)x}c_1 + x\,e^{(-1-i)x}c_2 + e^{(-1+i)x}c_3 + x\,e^{(-1+i)x}c_4 + e^{(1-i)x}c_5 + x\,e^{(1-i)x}c_6 + e^{(1+i)x}c_7 + x\,e^{(1+i)x}c_8$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{(-1-i)x}\\
y_2 &= x\,\mathrm{e}^{(-1-i)x}\\
y_3 &= \mathrm{e}^{(-1+i)x}\\
y_4 &= x\,\mathrm{e}^{(-1+i)x}\\
y_5 &= \mathrm{e}^{(1-i)x}\\
y_6 &= x\,\mathrm{e}^{(1-i)x}\\
y_7 &= \mathrm{e}^{(1+i)x}\\
y_8 &= x\,\mathrm{e}^{(1+i)x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \mathrm{e}^{(-1-i)x}c_1 + x\,\mathrm{e}^{(-1-i)x}c_2 + \mathrm{e}^{(-1+i)x}c_3 + x\,\mathrm{e}^{(-1+i)x}c_4 \\
& + \mathrm{e}^{(1-i)x}c_5 + x\,\mathrm{e}^{(1-i)x}c_6 + \mathrm{e}^{(1+i)x}c_7 + x\,\mathrm{e}^{(1+i)x}c_8
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & \mathrm{e}^{(-1-i)x}c_1 + x\,\mathrm{e}^{(-1-i)x}c_2 + \mathrm{e}^{(-1+i)x}c_3 + x\,\mathrm{e}^{(-1+i)x}c_4 \\
& + \mathrm{e}^{(1-i)x}c_5 + x\,\mathrm{e}^{(1-i)x}c_6 + \mathrm{e}^{(1+i)x}c_7 + x\,\mathrm{e}^{(1+i)x}c_8
\end{aligned}
$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 47

```
dsolve(diff(y(x),x$8)+8*diff(y(x),x$4)+16*y(x)=0,y(x), singsol=all)
```

$$y(x) = ((c_4x + c_2)\cos(x) + \sin(x)(c_3x + c_1))\,\mathrm{e}^{-x} + ((c_8x + c_6)\cos(x) + \sin(x)(c_7x + c_5))\,\mathrm{e}^{x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 238

```
DSolve[D[y[x],{x,8}]+8*y''''[x]+3*y'''[x]+16*y[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$
\begin{aligned}
y(x) \rightarrow\ & c_1 \exp\left(x\text{Root}\left[\#1^8+8\#1^4+3\#1^3+16\&,1\right]\right) \\
&+c_2 \exp\left(x\text{Root}\left[\#1^8+8\#1^4+3\#1^3+16\&,2\right]\right) \\
&+c_5 \exp\left(x\text{Root}\left[\#1^8+8\#1^4+3\#1^3+16\&,5\right]\right) \\
&+c_6 \exp\left(x\text{Root}\left[\#1^8+8\#1^4+3\#1^3+16\&,6\right]\right) \\
&+c_3 \exp\left(x\text{Root}\left[\#1^8+8\#1^4+3\#1^3+16\&,3\right]\right) \\
&+c_4 \exp\left(x\text{Root}\left[\#1^8+8\#1^4+3\#1^3+16\&,4\right]\right) \\
&+c_7 \exp\left(x\text{Root}\left[\#1^8+8\#1^4+3\#1^3+16\&,7\right]\right) \\
&+c_8 \exp\left(x\text{Root}\left[\#1^8+8\#1^4+3\#1^3+16\&,8\right]\right)
\end{aligned}
$$

2.9 problem 16

Internal problem ID [830]
Internal file name [`OUTPUT/830_Sunday_June_05_2022_01_50_42_AM_71165634/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 4.2, Higher order linear differential equations. Constant coefficients. page 180
Problem number: 16.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + 2y'' + y = 0}$$

The characteristic equation is

$$\lambda^4 + 2\lambda^2 + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = i$$
$$\lambda_2 = -i$$
$$\lambda_3 = i$$
$$\lambda_4 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = e^{-ix}c_1 + x\,e^{-ix}c_2 + e^{ix}c_3 + x\,e^{ix}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-ix}$$
$$y_2 = x\,e^{-ix}$$
$$y_3 = e^{ix}$$
$$y_4 = x\,e^{ix}$$

Summary

The solution(s) found are the following

$$y = e^{-ix} c_1 + x\,e^{-ix} c_2 + e^{ix} c_3 + x\,e^{ix} c_4 \tag{1}$$

Verification of solutions

$$y = e^{-ix} c_1 + x\,e^{-ix} c_2 + e^{ix} c_3 + x\,e^{ix} c_4$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$4)+2*diff(y(x),x$2)+y(x)=0,y(x), singsol=all)
```

$$y(x) = (c_4 x + c_2) \cos(x) + \sin(x) (c_3 x + c_1)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 26

```
DSolve[y''''[x]+2*y''[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (c_2 x + c_1) \cos(x) + (c_4 x + c_3) \sin(x)$$

2.10 problem 17

Internal problem ID [831]
Internal file name [`OUTPUT/831_Sunday_June_05_2022_01_50_43_AM_58316676/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 4.2, Higher order linear differential equations. Constant coefficients. page 180
Problem number: 17.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + 5y'' + 6y' + 2y = 0}$$

The characteristic equation is

$$\lambda^3 + 5\lambda^2 + 6\lambda + 2 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = -2 - \sqrt{2}$$
$$\lambda_3 = -2 + \sqrt{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + e^{\left(-2+\sqrt{2}\right)x} c_2 + e^{\left(-2-\sqrt{2}\right)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{\left(-2+\sqrt{2}\right)x}$$
$$y_3 = e^{\left(-2-\sqrt{2}\right)x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + \mathrm{e}^{\left(-2+\sqrt{2}\right)x} c_2 + \mathrm{e}^{\left(-2-\sqrt{2}\right)x} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-x} + \mathrm{e}^{\left(-2+\sqrt{2}\right)x} c_2 + \mathrm{e}^{\left(-2-\sqrt{2}\right)x} c_3$$

Verified OK.

2.10.1 Maple step by step solution

Let's solve

$y''' + 5y'' + 6y' + 2y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -5y_3(x) - 6y_2(x) - 2y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -5y_3(x) - 6y_2(x) - 2y_1(x)]$

- Define vector

  $$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -6 & -5 \end{bmatrix} \cdot \overrightarrow{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -6 & -5 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[-2-\sqrt{2}, \begin{bmatrix} \frac{1}{\left(-2-\sqrt{2}\right)^2} \\ \frac{1}{-2-\sqrt{2}} \\ 1 \end{bmatrix}\right], \left[-2+\sqrt{2}, \begin{bmatrix} \frac{1}{\left(-2+\sqrt{2}\right)^2} \\ \frac{1}{-2+\sqrt{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-2-\sqrt{2}, \begin{bmatrix} \frac{1}{\left(-2-\sqrt{2}\right)^2} \\ \frac{1}{-2-\sqrt{2}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{\left(-2-\sqrt{2}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(-2-\sqrt{2}\right)^2} \\ \frac{1}{-2-\sqrt{2}} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[-2+\sqrt{2}, \left[\begin{array}{c} \frac{1}{\left(-2+\sqrt{2}\right)^2} \\ \frac{1}{-2+\sqrt{2}} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{\left(-2+\sqrt{2}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(-2+\sqrt{2}\right)^2} \\ \frac{1}{-2+\sqrt{2}} \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-x} \cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right] + c_2 e^{\left(-2-\sqrt{2}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(-2-\sqrt{2}\right)^2} \\ \frac{1}{-2-\sqrt{2}} \\ 1 \end{array}\right] + c_3 e^{\left(-2+\sqrt{2}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(-2+\sqrt{2}\right)^2} \\ \frac{1}{-2+\sqrt{2}} \\ 1 \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{c_2\left(3-2\sqrt{2}\right)e^{-\left(2+\sqrt{2}\right)x}}{2} + \frac{c_3\left(2\sqrt{2}+3\right)e^{\left(-2+\sqrt{2}\right)x}}{2} + c_1 e^{-x}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve(diff(y(x),x$3)+5*diff(y(x),x$2)+6*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-x}c_1 + c_2e^{(\sqrt{2}-2)x} + c_3e^{-(2+\sqrt{2})x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 43

```
DSolve[y'''[x]+5*y''[x]+6*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}\left(c_1e^{-\left(\left(1+\sqrt{2}\right)x\right)} + c_2e^{\left(\sqrt{2}-1\right)x} + c_3\right)$$

2.11 problem 18

Internal problem ID [832]
Internal file name [`OUTPUT/832_Sunday_June_05_2022_01_50_44_AM_67612248/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 4.2, Higher order linear differential equations. Constant coefficients. page 180
Problem number: 18.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - 7y''' + 6y'' + 30y' - 36y = 0}$$

The characteristic equation is

$$\lambda^4 - 7\lambda^3 + 6\lambda^2 + 30\lambda - 36 = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= 3\\ \lambda_2 &= -2\\ \lambda_3 &= 3 - \sqrt{3}\\ \lambda_4 &= 3 + \sqrt{3}\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-2x} + c_2 e^{3x} + e^{\left(3-\sqrt{3}\right)x} c_3 + e^{\left(3+\sqrt{3}\right)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-2x} \\
y_2 &= \mathrm{e}^{3x} \\
y_3 &= \mathrm{e}^{\left(3-\sqrt{3}\right)x} \\
y_4 &= \mathrm{e}^{\left(3+\sqrt{3}\right)x}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{3x} + \mathrm{e}^{\left(3-\sqrt{3}\right)x}c_3 + \mathrm{e}^{\left(3+\sqrt{3}\right)x}c_4 \tag{1}$$

Verification of solutions

$$y = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{3x} + \mathrm{e}^{\left(3-\sqrt{3}\right)x}c_3 + \mathrm{e}^{\left(3+\sqrt{3}\right)x}c_4$$

Verified OK.

2.11.1 Maple step by step solution

Let's solve

$y'''' - 7y''' + 6y'' + 30y' - 36y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Define new variable $y_4(x)$

    $y_4(x) = y'''$

  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = 7y_4(x) - 6y_3(x) - 30y_2(x) + 36y_1(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 7y_4(x) - 6y_3(x) - 30y_2(x) + 36y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 36 & -30 & -6 & 7 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 36 & -30 & -6 & 7 \end{bmatrix}$$

- Rewrite the system as

$\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right], \left[3-\sqrt{3}, \begin{bmatrix} \frac{1}{(3-\sqrt{3})^3} \\ \frac{1}{(3-\sqrt{3})^2} \\ \frac{1}{3-\sqrt{3}} \\ 1 \end{bmatrix}\right], \left[3+\sqrt{3}, \begin{bmatrix} \frac{1}{(3+\sqrt{3})^3} \\ \frac{1}{(3+\sqrt{3})^2} \\ \frac{1}{3+\sqrt{3}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-2x} \cdot \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{3x} \cdot \begin{bmatrix} \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3-\sqrt{3}, \begin{bmatrix} \frac{1}{\left(3-\sqrt{3}\right)^3} \\ \frac{1}{\left(3-\sqrt{3}\right)^2} \\ \frac{1}{3-\sqrt{3}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \mathrm{e}^{\left(3-\sqrt{3}\right)x} \cdot \begin{bmatrix} \frac{1}{\left(3-\sqrt{3}\right)^3} \\ \frac{1}{\left(3-\sqrt{3}\right)^2} \\ \frac{1}{3-\sqrt{3}} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3+\sqrt{3}, \begin{bmatrix} \frac{1}{\left(3+\sqrt{3}\right)^3} \\ \frac{1}{\left(3+\sqrt{3}\right)^2} \\ \frac{1}{3+\sqrt{3}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_4 = \mathrm{e}^{\left(3+\sqrt{3}\right)x} \cdot \begin{bmatrix} \frac{1}{\left(3+\sqrt{3}\right)^3} \\ \frac{1}{\left(3+\sqrt{3}\right)^2} \\ \frac{1}{3+\sqrt{3}} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + c_4\vec{y}_4$$

- Substitute solutions into the general solution

$$\vec{y} = c_1\mathrm{e}^{-2x} \cdot \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_2\mathrm{e}^{3x} \cdot \begin{bmatrix} \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix} + \mathrm{e}^{\left(3-\sqrt{3}\right)x} c_3 \cdot \begin{bmatrix} \frac{1}{\left(3-\sqrt{3}\right)^3} \\ \frac{1}{\left(3-\sqrt{3}\right)^2} \\ \frac{1}{3-\sqrt{3}} \\ 1 \end{bmatrix} + \mathrm{e}^{\left(3+\sqrt{3}\right)x} c_4 \cdot \begin{bmatrix} \frac{1}{\left(3+\sqrt{3}\right)^3} \\ \frac{1}{\left(3+\sqrt{3}\right)^2} \\ \frac{1}{3+\sqrt{3}} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{5\left(c_3\left(\sqrt{3}+\frac{9}{5}\right)\mathrm{e}^{-x\left(-5+\sqrt{3}\right)} - \left(\sqrt{3}-\frac{9}{5}\right)c_4\mathrm{e}^{x\left(5+\sqrt{3}\right)} + \frac{4c_2\mathrm{e}^{5x}}{15} - \frac{9c_1}{10}\right)\mathrm{e}^{-2x}}{36}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 38

```
dsolve(diff(y(x),x$4)-7*diff(y(x),x$3)+6*diff(y(x),x$2)+30*diff(y(x),x)-36*y(x)=0,y(x), sing
```

$$y(x) = \left(c_1 e^{5x} + c_3 e^{x\left(5+\sqrt{3}\right)} + c_4 e^{-x\left(-5+\sqrt{3}\right)} + c_2\right) e^{-2x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 51

```
DSolve[y''''[x]-7*y'''[x]+6*y''[x]+30*y'[x]-36*y[x]==0,y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to c_1 e^{-\left(\left(\sqrt{3}-3\right)x\right)} + c_2 e^{\left(3+\sqrt{3}\right)x} + c_3 e^{-2x} + c_4 e^{3x}$$

3 Chapter 6.2, The Laplace Transform. Solution of Initial Value Problems. page 255

3.1 problem 8

Internal problem ID [833]
Internal file name [`OUTPUT/833_Sunday_June_05_2022_01_50_45_AM_6567445/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.2, The Laplace Transform. Solution of Initial Value Problems. page 255
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y' - 6y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = -1]$$

3.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$p(t) = -1$$
$$q(t) = -6$$
$$F = 0$$

Hence the ode is

$$y'' - y' - 6y = 0$$

The domain of $p(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - sY(s) + y(0) - 6Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1\\ y'(0) &= -1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 2 - s - sY(s) - 6Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s-2}{s^2 - s - 6}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{5s - 15} + \frac{4}{5(s+2)}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{1}{5s-15}\right) &= \frac{e^{3t}}{5}\\ \mathcal{L}^{-1}\left(\frac{4}{5(s+2)}\right) &= \frac{4\,e^{-2t}}{5}\end{aligned}$$

Adding the above results and simplifying gives

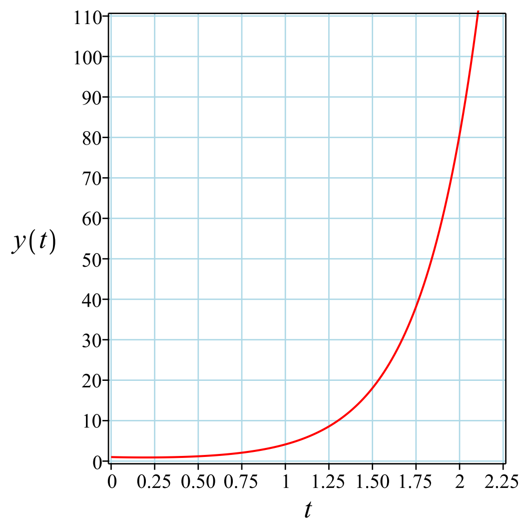
$$y = \frac{4\,\mathrm{e}^{-2t}}{5} + \frac{\mathrm{e}^{3t}}{5}$$

Simplifying the solution gives

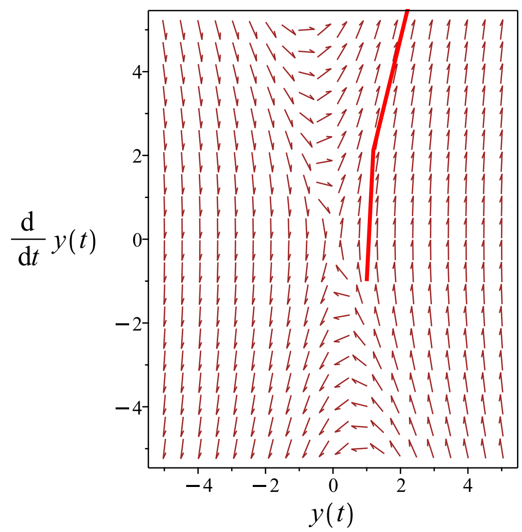
$$y = \frac{\left(\mathrm{e}^{5t} + 4\right)\mathrm{e}^{-2t}}{5}$$

Summary
The solution(s) found are the following

$$y = \frac{\left(\mathrm{e}^{5t} + 4\right)\mathrm{e}^{-2t}}{5} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\left(\mathrm{e}^{5t} + 4\right)\mathrm{e}^{-2t}}{5}$$

Verified OK.

3.1.2 Maple step by step solution

Let's solve

$\left[y''-y'-6y=0, y(0)=1, y'\Big|_{\{t=0\}}=-1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2-r-6=0$

- Factor the characteristic polynomial

  $(r+2)(r-3)=0$

- Roots of the characteristic polynomial

  $r=(-2,3)$

- 1st solution of the ODE

  $y_1(t)=\mathrm{e}^{-2t}$

- 2nd solution of the ODE

  $y_2(t)=\mathrm{e}^{3t}$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)$

- Substitute in solutions

  $y=c_1\mathrm{e}^{-2t}+c_2\mathrm{e}^{3t}$

- □ Check validity of solution $y=c_1\mathrm{e}^{-2t}+c_2\mathrm{e}^{3t}$

  - Use initial condition $y(0)=1$

    $1=c_1+c_2$

  - Compute derivative of the solution

    $y'=-2c_1\mathrm{e}^{-2t}+3c_2\mathrm{e}^{3t}$

  - Use the initial condition $y'\Big|_{\{t=0\}}=-1$

    $-1=-2c_1+3c_2$

  - Solve for c_1 and c_2

    $\left\{c_1=\frac{4}{5}, c_2=\frac{1}{5}\right\}$

- Substitute constant values into general solution and simplify

  $$y = \frac{(e^{5t}+4)e^{-2t}}{5}$$

- Solution to the IVP

  $$y = \frac{(e^{5t}+4)e^{-2t}}{5}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.609 (sec). Leaf size: 17

```
dsolve([diff(y(t),t$2)-diff(y(t),t)-6*y(t)=0,y(0) = 1, D(y)(0) = -1],y(t), singsol=all)
```

$$y(t) = \frac{(e^{5t} + 4)\,e^{-2t}}{5}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 21

```
DSolve[{y''[t]-y'[t]-6*y[t]==0,{y[0]==1,y'[0]==-1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{5}e^{-2t}\left(e^{5t} + 4\right)$$

3.2 problem 9

Internal problem ID [834]
Internal file name [`OUTPUT/834_Sunday_June_05_2022_01_50_46_AM_43487617/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.2, The Laplace Transform. Solution of Initial Value Problems. page 255
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 3y' + 2y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

3.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 3 \\ q(t) &= 2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 3y' + 2y = 0$$

The domain of $p(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 3sY(s) - 3y(0) + 2Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1 \\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 3 - s + 3sY(s) + 2Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s+3}{s^2+3s+2}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{1}{s+2} + \frac{2}{s+1}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}\mathcal{L}^{-1}\left(-\frac{1}{s+2}\right) &= -e^{-2t} \\ \mathcal{L}^{-1}\left(\frac{2}{s+1}\right) &= 2\,e^{-t}\end{aligned}$$

Adding the above results and simplifying gives

$$y = -e^{-2t} + 2e^{-t}$$

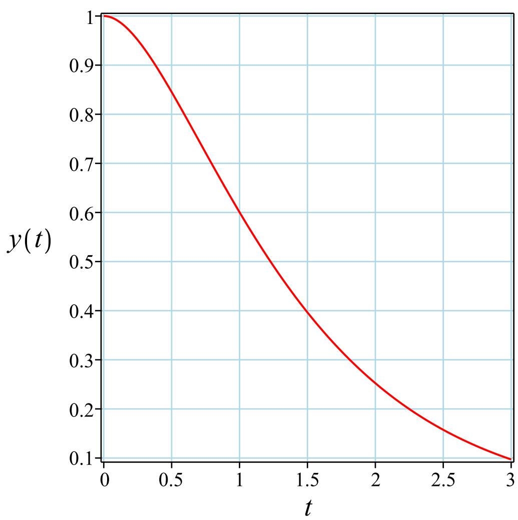
Simplifying the solution gives

$$y = -e^{-2t} + 2e^{-t}$$

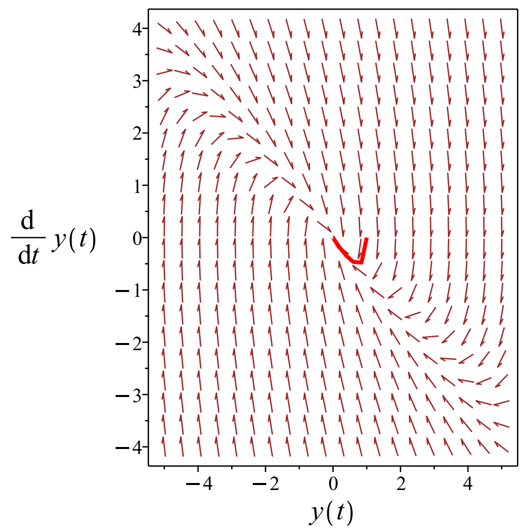
Summary

The solution(s) found are the following

$$y = -e^{-2t} + 2e^{-t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -e^{-2t} + 2e^{-t}$$

Verified OK.

3.2.2 Maple step by step solution

Let's solve

$$\left[y'' + 3y' + 2y = 0, y(0) = 1, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, -1)$

- 1st solution of the ODE

  $y_1(t) = e^{-2t}$

- 2nd solution of the ODE

  $y_2(t) = e^{-t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = c_1 e^{-2t} + c_2 e^{-t}$

☐ Check validity of solution $y = c_1 e^{-2t} + c_2 e^{-t}$

  - Use initial condition $y(0) = 1$

    $1 = c_1 + c_2$

  - Compute derivative of the solution

    $y' = -2c_1 e^{-2t} - c_2 e^{-t}$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 0$

    $0 = -2c_1 - c_2$

  - Solve for c_1 and c_2

    $\{c_1 = -1, c_2 = 2\}$

- Substitute constant values into general solution and simplify

  $y = -e^{-2t} + 2e^{-t}$

- Solution to the IVP

  $y = -e^{-2t} + 2e^{-t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.532 (sec). Leaf size: 17

```
dsolve([diff(y(t),t$2)+3*diff(y(t),t)+2*y(t)=0,y(0) = 1, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = 2e^{-t} - e^{-2t}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 18

```
DSolve[{y''[t]+3*y'[t]+2*y[t]==0,{y[0]==1,y'[0]==0}},y[t],t,IncludeSingularSolutions -> True
```

$$y(t) \to e^{-2t}(2e^t - 1)$$

3.3 problem 10

Internal problem ID [835]
Internal file name [`OUTPUT/835_Sunday_June_05_2022_01_50_47_AM_87709762/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.2, The Laplace Transform. Solution of Initial Value Problems. page 255
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' + 2y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

3.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -2 \\ q(t) &= 2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 2y' + 2y = 0$$

The domain of $p(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 2sY(s) + 2y(0) + 2Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0 \\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 - 2sY(s) + 2Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{1}{s^2 - 2s + 2}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{i}{2(s - 1 - i)} + \frac{i}{2s - 2 + 2i}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}\mathcal{L}^{-1}\left(-\frac{i}{2(s - 1 - i)}\right) &= -\frac{i\,\mathrm{e}^{(1+i)t}}{2} \\ \mathcal{L}^{-1}\left(\frac{i}{2s - 2 + 2i}\right) &= \frac{i\,\mathrm{e}^{(1-i)t}}{2}\end{aligned}$$

Adding the above results and simplifying gives

$$y = \sin(t)\,e^{t}$$

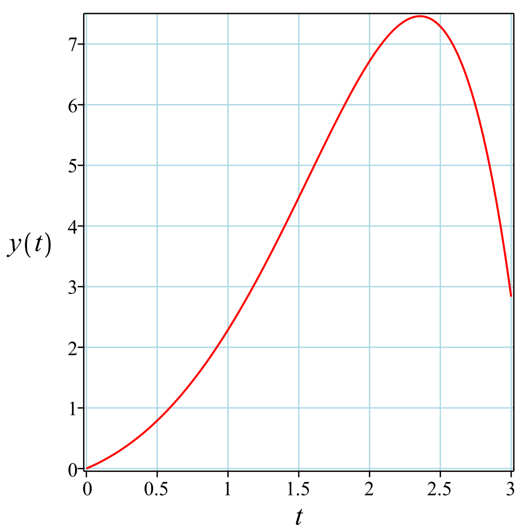
Simplifying the solution gives

$$y = \sin(t)\,e^{t}$$

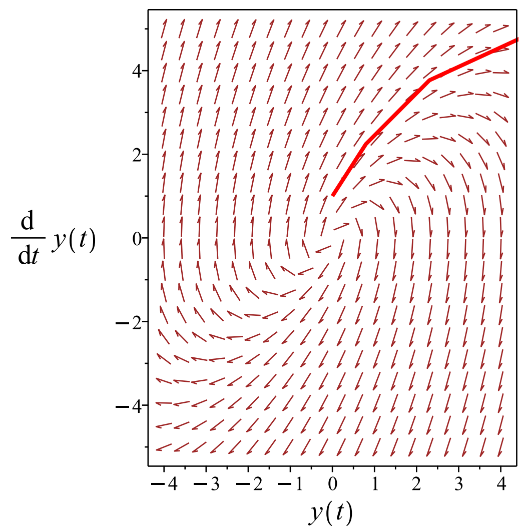
Summary

The solution(s) found are the following

$$y = \sin(t)\,e^{t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sin(t)\,e^{t}$$

Verified OK.

3.3.2 Maple step by step solution

Let's solve

$$\left[y'' - 2y' + 2y = 0, y(0) = 0, y'\Big|_{\{t=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 2r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (1 - \mathrm{I}, 1 + \mathrm{I})$

- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^t \cos(t)$

- 2nd solution of the ODE

  $y_2(t) = \sin(t)\, \mathrm{e}^t$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^t \cos(t) + c_2 \sin(t)\, \mathrm{e}^t$

☐ Check validity of solution $y = c_1 \mathrm{e}^t \cos(t) + c_2 \sin(t)\, \mathrm{e}^t$

  - Use initial condition $y(0) = 0$

    $0 = c_1$

  - Compute derivative of the solution

    $y' = c_1 \mathrm{e}^t \cos(t) - c_1 \mathrm{e}^t \sin(t) + c_2 \cos(t)\, \mathrm{e}^t + c_2 \sin(t)\, \mathrm{e}^t$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 1$

    $1 = c_1 + c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 0, c_2 = 1\}$

- Substitute constant values into general solution and simplify

  $y = \sin(t)\, e^{t}$

- Solution to the IVP

  $y = \sin(t)\, e^{t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.5 (sec). Leaf size: 9

```
dsolve([diff(y(t),t$2)-2*diff(y(t),t)+2*y(t)=0,y(0) = 0, D(y)(0) = 1],y(t), singsol=all)
```

$$y(t) = e^{t} \sin(t)$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 11

```
DSolve[{y''[t]-2*y'[t]+2*y[t]==0,{y[0]==0,y'[0]==1}},y[t],t,IncludeSingularSolutions -> True
```

$$y(t) \to e^{t} \sin(t)$$

3.4 problem 11

Internal problem ID [836]
Internal file name [`OUTPUT/836_Sunday_June_05_2022_01_50_48_AM_37859137/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.2, The Laplace Transform. Solution of Initial Value Problems. page 255
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' + 4y = 0}$$

With initial conditions

$$[y(0) = 2, y'(0) = 0]$$

3.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -2 \\ q(t) &= 4 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 2y' + 4y = 0$$

The domain of $p(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 2sY(s) + 2y(0) + 4Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 2\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 4 - 2s - 2sY(s) + 4Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{2s - 4}{s^2 - 2s + 4}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1 + \frac{i\sqrt{3}}{3}}{s - 1 - i\sqrt{3}} + \frac{1 - \frac{i\sqrt{3}}{3}}{s - 1 + i\sqrt{3}}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{1 + \frac{i\sqrt{3}}{3}}{s - 1 - i\sqrt{3}}\right) &= \frac{\left(i\sqrt{3} + 3\right) \mathrm{e}^{\left(1+i\sqrt{3}\right)t}}{3}\\ \mathcal{L}^{-1}\left(\frac{1 - \frac{i\sqrt{3}}{3}}{s - 1 + i\sqrt{3}}\right) &= \frac{\left(-i\sqrt{3} + 3\right) \mathrm{e}^{\left(1-i\sqrt{3}\right)t}}{3}\end{aligned}$$

Adding the above results and simplifying gives

$$y = \frac{2\,e^{t}\left(3\cos\left(\sqrt{3}\,t\right) - \sin\left(\sqrt{3}\,t\right)\sqrt{3}\right)}{3}$$

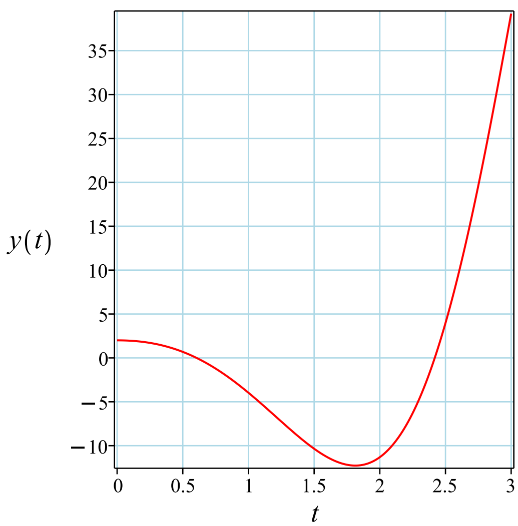
Simplifying the solution gives

$$y = -\frac{2\,e^{t}\left(\sin\left(\sqrt{3}\,t\right)\sqrt{3} - 3\cos\left(\sqrt{3}\,t\right)\right)}{3}$$

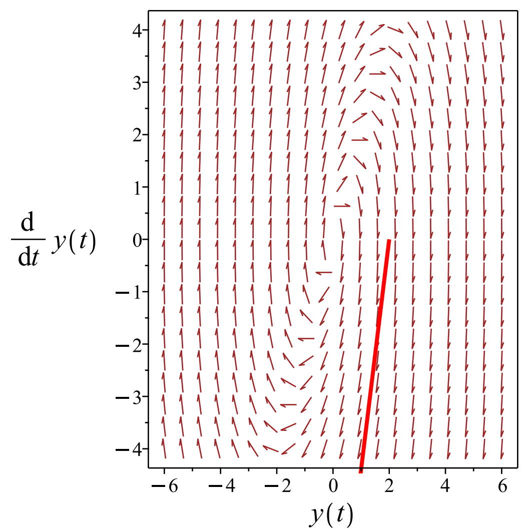
Summary

The solution(s) found are the following

$$y = -\frac{2\,e^{t}\left(\sin\left(\sqrt{3}\,t\right)\sqrt{3} - 3\cos\left(\sqrt{3}\,t\right)\right)}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{2\,e^{t}\left(\sin\left(\sqrt{3}\,t\right)\sqrt{3} - 3\cos\left(\sqrt{3}\,t\right)\right)}{3}$$

Verified OK.

3.4.2 Maple step by step solution

Let's solve

$\left[y''-2y'+4y=0, y(0)=2, y'\Big|_{\{t=0\}}=0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2-2r+4=0$

- Use quadratic formula to solve for r

  $r=\frac{2\pm(\sqrt{-12})}{2}$

- Roots of the characteristic polynomial

  $r=\left(1-\mathrm{I}\sqrt{3}, 1+\mathrm{I}\sqrt{3}\right)$

- 1st solution of the ODE

  $y_1(t)=\mathrm{e}^t\cos\left(\sqrt{3}\,t\right)$

- 2nd solution of the ODE

  $y_2(t)=\mathrm{e}^t\sin\left(\sqrt{3}\,t\right)$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)$

- Substitute in solutions

  $y=c_1\mathrm{e}^t\cos\left(\sqrt{3}\,t\right)+c_2\mathrm{e}^t\sin\left(\sqrt{3}\,t\right)$

□ Check validity of solution $y=c_1\mathrm{e}^t\cos\left(\sqrt{3}\,t\right)+c_2\mathrm{e}^t\sin\left(\sqrt{3}\,t\right)$

  - Use initial condition $y(0)=2$

    $2=c_1$

  - Compute derivative of the solution

    $y'=c_1\mathrm{e}^t\cos\left(\sqrt{3}\,t\right)-c_1\mathrm{e}^t\sin\left(\sqrt{3}\,t\right)\sqrt{3}+c_2\mathrm{e}^t\sin\left(\sqrt{3}\,t\right)+c_2\mathrm{e}^t\sqrt{3}\,\cos\left(\sqrt{3}\,t\right)$

  - Use the initial condition $y'\Big|_{\{t=0\}}=0$

    $0=c_1+\sqrt{3}\,c_2$

  - Solve for c_1 and c_2

$$\left\{c_1 = 2, c_2 = -\frac{2\sqrt{3}}{3}\right\}$$

- Substitute constant values into general solution and simplify

$$y = -\frac{2\,\mathrm{e}^{t}\left(\sin\left(\sqrt{3}\,t\right)\sqrt{3}-3\cos\left(\sqrt{3}\,t\right)\right)}{3}$$

- Solution to the IVP

$$y = -\frac{2\,\mathrm{e}^{t}\left(\sin\left(\sqrt{3}\,t\right)\sqrt{3}-3\cos\left(\sqrt{3}\,t\right)\right)}{3}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.547 (sec). Leaf size: 28

```
dsolve([diff(y(t),t$2)-2*diff(y(t),t)+4*y(t)=0,y(0) = 2, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = -\frac{2\left(\sqrt{3}\,\sin\left(\sqrt{3}\,t\right)-3\cos\left(\sqrt{3}\,t\right)\right)\mathrm{e}^{t}}{3}$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 37

```
DSolve[{y''[t]-2*y'[t]+4*y[t]==0,{y[0]==2,y'[0]==0}},y[t],t,IncludeSingularSolutions -> True
```

$$y(t) \to -\frac{2}{3}e^{t}\left(\sqrt{3}\sin\left(\sqrt{3}t\right)-3\cos\left(\sqrt{3}t\right)\right)$$

3.5 problem 12

Internal problem ID [837]
Internal file name [`OUTPUT/837_Sunday_June_05_2022_01_50_49_AM_17159875/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.2, The Laplace Transform. Solution of Initial Value Problems. page 255
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y' + 5y = 0}$$

With initial conditions

$$[y(0) = 2, y'(0) = -1]$$

3.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 5 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 2y' + 5y = 0$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 2sY(s) - 2y(0) + 5Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 2\\ y'(0) &= -1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 3 - 2s + 2sY(s) + 5Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{2s + 3}{s^2 + 2s + 5}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1 - \frac{i}{4}}{s + 1 - 2i} + \frac{1 + \frac{i}{4}}{s + 1 + 2i}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{1 - \frac{i}{4}}{s + 1 - 2i}\right) &= \left(1 - \frac{i}{4}\right) e^{(-1+2i)t}\\ \mathcal{L}^{-1}\left(\frac{1 + \frac{i}{4}}{s + 1 + 2i}\right) &= \left(1 + \frac{i}{4}\right) e^{(-1-2i)t}\end{aligned}$$

Adding the above results and simplifying gives

$$y = \frac{e^{-t}(4\cos(2t) + \sin(2t))}{2}$$

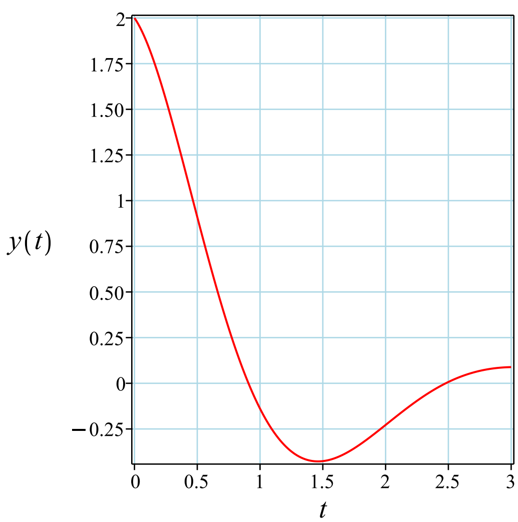
Simplifying the solution gives

$$y = \frac{e^{-t}(4\cos(2t) + \sin(2t))}{2}$$

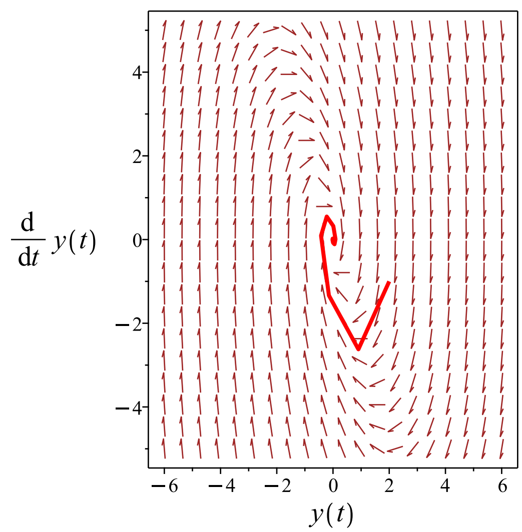
<u>Summary</u>

The solution(s) found are the following

$$y = \frac{e^{-t}(4\cos(2t) + \sin(2t))}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{e^{-t}(4\cos(2t) + \sin(2t))}{2}$$

Verified OK.

3.5.2 Maple step by step solution

Let's solve

$$\left[y''+2y'+5y=0, y(0)=2, y'\Big|_{\{t=0\}}=-1\right]$$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of ODE

  $r^2+2r+5=0$
- Use quadratic formula to solve for r

  $r=\frac{(-2)\pm(\sqrt{-16})}{2}$
- Roots of the characteristic polynomial

  $r=(-1-2\,\mathrm{I}, -1+2\,\mathrm{I})$
- 1st solution of the ODE

  $y_1(t)=\mathrm{e}^{-t}\cos(2t)$
- 2nd solution of the ODE

  $y_2(t)=\mathrm{e}^{-t}\sin(2t)$
- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)$
- Substitute in solutions

  $y=c_1\mathrm{e}^{-t}\cos(2t)+c_2\mathrm{e}^{-t}\sin(2t)$

□ Check validity of solution $y=c_1\mathrm{e}^{-t}\cos(2t)+c_2\mathrm{e}^{-t}\sin(2t)$

  - Use initial condition $y(0)=2$

    $2=c_1$
  - Compute derivative of the solution

    $y'=-c_1\mathrm{e}^{-t}\cos(2t)-2c_1\mathrm{e}^{-t}\sin(2t)-c_2\mathrm{e}^{-t}\sin(2t)+2c_2\mathrm{e}^{-t}\cos(2t)$
  - Use the initial condition $y'\Big|_{\{t=0\}}=-1$

    $-1=-c_1+2c_2$
  - Solve for c_1 and c_2

    $\left\{c_1=2, c_2=\frac{1}{2}\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{e^{-t}(4\cos(2t)+\sin(2t))}{2}$

- Solution to the IVP

  $y = \frac{e^{-t}(4\cos(2t)+\sin(2t))}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.563 (sec). Leaf size: 21

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)+5*y(t)=0,y(0) = 2, D(y)(0) = -1],y(t), singsol=all)
```

$$y(t) = \frac{e^{-t}(4\cos(2t) + \sin(2t))}{2}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 25

```
DSolve[{y''[t]+2*y'[t]+5*y[t]==0,{y[0]==2,y'[0]==-1}},y[t],t,IncludeSingularSolutions -> Tru
```

$$y(t) \to \frac{1}{2}e^{-t}(\sin(2t) + 4\cos(2t))$$

3.6 problem 13

Internal problem ID [838]
Internal file name [OUTPUT/838_Sunday_June_05_2022_01_50_50_AM_2433388/index.tex]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.2, The Laplace Transform. Solution of Initial Value Problems. page 255
Problem number: 13.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_laplace"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - 4y''' + 6y'' - 4y' + y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = 1]$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \\
\mathcal{L}(y''') &= s^3Y(s) - y''(0) - sy'(0) - s^2y(0) \\
\mathcal{L}(y'''') &= s^4Y(s) - y'''(0) - sy''(0) - s^2y'(0) - s^3y(0)
\end{aligned}$$

The given ode becomes an algebraic equation in the Laplace domain

$$s^4Y(s)-y'''(0)-sy''(0)-s^2y'(0)-s^3y(0)-4s^3Y(s)+4y''(0)+4sy'(0)+4s^2y(0)+6s^2Y(s)-6y'(0)-6sy(0)-4s \tag{1}$$

But the initial conditions are

$$
\begin{aligned}
y(0) &= 0 \\
y'(0) &= 1 \\
y''(0) &= 0 \\
y'''(0) &= 1
\end{aligned}
$$

Substituting these initial conditions in above in Eq (1) gives

$$
s^4 Y(s) - 7 - s^2 - 4s^3 Y(s) + 4s + 6s^2 Y(s) - 4sY(s) + Y(s) = 0
$$

Solving the above equation for $Y(s)$ results in

$$
Y(s) = \frac{s^2 - 4s + 7}{s^4 - 4s^3 + 6s^2 - 4s + 1}
$$

Applying partial fractions decomposition results in

$$
Y(s) = \frac{4}{(s-1)^4} - \frac{2}{(s-1)^3} + \frac{1}{(s-1)^2}
$$

The inverse Laplace of each term above is now found, which gives

$$
\begin{aligned}
\mathcal{L}^{-1}\left(\frac{4}{(s-1)^4}\right) &= \frac{2t^3 e^t}{3} \\
\mathcal{L}^{-1}\left(-\frac{2}{(s-1)^3}\right) &= -t^2 e^t \\
\mathcal{L}^{-1}\left(\frac{1}{(s-1)^2}\right) &= t\, e^t
\end{aligned}
$$

Adding the above results and simplifying gives

$$
y = \frac{e^t (2t^3 - 3t^2 + 3t)}{3}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \frac{e^t (2t^3 - 3t^2 + 3t)}{3} \tag{1}
$$

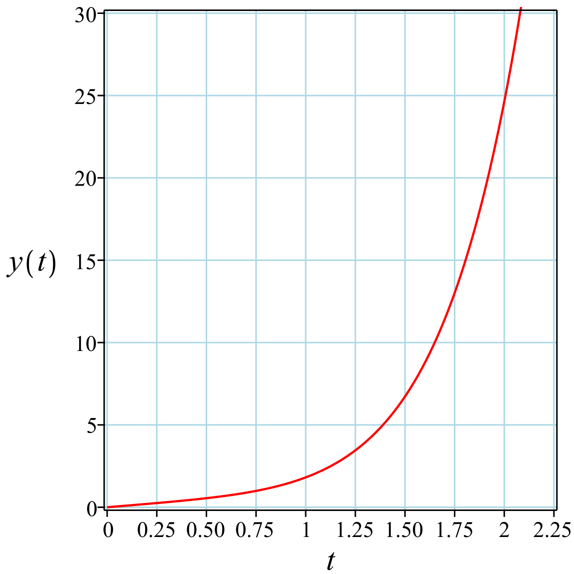


Figure 6: Solution plot

Verification of solutions

$$y = \frac{e^t(2t^3 - 3t^2 + 3t)}{3}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.579 (sec). Leaf size: 22

```
dsolve([diff(y(t),t$4)-4*diff(y(t),t$3)+6*diff(y(t),t$2)-4*diff(y(t),t)+y(t)=0,y(0) = 0, D(y
```

$$y(t) = \frac{e^t t(2t^2 - 3t + 3)}{3}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 23

```
DSolve[{y''''[t]-4*y'''[t]+6*y''[t]-4*y'[t]+y[t]==0,{y[0]==0,y'[0]==1,y''[0]==0,y'''[0]==1}}
```

$$y(t) \to \frac{1}{3}e^t t(2t^2 - 3t + 3)$$

3.7 problem 14

Internal problem ID [839]
Internal file name [`OUTPUT/839_Sunday_June_05_2022_01_50_52_AM_19041891/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.2, The Laplace Transform. Solution of Initial Value Problems. page 255
Problem number: 14.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_laplace"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - 4y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0, y''(0) = 1, y'''(0) = 0]$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \\
\mathcal{L}(y''') &= s^3Y(s) - y''(0) - sy'(0) - s^2y(0) \\
\mathcal{L}(y'''') &= s^4Y(s) - y'''(0) - sy''(0) - s^2y'(0) - s^3y(0)
\end{aligned}$$

The given ode becomes an algebraic equation in the Laplace domain

$$s^4Y(s) - y'''(0) - sy''(0) - s^2y'(0) - s^3y(0) - 4Y(s) = 0 \tag{1}$$

But the initial conditions are

$$
\begin{aligned}
y(0) &= 1 \\
y'(0) &= 0 \\
y''(0) &= 1 \\
y'''(0) &= 0
\end{aligned}
$$

Substituting these initial conditions in above in Eq (1) gives

$$
s^4Y(s) - s - s^3 - 4Y(s) = 0
$$

Solving the above equation for $Y(s)$ results in

$$
Y(s) = \frac{s(s^2+1)}{s^4-4}
$$

Applying partial fractions decomposition results in

$$
Y(s) = \frac{1}{8s-8i\sqrt{2}} + \frac{1}{8s+8i\sqrt{2}} + \frac{3}{8\left(s-\sqrt{2}\right)} + \frac{3}{8\left(s+\sqrt{2}\right)}
$$

The inverse Laplace of each term above is now found, which gives

$$
\begin{aligned}
\mathcal{L}^{-1}\left(\frac{1}{8s-8i\sqrt{2}}\right) &= \frac{e^{i\sqrt{2}\,t}}{8} \\
\mathcal{L}^{-1}\left(\frac{1}{8s+8i\sqrt{2}}\right) &= \frac{e^{-i\sqrt{2}\,t}}{8} \\
\mathcal{L}^{-1}\left(\frac{3}{8\left(s-\sqrt{2}\right)}\right) &= \frac{3\,e^{\sqrt{2}\,t}}{8} \\
\mathcal{L}^{-1}\left(\frac{3}{8\left(s+\sqrt{2}\right)}\right) &= \frac{3\,e^{-\sqrt{2}\,t}}{8}
\end{aligned}
$$

Adding the above results and simplifying gives

$$
y = \frac{\cos\left(\sqrt{2}\,t\right)}{4} + \frac{3\cosh\left(\sqrt{2}\,t\right)}{4}
$$

Summary

The solution(s) found are the following

$$
y = \frac{\cos\left(\sqrt{2}\,t\right)}{4} + \frac{3\cosh\left(\sqrt{2}\,t\right)}{4} \tag{1}
$$

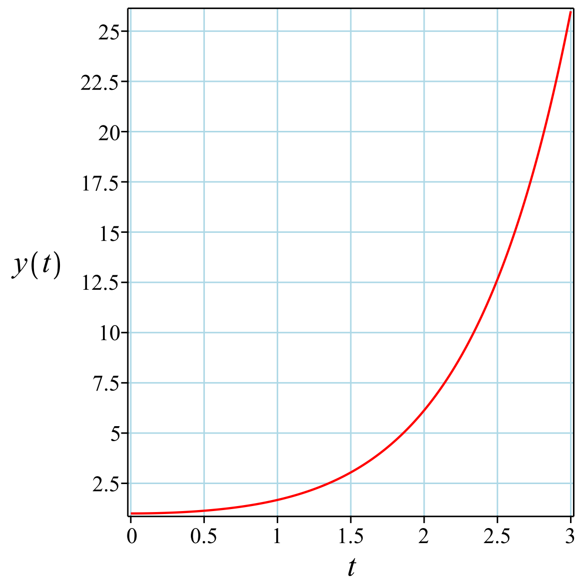


Figure 7: Solution plot

Verification of solutions

$$y = \frac{\cos\left(\sqrt{2}\, t\right)}{4} + \frac{3 \cosh\left(\sqrt{2}\, t\right)}{4}$$

Verified OK.

3.7.1 Maple step by step solution

Let's solve

$$\left[y'''' - 4y = 0, y(0) = 1, y'\Big|_{\{t=0\}} = 0, y''\Big|_{\{t=0\}} = 1, y'''\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(t)$

    $y_1(t) = y$
  - Define new variable $y_2(t)$

    $y_2(t) = y'$
  - Define new variable $y_3(t)$

$y_3(t) = y''$

- Define new variable $y_4(t)$

  $y_4(t) = y'''$

- Isolate for $y_4'(t)$ using original ODE

  $y_4'(t) = 4y_1(t)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(t) = y_1'(t), y_3(t) = y_2'(t), y_4(t) = y_3'(t), y_4'(t) = 4y_1(t)]$

- Define vector

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{bmatrix}$$

- System to solve

$$\vec{y}'(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(t) = A \cdot \vec{y}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[\sqrt{2}, \begin{bmatrix} \frac{\sqrt{2}}{4} \\ \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}\sqrt{2}, \begin{bmatrix} -\frac{\mathrm{I}}{4}\sqrt{2} \\ -\frac{1}{2} \\ \frac{\mathrm{I}}{2}\sqrt{2} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}\sqrt{2}, \begin{bmatrix} \frac{\mathrm{I}}{4}\sqrt{2} \\ -\frac{1}{2} \\ -\frac{\mathrm{I}}{2}\sqrt{2} \\ 1 \end{bmatrix}\right], \left[-\sqrt{2}, \begin{bmatrix} -\frac{\sqrt{2}}{4} \\ \frac{1}{2} \\ -\frac{\sqrt{2}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[\sqrt{2}, \begin{bmatrix} \frac{\sqrt{2}}{4} \\ \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{\sqrt{2}\,t} \cdot \begin{bmatrix} \frac{\sqrt{2}}{4} \\ \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}\sqrt{2}, \begin{bmatrix} -\frac{\mathrm{I}}{4}\sqrt{2} \\ -\frac{1}{2} \\ \frac{\mathrm{I}}{2}\sqrt{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I}\sqrt{2}\,t} \cdot \begin{bmatrix} -\frac{\mathrm{I}}{4}\sqrt{2} \\ -\frac{1}{2} \\ \frac{\mathrm{I}}{2}\sqrt{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\left(\cos\left(\sqrt{2}\,t\right) - \mathrm{I}\sin\left(\sqrt{2}\,t\right)\right) \cdot \begin{bmatrix} -\frac{\mathrm{I}}{4}\sqrt{2} \\ -\frac{1}{2} \\ \frac{\mathrm{I}}{2}\sqrt{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\frac{\mathrm{I}}{4}\left(\cos\left(\sqrt{2}\,t\right) - \mathrm{I}\sin\left(\sqrt{2}\,t\right)\right)\sqrt{2} \\ -\frac{\cos\left(\sqrt{2}\,t\right)}{2} + \frac{\mathrm{I}\sin\left(\sqrt{2}\,t\right)}{2} \\ \frac{\mathrm{I}}{2}\left(\cos\left(\sqrt{2}\,t\right) - \mathrm{I}\sin\left(\sqrt{2}\,t\right)\right)\sqrt{2} \\ \cos\left(\sqrt{2}\,t\right) - \mathrm{I}\sin\left(\sqrt{2}\,t\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(t) = \begin{bmatrix} -\frac{\sqrt{2}\sin\left(\sqrt{2}\,t\right)}{4} \\ -\frac{\cos\left(\sqrt{2}\,t\right)}{2} \\ \frac{\sqrt{2}\sin\left(\sqrt{2}\,t\right)}{2} \\ \cos\left(\sqrt{2}\,t\right) \end{bmatrix}, \vec{y}_3(t) = \begin{bmatrix} -\frac{\cos\left(\sqrt{2}\,t\right)\sqrt{2}}{4} \\ \frac{\sin\left(\sqrt{2}\,t\right)}{2} \\ \frac{\cos\left(\sqrt{2}\,t\right)\sqrt{2}}{2} \\ -\sin\left(\sqrt{2}\,t\right) \end{bmatrix}\right]$$

- Consider eigenpair

$$\left[-\sqrt{2}, \begin{bmatrix} -\frac{\sqrt{2}}{4} \\ \frac{1}{2} \\ -\frac{\sqrt{2}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_4 = \mathrm{e}^{-\sqrt{2}\,t} \cdot \begin{bmatrix} -\frac{\sqrt{2}}{4} \\ \frac{1}{2} \\ -\frac{\sqrt{2}}{2} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2(t) + c_3\vec{y}_3(t) + c_4\vec{y}_4$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 \mathrm{e}^{\sqrt{2}\,t} \cdot \left[\begin{array}{c} \frac{\sqrt{2}}{4} \\ \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ 1 \end{array}\right] + c_4 \mathrm{e}^{-\sqrt{2}\,t} \cdot \left[\begin{array}{c} -\frac{\sqrt{2}}{4} \\ \frac{1}{2} \\ -\frac{\sqrt{2}}{2} \\ 1 \end{array}\right] + \left[\begin{array}{c} -\frac{c_2\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)}{4} - \frac{c_3\cos\left(\sqrt{2}\,t\right)\sqrt{2}}{4} \\ -\frac{c_2\cos\left(\sqrt{2}\,t\right)}{2} + \frac{c_3\sin\left(\sqrt{2}\,t\right)}{2} \\ \frac{c_2\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)}{2} + \frac{c_3\cos\left(\sqrt{2}\,t\right)\sqrt{2}}{2} \\ c_2\cos\left(\sqrt{2}\,t\right) - c_3\sin\left(\sqrt{2}\,t\right) \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = -\frac{\sqrt{2}\left(c_3\cos\left(\sqrt{2}\,t\right)+c_2\sin\left(\sqrt{2}\,t\right)-c_1\mathrm{e}^{\sqrt{2}\,t}+c_4\mathrm{e}^{-\sqrt{2}\,t}\right)}{4}$$

- Use the initial condition $y(0) = 1$

$$1 = -\frac{\sqrt{2}\,(c_3-c_1+c_4)}{4}$$

- Calculate the 1st derivative of the solution

$$y' = -\frac{\sqrt{2}\left(-c_3\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)+c_2\cos\left(\sqrt{2}\,t\right)\sqrt{2}-c_1\mathrm{e}^{\sqrt{2}\,t}\sqrt{2}-c_4\mathrm{e}^{-\sqrt{2}\,t}\sqrt{2}\right)}{4}$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

$$0 = -\frac{\sqrt{2}\left(\sqrt{2}\,c_2-\sqrt{2}\,c_1-c_4\sqrt{2}\right)}{4}$$

- Calculate the 2nd derivative of the solution

$$y'' = -\frac{\sqrt{2}\left(-2c_3\cos\left(\sqrt{2}\,t\right)-2c_2\sin\left(\sqrt{2}\,t\right)-2c_1\mathrm{e}^{\sqrt{2}\,t}+2c_4\mathrm{e}^{-\sqrt{2}\,t}\right)}{4}$$

- Use the initial condition $y''\Big|_{\{t=0\}} = 1$

$$1 = -\frac{\sqrt{2}\,(-2c_3-2c_1+2c_4)}{4}$$

- Calculate the 3rd derivative of the solution

$$y''' = -\frac{\sqrt{2}\left(2c_3\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)-2c_2\cos\left(\sqrt{2}\,t\right)\sqrt{2}-2c_1\mathrm{e}^{\sqrt{2}\,t}\sqrt{2}-2c_4\mathrm{e}^{-\sqrt{2}\,t}\sqrt{2}\right)}{4}$$

- Use the initial condition $y'''\Big|_{\{t=0\}} = 0$

$$0 = -\frac{\sqrt{2}\left(-2\sqrt{2}\,c_2-2\sqrt{2}\,c_1-2c_4\sqrt{2}\right)}{4}$$

- Solve for the unknown coefficients

$$\left\{c_1 = \frac{3\sqrt{2}}{4}, c_2 = 0, c_3 = -\frac{\sqrt{2}}{2}, c_4 = -\frac{3\sqrt{2}}{4}\right\}$$

- Solution to the IVP

$$y = \frac{\cos\left(\sqrt{2}\, t\right)}{4} + \frac{3\, e^{\sqrt{2}\, t}}{8} + \frac{3\, e^{-\sqrt{2}\, t}}{8}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.516 (sec). Leaf size: 21

```
dsolve([diff(y(t),t$4)-4*y(t)=0,y(0) = 1, D(y)(0) = 0, (D@@2)(y)(0) = 1, (D@@3)(y)(0) = 0],y
```

$$y(t) = \frac{\cos\left(t\sqrt{2}\right)}{4} + \frac{3\cosh\left(t\sqrt{2}\right)}{4}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 43

```
DSolve[{y''''[t]-4*y[t]==0,{y[0]==1,y'[0]==0,y''[0]==1,y'''[0]==0}},y[t],t,IncludeSingularSo
```

$$y(t) \to \frac{1}{8}\left(3e^{-\sqrt{2}t} + 3e^{\sqrt{2}t} + 2\cos\left(\sqrt{2}t\right)\right)$$

3.8 problem 15

Internal problem ID [840]
Internal file name [`OUTPUT/840_Sunday_June_05_2022_01_50_53_AM_32515336/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.2, The Laplace Transform. Solution of Initial Value Problems. page 255
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + \omega^2 y = \cos(2t)}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

3.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= \omega^2 \\ F &= \cos(2t) \end{aligned}$$

Hence the ode is

$$y'' + \omega^2 y = \cos(2t)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \omega^2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \cos(2t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + \omega^2 Y(s) = \frac{s}{s^2+4} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - s + \omega^2 Y(s) = \frac{s}{s^2+4}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s(s^2+5)}{(s^2+4)(\omega^2+s^2)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{\omega^2-5}{(2\omega^2-8)\left(s-\sqrt{-\omega^2}\right)} + \frac{\omega^2-5}{(2\omega^2-8)\left(s+\sqrt{-\omega^2}\right)} + \frac{1}{2(\omega^2-4)(s-2i)} + \frac{1}{2(\omega^2-4)(s+2i)}$$

The inverse Laplace of each term above is now found, which gives

$$
\begin{aligned}
\mathcal{L}^{-1}\left(\frac{\omega^2-5}{(2\omega^2-8)\left(s-\sqrt{-\omega^2}\right)}\right) &= \frac{(\omega^2-5)\,\mathrm{e}^{i\,\mathrm{csgn}(i\omega)\omega t}}{2\omega^2-8} \\
\mathcal{L}^{-1}\left(\frac{\omega^2-5}{(2\omega^2-8)\left(s+\sqrt{-\omega^2}\right)}\right) &= \frac{(\omega^2-5)\,\mathrm{e}^{-i\,\mathrm{csgn}(i\omega)\omega t}}{2\omega^2-8} \\
\mathcal{L}^{-1}\left(\frac{1}{2\,(\omega^2-4)\,(s-2i)}\right) &= \frac{\mathrm{e}^{2it}}{2\omega^2-8} \\
\mathcal{L}^{-1}\left(\frac{1}{2\,(\omega^2-4)\,(s+2i)}\right) &= \frac{\mathrm{e}^{-2it}}{2\omega^2-8}
\end{aligned}
$$

Adding the above results and simplifying gives

$$
y = \frac{\cos(2t)+\cos(\omega t)\,(\omega^2-5)}{\omega^2-4}
$$

Simplifying the solution gives

$$
y = \frac{\cos(2t)+\cos(\omega t)\,(\omega^2-5)}{\omega^2-4}
$$

Summary

The solution(s) found are the following

$$
y = \frac{\cos(2t)+\cos(\omega t)\,(\omega^2-5)}{\omega^2-4} \tag{1}
$$

Verification of solutions

$$
y = \frac{\cos(2t)+\cos(\omega t)\,(\omega^2-5)}{\omega^2-4}
$$

Verified OK.

3.8.2 Maple step by step solution

Let's solve

$$
\left[y''+\omega^2 y=\cos(2t),\, y(0)=1,\, y'\Big|_{\{t=0\}}=0\right]
$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $\omega^2 + r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm \left(\sqrt{-4\omega^2}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(\sqrt{-\omega^2}, -\sqrt{-\omega^2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{\sqrt{-\omega^2}\,t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-\sqrt{-\omega^2}\,t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{\sqrt{-\omega^2}\,t} + c_2 \mathrm{e}^{-\sqrt{-\omega^2}\,t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \cos(2t)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{\sqrt{-\omega^2}\,t} & \mathrm{e}^{-\sqrt{-\omega^2}\,t} \\ \sqrt{-\omega^2}\,\mathrm{e}^{\sqrt{-\omega^2}\,t} & -\sqrt{-\omega^2}\,\mathrm{e}^{-\sqrt{-\omega^2}\,t} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = -2\sqrt{-\omega^2}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = \frac{\mathrm{e}^{\sqrt{-\omega^2}\,t}\left(\int \mathrm{e}^{-\sqrt{-\omega^2}\,t} \cos(2t) dt\right) - \mathrm{e}^{-\sqrt{-\omega^2}\,t}\left(\int \mathrm{e}^{\sqrt{-\omega^2}\,t} \cos(2t) dt\right)}{2\sqrt{-\omega^2}}$

  - Compute integrals

    $y_p(t) = \frac{\cos(2t)}{\omega^2 - 4}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{\sqrt{-\omega^2}\,t} + c_2 \mathrm{e}^{-\sqrt{-\omega^2}\,t} + \frac{\cos(2t)}{\omega^2 - 4}$

- ☐ Check validity of solution $y = c_1 e^{\sqrt{-\omega^2}\, t} + c_2 e^{-\sqrt{-\omega^2}\, t} + \frac{\cos(2t)}{\omega^2-4}$
  - Use initial condition $y(0) = 1$

    $1 = c_1 + c_2 + \frac{1}{\omega^2-4}$
  - Compute derivative of the solution

    $y' = c_1\sqrt{-\omega^2}\, e^{\sqrt{-\omega^2}\, t} - c_2\sqrt{-\omega^2}\, e^{-\sqrt{-\omega^2}\, t} - \frac{2\sin(2t)}{\omega^2-4}$
  - Use the initial condition $y'\Big|_{\{t=0\}} = 0$

    $0 = c_1\sqrt{-\omega^2} - c_2\sqrt{-\omega^2}$
  - Solve for c_1 and c_2

    $\left\{c_1 = \frac{\omega^2-5}{2(\omega^2-4)}, c_2 = \frac{\omega^2-5}{2(\omega^2-4)}\right\}$
  - Substitute constant values into general solution and simplify

    $y = \frac{2\cos(2t) + \left(e^{\sqrt{-\omega^2}\, t} + e^{-\sqrt{-\omega^2}\, t}\right)\omega^2 - 5\, e^{\sqrt{-\omega^2}\, t} - 5\, e^{-\sqrt{-\omega^2}\, t}}{2\omega^2-8}$
- Solution to the IVP

  $y = \frac{2\cos(2t) + \left(e^{\sqrt{-\omega^2}\, t} + e^{-\sqrt{-\omega^2}\, t}\right)\omega^2 - 5\, e^{\sqrt{-\omega^2}\, t} - 5\, e^{-\sqrt{-\omega^2}\, t}}{2\omega^2-8}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.562 (sec). Leaf size: 27

```
dsolve([diff(y(t),t$2)+omega^2*y(t)=cos(2*t),y(0) = 1, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = \frac{\cos(2t) + \cos(\omega t)(\omega^2 - 5)}{\omega^2 - 4}$$

✓ Solution by Mathematica

Time used: 0.209 (sec). Leaf size: 28

```
DSolve[{y''[t]+w^2*y[t]==Cos[2*t],{y[0]==1,y'[0]==0}},y[t],t,IncludeSingularSolutions -> Tru
```

$$y(t) \to \frac{(w^2 - 5)\cos(tw) + \cos(2t)}{w^2 - 4}$$

3.9 problem 16

Internal problem ID [841]
Internal file name [`OUTPUT/841_Sunday_June_05_2022_01_50_54_AM_6735311/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.2, The Laplace Transform. Solution of Initial Value Problems. page 255
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 2y' + 2y = e^{-t}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

3.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -2 \\ q(t) &= 2 \\ F &= e^{-t} \end{aligned}$$

Hence the ode is

$$y'' - 2y' + 2y = e^{-t}$$

The domain of $p(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \mathrm{e}^{-t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 2sY(s) + 2y(0) + 2Y(s) = \frac{1}{s+1} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 - 2sY(s) + 2Y(s) = \frac{1}{s+1}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{2+s}{(s+1)(s^2-2s+2)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{-\frac{1}{10} - \frac{7i}{10}}{s-1-i} + \frac{-\frac{1}{10} + \frac{7i}{10}}{s-1+i} + \frac{1}{5s+5}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{-\frac{1}{10}-\frac{7i}{10}}{s-1-i}\right) = \left(-\frac{1}{10}-\frac{7i}{10}\right) \mathrm{e}^{(1+i)t}$$
$$\mathcal{L}^{-1}\left(\frac{-\frac{1}{10}+\frac{7i}{10}}{s-1+i}\right) = \left(-\frac{1}{10}+\frac{7i}{10}\right) \mathrm{e}^{(1-i)t}$$
$$\mathcal{L}^{-1}\left(\frac{1}{5s+5}\right) = \frac{\mathrm{e}^{-t}}{5}$$

Adding the above results and simplifying gives

$$y = \frac{\mathrm{e}^{-t}}{5} + \frac{(-\cos(t) + 7\sin(t))\,\mathrm{e}^{t}}{5}$$

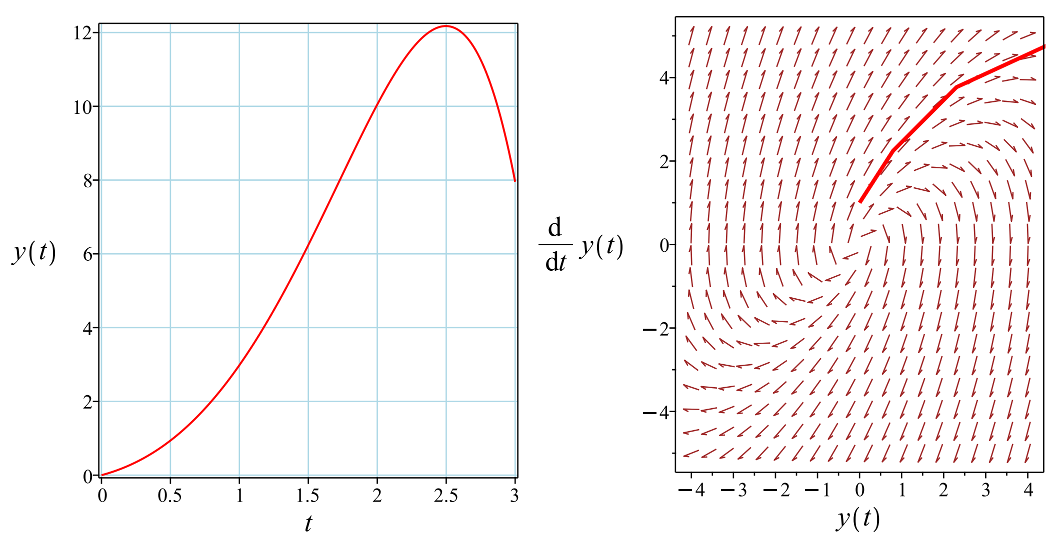
Simplifying the solution gives

$$y = \frac{\mathrm{e}^{-t}}{5} + \frac{(-\cos(t) + 7\sin(t))\,\mathrm{e}^{t}}{5}$$

Summary

The solution(s) found are the following

$$y = \frac{\mathrm{e}^{-t}}{5} + \frac{(-\cos(t) + 7\sin(t))\,\mathrm{e}^{t}}{5} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{e^{-t}}{5} + \frac{(-\cos(t) + 7\sin(t))\, e^{t}}{5}$$

Verified OK.

3.9.2 Maple step by step solution

Let's solve

$\left[y'' - 2y' + 2y = e^{-t}, y(0) = 0, y'\Big|_{\{t=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (1 - I, 1 + I)$

- 1st solution of the homogeneous ODE

  $y_1(t) = e^{t} \cos(t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(t)\, e^{t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{t} \cos(t) + c_2 \sin(t)\, e^{t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = e^{-t}\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \left[\begin{array}{cc} e^t \cos(t) & \sin(t) e^t \\ e^t \cos(t) - \sin(t) e^t & e^t \cos(t) + \sin(t) e^t \end{array}\right]$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = e^{2t}$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -e^t\left(\cos(t)\left(\int e^{-2t} \sin(t)\, dt\right) - \sin(t)\left(\int e^{-2t} \cos(t)\, dt\right)\right)$

- Compute integrals

  $y_p(t) = \frac{e^{-t}}{5}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^t \cos(t) + c_2 \sin(t) e^t + \frac{e^{-t}}{5}$

☐ Check validity of solution $y = c_1 e^t \cos(t) + c_2 \sin(t) e^t + \frac{e^{-t}}{5}$

- Use initial condition $y(0) = 0$

  $0 = c_1 + \frac{1}{5}$

- Compute derivative of the solution

  $y' = c_1 e^t \cos(t) - c_1 e^t \sin(t) + c_2 \cos(t) e^t + c_2 \sin(t) e^t - \frac{e^{-t}}{5}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 1$

  $1 = c_1 - \frac{1}{5} + c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = -\frac{1}{5}, c_2 = \frac{7}{5}\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{e^{-t}}{5} + \frac{(-\cos(t) + 7\sin(t))e^t}{5}$

- Solution to the IVP

  $y = \frac{e^{-t}}{5} + \frac{(-\cos(t) + 7\sin(t))e^t}{5}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.578 (sec). Leaf size: 24

```
dsolve([diff(y(t),t$2)-2*diff(y(t),t)+2*y(t)=exp(-t),y(0) = 0, D(y)(0) = 1],y(t), singsol=al
```

$$y(t) = \frac{e^{-t}}{5} + \frac{(-\cos(t) + 7\sin(t))\, e^{t}}{5}$$

✓ Solution by Mathematica

Time used: 0.071 (sec). Leaf size: 29

```
DSolve[{y''[t]-2*y'[t]+2*y[t]==Exp[-t],{y[0]==0,y'[0]==1}},y[t],t,IncludeSingularSolutions -
```

$$y(t) \to \frac{1}{5}\left(e^{-t} + 7e^{t}\sin(t) - e^{t}\cos(t)\right)$$

3.10 problem 17

Internal problem ID [842]
Internal file name [`OUTPUT/842_Sunday_June_05_2022_01_50_55_AM_70964363/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.2, The Laplace Transform. Solution of Initial Value Problems. page 255
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$y'' + 4y = \begin{cases} 1 & 0 \leq t < \pi \\ 0 & \pi \leq t < \infty \end{cases}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

3.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$p(t) = 0$$
$$q(t) = 4$$
$$F = \begin{cases} 0 & t < 0 \\ 1 & t < \pi \\ 0 & \pi \leq t \end{cases}$$

Hence the ode is

$$y'' + 4y = \begin{cases} 0 & t < 0 \\ 1 & t < \pi \\ 0 & \pi \le t \end{cases}$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \begin{cases} 0 & t < 0 \\ 1 & t < \pi \\ 0 & \pi \le t \end{cases}$ is

$$\{0 \le t \le \pi, \pi \le t \le \infty, -\infty \le t \le 0\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned} \mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 4Y(s) = \frac{1 - e^{-\pi s}}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned} y(0) &= 1 \\ y'(0) &= 0 \end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - s + 4Y(s) = \frac{1 - e^{-\pi s}}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{-s^2 + e^{-\pi s} - 1}{s(s^2 + 4)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(-\frac{-s^2 + e^{-\pi s} - 1}{s(s^2 + 4)}\right) \\ &= \frac{3\cos(2t)}{4} - \frac{\text{Heaviside}(t - \pi)\sin(t)^2}{2} + \frac{1}{4} \end{aligned}$$

Hence the final solution is

$$y = \frac{3\cos(2t)}{4} - \frac{\text{Heaviside}(t - \pi)\sin(t)^2}{2} + \frac{1}{4}$$

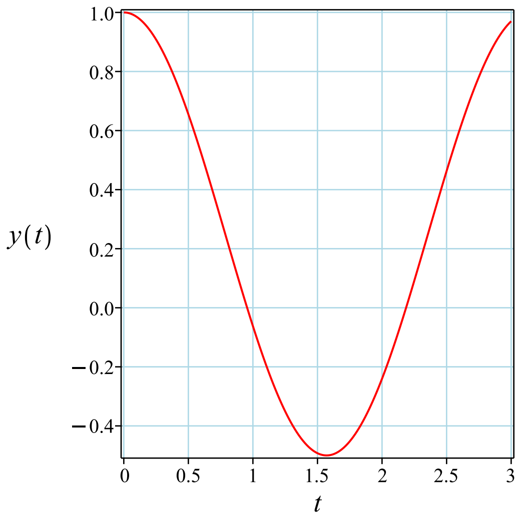
Simplifying the solution gives

$$y = -\frac{\text{Heaviside}(t - \pi)\sin(t)^2}{2} + \frac{3\cos(t)^2}{2} - \frac{1}{2}$$

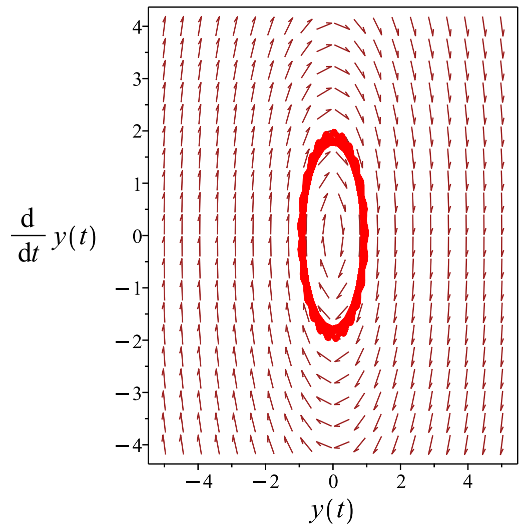
Summary

The solution(s) found are the following

$$y = -\frac{\text{Heaviside}(t - \pi)\sin(t)^2}{2} + \frac{3\cos(t)^2}{2} - \frac{1}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{\text{Heaviside}(t-\pi)\sin(t)^2}{2} + \frac{3\cos(t)^2}{2} - \frac{1}{2}$$

Verified OK.

3.10.2 Maple step by step solution

Let's solve

$$\left[y'' + 4y = \begin{cases} 0 & t < 0 \\ 1 & t < \pi \\ 0 & \pi \leq t \end{cases}, y(0) = 1, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

$r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(2t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(2t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- Use variation of parameters to find y_p here $f(t)$ is the forcing function

$$\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \begin{cases} 0 & t < 0 \\ 1 & t < \pi \\ 0 & \pi \leq t \end{cases}\right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 2$

- Substitute functions into equation for $y_p(t)$

$$y_p(t) = -\cos(2t)\left(\int \left(\begin{cases} 0 & t < 0 \\ \frac{\sin(2t)}{2} & t < \pi \\ 0 & \pi \leq t \end{cases}\right) dt\right) + \sin(2t)\left(\int \left(\begin{cases} 0 & t < 0 \\ \frac{\cos(2t)}{2} & t < \pi \\ 0 & \pi \leq t \end{cases}\right) dt\right)$$

- Compute integrals

$$y_p(t) = \begin{cases} 0 & t \leq 0 \\ -\frac{\cos(2t)}{4} + \frac{1}{4} & t \leq \pi \\ 0 & \pi < t \end{cases}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(2t) + c_2 \sin(2t) + \begin{cases} 0 & t \leq 0 \\ -\frac{\cos(2t)}{4} + \frac{1}{4} & t \leq \pi \\ 0 & \pi < t \end{cases}$$

☐ Check validity of solution $y = c_1 \cos(2t) + c_2 \sin(2t) + \begin{cases} 0 & t \leq 0 \\ -\frac{\cos(2t)}{4} + \frac{1}{4} & t \leq \pi \\ 0 & \pi < t \end{cases}$

- Use initial condition $y(0) = 1$

  $1 = c_1$
- Compute derivative of the solution

  $$y' = -2c_1 \sin(2t) + 2c_2 \cos(2t) + \begin{cases} 0 & t \leq 0 \\ \frac{\sin(2t)}{2} & t \leq \pi \\ 0 & \pi < t \end{cases}$$
- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = 2c_2$
- Solve for c_1 and c_2

  $\{c_1 = 1, c_2 = 0\}$
- Substitute constant values into general solution and simplify

  $$y = \cos(2t) - \left(\begin{cases} 0 & t \leq 0 \\ -\frac{1}{4} + \frac{\cos(2t)}{4} & t \leq \pi \\ 0 & \pi < t \end{cases}\right)$$

- Solution to the IVP

  $$y = \cos(2t) - \left(\begin{cases} 0 & t \leq 0 \\ -\frac{1}{4} + \frac{\cos(2t)}{4} & t \leq \pi \\ 0 & \pi < t \end{cases}\right)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.891 (sec). Leaf size: 33

```
dsolve([diff(y(t),t$2)+4*y(t)=piecewise(0<=t and t<Pi,1,Pi<=t and t<infinity,0),y(0) = 1, D(
```

$$y(t) = \begin{cases} \frac{3\cos(2t)}{4} + \frac{1}{4} & t < \pi \\ \cos(2t) & \pi \leq t \end{cases}$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 31

```
DSolve[{y''[t]+4*y[t]==Piecewise[{{1,0<t<Pi},{0,Pi<=t<Infinity}}],{y[0]==1,y'[0]==0}},y[t],t
```

$$y(t) \to \begin{cases} \cos(2t) & t > \pi \vee t \leq 0 \\ \frac{1}{4}(3\cos(2t) + 1) & \text{True} \end{cases}$$

3.11 problem 18

Internal problem ID [843]
Internal file name [`OUTPUT/843_Sunday_June_05_2022_01_50_58_AM_84931335/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.2, The Laplace Transform. Solution of Initial Value Problems. page 255
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$
\boxed{y'' + 4y = \begin{cases} 1 & 0 \le t < 1 \\ 0 & 1 \le t < \infty \end{cases}}
$$

With initial conditions

$$
[y(0) = 0, y'(0) = 0]
$$

3.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$
y'' + p(t)y' + q(t)y = F
$$

Where here

$$
\begin{aligned}
p(t) &= 0 \\
q(t) &= 4 \\
F &= \begin{cases} 0 & t < 0 \\ 1 & t < 1 \\ 0 & 1 \le t \end{cases}
\end{aligned}
$$

Hence the ode is

$$y'' + 4y = \begin{cases} 0 & t < 0 \\ 1 & t < 1 \\ 0 & 1 \le t \end{cases}$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \begin{cases} 0 & t < 0 \\ 1 & t < 1 \\ 0 & 1 \le t \end{cases}$ is

$$\{0 \le t \le 1, 1 \le t \le \infty, -\infty \le t \le 0\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$
$$\mathcal{L}(y'') = s^2Y(s) - y'(0) - sy(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 4Y(s) = \frac{1 - e^{-s}}{s} \tag{1}$$

But the initial conditions are

$$y(0) = 0$$
$$y'(0) = 0$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2 Y(s) + 4Y(s) = \frac{1 - \mathrm{e}^{-s}}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{-1 + \mathrm{e}^{-s}}{s\,(s^2 + 4)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(-\frac{-1 + \mathrm{e}^{-s}}{s\,(s^2 + 4)}\right) \\ &= -\frac{\cos(2t)}{4} + \frac{1}{4} - \frac{\mathrm{Heaviside}(-1 + t)\sin(-1 + t)^2}{2} \end{aligned}$$

Hence the final solution is

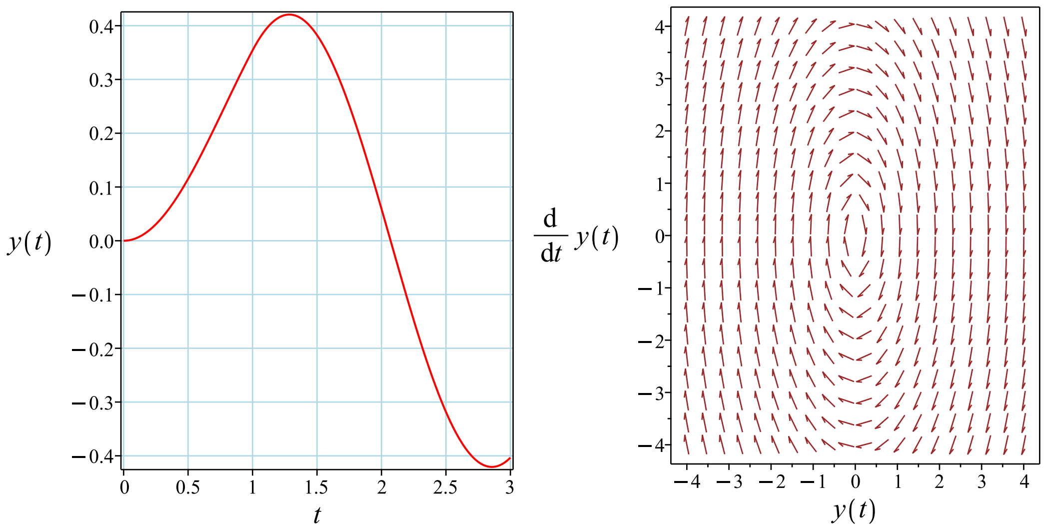
$$y = -\frac{\cos(2t)}{4} + \frac{1}{4} - \frac{\mathrm{Heaviside}(-1 + t)\sin(-1 + t)^2}{2}$$

Simplifying the solution gives

$$y = -\frac{\cos(2t)}{4} + \frac{1}{4} - \frac{\mathrm{Heaviside}(-1 + t)\sin(-1 + t)^2}{2}$$

Summary
The solution(s) found are the following

$$y = -\frac{\cos(2t)}{4} + \frac{1}{4} - \frac{\mathrm{Heaviside}(-1 + t)\sin(-1 + t)^2}{2} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{\cos(2t)}{4} + \frac{1}{4} - \frac{\text{Heaviside}(-1+t)\sin(-1+t)^2}{2}$$

Verified OK.

3.11.2 Maple step by step solution

Let's solve

$$\left[y'' + 4y = \begin{cases} 0 & t < 0 \\ 1 & t < 1 \\ 0 & 1 \leq t \end{cases}, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2
  y''
- Characteristic polynomial of homogeneous ODE
  $r^2 + 4 = 0$
- Use quadratic formula to solve for r
  $r = \frac{0 \pm (\sqrt{-16})}{2}$
- Roots of the characteristic polynomial

$r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(2t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(2t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- Use variation of parameters to find y_p here $f(t)$ is the forcing function

$$\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = \begin{cases} 0 & t<0 \\ 1 & t<1 \\ 0 & 1 \leq t \end{cases}\right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 2$

- Substitute functions into equation for $y_p(t)$

$$y_p(t) = -\cos(2t)\left(\int \left(\begin{cases} 0 & t<0 \\ \frac{\sin(2t)}{2} & t<1 \\ 0 & 1 \leq t \end{cases}\right)dt\right) + \sin(2t)\left(\int \left(\begin{cases} 0 & t<0 \\ \frac{\cos(2t)}{2} & t<1 \\ 0 & 1 \leq t \end{cases}\right)dt\right)$$

- Compute integrals

$$y_p(t) = -\frac{\left(\begin{cases} 0 & t \leq 0 \\ -1+\cos(2t) & t \leq 1 \\ 2\sin(1)^2\cos(2t) - \sin(2)\sin(2t) & 1<t \end{cases}\right)}{4}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(2t) + c_2 \sin(2t) - \frac{\left(\begin{cases} 0 & t \le 0 \\ -1 + \cos(2t) & t \le 1 \\ 2\sin(1)^2 \cos(2t) - \sin(2)\sin(2t) & 1 < t \end{cases}\right)}{4}$$

- ☐ Check validity of solution $y = c_1 \cos(2t) + c_2 \sin(2t) - \frac{\left(\begin{cases} 0 & t \le 0 \\ -1 + \cos(2t) & t \le 1 \\ 2\sin(1)^2 \cos(2t) - \sin(2)\sin(2t) & 1 < t \end{cases}\right)}{4}$
  - Use initial condition $y(0) = 0$

    $0 = c_1$
  - Compute derivative of the solution

    $$y' = -2c_1 \sin(2t) + 2c_2 \cos(2t) - \frac{\left(\begin{cases} 0 & t \le 0 \\ -2\sin(2t) & t \le 1 \\ -4\sin(1)^2 \sin(2t) - 2\sin(2)\cos(2t) & 1 < t \end{cases}\right)}{4}$$
  - Use the initial condition $y'\Big|_{\{t=0\}} = 0$

    $0 = 2c_2$
  - Solve for c_1 and c_2

    $\{c_1 = 0, c_2 = 0\}$
  - Substitute constant values into general solution and simplify

    $$y = -\frac{\left(\begin{cases} 0 & t \le 0 \\ -1 + \cos(2t) & t \le 1 \\ 2\sin(1)^2 \cos(2t) - \sin(2)\sin(2t) & 1 < t \end{cases}\right)}{4}$$
- Solution to the IVP

  $$y = -\frac{\left(\begin{cases} 0 & t \le 0 \\ -1 + \cos(2t) & t \le 1 \\ 2\sin(1)^2 \cos(2t) - \sin(2)\sin(2t) & 1 < t \end{cases}\right)}{4}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.875 (sec). Leaf size: 35

```
dsolve([diff(y(t),t$2)+4*y(t)=piecewise(0<=t and t<1,1,1<=t and t<infinity,0),y(0) = 0, D(y)
```

$$y(t) = \frac{\left(\begin{cases} 1 & t < 1 \\ \cos(2t-2) & 1 \leq t \end{cases}\right)}{4} - \frac{\cos(2t)}{4}$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 39

```
DSolve[{y''[t]+4*y[t]==Piecewise[{{1,0<t<1},{0,1<=t<Infinity}}],{y[0]==0,y'[0]==0}},y[t],t,I
```

$$y(t) \to \begin{cases} 0 & t \leq 0 \\ \frac{\sin^2(t)}{2} & 0 < t \leq 1 \\ -\frac{1}{2}\sin(1)\sin(1-2t) & \text{True} \end{cases}$$

3.12 problem 19

Internal problem ID [844]
Internal file name [`OUTPUT/844_Sunday_June_05_2022_01_51_01_AM_11854967/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.2, The Laplace Transform. Solution of Initial Value Problems. page 255
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$y'' + y = \begin{cases} t & 0 \leq t < 1 \\ -t+2 & 1 \leq t < 2 \\ 0 & 2 \leq t < \infty \end{cases}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

3.12.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$
\begin{aligned}
p(t) &= 0 \\
q(t) &= 1 \\
F &= \begin{cases} 0 & t < 0 \\ t & t < 1 \\ -t+2 & t < 2 \\ 0 & 2 \leq t \end{cases}
\end{aligned}
$$

Hence the ode is

$$
y'' + y = \begin{cases} 0 & t < 0 \\ t & t < 1 \\ -t+2 & t < 2 \\ 0 & 2 \leq t \end{cases}
$$

The domain of $p(t) = 0$ is

$$
\{-\infty < t < \infty\}
$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$
\{-\infty < t < \infty\}
$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \begin{cases} 0 & t < 0 \\ t & t < 1 \\ -t+2 & t < 2 \\ 0 & 2 \leq t \end{cases}$ is

$$
\{-\infty < t < \infty\}
$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$
\mathcal{L}(y) = Y(s)
$$

Taking the Laplace transform of the ode and using the relations that

$$
\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2 Y(s) - y'(0) - s y(0)
\end{aligned}
$$

The given ode now becomes an algebraic equation in the Laplace domain

$$
s^2 Y(s) - y'(0) - s y(0) + Y(s) = \frac{\mathrm{e}^{-2s} - 2\,\mathrm{e}^{-s} + 1}{s^2} \tag{1}
$$

But the initial conditions are

$$
\begin{aligned}
y(0) &= 1 \\
y'(0) &= 0
\end{aligned}
$$

Substituting these initial conditions in above in Eq (1) gives

$$
s^2 Y(s) - s + Y(s) = \frac{\mathrm{e}^{-2s} - 2\,\mathrm{e}^{-s} + 1}{s^2}
$$

Solving the above equation for $Y(s)$ results in

$$
Y(s) = \frac{s^3 + \mathrm{e}^{-2s} - 2\,\mathrm{e}^{-s} + 1}{s^2 \left(s^2 + 1\right)}
$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{s^3 + \mathrm{e}^{-2s} - 2\,\mathrm{e}^{-s} + 1}{s^2 \left(s^2 + 1\right)}\right) \\
&= \cos(t) + \mathrm{Heaviside}(t-2)\,(t - 2 - \sin(t-2)) - 2\,\mathrm{Heaviside}(-1+t)\,(t - 1 - \sin(-1+t)) + t - \sin
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y = {} & \cos(t) + \mathrm{Heaviside}(t-2)\,(t - 2 - \sin(t-2)) \\
& - 2\,\mathrm{Heaviside}(-1+t)\,(t - 1 - \sin(-1+t)) + t - \sin(t)
\end{aligned}
$$

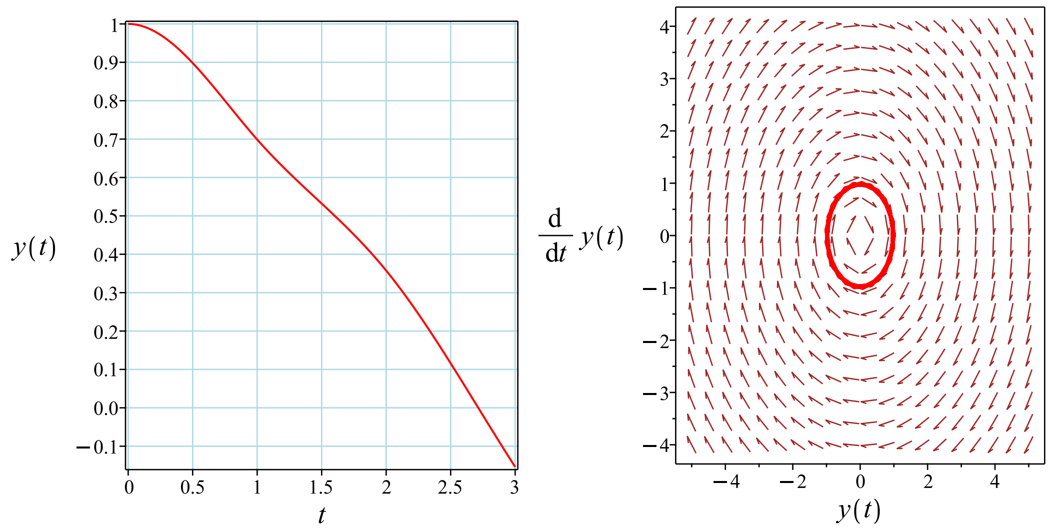
Simplifying the solution gives

$$
\begin{aligned}
y = {} & (-2t + 2 + 2\sin(-1+t))\,\mathrm{Heaviside}(-1+t) \\
& + \mathrm{Heaviside}(t-2)\,(t - 2 - \sin(t-2)) + t + \cos(t) - \sin(t)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & (-2t + 2 + 2\sin(-1+t))\,\mathrm{Heaviside}(-1+t) \\
& + \mathrm{Heaviside}(t-2)\,(t - 2 - \sin(t-2)) + t + \cos(t) - \sin(t)
\end{aligned} \tag{1}
$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = (-2t + 2 + 2\sin(-1 + t))\,\text{Heaviside}(-1 + t) + \text{Heaviside}(t - 2)(t - 2 - \sin(t - 2)) + t + \cos(t) - \sin(t)$$

Verified OK.

3.12.2 Maple step by step solution

Let's solve

$$\left[y'' + y = \begin{cases} 0 & t < 0 \\ t & t < 1 \\ -t + 2 & t < 2 \\ 0 & 2 \leq t \end{cases}, y(0) = 1, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(t) + c_2 \sin(t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

$$\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \begin{cases} 0 & t < 0 \\ t & t < 1 \\ -t+2 & t < 2 \\ 0 & 2 \leq t \end{cases}\right]$$

- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 1$

- ○ Substitute functions into equation for $y_p(t)$

$$y_p(t) = -\cos(t)\left(\int \sin(t)\left(\begin{cases} 0 & t < 0 \\ t & t < 1 \\ -t+2 & t < 2 \\ 0 & 2 \leq t \end{cases}\right) dt\right) + \sin(t)\left(\int \cos(t)\left(\begin{cases} 0 & t < 0 \\ t & t < 1 \\ -t+2 & t < 2 \\ 0 & 2 \leq t \end{cases}\right)\right.$$

- ○ Compute integrals

$$y_p(t) = \begin{cases} 0 & t \leq 0 \\ t - \sin(t) & t \leq 1 \\ (-1 + 2\cos(1))\sin(t) - 2\cos(t)\sin(1) - t + 2 & t \leq 2 \\ -2(\sin(t)\cos(1) - \cos(t)\sin(1))(\cos(1) - 1) & 2 < t \end{cases}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(t) + c_2 \sin(t) + \begin{cases} 0 & t \leq 0 \\ t - \sin(t) & t \leq 1 \\ (-1 + 2\cos(1))\sin(t) - 2\cos(t)\sin(1) - t + 2 & t \leq 2 \\ -2(\sin(t)\cos(1) - \cos(t)\sin(1))(\cos(1) - 1) & 2 < t \end{cases}$$

□ Check validity of solution $y = c_1 \cos(t) + c_2 \sin(t) + \begin{cases} 0 \\ t - \sin(t) \\ (-1 + 2\cos(1))\sin(t) - 2\cos(t)\sin(1 \\ -2(\sin(t)\cos(1) - \cos(t)\sin(1))(\cos \end{cases}$

- Use initial condition $y(0) = 1$

  $1 = c_1$

- Compute derivative of the solution

$$y' = -c_1 \sin(t) + c_2 \cos(t) + \begin{cases} 0 & t \leq 0 \\ 1 - \cos(t) & t \leq 1 \\ (-1 + 2\cos(1))\cos(t) + 2\sin(1)\sin(t) - 1 & t \leq 2 \\ -2(\cos(1)\cos(t) + \sin(1)\sin(t))(\cos(1) - 1) & 2 < t \end{cases}$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = c_2$

- Solve for c_1 and c_2

  $\{c_1 = 1, c_2 = 0\}$

- Substitute constant values into general solution and simplify

$$y = \cos(t) + \begin{cases} 0 & t \leq 0 \\ t - \sin(t) & t \leq 1 \\ (-1 + 2\cos(1))\sin(t) - 2\cos(t)\sin(1) - t + 2 & t \leq 2 \\ -2(\sin(t)\cos(1) - \cos(t)\sin(1))(\cos(1) - 1) & 2 < t \end{cases}$$

- Solution to the IVP

$$y = \cos(t) + \begin{cases} 0 & t \leq 0 \\ t - \sin(t) & t \leq 1 \\ (-1 + 2\cos(1))\sin(t) - 2\cos(t)\sin(1) - t + 2 & t \leq 2 \\ -2(\sin(t)\cos(1) - \cos(t)\sin(1))(\cos(1) - 1) & 2 < t \end{cases}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.907 (sec). Leaf size: 58

```
dsolve([diff(y(t),t$2)+y(t)=piecewise(0<=t and t<1,t,1<=t and t<2,2-t,2<=t and t<infinity,0)
```

$$y(t) = -\sin(t) + \cos(t) + \left(\begin{cases} t & t < 1 \\ 2 - t + 2\sin(t-1) & t < 2 \\ -\sin(t-2) + 2\sin(t-1) & 2 \leq t \end{cases} \right)$$

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 68

```
DSolve[{y''[t]+y[t]==Piecewise[{{t,0<t<1},{2-t,1<=t<2},{0,2<=t<Infinity}}],{y[0]==1,y'[0]==0
```

$$y(t)\to \begin{cases} \cos(t) & t\leq 0 \\ \cos(t)-4\sin^2\left(\frac{1}{2}\right)\sin(1-t) & t>2 \\ t+\cos(t)-\sin(t) & 0<t\leq 1 \\ -t+\cos(t)-2\sin(1-t)-\sin(t)+2 & \text{True} \end{cases}$$

4 Chapter 6.4, The Laplace Transform. Differential equations with discontinuous forcing functions. page 268

4.1 problem 1

Internal problem ID [845]
Internal file name [`OUTPUT/845_Sunday_June_05_2022_01_51_06_AM_6446313/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.4, The Laplace Transform. Differential equations with discontinuous forcing functions. page 268
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + y = \begin{cases} 1 & 0 \leq t < 3\pi \\ 0 & 3\pi \leq t < \infty \end{cases}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

4.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$
\begin{aligned}
p(t) &= 0 \\
q(t) &= 1 \\
F &= \begin{cases} 0 & t < 0 \\ 1 & t < 3\pi \\ 0 & 3\pi \leq t \end{cases}
\end{aligned}
$$

Hence the ode is

$$
y'' + y = \begin{cases} 0 & t < 0 \\ 1 & t < 3\pi \\ 0 & 3\pi \leq t \end{cases}
$$

The domain of $p(t) = 0$ is

$$
\{-\infty < t < \infty\}
$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$
\{-\infty < t < \infty\}
$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \begin{cases} 0 & t < 0 \\ 1 & t < 3\pi \\ 0 & 3\pi \leq t \end{cases}$ is

$$
\{0 \leq t \leq 3\pi, 3\pi \leq t \leq \infty, -\infty \leq t \leq 0\}
$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$
\mathcal{L}(y) = Y(s)
$$

Taking the Laplace transform of the ode and using the relations that

$$
\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2 Y(s) - y'(0) - sy(0)
\end{aligned}
$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + Y(s) = \frac{1 - e^{-3\pi s}}{s} \tag{1}$$

But the initial conditions are

$$y(0) = 0$$
$$y'(0) = 1$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 + Y(s) = \frac{1 - e^{-3\pi s}}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{-1 + e^{-3\pi s} - s}{s\,(s^2 + 1)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(-\frac{-1 + e^{-3\pi s} - s}{s\,(s^2 + 1)}\right) \\ &= -\cos(t) + 1 - 2\,\text{Heaviside}\,(t - 3\pi)\cos\left(\frac{t}{2}\right)^2 + \sin(t) \end{aligned}$$

Hence the final solution is

$$y = -\cos(t) + 1 - 2\,\text{Heaviside}\,(t - 3\pi)\cos\left(\frac{t}{2}\right)^2 + \sin(t)$$

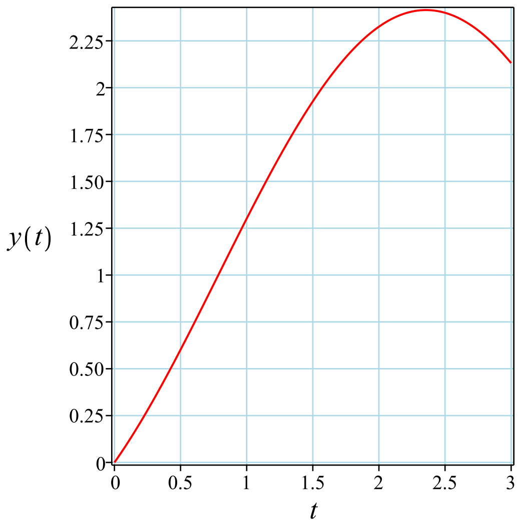
Simplifying the solution gives

$$y = -\cos(t) + 1 + (-1 - \cos(t))\,\text{Heaviside}\,(t - 3\pi) + \sin(t)$$

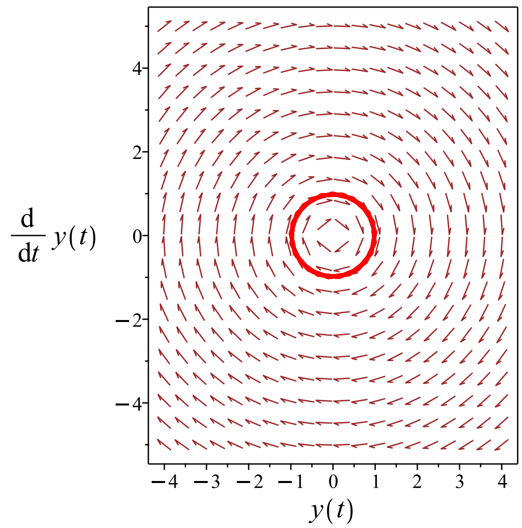
Summary

The solution(s) found are the following

$$y = -\cos(t) + 1 + (-1 - \cos(t))\,\text{Heaviside}\,(t - 3\pi) + \sin(t) \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\cos(t) + 1 + (-1 - \cos(t)) \operatorname{Heaviside}(t - 3\pi) + \sin(t)$$

Verified OK.

4.1.2 Maple step by step solution

Let's solve

$$\left[y'' + y = \begin{cases} 0 & t < 0 \\ 1 & t < 3\pi \\ 0 & 3\pi \leq t \end{cases}, y(0) = 0, y'\Big|_{\{t=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2
  y''
- Characteristic polynomial of homogeneous ODE
  $r^2 + 1 = 0$
- Use quadratic formula to solve for r
  $r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial
  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(t) + c_2 \sin(t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $$\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = \begin{cases} 0 & t < 0 \\ 1 & t < 3\pi \\ 0 & 3\pi \leq t \end{cases}\right]$$

  - Wronskian of solutions of the homogeneous equation

    $$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 1$

  - Substitute functions into equation for $y_p(t)$

    $$y_p(t) = -\cos(t)\left(\int \left(\begin{cases} 0 & t < 0 \\ \sin(t) & t < 3\pi \\ 0 & 3\pi \leq t \end{cases}\right) dt\right) + \sin(t)\left(\int \left(\begin{cases} 0 & t < 0 \\ \cos(t) & t < 3\pi \\ 0 & 3\pi \leq t \end{cases}\right) dt\right)$$

  - Compute integrals

    $$y_p(t) = \begin{cases} 0 & t \leq 0 \\ 1 - \cos(t) & t \leq 3\pi \\ -2\cos(t) & 3\pi < t \end{cases}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(t) + c_2 \sin(t) + \begin{cases} 0 & t \leq 0 \\ 1 - \cos(t) & t \leq 3\pi \\ -2\cos(t) & 3\pi < t \end{cases}$$

- ☐ Check validity of solution $y = c_1 \cos(t) + c_2 \sin(t) + \begin{cases} 0 & t \leq 0 \\ 1 - \cos(t) & t \leq 3\pi \\ -2\cos(t) & 3\pi < t \end{cases}$
  - Use initial condition $y(0) = 0$

    $0 = c_1$
  - Compute derivative of the solution

    $$y' = -c_1 \sin(t) + c_2 \cos(t) + \begin{cases} 0 & t \leq 0 \\ \sin(t) & t \leq 3\pi \\ 2\sin(t) & 3\pi < t \end{cases}$$
  - Use the initial condition $y'\Big|_{\{t=0\}} = 1$

    $1 = c_2$
  - Solve for c_1 and c_2

    $\{c_1 = 0, c_2 = 1\}$
  - Substitute constant values into general solution and simplify

    $$y = \sin(t) - \left(\begin{cases} 0 & t \leq 0 \\ \cos(t) - 1 & t \leq 3\pi \\ 2\cos(t) & 3\pi < t \end{cases} \right)$$

- Solution to the IVP

  $$y = \sin(t) - \left(\begin{cases} 0 & t \leq 0 \\ \cos(t) - 1 & t \leq 3\pi \\ 2\cos(t) & 3\pi < t \end{cases} \right)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.797 (sec). Leaf size: 39

```
dsolve([diff(y(t),t$2)+y(t)=piecewise(0<=t and t<3*Pi,1,3*Pi<=t and t<infinity,0),y(0) = 0,
```

$$y(t) = \sin(t) - \left(\begin{cases} \cos(t) - 1 & t < 3\pi \\ 2\cos(t) & 3\pi \leq t \end{cases}\right)$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 34

```
DSolve[{y''[t]+y[t]==Piecewise[{{1,0<=t<3*Pi},{0,3*Pi<=t<Infinity}}],{y[0]==0,y'[0]==1}},y[t
```

$$y(t) \to \left\{ \begin{array}{cc} \sin(t) & t \leq 0 \\ \sin(t) - 2\cos(t) & t > 3\pi \\ -\cos(t) + \sin(t) + 1 & \text{True} \end{array} \right.$$

4.2 problem 2

Internal problem ID [846]
Internal file name [`OUTPUT/846_Sunday_June_05_2022_01_51_09_AM_23570421/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.4, The Laplace Transform. Differential equations with discontinuous forcing functions. page 268
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$y'' + 2y' + 2y = \begin{cases} 1 & \pi \leq t < 2\pi \\ 0 & \text{otherwise} \end{cases}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

4.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$
\begin{aligned}
p(t) &= 2 \\
q(t) &= 2 \\
F &= \begin{cases} 0 & t < \pi \\ 1 & t < 2\pi \\ 0 & 2\pi \leq t \end{cases}
\end{aligned}
$$

Hence the ode is

$$
y'' + 2y' + 2y = \begin{cases} 0 & t < \pi \\ 1 & t < 2\pi \\ 0 & 2\pi \leq t \end{cases}
$$

The domain of $p(t) = 2$ is

$$
\{-\infty < t < \infty\}
$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$
\{-\infty < t < \infty\}
$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \begin{cases} 0 & t < \pi \\ 1 & t < 2\pi \\ 0 & 2\pi \leq t \end{cases}$ is

$$
\{\pi \leq t \leq 2\pi, 2\pi \leq t \leq \infty, -\infty \leq t \leq \pi\}
$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$
\mathcal{L}(y) = Y(s)
$$

Taking the Laplace transform of the ode and using the relations that

$$
\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)
\end{aligned}
$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 2sY(s) - 2y(0) + 2Y(s) = \frac{e^{-\pi s} - e^{-2\pi s}}{s} \tag{1}$$

But the initial conditions are

$$y(0) = 0$$
$$y'(0) = 1$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 + 2sY(s) + 2Y(s) = \frac{e^{-\pi s} - e^{-2\pi s}}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{e^{-\pi s} - e^{-2\pi s} + s}{s\,(s^2 + 2s + 2)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{e^{-\pi s} - e^{-2\pi s} + s}{s\,(s^2 + 2s + 2)}\right) \\
&= e^{-t}\sin(t) + \frac{(-1 + e^{2\pi - t}(\cos(t) + \sin(t)))\,\text{Heaviside}\,(t - 2\pi)}{2} + \frac{(1 + e^{\pi - t}(\cos(t) + \sin(t)))\,\text{Heaviside}}{2}
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y = e^{-t}\sin(t) &+ \frac{(-1 + e^{2\pi - t}(\cos(t) + \sin(t)))\,\text{Heaviside}\,(t - 2\pi)}{2} \\
&+ \frac{(1 + e^{\pi - t}(\cos(t) + \sin(t)))\,\text{Heaviside}\,(t - \pi)}{2}
\end{aligned}$$

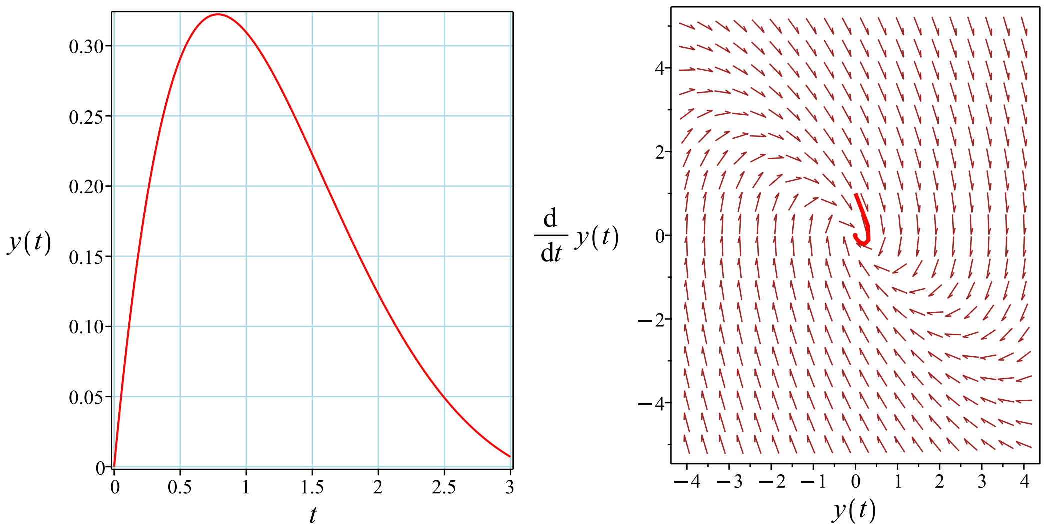
Simplifying the solution gives

$$\begin{aligned}
y = {}&\frac{\text{Heaviside}\,(t - 2\pi)\,(\cos(t) + \sin(t))\,e^{2\pi - t}}{2} - \frac{\text{Heaviside}\,(t - 2\pi)}{2} \\
&+ \frac{(1 + e^{\pi - t}(\cos(t) + \sin(t)))\,\text{Heaviside}\,(t - \pi)}{2} + e^{-t}\sin(t)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y = {}&\frac{\text{Heaviside}\,(t - 2\pi)\,(\cos(t) + \sin(t))\,e^{2\pi - t}}{2} - \frac{\text{Heaviside}\,(t - 2\pi)}{2} \\
&+ \frac{(1 + e^{\pi - t}(\cos(t) + \sin(t)))\,\text{Heaviside}\,(t - \pi)}{2} + e^{-t}\sin(t)
\end{aligned} \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\text{Heaviside}\,(t-2\pi)\,(\cos(t)+\sin(t))\,\mathrm{e}^{2\pi-t}}{2} - \frac{\text{Heaviside}\,(t-2\pi)}{2} + \frac{(1+\mathrm{e}^{\pi-t}(\cos(t)+\sin(t)))\,\text{Heaviside}\,(t-\pi)}{2} + \mathrm{e}^{-t}\sin(t)$$

Verified OK.

4.2.2 Maple step by step solution

Let's solve

$$\left[y''+2y'+2y = \begin{cases} 0 & t<\pi \\ 1 & t<2\pi \\ 0 & 2\pi \le t \end{cases}, y(0)=0, y'\Big|_{\{t=0\}}=1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2+2r+2=0$

- Use quadratic formula to solve for r

  $r=\frac{(-2)\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-1 - \mathrm{I}, -1 + \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-t}\cos(t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-t}\sin(t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-t}\cos(t) + c_2 \mathrm{e}^{-t}\sin(t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- Use variation of parameters to find y_p here $f(t)$ is the forcing function

$$\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \begin{cases} 0 & t < \pi \\ 1 & t < 2\pi \\ 0 & 2\pi \leq t \end{cases}\right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-t}\cos(t) & \mathrm{e}^{-t}\sin(t) \\ -\mathrm{e}^{-t}\cos(t) - \mathrm{e}^{-t}\sin(t) & -\mathrm{e}^{-t}\sin(t) + \mathrm{e}^{-t}\cos(t) \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = \mathrm{e}^{-2t}$

- Substitute functions into equation for $y_p(t)$

$$y_p(t) = -\mathrm{e}^{-t}\left(\cos(t)\left(\int \left(\begin{cases} 0 & t < \pi \\ \sin(t)\mathrm{e}^{t} & t < 2\pi \\ 0 & 2\pi \leq t \end{cases}\right) dt\right) - \sin(t)\left(\int \left(\begin{cases} 0 & t < \pi \\ \mathrm{e}^{t}\cos(t) & t < 2\pi \\ 0 & 2\pi \leq t \end{cases}\right) d$$

- Compute integrals

$$y_p(t) = \frac{\left(\begin{cases} 0 & t \leq \pi \\ 1 + \mathrm{e}^{\pi - t}(\cos(t) + \sin(t)) & t \leq 2\pi \\ \mathrm{e}^{\pi - t}(1 + \mathrm{e}^{\pi})(\cos(t) + \sin(t)) & 2\pi < t \end{cases}\right)}{2}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \mathrm{e}^{-t} \cos(t) + c_2 \mathrm{e}^{-t} \sin(t) + \frac{\left(\begin{cases} 0 & t \le \pi \\ 1 + \mathrm{e}^{\pi - t}(\cos(t) + \sin(t)) & t \le 2\pi \\ \mathrm{e}^{\pi - t}(1 + \mathrm{e}^{\pi})(\cos(t) + \sin(t)) & 2\pi < t \end{cases}\right)}{2}$$

☐ Check validity of solution $y = c_1 \mathrm{e}^{-t} \cos(t) + c_2 \mathrm{e}^{-t} \sin(t) + \frac{\left(\begin{cases} 0 & \\ 1 + \mathrm{e}^{\pi - t}(\cos(t) + \sin(t)) & t \\ \mathrm{e}^{\pi - t}(1 + \mathrm{e}^{\pi})(\cos(t) + \sin(t)) & 2 \end{cases}\right.}{2}$

  - Use initial condition $y(0) = 0$

    $0 = c_1$

  - Compute derivative of the solution

    $$y' = -c_1 \mathrm{e}^{-t} \cos(t) - c_1 \mathrm{e}^{-t} \sin(t) - c_2 \mathrm{e}^{-t} \sin(t) + c_2 \mathrm{e}^{-t} \cos(t) + \frac{\left(\begin{cases} \\ -\mathrm{e}^{\pi - t}(\cos(t) + \sin \\ -\mathrm{e}^{\pi - t}(1 + \mathrm{e}^{\pi})(\cos(t) + \sin \end{cases}\right.}{}$$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 1$

    $1 = -c_1 + c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 0, c_2 = 1\}$

  - Substitute constant values into general solution and simplify

    $$y = \mathrm{e}^{-t} \sin(t) + \frac{\left(\begin{cases} 0 & t \le \pi \\ 1 + \mathrm{e}^{\pi - t}(\cos(t) + \sin(t)) & t \le 2\pi \\ \mathrm{e}^{\pi - t}(1 + \mathrm{e}^{\pi})(\cos(t) + \sin(t)) & 2\pi < t \end{cases}\right)}{2}$$

- Solution to the IVP

$$y = \mathrm{e}^{-t} \sin(t) + \frac{\left(\begin{cases} 0 & t \le \pi \\ 1 + \mathrm{e}^{\pi - t}(\cos(t) + \sin(t)) & t \le 2\pi \\ \mathrm{e}^{\pi - t}(1 + \mathrm{e}^{\pi})(\cos(t) + \sin(t)) & 2\pi < t \end{cases}\right)}{2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.797 (sec). Leaf size: 83

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)+2*y(t)=piecewise(Pi<=t and t<2*Pi,1,true,0),y(0) = 0,
```

$$y(t) = \sin(t)\,e^{-t} + \frac{\left(\begin{cases} 0 & t < \pi \\ 1 + e^{\pi - t}(\cos(t) + \sin(t)) & t < 2\pi \\ (\cos(t) + \sin(t))\left(e^{\pi - t} + e^{2\pi - t}\right) & 2\pi \leq t \end{cases}\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 89

```
DSolve[{y''[t]+2*y'[t]+2*y[t]==Piecewise[{{1,Pi<=t<2*Pi},{0,True}}],{y[0]==0,y'[0]==1}},y[t]
```

$$y(t) \to \begin{cases} e^{-t}\sin(t) & t \leq \pi \\ \frac{1}{2}e^{-t}(e^{\pi}\cos(t) + e^{t} + (2 + e^{\pi})\sin(t)) & \pi < t \leq 2\pi \\ \frac{1}{2}e^{-t}(e^{\pi}(1 + e^{\pi})\cos(t) + (2 + e^{\pi} + e^{2\pi})\sin(t)) & \text{True} \end{cases}$$

4.3 problem 3

Internal problem ID [847]
Internal file name [`OUTPUT/847_Sunday_June_05_2022_01_51_16_AM_63013947/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.4, The Laplace Transform. Differential equations with discontinuous forcing functions. page 268
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = \sin(t) - \text{Heaviside}(t - 2\pi)\sin(t)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

4.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 4 \\ F &= \sin(t)\,(1 - \text{Heaviside}(t - 2\pi)) \end{aligned}$$

Hence the ode is

$$y'' + 4y = \sin(t)(1 - \text{Heaviside}(t - 2\pi))$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \sin(t)(1 - \text{Heaviside}(t - 2\pi))$ is

$$\{t < 2\pi \vee 2\pi < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 4Y(s) = \frac{-e^{-2\pi s} + 1}{s^2 + 1} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0 \\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 4Y(s) = \frac{-e^{-2\pi s} + 1}{s^2 + 1}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{e^{-2\pi s} - 1}{(s^2 + 1)(s^2 + 4)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(-\frac{e^{-2\pi s}-1}{(s^2+1)(s^2+4)}\right) \\ &= \frac{\text{Heaviside}(2\pi-t)(2\sin(t)-\sin(2t))}{6} \end{aligned}$$

Hence the final solution is

$$y = \frac{\text{Heaviside}(2\pi-t)(2\sin(t)-\sin(2t))}{6}$$

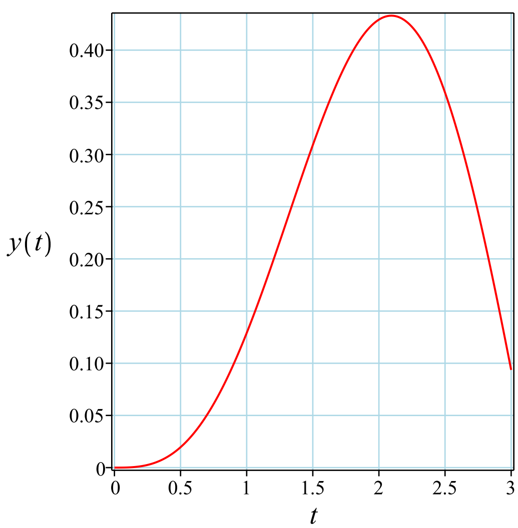
Simplifying the solution gives

$$y = \frac{\sin(t)(\cos(t)-1)(-1+\text{Heaviside}(t-2\pi))}{3}$$

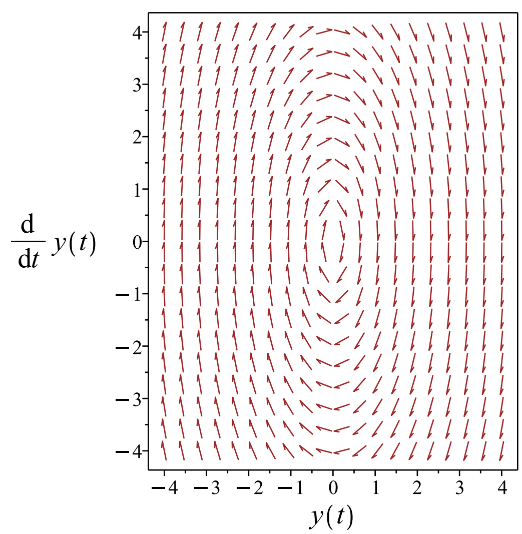
Summary

The solution(s) found are the following

$$y = \frac{\sin(t)(\cos(t)-1)(-1+\text{Heaviside}(t-2\pi))}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\sin(t)(\cos(t)-1)(-1+\text{Heaviside}(t-2\pi))}{3}$$

Verified OK.

4.3.2 Maple step by step solution

Let's solve

$\left[y'' + 4y = \sin(t)(1 - \mathit{Heaviside}(t - 2\pi)), y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -4y - \sin(t)(-1 + \mathit{Heaviside}(t - 2\pi))$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 4y = -\sin(t)(-1 + \mathit{Heaviside}(t - 2\pi))$

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(2t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(2t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = -\sin(t)(-1 + \mathit{Heaviside}(t\right.$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix}$$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 2$

- ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = \frac{\cos(2t)\left(\int \sin(2t)\sin(t)(-1+Heaviside(t-2\pi))dt\right)}{2} - \frac{\sin(2t)\left(\int \cos(2t)\sin(t)(-1+Heaviside(t-2\pi))dt\right)}{2}$

- ○ Compute integrals

  $y_p(t) = \frac{\sin(t)(1+(\cos(t)-1)Heaviside(t-2\pi))}{3}$

- ● Substitute particular solution into general solution to ODE

  $y = c_1\cos(2t) + c_2\sin(2t) + \frac{\sin(t)(1+(\cos(t)-1)Heaviside(t-2\pi))}{3}$

- □ Check validity of solution $y = c_1\cos(2t) + c_2\sin(2t) + \frac{\sin(t)(1+(\cos(t)-1)Heaviside(t-2\pi))}{3}$

- ○ Use initial condition $y(0) = 0$

  $0 = c_1$

- ○ Compute derivative of the solution

  $y' = -2c_1\sin(2t) + 2c_2\cos(2t) + \frac{\cos(t)(1+(\cos(t)-1)Heaviside(t-2\pi))}{3} + \frac{\sin(t)(-Heaviside(t-2\pi)\sin(t)+(\cos(t)-1}{3}$

- ○ Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = \frac{1}{3} + 2c_2$

- ○ Solve for c_1 and c_2

  $\left\{c_1 = 0, c_2 = -\frac{1}{6}\right\}$

- ○ Substitute constant values into general solution and simplify

  $y = \frac{\sin(t)(\cos(t)-1)(-1+Heaviside(t-2\pi))}{3}$

- ● Solution to the IVP

  $y = \frac{\sin(t)(\cos(t)-1)(-1+Heaviside(t-2\pi))}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.438 (sec). Leaf size: 25

```
dsolve([diff(y(t),t$2)+4*y(t)=sin(t)-Heaviside(t-2*Pi)*sin(t-2*Pi),y(0) = 0, D(y)(0) = 0],y(
```

$$y(t) = \frac{\sin(t)(\cos(t) - 1)(-1 + \text{Heaviside}(t - 2\pi))}{3}$$

✓ Solution by Mathematica

Time used: 0.061 (sec). Leaf size: 27

```
DSolve[{y''[t]+4*y[t]==Sin[t]-UnitStep[t-2*Pi]*Sin[t-2*Pi],{y[0]==0,y'[0]==0}},y[t],t,Includ
```

$$y(t) \to \frac{2}{3}\theta(2\pi - t)\sin^2\left(\frac{t}{2}\right)\sin(t)$$

4.4 problem 4

Internal problem ID [848]
Internal file name [`OUTPUT/848_Sunday_June_05_2022_01_51_18_AM_11805961/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.4, The Laplace Transform. Differential equations with discontinuous forcing functions. page 268
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 3y' + 2y = \begin{cases} 1 & 0 \leq t < 10 \\ 0 & \text{otherwise} \end{cases}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

4.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$
\begin{aligned}
p(t) &= 3 \\
q(t) &= 2 \\
F &= \begin{cases} 0 & t < 0 \\ 1 & t < 10 \\ 0 & 10 \leq t \end{cases}
\end{aligned}
$$

Hence the ode is

$$
y'' + 3y' + 2y = \begin{cases} 0 & t < 0 \\ 1 & t < 10 \\ 0 & 10 \leq t \end{cases}
$$

The domain of $p(t) = 3$ is

$$
\{-\infty < t < \infty\}
$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$
\{-\infty < t < \infty\}
$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \begin{cases} 0 & t < 0 \\ 1 & t < 10 \\ 0 & 10 \leq t \end{cases}$ is

$$
\{0 \leq t \leq 10, 10 \leq t \leq \infty, -\infty \leq t \leq 0\}
$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$
\mathcal{L}(y) = Y(s)
$$

Taking the Laplace transform of the ode and using the relations that

$$
\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)
\end{aligned}
$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 3sY(s) - 3y(0) + 2Y(s) = \frac{1 - e^{-10s}}{s} \tag{1}$$

But the initial conditions are

$$y(0) = 0$$
$$y'(0) = 0$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 3sY(s) + 2Y(s) = \frac{1 - e^{-10s}}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{-1 + e^{-10s}}{s(s^2 + 3s + 2)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(-\frac{-1 + e^{-10s}}{s(s^2 + 3s + 2)}\right) \\ &= \frac{\text{Heaviside}(10 - t)}{2} + \frac{e^{-2t}}{2} - e^{-t} + \frac{(-e^{-2t+20} + 2e^{10-t})\,\text{Heaviside}(t - 10)}{2} \end{aligned}$$

Hence the final solution is

$$y = \frac{\text{Heaviside}(10 - t)}{2} + \frac{e^{-2t}}{2} - e^{-t} + \frac{(-e^{-2t+20} + 2e^{10-t})\,\text{Heaviside}(t - 10)}{2}$$

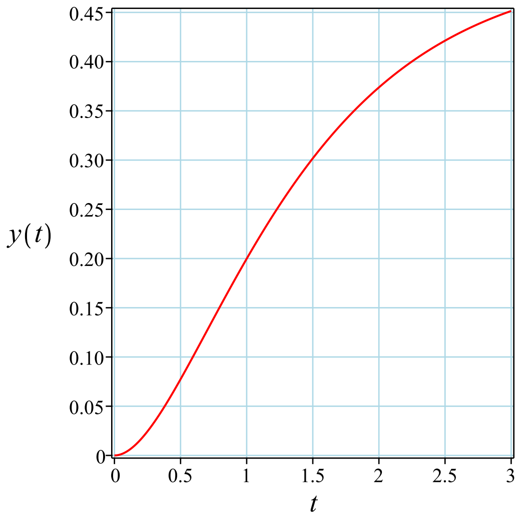
Simplifying the solution gives

$$\begin{aligned} y = \frac{1}{2} &- \frac{\text{Heaviside}(t - 10)}{2} + \frac{e^{-2t}}{2} - e^{-t} \\ &- \frac{\text{Heaviside}(t - 10)\,e^{-2t+20}}{2} + \text{Heaviside}(t - 10)\,e^{10-t} \end{aligned}$$

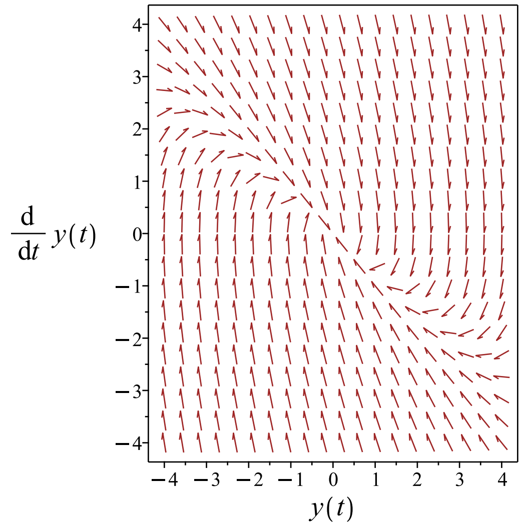
Summary

The solution(s) found are the following

$$\begin{aligned} y = \frac{1}{2} &- \frac{\text{Heaviside}(t - 10)}{2} + \frac{e^{-2t}}{2} - e^{-t} \\ &- \frac{\text{Heaviside}(t - 10)\,e^{-2t+20}}{2} + \text{Heaviside}(t - 10)\,e^{10-t} \end{aligned} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{1}{2} - \frac{\text{Heaviside}\,(t-10)}{2} + \frac{e^{-2t}}{2} - e^{-t} - \frac{\text{Heaviside}\,(t-10)\, e^{-2t+20}}{2} + \text{Heaviside}\,(t-10)\, e^{10-t}$$

Verified OK.

4.4.2 Maple step by step solution

Let's solve

$$\left[y'' + 3y' + 2y = \begin{cases} 0 & t < 0 \\ 1 & t < 10 \\ 0 & 10 \le t \end{cases}, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, -1)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

$$\left[y_p(t) = -y_1(t) \left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt \right) + y_2(t) \left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt \right), f(t) = \begin{cases} 0 & t < 0 \\ 1 & t < 10 \\ 0 & 10 \leq t \end{cases} \right]$$

- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-2t} & \mathrm{e}^{-t} \\ -2\,\mathrm{e}^{-2t} & -\mathrm{e}^{-t} \end{bmatrix}$$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = \mathrm{e}^{-3t}$

- ○ Substitute functions into equation for $y_p(t)$

$$y_p(t) = -\mathrm{e}^{-2t} \left(\int \left(\begin{cases} 0 & t < 0 \\ \mathrm{e}^{2t} & t < 10 \\ 0 & 10 \leq t \end{cases} \right) dt \right) + \mathrm{e}^{-t} \left(\int \left(\begin{cases} 0 & t < 0 \\ \mathrm{e}^{t} & t < 10 \\ 0 & 10 \leq t \end{cases} \right) dt \right)$$

- ○ Compute integrals

$$y_p(t) = \frac{\left(\begin{cases} 0 & t \leq 0 \\ \mathrm{e}^{-2t} - 2\,\mathrm{e}^{-t} + 1 & t \leq 10 \\ \mathrm{e}^{-2t} - 2\,\mathrm{e}^{-t} - \mathrm{e}^{-2t+20} + 2\,\mathrm{e}^{10-t} & 10 < t \end{cases} \right)}{2}$$

- Substitute particular solution into general solution to ODE

$$y = c_1\mathrm{e}^{-2t} + c_2\mathrm{e}^{-t} + \frac{\left(\begin{cases} 0 & t \leq 0 \\ \mathrm{e}^{-2t} - 2\,\mathrm{e}^{-t} + 1 & t \leq 10 \\ \mathrm{e}^{-2t} - 2\,\mathrm{e}^{-t} - \mathrm{e}^{-2t+20} + 2\,\mathrm{e}^{10-t} & 10 < t \end{cases}\right)}{2}$$

☐ Check validity of solution $y = c_1\mathrm{e}^{-2t} + c_2\mathrm{e}^{-t} + \frac{\left(\begin{cases} 0 & t \leq 0 \\ \mathrm{e}^{-2t} - 2\mathrm{e}^{-t} + 1 & t \leq 10 \\ \mathrm{e}^{-2t} - 2\mathrm{e}^{-t} - \mathrm{e}^{-2t+20} + 2\mathrm{e}^{10-t} & 10 < t \end{cases}\right)}{2}$

  - Use initial condition $y(0) = 0$

    $0 = c_1 + c_2$

  - Compute derivative of the solution

$$y' = -2c_1\mathrm{e}^{-2t} - c_2\mathrm{e}^{-t} + \frac{\left(\begin{cases} 0 & t \leq 0 \\ -2\,\mathrm{e}^{-2t} + 2\,\mathrm{e}^{-t} & t \leq 10 \\ -2\,\mathrm{e}^{-2t} + 2\,\mathrm{e}^{-t} + 2\,\mathrm{e}^{-2t+20} - 2\,\mathrm{e}^{10-t} & 10 < t \end{cases}\right)}{2}$$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 0$

    $0 = -2c_1 - c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 0, c_2 = 0\}$

  - Substitute constant values into general solution and simplify

$$y = \frac{\left(\begin{cases} 0 & t \leq 0 \\ \mathrm{e}^{-2t} - 2\,\mathrm{e}^{-t} + 1 & t \leq 10 \\ \mathrm{e}^{-2t} - 2\,\mathrm{e}^{-t} - \mathrm{e}^{-2t+20} + 2\,\mathrm{e}^{10-t} & 10 < t \end{cases}\right)}{2}$$

- Solution to the IVP

$$y = \frac{\left(\begin{cases} 0 & t \leq 0 \\ \mathrm{e}^{-2t} - 2\,\mathrm{e}^{-t} + 1 & t \leq 10 \\ \mathrm{e}^{-2t} - 2\,\mathrm{e}^{-t} - \mathrm{e}^{-2t+20} + 2\,\mathrm{e}^{10-t} & 10 < t \end{cases}\right)}{2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.438 (sec). Leaf size: 65

```
dsolve([diff(y(t),t$2)+3*diff(y(t),t)+2*y(t)=piecewise(0<=t and t<10,1,true,0),y(0) = 0, D(y
```

$$y(t) = \frac{\left(\begin{cases} 1-2\,\mathrm{e}^{-t}+\mathrm{e}^{-2t} & t<10 \\ -2\,\mathrm{e}^{-10}+\mathrm{e}^{-20}+2 & t=10 \\ 2\,\mathrm{e}^{10-t}-\mathrm{e}^{20-2t}-2\,\mathrm{e}^{-t}+\mathrm{e}^{-2t} & 10<t \end{cases}\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 61

```
DSolve[{y''[t]+3*y'[t]+2*y[t]==Piecewise[{{1,0<=t<10},{0,True}}],{y[0]==0,y'[0]==0}},y[t],t,
```

$$y(t) \to \begin{cases} 0 & t \leq 0 \\ \frac{1}{2}e^{-2t}(-1+e^t)^2 & 0<t\leq 10 \\ \frac{1}{2}e^{-2t}(-1+e^{10})\left(-1-e^{10}+2e^t\right) & \text{True} \end{cases}$$

4.5 problem 5

Internal problem ID [849]
Internal file name [`OUTPUT/849_Sunday_June_05_2022_01_51_22_AM_5247366/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.4, The Laplace Transform. Differential equations with discontinuous forcing functions. page 268
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y' + \frac{5y}{4} = t - \text{Heaviside}\left(t - \frac{\pi}{2}\right)\left(t - \frac{\pi}{2}\right)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

4.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 1 \\ q(t) &= \frac{5}{4} \\ F &= \frac{(-2t + \pi)\,\text{Heaviside}\left(t - \frac{\pi}{2}\right)}{2} + t \end{aligned}$$

Hence the ode is

$$y'' + y' + \frac{5y}{4} = \frac{(-2t+\pi)\,\text{Heaviside}\left(t-\frac{\pi}{2}\right)}{2} + t$$

The domain of $p(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \frac{5}{4}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \frac{(-2t+\pi)\,\text{Heaviside}\left(t-\frac{\pi}{2}\right)}{2} + t$ is

$$\left\{t < \frac{\pi}{2} \vee \frac{\pi}{2} < t\right\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + sY(s) - y(0) + \frac{5Y(s)}{4} = \frac{-\mathrm{e}^{-\frac{\pi s}{2}} + 1}{s^2} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + sY(s) + \frac{5Y(s)}{4} = \frac{-\mathrm{e}^{-\frac{\pi s}{2}} + 1}{s^2}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{4\left(\mathrm{e}^{-\frac{\pi s}{2}}-1\right)}{s^2\left(4s^2+4s+5\right)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(-\frac{4\left(\mathrm{e}^{-\frac{\pi s}{2}}-1\right)}{s^2\left(4s^2+4s+5\right)}\right) \\
&= \frac{4\,\mathrm{e}^{-\frac{t}{2}}\left(4\cos(t)-3\sin(t)\right)}{25} + \frac{4\,\mathrm{Heaviside}\left(-t+\frac{\pi}{2}\right)\left(-4+5t\right)}{25} + \frac{2\left(5\pi-2\,\mathrm{e}^{-\frac{t}{2}+\frac{\pi}{4}}\left(3\cos(t)+4\sin(t)\right)\right)}{25}
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= \frac{4\,\mathrm{e}^{-\frac{t}{2}}\left(4\cos(t)-3\sin(t)\right)}{25} + \frac{4\,\mathrm{Heaviside}\left(-t+\frac{\pi}{2}\right)\left(-4+5t\right)}{25} \\
&\quad + \frac{2\left(5\pi-2\,\mathrm{e}^{-\frac{t}{2}+\frac{\pi}{4}}\left(3\cos(t)+4\sin(t)\right)\right)\mathrm{Heaviside}\left(t-\frac{\pi}{2}\right)}{25}
\end{aligned}$$

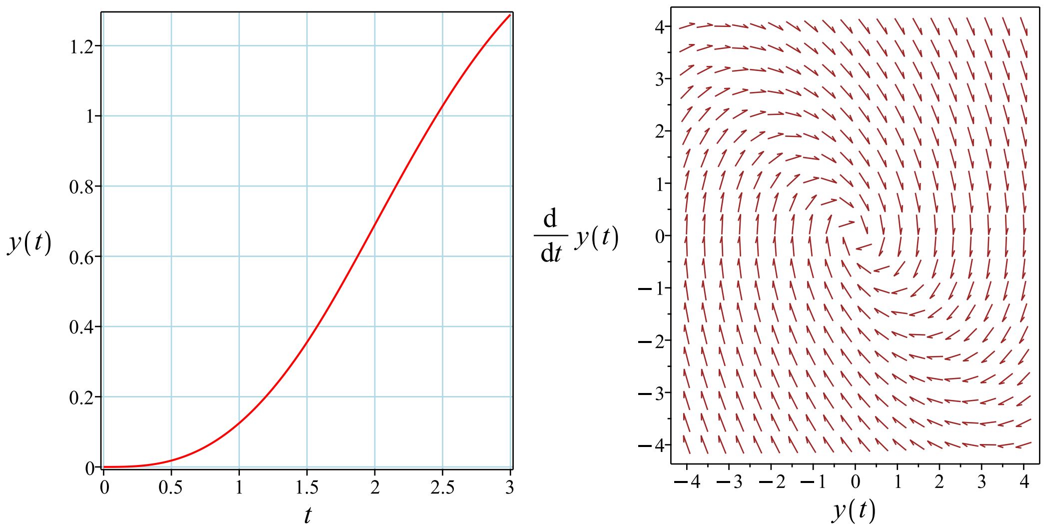
Simplifying the solution gives

$$\begin{aligned}
y &= -\frac{16}{25} - \frac{12\,\mathrm{Heaviside}\left(t-\frac{\pi}{2}\right)\left(\cos(t)+\frac{4\sin(t)}{3}\right)\mathrm{e}^{-\frac{t}{2}+\frac{\pi}{4}}}{25} \\
&\quad + \frac{2(8-10t+5\pi)\,\mathrm{Heaviside}\left(t-\frac{\pi}{2}\right)}{25} + \frac{4\,\mathrm{e}^{-\frac{t}{2}}\left(4\cos(t)-3\sin(t)\right)}{25} + \frac{4t}{5}
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= -\frac{16}{25} - \frac{12\,\mathrm{Heaviside}\left(t-\frac{\pi}{2}\right)\left(\cos(t)+\frac{4\sin(t)}{3}\right)\mathrm{e}^{-\frac{t}{2}+\frac{\pi}{4}}}{25} \\
&\quad + \frac{2(8-10t+5\pi)\,\mathrm{Heaviside}\left(t-\frac{\pi}{2}\right)}{25} + \frac{4\,\mathrm{e}^{-\frac{t}{2}}\left(4\cos(t)-3\sin(t)\right)}{25} + \frac{4t}{5}
\end{aligned} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{16}{25} - \frac{12\,\text{Heaviside}\left(t-\frac{\pi}{2}\right)\left(\cos(t)+\frac{4\sin(t)}{3}\right)e^{-\frac{t}{2}+\frac{\pi}{4}}}{25} + \frac{2(8-10t+5\pi)\,\text{Heaviside}\left(t-\frac{\pi}{2}\right)}{25} + \frac{4\,e^{-\frac{t}{2}}(4\cos(t)-3\sin(t))}{25} + \frac{4t}{5}$$

Verified OK.

4.5.2 Maple step by step solution

Let's solve

$$\left[y''+y'+\frac{5y}{4}=\frac{(-2t+\pi)\,\mathit{Heaviside}\left(t-\frac{\pi}{2}\right)}{2}+t, y(0)=0, y'\Big|_{\{t=0\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{\mathit{Heaviside}\left(t-\frac{\pi}{2}\right)\pi}{2}-\mathit{Heaviside}\left(t-\frac{\pi}{2}\right)t-y'-\frac{5y}{4}+t$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+y'+\frac{5y}{4}=t-\mathit{Heaviside}\left(t-\frac{\pi}{2}\right)t+\frac{\mathit{Heaviside}\left(t-\frac{\pi}{2}\right)\pi}{2}$

- Characteristic polynomial of homogeneous ODE

$r^2 + r + \frac{5}{4} = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1) \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \mathrm{I}, -\frac{1}{2} + \mathrm{I}\right)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(t)\, \mathrm{e}^{-\frac{t}{2}}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(t)\, \mathrm{e}^{-\frac{t}{2}}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(t)\, \mathrm{e}^{-\frac{t}{2}} + c_2 \sin(t)\, \mathrm{e}^{-\frac{t}{2}} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = t - \mathit{Heaviside}\left(t - \frac{\pi}{2}\right) t + \frac{\mathit{H}\text{...}}{}\right.$

- Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(t)\, \mathrm{e}^{-\frac{t}{2}} & \sin(t)\, \mathrm{e}^{-\frac{t}{2}} \\ -\sin(t)\, \mathrm{e}^{-\frac{t}{2}} - \frac{\cos(t) \mathrm{e}^{-\frac{t}{2}}}{2} & \cos(t)\, \mathrm{e}^{-\frac{t}{2}} - \frac{\sin(t) \mathrm{e}^{-\frac{t}{2}}}{2} \end{bmatrix}$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = \mathrm{e}^{-t}$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{\mathrm{e}^{-\frac{t}{2}}\left(\cos(t)\left(\int -2\left(-t + \mathit{Heaviside}\left(t - \frac{\pi}{2}\right)\left(t - \frac{\pi}{2}\right)\right) \mathrm{e}^{\frac{t}{2}} \sin(t) dt\right) - \sin(t)\left(\int -2\left(-t + \mathit{Heaviside}\left(t - \frac{\pi}{2}\right)\left(t - \frac{\pi}{2}\right)\right) \mathrm{e}^{\frac{t}{2}} \cos(t) dt\right)\right)}{2}$

- Compute integrals

  $y_p(t) = -\frac{16}{25} - \frac{12 \mathit{Heaviside}\left(t - \frac{\pi}{2}\right)\left(\cos(t) + \frac{4 \sin(t)}{3}\right) \mathrm{e}^{-\frac{t}{2} + \frac{\pi}{4}}}{25} + \frac{2(8 - 10t + 5\pi) \mathit{Heaviside}\left(t - \frac{\pi}{2}\right)}{25} + \frac{4t}{5}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(t)\, \mathrm{e}^{-\frac{t}{2}} + c_2 \sin(t)\, \mathrm{e}^{-\frac{t}{2}} - \frac{16}{25} - \frac{12 \mathit{Heaviside}\left(t - \frac{\pi}{2}\right)\left(\cos(t) + \frac{4 \sin(t)}{3}\right) \mathrm{e}^{-\frac{t}{2} + \frac{\pi}{4}}}{25} + \frac{2(8 - 10t + 5\pi) \mathit{Heaviside}\left(t - \text{...}\right.}{25}$

- ☐ Check validity of solution $y = c_1 \cos(t)\, e^{-\frac{t}{2}} + c_2 \sin(t)\, e^{-\frac{t}{2}} - \frac{16}{25} - \frac{12 \mathit{Heaviside}\left(t-\frac{\pi}{2}\right)\left(\cos(t)+\frac{4\sin(t)}{3}\right) e^{-\frac{t}{2}}}{25}$
  - Use initial condition $y(0) = 0$

    $0 = c_1 - \frac{16}{25}$
  - Compute derivative of the solution

    $y' = -c_1 \sin(t)\, e^{-\frac{t}{2}} - \frac{c_1 \cos(t) e^{-\frac{t}{2}}}{2} + c_2 \cos(t)\, e^{-\frac{t}{2}} - \frac{c_2 \sin(t) e^{-\frac{t}{2}}}{2} - \frac{12 \mathit{Dirac}\left(t-\frac{\pi}{2}\right)\left(\cos(t)+\frac{4\sin(t)}{3}\right) e^{-\frac{t}{2}+\frac{\pi}{4}}}{25} - \frac{1}{}$
  - Use the initial condition $y'\Big|_{\{t=0\}} = 0$

    $0 = \frac{4}{5} - \frac{c_1}{2} + c_2$
  - Solve for c_1 and c_2

    $\left\{c_1 = \frac{16}{25}, c_2 = -\frac{12}{25}\right\}$
  - Substitute constant values into general solution and simplify

    $y = -\frac{16}{25} - \frac{12 \mathit{Heaviside}\left(t-\frac{\pi}{2}\right)\left(\cos(t)+\frac{4\sin(t)}{3}\right) e^{-\frac{t}{2}+\frac{\pi}{4}}}{25} + \frac{2(8-10t+5\pi) \mathit{Heaviside}\left(t-\frac{\pi}{2}\right)}{25} + \frac{4\, e^{-\frac{t}{2}} (4\cos(t)-3\sin(t))}{25} + \frac{4t}{5}$
- Solution to the IVP

  $y = -\frac{16}{25} - \frac{12 \mathit{Heaviside}\left(t-\frac{\pi}{2}\right)\left(\cos(t)+\frac{4\sin(t)}{3}\right) e^{-\frac{t}{2}+\frac{\pi}{4}}}{25} + \frac{2(8-10t+5\pi) \mathit{Heaviside}\left(t-\frac{\pi}{2}\right)}{25} + \frac{4\, e^{-\frac{t}{2}} (4\cos(t)-3\sin(t))}{25} + \frac{4t}{5}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.422 (sec). Leaf size: 66

```
dsolve([diff(y(t),t$2)+diff(y(t),t)+5/4*y(t)=t-Heaviside(t-Pi/2)*(t-Pi/2),y(0) = 0, D(y)(0)
```

$$y(t) = -\frac{16}{25} - \frac{12\,\mathrm{Heaviside}\left(t-\frac{\pi}{2}\right)\left(\cos\left(t\right)+\frac{4\sin\left(t\right)}{3}\right)\mathrm{e}^{-\frac{t}{2}+\frac{\pi}{4}}}{25} + \frac{2(8-10t+5\pi)\,\mathrm{Heaviside}\left(t-\frac{\pi}{2}\right)}{25} + \frac{4(4\cos\left(t\right)-3\sin\left(t\right))\,\mathrm{e}^{-\frac{t}{2}}}{25} + \frac{4t}{5}$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 96

```
DSolve[{y''[t]+y'[t]+5/4*y[t]==t-UnitStep[t-Pi/2]*(t-Pi/2),{y[0]==0,y'[0]==0}},y[t],t,Includ
```

$$y(t) \to \begin{cases} \frac{4}{25}e^{-t/2}\left(e^{t/2}(5t-4)+4\cos(t)-3\sin(t)\right) & 2t \leq \pi \\ -\frac{2}{25}e^{-t/2}\left(\left(-8+6e^{\pi/4}\right)\cos(t)+\left(6+8e^{\pi/4}\right)\sin(t)-5e^{t/2}\pi\right) & \text{True} \end{cases}$$

4.6 problem 6

Internal problem ID [850]
Internal file name [`OUTPUT/850_Sunday_June_05_2022_01_51_27_AM_71448528/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.4, The Laplace Transform. Differential equations with discontinuous forcing functions. page 268
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$y'' + y' + \frac{5y}{4} = \left\{ \begin{array}{cc} \sin(t) & 0 \le t < \pi \\ 0 & \text{otherwise} \end{array} \right.$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

4.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$
\begin{aligned}
p(t) &= 1 \\
q(t) &= \frac{5}{4} \\
F &= \left\{ \begin{array}{cc} 0 & t<0 \\ \sin \left(t \right) & t<\pi \\ 0 & \pi \le t \end{array} \right.
\end{aligned}
$$

Hence the ode is

$$
y'' + y' + \frac{5y}{4} = \left\{ \begin{array}{cc} 0 & t<0 \\ \sin \left(t \right) & t<\pi \\ 0 & \pi \le t \end{array} \right.
$$

The domain of $p(t) = 1$ is

$$
\{-\infty < t < \infty\}
$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \frac{5}{4}$ is

$$
\{-\infty < t < \infty\}
$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \left\{ \begin{array}{cc} 0 & t<0 \\ \sin \left(t \right) & t<\pi \\ 0 & \pi \le t \end{array} \right.$ is

$$
\{0 \le t \le \pi, \pi \le t \le \infty, -\infty \le t \le 0\}
$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$
\mathcal{L}(y) = Y(s)
$$

Taking the Laplace transform of the ode and using the relations that

$$
\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2 Y(s) - y'(0) - sy(0)
\end{aligned}
$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + sY(s) - y(0) + \frac{5Y(s)}{4} = \frac{1 + \mathrm{e}^{-\pi s}}{s^2 + 1} \tag{1}$$

But the initial conditions are

$$\begin{aligned} y(0) &= 0 \\ y'(0) &= 0 \end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + sY(s) + \frac{5Y(s)}{4} = \frac{1 + \mathrm{e}^{-\pi s}}{s^2 + 1}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{4 + 4\,\mathrm{e}^{-\pi s}}{(s^2 + 1)(4s^2 + 4s + 5)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{4 + 4\,\mathrm{e}^{-\pi s}}{(s^2 + 1)(4s^2 + 4s + 5)}\right) \\ &= \frac{8\,\mathrm{Heaviside}(t - \pi)\,\mathrm{e}^{\frac{\pi}{4} - \frac{t}{4}}\left(4\cos(t)\sinh\left(-\frac{\pi}{4} + \frac{t}{4}\right) - \sin(t)\cosh\left(-\frac{\pi}{4} + \frac{t}{4}\right)\right)}{17} + \frac{8\left(-4\cos(t)\sinh\left(\frac{t}{4}\right) + \mathrm{si}\right.}{17} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y = {} & \frac{8\,\mathrm{Heaviside}(t - \pi)\,\mathrm{e}^{\frac{\pi}{4} - \frac{t}{4}}\left(4\cos(t)\sinh\left(-\frac{\pi}{4} + \frac{t}{4}\right) - \sin(t)\cosh\left(-\frac{\pi}{4} + \frac{t}{4}\right)\right)}{17} \\ & + \frac{8\left(-4\cos(t)\sinh\left(\frac{t}{4}\right) + \sin(t)\cosh\left(\frac{t}{4}\right)\right)\mathrm{e}^{-\frac{t}{4}}}{17} \end{aligned}$$

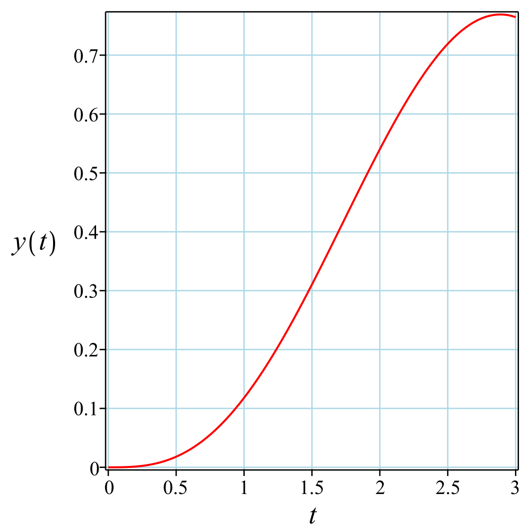
Simplifying the solution gives

$$\begin{aligned} y = {} & -\frac{8\,\mathrm{Heaviside}(t - \pi)\left(-4\cos(t)\sinh\left(-\frac{\pi}{4} + \frac{t}{4}\right) + \sin(t)\cosh\left(-\frac{\pi}{4} + \frac{t}{4}\right)\right)\mathrm{e}^{\frac{\pi}{4} - \frac{t}{4}}}{17} \\ & + \frac{8\left(-4\cos(t)\sinh\left(\frac{t}{4}\right) + \sin(t)\cosh\left(\frac{t}{4}\right)\right)\mathrm{e}^{-\frac{t}{4}}}{17} \end{aligned}$$

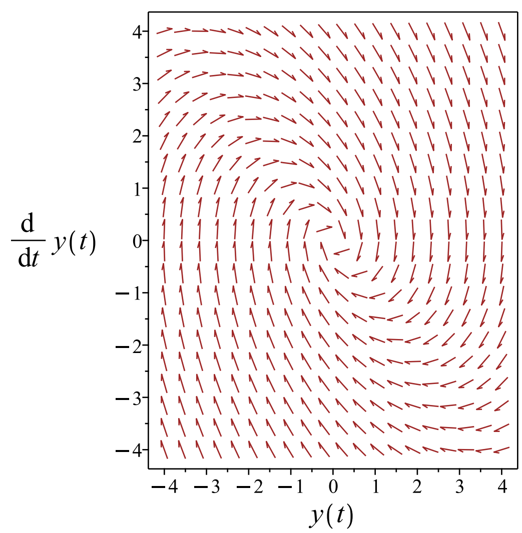
Summary

The solution(s) found are the following

$$
\begin{align*}
y &= -\frac{8\,\text{Heaviside}\left(t-\pi\right)\left(-4\cos\left(t\right)\sinh\left(-\frac{\pi}{4}+\frac{t}{4}\right)+\sin\left(t\right)\cosh\left(-\frac{\pi}{4}+\frac{t}{4}\right)\right)\mathrm{e}^{\frac{\pi}{4}-\frac{t}{4}}}{17} \tag{1}\\
&\quad + \frac{8\left(-4\cos\left(t\right)\sinh\left(\frac{t}{4}\right)+\sin\left(t\right)\cosh\left(\frac{t}{4}\right)\right)\mathrm{e}^{-\frac{t}{4}}}{17}
\end{align*}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$
\begin{align*}
y &= -\frac{8\,\text{Heaviside}\left(t-\pi\right)\left(-4\cos\left(t\right)\sinh\left(-\frac{\pi}{4}+\frac{t}{4}\right)+\sin\left(t\right)\cosh\left(-\frac{\pi}{4}+\frac{t}{4}\right)\right)\mathrm{e}^{\frac{\pi}{4}-\frac{t}{4}}}{17}\\
&\quad + \frac{8\left(-4\cos\left(t\right)\sinh\left(\frac{t}{4}\right)+\sin\left(t\right)\cosh\left(\frac{t}{4}\right)\right)\mathrm{e}^{-\frac{t}{4}}}{17}
\end{align*}
$$

Verified OK.

4.6.2 Maple step by step solution

Let's solve

$$
\left[y''+y'+\frac{5y}{4}=\left\{\begin{array}{cc} 0 & t<0 \\ \sin\left(t\right) & t<\pi \\ 0 & \pi\le t \end{array}\right., y(0)=0, y'\Big|_{\{t=0\}}=0\right]
$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + \frac{5}{4} = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1) \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \mathrm{I}, -\frac{1}{2} + \mathrm{I}\right)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(t)\,\mathrm{e}^{-\frac{t}{2}}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(t)\,\mathrm{e}^{-\frac{t}{2}}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(t)\,\mathrm{e}^{-\frac{t}{2}} + c_2 \sin(t)\,\mathrm{e}^{-\frac{t}{2}} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $$\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \left\{\begin{array}{cc} 0 & t < 0 \\ \sin(t) & t < \pi \\ 0 & \pi \leq t \end{array}\right.\right]$$

  - Wronskian of solutions of the homogeneous equation

    $$W(y_1(t), y_2(t)) = \left[\begin{array}{cc} \cos(t)\,\mathrm{e}^{-\frac{t}{2}} & \sin(t)\,\mathrm{e}^{-\frac{t}{2}} \\ -\sin(t)\,\mathrm{e}^{-\frac{t}{2}} - \frac{\cos(t)\mathrm{e}^{-\frac{t}{2}}}{2} & \cos(t)\,\mathrm{e}^{-\frac{t}{2}} - \frac{\sin(t)\mathrm{e}^{-\frac{t}{2}}}{2} \end{array}\right]$$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = \mathrm{e}^{-t}$

  - Substitute functions into equation for $y_p(t)$

$$y_p(t) = e^{-\frac{t}{2}} \left(-\cos(t) \left(\int \left(\begin{cases} 0 & t < 0 \\ \sin(t)^2 e^{\frac{t}{2}} & t < \pi \\ 0 & \pi \le t \end{cases} \right) dt \right) + \sin(t) \left(\int \left(\begin{cases} 0 & t < 0 \\ \frac{\sin(2t) e^{\frac{t}{2}}}{2} & t < \pi \\ 0 & \pi \le t \end{cases} \right) dt \right.\right.$$

- Compute integrals

$$y_p(t) = \frac{4 \left(\begin{cases} 0 & t \le 0 \\ (\sin(t) + 4\cos(t)) e^{-\frac{t}{2}} - 4\cos(t) + \sin(t) & t \le \pi \\ -\left(-1 + e^{\frac{\pi}{2}}\right)(\sin(t) + 4\cos(t)) e^{-\frac{t}{2}} & \pi < t \end{cases} \right)}{17}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(t) e^{-\frac{t}{2}} + c_2 \sin(t) e^{-\frac{t}{2}} + \frac{4 \left(\begin{cases} 0 & t \le 0 \\ (\sin(t) + 4\cos(t)) e^{-\frac{t}{2}} - 4\cos(t) + \sin(t) & t \le \pi \\ -\left(-1 + e^{\frac{\pi}{2}}\right)(\sin(t) + 4\cos(t)) e^{-\frac{t}{2}} & \pi < t \end{cases} \right)}{17}$$

☐ Check validity of solution $y = c_1 \cos(t) e^{-\frac{t}{2}} + c_2 \sin(t) e^{-\frac{t}{2}} + \frac{4 \left(\begin{cases} 0 \\ (\sin(t) + 4\cos(t)) e^{-\frac{t}{2}} - 4\cos \\ -\left(-1 + e^{\frac{\pi}{2}}\right)(\sin(t) + 4\cos \end{cases} \right)}{17}$

- Use initial condition $y(0) = 0$

  $0 = c_1$

- Compute derivative of the solution

$$y' = -c_1 \sin(t) e^{-\frac{t}{2}} - \frac{c_1 \cos(t) e^{-\frac{t}{2}}}{2} + c_2 \cos(t) e^{-\frac{t}{2}} - \frac{c_2 \sin(t) e^{-\frac{t}{2}}}{2} + \frac{4 \left(\begin{cases} (\cos(t) - 4\sin(t)) e^{-\frac{t}{2}} - \frac{(\sin}{} \\ -\left(-1 + e^{\frac{\pi}{2}}\right)(\cos(t) - 4\sin(\end{cases} \right)}{}$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = -\frac{c_1}{2} + c_2$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 0\}$

- Substitute constant values into general solution and simplify

$$y = \frac{4\left(\left\{\begin{array}{cc} 0 & t \le 0 \\ \left(\sin\left(t\right)+4\cos\left(t\right)\right)\mathrm{e}^{-\frac{t}{2}}-4\cos\left(t\right)+\sin\left(t\right) & t \le \pi \\ -\left(-1+\mathrm{e}^{\frac{\pi}{2}}\right)\left(\sin\left(t\right)+4\cos\left(t\right)\right)\mathrm{e}^{-\frac{t}{2}} & \pi < t \end{array}\right.\right)}{17}$$

- Solution to the IVP

$$y = \frac{4\left(\left\{\begin{array}{cc} 0 & t \le 0 \\ \left(\sin\left(t\right)+4\cos\left(t\right)\right)\mathrm{e}^{-\frac{t}{2}}-4\cos\left(t\right)+\sin\left(t\right) & t \le \pi \\ -\left(-1+\mathrm{e}^{\frac{\pi}{2}}\right)\left(\sin\left(t\right)+4\cos\left(t\right)\right)\mathrm{e}^{-\frac{t}{2}} & \pi < t \end{array}\right.\right)}{17}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.532 (sec). Leaf size: 91

```
dsolve([diff(y(t),t$2)+diff(y(t),t)+5/4*y(t)=piecewise(0<=t and t<Pi,sin(t),true,0),y(0) = 0
```

$$y(t) = \frac{4\left(\left\{\begin{array}{cc} -8\,\mathrm{e}^{-\frac{t}{4}}\left(\cos\left(t\right)\sinh\left(\frac{t}{4}\right)-\frac{\sin(t)\cosh\left(\frac{t}{4}\right)}{4}\right) & t < \pi \\ \left(-\mathrm{e}^{-\frac{t}{2}+\frac{\pi}{2}}+\mathrm{e}^{-\frac{t}{2}}\right)\left(4\cos\left(t\right)+\sin\left(t\right)\right) & \pi \le t \end{array}\right.\right)}{17}$$

✓ Solution by Mathematica

Time used: 0.129 (sec). Leaf size: 77

```
DSolve[{y''[t]+y'[t]+5/4*y[t]==Piecewise[{{Sin[t],0<=t<Pi},{0,True}}],{y[0]==0,y'[0]==0}},y[
```

$$y(t) \to \begin{cases} 0 & t \leq 0 \\ \frac{4}{17}\left(\left(-4+4e^{-t/2}\right)\cos(t)+\left(1+e^{-t/2}\right)\sin(t)\right) & 0 < t \leq \pi \\ -\frac{4}{17}e^{-t/2}\left(-1+e^{\pi/2}\right)\left(4\cos(t)+\sin(t)\right) & \text{True} \end{cases}$$

4.7 problem 7

Internal problem ID [851]
Internal file name [`OUTPUT/851_Sunday_June_05_2022_01_51_34_AM_88490583/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.4, The Laplace Transform. Differential equations with discontinuous forcing functions. page 268
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = \text{Heaviside}\,(t-\pi) - \text{Heaviside}\,(t-3\pi)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

4.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 4 \\ F &= \text{Heaviside}\,(t-\pi) - \text{Heaviside}\,(t-3\pi) \end{aligned}$$

Hence the ode is

$$y'' + 4y = \text{Heaviside}\,(t - \pi) - \text{Heaviside}\,(t - 3\pi)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \text{Heaviside}\,(t - \pi) - \text{Heaviside}\,(t - 3\pi)$ is

$$\{\pi \leq t \leq 3\pi, 3\pi \leq t \leq \infty, -\infty \leq t \leq \pi\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 4Y(s) = \frac{e^{-\pi s} - e^{-3\pi s}}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 4Y(s) = \frac{e^{-\pi s} - e^{-3\pi s}}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{e^{-\pi s} - e^{-3\pi s}}{s\,(s^2+4)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{e^{-\pi s} - e^{-3\pi s}}{s\,(s^2+4)}\right) \\ &= \frac{\sin(t)^2\,(\text{Heaviside}(t-\pi) - \text{Heaviside}(t-3\pi))}{2} \end{aligned}$$

Hence the final solution is

$$y = \frac{\sin(t)^2\,(\text{Heaviside}(t-\pi) - \text{Heaviside}(t-3\pi))}{2}$$

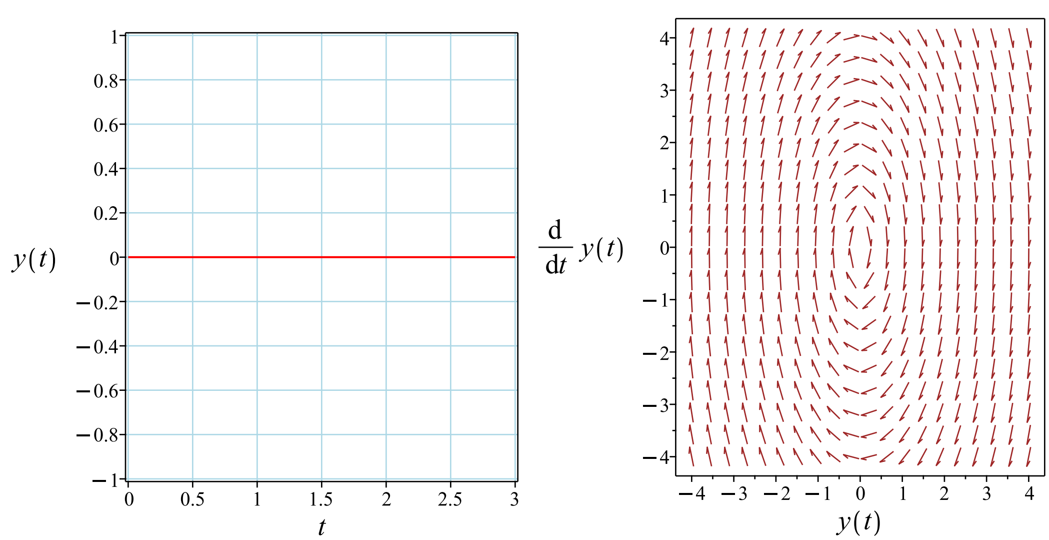
Simplifying the solution gives

$$y = \frac{\sin(t)^2\,(\text{Heaviside}(t-\pi) - \text{Heaviside}(t-3\pi))}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\sin(t)^2\,(\text{Heaviside}(t-\pi) - \text{Heaviside}(t-3\pi))}{2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{\sin(t)^2 (\text{Heaviside}(t-\pi) - \text{Heaviside}(t-3\pi))}{2}$$

Verified OK.

4.7.2 Maple step by step solution

Let's solve

$$\left[y'' + 4y = Heaviside(t-\pi) - Heaviside(t-3\pi), y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$
- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$
- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$
- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(2t)$
- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(2t)$
- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = Heaviside(t-\pi) - Heavisi\right.$
- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 2$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{\cos(2t)\left(\int \sin(2t)(\mathit{Heaviside}(t-\pi)-\mathit{Heaviside}(t-3\pi))dt\right)}{2} + \frac{\sin(2t)\left(\int \cos(2t)(\mathit{Heaviside}(t-\pi)-\mathit{Heaviside}(t-3\pi))dt\right)}{2}$

- Compute integrals

  $y_p(t) = -\frac{(\mathit{Heaviside}(t-\pi)-\mathit{Heaviside}(t-3\pi))(-1+\cos(2t))}{4}$

- Substitute particular solution into general solution to ODE

  $y = c_1\cos(2t) + c_2\sin(2t) - \frac{(\mathit{Heaviside}(t-\pi)-\mathit{Heaviside}(t-3\pi))(-1+\cos(2t))}{4}$

☐ Check validity of solution $y = c_1\cos(2t) + c_2\sin(2t) - \frac{(\mathit{Heaviside}(t-\pi)-\mathit{Heaviside}(t-3\pi))(-1+\cos(2t))}{4}$

- Use initial condition $y(0) = 0$

  $0 = c_1$

- Compute derivative of the solution

  $y' = -2c_1\sin(2t) + 2c_2\cos(2t) - \frac{(\mathit{Dirac}(t-\pi)-\mathit{Dirac}(t-3\pi))(-1+\cos(2t))}{4} + \frac{\sin(2t)(\mathit{Heaviside}(t-\pi)-\mathit{Heaviside}(t-3}{2}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = 2c_2$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $y = -\frac{(\mathit{Heaviside}(t-\pi)-\mathit{Heaviside}(t-3\pi))(-1+\cos(2t))}{4}$

- Solution to the IVP

  $y = -\frac{(\mathit{Heaviside}(t-\pi)-\mathit{Heaviside}(t-3\pi))(-1+\cos(2t))}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.297 (sec). Leaf size: 25

```
dsolve([diff(y(t),t$2)+4*y(t)=Heaviside(t-Pi)-Heaviside(t-3*Pi),y(0) = 0, D(y)(0) = 0],y(t),
```

$$y(t) = \frac{(\text{Heaviside}\,(t-\pi) - \text{Heaviside}\,(t-3\pi))\sin\,(t)^2}{2}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 25

```
DSolve[{y''[t]+4*y[t]==UnitStep[t-Pi]-UnitStep[t-3*Pi],{y[0]==0,y'[0]==0}},y[t],t,IncludeSin
```

$$y(t) \to \begin{cases} \frac{\sin^2(t)}{2} & \pi < t \leq 3\pi \\ 0 & \text{True} \end{cases}$$

4.8 problem 8

Internal problem ID [852]
Internal file name [OUTPUT/852_Sunday_June_05_2022_01_51_37_AM_29676591/index.tex]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.4, The Laplace Transform. Differential equations with discontinuous forcing functions. page 268
Problem number: 8.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_laplace"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' + 5y'' + 4y = 1 - \text{Heaviside}\left(t - \pi\right)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0, y''(0) = 0, y'''(0) = 0]$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \\
\mathcal{L}(y''') &= s^3Y(s) - y''(0) - sy'(0) - s^2y(0) \\
\mathcal{L}(y'''') &= s^4Y(s) - y'''(0) - sy''(0) - s^2y'(0) - s^3y(0)
\end{aligned}$$

The given ode becomes an algebraic equation in the Laplace domain

$$s^4Y(s) - y'''(0) - sy''(0) - s^2y'(0) - s^3y(0) + 5s^2Y(s) - 5y'(0) - 5sy(0) + 4Y(s) = \frac{1 - e^{-\pi s}}{s} \tag{1}$$

But the initial conditions are

$$
\begin{aligned}
y(0) &= 0 \\
y'(0) &= 0 \\
y''(0) &= 0 \\
y'''(0) &= 0
\end{aligned}
$$

Substituting these initial conditions in above in Eq (1) gives

$$
s^4 Y(s) + 5s^2 Y(s) + 4Y(s) = \frac{1 - \mathrm{e}^{-\pi s}}{s}
$$

Solving the above equation for $Y(s)$ results in

$$
Y(s) = -\frac{-1 + \mathrm{e}^{-\pi s}}{s\,(s^4 + 5s^2 + 4)}
$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(-\frac{-1 + \mathrm{e}^{-\pi s}}{s\,(s^4 + 5s^2 + 4)}\right) \\
&= -\frac{2\cos(t)}{3} + \frac{\text{Heaviside}\,(\pi - t)\,(1 + \cos(t))^2}{6}
\end{aligned}
$$

Hence the final solution is

$$
y = -\frac{2\cos(t)}{3} + \frac{\text{Heaviside}\,(\pi - t)\,(1 + \cos(t))^2}{6}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{2\cos(t)}{3} + \frac{\text{Heaviside}\,(\pi - t)\,(1 + \cos(t))^2}{6} \tag{1}
$$

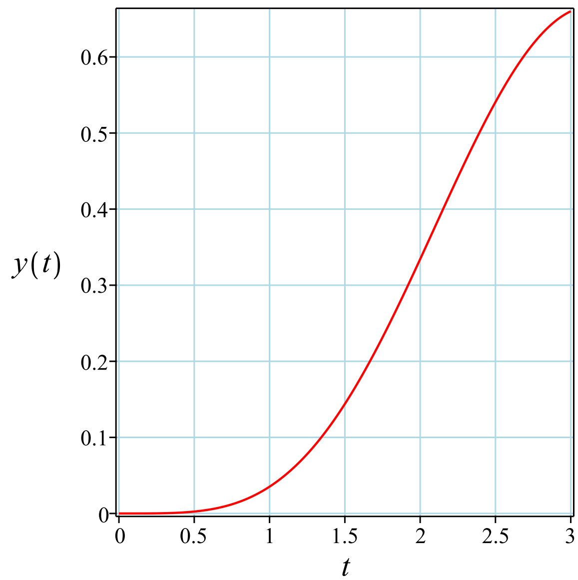


Figure 19: Solution plot

Verification of solutions

$$y = -\frac{2\cos(t)}{3} + \frac{\text{Heaviside}(\pi - t)(1 + \cos(t))^2}{6}$$

Verified OK.

4.8.1 Maple step by step solution

Let's solve

$$\left[y'''' + 5y'' + 4y = 1 - \mathit{Heaviside}(t - \pi), y(0) = 0, y'\Big|_{\{t=0\}} = 0, y''\Big|_{\{t=0\}} = 0, y'''\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 4

  y''''

□ Convert linear ODE into a system of first order ODEs

- ○ Define new variable $y_1(t)$

  $y_1(t) = y$

- ○ Define new variable $y_2(t)$

  $y_2(t) = y'$

- ○ Define new variable $y_3(t)$

$y_3(t) = y''$

- Define new variable $y_4(t)$

  $y_4(t) = y'''$

- Isolate for $y_4'(t)$ using original ODE

  $y_4'(t) = 1 - \mathit{Heaviside}(t - \pi) - 5y_3(t) - 4y_1(t)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(t) = y_1'(t), y_3(t) = y_2'(t), y_4(t) = y_3'(t), y_4'(t) = 1 - \mathit{Heaviside}(t - \pi) - 5y_3(t) - 4y_1(t)]$

- Define vector

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{bmatrix}$$

- System to solve

$$\vec{y}'(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & -5 & 0 \end{bmatrix} \cdot \vec{y}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 - \mathit{Heaviside}(t - \pi) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 - \mathit{Heaviside}(t - \pi) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & -5 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(t) = A \cdot \overrightarrow{y}(t) + \overrightarrow{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} \mathrm{I} \\ -1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right], \left[2\,\mathrm{I}, \begin{bmatrix} \frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ -\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-2\,\mathrm{I}t} \cdot \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(2t) - \mathrm{I}\sin(2t)) \cdot \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\frac{\mathrm{I}}{8}(\cos(2t) - \mathrm{I}\sin(2t)) \\ -\frac{\cos(2t)}{4} + \frac{\mathrm{I}\sin(2t)}{4} \\ \frac{\mathrm{I}}{2}(\cos(2t) - \mathrm{I}\sin(2t)) \\ \cos(2t) - \mathrm{I}\sin(2t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_1(t) = \begin{bmatrix} -\frac{\sin(2t)}{8} \\ -\frac{\cos(2t)}{4} \\ \frac{\sin(2t)}{2} \\ \cos(2t) \end{bmatrix}, \vec{y}_2(t) = \begin{bmatrix} -\frac{\cos(2t)}{8} \\ \frac{\sin(2t)}{4} \\ \frac{\cos(2t)}{2} \\ -\sin(2t) \end{bmatrix}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I}t} \cdot \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(t) - \mathrm{I}\sin(t)) \cdot \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\mathrm{I}(\cos(t) - \mathrm{I}\sin(t)) \\ -\cos(t) + \mathrm{I}\sin(t) \\ \mathrm{I}(\cos(t) - \mathrm{I}\sin(t)) \\ \cos(t) - \mathrm{I}\sin(t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(t) = \begin{bmatrix} -\sin(t) \\ -\cos(t) \\ \sin(t) \\ \cos(t) \end{bmatrix}, \vec{y}_4(t) = \begin{bmatrix} -\cos(t) \\ \sin(t) \\ \cos(t) \\ -\sin(t) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

  $\vec{y}(t) = c_1\vec{y}_1(t) + c_2\vec{y}_2(t) + c_3\vec{y}_3(t) + c_4\vec{y}_4(t) + \vec{y}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syste

$$\phi(t) = \begin{bmatrix} -\frac{\sin(2t)}{8} & -\frac{\cos(2t)}{8} & -\sin(t) & -\cos(t) \\ -\frac{\cos(2t)}{4} & \frac{\sin(2t)}{4} & -\cos(t) & \sin(t) \\ \frac{\sin(2t)}{2} & \frac{\cos(2t)}{2} & \sin(t) & \cos(t) \\ \cos(2t) & -\sin(2t) & \cos(t) & -\sin(t) \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} -\frac{\sin(2t)}{8} & -\frac{\cos(2t)}{8} & -\sin(t) & -\cos(t) \\ -\frac{\cos(2t)}{4} & \frac{\sin(2t)}{4} & -\cos(t) & \sin(t) \\ \frac{\sin(2t)}{2} & \frac{\cos(2t)}{2} & \sin(t) & \cos(t) \\ \cos(2t) & -\sin(2t) & \cos(t) & -\sin(t) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 0 & -\frac{1}{8} & 0 & -1 \\ -\frac{1}{4} & 0 & -1 & 0 \\ 0 & \frac{1}{2} & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} -\frac{2\cos(t)^2}{3} + \frac{1}{3} + \frac{4\cos(t)}{3} & -\frac{\sin(2t)}{6} + \frac{4\sin(t)}{3} & -\frac{2\cos(t)^2}{3} + \frac{1}{3} + \frac{\cos(t)}{3} & \frac{\sin(t)}{3} - \frac{\sin(2t)}{6} \\ \frac{2\sin(2t)}{3} - \frac{4\sin(t)}{3} & -\frac{2\cos(t)^2}{3} + \frac{1}{3} + \frac{4\cos(t)}{3} & \frac{2\sin(2t)}{3} - \frac{\sin(t)}{3} & -\frac{2\cos(t)^2}{3} + \frac{1}{3} + \cdots \\ \frac{8\cos(t)^2}{3} - \frac{4}{3} - \frac{4\cos(t)}{3} & \frac{2\sin(2t)}{3} - \frac{4\sin(t)}{3} & \frac{8\cos(t)^2}{3} - \frac{4}{3} - \frac{\cos(t)}{3} & \frac{2\sin(2t)}{3} - \frac{\sin(t)}{3} \\ -\frac{8\sin(2t)}{3} + \frac{4\sin(t)}{3} & \frac{8\cos(t)^2}{3} - \frac{4}{3} - \frac{4\cos(t)}{3} & -\frac{8\sin(2t)}{3} + \frac{\sin(t)}{3} & \frac{8\cos(t)^2}{3} - \frac{4}{3} - \cdots \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

  $\vec{y}_p(t) = \Phi(t) \cdot \vec{v}(t)$
- Take the derivative of the particular solution

  $\vec{y}'_p(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$
- Cancel like terms

  $\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$
- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$
- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$
- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{y}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(t) = \begin{bmatrix} -\frac{(1+\cos(t))^2 \mathit{Heaviside}(t-\pi)}{6} + \frac{(\cos(t)-1)^2}{6} \\ \frac{\sin(t)((1+\cos(t))\mathit{Heaviside}(t-\pi)-\cos(t)+1)}{3} \\ \frac{\left(2\cos(t)^2+\cos(t)-1\right)\mathit{Heaviside}(t-\pi)}{3} - \frac{2\cos(t)^2}{3} + \frac{\cos(t)}{3} + \frac{1}{3} \\ -\frac{\sin(t)((4\cos(t)+1)\mathit{Heaviside}(t-\pi)-4\cos(t)+1)}{3} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(t) = c_1\vec{y}_1(t) + c_2\vec{y}_2(t) + c_3\vec{y}_3(t) + c_4\vec{y}_4(t) + \begin{bmatrix} -\frac{(1+\cos(t))^2 \mathit{Heaviside}(t-\pi)}{6} + \frac{(\cos(t)-1)^2}{6} \\ \frac{\sin(t)((1+\cos(t))\mathit{Heaviside}(t-\pi)-\cos(t)+1)}{3} \\ \frac{\left(2\cos(t)^2+\cos(t)-1\right)\mathit{Heaviside}(t-\pi)}{3} - \frac{2\cos(t)^2}{3} + \frac{\cos(t)}{3} \\ -\frac{\sin(t)((4\cos(t)+1)\mathit{Heaviside}(t-\pi)-4\cos(t)+1)}{3} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$y = -\frac{(1+\cos(t))^2 \mathit{Heaviside}(t-\pi)}{6} + \frac{(-6c_2+4)\cos(t)^2}{24} + \frac{(-6c_1\sin(t)-24c_4-8)\cos(t)}{24} - c_3 \sin(t) + \frac{c_2}{8} + \frac{1}{6}$

- Use the initial condition $y(0) = 0$

$0 = -\frac{c_2}{8} - c_4$

- Calculate the 1st derivative of the solution

$y' = \frac{\sin(t)(1+\cos(t))\mathit{Heaviside}(t-\pi)}{3} - \frac{\mathit{Dirac}(t-\pi)(1+\cos(t))^2}{6} - \frac{(-6c_2+4)\cos(t)\sin(t)}{12} - \frac{c_1\cos(t)^2}{4} - \frac{(-6c_1\sin(t)-24c_4-}{24}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

$0 = -\frac{c_1}{4} - c_3$

- Calculate the 2nd derivative of the solution

$y'' = \frac{\cos(t)(1+\cos(t))\mathit{Heaviside}(t-\pi)}{3} - \frac{\mathit{Heaviside}(t-\pi)\sin(t)^2}{3} + \frac{2\mathit{Dirac}(t-\pi)(1+\cos(t))\sin(t)}{3} - \frac{\mathit{Dirac}(1,t-\pi)(1+\cos(t))^2}{6}$

- Use the initial condition $y''\Big|_{\{t=0\}} = 0$

$0 = \frac{c_2}{2} + c_4$

- Calculate the 3rd derivative of the solution

$y''' = -\frac{\sin(t)(1+\cos(t))\mathit{Heaviside}(t-\pi)}{3} - \mathit{Heaviside}(t-\pi)\sin(t)\cos(t) + \mathit{Dirac}(t-\pi)(1+\cos(t))\cos$

- Use the initial condition $y'''\Big|_{\{t=0\}} = 0$

$0 = c_1 + c_3$

- Solve for the unknown coefficients

$\{c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 0\}$

- Solution to the IVP

$y = -\frac{(1+\cos(t))^2 \mathit{Heaviside}(t-\pi)}{6} - \frac{4(\cos(t)-1)\left(\frac{3}{8} - \frac{3\cos(t)}{8}\right)}{9}$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.328 (sec). Leaf size: 23

```
dsolve([diff(y(t),t$4)+5*diff(y(t),t$2)+4*y(t)=1-Heaviside(t-Pi),y(0) = 0, D(y)(0) = 0, (D@@
```

$$y(t) = -\frac{(\cos(t)+1)^2 \operatorname{Heaviside}(t-\pi)}{6} + \frac{(\cos(t)-1)^2}{6}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 29

```
DSolve[{y''''[t]+5*y''[t]+4*y[t]==1-UnitStep[t-Pi],{y[0]==0,y'[0]==0,y''[0]==0,y'''[0]==0}},
```

$$y(t) \to \begin{cases} \frac{2}{3}\sin^4\left(\frac{t}{2}\right) & t \leq \pi \\ -\frac{2\cos(t)}{3} & \text{True} \end{cases}$$

4.9 problem 11(b)

Internal problem ID [853]
Internal file name [`OUTPUT/853_Sunday_June_05_2022_01_51_40_AM_13796087/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.4, The Laplace Transform. Differential equations with discontinuous forcing functions. page 268
Problem number: 11(b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{u'' + \frac{u'}{4} + u = k\left(\text{Heaviside}\left(t - \frac{3}{2}\right) - \text{Heaviside}\left(t - \frac{5}{2}\right)\right)}$$

With initial conditions

$$[u(0) = 0, u'(0) = 0]$$

4.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$u'' + p(t)u' + q(t)u = F$$

Where here

$$\begin{aligned} p(t) &= \frac{1}{4} \\ q(t) &= 1 \\ F &= k\left(\text{Heaviside}\left(t - \frac{3}{2}\right) - \text{Heaviside}\left(t - \frac{5}{2}\right)\right) \end{aligned}$$

Hence the ode is

$$u'' + \frac{u'}{4} + u = k\left(\text{Heaviside}\left(t - \frac{3}{2}\right) - \text{Heaviside}\left(t - \frac{5}{2}\right)\right)$$

The domain of $p(t) = \frac{1}{4}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = k\left(\text{Heaviside}\left(t - \frac{3}{2}\right) - \text{Heaviside}\left(t - \frac{5}{2}\right.\right.$
is

$$\left\{\frac{3}{2} \leq t \leq \frac{5}{2}, \frac{5}{2} \leq t \leq \infty, -\infty \leq t \leq \frac{3}{2}\right\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(u) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(u') &= sY(s) - u(0)\\ \mathcal{L}(u'') &= s^2Y(s) - u'(0) - su(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - u'(0) - su(0) + \frac{sY(s)}{4} - \frac{u(0)}{4} + Y(s) = \frac{k\left(\mathrm{e}^{-\frac{3s}{2}} - \mathrm{e}^{-\frac{5s}{2}}\right)}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}u(0) &= 0\\ u'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + \frac{sY(s)}{4} + Y(s) = \frac{k\left(\mathrm{e}^{-\frac{3s}{2}} - \mathrm{e}^{-\frac{5s}{2}}\right)}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{4k\left(\mathrm{e}^{-\frac{3s}{2}} - \mathrm{e}^{-\frac{5s}{2}}\right)}{s\left(4s^2+s+4\right)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
u &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{4k\left(\mathrm{e}^{-\frac{3s}{2}} - \mathrm{e}^{-\frac{5s}{2}}\right)}{s\left(4s^2+s+4\right)}\right) \\
&= \frac{\left(i\sqrt{7}+21\right)\left(\left(-63+3i\sqrt{7}+32\,\mathrm{e}^{-\frac{\left(3i\sqrt{7}+1\right)(2t-5)}{16}} + \left(31-3i\sqrt{7}\right)\mathrm{e}^{-\frac{\left(-3i\sqrt{7}+1\right)(2t-5)}{16}}\right)\mathrm{Heaviside}\left(t-\frac{5}{2}\right) + \right.}{1344}
\end{aligned}$$

Hence the final solution is

$$u = \frac{\left(i\sqrt{7}+21\right)\left(\left(-63+3i\sqrt{7}+32\,\mathrm{e}^{-\frac{\left(3i\sqrt{7}+1\right)(2t-5)}{16}} + \left(31-3i\sqrt{7}\right)\mathrm{e}^{-\frac{\left(-3i\sqrt{7}+1\right)(2t-5)}{16}}\right)\mathrm{Heaviside}\left(t-\frac{5}{2}\right) + \left(6\right.\right.}{1344}$$

Simplifying the solution gives

$$u = -\frac{k\left(\left(-21+i\sqrt{7}\right)\mathrm{Heaviside}\left(t-\frac{5}{2}\right)\mathrm{e}^{\frac{3i(2t-5)\sqrt{7}}{16}-\frac{t}{8}+\frac{5}{16}} + \left(i\sqrt{7}+21\right)\mathrm{Heaviside}\left(t-\frac{3}{2}\right)\mathrm{e}^{\frac{3}{16}+\frac{3i(-2t+3)\sqrt{7}}{16}-\frac{t}{8}} + \right.}{}$$

Summary

The solution(s) found are the following

$$u = -\frac{k\left(\left(-21+i\sqrt{7}\right)\mathrm{Heaviside}\left(t-\frac{5}{2}\right)\mathrm{e}^{\frac{3i(2t-5)\sqrt{7}}{16}-\frac{t}{8}+\frac{5}{16}} + \left(i\sqrt{7}+21\right)\mathrm{Heaviside}\left(t-\frac{3}{2}\right)\mathrm{e}^{\frac{3}{16}+\frac{3i(-2t+3)\sqrt{7}}{16}-\frac{t}{8}} + \right.}{} \tag{1}$$

Verification of solutions

$$u = -\frac{k\left(\left(-21+i\sqrt{7}\right)\mathrm{Heaviside}\left(t-\frac{5}{2}\right)\mathrm{e}^{\frac{3i(2t-5)\sqrt{7}}{16}-\frac{t}{8}+\frac{5}{16}} + \left(i\sqrt{7}+21\right)\mathrm{Heaviside}\left(t-\frac{3}{2}\right)\mathrm{e}^{\frac{3}{16}+\frac{3i(-2t+3)\sqrt{7}}{16}-\frac{t}{8}} + \right.}{}$$

Verified OK.

4.9.2 Maple step by step solution

Let's solve

$$\left[u'' + \frac{u'}{4} + u = k\left(Heaviside\left(t - \frac{3}{2}\right) - Heaviside\left(t - \frac{5}{2}\right)\right), u(0) = 0, u'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  u''

- Characteristic polynomial of homogeneous ODE

  $r^2 + \frac{1}{4}r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{\left(-\frac{1}{4}\right) \pm \left(\sqrt{-\frac{63}{16}}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{8} - \frac{3\,\mathrm{I}\sqrt{7}}{8}, -\frac{1}{8} + \frac{3\,\mathrm{I}\sqrt{7}}{8}\right)$

- 1st solution of the homogeneous ODE

  $u_1(t) = \mathrm{e}^{-\frac{t}{8}} \cos\left(\frac{3\sqrt{7}\,t}{8}\right)$

- 2nd solution of the homogeneous ODE

  $u_2(t) = \mathrm{e}^{-\frac{t}{8}} \sin\left(\frac{3\sqrt{7}\,t}{8}\right)$

- General solution of the ODE

  $u = c_1 u_1(t) + c_2 u_2(t) + u_p(t)$

- Substitute in solutions of the homogeneous ODE

  $u = c_1 \mathrm{e}^{-\frac{t}{8}} \cos\left(\frac{3\sqrt{7}\,t}{8}\right) + c_2 \mathrm{e}^{-\frac{t}{8}} \sin\left(\frac{3\sqrt{7}\,t}{8}\right) + u_p(t)$

□ Find a particular solution $u_p(t)$ of the ODE

  ○ Use variation of parameters to find u_p here $f(t)$ is the forcing function

  $\left[u_p(t) = -u_1(t)\left(\int \frac{u_2(t) f(t)}{W(u_1(t),u_2(t))} dt\right) + u_2(t)\left(\int \frac{u_1(t) f(t)}{W(u_1(t),u_2(t))} dt\right), f(t) = k\left(Heaviside\left(t - \frac{3}{2}\right) - Hea\right.\right.$

  ○ Wronskian of solutions of the homogeneous equation

  $$W(u_1(t), u_2(t)) = \begin{bmatrix} \mathrm{e}^{-\frac{t}{8}} \cos\left(\frac{3\sqrt{7}\,t}{8}\right) & \mathrm{e}^{-\frac{t}{8}} \sin\left(\frac{3\sqrt{7}\,t}{8}\right) \\ -\frac{\mathrm{e}^{-\frac{t}{8}} \cos\left(\frac{3\sqrt{7}\,t}{8}\right)}{8} - \frac{3\,\mathrm{e}^{-\frac{t}{8}}\sqrt{7}\sin\left(\frac{3\sqrt{7}\,t}{8}\right)}{8} & -\frac{\mathrm{e}^{-\frac{t}{8}} \sin\left(\frac{3\sqrt{7}\,t}{8}\right)}{8} + \frac{3\,\mathrm{e}^{-\frac{t}{8}}\sqrt{7}\cos\left(\frac{3\sqrt{7}\,t}{8}\right)}{8} \end{bmatrix}$$

- Compute Wronskian

  $W(u_1(t), u_2(t)) = \frac{3\sqrt{7}\,e^{-\frac{t}{4}}}{8}$

- Substitute functions into equation for $u_p(t)$

  $u_p(t) = -\frac{8k\sqrt{7}\,e^{-\frac{t}{8}} (\cos(\frac{3\sqrt{7}\,t}{8}) (\int e^{\frac{t}{8}} \sin(\frac{3\sqrt{7}\,t}{8}) (Heaviside(t-\frac{3}{2}) - Heaviside(t-\frac{5}{2})) dt) - \sin(\frac{3\sqrt{7}\,t}{8}) (\int e^{\frac{t}{8}} \cos(\frac{3\sqrt{7}\,t}{8}) (Heavi}{21}$

- Compute integrals

  $u_p(t) = -\frac{(-((\sqrt{7} \sin(\frac{9\sqrt{7}}{16}) - 21 \cos(\frac{9\sqrt{7}}{16})) \cos(\frac{3\sqrt{7}\,t}{8}) - \sin(\frac{3\sqrt{7}\,t}{8}) (\sqrt{7} \cos(\frac{9\sqrt{7}}{16}) + 21 \sin(\frac{9\sqrt{7}}{16}))) Heaviside(t-\frac{3}{2}) e^{\frac{3}{16}-\frac{t}{8}} +}{}$

- Substitute particular solution into general solution to ODE

  $u = c_1 e^{-\frac{t}{8}} \cos\left(\frac{3\sqrt{7}\,t}{8}\right) + c_2 e^{-\frac{t}{8}} \sin\left(\frac{3\sqrt{7}\,t}{8}\right) - \frac{(-((\sqrt{7} \sin(\frac{9\sqrt{7}}{16}) - 21 \cos(\frac{9\sqrt{7}}{16})) \cos(\frac{3\sqrt{7}\,t}{8}) - \sin(\frac{3\sqrt{7}\,t}{8}) (\sqrt{7} \cos(\frac{9}{}}{}$

- ☐ Check validity of solution $u = c_1 e^{-\frac{t}{8}} \cos\left(\frac{3\sqrt{7}\,t}{8}\right) + c_2 e^{-\frac{t}{8}} \sin\left(\frac{3\sqrt{7}\,t}{8}\right) - \frac{(-((\sqrt{7} \sin(\frac{9\sqrt{7}}{16}) - 21 \cos(\frac{9\sqrt{7}}{16})}{}$

  - Use initial condition $u(0) = 0$

    $0 = c_1$

  - Compute derivative of the solution

    $u' = -\frac{c_1 e^{-\frac{t}{8}} \cos(\frac{3\sqrt{7}\,t}{8})}{8} - \frac{3c_1 e^{-\frac{t}{8}} \sqrt{7} \sin(\frac{3\sqrt{7}\,t}{8})}{8} - \frac{c_2 e^{-\frac{t}{8}} \sin(\frac{3\sqrt{7}\,t}{8})}{8} + \frac{3c_2 e^{-\frac{t}{8}} \sqrt{7} \cos(\frac{3\sqrt{7}\,t}{8})}{8} - \frac{(-(-\frac{3(\sqrt{7} \sin(\frac{9\sqrt{7}}{16}}{}}{}$

  - Use the initial condition $u'\Big|_{\{t=0\}} = 0$

    $0 = -\frac{c_1}{8} + \frac{3\sqrt{7}\,c_2}{8}$

  - Solve for c_1 and c_2

    $\{c_1 = 0, c_2 = 0\}$

  - Substitute constant values into general solution and simplify

    $u = -\frac{(-((\sqrt{7} \sin(\frac{9\sqrt{7}}{16}) - 21 \cos(\frac{9\sqrt{7}}{16})) \cos(\frac{3\sqrt{7}\,t}{8}) - \sin(\frac{3\sqrt{7}\,t}{8}) (\sqrt{7} \cos(\frac{9\sqrt{7}}{16}) + 21 \sin(\frac{9\sqrt{7}}{16}))) Heaviside(t-\frac{3}{2}) e^{\frac{3}{16}-\frac{t}{8}} + ((\sqrt{}}{}$

- Solution to the IVP

  $u = -\frac{(-((\sqrt{7} \sin(\frac{9\sqrt{7}}{16}) - 21 \cos(\frac{9\sqrt{7}}{16})) \cos(\frac{3\sqrt{7}\,t}{8}) - \sin(\frac{3\sqrt{7}\,t}{8}) (\sqrt{7} \cos(\frac{9\sqrt{7}}{16}) + 21 \sin(\frac{9\sqrt{7}}{16}))) Heaviside(t-\frac{3}{2}) e^{\frac{3}{16}-\frac{t}{8}} + ((\sqrt{}}{}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 1.344 (sec). Leaf size: 129

```
dsolve([diff(u(t),t$2)+1/4*diff(u(t),t)+u(t)=k*(Heaviside(t-3/2)-Heaviside(t-5/2)),u(0) = 0,
```

$$u(t) = -\frac{k\left(\text{Heaviside}\left(t-\frac{5}{2}\right)\left(-21+i\sqrt{7}\right)\mathrm{e}^{\frac{3i\sqrt{7}\,(2t-5)}{16}-\frac{t}{8}+\frac{5}{16}}+\left(-i\sqrt{7}-21\right)\text{Heaviside}\left(t-\frac{5}{2}\right)\mathrm{e}^{-\frac{3i\sqrt{7}\,(2t-5)}{16}-\frac{t}{8}+\frac{5}{16}}\right.}{}$$

✓ Solution by Mathematica

Time used: 0.163 (sec). Leaf size: 192

```
DSolve[{u''[t]+1/4*u'[t]+u[t]==k*(UnitStep[t-3/2]-UnitStep[t-5/2]),{u[0]==0,u'[0]==0}},u[t],
```

$u(t)$

$$\to \left\{ \begin{array}{c} -e^{\frac{3}{16}-\frac{t}{8}}\cos\left(\frac{3}{16}\sqrt{7}(3-2t)\right)k+\frac{e^{\frac{3}{16}-\frac{t}{8}}\sin\left(\frac{3}{16}\sqrt{7}(3-2t)\right)k}{3\sqrt{7}}+k \\ \frac{1}{21}e^{\frac{3}{16}-\frac{t}{8}}k\left(-21\cos\left(\frac{3}{16}\sqrt{7}(3-2t)\right)+21\sqrt[8]{e}\cos\left(\frac{3}{16}\sqrt{7}(5-2t)\right)+\sqrt{7}\left(\sin\left(\frac{3}{16}\sqrt{7}(3-2t)\right)-\sqrt[8]{e}\sin\right.\right. \end{array}\right.$$

4.10 problem 11(c) k=1/2

Internal problem ID [854]
Internal file name [`OUTPUT/854_Sunday_June_05_2022_01_51_52_AM_90207999/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.4, The Laplace Transform. Differential equations with discontinuous forcing functions. page 268
Problem number: 11(c) k=1/2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{u'' + \frac{u'}{4} + u = \frac{\text{Heaviside}\left(t - \frac{3}{2}\right)}{2} - \frac{\text{Heaviside}\left(t - \frac{5}{2}\right)}{2}}$$

With initial conditions

$$[u(0) = 0, u'(0) = 0]$$

4.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$u'' + p(t)u' + q(t)u = F$$

Where here

$$\begin{aligned} p(t) &= \frac{1}{4} \\ q(t) &= 1 \\ F &= \frac{\text{Heaviside}\left(t - \frac{3}{2}\right)}{2} - \frac{\text{Heaviside}\left(t - \frac{5}{2}\right)}{2} \end{aligned}$$

Hence the ode is

$$u'' + \frac{u'}{4} + u = \frac{\text{Heaviside}\left(t - \frac{3}{2}\right)}{2} - \frac{\text{Heaviside}\left(t - \frac{5}{2}\right)}{2}$$

The domain of $p(t) = \frac{1}{4}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \frac{\text{Heaviside}\left(t - \frac{3}{2}\right)}{2} - \frac{\text{Heaviside}\left(t - \frac{5}{2}\right)}{2}$ is

$$\left\{\frac{3}{2} \le t \le \frac{5}{2}, \frac{5}{2} \le t \le \infty, -\infty \le t \le \frac{3}{2}\right\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(u) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(u') &= sY(s) - u(0)\\ \mathcal{L}(u'') &= s^2Y(s) - u'(0) - su(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - u'(0) - su(0) + \frac{sY(s)}{4} - \frac{u(0)}{4} + Y(s) = \frac{e^{-\frac{3s}{2}} - e^{-\frac{5s}{2}}}{2s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}u(0) &= 0\\ u'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + \frac{sY(s)}{4} + Y(s) = \frac{e^{-\frac{3s}{2}} - e^{-\frac{5s}{2}}}{2s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{2\,\mathrm{e}^{-\frac{3s}{2}} - 2\,\mathrm{e}^{-\frac{5s}{2}}}{s\,(4s^2+s+4)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} u &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{2\,\mathrm{e}^{-\frac{3s}{2}} - 2\,\mathrm{e}^{-\frac{5s}{2}}}{s\,(4s^2+s+4)}\right) \\ &= \frac{\left(i\sqrt{7}+21\right)\left(\left(-63+3i\sqrt{7}+32\,\mathrm{e}^{-\frac{\left(3i\sqrt{7}+1\right)(2t-5)}{16}} + \left(31-3i\sqrt{7}\right)\mathrm{e}^{-\frac{\left(-3i\sqrt{7}+1\right)(2t-5)}{16}}\right)\mathrm{Heaviside}\left(t-\frac{5}{2}\right) + \right.}{2688} \end{aligned}$$

Hence the final solution is

$$u = \frac{\left(i\sqrt{7}+21\right)\left(\left(-63+3i\sqrt{7}+32\,\mathrm{e}^{-\frac{\left(3i\sqrt{7}+1\right)(2t-5)}{16}} + \left(31-3i\sqrt{7}\right)\mathrm{e}^{-\frac{\left(-3i\sqrt{7}+1\right)(2t-5)}{16}}\right)\mathrm{Heaviside}\left(t-\frac{5}{2}\right) + \left(\right.\right.}{2688}$$

Simplifying the solution gives

$$\begin{aligned} u = {} & \frac{\left(-i\sqrt{7}+21\right)\mathrm{Heaviside}\left(t-\frac{5}{2}\right)\mathrm{e}^{\frac{3i(2t-5)\sqrt{7}}{16}-\frac{t}{8}+\frac{5}{16}}}{84} \\ & + \frac{\left(-i\sqrt{7}-21\right)\mathrm{Heaviside}\left(t-\frac{3}{2}\right)\mathrm{e}^{\frac{3}{16}+\frac{3i(-2t+3)\sqrt{7}}{16}-\frac{t}{8}}}{84} \\ & + \frac{\left(i\sqrt{7}+21\right)\mathrm{Heaviside}\left(t-\frac{5}{2}\right)\mathrm{e}^{-\frac{3i(2t-5)\sqrt{7}}{16}-\frac{t}{8}+\frac{5}{16}}}{84} \\ & + \frac{\left(-21+i\sqrt{7}\right)\mathrm{Heaviside}\left(t-\frac{3}{2}\right)\mathrm{e}^{\frac{\left(3i\sqrt{7}-1\right)(-3+2t)}{16}}}{84} \\ & - \frac{\mathrm{Heaviside}\left(t-\frac{5}{2}\right)}{2} + \frac{\mathrm{Heaviside}\left(t-\frac{3}{2}\right)}{2} \end{aligned}$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
u = {} & \frac{\left(-i\sqrt{7}+21\right) \operatorname{Heaviside}\left(t-\frac{5}{2}\right) \mathrm{e}^{\frac{3 i (2 t-5) \sqrt{7}}{16}-\frac{t}{8}+\frac{5}{16}}}{84} \\
& + \frac{\left(-i\sqrt{7}-21\right) \operatorname{Heaviside}\left(t-\frac{3}{2}\right) \mathrm{e}^{\frac{3}{16}+\frac{3 i (-2 t+3) \sqrt{7}}{16}-\frac{t}{8}}}{84} \\
& + \frac{\left(i\sqrt{7}+21\right) \operatorname{Heaviside}\left(t-\frac{5}{2}\right) \mathrm{e}^{-\frac{3 i (2 t-5) \sqrt{7}}{16}-\frac{t}{8}+\frac{5}{16}}}{84} \\
& + \frac{\left(-21+i\sqrt{7}\right) \operatorname{Heaviside}\left(t-\frac{3}{2}\right) \mathrm{e}^{\frac{\left(3 i \sqrt{7}-1\right) (-3+2 t)}{16}}}{84} \\
& - \frac{\operatorname{Heaviside}\left(t-\frac{5}{2}\right)}{2} + \frac{\operatorname{Heaviside}\left(t-\frac{3}{2}\right)}{2}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
u = {} & \frac{\left(-i\sqrt{7}+21\right) \operatorname{Heaviside}\left(t-\frac{5}{2}\right) \mathrm{e}^{\frac{3 i (2 t-5) \sqrt{7}}{16}-\frac{t}{8}+\frac{5}{16}}}{84} \\
& + \frac{\left(-i\sqrt{7}-21\right) \operatorname{Heaviside}\left(t-\frac{3}{2}\right) \mathrm{e}^{\frac{3}{16}+\frac{3 i (-2 t+3) \sqrt{7}}{16}-\frac{t}{8}}}{84} \\
& + \frac{\left(i\sqrt{7}+21\right) \operatorname{Heaviside}\left(t-\frac{5}{2}\right) \mathrm{e}^{-\frac{3 i (2 t-5) \sqrt{7}}{16}-\frac{t}{8}+\frac{5}{16}}}{84} \\
& + \frac{\left(-21+i\sqrt{7}\right) \operatorname{Heaviside}\left(t-\frac{3}{2}\right) \mathrm{e}^{\frac{\left(3 i \sqrt{7}-1\right) (-3+2 t)}{16}}}{84} \\
& - \frac{\operatorname{Heaviside}\left(t-\frac{5}{2}\right)}{2} + \frac{\operatorname{Heaviside}\left(t-\frac{3}{2}\right)}{2}
\end{aligned}
$$

Verified OK.

4.10.2 Maple step by step solution

Let's solve

$$\left[u''+\frac{u'}{4}+u=\frac{\mathit{Heaviside}\left(t-\frac{3}{2}\right)}{2}-\frac{\mathit{Heaviside}\left(t-\frac{5}{2}\right)}{2}, u(0)=0, u'\Big|_{\{t=0\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  u''

- Characteristic polynomial of homogeneous ODE

  $r^2+\frac{1}{4}r+1=0$

- Use quadratic formula to solve for r

  $r = \frac{(-\frac{1}{4}) \pm \left(\sqrt{-\frac{63}{16}}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{8} - \frac{3\,\mathrm{I}\sqrt{7}}{8}, -\frac{1}{8} + \frac{3\,\mathrm{I}\sqrt{7}}{8}\right)$

- 1st solution of the homogeneous ODE

  $u_1(t) = \mathrm{e}^{-\frac{t}{8}} \cos\left(\frac{3\sqrt{7}\,t}{8}\right)$

- 2nd solution of the homogeneous ODE

  $u_2(t) = \mathrm{e}^{-\frac{t}{8}} \sin\left(\frac{3\sqrt{7}\,t}{8}\right)$

- General solution of the ODE

  $u = c_1 u_1(t) + c_2 u_2(t) + u_p(t)$

- Substitute in solutions of the homogeneous ODE

  $u = c_1 \mathrm{e}^{-\frac{t}{8}} \cos\left(\frac{3\sqrt{7}\,t}{8}\right) + c_2 \mathrm{e}^{-\frac{t}{8}} \sin\left(\frac{3\sqrt{7}\,t}{8}\right) + u_p(t)$

☐ Find a particular solution $u_p(t)$ of the ODE

  - Use variation of parameters to find u_p here $f(t)$ is the forcing function

    $\left[u_p(t) = -u_1(t)\left(\int \frac{u_2(t) f(t)}{W(u_1(t),u_2(t))} dt\right) + u_2(t)\left(\int \frac{u_1(t) f(t)}{W(u_1(t),u_2(t))} dt\right), f(t) = \frac{Heaviside(t-\frac{3}{2})}{2} - \frac{Heaviside(t-\frac{5}{2})}{2}\right.$

  - Wronskian of solutions of the homogeneous equation

    $$W(u_1(t), u_2(t)) = \begin{bmatrix} \mathrm{e}^{-\frac{t}{8}} \cos\left(\frac{3\sqrt{7}\,t}{8}\right) & \mathrm{e}^{-\frac{t}{8}} \sin\left(\frac{3\sqrt{7}\,t}{8}\right) \\ -\frac{\mathrm{e}^{-\frac{t}{8}} \cos\left(\frac{3\sqrt{7}\,t}{8}\right)}{8} - \frac{3\,\mathrm{e}^{-\frac{t}{8}}\sqrt{7}\sin\left(\frac{3\sqrt{7}\,t}{8}\right)}{8} & -\frac{\mathrm{e}^{-\frac{t}{8}} \sin\left(\frac{3\sqrt{7}\,t}{8}\right)}{8} + \frac{3\,\mathrm{e}^{-\frac{t}{8}}\sqrt{7}\cos\left(\frac{3\sqrt{7}\,t}{8}\right)}{8} \end{bmatrix}$$

  - Compute Wronskian

    $W(u_1(t), u_2(t)) = \frac{3\sqrt{7}\,\mathrm{e}^{-\frac{t}{4}}}{8}$

  - Substitute functions into equation for $u_p(t)$

    $u_p(t) = -\frac{4\sqrt{7}\,\mathrm{e}^{-\frac{t}{8}}\left(\cos\left(\frac{3\sqrt{7}\,t}{8}\right)\left(\int \mathrm{e}^{\frac{t}{8}} \sin\left(\frac{3\sqrt{7}\,t}{8}\right)(Heaviside(t-\frac{3}{2}) - Heaviside(t-\frac{5}{2}))dt\right) - \sin\left(\frac{3\sqrt{7}\,t}{8}\right)\left(\int \mathrm{e}^{\frac{t}{8}} \cos\left(\frac{3\sqrt{7}\,t}{8}\right)(Heavis\ldots\right.\right.}{21}$

  - Compute integrals

    $u_p(t) = \frac{\left(\left(\sqrt{7}\sin\left(\frac{9\sqrt{7}}{16}\right) - 21\cos\left(\frac{9\sqrt{7}}{16}\right)\right)\cos\left(\frac{3\sqrt{7}\,t}{8}\right) - \sin\left(\frac{3\sqrt{7}\,t}{8}\right)\left(\sqrt{7}\cos\left(\frac{9\sqrt{7}}{16}\right) + 21\sin\left(\frac{9\sqrt{7}}{16}\right)\right)\right) Heaviside(t-\frac{3}{2})\,\mathrm{e}^{\frac{3}{16}-\frac{t}{8}}}{42} - \ldots$

- Substitute particular solution into general solution to ODE

$$u = c_1 e^{-\frac{t}{8}} \cos\left(\frac{3\sqrt{7}\,t}{8}\right) + c_2 e^{-\frac{t}{8}} \sin\left(\frac{3\sqrt{7}\,t}{8}\right) + \frac{\left(\left(\sqrt{7}\sin\left(\frac{9\sqrt{7}}{16}\right) - 21\cos\left(\frac{9\sqrt{7}}{16}\right)\right)\cos\left(\frac{3\sqrt{7}\,t}{8}\right) - \sin\left(\frac{3\sqrt{7}\,t}{8}\right)\left(\sqrt{7}\cos\left(\frac{9\sqrt{7}}{16}\right)\right.}{42}$$

□ Check validity of solution $u = c_1 e^{-\frac{t}{8}} \cos\left(\frac{3\sqrt{7}\,t}{8}\right) + c_2 e^{-\frac{t}{8}} \sin\left(\frac{3\sqrt{7}\,t}{8}\right) + \frac{\left(\left(\sqrt{7}\sin\left(\frac{9\sqrt{7}}{16}\right) - 21\cos\left(\frac{9\sqrt{7}}{16}\right)\right)\cos\right.}{}$

- Use initial condition $u(0) = 0$

  $0 = c_1$

- Compute derivative of the solution

$$u' = -\frac{c_1 e^{-\frac{t}{8}} \cos\left(\frac{3\sqrt{7}\,t}{8}\right)}{8} - \frac{3c_1 e^{-\frac{t}{8}} \sqrt{7}\sin\left(\frac{3\sqrt{7}\,t}{8}\right)}{8} - \frac{c_2 e^{-\frac{t}{8}} \sin\left(\frac{3\sqrt{7}\,t}{8}\right)}{8} + \frac{3c_2 e^{-\frac{t}{8}} \sqrt{7}\cos\left(\frac{3\sqrt{7}\,t}{8}\right)}{8} + \frac{\left(-\frac{3\left(\sqrt{7}\sin\left(\frac{9\sqrt{7}}{16}\right) - \right.}{}\right.}{}$$

- Use the initial condition $u'\Big|_{\{t=0\}} = 0$

  $0 = -\frac{c_1}{8} + \frac{3\sqrt{7}\,c_2}{8}$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 0\}$

- Substitute constant values into general solution and simplify

$$u = \frac{\left(\left(\sqrt{7}\sin\left(\frac{9\sqrt{7}}{16}\right) - 21\cos\left(\frac{9\sqrt{7}}{16}\right)\right)\cos\left(\frac{3\sqrt{7}\,t}{8}\right) - \sin\left(\frac{3\sqrt{7}\,t}{8}\right)\left(\sqrt{7}\cos\left(\frac{9\sqrt{7}}{16}\right) + 21\sin\left(\frac{9\sqrt{7}}{16}\right)\right)\right) Heaviside\left(t - \frac{3}{2}\right) e^{\frac{3}{16} - \frac{t}{8}}}{42} - \frac{\left(\left(\sqrt{7}\,s\right.\right.}{}$$

- Solution to the IVP

$$u = \frac{\left(\left(\sqrt{7}\sin\left(\frac{9\sqrt{7}}{16}\right) - 21\cos\left(\frac{9\sqrt{7}}{16}\right)\right)\cos\left(\frac{3\sqrt{7}\,t}{8}\right) - \sin\left(\frac{3\sqrt{7}\,t}{8}\right)\left(\sqrt{7}\cos\left(\frac{9\sqrt{7}}{16}\right) + 21\sin\left(\frac{9\sqrt{7}}{16}\right)\right)\right) Heaviside\left(t - \frac{3}{2}\right) e^{\frac{3}{16} - \frac{t}{8}}}{42} - \frac{\left(\left(\sqrt{7}\,s\right.\right.}{}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.422 (sec). Leaf size: 128

```
dsolve([diff(u(t),t$2)+1/4*diff(u(t),t)+u(t)=1/2*(Heaviside(t-3/2)-Heaviside(t-5/2)),u(0) =
```

$$
\begin{aligned}
u(t) = &\frac{\left(-i\sqrt{7}+21\right) \mathrm{Heaviside}\left(t-\frac{5}{2}\right) \mathrm{e}^{\frac{3i\sqrt{7}\,(2t-5)}{16}-\frac{t}{8}+\frac{5}{16}}}{84} \\
&+ \frac{\mathrm{Heaviside}\left(t-\frac{5}{2}\right) \mathrm{e}^{-\frac{3i\sqrt{7}\,(2t-5)}{16}-\frac{t}{8}+\frac{5}{16}} \left(i\sqrt{7}+21\right)}{84} \\
&+ \frac{\left(-i\sqrt{7}-21\right) \mathrm{Heaviside}\left(t-\frac{3}{2}\right) \mathrm{e}^{\frac{3}{16}+\frac{3i(-2t+3)\sqrt{7}}{16}-\frac{t}{8}}}{84} \\
&+ \frac{\left(-21+i\sqrt{7}\right) \mathrm{Heaviside}\left(t-\frac{3}{2}\right) \mathrm{e}^{\frac{\left(3i\sqrt{7}-1\right)(2t-3)}{16}}}{84} \\
&- \frac{\mathrm{Heaviside}\left(t-\frac{5}{2}\right)}{2} + \frac{\mathrm{Heaviside}\left(t-\frac{3}{2}\right)}{2}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.115 (sec). Leaf size: 190

```
DSolve[{u''[t]+1/4*u'[t]+u[t]==1/2*(UnitStep[t-3/2]-UnitStep[t-5/2]),{u[0]==0,u'[0]==0}},u[t
```

$$u(t)$$

$$
\to \{ \begin{array}{c}
\frac{1}{42}\left(-21e^{\frac{3}{16}-\frac{t}{8}}\cos\left(\frac{3}{16}\sqrt{7}(3-2t)\right)+\sqrt{7}e^{\frac{3}{16}-\frac{t}{8}}\sin\left(\frac{3}{16}\sqrt{7}(3-2t)\right)+21\right) \\
\frac{1}{42}e^{\frac{3}{16}-\frac{t}{8}}\left(-21\cos\left(\frac{3}{16}\sqrt{7}(3-2t)\right)+21\sqrt[8]{e}\cos\left(\frac{3}{16}\sqrt{7}(5-2t)\right)+\sqrt{7}\left(\sin\left(\frac{3}{16}\sqrt{7}(3-2t)\right)-\sqrt[8]{e}\sin\left(\right.\right.\right.
\end{array}
$$

4.11 problem 12

Internal problem ID [855]
Internal file name [OUTPUT/855_Sunday_June_05_2022_01_52_00_AM_53461595/index.tex]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.4, The Laplace Transform. Differential equations with discontinuous forcing functions. page 268
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$u'' + \frac{u'}{4} + u = \frac{\text{Heaviside}\,(t-5)\,(t-5) - \text{Heaviside}\,(t-5-k)\,(t-5-k)}{k}$$

With initial conditions

$$[u(0) = 0, u'(0) = 0]$$

4.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$u'' + p(t)u' + q(t)u = F$$

Where here

$$p(t) = \frac{1}{4}$$
$$q(t) = 1$$
$$F = \frac{(-t+5+k)\,\text{Heaviside}\,(t-5-k) + \text{Heaviside}\,(t-5)\,(t-5)}{k}$$

Hence the ode is

$$u'' + \frac{u'}{4} + u = \frac{(-t+5+k)\,\text{Heaviside}\,(t-5-k) + \text{Heaviside}\,(t-5)\,(t-5)}{k}$$

The domain of $p(t) = \frac{1}{4}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \frac{(-t+5+k)\,\text{Heaviside}(t-5-k)+\text{Heaviside}(t-5)(t-5)}{k}$ is

$$\{5 \le t \le 5+k, -\infty \le t \le 5, 5+k \le t \le \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(u) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(u') = sY(s) - u(0)$$
$$\mathcal{L}(u'') = s^2Y(s) - u'(0) - su(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - u'(0) - su(0) + \frac{sY(s)}{4} - \frac{u(0)}{4} + Y(s) = -\frac{\text{laplace}\,(\text{Heaviside}\,(t-5-k)\,t, t, s)}{k} + \text{laplace}\,(\text{Heav} \tag{1}$$

But the initial conditions are

$$u(0) = 0$$
$$u'(0) = 0$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + \frac{sY(s)}{4} + Y(s) = -\frac{\text{laplace}\,(\text{Heaviside}\,(t-5-k)\,t, t, s)}{k} + \text{laplace}\,(\text{Heaviside}\,(t-5-k)\,, t, s) +$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{4(-\operatorname{laplace}(\operatorname{Heaviside}(t-5-k), t, s)\, k\, s^2 + \operatorname{laplace}(\operatorname{Heaviside}(t-5-k)\, t, t, s)\, s^2 - 5\operatorname{laplace}}{k\, s^2 (4s^2 + s + 4)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} u &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(-\frac{4(-\operatorname{laplace}(\operatorname{Heaviside}(t-5-k), t, s)\, k\, s^2 + \operatorname{laplace}(\operatorname{Heaviside}(t-5-k)\, t, t, s)\, s^2 - 5\operatorname{lapl}}{k\, s^2 (4s^2 + s + 4)}\right. \\ &= \frac{-\left(21\cos\left(\frac{3\sqrt{7}\, t}{8}\right)(4k+21) + \sin\left(\frac{3\sqrt{7}\, t}{8}\right)\sqrt{7}\,(4k-11)\right) e^{-\frac{t}{8}} \operatorname{Heaviside}(-5-k) + (-\operatorname{Heaviside}(t-5}{} \end{aligned}$$

Simplifying the solution gives

$$u = \frac{-21\left(\frac{31\sqrt{7}\sin\left(\frac{3\sqrt{7}(-t+5+k)}{8}\right)}{21} + \cos\left(\frac{3\sqrt{7}(-t+5+k)}{8}\right)\right)(\operatorname{Heaviside}(5+k) + \operatorname{Heaviside}(t-5-k) - 1)\, e^{-\frac{t}{8}+\frac{5}{8}+}}{}$$

Summary

The solution(s) found are the following

$$u = \frac{-21\left(\frac{31\sqrt{7}\sin\left(\frac{3\sqrt{7}(-t+5+k)}{8}\right)}{21} + \cos\left(\frac{3\sqrt{7}(-t+5+k)}{8}\right)\right)(\operatorname{Heaviside}(5+k) + \operatorname{Heaviside}(t-5-k) - 1)\, e^{-\frac{t}{8}+\frac{5}{8}+}}{} \tag{1}$$

Verification of solutions

$$u = \frac{-21\left(\frac{31\sqrt{7}\sin\left(\frac{3\sqrt{7}(-t+5+k)}{8}\right)}{21} + \cos\left(\frac{3\sqrt{7}(-t+5+k)}{8}\right)\right)(\operatorname{Heaviside}(5+k) + \operatorname{Heaviside}(t-5-k) - 1)\, e^{-\frac{t}{8}+\frac{5}{8}+}}{}$$

Verified OK.

4.11.2 Maple step by step solution

Let's solve

$$\left[u'' + \frac{u'}{4} + u = \frac{(-t+5+k)\mathit{Heaviside}(t-5-k) + \mathit{Heaviside}(t-5)(t-5)}{k}, u(0) = 0, u'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

u''

- Isolate 2nd derivative

  $u'' = -u + \frac{4 Heaviside(t-5)t + 4 Heaviside(t-5-k)k - 4 Heaviside(t-5-k)t - u'k - 20 Heaviside(t-5) + 20 Heaviside(t-5-k)}{4k}$

- Group terms with u on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $u'' + \frac{u'}{4} + u = \frac{Heaviside(t-5)t + Heaviside(t-5-k)k - Heaviside(t-5-k)t - 5 Heaviside(t-5) + 5 Heaviside(t-5-k)}{k}$

- Characteristic polynomial of homogeneous ODE

  $r^2 + \frac{1}{4}r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{\left(-\frac{1}{4}\right) \pm \left(\sqrt{-\frac{63}{16}}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{8} - \frac{3\,\mathrm{I}\sqrt{7}}{8}, -\frac{1}{8} + \frac{3\,\mathrm{I}\sqrt{7}}{8}\right)$

- 1st solution of the homogeneous ODE

  $u_1(t) = \mathrm{e}^{-\frac{t}{8}} \cos\left(\frac{3\sqrt{7}\,t}{8}\right)$

- 2nd solution of the homogeneous ODE

  $u_2(t) = \mathrm{e}^{-\frac{t}{8}} \sin\left(\frac{3\sqrt{7}\,t}{8}\right)$

- General solution of the ODE

  $u = c_1 u_1(t) + c_2 u_2(t) + u_p(t)$

- Substitute in solutions of the homogeneous ODE

  $u = c_1 \mathrm{e}^{-\frac{t}{8}} \cos\left(\frac{3\sqrt{7}\,t}{8}\right) + c_2 \mathrm{e}^{-\frac{t}{8}} \sin\left(\frac{3\sqrt{7}\,t}{8}\right) + u_p(t)$

☐ Find a particular solution $u_p(t)$ of the ODE

  - Use variation of parameters to find u_p here $f(t)$ is the forcing function

    $\left[u_p(t) = -u_1(t)\left(\int \frac{u_2(t) f(t)}{W(u_1(t), u_2(t))} dt\right) + u_2(t)\left(\int \frac{u_1(t) f(t)}{W(u_1(t), u_2(t))} dt\right), f(t) = \frac{Heaviside(t-5)t + Heaviside(t-5-k}{}\right.$

  - Wronskian of solutions of the homogeneous equation

    $W(u_1(t), u_2(t)) = \begin{bmatrix} \mathrm{e}^{-\frac{t}{8}} \cos\left(\frac{3\sqrt{7}\,t}{8}\right) & \mathrm{e}^{-\frac{t}{8}} \sin\left(\frac{3\sqrt{7}\,t}{8}\right) \\ -\frac{\mathrm{e}^{-\frac{t}{8}} \cos\left(\frac{3\sqrt{7}\,t}{8}\right)}{8} - \frac{3\,\mathrm{e}^{-\frac{t}{8}} \sqrt{7} \sin\left(\frac{3\sqrt{7}\,t}{8}\right)}{8} & -\frac{\mathrm{e}^{-\frac{t}{8}} \sin\left(\frac{3\sqrt{7}\,t}{8}\right)}{8} + \frac{3\,\mathrm{e}^{-\frac{t}{8}} \sqrt{7} \cos\left(\frac{3\sqrt{7}\,t}{8}\right)}{8} \end{bmatrix}$

  - Compute Wronskian

$W(u_1(t), u_2(t)) = \frac{3\sqrt{7}\,e^{-\frac{t}{4}}}{8}$

- Substitute functions into equation for $u_p(t)$

$$u_p(t) = -\frac{8\sqrt{7}\,e^{-\frac{t}{8}}\left(\cos\left(\frac{3\sqrt{7}\,t}{8}\right)\left(\int ((-t+5+k)Heaviside(t-5-k)+Heaviside(t-5)(t-5))\sin\left(\frac{3\sqrt{7}\,t}{8}\right)e^{\frac{t}{8}}dt\right)-\sin\left(\frac{3\sqrt{7}\,t}{8}\right)\left(\int ((-t+\right.\right.}{21k}$$

- Compute integrals

$$u_p(t) = \frac{-21\left(\frac{31\sqrt{7}\,\sin\left(\frac{3\sqrt{7}\,(-t+5+k)}{8}\right)}{21}+\cos\left(\frac{3\sqrt{7}\,(-t+5+k)}{8}\right)\right)Heaviside(t-5-k)e^{-\frac{t}{8}+\frac{5}{8}+\frac{k}{8}}+21\cos\left(\frac{3\sqrt{7}\,(t-5)}{8}\right)e^{-\frac{t}{8}+\frac{5}{8}}Heavisid}{84k}$$

- Substitute particular solution into general solution to ODE

$$u = c_1 e^{-\frac{t}{8}}\cos\left(\frac{3\sqrt{7}\,t}{8}\right)+c_2 e^{-\frac{t}{8}}\sin\left(\frac{3\sqrt{7}\,t}{8}\right)+\frac{-21\left(\frac{31\sqrt{7}\,\sin\left(\frac{3\sqrt{7}\,(-t+5+k)}{8}\right)}{21}+\cos\left(\frac{3\sqrt{7}\,(-t+5+k)}{8}\right)\right)Heaviside(t-5-}{}$$

☐ Check validity of solution $u = c_1 e^{-\frac{t}{8}}\cos\left(\frac{3\sqrt{7}\,t}{8}\right)+c_2 e^{-\frac{t}{8}}\sin\left(\frac{3\sqrt{7}\,t}{8}\right)+\frac{-21\left(\frac{31\sqrt{7}\,\sin\left(\frac{3\sqrt{7}\,(-t+5+k)}{8}\right)}{21}+co\right.}{}$

- Use initial condition $u(0) = 0$

$$0 = c_1 + \frac{-21\left(\frac{31\sqrt{7}\,\sin\left(\frac{3\sqrt{7}\,(5+k)}{8}\right)}{21}+\cos\left(\frac{3\sqrt{7}\,(5+k)}{8}\right)\right)Heaviside(-5-k)e^{\frac{5}{8}+\frac{k}{8}}+(84k+441)Heaviside(-5-k)}{84k}$$

- Compute derivative of the solution

$$u' = -\frac{c_1 e^{-\frac{t}{8}}\cos\left(\frac{3\sqrt{7}\,t}{8}\right)}{8}-\frac{3c_1 e^{-\frac{t}{8}}\sqrt{7}\,\sin\left(\frac{3\sqrt{7}\,t}{8}\right)}{8}-\frac{c_2 e^{-\frac{t}{8}}\sin\left(\frac{3\sqrt{7}\,t}{8}\right)}{8}+\frac{3c_2 e^{-\frac{t}{8}}\sqrt{7}\,\cos\left(\frac{3\sqrt{7}\,t}{8}\right)}{8}+\frac{-21\left(-\frac{31\cos\left(\frac{3\sqrt{7}\,(-}{8}\right)}{8}\right.}{}$$

- Use the initial condition $u'\Big|_{\{t=0\}} = 0$

$$0 = -\frac{c_1}{8}+\frac{3\sqrt{7}\,c_2}{8}+\frac{-21\left(-\frac{31\cos\left(\frac{3\sqrt{7}\,(5+k)}{8}\right)}{8}+\frac{3\sqrt{7}\,\sin\left(\frac{3\sqrt{7}\,(5+k)}{8}\right)}{8}\right)Heaviside(-5-k)e^{\frac{5}{8}+\frac{k}{8}}-21\left(\frac{31\sqrt{7}\,\sin\left(\frac{3\sqrt{7}\,(5+k)}{8}\right)}{21}+co\right.}{}$$

- Solve for c_1 and c_2

$$\left\{c_1 = \frac{Heaviside(-5-k)\left(31\,e^{\frac{5}{8}+\frac{k}{8}}\sqrt{7}\,\sin\left(\frac{3\sqrt{7}\,(5+k)}{8}\right)+21\,e^{\frac{5}{8}+\frac{k}{8}}\cos\left(\frac{3\sqrt{7}\,(5+k)}{8}\right)-84k-441\right)}{84k}, c_2 = \frac{\left(63\,e^{\frac{5}{8}+\frac{k}{8}}\sqrt{7}\,\sin\left(\frac{3\sqrt{7}\,(5+\right.\right.}{}\right.$$

- Substitute constant values into general solution and simplify

$$u = \frac{-21\left(\frac{31\sqrt{7}\,\sin\left(\frac{3\sqrt{7}\,(-t+5+k)}{8}\right)}{21}+\cos\left(\frac{3\sqrt{7}\,(-t+5+k)}{8}\right)\right)(Heaviside(5+k)+Heaviside(t-5-k)-1)e^{-\frac{t}{8}+\frac{5}{8}+\frac{k}{8}}+21\cos\left(\frac{3\sqrt{7}\,(t-5)}{8}\right.}{}$$

- Solution to the IVP

$$u = \frac{-21\left(\frac{31\sqrt{7}\,\sin\left(\frac{3\sqrt{7}\,(-t+5+k)}{8}\right)}{21}+\cos\left(\frac{3\sqrt{7}\,(-t+5+k)}{8}\right)\right)(Heaviside(5+k)+Heaviside(t-5-k)-1)\mathrm{e}^{-\frac{t}{8}+\frac{5}{8}+\frac{k}{8}}+21\cos\left(\frac{3\sqrt{7}\,(t-5)}{8}\right.}{}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 1.938 (sec). Leaf size: 216

```
dsolve([diff(u(t),t$2)+1/4*diff(u(t),t)+u(t)=1/k*(Heaviside(t-5)*(t-5)-Heaviside(t-(5+k))*(t
```

$$u(t) = \frac{-21\left(\frac{31\sin\left(\frac{3\sqrt{7}\,(-t+5+k)}{8}\right)\sqrt{7}}{21}+\cos\left(\frac{3\sqrt{7}\,(-t+5+k)}{8}\right)\right)(\mathrm{Heaviside}\,(5+k)+\mathrm{Heaviside}\,(t-5-k)-1)\,\mathrm{e}^{-\frac{t}{8}+\frac{5}{8}+}}{}$$

✓ Solution by Mathematica

Time used: 13.449 (sec). Leaf size: 486

```
DSolve[{u''[t]+1/4*u'[t]+u[t]==1/k*(UnitStep[t-5]*(t-5)-UnitStep[t-(5+k)]*(t-(5+k)) ),{u[0]=
```

$u(t)$

$$\rightarrow \frac{e^{-t/8}\left(21e^{\frac{k+5}{8}}\cos\left(\frac{3}{8}\sqrt{7}(k-t+5)\right)-84k\cos\left(\frac{3\sqrt{7}t}{8}\right)-441\cos\left(\frac{3\sqrt{7}t}{8}\right)+31\sqrt{7}e^{\frac{k+5}{8}}\sin\left(\frac{3}{8}\sqrt{7}(k-t+5)\right)-4\sqrt{7}k\sin\left(\frac{3\sqrt{7}t}{8}\right)+11\sqrt{7}\sin\left(\frac{3\sqrt{7}t}{8}\right)+\right.}{}$$

$u(t)$

$$\rightarrow \frac{e^{-t/8}\left(\left(3\sqrt{7}e^{t/8}(4t-21)+3\sqrt{7}e^{5/8}\cos\left(\frac{3}{8}\sqrt{7}(t-5)\right)-31e^{5/8}\sin\left(\frac{3}{8}\sqrt{7}(t-5)\right)\right)\theta(t-5)-\left(3\sqrt{7}e^{t/8}(-4k+4t-21)+3\sqrt{7}e^{\frac{k+5}{8}}\cos\left(\frac{3}{8}\sqrt{7}(k-t+\right.\right.\right.}{12\sqrt{7}k}$$

5 Chapter 6.5, The Laplace Transform. Impulse functions. page 273

5.1 problem 1

Internal problem ID [856]
Internal file name [`OUTPUT/856_Sunday_June_05_2022_01_52_05_AM_60206903/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.5, The Laplace Transform. Impulse functions. page 273
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + 2y = \delta(t - \pi)}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

5.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 2 \\ F &= \delta(t - \pi) \end{aligned}$$

Hence the ode is

$$y'' + 2y' + 2y = \delta(t - \pi)$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \delta(t - \pi)$ is

$$\{t < \pi \vee \pi < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)
\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 2sY(s) - 2y(0) + 2Y(s) = \mathrm{e}^{-\pi s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}
y(0) &= 1 \\
y'(0) &= 0
\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 2 - s + 2sY(s) + 2Y(s) = \mathrm{e}^{-\pi s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{\mathrm{e}^{-\pi s} + s + 2}{s^2 + 2s + 2}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{\mathrm{e}^{-\pi s} + s + 2}{s^2 + 2s + 2}\right) \\
&= \mathrm{e}^{-t}(\cos(t) + \sin(t)) - \sin(t) \operatorname{Heaviside}(t - \pi)\, \mathrm{e}^{\pi - t}
\end{aligned}$$

Hence the final solution is

$$y = e^{-t}(\cos(t) + \sin(t)) - \sin(t)\,\text{Heaviside}(t - \pi)\,e^{\pi - t}$$

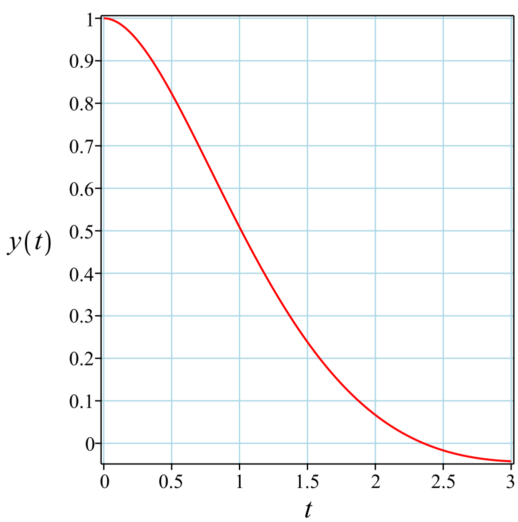
Simplifying the solution gives

$$y = e^{-t}(\cos(t) + \sin(t)) - \sin(t)\,\text{Heaviside}(t - \pi)\,e^{\pi - t}$$

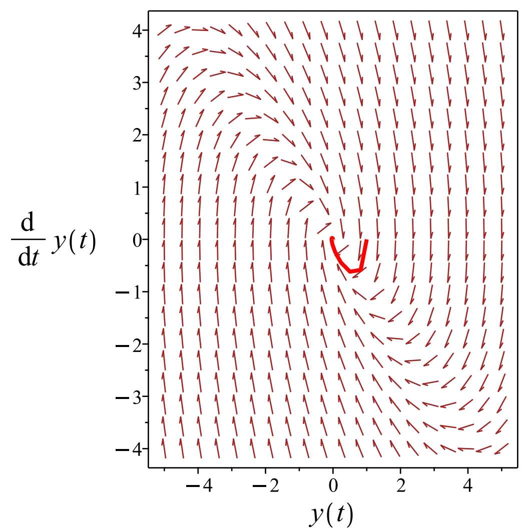
<u>Summary</u>

The solution(s) found are the following

$$y = e^{-t}(\cos(t) + \sin(t)) - \sin(t)\,\text{Heaviside}(t - \pi)\,e^{\pi - t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = e^{-t}(\cos(t) + \sin(t)) - \sin(t)\,\text{Heaviside}(t - \pi)\,e^{\pi - t}$$

Verified OK.

5.1.2 Maple step by step solution

Let's solve

$$\left[y'' + 2y' + 2y = Dirac(t - \pi), y(0) = 1, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-1 - \mathrm{I}, -1 + \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-t}\cos(t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-t}\sin(t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-t}\cos(t) + c_2 \mathrm{e}^{-t}\sin(t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = Dirac(t - \pi)\right]$

  - Wronskian of solutions of the homogeneous equation

    $$W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-t}\cos(t) & \mathrm{e}^{-t}\sin(t) \\ -\mathrm{e}^{-t}\cos(t) - \mathrm{e}^{-t}\sin(t) & -\mathrm{e}^{-t}\sin(t) + \mathrm{e}^{-t}\cos(t) \end{bmatrix}$$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = \mathrm{e}^{-2t}$

  - Substitute functions into equation for $y_p(t)$

$y_p(t) = -\left(\int Dirac(t-\pi)\, dt\right) \sin(t)\, e^{\pi - t}$

- Compute integrals

  $y_p(t) = -\sin(t)\, Heaviside(t-\pi)\, e^{\pi - t}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-t} \cos(t) + c_2 e^{-t} \sin(t) - \sin(t)\, Heaviside(t-\pi)\, e^{\pi - t}$

☐ Check validity of solution $y = c_1 e^{-t} \cos(t) + c_2 e^{-t} \sin(t) - \sin(t)\, Heaviside(t-\pi)\, e^{\pi - t}$

- Use initial condition $y(0) = 1$

  $1 = c_1$

- Compute derivative of the solution

  $y' = -c_1 e^{-t} \cos(t) - c_1 e^{-t} \sin(t) - c_2 e^{-t} \sin(t) + c_2 e^{-t} \cos(t) - \cos(t)\, Heaviside(t-\pi)\, e^{\pi - t} - 1$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = -c_1 + c_2$

- Solve for c_1 and c_2

  $\{c_1 = 1, c_2 = 1\}$

- Substitute constant values into general solution and simplify

  $y = e^{-t}(\cos(t) + \sin(t)) - \sin(t)\, Heaviside(t-\pi)\, e^{\pi - t}$

- Solution to the IVP

  $y = e^{-t}(\cos(t) + \sin(t)) - \sin(t)\, Heaviside(t-\pi)\, e^{\pi - t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.375 (sec). Leaf size: 31

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)+2*y(t)=Dirac(t-Pi),y(0) = 1, D(y)(0) = 0],y(t), singso
```

$$y(t) = e^{-t}(\cos(t) + \sin(t)) - \sin(t)\,\text{Heaviside}(t - \pi)\,e^{\pi - t}$$

✓ Solution by Mathematica

Time used: 0.069 (sec). Leaf size: 29

```
DSolve[{y''[t]+2*y'[t]+2*y[t]==DiracDelta[t-Pi],{y[0]==1,y'[0]==0}},y[t],t,IncludeSingularSo
```

$$y(t) \to e^{-t}(-e^{\pi}\theta(t - \pi)\sin(t) + \sin(t) + \cos(t))$$

5.2 problem 2

Internal problem ID [857]
Internal file name [`OUTPUT/857_Sunday_June_05_2022_01_52_08_AM_93087777/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.5, The Laplace Transform. Impulse functions. page 273
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = \delta(t - \pi) - \delta(t - 2\pi)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

5.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 4 \\ F &= \delta(t - \pi) - \delta(t - 2\pi) \end{aligned}$$

Hence the ode is

$$y'' + 4y = \delta(t - \pi) - \delta(t - 2\pi)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \delta(t-\pi) - \delta(t-2\pi)$ is

$$\{\pi \leq t \leq 2\pi, 2\pi \leq t \leq \infty, -\infty \leq t \leq \pi\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 4Y(s) = e^{-\pi s} - e^{-2\pi s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 4Y(s) = e^{-\pi s} - e^{-2\pi s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{e^{-\pi s} - e^{-2\pi s}}{s^2 + 4}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{\mathrm{e}^{-\pi s}-\mathrm{e}^{-2\pi s}}{s^2+4}\right) \\ &= \frac{\sin(2t)\left(-\operatorname{Heaviside}(t-2\pi)+\operatorname{Heaviside}(t-\pi)\right)}{2} \end{aligned}$$

Hence the final solution is

$$y = \frac{\sin(2t)\left(-\operatorname{Heaviside}(t-2\pi)+\operatorname{Heaviside}(t-\pi)\right)}{2}$$

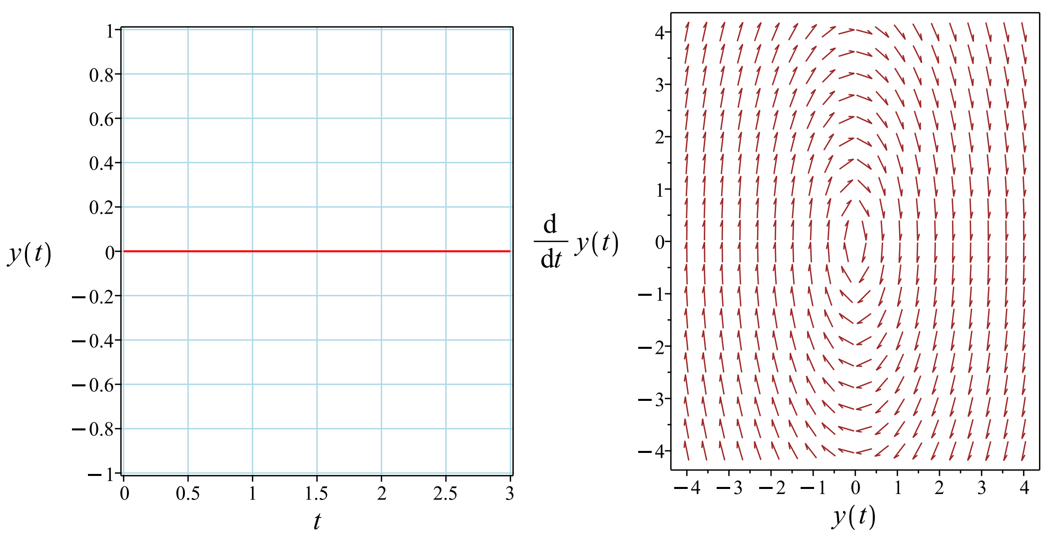
Simplifying the solution gives

$$y = -\frac{\sin(2t)\left(\operatorname{Heaviside}(t-2\pi)-\operatorname{Heaviside}(t-\pi)\right)}{2}$$

Summary

The solution(s) found are the following

$$y = -\frac{\sin(2t)\left(\operatorname{Heaviside}(t-2\pi)-\operatorname{Heaviside}(t-\pi)\right)}{2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{\sin(2t)\left(\operatorname{Heaviside}(t-2\pi)-\operatorname{Heaviside}(t-\pi)\right)}{2}$$

Verified OK.

5.2.2 Maple step by step solution

Let's solve

$\left[y'' + 4y = Dirac(t-\pi) - Dirac(t-2\pi), y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(2t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(2t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = Dirac(t-\pi) - Dirac(t-2\right.$

- ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix}$

- ◦ Compute Wronskian

  $W(y_1(t), y_2(t)) = 2$

- ◦ Substitute functions into equation for $y_p(t)$

$y_p(t) = \frac{\sin(2t)\left(\int (Dirac(t-\pi)-Dirac(t-2\pi))dt\right)}{2}$

- Compute integrals

  $y_p(t) = -\frac{\sin(2t)(Heaviside(t-2\pi)-Heaviside(t-\pi))}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) - \frac{\sin(2t)(Heaviside(t-2\pi)-Heaviside(t-\pi))}{2}$

□ Check validity of solution $y = c_1 \cos(2t) + c_2 \sin(2t) - \frac{\sin(2t)(Heaviside(t-2\pi)-Heaviside(t-\pi))}{2}$

- Use initial condition $y(0) = 0$

  $0 = c_1$

- Compute derivative of the solution

  $y' = -2c_1 \sin(2t) + 2c_2 \cos(2t) - \cos(2t)(Heaviside(t-2\pi) - Heaviside(t-\pi)) - \frac{\sin(2t)(-Dirac(}{}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = 2c_2$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $y = -\frac{\sin(2t)(Heaviside(t-2\pi)-Heaviside(t-\pi))}{2}$

- Solution to the IVP

  $y = -\frac{\sin(2t)(Heaviside(t-2\pi)-Heaviside(t-\pi))}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.359 (sec). Leaf size: 25

```
dsolve([diff(y(t),t$2)+4*y(t)=Dirac(t-Pi)-Dirac(t-2*Pi),y(0) = 0, D(y)(0) = 0],y(t), singsol
```

$$y(t) = -\frac{(\text{Heaviside}(t-2\pi) - \text{Heaviside}(t-\pi))\sin(2t)}{2}$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 26

```
DSolve[{y''[t]+4*y[t]==DiracDelta[t-Pi]-DiracDelta[t-2*Pi],{y[0]==0,y'[0]==0}},y[t],t,Includ
```

$$y(t) \to (\theta(t-2\pi) - \theta(t-\pi))\sin(t)(-\cos(t))$$

5.3 problem 3

Internal problem ID [858]
Internal file name [`OUTPUT/858_Sunday_June_05_2022_01_52_10_AM_85375152/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.5, The Laplace Transform. Impulse functions. page 273
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 3y' + 2y = \delta(t-5) + \text{Heaviside}\,(t-10)}$$

With initial conditions

$$\left[y(0) = 0, y'(0) = \frac{1}{2}\right]$$

5.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 3 \\ q(t) &= 2 \\ F &= \delta(t-5) + \text{Heaviside}\,(t-10) \end{aligned}$$

Hence the ode is

$$y'' + 3y' + 2y = \delta(t-5) + \text{Heaviside}\,(t-10)$$

The domain of $p(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \delta(t-5) + \text{Heaviside}\,(t-10)$ is

$$\{5 \leq t \leq 10, 10 \leq t \leq \infty, -\infty \leq t \leq 5\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 3sY(s) - 3y(0) + 2Y(s) = \mathrm{e}^{-5s} + \frac{\mathrm{e}^{-10s}}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0 \\ y'(0) &= \frac{1}{2}\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - \frac{1}{2} + 3sY(s) + 2Y(s) = \mathrm{e}^{-5s} + \frac{\mathrm{e}^{-10s}}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{2\,\mathrm{e}^{-5s}s + 2\,\mathrm{e}^{-10s} + s}{2s\,(s^2 + 3s + 2)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{2\,\mathrm{e}^{-5s}s + 2\,\mathrm{e}^{-10s} + s}{2s\,(s^2 + 3s + 2)}\right) \\
&= -\frac{\mathrm{e}^{-2t}}{2} + \frac{\mathrm{e}^{-t}}{2} + \frac{\mathrm{Heaviside}\,(t-10)\,(1 + \mathrm{e}^{-2t+20} - 2\,\mathrm{e}^{10-t})}{2} + \mathrm{Heaviside}\,(t-5)\,\left(-\mathrm{e}^{-2t+10} + \mathrm{e}^{-t+5}\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y = {} & -\frac{\mathrm{e}^{-2t}}{2} + \frac{\mathrm{e}^{-t}}{2} + \frac{\mathrm{Heaviside}\,(t-10)\,(1 + \mathrm{e}^{-2t+20} - 2\,\mathrm{e}^{10-t})}{2} \\
& + \mathrm{Heaviside}\,(t-5)\,\left(-\mathrm{e}^{-2t+10} + \mathrm{e}^{-t+5}\right)
\end{aligned}$$

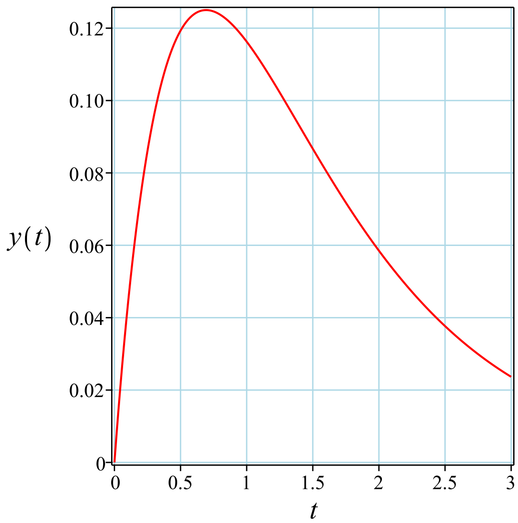
Simplifying the solution gives

$$\begin{aligned}
y = {} & -\frac{\mathrm{e}^{-2t}}{2} + \frac{\mathrm{e}^{-t}}{2} + \frac{\mathrm{Heaviside}\,(t-10)\,\mathrm{e}^{-2t+20}}{2} - \mathrm{Heaviside}\,(t-10)\,\mathrm{e}^{10-t} \\
& + \frac{\mathrm{Heaviside}\,(t-10)}{2} - \mathrm{Heaviside}\,(t-5)\,\mathrm{e}^{-2t+10} + \mathrm{Heaviside}\,(t-5)\,\mathrm{e}^{-t+5}
\end{aligned}$$

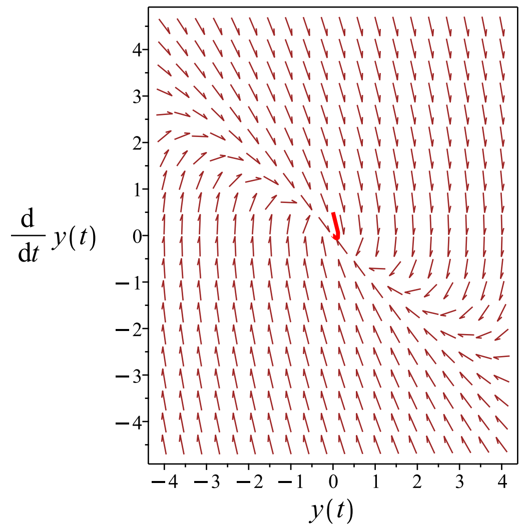
<u>Summary</u>

The solution(s) found are the following

$$\begin{aligned}
y = {} & -\frac{\mathrm{e}^{-2t}}{2} + \frac{\mathrm{e}^{-t}}{2} + \frac{\mathrm{Heaviside}\,(t-10)\,\mathrm{e}^{-2t+20}}{2} - \mathrm{Heaviside}\,(t-10)\,\mathrm{e}^{10-t} \\
& + \frac{\mathrm{Heaviside}\,(t-10)}{2} - \mathrm{Heaviside}\,(t-5)\,\mathrm{e}^{-2t+10} + \mathrm{Heaviside}\,(t-5)\,\mathrm{e}^{-t+5}
\end{aligned} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{e^{-2t}}{2} + \frac{e^{-t}}{2} + \frac{\text{Heaviside}\,(t-10)\,e^{-2t+20}}{2} - \text{Heaviside}\,(t-10)\,e^{10-t} + \frac{\text{Heaviside}\,(t-10)}{2} - \text{Heaviside}\,(t-5)\,e^{-2t+10} + \text{Heaviside}\,(t-5)\,e^{-t+5}$$

Verified OK.

5.3.2 Maple step by step solution

Let's solve

$$\left[y'' + 3y' + 2y = \mathit{Dirac}(t-5) + \mathit{Heaviside}(t-10)\,, y(0) = 0, y'\Big|_{\{t=0\}} = \frac{1}{2}\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, -1)$

- 1st solution of the homogeneous ODE

  $y_1(t) = e^{-2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{-t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2t} + c_2 e^{-t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = Dirac(t-5) + Heaviside(t\right.$

  ◦ Wronskian of solutions of the homogeneous equation

  $$W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-2t} & e^{-t} \\ -2\,e^{-2t} & -e^{-t} \end{bmatrix}$$

  ◦ Compute Wronskian

  $W(y_1(t), y_2(t)) = e^{-3t}$

  ◦ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -e^{-2t}\left(\int (Dirac(t-5)\,e^{10} + Heaviside(t-10)\,e^{2t})\,dt\right) + e^{-t}\left(\int (Dirac(t-5) + Heaviside\right.$

  ◦ Compute integrals

  $y_p(t) = -Heaviside(t-5)\,e^{-2t+10} + \frac{Heaviside(t-10)}{2} + \frac{Heaviside(t-10)e^{-2t+20}}{2} - Heaviside(t-10)\,e^{10-t}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-2t} + c_2 e^{-t} - Heaviside(t-5)\,e^{-2t+10} + \frac{Heaviside(t-10)}{2} + \frac{Heaviside(t-10)e^{-2t+20}}{2} - Heaviside(t$

☐ Check validity of solution $y = c_1 e^{-2t} + c_2 e^{-t} - Heaviside(t-5)\,e^{-2t+10} + \frac{Heaviside(t-10)}{2} + \frac{Heavisic}{}$

  ◦ Use initial condition $y(0) = 0$

  $0 = c_1 + c_2$

  ◦ Compute derivative of the solution

  $y' = -2c_1 e^{-2t} - c_2 e^{-t} - Dirac(t-5)\,e^{-2t+10} + 2Heaviside(t-5)\,e^{-2t+10} + \frac{Dirac(t-10)}{2} + \frac{Dirac(t-10}{2}$

  ◦ Use the initial condition $y'\Big|_{\{t=0\}} = \frac{1}{2}$

$\frac{1}{2} = -2c_1 - c_2$

- Solve for c_1 and c_2

  $\{c_1 = -\frac{1}{2}, c_2 = \frac{1}{2}\}$

- Substitute constant values into general solution and simplify

  $y = -\frac{e^{-2t}}{2} + \frac{e^{-t}}{2} + \frac{Heaviside(t-10)e^{-2t+20}}{2} - Heaviside(t-10)\,e^{10-t} + \frac{Heaviside(t-10)}{2} - Heaviside(t-5$

- Solution to the IVP

  $y = -\frac{e^{-2t}}{2} + \frac{e^{-t}}{2} + \frac{Heaviside(t-10)e^{-2t+20}}{2} - Heaviside(t-10)\,e^{10-t} + \frac{Heaviside(t-10)}{2} - Heaviside(t-5$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.406 (sec). Leaf size: 59

```
dsolve([diff(y(t),t$2)+3*diff(y(t),t)+2*y(t)=Dirac(t-5)+Heaviside(t-10),y(0) = 0, D(y)(0) =
```

$$y(t) = \frac{e^{-t}}{2} - \frac{e^{-2t}}{2} - \text{Heaviside}(t-10)\,e^{10-t} + \frac{\text{Heaviside}(t-10)\,e^{20-2t}}{2} + \frac{\text{Heaviside}(t-10)}{2} + \text{Heaviside}(t-5)\,e^{-t+5} - \text{Heaviside}(t-5)\,e^{10-2t}$$

✓ Solution by Mathematica

Time used: 0.226 (sec). Leaf size: 71

```
DSolve[{y''[t]+3*y'[t]+2*y[t]==DiracDelta[t-5]+UnitStep[t-10],{y[0]==0,y'[0]==1/2}},y[t],t,I
```

$$y(t) \to \frac{1}{2}e^{-2t}\left(2e^5\left(e^t-e^5\right)\theta(t-5) + \left(e^{10}-e^t\right)^2(-\theta(10-t)) + e^t + e^{2t} - 2e^{t+10} + e^{20} - 1\right)$$

5.4 problem 4

Internal problem ID [859]
Internal file name [`OUTPUT/859_Sunday_June_05_2022_01_52_17_AM_59275757/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.5, The Laplace Transform. Impulse functions. page 273
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + 3y = \sin(t) + \delta(t - 3\pi)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

5.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 3 \\ F &= \sin(t) + \delta(t - 3\pi) \end{aligned}$$

Hence the ode is

$$y'' + 2y' + 3y = \sin(t) + \delta(t - 3\pi)$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \sin(t) + \delta(t - 3\pi)$ is

$$\{t < 3\pi \lor 3\pi < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 2sY(s) - 2y(0) + 3Y(s) = \frac{1}{s^2+1} + e^{-3\pi s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 2sY(s) + 3Y(s) = \frac{1}{s^2+1} + e^{-3\pi s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{e^{-3\pi s}s^2 + e^{-3\pi s} + 1}{(s^2+1)(s^2+2s+3)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{e^{-3\pi s} s^2 + e^{-3\pi s} + 1}{(s^2+1)(s^2+2s+3)}\right) \\
&= \frac{\text{Heaviside}(t-3\pi)\sqrt{2}\, e^{3\pi - t} \sin\left(\sqrt{2}\,(t-3\pi)\right)}{2} + \frac{e^{-t}\cos\left(\sqrt{2}\, t\right)}{4} - \frac{\cos(t)}{4} + \frac{\sin(t)}{4}
\end{aligned}
$$

Hence the final solution is

$$
y = \frac{\text{Heaviside}(t-3\pi)\sqrt{2}\, e^{3\pi - t} \sin\left(\sqrt{2}\,(t-3\pi)\right)}{2} + \frac{e^{-t}\cos\left(\sqrt{2}\, t\right)}{4} - \frac{\cos(t)}{4} + \frac{\sin(t)}{4}
$$

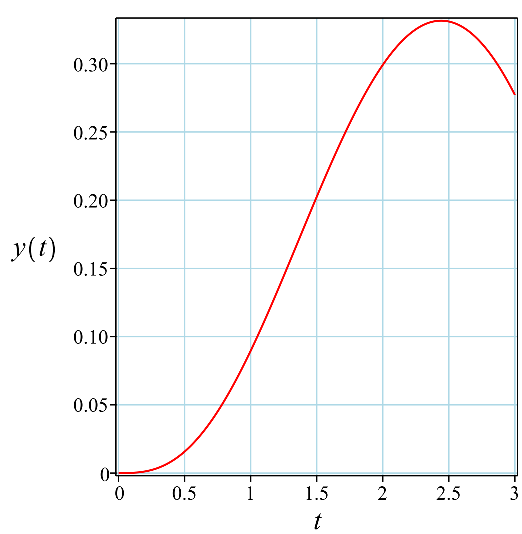
Simplifying the solution gives

$$
y = \frac{\text{Heaviside}(t-3\pi)\sqrt{2}\, e^{3\pi - t} \sin\left(\sqrt{2}\,(t-3\pi)\right)}{2} + \frac{e^{-t}\cos\left(\sqrt{2}\, t\right)}{4} - \frac{\cos(t)}{4} + \frac{\sin(t)}{4}
$$

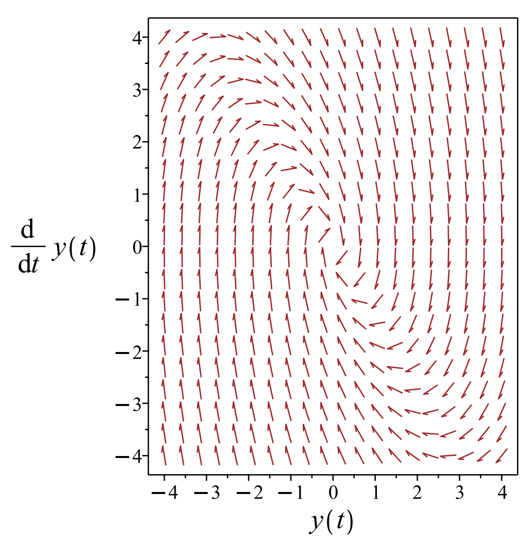
Summary

The solution(s) found are the following

$$
y = \frac{\text{Heaviside}(t-3\pi)\sqrt{2}\, e^{3\pi - t} \sin\left(\sqrt{2}\,(t-3\pi)\right)}{2} + \frac{e^{-t}\cos\left(\sqrt{2}\, t\right)}{4} - \frac{\cos(t)}{4} + \frac{\sin(t)}{4} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\text{Heaviside}\left(t-3\pi\right)\sqrt{2}\,\mathrm{e}^{3\pi-t}\sin\left(\sqrt{2}\,(t-3\pi)\right)}{2}+\frac{\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{4}-\frac{\cos\left(t\right)}{4}+\frac{\sin\left(t\right)}{4}$$

Verified OK.

5.4.2 Maple step by step solution

Let's solve

$$\left[y''+2y'+3y=\sin\left(t\right)+Dirac(t-3\pi)\,, y(0)=0, y'\Big|_{\{t=0\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of homogeneous ODE

  $r^2+2r+3=0$
- Use quadratic formula to solve for r

  $r=\frac{(-2)\pm(\sqrt{-8})}{2}$
- Roots of the characteristic polynomial

  $r=\left(-1-\mathrm{I}\sqrt{2}, -1+\mathrm{I}\sqrt{2}\right)$
- 1st solution of the homogeneous ODE

  $y_1(t)=\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)$
- 2nd solution of the homogeneous ODE

  $y_2(t)=\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right)$
- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$
- Substitute in solutions of the homogeneous ODE

  $y=c_1\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)+c_2\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right)+y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t)=-y_1(t)\left(\int\frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int\frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t)=\sin\left(t\right)+Dirac(t-3\pi)\right]$
- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \left[\begin{array}{cc} \mathrm{e}^{-t} \cos\left(\sqrt{2}\, t\right) & \mathrm{e}^{-t} \sin\left(\sqrt{2}\, t\right) \\ -\mathrm{e}^{-t} \cos\left(\sqrt{2}\, t\right) - \sqrt{2}\, \mathrm{e}^{-t} \sin\left(\sqrt{2}\, t\right) & -\mathrm{e}^{-t} \sin\left(\sqrt{2}\, t\right) + \sqrt{2}\, \mathrm{e}^{-t} \cos\left(\sqrt{2}\, t\right) \end{array}\right]$$

- Compute Wronskian

$$W(y_1(t), y_2(t)) = \sqrt{2}\, \mathrm{e}^{-2t}$$

- Substitute functions into equation for $y_p(t)$

$$y_p(t) = -\frac{\sqrt{2}\, \mathrm{e}^{-t} \left(\cos\left(\sqrt{2}\, t\right) \left(\int \mathrm{e}^{t} \sin\left(\sqrt{2}\, t\right) (\sin(t) + \mathit{Dirac}(t - 3\pi)) dt\right) - \sin\left(\sqrt{2}\, t\right) \left(\int \mathrm{e}^{t} \cos\left(\sqrt{2}\, t\right) (\sin(t) + \mathit{Dirac}(t - 3\pi)) dt\right)\right)}{2}$$

- Compute integrals

$$y_p(t) = \frac{\mathit{Heaviside}(t - 3\pi) \sqrt{2}\, \mathrm{e}^{3\pi - t} \sin\left(\sqrt{2}\, (t - 3\pi)\right)}{2} + \frac{\sin(t)}{4} - \frac{\cos(t)}{4}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \mathrm{e}^{-t} \cos\left(\sqrt{2}\, t\right) + c_2 \mathrm{e}^{-t} \sin\left(\sqrt{2}\, t\right) + \frac{\mathit{Heaviside}(t - 3\pi) \sqrt{2}\, \mathrm{e}^{3\pi - t} \sin\left(\sqrt{2}\, (t - 3\pi)\right)}{2} + \frac{\sin(t)}{4} - \frac{\cos(t)}{4}$$

☐ Check validity of solution $y = c_1 \mathrm{e}^{-t} \cos\left(\sqrt{2}\, t\right) + c_2 \mathrm{e}^{-t} \sin\left(\sqrt{2}\, t\right) + \frac{\mathit{Heaviside}(t - 3\pi) \sqrt{2}\, \mathrm{e}^{3\pi - t} \sin\left(\sqrt{2}\, (t - 3\right.}{2}$

- Use initial condition $y(0) = 0$

$$0 = c_1 - \frac{1}{4}$$

- Compute derivative of the solution

$$y' = -c_1 \mathrm{e}^{-t} \cos\left(\sqrt{2}\, t\right) - c_1 \sqrt{2}\, \mathrm{e}^{-t} \sin\left(\sqrt{2}\, t\right) - c_2 \mathrm{e}^{-t} \sin\left(\sqrt{2}\, t\right) + c_2 \sqrt{2}\, \mathrm{e}^{-t} \cos\left(\sqrt{2}\, t\right) + \frac{\mathit{Dirac}(t - 3\pi}{}$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

$$0 = -c_1 + \frac{1}{4} + \sqrt{2}\, c_2$$

- Solve for c_1 and c_2

$$\left\{c_1 = \frac{1}{4}, c_2 = 0\right\}$$

- Substitute constant values into general solution and simplify

$$y = \frac{\mathit{Heaviside}(t - 3\pi) \sqrt{2}\, \mathrm{e}^{3\pi - t} \sin\left(\sqrt{2}\, (t - 3\pi)\right)}{2} + \frac{\mathrm{e}^{-t} \cos\left(\sqrt{2}\, t\right)}{4} - \frac{\cos(t)}{4} + \frac{\sin(t)}{4}$$

- Solution to the IVP

$$y = \frac{\mathit{Heaviside}(t - 3\pi) \sqrt{2}\, \mathrm{e}^{3\pi - t} \sin\left(\sqrt{2}\, (t - 3\pi)\right)}{2} + \frac{\mathrm{e}^{-t} \cos\left(\sqrt{2}\, t\right)}{4} - \frac{\cos(t)}{4} + \frac{\sin(t)}{4}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.938 (sec). Leaf size: 54

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)+3*y(t)=sin(t)+Dirac(t-3*Pi),y(0) = 0, D(y)(0) = 0],y(t
```

$$y(t)=\frac{\sqrt{2}\,\mathrm{e}^{3\pi-t}\,\mathrm{Heaviside}\left(t-3\pi\right)\sin\left(\sqrt{2}\left(t-3\pi\right)\right)}{2}-\frac{\cos\left(t\right)}{4}+\frac{\sin\left(t\right)}{4}+\frac{\mathrm{e}^{-t}\cos\left(t\sqrt{2}\right)}{4}$$

✓ Solution by Mathematica

Time used: 1.726 (sec). Leaf size: 82

```
DSolve[{y''[t]+2*y'[t]+3*y[t]==Sin[t]+DiracDelta[t-3*Pi],{y[0]==0,y'[0]==1/2}},y[t],t,Includ
```

$$y(t)\to\frac{1}{4}e^{-t}\left(-2\sqrt{2}e^{3\pi}\theta(t-3\pi)\sin\left(\sqrt{2}(3\pi-t)\right)+e^{t}\sin(t)+\sqrt{2}\sin\left(\sqrt{2}t\right)-e^{t}\cos(t)+\cos\left(\sqrt{2}t\right)\right)$$

5.5 problem 5

Internal problem ID [860]
Internal file name [`OUTPUT/860_Sunday_June_05_2022_01_52_27_AM_34799035/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.5, The Laplace Transform. Impulse functions. page 273
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \delta(t - 2\pi)\cos(t)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

5.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 1 \\ F &= \delta(t - 2\pi) \end{aligned}$$

Hence the ode is

$$y'' + y = \delta(t - 2\pi)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \delta(t - 2\pi)$ is

$$\{t < 2\pi \lor 2\pi < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2 Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2 Y(s) - y'(0) - sy(0) + Y(s) = \mathrm{e}^{-2\pi s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2 Y(s) - 1 + Y(s) = \mathrm{e}^{-2\pi s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{\mathrm{e}^{-2\pi s} + 1}{s^2 + 1}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{e^{-2\pi s}+1}{s^2+1}\right) \\ &= \sin(t)\,(\text{Heaviside}(t-2\pi)+1) \end{aligned}$$

Hence the final solution is

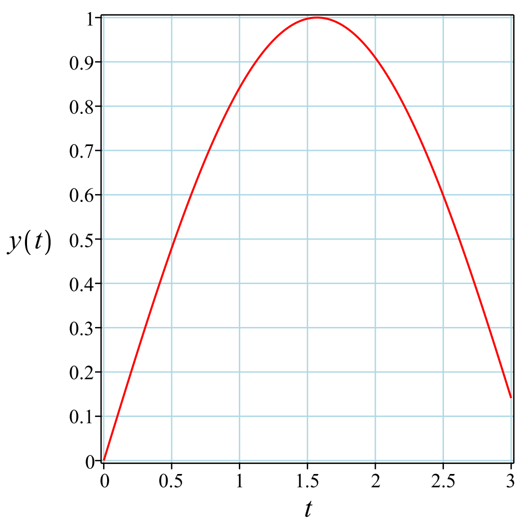
$$y = \sin(t)\,(\text{Heaviside}(t-2\pi)+1)$$

Simplifying the solution gives

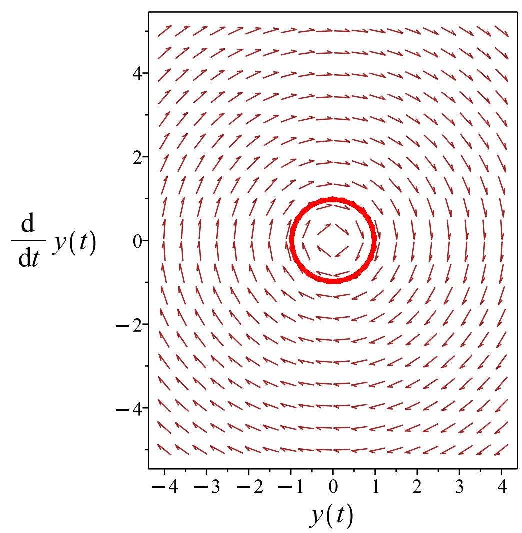
$$y = \sin(t)\,(\text{Heaviside}(t-2\pi)+1)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \sin(t)\,(\text{Heaviside}(t-2\pi)+1) \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \sin(t)\,(\text{Heaviside}(t-2\pi)+1)$$

Verified OK.

5.5.2 Maple step by step solution

Let's solve

$$\left[y'' + y = Dirac(t - 2\pi)\,, y(0) = 0, y'\Big|_{\{t=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(t) + c_2 \sin(t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $$\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = Dirac(t - 2\pi)\right]$$

- ○ Wronskian of solutions of the homogeneous equation

  $$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 1$

- ○ Substitute functions into equation for $y_p(t)$

$y_p(t) = \sin(t) \left(\int Dirac(t - 2\pi)\, dt\right)$

- Compute integrals

  $y_p(t) = Heaviside(t - 2\pi) \sin(t)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(t) + c_2 \sin(t) + Heaviside(t - 2\pi) \sin(t)$

□ Check validity of solution $y = c_1 \cos(t) + c_2 \sin(t) + Heaviside(t - 2\pi) \sin(t)$

- Use initial condition $y(0) = 0$

  $0 = c_1$

- Compute derivative of the solution

  $y' = -c_1 \sin(t) + c_2 \cos(t) + \sin(t)\, Dirac(t - 2\pi) + \cos(t)\, Heaviside(t - 2\pi)$

- Use the initial condition $y'\Big|_{\{t=0\}} = 1$

  $1 = c_2$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 1\}$

- Substitute constant values into general solution and simplify

  $y = \sin(t) \left(Heaviside(t - 2\pi) + 1\right)$

- Solution to the IVP

  $y = \sin(t) \left(Heaviside(t - 2\pi) + 1\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.375 (sec). Leaf size: 15

```
dsolve([diff(y(t),t$2)+y(t)=Dirac(t-2*Pi)*cos(t),y(0) = 0, D(y)(0) = 1],y(t), singsol=all)
```

$$y(t) = \sin(t)\left(\text{Heaviside}(t - 2\pi) + 1\right)$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 16

```
DSolve[{y''[t]+y[t]==DiracDelta[t-2*Pi]*Cos[t],{y[0]==0,y'[0]==1}},y[t],t,IncludeSingularSol
```

$$y(t) \to (\theta(t - 2\pi) + 1)\sin(t)$$

5.6 problem 6

Internal problem ID [861]
Internal file name [`OUTPUT/861_Sunday_June_05_2022_01_52_29_AM_16499628/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.5, The Laplace Transform. Impulse functions. page 273
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = 2\delta\left(t - \frac{\pi}{4}\right)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

5.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 4 \\ F &= 2\delta\left(t - \frac{\pi}{4}\right) \end{aligned}$$

Hence the ode is

$$y'' + 4y = 2\delta\left(t - \frac{\pi}{4}\right)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 2\delta\left(t - \frac{\pi}{4}\right)$ is

$$\left\{t < \frac{\pi}{4} \vee \frac{\pi}{4} < t\right\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 4Y(s) = 2\,\mathrm{e}^{-\frac{\pi s}{4}} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 4Y(s) = 2\,\mathrm{e}^{-\frac{\pi s}{4}}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{2\,\mathrm{e}^{-\frac{\pi s}{4}}}{s^2 + 4}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{2\,\mathrm{e}^{-\frac{\pi s}{4}}}{s^2+4}\right) \\
&= -\operatorname{Heaviside}\left(t-\frac{\pi}{4}\right)\cos(2t)
\end{aligned}
$$

Hence the final solution is

$$
y = -\operatorname{Heaviside}\left(t-\frac{\pi}{4}\right)\cos(2t)
$$

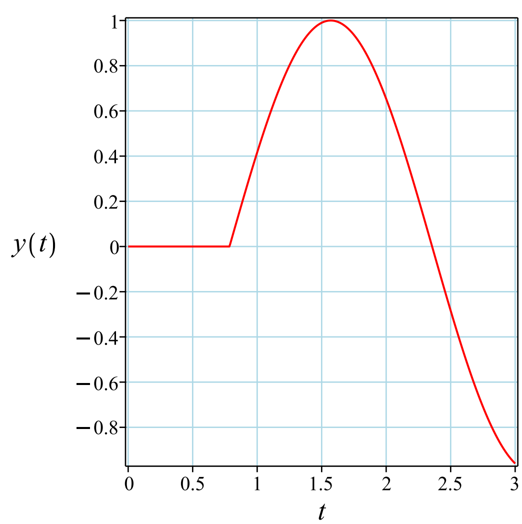
Simplifying the solution gives

$$
y = -\operatorname{Heaviside}\left(t-\frac{\pi}{4}\right)\cos(2t)
$$

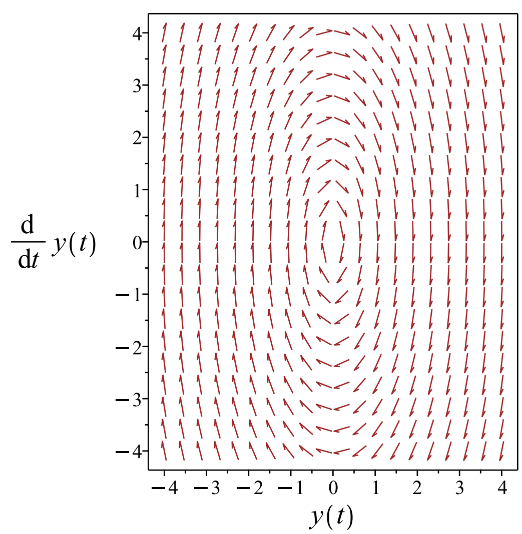
Summary

The solution(s) found are the following

$$
y = -\operatorname{Heaviside}\left(t-\frac{\pi}{4}\right)\cos(2t) \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$
y = -\operatorname{Heaviside}\left(t-\frac{\pi}{4}\right)\cos(2t)
$$

Verified OK.

5.6.2 Maple step by step solution

Let's solve

$\left[y''+4y=2Dirac\left(t-\frac{\pi}{4}\right), y(0)=0, y'\Big|_{\{t=0\}}=0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2+4=0$

- Use quadratic formula to solve for r

  $r=\frac{0\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r=(-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t)=\cos(2t)$

- 2nd solution of the homogeneous ODE

  $y_2(t)=\sin(2t)$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\cos(2t)+c_2\sin(2t)+y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t)=-y_1(t)\left(\int\frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int\frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t)=2Dirac\left(t-\frac{\pi}{4}\right)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t))=\begin{bmatrix}\cos(2t) & \sin(2t)\\ -2\sin(2t) & 2\cos(2t)\end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t))=2$

  - Substitute functions into equation for $y_p(t)$

$y_p(t) = -\cos(2t)\left(\int \mathit{Dirac}\left(t - \frac{\pi}{4}\right) dt\right)$

- Compute integrals

  $y_p(t) = -\mathit{Heaviside}\left(t - \frac{\pi}{4}\right)\cos(2t)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) - \mathit{Heaviside}\left(t - \frac{\pi}{4}\right)\cos(2t)$

- □ Check validity of solution $y = c_1 \cos(2t) + c_2 \sin(2t) - \mathit{Heaviside}\left(t - \frac{\pi}{4}\right)\cos(2t)$
  - Use initial condition $y(0) = 0$

    $0 = c_1$

  - Compute derivative of the solution

    $y' = -2c_1 \sin(2t) + 2c_2 \cos(2t) - \mathit{Dirac}\left(t - \frac{\pi}{4}\right)\cos(2t) + 2\mathit{Heaviside}\left(t - \frac{\pi}{4}\right)\sin(2t)$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 0$

    $0 = 2c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 0, c_2 = 0\}$

  - Substitute constant values into general solution and simplify

    $y = -\mathit{Heaviside}\left(t - \frac{\pi}{4}\right)\cos(2t)$

- Solution to the IVP

  $y = -\mathit{Heaviside}\left(t - \frac{\pi}{4}\right)\cos(2t)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.406 (sec). Leaf size: 16

```
dsolve([diff(y(t),t$2)+4*y(t)=2*Dirac(t-Pi/4),y(0) = 0, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = -\operatorname{Heaviside}\left(t - \frac{\pi}{4}\right)\cos(2t)$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 28

```
DSolve[{y''[t]+4*y[t]==2*DiracDelta[t-Pi/4],{y[0]==0,y'[0]==1}},y[t],t,IncludeSingularSoluti
```

$$y(t) \to \frac{1}{2}(\sin(2t) - 2\theta(4t - \pi)\cos(2t))$$

5.7 problem 7

Internal problem ID [862]
Internal file name [`OUTPUT/862_Sunday_June_05_2022_01_52_31_AM_84328360/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.5, The Laplace Transform. Impulse functions. page 273
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + 2y = \cos(t) + \delta\left(t - \frac{\pi}{2}\right)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

5.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 2 \\ F &= \cos(t) + \delta\left(t - \frac{\pi}{2}\right) \end{aligned}$$

Hence the ode is

$$y'' + 2y' + 2y = \cos(t) + \delta\left(t - \frac{\pi}{2}\right)$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \cos(t) + \delta\left(t - \frac{\pi}{2}\right)$ is

$$\left\{t < \frac{\pi}{2} \vee \frac{\pi}{2} < t\right\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 2sY(s) - 2y(0) + 2Y(s) = \frac{s}{s^2+1} + e^{-\frac{\pi s}{2}} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0 \\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 2sY(s) + 2Y(s) = \frac{s}{s^2+1} + e^{-\frac{\pi s}{2}}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{e^{-\frac{\pi s}{2}}s^2 + e^{-\frac{\pi s}{2}} + s}{(s^2+1)(s^2+2s+2)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{e^{-\frac{\pi s}{2}} s^2 + e^{-\frac{\pi s}{2}} + s}{(s^2+1)(s^2+2s+2)}\right) \\
&= \frac{\cos(t)}{5} + \frac{2\sin(t)}{5} - \frac{e^{-t}(\cos(t)+3\sin(t))}{5} + \frac{\left(\cos(t) + 2\sin(t) - 2\,e^{-t+\frac{\pi}{2}}(2\cos(t)+\sin(t)) - 2\,e^{-\frac{t}{2}}\right.}{}
\end{aligned}
$$

Hence the final solution is

$$
y = \frac{\cos(t)}{5} + \frac{2\sin(t)}{5} - \frac{e^{-t}(\cos(t)+3\sin(t))}{5} + \frac{\left(\cos(t) + 2\sin(t) - 2\,e^{-t+\frac{\pi}{2}}(2\cos(t)+\sin(t)) - 2\,e^{-\frac{t}{2}+\frac{\pi}{4}}\left(2\sin(t)\sinh\left(\frac{t}{2}-\frac{\pi}{4}\right) + \cos(t)\cosh\left(\frac{t}{2}-\frac{\pi}{4}\right.\right.\right.}{5}
$$

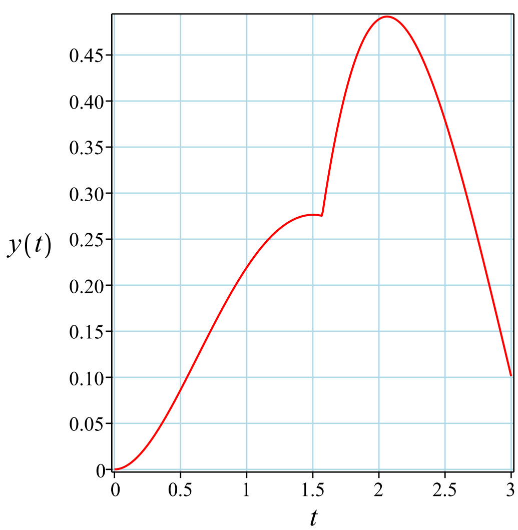
Simplifying the solution gives

$$
y = -\cos(t)\,e^{-t+\frac{\pi}{2}}\,\mathrm{Heaviside}\left(t-\frac{\pi}{2}\right) + \frac{(-\cos(t)-3\sin(t))\,e^{-t}}{5} + \frac{\cos(t)}{5} + \frac{2\sin(t)}{5}
$$

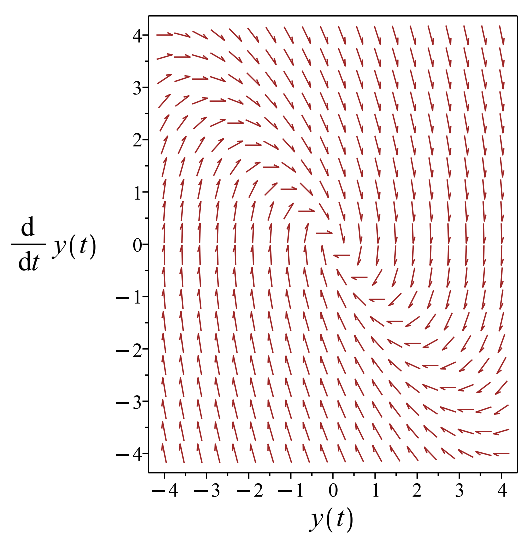
<u>Summary</u>

The solution(s) found are the following

$$
y = -\cos(t)\,e^{-t+\frac{\pi}{2}}\,\mathrm{Heaviside}\left(t-\frac{\pi}{2}\right) + \frac{(-\cos(t)-3\sin(t))\,e^{-t}}{5} + \frac{\cos(t)}{5} + \frac{2\sin(t)}{5} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\cos(t)\, e^{-t+\frac{\pi}{2}} \,\text{Heaviside}\left(t - \frac{\pi}{2}\right) + \frac{(-\cos(t) - 3\sin(t))\, e^{-t}}{5} + \frac{\cos(t)}{5} + \frac{2\sin(t)}{5}$$

Verified OK.

5.7.2 Maple step by step solution

Let's solve

$$\left[y'' + 2y' + 2y = \cos(t) + \mathit{Dirac}\left(t - \frac{\pi}{2}\right), y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-1 - \mathrm{I}, -1 + \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = e^{-t}\cos(t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{-t}\sin(t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t) + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \cos(t) + \mathit{Dirac}\left(t - \frac{\pi}{2}\right)\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-t}\cos(t) & e^{-t}\sin(t) \\ -e^{-t}\cos(t) - e^{-t}\sin(t) & -e^{-t}\sin(t) + e^{-t}\cos(t) \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = e^{-2t}$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -e^{-t}\left(\cos(t)\left(\int \sin(t)\left(\cos(t) + Dirac\left(t - \frac{\pi}{2}\right)\right)e^t dt\right) - \sin(t)\left(\int \cos(t)^2 e^t dt\right)\right)$

- Compute integrals

  $y_p(t) = -\cos(t)\,e^{-t+\frac{\pi}{2}}\,Heaviside\left(t - \frac{\pi}{2}\right) + \frac{2\sin(t)}{5} + \frac{\cos(t)}{5}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t) - \cos(t)\,e^{-t+\frac{\pi}{2}}\,Heaviside\left(t - \frac{\pi}{2}\right) + \frac{2\sin(t)}{5} + \frac{\cos(t)}{5}$

☐ Check validity of solution $y = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t) - \cos(t)\,e^{-t+\frac{\pi}{2}}\,Heaviside\left(t - \frac{\pi}{2}\right) + \frac{2\sin(t)}{5}$

- Use initial condition $y(0) = 0$

  $0 = c_1 + \frac{1}{5}$

- Compute derivative of the solution

  $y' = -c_1 e^{-t}\cos(t) - c_1 e^{-t}\sin(t) - c_2 e^{-t}\sin(t) + c_2 e^{-t}\cos(t) + \sin(t)\,e^{-t+\frac{\pi}{2}}\,Heaviside\left(t - \frac{\pi}{2}\right) +$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = -c_1 + \frac{2}{5} + c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = -\frac{1}{5}, c_2 = -\frac{3}{5}\right\}$

- Substitute constant values into general solution and simplify

  $y = -\cos(t)\,e^{-t+\frac{\pi}{2}}\,Heaviside\left(t - \frac{\pi}{2}\right) + \frac{(-\cos(t) - 3\sin(t))e^{-t}}{5} + \frac{\cos(t)}{5} + \frac{2\sin(t)}{5}$

- Solution to the IVP

  $y = -\cos(t)\,e^{-t+\frac{\pi}{2}}\,Heaviside\left(t - \frac{\pi}{2}\right) + \frac{(-\cos(t) - 3\sin(t))e^{-t}}{5} + \frac{\cos(t)}{5} + \frac{2\sin(t)}{5}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.562 (sec). Leaf size: 92

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)+2*y(t)=cos(t)+Dirac(t-Pi/2),y(0) = 0, D(y)(0) = 0],y(t
```

$$y(t) = -\cos(t)\,\mathrm{Heaviside}\left(t-\frac{\pi}{2}\right)\mathrm{e}^{-t+\frac{\pi}{2}} + \frac{(-\cos(t)-3\sin(t))\,\mathrm{e}^{-t}}{5} + \frac{\cos(t)}{5} + \frac{2\sin(t)}{5}$$

✓ Solution by Mathematica

Time used: 0.176 (sec). Leaf size: 52

```
DSolve[{y''[t]+2*y'[t]+2*y[t]==Cos[t]+DiracDelta[t-Pi/2],{y[0]==0,y'[0]==0}},y[t],t,IncludeS
```

$$y(t) \to \frac{1}{5}e^{-t}\left(-5e^{\pi/2}\theta(2t-\pi)\cos(t) + \left(2e^t-3\right)\sin(t) + \left(e^t-1\right)\cos(t)\right)$$

5.8 problem 8

Internal problem ID [863]
Internal file name [`OUTPUT/863_Sunday_June_05_2022_01_52_35_AM_5666023/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.5, The Laplace Transform. Impulse functions. page 273
Problem number: 8.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_laplace"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' - y = \delta(-1+t)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0, y''(0) = 0, y'''(0) = 0]$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \\
\mathcal{L}(y''') &= s^3Y(s) - y''(0) - sy'(0) - s^2y(0) \\
\mathcal{L}(y'''') &= s^4Y(s) - y'''(0) - sy''(0) - s^2y'(0) - s^3y(0)
\end{aligned}$$

The given ode becomes an algebraic equation in the Laplace domain

$$s^4Y(s) - y'''(0) - sy''(0) - s^2y'(0) - s^3y(0) - Y(s) = \mathrm{e}^{-s} \tag{1}$$

But the initial conditions are

$$
\begin{aligned}
y(0) &= 0 \\
y'(0) &= 0 \\
y''(0) &= 0 \\
y'''(0) &= 0
\end{aligned}
$$

Substituting these initial conditions in above in Eq (1) gives

$$s^4 Y(s) - Y(s) = \mathrm{e}^{-s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{\mathrm{e}^{-s}}{s^4 - 1}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{\mathrm{e}^{-s}}{s^4 - 1}\right) \\
&= \frac{\text{Heaviside}\,(-1+t)\,(-\sin\,(-1+t) + \sinh\,(-1+t))}{2}
\end{aligned}
$$

Hence the final solution is

$$y = \frac{\text{Heaviside}\,(-1+t)\,(-\sin\,(-1+t) + \sinh\,(-1+t))}{2}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{\text{Heaviside}\,(-1+t)\,(-\sin\,(-1+t) + \sinh\,(-1+t))}{2} \tag{1}$$

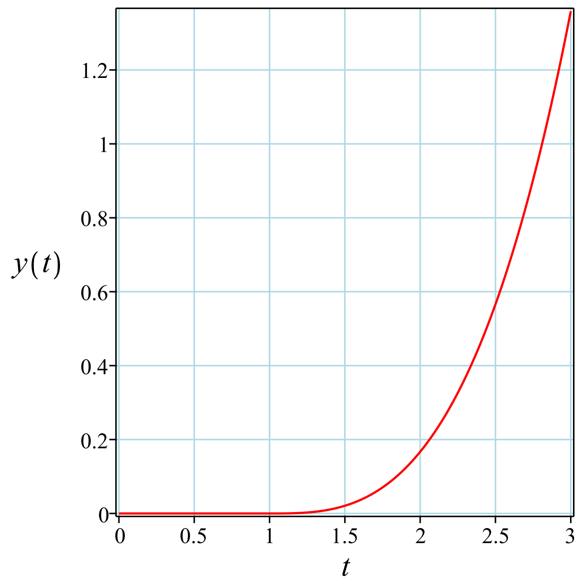


Figure 27: Solution plot

Verification of solutions

$$y = \frac{\text{Heaviside}(-1+t)(-\sin(-1+t)+\sinh(-1+t))}{2}$$

Verified OK.

5.8.1 Maple step by step solution

Let's solve

$$\left[y'''' - y = Dirac(-1+t), y(0) = 0, y'\Big|_{\{t=0\}} = 0, y''\Big|_{\{t=0\}} = 0, y'''\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 4

  y''''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(t)$

    $y_1(t) = y$
  - Define new variable $y_2(t)$

    $y_2(t) = y'$
  - Define new variable $y_3(t)$

$y_3(t) = y''$

- Define new variable $y_4(t)$

  $y_4(t) = y'''$

- Isolate for $y_4'(t)$ using original ODE

  $y_4'(t) = Dirac(-1+t) + y_1(t)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(t) = y_1'(t), y_3(t) = y_2'(t), y_4(t) = y_3'(t), y_4'(t) = Dirac(-1+t) + y_1(t)]$

- Define vector

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{bmatrix}$$

- System to solve

$$\vec{y}'(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ Dirac(-1+t) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ Dirac(-1+t) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(t) = A \cdot \vec{y}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} \mathrm{I} \\ -1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-t} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \mathrm{e}^{t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-I, \begin{bmatrix} -I \\ -1 \\ I \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-It} \cdot \begin{bmatrix} -I \\ -1 \\ I \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(t) - I\sin(t)) \cdot \begin{bmatrix} -I \\ -1 \\ I \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -I(\cos(t) - I\sin(t)) \\ -\cos(t) + I\sin(t) \\ I(\cos(t) - I\sin(t)) \\ \cos(t) - I\sin(t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(t) = \begin{bmatrix} -\sin(t) \\ -\cos(t) \\ \sin(t) \\ \cos(t) \end{bmatrix}, \vec{y}_4(t) = \begin{bmatrix} -\cos(t) \\ \sin(t) \\ \cos(t) \\ -\sin(t) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(t) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(t) + c_4\vec{y}_4(t) + \vec{y}_p(t)$

□ Fundamental matrix

  ◦ Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t)=\begin{bmatrix}-e^{-t} & e^{t} & -\sin(t) & -\cos(t)\\ e^{-t} & e^{t} & -\cos(t) & \sin(t)\\ -e^{-t} & e^{t} & \sin(t) & \cos(t)\\ e^{-t} & e^{t} & \cos(t) & -\sin(t)\end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0)=I$ where I is th
  $\Phi(t)=\phi(t)\cdot\frac{1}{\phi(0)}$
- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t)=\begin{bmatrix}-e^{-t} & e^{t} & -\sin(t) & -\cos(t)\\ e^{-t} & e^{t} & -\cos(t) & \sin(t)\\ -e^{-t} & e^{t} & \sin(t) & \cos(t)\\ e^{-t} & e^{t} & \cos(t) & -\sin(t)\end{bmatrix}\cdot\frac{1}{\begin{bmatrix}-1 & 1 & 0 & -1\\ 1 & 1 & -1 & 0\\ -1 & 1 & 0 & 1\\ 1 & 1 & 1 & 0\end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t)=\begin{bmatrix}\frac{e^{-t}}{4}+\frac{e^{t}}{4}+\frac{\cos(t)}{2} & -\frac{e^{-t}}{4}+\frac{e^{t}}{4}+\frac{\sin(t)}{2} & \frac{e^{-t}}{4}+\frac{e^{t}}{4}-\frac{\cos(t)}{2} & -\frac{e^{-t}}{4}+\frac{e^{t}}{4}-\frac{\sin(t)}{2}\\ -\frac{e^{-t}}{4}+\frac{e^{t}}{4}-\frac{\sin(t)}{2} & \frac{e^{-t}}{4}+\frac{e^{t}}{4}+\frac{\cos(t)}{2} & -\frac{e^{-t}}{4}+\frac{e^{t}}{4}+\frac{\sin(t)}{2} & \frac{e^{-t}}{4}+\frac{e^{t}}{4}-\frac{\cos(t)}{2}\\ \frac{e^{-t}}{4}+\frac{e^{t}}{4}-\frac{\cos(t)}{2} & -\frac{e^{-t}}{4}+\frac{e^{t}}{4}-\frac{\sin(t)}{2} & \frac{e^{-t}}{4}+\frac{e^{t}}{4}+\frac{\cos(t)}{2} & -\frac{e^{-t}}{4}+\frac{e^{t}}{4}+\frac{\sin(t)}{2}\\ -\frac{e^{-t}}{4}+\frac{e^{t}}{4}+\frac{\sin(t)}{2} & \frac{e^{-t}}{4}+\frac{e^{t}}{4}-\frac{\cos(t)}{2} & -\frac{e^{-t}}{4}+\frac{e^{t}}{4}-\frac{\sin(t)}{2} & \frac{e^{-t}}{4}+\frac{e^{t}}{4}+\frac{\cos(t)}{2}\end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(t)$ and solve for $\overrightarrow{v}(t)$
  $\overrightarrow{y}_p(t)=\Phi(t)\cdot\overrightarrow{v}(t)$
- Take the derivative of the particular solution
  $\overrightarrow{y}'_p(t)=\Phi'(t)\cdot\overrightarrow{v}(t)+\Phi(t)\cdot\overrightarrow{v}'(t)$
- Substitute particular solution and its derivative into the system of ODEs
  $\Phi'(t)\cdot\overrightarrow{v}(t)+\Phi(t)\cdot\overrightarrow{v}'(t)=A\cdot\Phi(t)\cdot\overrightarrow{v}(t)+\overrightarrow{f}(t)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der
  $A\cdot\Phi(t)\cdot\overrightarrow{v}(t)+\Phi(t)\cdot\overrightarrow{v}'(t)=A\cdot\Phi(t)\cdot\overrightarrow{v}(t)+\overrightarrow{f}(t)$
- Cancel like terms

$\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{y}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(t) = \begin{bmatrix} -\frac{Heaviside(-1+t)\left(-2\cos(t)\sin(1)+2\sin(t)\cos(1)+e^{1-t}-e^{-1+t}\right)}{4} \\ -\frac{Heaviside(-1+t)\left(2\sin(1)\sin(t)+2\cos(1)\cos(t)-e^{1-t}-e^{-1+t}\right)}{4} \\ \frac{Heaviside(-1+t)\left(2\sin(t)\cos(1)-2\cos(t)\sin(1)-e^{1-t}+e^{-1+t}\right)}{4} \\ \frac{Heaviside(-1+t)\left(2\sin(1)\sin(t)+2\cos(1)\cos(t)+e^{1-t}+e^{-1+t}\right)}{4} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(t) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(t) + c_4\vec{y}_4(t) + \begin{bmatrix} -\frac{Heaviside(-1+t)\left(-2\cos(t)\sin(1)+2\sin(t)\cos(1)+e^{1-t}-e^{-1+t}\right)}{4} \\ -\frac{Heaviside(-1+t)\left(2\sin(1)\sin(t)+2\cos(1)\cos(t)-e^{1-t}-e^{-1+t}\right)}{4} \\ \frac{Heaviside(-1+t)\left(2\sin(t)\cos(1)-2\cos(t)\sin(1)-e^{1-t}+e^{-1+t}\right)}{4} \\ \frac{Heaviside(-1+t)\left(2\sin(1)\sin(t)+2\cos(1)\cos(t)+e^{1-t}+e^{-1+t}\right)}{4} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y = -\frac{Heaviside(-1+t)e^{1-t}}{4} + \frac{\left(-2\sin(t)\cos(1)+2\cos(t)\sin(1)+e^{-1+t}\right)Heaviside(-1+t)}{4} - c_3\sin(t) - c_1e^{-t} + c_2e^{t} -$

- Use the initial condition $y(0) = 0$

  $0 = -c_1 + c_2 - c_4$

- Calculate the 1st derivative of the solution

  $y' = -\frac{Dirac(-1+t)e^{1-t}}{4} + \frac{Heaviside(-1+t)e^{1-t}}{4} + \frac{\left(-2\sin(1)\sin(t)-2\cos(1)\cos(t)+e^{-1+t}\right)Heaviside(-1+t)}{4} + \frac{(-2\sin(t)\,c}{}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = -c_3 + c_1 + c_2$

- Calculate the 2nd derivative of the solution

$$y'' = -\frac{Dirac(1,-1+t)\mathrm{e}^{1-t}}{4} + \frac{Dirac(-1+t)\mathrm{e}^{1-t}}{2} - \frac{Heaviside(-1+t)\mathrm{e}^{1-t}}{4} + \frac{(2\sin(t)\cos(1)-2\cos(t)\sin(1)+\mathrm{e}^{-1+t})\,Heaviside}{4}$$

- Use the initial condition $y''\Big|_{\{t=0\}} = 0$

$$0 = -c_1 + c_2 + c_4$$

- Calculate the 3rd derivative of the solution

$$y''' = -\frac{Dirac(2,-1+t)\mathrm{e}^{1-t}}{4} + \frac{3Dirac(1,-1+t)\mathrm{e}^{1-t}}{4} - \frac{3Dirac(-1+t)\mathrm{e}^{1-t}}{4} + \frac{Heaviside(-1+t)\mathrm{e}^{1-t}}{4} + \frac{(2\sin(1)\sin(t)+2\cos(}{}$$

- Use the initial condition $y'''\Big|_{\{t=0\}} = 0$

$$0 = c_1 + c_2 + c_3$$

- Solve for the unknown coefficients

$$\{c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 0\}$$

- Solution to the IVP

$$y = -\frac{Heaviside(-1+t)\mathrm{e}^{1-t}}{4} + \frac{(-2\sin(t)\cos(1)+2\cos(t)\sin(1)+\mathrm{e}^{-1+t})\,Heaviside(-1+t)}{4}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.406 (sec). Leaf size: 21

```
dsolve([diff(y(t),t$4)-y(t)=Dirac(t-1),y(0) = 0, D(y)(0) = 0, (D@@2)(y)(0) = 0, (D@@3)(y)(0)
```

$$y(t) = -\frac{\text{Heaviside}\,(t-1)\,(\sin\,(t-1) - \sinh\,(t-1))}{2}$$

✓ Solution by Mathematica

Time used: 0.11 (sec). Leaf size: 44

```
DSolve[{y''''[t]-y[t]==DiracDelta[t-1],{y[0]==0,y'[0]==0,y''[0]==0,y'''[0]==0}},y[t],t,Inclu
```

$$y(t) \to \frac{1}{4}e^{-t-1}\theta(t-1)\left(e^{2t}+2e^{t+1}\sin(1-t)-e^{2}\right)$$

5.9 problem 10(a)

Internal problem ID [864]
Internal file name [`OUTPUT/864_Sunday_June_05_2022_01_52_38_AM_14047982/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.5, The Laplace Transform. Impulse functions. page 273
Problem number: 10(a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + \frac{y'}{2} + y = \delta(-1+t)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

5.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= \frac{1}{2} \\ q(t) &= 1 \\ F &= \delta(-1+t) \end{aligned}$$

Hence the ode is

$$y'' + \frac{y'}{2} + y = \delta(-1+t)$$

The domain of $p(t) = \frac{1}{2}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \delta(-1+t)$ is

$$\{t < 1 \vee 1 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + \frac{sY(s)}{2} - \frac{y(0)}{2} + Y(s) = \mathrm{e}^{-s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0 \\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + \frac{sY(s)}{2} + Y(s) = \mathrm{e}^{-s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{2\,\mathrm{e}^{-s}}{2s^2 + s + 2}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{2\,\mathrm{e}^{-s}}{2s^2+s+2}\right) \\
&= \frac{4\,\text{Heaviside}\,(-1+t)\,\sqrt{15}\,\mathrm{e}^{\frac{1}{4}-\frac{t}{4}}\sin\left(\frac{\sqrt{15}\,(-1+t)}{4}\right)}{15}
\end{aligned}
$$

Hence the final solution is

$$
y = \frac{4\,\text{Heaviside}\,(-1+t)\,\sqrt{15}\,\mathrm{e}^{\frac{1}{4}-\frac{t}{4}}\sin\left(\frac{\sqrt{15}\,(-1+t)}{4}\right)}{15}
$$

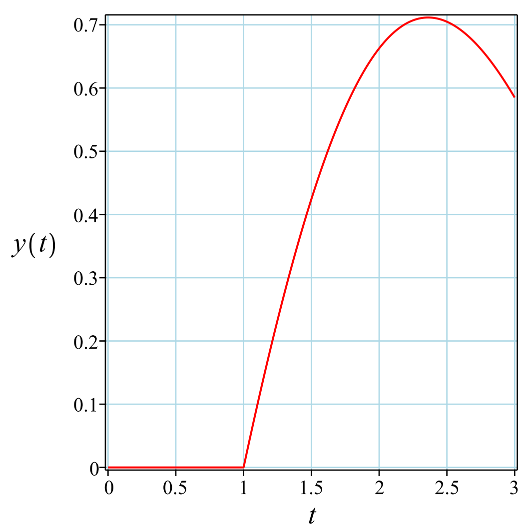
Simplifying the solution gives

$$
y = \frac{4\,\text{Heaviside}\,(-1+t)\,\sqrt{15}\,\mathrm{e}^{\frac{1}{4}-\frac{t}{4}}\sin\left(\frac{\sqrt{15}\,(-1+t)}{4}\right)}{15}
$$

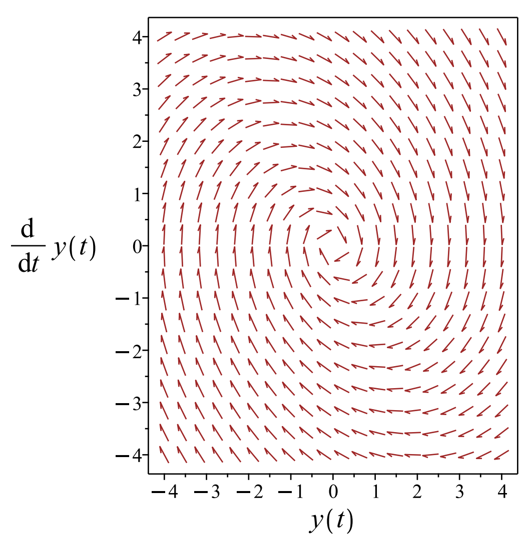
Summary

The solution(s) found are the following

$$
y = \frac{4\,\text{Heaviside}\,(-1+t)\,\sqrt{15}\,\mathrm{e}^{\frac{1}{4}-\frac{t}{4}}\sin\left(\frac{\sqrt{15}\,(-1+t)}{4}\right)}{15} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{4\,\mathrm{Heaviside}\left(-1+t\right)\sqrt{15}\,\mathrm{e}^{\frac{1}{4}-\frac{t}{4}}\sin\left(\frac{\sqrt{15}\,(-1+t)}{4}\right)}{15}$$

Verified OK.

5.9.2 Maple step by step solution

Let's solve

$\left[y''+\frac{y'}{2}+y=\mathit{Dirac}(-1+t)\,,y(0)=0,y'\Big|_{\{t=0\}}=0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2+\frac{1}{2}r+1=0$

- Use quadratic formula to solve for r

  $r=\frac{(-\frac{1}{2})\pm\left(\sqrt{-\frac{15}{4}}\right)}{2}$

- Roots of the characteristic polynomial

  $r=\left(-\frac{1}{4}-\frac{\mathrm{I}\sqrt{15}}{4},-\frac{1}{4}+\frac{\mathrm{I}\sqrt{15}}{4}\right)$

- 1st solution of the homogeneous ODE

  $y_1(t)=\mathrm{e}^{-\frac{t}{4}}\cos\left(\frac{\sqrt{15}\,t}{4}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(t)=\mathrm{e}^{-\frac{t}{4}}\sin\left(\frac{\sqrt{15}\,t}{4}\right)$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\mathrm{e}^{-\frac{t}{4}}\cos\left(\frac{\sqrt{15}\,t}{4}\right)+c_2\mathrm{e}^{-\frac{t}{4}}\sin\left(\frac{\sqrt{15}\,t}{4}\right)+y_p(t)$

- ☐ Find a particular solution $y_p(t)$ of the ODE

  - ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t)=-y_1(t)\left(\int\frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int\frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right),f(t)=\mathit{Dirac}(-1+t)\right]$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-\frac{t}{4}} \cos\left(\frac{\sqrt{15}\, t}{4}\right) & e^{-\frac{t}{4}} \sin\left(\frac{\sqrt{15}\, t}{4}\right) \\ -\frac{e^{-\frac{t}{4}} \cos\left(\frac{\sqrt{15}\, t}{4}\right)}{4} - \frac{e^{-\frac{t}{4}} \sqrt{15}\, \sin\left(\frac{\sqrt{15}\, t}{4}\right)}{4} & -\frac{e^{-\frac{t}{4}} \sin\left(\frac{\sqrt{15}\, t}{4}\right)}{4} + \frac{e^{-\frac{t}{4}} \sqrt{15}\, \cos\left(\frac{\sqrt{15}\, t}{4}\right)}{4} \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(t), y_2(t)) = \frac{\sqrt{15}\, e^{-\frac{t}{2}}}{4}$$

- Substitute functions into equation for $y_p(t)$

$$y_p(t) = -\frac{4\sqrt{15}\, e^{\frac{1}{4}-\frac{t}{4}} \left(\int Dirac(-1+t)dt\right) \left(\cos\left(\frac{\sqrt{15}\, t}{4}\right) \sin\left(\frac{\sqrt{15}}{4}\right) - \sin\left(\frac{\sqrt{15}\, t}{4}\right) \cos\left(\frac{\sqrt{15}}{4}\right)\right)}{15}$$

- Compute integrals

$$y_p(t) = -\frac{4\sqrt{15}\, e^{\frac{1}{4}-\frac{t}{4}}\, Heaviside(-1+t) \left(\cos\left(\frac{\sqrt{15}\, t}{4}\right) \sin\left(\frac{\sqrt{15}}{4}\right) - \sin\left(\frac{\sqrt{15}\, t}{4}\right) \cos\left(\frac{\sqrt{15}}{4}\right)\right)}{15}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{-\frac{t}{4}} \cos\left(\frac{\sqrt{15}\, t}{4}\right) + c_2 e^{-\frac{t}{4}} \sin\left(\frac{\sqrt{15}\, t}{4}\right) - \frac{4\sqrt{15}\, e^{\frac{1}{4}-\frac{t}{4}}\, Heaviside(-1+t) \left(\cos\left(\frac{\sqrt{15}\, t}{4}\right) \sin\left(\frac{\sqrt{15}}{4}\right) - \sin\left(\frac{\sqrt{15}\, t}{4}\right) \cos\left(\frac{\sqrt{15}}{4}\right)\right)}{15}$$

☐ Check validity of solution $y = c_1 e^{-\frac{t}{4}} \cos\left(\frac{\sqrt{15}\, t}{4}\right) + c_2 e^{-\frac{t}{4}} \sin\left(\frac{\sqrt{15}\, t}{4}\right) - \frac{4\sqrt{15}\, e^{\frac{1}{4}-\frac{t}{4}}\, Heaviside(-1+t) \left(\cos\left(\right.\right.}{}$

- Use initial condition $y(0) = 0$

$$0 = c_1$$

- Compute derivative of the solution

$$y' = -\frac{c_1 e^{-\frac{t}{4}} \cos\left(\frac{\sqrt{15}\, t}{4}\right)}{4} - \frac{c_1 e^{-\frac{t}{4}} \sqrt{15}\, \sin\left(\frac{\sqrt{15}\, t}{4}\right)}{4} - \frac{c_2 e^{-\frac{t}{4}} \sin\left(\frac{\sqrt{15}\, t}{4}\right)}{4} + \frac{c_2 e^{-\frac{t}{4}} \sqrt{15}\, \cos\left(\frac{\sqrt{15}\, t}{4}\right)}{4} + \frac{\sqrt{15}\, e^{\frac{1}{4}-\frac{t}{4}}\, Heaviside(\cdots}{}$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

$$0 = -\frac{c_1}{4} + \frac{c_2\sqrt{15}}{4}$$

- Solve for c_1 and c_2

$$\{c_1 = 0, c_2 = 0\}$$

- Substitute constant values into general solution and simplify

$$y = -\frac{4\sqrt{15}\, e^{\frac{1}{4}-\frac{t}{4}}\, Heaviside(-1+t) \left(\cos\left(\frac{\sqrt{15}\, t}{4}\right) \sin\left(\frac{\sqrt{15}}{4}\right) - \sin\left(\frac{\sqrt{15}\, t}{4}\right) \cos\left(\frac{\sqrt{15}}{4}\right)\right)}{15}$$

- Solution to the IVP

$$y = -\frac{4\sqrt{15}\, e^{\frac{1}{4}-\frac{t}{4}}\, Heaviside(-1+t) \left(\cos\left(\frac{\sqrt{15}\, t}{4}\right) \sin\left(\frac{\sqrt{15}}{4}\right) - \sin\left(\frac{\sqrt{15}\, t}{4}\right) \cos\left(\frac{\sqrt{15}}{4}\right)\right)}{15}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.562 (sec). Leaf size: 28

```
dsolve([diff(y(t),t$2)+1/2*diff(y(t),t)+y(t)=Dirac(t-1),y(0) = 0, D(y)(0) = 0],y(t), singsol
```

$$y(t) = \frac{4\,\mathrm{e}^{\frac{1}{4}-\frac{t}{4}}\,\mathrm{Heaviside}\left(t-1\right)\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,(t-1)}{4}\right)}{15}$$

✓ Solution by Mathematica

Time used: 0.097 (sec). Leaf size: 40

```
DSolve[{y''[t]+1/2*y'[t]+y[t]==DiracDelta[t-1],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSol
```

$$y(t) \to \frac{4e^{\frac{1}{4}-\frac{t}{4}}\theta(t-1)\sin\left(\frac{1}{4}\sqrt{15}(t-1)\right)}{\sqrt{15}}$$

5.10 problem 10(c)

Internal problem ID [865]
Internal file name [`OUTPUT/865_Sunday_June_05_2022_01_52_42_AM_21475340/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.5, The Laplace Transform. Impulse functions. page 273
Problem number: 10(c).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + \frac{y'}{4} + y = \delta(-1+t)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

5.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= \frac{1}{4} \\ q(t) &= 1 \\ F &= \delta(-1+t) \end{aligned}$$

Hence the ode is

$$y'' + \frac{y'}{4} + y = \delta(-1+t)$$

The domain of $p(t) = \frac{1}{4}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \delta(-1+t)$ is

$$\{t < 1 \vee 1 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$
$$\mathcal{L}(y'') = s^2Y(s) - y'(0) - sy(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + \frac{sY(s)}{4} - \frac{y(0)}{4} + Y(s) = \mathrm{e}^{-s} \tag{1}$$

But the initial conditions are

$$y(0) = 0$$
$$y'(0) = 0$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + \frac{sY(s)}{4} + Y(s) = \mathrm{e}^{-s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{4\,\mathrm{e}^{-s}}{4s^2 + s + 4}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{4\,\mathrm{e}^{-s}}{4s^2+s+4}\right) \\ &= \frac{8\,\mathrm{Heaviside}\left(-1+t\right)\sqrt{7}\,\mathrm{e}^{\frac{1}{8}-\frac{t}{8}}\sin\left(\frac{3\sqrt{7}\,(-1+t)}{8}\right)}{21} \end{aligned}$$

Hence the final solution is

$$y = \frac{8\,\mathrm{Heaviside}\left(-1+t\right)\sqrt{7}\,\mathrm{e}^{\frac{1}{8}-\frac{t}{8}}\sin\left(\frac{3\sqrt{7}\,(-1+t)}{8}\right)}{21}$$

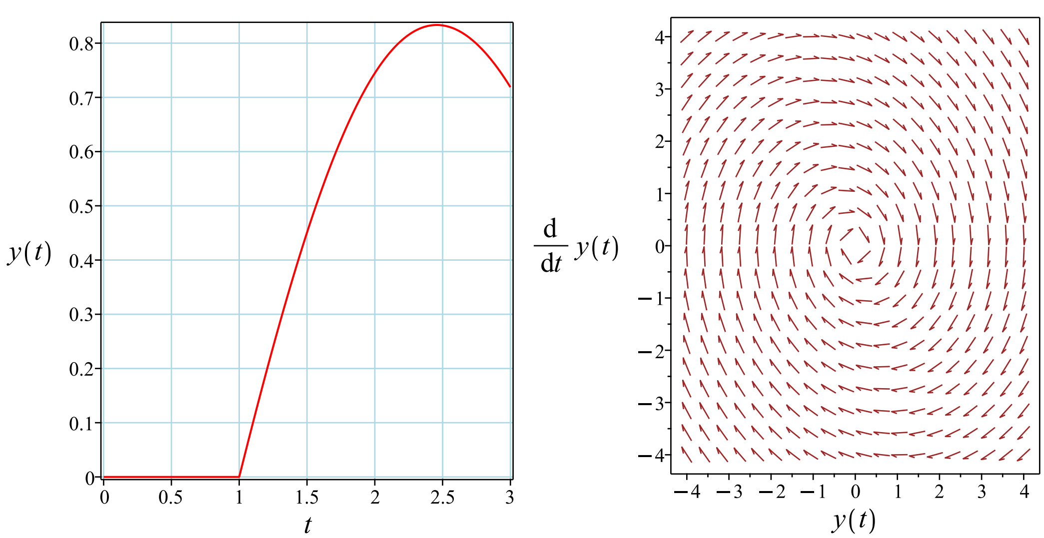
Simplifying the solution gives

$$y = \frac{8\,\mathrm{Heaviside}\left(-1+t\right)\sqrt{7}\,\mathrm{e}^{\frac{1}{8}-\frac{t}{8}}\sin\left(\frac{3\sqrt{7}\,(-1+t)}{8}\right)}{21}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{8\,\mathrm{Heaviside}\left(-1+t\right)\sqrt{7}\,\mathrm{e}^{\frac{1}{8}-\frac{t}{8}}\sin\left(\frac{3\sqrt{7}\,(-1+t)}{8}\right)}{21} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{8\,\mathrm{Heaviside}\left(-1+t\right)\sqrt{7}\,\mathrm{e}^{\frac{1}{8}-\frac{t}{8}}\sin\left(\frac{3\sqrt{7}\,(-1+t)}{8}\right)}{21}$$

Verified OK.

5.10.2 Maple step by step solution

Let's solve

$\left[y''+\frac{y'}{4}+y=\mathit{Dirac}(-1+t)\,, y(0)=0, y'\Big|_{\{t=0\}}=0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2+\frac{1}{4}r+1=0$

- Use quadratic formula to solve for r

  $r=\frac{\left(-\frac{1}{4}\right)\pm\left(\sqrt{-\frac{63}{16}}\right)}{2}$

- Roots of the characteristic polynomial

  $r=\left(-\frac{1}{8}-\frac{3\,\mathrm{I}\sqrt{7}}{8}, -\frac{1}{8}+\frac{3\,\mathrm{I}\sqrt{7}}{8}\right)$

- 1st solution of the homogeneous ODE

  $y_1(t)=\mathrm{e}^{-\frac{t}{8}}\cos\left(\frac{3\sqrt{7}\,t}{8}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(t)=\mathrm{e}^{-\frac{t}{8}}\sin\left(\frac{3\sqrt{7}\,t}{8}\right)$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\mathrm{e}^{-\frac{t}{8}}\cos\left(\frac{3\sqrt{7}\,t}{8}\right)+c_2\mathrm{e}^{-\frac{t}{8}}\sin\left(\frac{3\sqrt{7}\,t}{8}\right)+y_p(t)$

- □ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t)=-y_1(t)\left(\int\frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int\frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t)=\mathit{Dirac}(-1+t)\right]$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-\frac{t}{8}} \cos\left(\frac{3\sqrt{7}\,t}{8}\right) & e^{-\frac{t}{8}} \sin\left(\frac{3\sqrt{7}\,t}{8}\right) \\ -\frac{e^{-\frac{t}{8}} \cos\left(\frac{3\sqrt{7}\,t}{8}\right)}{8} - \frac{3\,e^{-\frac{t}{8}}\sqrt{7}\sin\left(\frac{3\sqrt{7}\,t}{8}\right)}{8} & -\frac{e^{-\frac{t}{8}} \sin\left(\frac{3\sqrt{7}\,t}{8}\right)}{8} + \frac{3\,e^{-\frac{t}{8}}\sqrt{7}\cos\left(\frac{3\sqrt{7}\,t}{8}\right)}{8} \end{bmatrix}$$

- Compute Wronskian

$W(y_1(t), y_2(t)) = \frac{3\sqrt{7}\,e^{-\frac{t}{4}}}{8}$

- Substitute functions into equation for $y_p(t)$

$y_p(t) = \frac{8\sqrt{7}\,e^{\frac{1}{8}-\frac{t}{8}}\left(\int Dirac(-1+t)dt\right)\left(\sin\left(\frac{3\sqrt{7}\,t}{8}\right)\cos\left(\frac{3\sqrt{7}}{8}\right)-\cos\left(\frac{3\sqrt{7}\,t}{8}\right)\sin\left(\frac{3\sqrt{7}}{8}\right)\right)}{21}$

- Compute integrals

$y_p(t) = \frac{8\sqrt{7}\,e^{\frac{1}{8}-\frac{t}{8}}\,Heaviside(-1+t)\left(\sin\left(\frac{3\sqrt{7}\,t}{8}\right)\cos\left(\frac{3\sqrt{7}}{8}\right)-\cos\left(\frac{3\sqrt{7}\,t}{8}\right)\sin\left(\frac{3\sqrt{7}}{8}\right)\right)}{21}$

- Substitute particular solution into general solution to ODE

$y = c_1 e^{-\frac{t}{8}} \cos\left(\frac{3\sqrt{7}\,t}{8}\right) + c_2 e^{-\frac{t}{8}} \sin\left(\frac{3\sqrt{7}\,t}{8}\right) + \frac{8\sqrt{7}\,e^{\frac{1}{8}-\frac{t}{8}}\,Heaviside(-1+t)\left(\sin\left(\frac{3\sqrt{7}\,t}{8}\right)\cos\left(\frac{3\sqrt{7}}{8}\right)-\cos\left(\frac{3\sqrt{7}\,t}{8}\right)\sin\left(\frac{3\sqrt{7}}{8}\right)\right)}{21}$

□ Check validity of solution $y = c_1 e^{-\frac{t}{8}} \cos\left(\frac{3\sqrt{7}\,t}{8}\right) + c_2 e^{-\frac{t}{8}} \sin\left(\frac{3\sqrt{7}\,t}{8}\right) + \frac{8\sqrt{7}\,e^{\frac{1}{8}-\frac{t}{8}}\,Heaviside(-1+t)\left(\sin\left(3\right.\right.}{}$

- Use initial condition $y(0) = 0$

$0 = c_1$

- Compute derivative of the solution

$y' = -\frac{c_1 e^{-\frac{t}{8}} \cos\left(\frac{3\sqrt{7}\,t}{8}\right)}{8} - \frac{3c_1 e^{-\frac{t}{8}}\sqrt{7}\sin\left(\frac{3\sqrt{7}\,t}{8}\right)}{8} - \frac{c_2 e^{-\frac{t}{8}} \sin\left(\frac{3\sqrt{7}\,t}{8}\right)}{8} + \frac{3c_2 e^{-\frac{t}{8}}\sqrt{7}\cos\left(\frac{3\sqrt{7}\,t}{8}\right)}{8} - \frac{\sqrt{7}\,e^{\frac{1}{8}-\frac{t}{8}}\,Heaviside(-}{}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

$0 = -\frac{c_1}{8} + \frac{3\sqrt{7}\,c_2}{8}$

- Solve for c_1 and c_2

$\{c_1 = 0, c_2 = 0\}$

- Substitute constant values into general solution and simplify

$y = \frac{8\sqrt{7}\,e^{\frac{1}{8}-\frac{t}{8}}\,Heaviside(-1+t)\left(\sin\left(\frac{3\sqrt{7}\,t}{8}\right)\cos\left(\frac{3\sqrt{7}}{8}\right)-\cos\left(\frac{3\sqrt{7}\,t}{8}\right)\sin\left(\frac{3\sqrt{7}}{8}\right)\right)}{21}$

- Solution to the IVP

$y = \frac{8\sqrt{7}\,e^{\frac{1}{8}-\frac{t}{8}}\,Heaviside(-1+t)\left(\sin\left(\frac{3\sqrt{7}\,t}{8}\right)\cos\left(\frac{3\sqrt{7}}{8}\right)-\cos\left(\frac{3\sqrt{7}\,t}{8}\right)\sin\left(\frac{3\sqrt{7}}{8}\right)\right)}{21}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 1.907 (sec). Leaf size: 28

```
dsolve([diff(y(t),t$2)+1/4*diff(y(t),t)+y(t)=Dirac(t-1),y(0) = 0, D(y)(0) = 0],y(t), singsol
```

$$y(t) = \frac{8\,\mathrm{e}^{\frac{1}{8}-\frac{t}{8}}\,\mathrm{Heaviside}\left(t-1\right)\sqrt{7}\,\sin\left(\frac{3\sqrt{7}\,(t-1)}{8}\right)}{21}$$

✓ Solution by Mathematica
Time used: 0.075 (sec). Leaf size: 42

```
DSolve[{y''[t]+1/4*y'[t]+y[t]==DiracDelta[t-1],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSol
```

$$y(t) \to \frac{8e^{\frac{1}{8}-\frac{t}{8}}\theta(t-1)\sin\left(\frac{3}{8}\sqrt{7}(t-1)\right)}{3\sqrt{7}}$$

5.11 problem 12

Internal problem ID [866]
Internal file name [`OUTPUT/866_Sunday_June_05_2022_01_52_46_AM_63188027/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.5, The Laplace Transform. Impulse functions. page 273
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + y = \frac{\text{Heaviside}\left(t-4+k\right) - \text{Heaviside}\left(t-4-k\right)}{2k}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

5.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 1 \\ F &= \frac{\text{Heaviside}\left(t-4+k\right) - \text{Heaviside}\left(t-4-k\right)}{2k} \end{aligned}$$

Hence the ode is

$$y'' + y = \frac{\text{Heaviside}\,(t-4+k) - \text{Heaviside}\,(t-4-k)}{2k}$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \frac{\text{Heaviside}(t-4+k) - \text{Heaviside}(t-4-k)}{2k}$ is

$$\{-\infty \leq t \leq 4-k, 4-k \leq t \leq 4+k, 4+k \leq t \leq \infty\}$$

But the point $t_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + Y(s) = \frac{\text{laplace}\,(\text{Heaviside}\,(t-4+k)\,,t,s)}{2k} - \frac{\text{laplace}\,(\text{Heaviside}\,(t-4-k)\,,t,s)}{2k} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + Y(s) = \frac{\text{laplace}\,(\text{Heaviside}\,(t-4+k)\,,t,s)}{2k} - \frac{\text{laplace}\,(\text{Heaviside}\,(t-4-k)\,,t,s)}{2k}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{\text{laplace}\left(\text{Heaviside}\left(t-4+k\right), t, s\right) - \text{laplace}\left(\text{Heaviside}\left(t-4-k\right), t, s\right)}{2k\left(s^2+1\right)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{\text{laplace}\left(\text{Heaviside}\left(t-4+k\right), t, s\right) - \text{laplace}\left(\text{Heaviside}\left(t-4-k\right), t, s\right)}{2k\left(s^2+1\right)}\right) \\ &= \frac{\text{Heaviside}\left(-4-k\right)\left(\cos\left(t\right)-\cos\left(-t+4+k\right)\right) + \text{Heaviside}\left(-4+k\right)\left(\cos\left(t-4+k\right)-\cos\left(t\right)\right) + \text{He}}{2k} \end{aligned}$$

Simplifying the solution gives

$$y = \frac{\left(\text{Heaviside}\left(4+k\right)+\text{Heaviside}\left(t-4-k\right)-1\right)\cos\left(-t+4+k\right)-\text{Heaviside}\left(t-4-k\right)+\left(-\cos\left(t-4\right.\right.}{}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(\text{Heaviside}\left(4+k\right)+\text{Heaviside}\left(t-4-k\right)-1\right)\cos\left(-t+4+k\right)-\text{Heaviside}\left(t-4-k\right)+\left(-\cos\left(t-4\right.\right.}{} \tag{1}$$

Verification of solutions

$$y = \frac{\left(\text{Heaviside}\left(4+k\right)+\text{Heaviside}\left(t-4-k\right)-1\right)\cos\left(-t+4+k\right)-\text{Heaviside}\left(t-4-k\right)+\left(-\cos\left(t-4\right.\right.}{}$$

Verified OK.

5.11.2 Maple step by step solution

Let's solve

$$\left[y''+y=\frac{\mathit{Heaviside}(t-4+k)-\mathit{Heaviside}(t-4-k)}{2k}, y(0)=0, y'\Big|_{\{t=0\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2+1=0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(t) + c_2 \sin(t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \frac{\mathit{Heaviside}(t-4+k) - \mathit{Heaviside}(t-4-}{2k}\right.$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 1$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = \frac{-\cos(t)\left(\int \sin(t)(\mathit{Heaviside}(t-4+k) - \mathit{Heaviside}(t-4-k))dt\right) + \sin(t)\left(\int \cos(t)(\mathit{Heaviside}(t-4+k) - \mathit{Heaviside}(t-4-k))dt\right)}{2k}$

  - Compute integrals

    $y_p(t) = \frac{\mathit{Heaviside}(t-4-k)(\cos(t)\cos(4+k) + \sin(t)\sin(4+k) - 1) - \mathit{Heaviside}(t-4+k)(\cos(t)\cos(-4+k) - \sin(t)\sin(-4+k) - 1)}{2k}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(t) + c_2 \sin(t) + \frac{\mathit{Heaviside}(t-4-k)(\cos(t)\cos(4+k) + \sin(t)\sin(4+k) - 1) - \mathit{Heaviside}(t-4+k)(\cos(t)\cos(-4+k) -}{2k}$

☐ Check validity of solution $y = c_1 \cos(t) + c_2 \sin(t) + \frac{\mathit{Heaviside}(t-4-k)(\cos(t)\cos(4+k) + \sin(t)\sin(4+k) - 1) - }{2}$

  - Use initial condition $y(0) = 0$

    $0 = c_1 + \frac{\mathit{Heaviside}(-4-k)(\cos(4+k) - 1) - \mathit{Heaviside}(-4+k)(\cos(-4+k) - 1)}{2k}$

- Compute derivative of the solution

  $y' = -c_1 \sin(t) + c_2 \cos(t) + \frac{Dirac(-t+4+k)(\cos(t)\cos(4+k)+\sin(t)\sin(4+k)-1)+Heaviside(t-4-k)(-\sin(t)\cos(4+k)}{}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = c_2 + \frac{Dirac(4+k)(\cos(4+k)-1)+\sin(4+k)Heaviside(-4-k)-Dirac(-4+k)(\cos(-4+k)-1)+\sin(-4+k)Heaviside(-4+k)}{2k}$

- Solve for c_1 and c_2

  $\left\{c_1 = \frac{Heaviside(-4+k)\cos(-4+k)-\cos(4+k)Heaviside(-4-k)-Heaviside(-4+k)+Heaviside(-4-k)}{2k}, c_2 = -\frac{\sin(-4+k)Hea}{}\right.$

- Substitute constant values into general solution and simplify

  $y = \frac{Heaviside(t-4-k)(\cos(t)\cos(4+k)+\sin(t)\sin(4+k)-1)+(-\cos(t)\cos(-4+k)+\sin(t)\sin(-4+k)+1)Heaviside(t-4+k)+Heavi}{2k}$

- Solution to the IVP

  $y = \frac{Heaviside(t-4-k)(\cos(t)\cos(4+k)+\sin(t)\sin(4+k)-1)+(-\cos(t)\cos(-4+k)+\sin(t)\sin(-4+k)+1)Heaviside(t-4+k)+Heavi}{2k}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.641 (sec). Leaf size: 76

```
dsolve([diff(y(t),t$2)+y(t)=1/(2*k)*(Heaviside(t-(4-k)) -  Heaviside(t-(4+k)) ),y(0) = 0, D(
```

$y(t) = \frac{(\mathrm{Heaviside}(4+k) + \mathrm{Heaviside}(t-4-k) - 1)\cos(-t+4+k) - \mathrm{Heaviside}(t-4-k) + (-\cos(t-4}{}$

✓ Solution by Mathematica

Time used: 1.204 (sec). Leaf size: 181

```
DSolve[{y''[t]+y[t]==1/(2*k)*(UnitStep[t-(4-k)] -  UnitStep[t-(4+k)] ),{y[0]==0,y'[0]==0}},y
```

$$y(t) \to \boxed{\frac{(\cos(k-t+4)-1)\theta(-k+t-4)-(\cos(-k-t+4)-1)\theta(k+t-4)}{2k} \text{ if } -4<k<4}$$

$$y(t) \to \boxed{\frac{\cos(-k-t+4)-\cos(t)+(\cos(k-t+4)-1)\theta(-k+t-4)-(\cos(-k-t+4)-1)\theta(k+t-4)}{2k} \text{ if } k>4}$$

$$y(t) \to \boxed{\frac{-\cos(k-t+4)+\cos(t)+(\cos(k-t+4)-1)\theta(-k+t-4)-(\cos(-k-t+4)-1)\theta(k+t-4)}{2k} \text{ if } k<-4}$$

5.12 problem 19(a)

Internal problem ID [867]
Internal file name [`OUTPUT/867_Sunday_June_05_2022_01_52_48_AM_8947288/index.tex`]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.5, The Laplace Transform. Impulse functions. page 273
Problem number: 19(a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + 2y = f(t)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

5.12.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 2, C = 2, f(t) = f(t)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 2y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 2, C = 2$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 2\lambda \, e^{\lambda t} + 2 \, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 2\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{2^2 - (4)\,(1)\,(2)} \\ &= -1 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + i \\ \lambda_2 &= -1 - i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 + i \\ \lambda_2 &= -1 - i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^{-t}(c_1 \cos(t) + c_2 \sin(t))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-t}(c_1 \cos(t) + c_2 \sin(t))$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-t} \cos(t)$$
$$y_2 = e^{-t} \sin(t)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{a W(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{a W(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-t} \cos(t) & e^{-t} \sin(t) \\ \frac{d}{dt}(e^{-t} \cos(t)) & \frac{d}{dt}(e^{-t} \sin(t)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-t} \cos(t) & e^{-t} \sin(t) \\ -e^{-t} \cos(t) - e^{-t} \sin(t) & -e^{-t} \sin(t) + e^{-t} \cos(t) \end{vmatrix}$$

Therefore

$$W = \left(e^{-t} \cos(t)\right) \left(-e^{-t} \sin(t) + e^{-t} \cos(t)\right) - \left(e^{-t} \sin(t)\right) \left(-e^{-t} \cos(t) - e^{-t} \sin(t)\right)$$

Which simplifies to

$$W = e^{-2t} \cos(t)^2 + e^{-2t} \sin(t)^2$$

Which simplifies to

$$W = e^{-2t}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{-t} \sin(t) f(t)}{e^{-2t}} dt$$

Which simplifies to

$$u_1 = -\int f(t) \sin(t) e^{t} dt$$

Hence

$$u_1 = -\left(\int_0^t f(\alpha) \sin(\alpha) e^{\alpha} d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-t} \cos(t) f(t)}{e^{-2t}} dt$$

Which simplifies to

$$u_2 = \int f(t) \cos(t) e^{t} dt$$

Hence

$$u_2 = \int_0^t f(\alpha) \cos(\alpha) e^{\alpha} d\alpha$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -\left(\int_0^t f(\alpha) \sin(\alpha) e^{\alpha} d\alpha\right) e^{-t} \cos(t) + \left(\int_0^t f(\alpha) \cos(\alpha) e^{\alpha} d\alpha\right) e^{-t} \sin(t)$$

Which simplifies to

$$y_p(t) = e^{-t}\left(-\left(\int_0^t f(\alpha) \sin(\alpha) e^{\alpha} d\alpha\right) \cos(t) + \left(\int_0^t f(\alpha) \cos(\alpha) e^{\alpha} d\alpha\right) \sin(t)\right)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(e^{-t}(c_1 \cos(t) + c_2 \sin(t))\right) \\
&\quad + \left(e^{-t}\left(-\left(\int_0^t f(\alpha) \sin(\alpha) e^{\alpha} d\alpha\right) \cos(t) + \left(\int_0^t f(\alpha) \cos(\alpha) e^{\alpha} d\alpha\right) \sin(t)\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = e^{-t}(c_1 \cos(t) + c_2 \sin(t)) + e^{-t}\left(-\left(\int_0^t f(\alpha) \sin(\alpha) e^{\alpha} d\alpha\right) \cos(t) + \left(\int_0^t f(\alpha) \cos(\alpha) e^{\alpha} d\alpha\right) \sin(t)\right) \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$
0 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -e^{-t}(c_1 \cos(t) + c_2 \sin(t)) + e^{-t}(-c_1 \sin(t) + c_2 \cos(t)) - e^{-t}\left(-\left(\int_0^t f(\alpha) \sin(\alpha) e^{\alpha} d\alpha\right) \cos(t) + \left(
$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$
0 = -c_1 + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 0 \\
c_2 &= 0
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = -\left(\int_0^t f(\alpha) \sin(\alpha) e^{\alpha} d\alpha\right) e^{-t} \cos(t) + \left(\int_0^t f(\alpha) \cos(\alpha) e^{\alpha} d\alpha\right) e^{-t} \sin(t)
$$

Which simplifies to

$$
y = e^{-t}\left(-\left(\int_0^t f(\alpha) \sin(\alpha) e^{\alpha} d\alpha\right) \cos(t) + \left(\int_0^t f(\alpha) \cos(\alpha) e^{\alpha} d\alpha\right) \sin(t)\right)
$$

Summary
The solution(s) found are the following

$$y = e^{-t}\left(-\left(\int_0^t f(\alpha)\sin(\alpha)\,e^{\alpha}d\alpha\right)\cos(t) + \left(\int_0^t f(\alpha)\cos(\alpha)\,e^{\alpha}d\alpha\right)\sin(t)\right) \tag{1}$$

Verification of solutions

$$y = e^{-t}\left(-\left(\int_0^t f(\alpha)\sin(\alpha)\,e^{\alpha}d\alpha\right)\cos(t) + \left(\int_0^t f(\alpha)\cos(\alpha)\,e^{\alpha}d\alpha\right)\sin(t)\right)$$

Verified OK.

5.12.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(t) = -z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 51: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dt} \\ &= z_1 e^{-t} \\ &= z_1\left(\mathrm{e}^{-t}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-t}\cos(t)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{-2t}}{(y_1)^2}\,dt \\ &= y_1(\tan(t)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-t}\cos(t)\right) + c_2\left(\mathrm{e}^{-t}\cos(t)(\tan(t))\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t)+By'(t)+Cy(t)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t)+By'(t)+Cy(t)=f(t)$. y_h is the solution to

$$y''+2y'+2y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-t}\cos(t) + c_2 \mathrm{e}^{-t}\sin(t)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \mathrm{e}^{-t}\cos(t) \\ y_2 &= \mathrm{e}^{-t}\sin(t) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-t}\cos(t) & \mathrm{e}^{-t}\sin(t) \\ \frac{d}{dt}(\mathrm{e}^{-t}\cos(t)) & \frac{d}{dt}(\mathrm{e}^{-t}\sin(t)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-t}\cos(t) & \mathrm{e}^{-t}\sin(t) \\ -\mathrm{e}^{-t}\cos(t) - \mathrm{e}^{-t}\sin(t) & -\mathrm{e}^{-t}\sin(t) + \mathrm{e}^{-t}\cos(t) \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-t}\cos(t)\right)\left(-\mathrm{e}^{-t}\sin(t) + \mathrm{e}^{-t}\cos(t)\right) - \left(\mathrm{e}^{-t}\sin(t)\right)\left(-\mathrm{e}^{-t}\cos(t) - \mathrm{e}^{-t}\sin(t)\right)$$

Which simplifies to

$$W = \mathrm{e}^{-2t}\cos(t)^2 + \mathrm{e}^{-2t}\sin(t)^2$$

Which simplifies to

$$W = \mathrm{e}^{-2t}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\mathrm{e}^{-t}\sin(t)\, f(t)}{\mathrm{e}^{-2t}}\, dt$$

Which simplifies to

$$u_1 = -\int f(t)\sin(t)\,\mathrm{e}^{t} dt$$

Hence

$$u_1 = -\left(\int_0^t f(\alpha)\sin(\alpha)\,\mathrm{e}^{\alpha} d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{-t}\cos(t)\, f(t)}{\mathrm{e}^{-2t}}\, dt$$

Which simplifies to

$$u_2 = \int f(t)\cos(t)\,\mathrm{e}^{t} dt$$

Hence

$$u_2 = \int_0^t f(\alpha)\cos(\alpha)\,\mathrm{e}^{\alpha} d\alpha$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -\left(\int_0^t f(\alpha)\sin(\alpha)\,\mathrm{e}^{\alpha}d\alpha\right)\mathrm{e}^{-t}\cos(t) + \left(\int_0^t f(\alpha)\cos(\alpha)\,\mathrm{e}^{\alpha}d\alpha\right)\mathrm{e}^{-t}\sin(t)$$

Which simplifies to

$$y_p(t) = \mathrm{e}^{-t}\left(-\left(\int_0^t f(\alpha)\sin(\alpha)\,\mathrm{e}^{\alpha}d\alpha\right)\cos(t) + \left(\int_0^t f(\alpha)\cos(\alpha)\,\mathrm{e}^{\alpha}d\alpha\right)\sin(t)\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-t}\cos(t) + c_2\mathrm{e}^{-t}\sin(t)\right) \\ &\quad + \left(\mathrm{e}^{-t}\left(-\left(\int_0^t f(\alpha)\sin(\alpha)\,\mathrm{e}^{\alpha}d\alpha\right)\cos(t) + \left(\int_0^t f(\alpha)\cos(\alpha)\,\mathrm{e}^{\alpha}d\alpha\right)\sin(t)\right)\right) \end{aligned}$$

Which simplifies to

$$\begin{aligned} y &= \mathrm{e}^{-t}(c_1\cos(t) + c_2\sin(t)) \\ &\quad + \mathrm{e}^{-t}\left(-\left(\int_0^t f(\alpha)\sin(\alpha)\,\mathrm{e}^{\alpha}d\alpha\right)\cos(t) + \left(\int_0^t f(\alpha)\cos(\alpha)\,\mathrm{e}^{\alpha}d\alpha\right)\sin(t)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^{-t}(c_1\cos(t) + c_2\sin(t)) + \mathrm{e}^{-t}\left(-\left(\int_0^t f(\alpha)\sin(\alpha)\,\mathrm{e}^{\alpha}d\alpha\right)\cos(t) + \left(\int_0^t f(\alpha)\cos(\alpha)\,\mathrm{e}^{\alpha}d\alpha\right)\sin(t)\right) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\mathrm{e}^{-t}(c_1\cos(t) + c_2\sin(t)) + \mathrm{e}^{-t}(-c_1\sin(t) + c_2\cos(t)) - \mathrm{e}^{-t}\left(-\left(\int_0^t f(\alpha)\sin(\alpha)\,\mathrm{e}^{\alpha}d\alpha\right)\cos(t) + \left(\right.\right.$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = -\left(\int_0^t f(\alpha)\sin(\alpha)\,\mathrm{e}^{\alpha}d\alpha\right)\mathrm{e}^{-t}\cos(t) + \left(\int_0^t f(\alpha)\cos(\alpha)\,\mathrm{e}^{\alpha}d\alpha\right)\mathrm{e}^{-t}\sin(t)$$

Which simplifies to

$$y = \mathrm{e}^{-t}\left(-\left(\int_0^t f(\alpha)\sin(\alpha)\,\mathrm{e}^{\alpha}d\alpha\right)\cos(t) + \left(\int_0^t f(\alpha)\cos(\alpha)\,\mathrm{e}^{\alpha}d\alpha\right)\sin(t)\right)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-t}\left(-\left(\int_0^t f(\alpha)\sin(\alpha)\,\mathrm{e}^{\alpha}d\alpha\right)\cos(t) + \left(\int_0^t f(\alpha)\cos(\alpha)\,\mathrm{e}^{\alpha}d\alpha\right)\sin(t)\right) \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{-t}\left(-\left(\int_0^t f(\alpha)\sin(\alpha)\,\mathrm{e}^{\alpha}d\alpha\right)\cos(t) + \left(\int_0^t f(\alpha)\cos(\alpha)\,\mathrm{e}^{\alpha}d\alpha\right)\sin(t)\right)$$

Verified OK.

5.12.3 Maple step by step solution

Let's solve

$$\left[y'' + 2y' + 2y = f(t), y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-1 - \mathrm{I}, -1 + \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-t}\cos(t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-t}\sin(t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1\mathrm{e}^{-t}\cos(t) + c_2\mathrm{e}^{-t}\sin(t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = f(t)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-t}\cos(t) & \mathrm{e}^{-t}\sin(t) \\ -\mathrm{e}^{-t}\cos(t) - \mathrm{e}^{-t}\sin(t) & -\mathrm{e}^{-t}\sin(t) + \mathrm{e}^{-t}\cos(t) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = \mathrm{e}^{-2t}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = -\mathrm{e}^{-t}\left(\cos(t)\left(\int f(t)\sin(t)\mathrm{e}^{t}dt\right) - \sin(t)\left(\int f(t)\cos(t)\mathrm{e}^{t}dt\right)\right)$

  - Compute integrals

    $y_p(t) = -\mathrm{e}^{-t}\left(\cos(t)\left(\int f(t)\sin(t)\mathrm{e}^{t}dt\right) - \sin(t)\left(\int f(t)\cos(t)\mathrm{e}^{t}dt\right)\right)$

- Substitute particular solution into general solution to ODE

  $y = c_1\mathrm{e}^{-t}\cos(t) + c_2\mathrm{e}^{-t}\sin(t) - \mathrm{e}^{-t}\left(\cos(t)\left(\int f(t)\sin(t)\mathrm{e}^{t}dt\right) - \sin(t)\left(\int f(t)\cos(t)\mathrm{e}^{t}dt\right)\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 43

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)+2*y(t)=f(t),y(0) = 0, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = \left(-\cos(t)\left(\int_0^t f(\_z1)\sin(\_z1)\,\mathrm{e}^{\_z1}d\_z1\right) + \sin(t)\left(\int_0^t f(\_z1)\cos(\_z1)\,\mathrm{e}^{\_z1}d\_z1\right)\right)\mathrm{e}^{-t}$$

✓ Solution by Mathematica

Time used: 0.104 (sec). Leaf size: 99

```
DSolve[{y''[t]+2*y'[t]+2*y[t]==f[t],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutions -> T
```

$$y(t) \to e^{-t}\left(-\sin(t)\int_1^0 e^{K[1]}\cos(K[1])f(K[1])dK[1] + \sin(t)\int_1^t e^{K[1]}\cos(K[1])f(K[1])dK[1] + \cos(t)\left(\int_1^t -e^{K[2]}f(K[2])\sin(K[2])dK[2] - \int_1^0 -e^{K[2]}f(K[2])\sin(K[2])dK[2]\right)\right)$$

5.13 problem 19(b)

Internal problem ID [868]
Internal file name [OUTPUT/868_Sunday_June_05_2022_01_52_50_AM_65372010/index.tex]

Book: Elementary differential equations and boundary value problems, 11th ed., Boyce, DiPrima, Meade
Section: Chapter 6.5, The Laplace Transform. Impulse functions. page 273
Problem number: 19(b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + 2y = \delta(t - \pi)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

5.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 2 \\ F &= \delta(t - \pi) \end{aligned}$$

Hence the ode is

$$y'' + 2y' + 2y = \delta(t - \pi)$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \delta(t - \pi)$ is

$$\{t < \pi \vee \pi < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$
$$\mathcal{L}(y'') = s^2Y(s) - y'(0) - sy(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 2sY(s) - 2y(0) + 2Y(s) = e^{-\pi s} \tag{1}$$

But the initial conditions are

$$y(0) = 0$$
$$y'(0) = 0$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 2sY(s) + 2Y(s) = e^{-\pi s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{e^{-\pi s}}{s^2 + 2s + 2}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{e^{-\pi s}}{s^2 + 2s + 2}\right) \\ &= -\sin(t)\,\text{Heaviside}(t - \pi)\,e^{\pi - t} \end{aligned}$$

Hence the final solution is

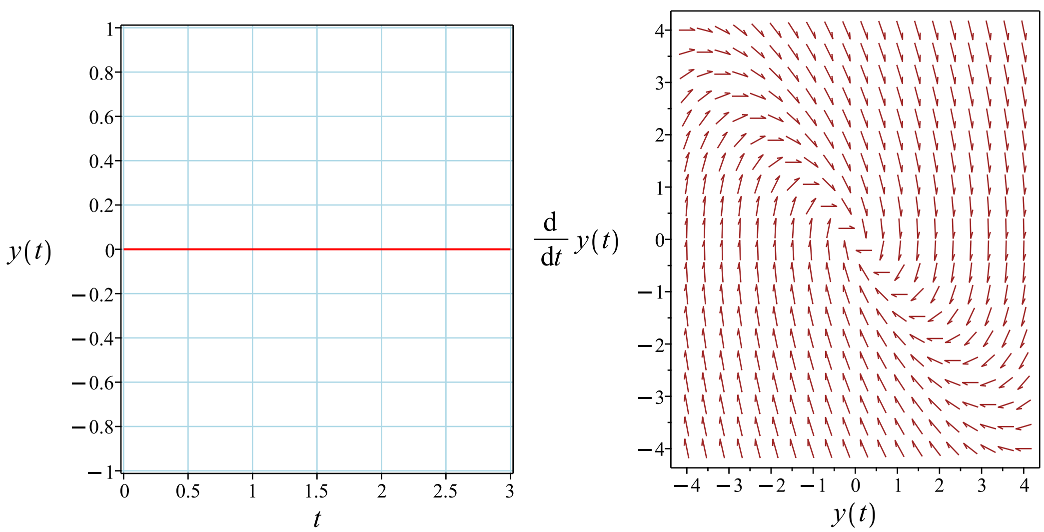
$$y = -\sin(t)\,\mathrm{Heaviside}(t - \pi)\,\mathrm{e}^{\pi - t}$$

Simplifying the solution gives

$$y = -\sin(t)\,\mathrm{Heaviside}(t - \pi)\,\mathrm{e}^{\pi - t}$$

Summary
The solution(s) found are the following

$$y = -\sin(t)\,\mathrm{Heaviside}(t - \pi)\,\mathrm{e}^{\pi - t} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\sin(t)\,\mathrm{Heaviside}(t - \pi)\,\mathrm{e}^{\pi - t}$$

Verified OK.

5.13.2 Maple step by step solution

Let's solve

$\left[y'' + 2y' + 2y = \mathit{Dirac}(t - \pi), y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2) \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-1 - \mathrm{I}, -1 + \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-t} \cos(t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-t} \sin(t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-t} \cos(t) + c_2 \mathrm{e}^{-t} \sin(t) + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t) \left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t) \left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \mathit{Dirac}(t - \pi)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-t} \cos(t) & \mathrm{e}^{-t} \sin(t) \\ -\mathrm{e}^{-t} \cos(t) - \mathrm{e}^{-t} \sin(t) & -\mathrm{e}^{-t} \sin(t) + \mathrm{e}^{-t} \cos(t) \end{bmatrix}$

  ◦ Compute Wronskian

    $W(y_1(t), y_2(t)) = \mathrm{e}^{-2t}$

  ◦ Substitute functions into equation for $y_p(t)$

$$y_p(t) = -\left(\int Dirac(t-\pi)\, dt\right) \sin(t)\, e^{\pi - t}$$

- Compute integrals

  $$y_p(t) = -\sin(t)\, Heaviside(t-\pi)\, e^{\pi-t}$$

- Substitute particular solution into general solution to ODE

  $$y = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t) - \sin(t)\, Heaviside(t-\pi)\, e^{\pi-t}$$

- ☐ Check validity of solution $y = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t) - \sin(t)\, Heaviside(t-\pi)\, e^{\pi-t}$
  - Use initial condition $y(0) = 0$

    $$0 = c_1$$
  - Compute derivative of the solution

    $$y' = -c_1 e^{-t}\cos(t) - c_1 e^{-t}\sin(t) - c_2 e^{-t}\sin(t) + c_2 e^{-t}\cos(t) - \cos(t)\, Heaviside(t-\pi)\, e^{\pi-t} - I$$
  - Use the initial condition $y'\Big|_{\{t=0\}} = 0$

    $$0 = -c_1 + c_2$$
  - Solve for c_1 and c_2

    $$\{c_1 = 0, c_2 = 0\}$$
  - Substitute constant values into general solution and simplify

    $$y = -\sin(t)\, Heaviside(t-\pi)\, e^{\pi-t}$$

- Solution to the IVP

  $$y = -\sin(t)\, Heaviside(t-\pi)\, e^{\pi-t}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.281 (sec). Leaf size: 20

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)+2*y(t)=Dirac(t-Pi),y(0) = 0, D(y)(0) = 0],y(t), singso
```

$$y(t) = -\sin(t)\,\mathrm{Heaviside}(t-\pi)\,\mathrm{e}^{\pi-t}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 22

```
DSolve[{y''[t]+2*y'[t]+2*y[t]==DiracDelta[t-Pi],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSo
```

$$y(t) \to -e^{\pi-t}\theta(t-\pi)\sin(t)$$