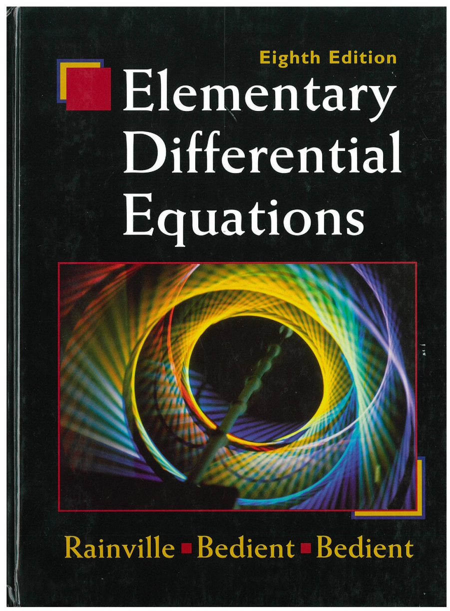
A Solution Manual For

Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.



Nasser M. Abbasi

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Contents

1 CHAPTER 8. Nonhomogeneous Equations: Undetermined Coefficients. Exercises Page 142

1.1 problem 1

Internal problem ID [6861]
Internal file name [`OUTPUT/6108_Saturday_September_10_2022_01_23_03_AM_9550685/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 8. Nonhomogeneous Equations: Undetermined Coefficients. Exercises Page 142
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + y = -\cos(x)}$$

1.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = -\cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\cos(x), x\sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \cos(x) + A_2 x \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x) + 2A_2 \cos(x) = -\cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\sin(x)}{2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) + \left(-\frac{x \sin(x)}{2}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(x) + c_2 \sin(x) - \frac{x \sin(x)}{2} \tag{1}
$$

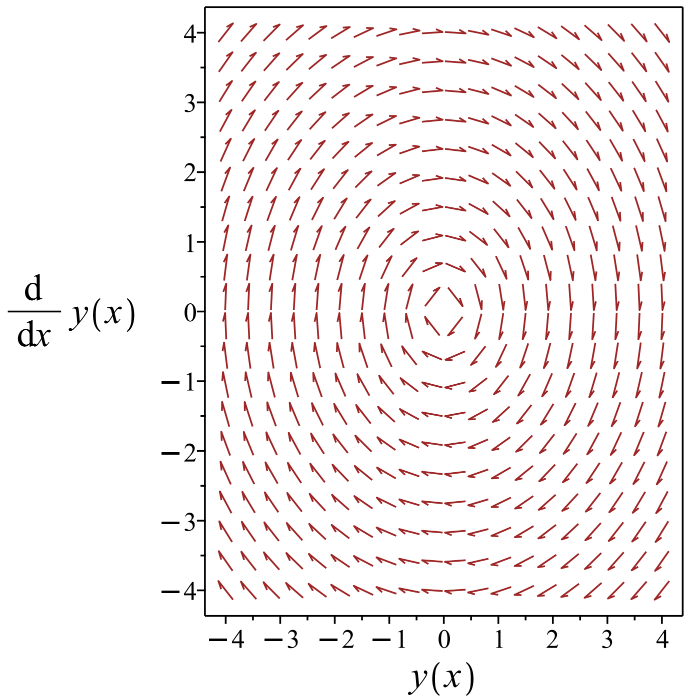

Figure 1: Slope field plot

Verification of solutions

$$
y = c_1 \cos(x) + c_2 \sin(x) - \frac{x \sin(x)}{2}
$$

Verified OK.

1.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 1: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\ &= \cos(x) (\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x) (\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\cos(x), x\sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\cos(x) + A_2 x\sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1\sin(x) + 2A_2\cos(x) = -\cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\sin(x)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(x) + c_2\sin(x)) + \left(-\frac{x\sin(x)}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) - \frac{x \sin(x)}{2} \tag{1}$$

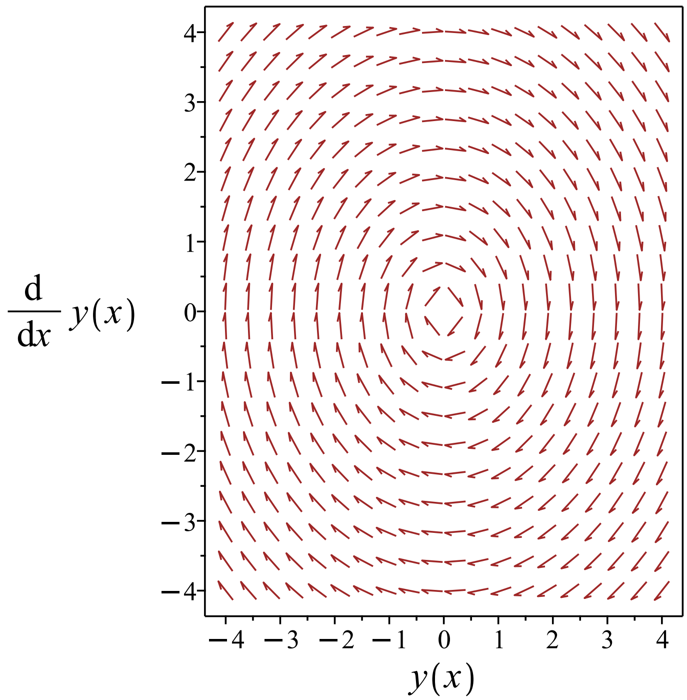

Figure 2: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) - \frac{x \sin(x)}{2}$$

Verified OK.

1.1.3 Maple step by step solution

Let's solve

$y'' + y = -\cos(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = -\cos(x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = \frac{\cos(x)\left(\int \sin(2x) dx\right)}{2} - \sin(x)\left(\int \cos(x)^2 dx\right)$

  - Compute integrals

    $y_p(x) = -\frac{\cos(x)}{4} - \frac{x \sin(x)}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) - \frac{\cos(x)}{4} - \frac{x \sin(x)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x$2)+y(x)=-cos(x),y(x), singsol=all)
```

$$y(x) = \frac{(2c_2 - x)\sin(x)}{2} + \cos(x)\, c_1$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 27

```
DSolve[y''[x]+y[x]==-Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(-\frac{1}{2} + c_1\right)\cos(x) - \frac{1}{2}(x - 2c_2)\sin(x)$$

1.2 problem 2

Internal problem ID [6862]
Internal file name [`OUTPUT/6109_Saturday_September_10_2022_01_23_03_AM_44998391/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 8. Nonhomogeneous Equations: Undetermined Coefficients. Exercises Page 142
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 6y' + 9y = e^x}$$

1.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -6, C = 9, f(x) = e^x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 6y' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -6, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 6\lambda\, e^{\lambda x} + 9\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 6\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -6, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{6}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(-6)^2 - (4)\,(1)\,(9)}\\ &= 3\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -3$. Therefore the solution is

$$y = c_1 e^{3x} + c_2 x\, e^{3x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{3x} + c_2 x\, e^{3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{3x}, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 \mathrm{e}^x = \mathrm{e}^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\mathrm{e}^x}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{3x} + c_2 x\, \mathrm{e}^{3x}\right) + \left(\frac{\mathrm{e}^x}{4}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{3x}(c_2 x + c_1) + \frac{\mathrm{e}^x}{4}$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{3x}(c_2 x + c_1) + \frac{\mathrm{e}^x}{4} \tag{1}$$

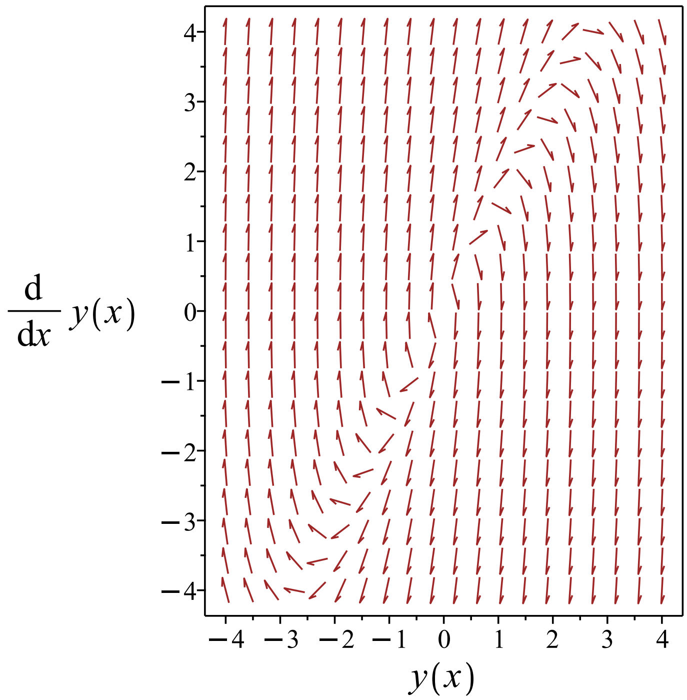

Figure 3: Slope field plot

Verification of solutions

$$y = e^{3x}(c_2 x + c_1) + \frac{e^x}{4}$$

Verified OK.

1.2.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -6$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int -6\, dx} \\ &= e^{-3x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= e^{-3x} e^{x} \\
\left(e^{-3x} y\right)'' &= e^{-3x} e^{x}
\end{aligned}
$$

Integrating once gives

$$
\left(e^{-3x} y\right)' = -\frac{e^{-2x}}{2} + c_1
$$

Integrating again gives

$$
\left(e^{-3x} y\right) = c_1 x + \frac{e^{-2x}}{4} + c_2
$$

Hence the solution is

$$
y = \frac{c_1 x + \frac{e^{-2x}}{4} + c_2}{e^{-3x}}
$$

Or

$$
y = c_1 x\, e^{3x} + e^{3x} c_2 + \frac{e^{x}}{4}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\, e^{3x} + e^{3x} c_2 + \frac{e^{x}}{4} \tag{1}
$$

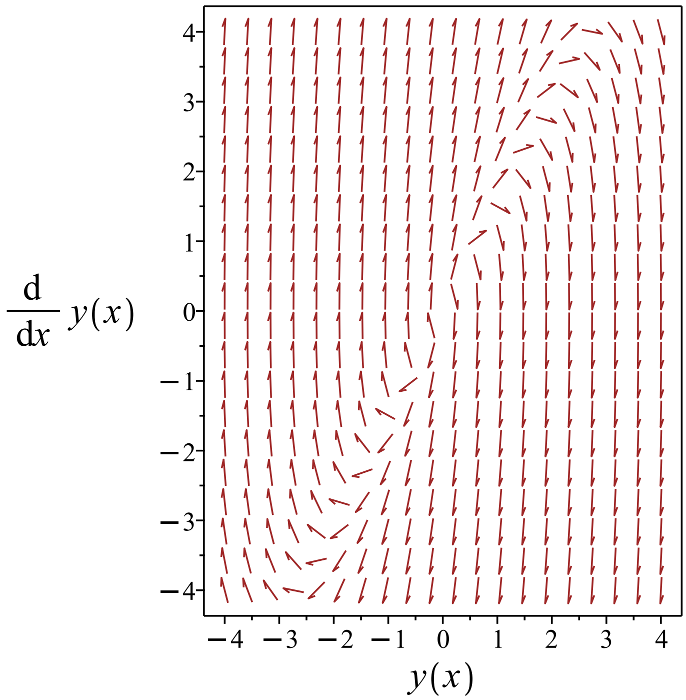

Figure 4: Slope field plot

Verification of solutions

$$y = c_1 x \, e^{3x} + e^{3x} c_2 + \frac{e^x}{4}$$

Verified OK.

1.2.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 6y' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -6 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 3: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-6}{1}\,dx} \\ &= z_1 e^{3x} \\ &= z_1 \left(\mathrm{e}^{3x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-6}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{6x}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{3x}\right) + c_2\left(\mathrm{e}^{3x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 6y' + 9y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1\mathrm{e}^{3x} + c_2 x\,\mathrm{e}^{3x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\mathrm{e}^{x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{x\,\mathrm{e}^{3x}, \mathrm{e}^{3x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1\mathrm{e}^{x}
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
4A_1\mathrm{e}^{x} = \mathrm{e}^{x}
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^x}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{3x} + c_2 x\, e^{3x}\right) + \left(\frac{e^x}{4}\right) \end{aligned}$$

Which simplifies to

$$y = e^{3x}(c_2 x + c_1) + \frac{e^x}{4}$$

<u>Summary</u>
The solution(s) found are the following

$$y = e^{3x}(c_2 x + c_1) + \frac{e^x}{4} \tag{1}$$

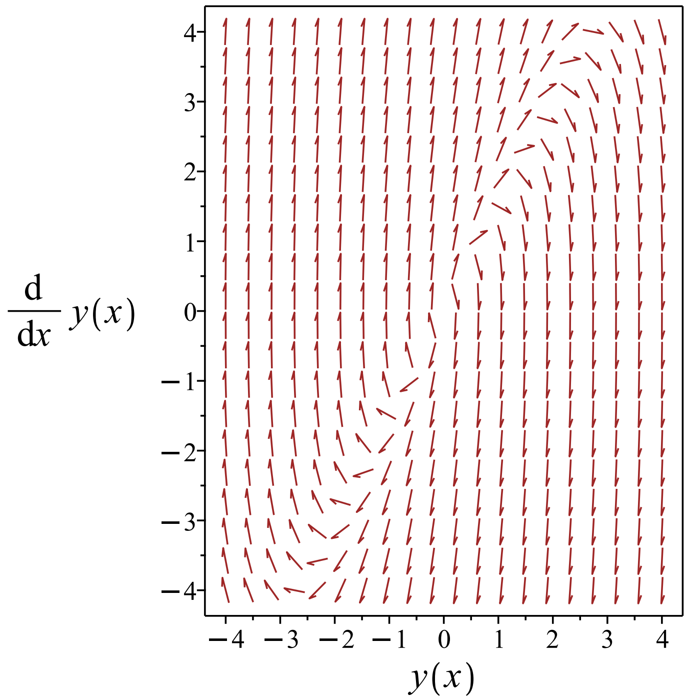

Figure 5: Slope field plot

<u>Verification of solutions</u>

$$y = e^{3x}(c_2x + c_1) + \frac{e^x}{4}$$

Verified OK.

1.2.4 Maple step by step solution

Let's solve

$y'' - 6y' + 9y = e^x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 6r + 9 = 0$

- Factor the characteristic polynomial

  $(r-3)^2 = 0$

- Root of the characteristic polynomial

$r = 3$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{3x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{3x} + c_2 x\,\mathrm{e}^{3x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \mathrm{e}^{x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{3x} & x\,\mathrm{e}^{3x} \\ 3\,\mathrm{e}^{3x} & \mathrm{e}^{3x} + 3x\,\mathrm{e}^{3x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{6x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = \mathrm{e}^{3x}\left(-\left(\int x\,\mathrm{e}^{-2x}dx\right) + \left(\int \mathrm{e}^{-2x}dx\right)x\right)$

  - Compute integrals

    $y_p(x) = \frac{\mathrm{e}^{x}}{4}$

- Substitute particular solution into general solution to ODE

  $y = c_2 x\,\mathrm{e}^{3x} + c_1 \mathrm{e}^{3x} + \frac{\mathrm{e}^{x}}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)-6*diff(y(x),x)+9*y(x)=exp(x),y(x), singsol=all)
```

$$y(x) = (c_1 x + c_2) e^{3x} + \frac{e^x}{4}$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 26

```
DSolve[y''[x]-6*y'[x]+9*y[x]==Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{e^x}{4} + e^{3x}(c_2 x + c_1)$$

1.3 problem 3

Internal problem ID [6863]
Internal file name [`OUTPUT/6110_Saturday_September_10_2022_01_23_04_AM_30605318/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 8. Nonhomogeneous Equations: Undetermined Coefficients. Exercises Page 142
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 3y' + 2y = 12x^2}$$

1.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 3, C = 2, f(x) = 12x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 3\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{3^2 - (4)\,(1)\,(2)} \\ &= -\frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{3}{2} + \frac{1}{2} \\ \lambda_2 &= -\frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 \\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(-1)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 e^{-x} + c_2 e^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-x} + c_2 e^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_3x^2 + 2A_2x + 6xA_3 + 2A_1 + 3A_2 + 2A_3 = 12x^2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 21, A_2 = -18, A_3 = 6]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 6x^2 - 18x + 21$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 e^{-2x}\right) + \left(6x^2 - 18x + 21\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{-2x} + 6x^2 - 18x + 21 \tag{1}$$

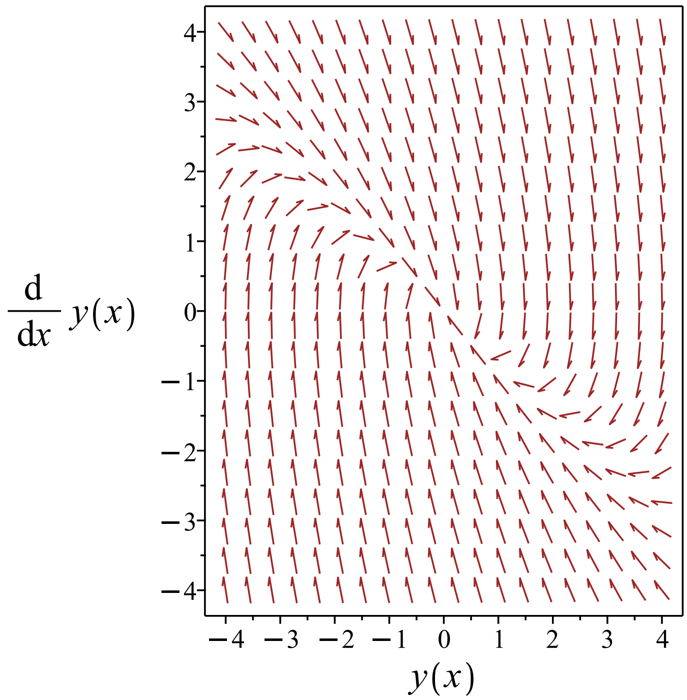

Figure 6: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^{-x} + c_2 e^{-2x} + 6x^2 - 18x + 21$$

Verified OK.

1.3.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 5: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\, dx} \\ &= z_1 e^{-\frac{3x}{2}} \\ &= z_1\left(\mathrm{e}^{-\frac{3x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-3x}}{(y_1)^2}\, dx \\ &= y_1(\mathrm{e}^{x}) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}\left(\mathrm{e}^{x}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 3y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2x} + \mathrm{e}^{-x} c_2
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
x^2
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1, x, x^2\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\mathrm{e}^{-2x}, \mathrm{e}^{-x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_3 x^2 + A_2 x + A_1
$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
2A_3x^2 + 2A_2x + 6xA_3 + 2A_1 + 3A_2 + 2A_3 = 12x^2
$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 21, A_2 = -18, A_3 = 6]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 6x^2 - 18x + 21$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + e^{-x} c_2\right) + \left(6x^2 - 18x + 21\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-2x} + e^{-x} c_2 + 6x^2 - 18x + 21 \tag{1}$$

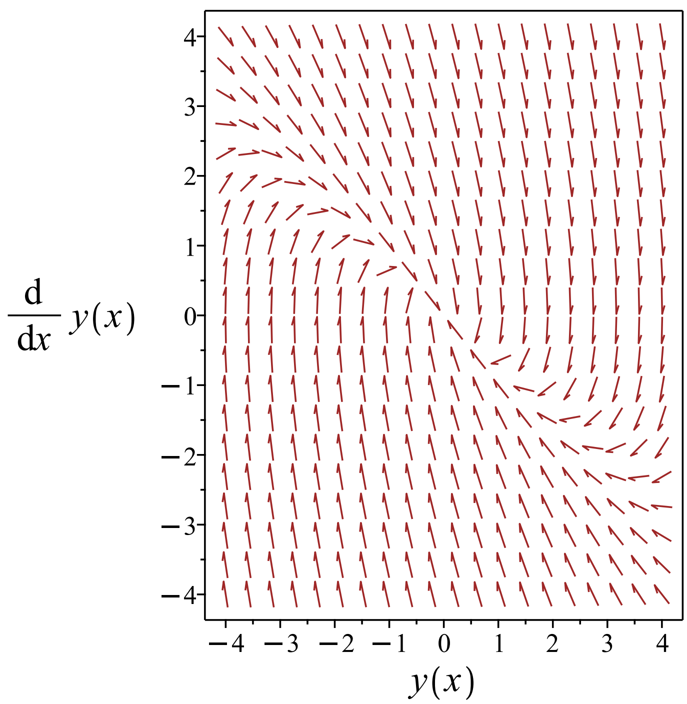

Figure 7: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + e^{-x} c_2 + 6x^2 - 18x + 21$$

Verified OK.

1.3.3 Maple step by step solution

Let's solve

$y'' + 3y' + 2y = 12x^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, -1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2x} + e^{-x} c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 12x^2\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-2x} & e^{-x} \\ -2\,e^{-2x} & -e^{-x} \end{bmatrix}$$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{-3x}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -12\,e^{-2x}\left(\int x^2 e^{2x} dx\right) + 12\,e^{-x}\left(\int x^2 e^{x} dx\right)$

- Compute integrals

  $y_p(x) = 6x^2 - 18x + 21$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-2x} + e^{-x} c_2 + 6x^2 - 18x + 21$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)+3*diff(y(x),x)+2*y(x)=12*x^2,y(x), singsol=all)
```

$$y(x) = -e^{-2x} c_1 + c_2 e^{-x} + 6x^2 - 18x + 21$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 31

```
DSolve[y''[x]+3*y'[x]+2*y[x]==12*x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 6x^2 - 18x + c_1 e^{-2x} + c_2 e^{-x} + 21$$

1.4 problem 4

Internal problem ID [6864]
Internal file name [`OUTPUT/6111_Saturday_September_10_2022_01_23_04_AM_9809890/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 8. Nonhomogeneous Equations: Undetermined Coefficients. Exercises Page 142
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 3y' + 2y = x^2 + 2x + 1}$$

1.4.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 3, C = 2, f(x) = (x + 1)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 3\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{3^2 - (4)\,(1)\,(2)}\\ &= -\frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{3}{2} + \frac{1}{2}\\ \lambda_2 &= -\frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(-1)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 e^{-x} + c_2 e^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-x} + c_2 e^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_3 x^2 + 2A_2 x + 6xA_3 + 2A_1 + 3A_2 + 2A_3 = (x+1)^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{3}{4}, A_2 = -\frac{1}{2}, A_3 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{3}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 e^{-2x}\right) + \left(\frac{1}{2}x^2 - \frac{1}{2}x + \frac{3}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{-2x} + \frac{x^2}{2} - \frac{x}{2} + \frac{3}{4} \tag{1}$$

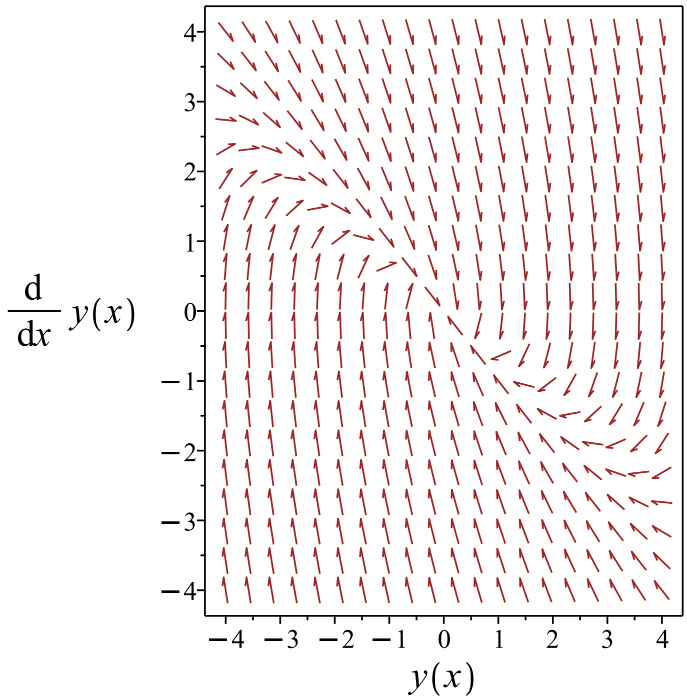

Figure 8: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{-2x} + \frac{x^2}{2} - \frac{x}{2} + \frac{3}{4}$$

Verified OK.

1.4.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 7: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\,dx} \\ &= z_1 e^{-\frac{3x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{3x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-3x}}{(y_1)^2}\,dx \\ &= y_1 (\mathrm{e}^{x}) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x} \left(\mathrm{e}^{x}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 3y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2x} + \mathrm{e}^{-x} c_2
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
x^2 + x + 1
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1, x, x^2\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\mathrm{e}^{-2x}, \mathrm{e}^{-x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_3 x^2 + A_2 x + A_1
$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
2A_3 x^2 + 2A_2 x + 6x A_3 + 2A_1 + 3A_2 + 2A_3 = (x + 1)^2
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{3}{4}, A_2 = -\frac{1}{2}, A_3 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{3}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-2x} + \mathrm{e}^{-x} c_2\right) + \left(\frac{1}{2}x^2 - \frac{1}{2}x + \frac{3}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-2x} + \mathrm{e}^{-x} c_2 + \frac{x^2}{2} - \frac{x}{2} + \frac{3}{4} \tag{1}$$

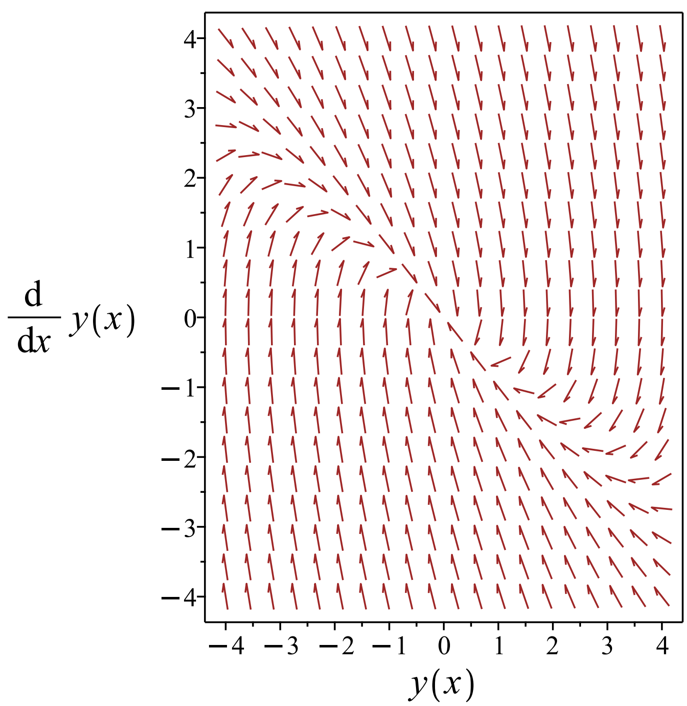

Figure 9: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + e^{-x} c_2 + \frac{x^2}{2} - \frac{x}{2} + \frac{3}{4}$$

Verified OK.

1.4.3 Maple step by step solution

Let's solve

$y'' + 3y' + 2y = (x+1)^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, -1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2x} + e^{-x} c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = (x+1)^2\right]$

  ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(x)\,,y_2(x)) = \begin{bmatrix} e^{-2x} & e^{-x} \\ -2\,e^{-2x} & -e^{-x} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x)\,,y_2(x)) = e^{-3x}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -e^{-2x}\left(\int (x+1)^2\, e^{2x} dx\right) + e^{-x}\left(\int (x+1)^2\, e^{x} dx\right)$

- Compute integrals

  $y_p(x) = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{3}{4}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-2x} + e^{-x} c_2 + \frac{x^2}{2} - \frac{x}{2} + \frac{3}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)+3*diff(y(x),x)+2*y(x)=1+2*x+x^2,y(x), singsol=all)
```

$$y(x) = \frac{3}{4} - \frac{x}{2} + \frac{x^2}{2} - e^{-2x} c_1 + c_2 e^{-x}$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 36

```
DSolve[y''[x]+3*y'[x]+2*y[x]==1+2*x+x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}\left(2x^2 - 2x + 3\right) + c_1 e^{-2x} + c_2 e^{-x}$$

2 CHAPTER 16. Nonlinear equations. Miscellaneous Exercises. Page 340

2.1 problem 1

Internal problem ID [6865]
Internal file name [`OUTPUT/6108_Friday_August_05_2022_02_14_16_AM_9550685/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 16. Nonlinear equations. Miscellaneous Exercises. Page 340
Problem number: 1.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`]]
```

$$\boxed{x^3 y'^2 + y y' x^2 = -4}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{-yx + \sqrt{y^2x^2 - 16x}}{2x^2} \tag{1}$$

$$y' = -\frac{yx + \sqrt{y^2x^2 - 16x}}{2x^2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$\begin{aligned} y' &= \frac{-xy + \sqrt{x^2y^2 - 16x}}{2x^2} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{\left(-xy + \sqrt{x^2y^2 - 16x}\right)(b_3 - a_2)}{2x^2} - \frac{\left(-xy + \sqrt{x^2y^2 - 16x}\right)^2 a_3}{4x^4} - \left(-\frac{-xy + \sqrt{x^2y^2 - 16x}}{x^3} + \frac{-y + \frac{2x\,y^2 - 16}{2\sqrt{x^2y^2 - 16x}}}{2x^2}\right)(xa_2 + ya_3 + a_1) - \frac{\left(-x + \frac{y\,x^2}{\sqrt{x^2y^2 - 16x}}\right)(xb_2 + yb_3 + b_1)}{2x^2} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2x^5yb_2 - 4x^3y^3a_3 - 6b_2x^4\sqrt{x^2y^2 - 16x} + 3\sqrt{x^2y^2 - 16x}\,x^2y^2a_3 + 2x^4yb_1 - 2x^3y^2a_1 - 2\sqrt{x^2y^2 - 16x}\,x^3}{4x^4\sqrt{x^2y^2 - 16x}} = 0$$

Setting the numerator to zero gives

$$-2x^5yb_2 + 4x^3y^3a_3 + 6b_2x^4\sqrt{x^2y^2 - 16x} - 3\sqrt{x^2y^2 - 16x}\,x^2y^2a_3 - 2x^4yb_1 + 2x^3y^2a_1 + 2\sqrt{x^2y^2 - 16x}\,x^3b_1 - 2\sqrt{x^2y^2 - 16x}\,x^2ya_1 - \left(x^2y^2 - 16x\right)^{\frac{3}{2}} a_3 - 16x^3a_2 - 32x^3b_3 - 80x^2ya_3 - 48x^2a_1 = 0 \tag{6E}$$

Simplifying the above gives

$$-2x^5yb_2 - 2x^4y^2a_2 - 2x^4y^2b_3 - 2x^3y^3a_3 + 6b_2x^4\sqrt{x^2y^2 - 16x} - 3\sqrt{x^2y^2 - 16x}\,x^2y^2a_3 - 2x^4yb_1 - 2x^3y^2a_1 + 2\left(x^2y^2 - 16x\right)x^2a_2 + 2\left(x^2y^2 - 16x\right)x^2b_3 + 6\left(x^2y^2 - 16x\right)xya_3 + 2\sqrt{x^2y^2 - 16x}\,x^3b_1 - 2\sqrt{x^2y^2 - 16x}\,x^2ya_1 - \left(x^2y^2 - 16x\right)^{\frac{3}{2}} a_3 + 4\left(x^2y^2 - 16x\right)xa_1 + 16x^3a_2 + 16x^2ya_3 + 16x^2a_1 = 0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$2x\left(-x^4yb_2 + 2x^2y^3a_3 + 3\sqrt{x\,(x\,y^2-16)}\,x^3b_2 - 2\sqrt{x\,(x\,y^2-16)}\,x\,y^2a_3 - x^3yb_1 + x^2y^2a_1 + \sqrt{x\,(x\,y^2-16)}\,x^2b_1 - \sqrt{x\,(x\,y^2-16)}\,xya_1 - 8x^2a_2 - 16x^2b_3 - 40xya_3 + 8\sqrt{x\,(x\,y^2-16)}\,a_3 - 24xa_1\right) = 0$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{x\,(x\,y^2-16)}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{x\,(x\,y^2-16)} = v_3\right\}$$

The above PDE (6E) now becomes

$$2v_1\left(2v_1^2v_2^3a_3 - v_1^4v_2b_2 + v_1^2v_2^2a_1 - 2v_3v_1v_2^2a_3 - v_1^3v_2b_1 + 3v_3v_1^3b_2 - v_3v_1v_2a_1 + v_3v_1^2b_1 - 8v_1^2a_2 - 40v_1v_2a_3 - 16v_1^2b_3 - 24v_1a_1 + 8v_3a_3\right) = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$-2b_2v_2v_1^5 - 2b_1v_2v_1^4 + 6b_2v_3v_1^4 + 4a_3v_2^3v_1^3 + 2a_1v_2^2v_1^3 + 2b_1v_3v_1^3 + (-16a_2 - 32b_3)\,v_1^3 - 4a_3v_2^2v_3v_1^2 - 2a_1v_2v_3v_1^2 - 80a_3v_2v_1^2 - 48a_1v_1^2 + 16v_3a_3v_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-48a_1 &= 0 \\
-2a_1 &= 0 \\
2a_1 &= 0 \\
-80a_3 &= 0 \\
-4a_3 &= 0 \\
4a_3 &= 0 \\
16a_3 &= 0 \\
-2b_1 &= 0 \\
2b_1 &= 0 \\
-2b_2 &= 0 \\
6b_2 &= 0 \\
-16a_2 - 32b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -2b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -2x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{-xy + \sqrt{x^2y^2 - 16x}}{2x^2}\right)(-2x) \\
&= \frac{\sqrt{x^2y^2 - 16x}}{x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{\sqrt{x^2y^2-16x}}{x}} dy \end{aligned}$$

Which results in

$$S = \frac{x \ln\left(\frac{y\,x^2}{\sqrt{x^2}} + \sqrt{x^2y^2 - 16x}\right)}{\sqrt{x^2}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-xy + \sqrt{x^2y^2 - 16x}}{2x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y\sqrt{x}\,\sqrt{x\,y^2 - 16} + x\,y^2 - 8}{\sqrt{x}\,\sqrt{x\,y^2 - 16}\,\left(xy + \sqrt{x}\,\sqrt{x\,y^2 - 16}\right)} \\ S_y &= \frac{x\left(y\sqrt{x} + \sqrt{x\,y^2 - 16}\right)}{\sqrt{x\,y^2 - 16}\,\left(xy + \sqrt{x}\,\sqrt{x\,y^2 - 16}\right)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\left(xy + \sqrt{x}\,\sqrt{x\,y^2 - 16}\right)\sqrt{x\,(x\,y^2 - 16)} + x^2y^2 + x^{\frac{3}{2}}\sqrt{x\,y^2 - 16}\,y - 16x}{\sqrt{x\,y^2 - 16}\,x^{\frac{3}{2}}\left(2xy + 2\sqrt{x}\,\sqrt{x\,y^2 - 16}\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln\left(yx + \sqrt{x}\,\sqrt{xy^2 - 16}\right) = \ln(x) + c_1$$

Which simplifies to

$$\ln\left(yx + \sqrt{x}\,\sqrt{xy^2 - 16}\right) = \ln(x) + c_1$$

Which gives

$$y = \frac{(e^{2c_1}x + 16)\,e^{-c_1}}{2x}$$

Summary

The solution(s) found are the following

$$y = \frac{(e^{2c_1}x + 16)\,e^{-c_1}}{2x} \tag{1}$$

Verification of solutions

$$y = \frac{(e^{2c_1}x + 16)\,e^{-c_1}}{2x}$$

Verified OK.

Solving equation (2)

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{xy+\sqrt{x^2y^2-16x}}{2x^2} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2 - \frac{\left(xy+\sqrt{x^2y^2-16x}\right)(b_3-a_2)}{2x^2} - \frac{\left(xy+\sqrt{x^2y^2-16x}\right)^2 a_3}{4x^4} \\
& - \left(-\frac{y+\frac{2x\,y^2-16}{2\sqrt{x^2y^2-16x}}}{2x^2} + \frac{xy+\sqrt{x^2y^2-16x}}{x^3}\right)(x a_2 + y a_3 + a_1) \\
& + \frac{\left(x+\frac{y\,x^2}{\sqrt{x^2y^2-16x}}\right)(x b_2 + y b_3 + b_1)}{2x^2} = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{-2x^5 y b_2 + 4x^3y^3a_3 - 6b_2x^4\sqrt{x^2y^2-16x} + 3\sqrt{x^2y^2-16x}\,x^2y^2a_3 - 2x^4yb_1 + 2x^3y^2a_1 - 2\sqrt{x^2y^2-16x}\,x}{4x^4\sqrt{x^2y^2-16x}}
$$

$= 0$

Setting the numerator to zero gives

$$
\begin{aligned}
& 2x^5yb_2 - 4x^3y^3a_3 + 6b_2x^4\sqrt{x^2y^2-16x} - 3\sqrt{x^2y^2-16x}\,x^2y^2a_3 \\
& + 2x^4yb_1 - 2x^3y^2a_1 + 2\sqrt{x^2y^2-16x}\,x^3b_1 - 2\sqrt{x^2y^2-16x}\,x^2ya_1 \\
& - \left(x^2y^2-16x\right)^{\frac{3}{2}} a_3 + 16x^3a_2 + 32x^3b_3 + 80x^2ya_3 + 48x^2a_1 = 0
\end{aligned} \tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
&2x^5yb_2+2x^4y^2a_2+2x^4y^2b_3+2x^3y^3a_3+6b_2x^4\sqrt{x^2y^2-16x}\\
&-3\sqrt{x^2y^2-16x}\,x^2y^2a_3+2x^4yb_1+2x^3y^2a_1\\
&-2\left(x^2y^2-16x\right)x^2a_2-2\left(x^2y^2-16x\right)x^2b_3-6\left(x^2y^2-16x\right)xya_3\\
&+2\sqrt{x^2y^2-16x}\,x^3b_1-2\sqrt{x^2y^2-16x}\,x^2ya_1-\left(x^2y^2-16x\right)^{\frac{3}{2}}a_3\\
&-4\left(x^2y^2-16x\right)xa_1-16x^3a_2-16x^2ya_3-16x^2a_1=0
\end{aligned}
\tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&2x\Big(x^4yb_2-2x^2y^3a_3+3\sqrt{x\,(x\,y^2-16)}\,x^3b_2-2\sqrt{x\,(x\,y^2-16)}\,x\,y^2a_3\\
&+x^3yb_1-x^2y^2a_1+\sqrt{x\,(x\,y^2-16)}\,x^2b_1-\sqrt{x\,(x\,y^2-16)}\,xya_1\\
&+8x^2a_2+16x^2b_3+40xya_3+8\sqrt{x\,(x\,y^2-16)}\,a_3+24xa_1\Big)=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$
\left\{x,y,\sqrt{x\,(x\,y^2-16)}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$
\left\{x=v_1,y=v_2,\sqrt{x\,(x\,y^2-16)}=v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&2v_1\left(-2v_1^2v_2^3a_3+v_1^4v_2b_2-v_1^2v_2^2a_1-2v_3v_1v_2^2a_3+v_1^3v_2b_1+3v_3v_1^3b_2\right.\\
&\left.-v_3v_1v_2a_1+v_3v_1^2b_1+8v_1^2a_2+40v_1v_2a_3+16v_1^2b_3+24v_1a_1+8v_3a_3\right)=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1,v_2,v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&2b_2v_2v_1^5+2b_1v_2v_1^4+6b_2v_3v_1^4-4a_3v_2^3v_1^3-2a_1v_2^2v_1^3+2b_1v_3v_1^3+(16a_2+32b_3)\,v_1^3\\
&-4a_3v_2^2v_3v_1^2-2a_1v_2v_3v_1^2+80a_3v_2v_1^2+48a_1v_1^2+16v_3a_3v_1=0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-2a_1 &= 0 \\
48a_1 &= 0 \\
-4a_3 &= 0 \\
16a_3 &= 0 \\
80a_3 &= 0 \\
2b_1 &= 0 \\
2b_2 &= 0 \\
6b_2 &= 0 \\
16a_2 + 32b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -2b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -2x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(-\frac{xy + \sqrt{x^2y^2 - 16x}}{2x^2}\right)(-2x) \\
&= -\frac{\sqrt{x^2y^2 - 16x}}{x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\frac{\sqrt{x^2y^2-16x}}{x}} dy \end{aligned}$$

Which results in

$$S = -\frac{x \ln\left(\frac{y\,x^2}{\sqrt{x^2}} + \sqrt{x^2y^2 - 16x}\right)}{\sqrt{x^2}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{xy + \sqrt{x^2y^2 - 16x}}{2x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y\sqrt{x}\,\sqrt{x\,y^2-16} + x\,y^2 - 8}{\sqrt{x}\,\sqrt{x\,y^2-16}\,\left(xy + \sqrt{x}\,\sqrt{x\,y^2-16}\right)} \\ S_y &= \frac{\left(-y\sqrt{x} - \sqrt{x\,y^2-16}\right)x}{\sqrt{x\,y^2-16}\,\left(xy + \sqrt{x}\,\sqrt{x\,y^2-16}\right)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{\left(-xy-\sqrt{x}\,\sqrt{x\,y^2-16}\right)\sqrt{x\,(x\,y^2-16)}+x^2y^2+x^{\frac{3}{2}}\sqrt{x\,y^2-16}\,y-16x}{\sqrt{x\,y^2-16}\,x^{\frac{3}{2}}\left(2xy+2\sqrt{x}\,\sqrt{x\,y^2-16}\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln\left(yx+\sqrt{x}\,\sqrt{xy^2-16}\right) = c_1$$

Which simplifies to

$$-\ln\left(yx+\sqrt{x}\,\sqrt{xy^2-16}\right) = c_1$$

Which gives

$$y = \frac{(16\,\mathrm{e}^{2c_1}x+1)\,\mathrm{e}^{-c_1}}{2x}$$

Summary

The solution(s) found are the following

$$y = \frac{(16\,\mathrm{e}^{2c_1}x+1)\,\mathrm{e}^{-c_1}}{2x} \tag{1}$$

Verification of solutions

$$y = \frac{(16\,\mathrm{e}^{2c_1}x+1)\,\mathrm{e}^{-c_1}}{2x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      1st order, trying the canonical coordinates of the invariance group
      <- 1st order, canonical coordinates successful
      <- homogeneous successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      1st order, trying the canonical coordinates of the invariance group
      <- 1st order, canonical coordinates successful
      <- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.156 (sec). Leaf size: 53

```
dsolve(x^3*diff(y(x),x)^2+x^2*y(x)*diff(y(x),x)+4=0,y(x), singsol=all)
```

$$
\begin{aligned}
y(x) &= -\frac{4}{\sqrt{x}} \\
y(x) &= \frac{4}{\sqrt{x}} \\
y(x) &= \frac{c_1^2 x + 16}{2 x c_1} \\
y(x) &= \frac{c_1^2 + 16 x}{2 x c_1}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.558 (sec). Leaf size: 77

```
DSolve[x^3*(y'[x])^2+x^2*y[x]*y'[x]+4==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to -\frac{e^{-\frac{c_1}{2}}\left(x + 64 e^{c_1}\right)}{4x} \\
y(x) &\to \frac{e^{-\frac{c_1}{2}}\left(x + 64 e^{c_1}\right)}{4x} \\
y(x) &\to -\frac{4}{\sqrt{x}} \\
y(x) &\to \frac{4}{\sqrt{x}}
\end{aligned}
$$

2.2 problem 2

Internal problem ID [6866]
Internal file name [`OUTPUT/6109_Friday_August_05_2022_02_19_54_AM_48557998/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 16. Nonlinear equations. Miscellaneous Exercises. Page 340
Problem number: 2.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"exact", "linear", "quadrature", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{6xy'^2-(3x+2y)\,y'+y=0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y'=\frac{1}{2} \tag{1}$$
$$y'=\frac{y}{3x} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{1}{2}\,\mathrm{d}x \\ &= \frac{x}{2}+c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y=\frac{x}{2}+c_1 \tag{1}$$

Verification of solutions

$$y = \frac{x}{2} + c_1$$

Verified OK.

Solving equation (2)

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{y}{3x} \end{aligned}$$

Where $f(x) = \frac{1}{3x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y}\,dy &= \frac{1}{3x}\,dx \\ \int \frac{1}{y}\,dy &= \int \frac{1}{3x}\,dx \\ \ln(y) &= \frac{\ln(x)}{3} + c_2 \\ y &= e^{\frac{\ln(x)}{3} + c_2} \\ &= c_2 x^{\frac{1}{3}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2 x^{\frac{1}{3}} \tag{1}$$

Verification of solutions

$$y = c_2 x^{\frac{1}{3}}$$

Verified OK.

2.2.1 Maple step by step solution

Let's solve

$6xy'^2 - (3x + 2y)\,y' + y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \left(6xy'^2 - (3x+2y)\,y' + y\right) dx = \int 0 dx + c_1$
- Cannot compute integral

  $\int \left(6xy'^2 - (3x+2y)\,y' + y\right) dx = c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(6*x*diff(y(x),x)^2-(3*x+2*y(x))*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^{\frac{1}{3}}$$
$$y(x) = \frac{x}{2} + c_1$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 30

```
DSolve[6*x*(y'[x])^2-(3*x+2*y[x])*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \sqrt[3]{x}$$
$$y(x) \to \frac{x}{2} + c_1$$
$$y(x) \to 0$$

2.3 problem 3

Internal problem ID [6867]
Internal file name [`OUTPUT/6110_Friday_August_05_2022_02_19_55_AM_18144345/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 16. Nonlinear equations. Miscellaneous Exercises. Page 340
Problem number: 3.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries]]
```

$$\boxed{9y'^2 + 3xy^4y' + y^5 = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\left(-\frac{xy^2}{2} + \frac{\sqrt{x^2y^4-4y}}{2}\right)y^2}{3} \tag{1}$$

$$y' = \frac{\left(-\frac{xy^2}{2} - \frac{\sqrt{x^2y^4-4y}}{2}\right)y^2}{3} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = -\frac{\left(x\,y^2 - \sqrt{y^4x^2-4y}\right)y^2}{6}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{\left(x\,y^2 - \sqrt{y^4x^2 - 4y}\right) y^2 (b_3 - a_2)}{6} - \frac{\left(x\,y^2 - \sqrt{y^4x^2 - 4y}\right)^2 y^4 a_3}{36} + \frac{\left(y^2 - \frac{y^4 x}{\sqrt{y^4x^2-4y}}\right) y^2 (x a_2 + y a_3 + a_1)}{6} - \left(-\frac{\left(2xy - \frac{4x^2y^3-4}{2\sqrt{y^4x^2-4y}}\right) y^2}{6} - \frac{\left(x\,y^2 - \sqrt{y^4x^2 - 4y}\right) y}{3}\right) (x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-2x^3y^{10}a_3 + \sqrt{y^4x^2 - 4y}\, x^2y^8a_3 + 24x^3y^5b_2 + 12x^2y^6a_2 + 18x^2y^6b_3 + 14x\,y^7a_3 + \left(y^4x^2 - 4y\right)^{\frac{3}{2}} y^4a_3 + 2}{}$$

$$= 0$$

Setting the numerator to zero gives

$$2x^3y^{10}a_3 - \sqrt{y^4x^2 - 4y}\, x^2y^8a_3 - 24x^3y^5b_2 - 12x^2y^6a_2 - 18x^2y^6b_3 - 14x\,y^7a_3 - \left(y^4x^2 - 4y\right)^{\frac{3}{2}} y^4a_3 - 24x^2y^5b_1 - 6x\,y^6a_1 + 24\sqrt{y^4x^2 - 4y}\, x^2y^3b_2 + 12\sqrt{y^4x^2 - 4y}\, x\,y^4a_2 + 18\sqrt{y^4x^2 - 4y}\, x\,y^4b_3 + 6\sqrt{y^4x^2 - 4y}\, y^5a_3 + 24\sqrt{y^4x^2 - 4y}\, x\,y^3b_1 + 6\sqrt{y^4x^2 - 4y}\, y^4a_1 + 60x\,y^2b_2 + 24y^3a_2 + 36y^3b_3 + 60y^2b_1 + 36b_2\sqrt{y^4x^2 - 4y} = 0 \tag{6E}$$

Simplifying the above gives

$$
\begin{aligned}
&-\sqrt{y^4x^2-4y}\,x^2y^8a_3+2\left(y^4x^2-4y\right)x\,y^6a_3-12x^3y^5b_2-6x^2y^6a_2\\
&-12x^2y^6b_3-6x\,y^7a_3-\left(y^4x^2-4y\right)^{\frac{3}{2}}y^4a_3-12x^2y^5b_1-6x\,y^6a_1\\
&+24\sqrt{y^4x^2-4y}\,x^2y^3b_2+12\sqrt{y^4x^2-4y}\,x\,y^4a_2+18\sqrt{y^4x^2-4y}\,x\,y^4b_3\\
&+6\sqrt{y^4x^2-4y}\,y^5a_3+24\sqrt{y^4x^2-4y}\,x\,y^3b_1+6\sqrt{y^4x^2-4y}\,y^4a_1\\
&-12\left(y^4x^2-4y\right)xyb_2-6\left(y^4x^2-4y\right)y^2a_2-6\left(y^4x^2-4y\right)y^2b_3\\
&-12\left(y^4x^2-4y\right)yb_1+12x\,y^2b_2+12y^3b_3+12y^2b_1+36b_2\sqrt{y^4x^2-4y}=0
\end{aligned}
\tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&2x^3y^{10}a_3-2\sqrt{y\left(x^2y^3-4\right)}\,x^2y^8a_3-24x^3y^5b_2-12x^2y^6a_2-18x^2y^6b_3\\
&-14x\,y^7a_3-24x^2y^5b_1-6x\,y^6a_1+24\sqrt{y\left(x^2y^3-4\right)}\,x^2y^3b_2\\
&+12\sqrt{y\left(x^2y^3-4\right)}\,x\,y^4a_2+18\sqrt{y\left(x^2y^3-4\right)}\,x\,y^4b_3+10\sqrt{y\left(x^2y^3-4\right)}\,y^5a_3\\
&+24\sqrt{y\left(x^2y^3-4\right)}\,x\,y^3b_1+6\sqrt{y\left(x^2y^3-4\right)}\,y^4a_1+60x\,y^2b_2\\
&+24y^3a_2+36y^3b_3+60y^2b_1+36b_2\sqrt{y\left(x^2y^3-4\right)}=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{y\left(x^2y^3-4\right)}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x=v_1, y=v_2, \sqrt{y\left(x^2y^3-4\right)}=v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&2v_1^3v_2^{10}a_3-2v_3v_1^2v_2^8a_3-12v_1^2v_2^6a_2-14v_1v_2^7a_3-24v_1^3v_2^5b_2-18v_1^2v_2^6b_3\\
&-6v_1v_2^6a_1-24v_1^2v_2^5b_1+12v_3v_1v_2^4a_2+10v_3v_2^5a_3+24v_3v_1^2v_2^3b_2+18v_3v_1v_2^4b_3\\
&+6v_3v_2^4a_1+24v_3v_1v_2^3b_1+24v_2^3a_2+60v_1v_2^2b_2+36v_2^3b_3+60v_2^2b_1+36b_2v_3=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&2v_1^3v_2^{10}a_3 - 24v_1^3v_2^5b_2 - 2v_3v_1^2v_2^8a_3 + (-12a_2 - 18b_3)\, v_1^2v_2^6 - 24v_1^2v_2^5b_1 \\
&+ 24v_3v_1^2v_2^3b_2 - 14v_1v_2^7a_3 - 6v_1v_2^6a_1 + (12a_2 + 18b_3)\, v_1v_2^4v_3 + 24v_3v_1v_2^3b_1 \\
&+ 60v_1v_2^2b_2 + 10v_3v_2^5a_3 + 6v_3v_2^4a_1 + (24a_2 + 36b_3)\, v_2^3 + 60v_2^2b_1 + 36b_2v_3 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-6a_1 &= 0 \\
6a_1 &= 0 \\
-14a_3 &= 0 \\
-2a_3 &= 0 \\
2a_3 &= 0 \\
10a_3 &= 0 \\
-24b_1 &= 0 \\
24b_1 &= 0 \\
60b_1 &= 0 \\
-24b_2 &= 0 \\
24b_2 &= 0 \\
36b_2 &= 0 \\
60b_2 &= 0 \\
-12a_2 - 18b_3 &= 0 \\
12a_2 + 18b_3 &= 0 \\
24a_2 + 36b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -\frac{3b_3}{2} \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = -\frac{3x}{2}$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{\left(x\,y^{2}-\sqrt{y^{4}x^{2}-4y}\right)y^{2}}{6}\right)\left(-\frac{3x}{2}\right) \\
&= -\frac{y^{4}x^{2}}{4}+\frac{x\,y^{2}\sqrt{y^{4}x^{2}-4y}}{4}+y \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\frac{y^{4}x^{2}}{4}+\frac{x\,y^{2}\sqrt{y^{4}x^{2}-4y}}{4}+y} dy
\end{aligned}$$

Which results in

$$S = \ln(y) + \frac{\ln\left(\frac{x\,y^{2}+\sqrt{y(x^{2}y^{3}-4)}}{y^{2}}\right)}{3} - \frac{\ln\left(\frac{-x\,y^{2}+\sqrt{y(x^{2}y^{3}-4)}}{y^{2}}\right)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{\left(x\,y^2 - \sqrt{y^4x^2 - 4y}\right)y^2}{6}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1\\
R_y &= 0\\
S_x &= \frac{2y^{\frac{3}{2}}}{3\sqrt{x^2y^3 - 4}}\\
S_y &= -\frac{4}{\sqrt{y}\,\sqrt{x^2y^3 - 4}\,\left(-x\,y^2 + \sqrt{y}\,\sqrt{x^2y^3 - 4}\right)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{2y^{\frac{3}{2}}\left(\sqrt{y}\,\sqrt{x^2y^3 - 4} - \sqrt{y\,(x^2y^3 - 4)}\right)}{\sqrt{x^2y^3 - 4}\,\left(3x\,y^2 - 3\sqrt{y}\,\sqrt{x^2y^3 - 4}\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(y) + \frac{\ln\left(xy^2 + \sqrt{y}\,\sqrt{y^3x^2 - 4}\right)}{3} - \frac{\ln\left(-xy^2 + \sqrt{y}\,\sqrt{y^3x^2 - 4}\right)}{3} = c_1$$

Which simplifies to

$$\ln(y) + \frac{\ln\left(x y^2 + \sqrt{y}\,\sqrt{y^3 x^2 - 4}\right)}{3} - \frac{\ln\left(-x y^2 + \sqrt{y}\,\sqrt{y^3 x^2 - 4}\right)}{3} = c_1$$

Summary

The solution(s) found are the following

$$\ln(y) + \frac{\ln\left(x y^2 + \sqrt{y}\,\sqrt{y^3 x^2 - 4}\right)}{3} - \frac{\ln\left(-x y^2 + \sqrt{y}\,\sqrt{y^3 x^2 - 4}\right)}{3} = c_1 \tag{1}$$

Verification of solutions

$$\ln(y) + \frac{\ln\left(x y^2 + \sqrt{y}\,\sqrt{y^3 x^2 - 4}\right)}{3} - \frac{\ln\left(-x y^2 + \sqrt{y}\,\sqrt{y^3 x^2 - 4}\right)}{3} = c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$\begin{aligned} y' &= -\frac{y^2\left(x\,y^2 + \sqrt{y^4 x^2 - 4y}\right)}{6} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{y^2\left(x\,y^2+\sqrt{y^4x^2-4y}\right)(b_3-a_2)}{6} - \frac{y^4\left(x\,y^2+\sqrt{y^4x^2-4y}\right)^2 a_3}{36} \\ &+ \frac{y^2\left(y^2+\frac{y^4x}{\sqrt{y^4x^2-4y}}\right)(xa_2+ya_3+a_1)}{6} \\ &- \left(-\frac{y\left(x\,y^2+\sqrt{y^4x^2-4y}\right)}{3} - \frac{y^2\left(2xy+\frac{4x^2y^3-4}{2\sqrt{y^4x^2-4y}}\right)}{6}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2x^3y^{10}a_3+\sqrt{y^4x^2-4y}\,x^2y^8a_3-24x^3y^5b_2-12x^2y^6a_2-18x^2y^6b_3-14x\,y^7a_3+\left(y^4x^2-4y\right)^{\frac{3}{2}}y^4a_3-24}{}$$

$$=0$$

Setting the numerator to zero gives

$$\begin{aligned} &-2x^3y^{10}a_3-\sqrt{y^4x^2-4y}\,x^2y^8a_3+24x^3y^5b_2+12x^2y^6a_2 \\ &+18x^2y^6b_3+14x\,y^7a_3-\left(y^4x^2-4y\right)^{\frac{3}{2}}y^4a_3+24x^2y^5b_1+6x\,y^6a_1 \\ &+24\sqrt{y^4x^2-4y}\,x^2y^3b_2+12\sqrt{y^4x^2-4y}\,x\,y^4a_2+18\sqrt{y^4x^2-4y}\,x\,y^4b_3 \\ &+6\sqrt{y^4x^2-4y}\,y^5a_3+24\sqrt{y^4x^2-4y}\,x\,y^3b_1+6\sqrt{y^4x^2-4y}\,y^4a_1 \\ &-60x\,y^2b_2-24y^3a_2-36y^3b_3-60y^2b_1+36b_2\sqrt{y^4x^2-4y}=0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &-\sqrt{y^4x^2-4y}\,x^2y^8a_3-2\left(y^4x^2-4y\right)x\,y^6a_3+12x^3y^5b_2+6x^2y^6a_2 \\ &+12x^2y^6b_3+6x\,y^7a_3-\left(y^4x^2-4y\right)^{\frac{3}{2}}y^4a_3+12x^2y^5b_1+6x\,y^6a_1 \\ &+24\sqrt{y^4x^2-4y}\,x^2y^3b_2+12\sqrt{y^4x^2-4y}\,x\,y^4a_2+18\sqrt{y^4x^2-4y}\,x\,y^4b_3 \\ &+6\sqrt{y^4x^2-4y}\,y^5a_3+24\sqrt{y^4x^2-4y}\,x\,y^3b_1+6\sqrt{y^4x^2-4y}\,y^4a_1 \\ &+12\left(y^4x^2-4y\right)xyb_2+6\left(y^4x^2-4y\right)y^2a_2+6\left(y^4x^2-4y\right)y^2b_3 \\ &+12\left(y^4x^2-4y\right)yb_1-12x\,y^2b_2-12y^3b_3-12y^2b_1+36b_2\sqrt{y^4x^2-4y}=0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&-2x^{3}y^{10}a_{3}-2\sqrt{y\left(x^{2}y^{3}-4\right)}\,x^{2}y^{8}a_{3}+24x^{3}y^{5}b_{2}+12x^{2}y^{6}a_{2}\\
&+18x^{2}y^{6}b_{3}+14x\,y^{7}a_{3}+24x^{2}y^{5}b_{1}+6x\,y^{6}a_{1}+24\sqrt{y\left(x^{2}y^{3}-4\right)}\,x^{2}y^{3}b_{2}\\
&+12\sqrt{y\left(x^{2}y^{3}-4\right)}\,x\,y^{4}a_{2}+18\sqrt{y\left(x^{2}y^{3}-4\right)}\,x\,y^{4}b_{3}+10\sqrt{y\left(x^{2}y^{3}-4\right)}\,y^{5}a_{3}\\
&+24\sqrt{y\left(x^{2}y^{3}-4\right)}\,x\,y^{3}b_{1}+6\sqrt{y\left(x^{2}y^{3}-4\right)}\,y^{4}a_{1}-60x\,y^{2}b_{2}\\
&-24y^{3}a_{2}-36y^{3}b_{3}-60y^{2}b_{1}+36b_{2}\sqrt{y\left(x^{2}y^{3}-4\right)}=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$
\left\{x,y,\sqrt{y\left(x^{2}y^{3}-4\right)}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$
\left\{x=v_{1},y=v_{2},\sqrt{y\left(x^{2}y^{3}-4\right)}=v_{3}\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-2v_{1}^{3}v_{2}^{10}a_{3}-2v_{3}v_{1}^{2}v_{2}^{8}a_{3}+12v_{1}^{2}v_{2}^{6}a_{2}+14v_{1}v_{2}^{7}a_{3}+24v_{1}^{3}v_{2}^{5}b_{2}+18v_{1}^{2}v_{2}^{6}b_{3}\\
&+6v_{1}v_{2}^{6}a_{1}+24v_{1}^{2}v_{2}^{5}b_{1}+12v_{3}v_{1}v_{2}^{4}a_{2}+10v_{3}v_{2}^{5}a_{3}+24v_{3}v_{1}^{2}v_{2}^{3}b_{2}+18v_{3}v_{1}v_{2}^{4}b_{3}\\
&+6v_{3}v_{2}^{4}a_{1}+24v_{3}v_{1}v_{2}^{3}b_{1}-24v_{2}^{3}a_{2}-60v_{1}v_{2}^{2}b_{2}-36v_{2}^{3}b_{3}-60v_{2}^{2}b_{1}+36b_{2}v_{3}=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_{1},v_{2},v_{3}\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&-2v_{1}^{3}v_{2}^{10}a_{3}+24v_{1}^{3}v_{2}^{5}b_{2}-2v_{3}v_{1}^{2}v_{2}^{8}a_{3}+\left(12a_{2}+18b_{3}\right)v_{1}^{2}v_{2}^{6}+24v_{1}^{2}v_{2}^{5}b_{1}\\
&+24v_{3}v_{1}^{2}v_{2}^{3}b_{2}+14v_{1}v_{2}^{7}a_{3}+6v_{1}v_{2}^{6}a_{1}+\left(12a_{2}+18b_{3}\right)v_{1}v_{2}^{4}v_{3}+24v_{3}v_{1}v_{2}^{3}b_{1}\\
&-60v_{1}v_{2}^{2}b_{2}+10v_{3}v_{2}^{5}a_{3}+6v_{3}v_{2}^{4}a_{1}+\left(-24a_{2}-36b_{3}\right)v_{2}^{3}-60v_{2}^{2}b_{1}+36b_{2}v_{3}=0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
6a_1 &= 0 \\
-2a_3 &= 0 \\
10a_3 &= 0 \\
14a_3 &= 0 \\
-60b_1 &= 0 \\
24b_1 &= 0 \\
-60b_2 &= 0 \\
24b_2 &= 0 \\
36b_2 &= 0 \\
-24a_2 - 36b_3 &= 0 \\
12a_2 + 18b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -\frac{3b_3}{2} \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -\frac{3x}{2} \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(-\frac{y^2\left(x\,y^2 + \sqrt{y^4x^2 - 4y}\right)}{6}\right)\left(-\frac{3x}{2}\right) \\
&= -\frac{y^4x^2}{4} - \frac{x\,y^2\sqrt{y^4x^2 - 4y}}{4} + y \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\frac{y^4 x^2}{4} - \frac{x\,y^2 \sqrt{y^4 x^2 - 4y}}{4} + y} dy \end{aligned}$$

Which results in

$$S = \ln(y) - \frac{\ln\left(\frac{x\,y^2 + \sqrt{y(x^2 y^3 - 4)}}{y^2}\right)}{3} + \frac{\ln\left(\frac{-x\,y^2 + \sqrt{y(x^2 y^3 - 4)}}{y^2}\right)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y^2 \left(x\,y^2 + \sqrt{y^4 x^2 - 4y}\right)}{6}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2y^{\frac{3}{2}}}{3\sqrt{x^2 y^3 - 4}} \\ S_y &= -\frac{4}{\left(x\,y^2 + \sqrt{y}\,\sqrt{x^2 y^3 - 4}\right) \sqrt{x^2 y^3 - 4}\,\sqrt{y}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{2y^{\frac{3}{2}}\left(\sqrt{y}\,\sqrt{x^2y^3-4}-\sqrt{y\,(x^2y^3-4)}\right)}{\sqrt{x^2y^3-4}\,\left(3x\,y^2+3\sqrt{y}\,\sqrt{x^2y^3-4}\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(y) - \frac{\ln\left(xy^2+\sqrt{y}\,\sqrt{y^3x^2-4}\right)}{3} + \frac{\ln\left(-xy^2+\sqrt{y}\,\sqrt{y^3x^2-4}\right)}{3} = c_1$$

Which simplifies to

$$\ln(y) - \frac{\ln\left(xy^2+\sqrt{y}\,\sqrt{y^3x^2-4}\right)}{3} + \frac{\ln\left(-xy^2+\sqrt{y}\,\sqrt{y^3x^2-4}\right)}{3} = c_1$$

Summary

The solution(s) found are the following

$$\ln(y) - \frac{\ln\left(xy^2+\sqrt{y}\,\sqrt{y^3x^2-4}\right)}{3} + \frac{\ln\left(-xy^2+\sqrt{y}\,\sqrt{y^3x^2-4}\right)}{3} = c_1 \tag{1}$$

Verification of solutions

$$\ln(y) - \frac{\ln\left(xy^2+\sqrt{y}\,\sqrt{y^3x^2-4}\right)}{3} + \frac{\ln\left(-xy^2+\sqrt{y}\,\sqrt{y^3x^2-4}\right)}{3} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
   -> Solving 1st order ODE of high degree, Lie methods, 1st trial
   `, `-> Computing symmetries using: way = 3`[-3/2*x, y]
```

✓ Solution by Maple

Time used: 0.204 (sec). Leaf size: 102

```
dsolve(9*diff(y(x),x)^2+3*x*y(x)^4*diff(y(x),x)+y(x)^5=0,y(x), singsol=all)
```

$$y(x) = \frac{2^{\frac{2}{3}}}{x^{\frac{2}{3}}}$$
$$y(x) = -\frac{2^{\frac{2}{3}}\left(1+i\sqrt{3}\right)}{2x^{\frac{2}{3}}}$$
$$y(x) = \frac{2^{\frac{2}{3}}\left(i\sqrt{3}-1\right)}{2x^{\frac{2}{3}}}$$
$$y(x) = 0$$
$$y(x) = \frac{\text{RootOf}\left(-2\ln(x)+3\left(\int^{\_Z}\frac{\_a^3+\sqrt{\_a^3(\_a^3-4)}-4}{\_a(\_a^3-4)}d\_a\right)+2c_1\right)}{x^{\frac{2}{3}}}$$

✓ Solution by Mathematica

Time used: 1.025 (sec). Leaf size: 212

```
DSolve[9*(y'[x])^2+3*x*y[x]^4*y'[x]+y[x]^5==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[-\frac{\sqrt{x^2y(x)^3-4}y(x)^{5/2}\text{arctanh}\left(\frac{xy(x)^{3/2}}{\sqrt{x^2y(x)^3-4}}\right)}{\sqrt{y(x)^5\left(x^2y(x)^3-4\right)}}-\frac{3}{2}\log(y(x))=c_1,y(x)\right]$$

$$\text{Solve}\left[\frac{y(x)^{5/2}\sqrt{x^2y(x)^3-4}\text{arctanh}\left(\frac{xy(x)^{3/2}}{\sqrt{x^2y(x)^3-4}}\right)}{\sqrt{y(x)^5\left(x^2y(x)^3-4\right)}}-\frac{3}{2}\log(y(x))=c_1,y(x)\right]$$

$$y(x)\to 0$$

$$y(x)\to \frac{(-2)^{2/3}}{x^{2/3}}$$

$$y(x)\to \frac{2^{2/3}}{x^{2/3}}$$

$$y(x)\to -\frac{\sqrt[3]{-1}2^{2/3}}{x^{2/3}}$$

2.4 problem 4

Internal problem ID [6868]
Internal file name [`OUTPUT/6111_Friday_August_05_2022_02_19_59_AM_2469170/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 16. Nonlinear equations. Miscellaneous Exercises. Page 340
Problem number: 4.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _rational]`

$$\boxed{4y^3y'^2 - 4xy' + y = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{x + \sqrt{x^2 - y^4}}{2y^3} \tag{1}$$

$$y' = -\frac{-x + \sqrt{x^2 - y^4}}{2y^3} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = \frac{x + \sqrt{-y^4 + x^2}}{2y^3}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{\left(x+\sqrt{-y^4+x^2}\right)(b_3-a_2)}{2y^3} - \frac{\left(x+\sqrt{-y^4+x^2}\right)^2 a_3}{4y^6} \\ & - \frac{\left(1+\frac{x}{\sqrt{-y^4+x^2}}\right)(x a_2 + y a_3 + a_1)}{2y^3} \\ & - \left(-\frac{1}{\sqrt{-y^4+x^2}} - \frac{3\left(x+\sqrt{-y^4+x^2}\right)}{2y^4}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-4b_2\sqrt{-y^4+x^2}\,y^6 + 2x\,y^6 b_2 - 2y^7 a_2 + 4y^7 b_3 + 2y^6 b_1 - 6\sqrt{-y^4+x^2}\,x^2 y^2 b_2 + 4\sqrt{-y^4+x^2}\,x\,y^3 a_2 - 8\sqrt{}}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & 4b_2\sqrt{-y^4+x^2}\,y^6 - 2x\,y^6 b_2 + 2y^7 a_2 - 4y^7 b_3 - 2y^6 b_1 + 6\sqrt{-y^4+x^2}\,x^2 y^2 b_2 \\ & - 4\sqrt{-y^4+x^2}\,x\,y^3 a_2 + 8\sqrt{-y^4+x^2}\,x\,y^3 b_3 - 2\sqrt{-y^4+x^2}\,y^4 a_3 \\ & + 6x^3 y^2 b_2 - 4x^2 y^3 a_2 + 8x^2 y^3 b_3 + 6\sqrt{-y^4+x^2}\,x\,y^2 b_1 - 2\sqrt{-y^4+x^2}\,y^3 a_1 \\ & + 6x^2 y^2 b_1 - 2x\,y^3 a_1 - \left(-y^4+x^2\right)^{\frac{3}{2}} a_3 - \sqrt{-y^4+x^2}\,x^2 a_3 - 2x^3 a_3 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$
\begin{aligned}
&4b_2\sqrt{-y^4+x^2}\,y^6+4x\,y^6b_2+4y^7b_3+4y^6b_1+6\left(-y^4+x^2\right)x\,y^2b_2\\
&-2\left(-y^4+x^2\right)y^3a_2+8\left(-y^4+x^2\right)y^3b_3+6\sqrt{-y^4+x^2}\,x^2y^2b_2\\
&-4\sqrt{-y^4+x^2}\,x\,y^3a_2+8\sqrt{-y^4+x^2}\,x\,y^3b_3\\
&-2\sqrt{-y^4+x^2}\,y^4a_3-2x^2y^3a_2-2x\,y^4a_3+6\left(-y^4+x^2\right)y^2b_1\\
&+6\sqrt{-y^4+x^2}\,x\,y^2b_1-2\sqrt{-y^4+x^2}\,y^3a_1-2x\,y^3a_1\\
&-\left(-y^4+x^2\right)^{\frac{3}{2}}a_3-2\left(-y^4+x^2\right)xa_3-\sqrt{-y^4+x^2}\,x^2a_3=0
\end{aligned}
\tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&-2x\,y^6b_2+4b_2\sqrt{-y^4+x^2}\,y^6+2y^7a_2-4y^7b_3-2y^6b_1+6x^3y^2b_2\\
&+6\sqrt{-y^4+x^2}\,x^2y^2b_2-4x^2y^3a_2+8x^2y^3b_3-4\sqrt{-y^4+x^2}\,x\,y^3a_2\\
&+8\sqrt{-y^4+x^2}\,x\,y^3b_3-\sqrt{-y^4+x^2}\,y^4a_3+6x^2y^2b_1+6\sqrt{-y^4+x^2}\,x\,y^2b_1\\
&-2x\,y^3a_1-2\sqrt{-y^4+x^2}\,y^3a_1-2x^3a_3-2\sqrt{-y^4+x^2}\,x^2a_3=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{-y^4+x^2}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x=v_1, y=v_2, \sqrt{-y^4+x^2}=v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&2v_2^7a_2-2v_1v_2^6b_2+4b_2v_3v_2^6-4v_2^7b_3-2v_2^6b_1-4v_1^2v_2^3a_2-4v_3v_1v_2^3a_2\\
&-v_3v_2^4a_3+6v_1^3v_2^2b_2+6v_3v_1^2v_2^2b_2+8v_1^2v_2^3b_3+8v_3v_1v_2^3b_3-2v_1v_2^3a_1\\
&-2v_3v_2^3a_1+6v_1^2v_2^2b_1+6v_3v_1v_2^2b_1-2v_1^3a_3-2v_3v_1^2a_3=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$6v_1^3v_2^2b_2 - 2v_1^3a_3 + (-4a_2 + 8b_3)\,v_1^2v_2^3 + 6v_3v_1^2v_2^2b_2 + 6v_1^2v_2^2b_1 \\ - 2v_3v_1^2a_3 - 2v_1v_2^6b_2 + (-4a_2 + 8b_3)\,v_1v_2^3v_3 - 2v_1v_2^3a_1 + 6v_3v_1v_2^2b_1 \\ + (2a_2 - 4b_3)\,v_2^7 + 4b_2v_3v_2^6 - 2v_2^6b_1 - v_3v_2^4a_3 - 2v_3v_2^3a_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -2a_1 &= 0 \\ -2a_3 &= 0 \\ -a_3 &= 0 \\ -2b_1 &= 0 \\ 6b_1 &= 0 \\ -2b_2 &= 0 \\ 4b_2 &= 0 \\ 6b_2 &= 0 \\ -4a_2 + 8b_3 &= 0 \\ 2a_2 - 4b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 2b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 2x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{x+\sqrt{-y^4+x^2}}{2y^3}\right)(2x) \\
&= \frac{y^4 - \sqrt{-y^4+x^2}\,x - x^2}{y^3} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta}dy \\
&= \int \frac{1}{\frac{y^4-\sqrt{-y^4+x^2}\,x-x^2}{y^3}}dy
\end{aligned}$$

Which results in

$$S = -\frac{\ln(y^2-x)}{4} + \ln(y) - \frac{\ln(y^2+x)}{4} + \frac{\ln(y^4-x^2)}{4} + \frac{x\ln\left(\frac{2x^2+2\sqrt{x^2}\,\sqrt{-y^4+x^2}}{y^2}\right)}{2\sqrt{x^2}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x+\sqrt{-y^4+x^2}}{2y^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x+\sqrt{-y^4+x^2}}{2\sqrt{-y^4+x^2}\,x} \\
S_y &= -\frac{y^3}{\sqrt{-y^4+x^2}\,\left(x+\sqrt{-y^4+x^2}\right)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{2x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R)}{2} + c_1 \tag{4}$$

Summary

The solution(s) found are the following

$$\frac{\ln(2)}{2} + \frac{\ln(x)}{2} + \frac{\ln\left(x+\sqrt{x^2-y^4}\right)}{2} = \frac{\ln(x)}{2} + c_1 \tag{1}$$

Verification of solutions

$$\frac{\ln(2)}{2} + \frac{\ln(x)}{2} + \frac{\ln\left(x+\sqrt{x^2-y^4}\right)}{2} = \frac{\ln(x)}{2} + c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{\sqrt{-y^4+x^2}-x}{2y^3} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 - \frac{\left(\sqrt{-y^4+x^2}-x\right)(b_3-a_2)}{2y^3} - \frac{\left(\sqrt{-y^4+x^2}-x\right)^2 a_3}{4y^6} \\
&+ \frac{\left(-1+\frac{x}{\sqrt{-y^4+x^2}}\right)(x a_2 + y a_3 + a_1)}{2y^3} \\
&- \left(\frac{1}{\sqrt{-y^4+x^2}} + \frac{\frac{3\sqrt{-y^4+x^2}}{2} - \frac{3x}{2}}{y^4}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{-4b_2\sqrt{-y^4+x^2}\,y^6 - 2x\,y^6 b_2 + 2y^7 a_2 - 4y^7 b_3 - 2y^6 b_1 - 6\sqrt{-y^4+x^2}\,x^2 y^2 b_2 + 4\sqrt{-y^4+x^2}\,x\,y^3 a_2 - 8\surd}{}
$$

$= 0$

Setting the numerator to zero gives

$$\begin{aligned} &4b_2\sqrt{-y^4+x^2}\,y^6+2x\,y^6b_2-2y^7a_2+4y^7b_3+2y^6b_1+6\sqrt{-y^4+x^2}\,x^2y^2b_2\\ &-4\sqrt{-y^4+x^2}\,x\,y^3a_2+8\sqrt{-y^4+x^2}\,x\,y^3b_3-2\sqrt{-y^4+x^2}\,y^4a_3\\ &-6x^3y^2b_2+4x^2y^3a_2-8x^2y^3b_3+6\sqrt{-y^4+x^2}\,x\,y^2b_1-2\sqrt{-y^4+x^2}\,y^3a_1\\ &-6x^2y^2b_1+2x\,y^3a_1-\left(-y^4+x^2\right)^{\frac{3}{2}}a_3-\sqrt{-y^4+x^2}\,x^2a_3+2x^3a_3=0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &4b_2\sqrt{-y^4+x^2}\,y^6-4x\,y^6b_2-4y^7b_3-4y^6b_1-6\left(-y^4+x^2\right)x\,y^2b_2\\ &+2\left(-y^4+x^2\right)y^3a_2-8\left(-y^4+x^2\right)y^3b_3+6\sqrt{-y^4+x^2}\,x^2y^2b_2\\ &-4\sqrt{-y^4+x^2}\,x\,y^3a_2+8\sqrt{-y^4+x^2}\,x\,y^3b_3\\ &-2\sqrt{-y^4+x^2}\,y^4a_3+2x^2y^3a_2+2x\,y^4a_3-6\left(-y^4+x^2\right)y^2b_1\\ &+6\sqrt{-y^4+x^2}\,x\,y^2b_1-2\sqrt{-y^4+x^2}\,y^3a_1+2x\,y^3a_1\\ &-\left(-y^4+x^2\right)^{\frac{3}{2}}a_3+2\left(-y^4+x^2\right)xa_3-\sqrt{-y^4+x^2}\,x^2a_3=0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned} &2x\,y^6b_2+4b_2\sqrt{-y^4+x^2}\,y^6-2y^7a_2+4y^7b_3+2y^6b_1-6x^3y^2b_2\\ &+6\sqrt{-y^4+x^2}\,x^2y^2b_2+4x^2y^3a_2-8x^2y^3b_3-4\sqrt{-y^4+x^2}\,x\,y^3a_2\\ &+8\sqrt{-y^4+x^2}\,x\,y^3b_3-\sqrt{-y^4+x^2}\,y^4a_3-6x^2y^2b_1+6\sqrt{-y^4+x^2}\,x\,y^2b_1\\ &+2x\,y^3a_1-2\sqrt{-y^4+x^2}\,y^3a_1+2x^3a_3-2\sqrt{-y^4+x^2}\,x^2a_3=0 \end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{-y^4+x^2}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x=v_1, y=v_2, \sqrt{-y^4+x^2}=v_3\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-2v_2^7a_2+2v_1v_2^6b_2+4b_2v_3v_2^6+4v_2^7b_3+2v_2^6b_1+4v_1^2v_2^3a_2-4v_3v_1v_2^3a_2\\ &-v_3v_2^4a_3-6v_1^3v_2^2b_2+6v_3v_1^2v_2^2b_2-8v_1^2v_2^3b_3+8v_3v_1v_2^3b_3+2v_1v_2^3a_1\\ &-2v_3v_2^3a_1-6v_1^2v_2^2b_1+6v_3v_1v_2^2b_1+2v_1^3a_3-2v_3v_1^2a_3=0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned} & -6v_1^3v_2^2b_2 + 2v_1^3a_3 + (4a_2 - 8b_3)\, v_1^2v_2^3 + 6v_3v_1^2v_2^2b_2 - 6v_1^2v_2^2b_1 \\ & - 2v_3v_1^2a_3 + 2v_1v_2^6b_2 + (-4a_2 + 8b_3)\, v_1v_2^3v_3 + 2v_1v_2^3a_1 + 6v_3v_1v_2^2b_1 \\ & + (-2a_2 + 4b_3)\, v_2^7 + 4b_2v_3v_2^6 + 2v_2^6b_1 - v_3v_2^4a_3 - 2v_3v_2^3a_1 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -2a_1 &= 0 \\ 2a_1 &= 0 \\ -2a_3 &= 0 \\ -a_3 &= 0 \\ 2a_3 &= 0 \\ -6b_1 &= 0 \\ 2b_1 &= 0 \\ 6b_1 &= 0 \\ -6b_2 &= 0 \\ 2b_2 &= 0 \\ 4b_2 &= 0 \\ 6b_2 &= 0 \\ -4a_2 + 8b_3 &= 0 \\ -2a_2 + 4b_3 &= 0 \\ 4a_2 - 8b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 2b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 2x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{\sqrt{-y^4 + x^2} - x}{2y^3}\right)(2x) \\
&= \frac{y^4 + \sqrt{-y^4 + x^2}\, x - x^2}{y^3} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{y^4 + \sqrt{-y^4 + x^2}\, x - x^2}{y^3}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{\ln(y^4 - x^2)}{4} - \frac{\ln(y^2 - x)}{4} + \ln(y) - \frac{\ln(y^2 + x)}{4} - \frac{x \ln\left(\frac{2x^2 + 2\sqrt{x^2}\,\sqrt{-y^4 + x^2}}{y^2}\right)}{2\sqrt{x^2}}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{\sqrt{-y^4 + x^2} - x}{2y^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{x + \sqrt{-y^4 + x^2}}{2\sqrt{-y^4 + x^2}\, x} \\ S_y &= \frac{-y^4 + 2x^2 + 2\sqrt{-y^4 + x^2}\, x}{y\sqrt{-y^4 + x^2}\, \left(x + \sqrt{-y^4 + x^2}\right)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{-y^4 + \sqrt{-y^4 + x^2}\, x + x^2}{2\sqrt{-y^4 + x^2}\, x \left(x + \sqrt{-y^4 + x^2}\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\ln(R)}{2} + c_1 \tag{4}$$

Which gives

$$y = e^{\frac{\ln(2)}{4} + \frac{\ln\left(-2\,e^{-2c_1}e^{4c_1} + 2\,e^{-2c_1}e^{2c_1}x\right)}{4} + \frac{c_1}{2}}$$

Summary
The solution(s) found are the following

$$y = e^{\frac{\ln(2)}{4}+\frac{\ln\left(-2\,e^{-2c_1}\,e^{4c_1}+2\,e^{-2c_1}\,e^{2c_1}x\right)}{4}+\frac{c_1}{2}} \tag{1}$$

Verification of solutions

$$y = e^{\frac{\ln(2)}{4}+\frac{\ln\left(-2\,e^{-2c_1}\,e^{4c_1}+2\,e^{-2c_1}\,e^{2c_1}x\right)}{4}+\frac{c_1}{2}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
   -> Solving 1st order ODE of high degree, Lie methods, 1st trial
   `, `-> Computing symmetries using: way = 3`[2*x, y]
```

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 85

```
dsolve(4*y(x)^3*diff(y(x),x)^2-4*x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{-x}$$
$$y(x) = -\sqrt{-x}$$
$$y(x) = \sqrt{x}$$
$$y(x) = -\sqrt{x}$$
$$y(x) = 0$$
$$y(x) = \text{RootOf}\left(-\ln(x) + 2\left(\int^{\_Z} -\frac{\_a^4 - \sqrt{-\_a^4+1} - 1}{\_a\,(\_a^4-1)} d\_a\right) + c_1\right)\sqrt{x}$$

✓ Solution by Mathematica

Time used: 0.581 (sec). Leaf size: 282

```
DSolve[4*y[x]^3*(y'[x])^2-4*x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -e^{\frac{c_1}{4}}\sqrt[4]{e^{c_1} - 2ix}$$
$$y(x) \to -ie^{\frac{c_1}{4}}\sqrt[4]{e^{c_1} - 2ix}$$
$$y(x) \to ie^{\frac{c_1}{4}}\sqrt[4]{e^{c_1} - 2ix}$$
$$y(x) \to e^{\frac{c_1}{4}}\sqrt[4]{e^{c_1} - 2ix}$$
$$y(x) \to -e^{\frac{c_1}{4}}\sqrt[4]{2ix + e^{c_1}}$$
$$y(x) \to -ie^{\frac{c_1}{4}}\sqrt[4]{2ix + e^{c_1}}$$
$$y(x) \to ie^{\frac{c_1}{4}}\sqrt[4]{2ix + e^{c_1}}$$
$$y(x) \to e^{\frac{c_1}{4}}\sqrt[4]{2ix + e^{c_1}}$$
$$y(x) \to 0$$
$$y(x) \to -\sqrt{x}$$
$$y(x) \to -i\sqrt{x}$$
$$y(x) \to i\sqrt{x}$$
$$y(x) \to \sqrt{x}$$

2.5 problem 5

Internal problem ID [6869]
Internal file name [`OUTPUT/6112_Friday_August_05_2022_02_20_01_AM_8646840/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 16. Nonlinear equations. Miscellaneous Exercises. Page 340
Problem number: 5.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{x^6 {y'}^2 - 2xy' - 4y = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{1 + \sqrt{1 + 4yx^4}}{x^5} \tag{1}$$

$$y' = -\frac{-1 + \sqrt{1 + 4yx^4}}{x^5} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$\begin{aligned} y' &= \frac{1 + \sqrt{4y\,x^4 + 1}}{x^5} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{\left(1+\sqrt{4y\,x^4+1}\right)(b_3-a_2)}{x^5} - \frac{\left(1+\sqrt{4y\,x^4+1}\right)^2 a_3}{x^{10}} \\ &- \left(-\frac{5\left(1+\sqrt{4y\,x^4+1}\right)}{x^6} + \frac{8y}{x^2\sqrt{4y\,x^4+1}}\right)(xa_2+ya_3+a_1) \\ &- \frac{2(xb_2+yb_3+b_1)}{x\sqrt{4y\,x^4+1}} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-b_2x^{10}\sqrt{4y\,x^4+1} + 2x^{10}b_2 - 8x^9ya_2 - 2x^9yb_3 - 12x^8y^2a_3 + 2x^9b_1 - 12x^8ya_1 - 4\sqrt{4y\,x^4+1}\,x^5a_2 - \sqrt{4}}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &b_2x^{10}\sqrt{4y\,x^4+1} - 2x^{10}b_2 + 8x^9ya_2 + 2x^9yb_3 + 12x^8y^2a_3 \\ &- 2x^9b_1 + 12x^8ya_1 + 4\sqrt{4y\,x^4+1}\,x^5a_2 + \sqrt{4y\,x^4+1}\,x^5b_3 \\ &+ 5\sqrt{4y\,x^4+1}\,x^4ya_3 + 5\sqrt{4y\,x^4+1}\,x^4a_1 + 4x^5a_2 + x^5b_3 \\ &- 3x^4ya_3 + 5x^4a_1 - \left(4y\,x^4+1\right)^{\frac{3}{2}} a_3 - a_3\sqrt{4y\,x^4+1} - 2a_3 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &b_2x^{10}\sqrt{4y\,x^4+1} - 2x^{10}b_2 - 8x^9ya_2 - 2x^9yb_3 - 8x^8y^2a_3 \\ &- 2x^9b_1 - 8x^8ya_1 + 4\left(4y\,x^4+1\right)x^5a_2 + \left(4y\,x^4+1\right)x^5b_3 \\ &+ 5\left(4y\,x^4+1\right)x^4ya_3 + 5\left(4y\,x^4+1\right)x^4a_1 + 4\sqrt{4y\,x^4+1}\,x^5a_2 \\ &+ \sqrt{4y\,x^4+1}\,x^5b_3 + 5\sqrt{4y\,x^4+1}\,x^4ya_3 + 5\sqrt{4y\,x^4+1}\,x^4a_1 \\ &- \left(4y\,x^4+1\right)^{\frac{3}{2}} a_3 - 2\left(4y\,x^4+1\right)a_3 - a_3\sqrt{4y\,x^4+1} = 0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$b_2x^{10}\sqrt{4y\,x^4+1}-2x^{10}b_2+8x^9ya_2+2x^9yb_3+12x^8y^2a_3-2x^9b_1+12x^8ya_1 \\ +4\sqrt{4y\,x^4+1}\,x^5a_2+\sqrt{4y\,x^4+1}\,x^5b_3+\sqrt{4y\,x^4+1}\,x^4ya_3+4x^5a_2 \\ +x^5b_3+5\sqrt{4y\,x^4+1}\,x^4a_1-3x^4ya_3+5x^4a_1-2a_3\sqrt{4y\,x^4+1}-2a_3=0$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$\left\{x,y,\sqrt{4y\,x^4+1}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$\left\{x=v_1,y=v_2,\sqrt{4y\,x^4+1}=v_3\right\}$$

The above PDE (6E) now becomes

$$b_2v_1^{10}v_3+8v_1^9v_2a_2+12v_1^8v_2^2a_3-2v_1^{10}b_2+2v_1^9v_2b_3+12v_1^8v_2a_1-2v_1^9b_1+4v_3v_1^5a_2 \\ +v_3v_1^4v_2a_3+v_3v_1^5b_3+5v_3v_1^4a_1+4v_1^5a_2-3v_1^4v_2a_3+v_1^5b_3+5v_1^4a_1-2a_3v_3-2a_3=0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1,v_2,v_3\}$$

Equation (7E) now becomes

$$b_2v_1^{10}v_3-2v_1^{10}b_2+(8a_2+2b_3)\,v_1^9v_2-2v_1^9b_1+12v_1^8v_2^2a_3+12v_1^8v_2a_1+(4a_2+b_3)\,v_1^5v_3 \\ +(4a_2+b_3)\,v_1^5+v_3v_1^4v_2a_3-3v_1^4v_2a_3+5v_3v_1^4a_1+5v_1^4a_1-2a_3v_3-2a_3=0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_3 &= 0 \\
b_2 &= 0 \\
5a_1 &= 0 \\
12a_1 &= 0 \\
-3a_3 &= 0 \\
-2a_3 &= 0 \\
12a_3 &= 0 \\
-2b_1 &= 0 \\
-2b_2 &= 0 \\
4a_2 + b_3 &= 0 \\
8a_2 + 2b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= -4a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= -4y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= -4y - \left(\frac{1 + \sqrt{4y\,x^4 + 1}}{x^5}\right)(x) \\
&= \frac{-4y\,x^4 - \sqrt{4y\,x^4 + 1} - 1}{x^4} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-4y\,x^4 - \sqrt{4y\,x^4+1}-1}{x^4}} dy \end{aligned}$$

Which results in

$$S = -\frac{\ln (y)}{4} - \frac{\operatorname{arctanh}\left(\sqrt{4y\,x^4+1}\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{1 + \sqrt{4y\,x^4+1}}{x^5}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{x\sqrt{4y\,x^4+1}} \\ S_y &= \frac{-1 + \frac{1}{\sqrt{4y\,x^4+1}}}{4y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(y)}{4} - \frac{\operatorname{arctanh}\left(\sqrt{1+4yx^4}\right)}{2} = c_1$$

Which simplifies to

$$-\frac{\ln(y)}{4} - \frac{\operatorname{arctanh}\left(\sqrt{1+4yx^4}\right)}{2} = c_1$$

Summary

The solution(s) found are the following

$$-\frac{\ln(y)}{4} - \frac{\operatorname{arctanh}\left(\sqrt{1+4yx^4}\right)}{2} = c_1 \tag{1}$$

Verification of solutions

$$-\frac{\ln(y)}{4} - \frac{\operatorname{arctanh}\left(\sqrt{1+4yx^4}\right)}{2} = c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = -\frac{-1+\sqrt{4y\,x^4+1}}{x^5}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{\left(-1+\sqrt{4y\,x^4+1}\right)(b_3-a_2)}{x^5} - \frac{\left(-1+\sqrt{4y\,x^4+1}\right)^2 a_3}{x^{10}} \\ &- \left(-\frac{8y}{x^2\sqrt{4y\,x^4+1}} + \frac{-5+5\sqrt{4y\,x^4+1}}{x^6}\right)(x a_2 + y a_3 + a_1) \\ &+ \frac{2x b_2 + 2y b_3 + 2b_1}{\sqrt{4y\,x^4+1}\,x} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-b_2 x^{10}\sqrt{4y\,x^4+1} - 2x^{10} b_2 + 8x^9 y a_2 + 2x^9 y b_3 + 12x^8 y^2 a_3 - 2x^9 b_1 + 12x^8 y a_1 - 4\sqrt{4y\,x^4+1}\,x^5 a_2 - \sqrt{4}}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &b_2 x^{10}\sqrt{4y\,x^4+1} + 2x^{10} b_2 - 8x^9 y a_2 - 2x^9 y b_3 - 12x^8 y^2 a_3 \\ &+ 2x^9 b_1 - 12x^8 y a_1 + 4\sqrt{4y\,x^4+1}\,x^5 a_2 + \sqrt{4y\,x^4+1}\,x^5 b_3 \\ &+ 5\sqrt{4y\,x^4+1}\,x^4 y a_3 + 5\sqrt{4y\,x^4+1}\,x^4 a_1 - 4x^5 a_2 - x^5 b_3 \\ &+ 3x^4 y a_3 - 5x^4 a_1 - \left(4y\,x^4+1\right)^{\frac{3}{2}} a_3 - a_3\sqrt{4y\,x^4+1} + 2a_3 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$
\begin{aligned}
&b_2x^{10}\sqrt{4y\,x^4+1}+2x^{10}b_2+8x^9ya_2+2x^9yb_3+8x^8y^2a_3\\
&+2x^9b_1+8x^8ya_1-4\left(4y\,x^4+1\right)x^5a_2-\left(4y\,x^4+1\right)x^5b_3\\
&-5\left(4y\,x^4+1\right)x^4ya_3-5\left(4y\,x^4+1\right)x^4a_1+4\sqrt{4y\,x^4+1}\,x^5a_2\\
&+\sqrt{4y\,x^4+1}\,x^5b_3+5\sqrt{4y\,x^4+1}\,x^4ya_3+5\sqrt{4y\,x^4+1}\,x^4a_1\\
&-\left(4y\,x^4+1\right)^{\frac{3}{2}}a_3+2\left(4y\,x^4+1\right)a_3-a_3\sqrt{4y\,x^4+1}=0
\end{aligned}
\tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&b_2x^{10}\sqrt{4y\,x^4+1}+2x^{10}b_2-8x^9ya_2-2x^9yb_3-12x^8y^2a_3+2x^9b_1-12x^8ya_1\\
&+4\sqrt{4y\,x^4+1}\,x^5a_2+\sqrt{4y\,x^4+1}\,x^5b_3+\sqrt{4y\,x^4+1}\,x^4ya_3-4x^5a_2\\
&-x^5b_3+5\sqrt{4y\,x^4+1}\,x^4a_1+3x^4ya_3-5x^4a_1-2a_3\sqrt{4y\,x^4+1}+2a_3=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{4y\,x^4+1}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x=v_1, y=v_2, \sqrt{4y\,x^4+1}=v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&b_2v_1^{10}v_3-8v_1^9v_2a_2-12v_1^8v_2^2a_3+2v_1^{10}b_2-2v_1^9v_2b_3-12v_1^8v_2a_1+2v_1^9b_1+4v_3v_1^5a_2\\
&+v_3v_1^4v_2a_3+v_3v_1^5b_3+5v_3v_1^4a_1-4v_1^5a_2+3v_1^4v_2a_3-v_1^5b_3-5v_1^4a_1-2a_3v_3+2a_3=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&b_2v_1^{10}v_3+2v_1^{10}b_2+(-8a_2-2b_3)\,v_1^9v_2+2v_1^9b_1-12v_1^8v_2^2a_3\\
&-12v_1^8v_2a_1+(4a_2+b_3)\,v_1^5v_3+(-4a_2-b_3)\,v_1^5+v_3v_1^4v_2a_3\\
&+3v_1^4v_2a_3+5v_3v_1^4a_1-5v_1^4a_1-2a_3v_3+2a_3=0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_3 &= 0 \\
b_2 &= 0 \\
-12a_1 &= 0 \\
-5a_1 &= 0 \\
5a_1 &= 0 \\
-12a_3 &= 0 \\
-2a_3 &= 0 \\
2a_3 &= 0 \\
3a_3 &= 0 \\
2b_1 &= 0 \\
2b_2 &= 0 \\
-8a_2 - 2b_3 &= 0 \\
-4a_2 - b_3 &= 0 \\
4a_2 + b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= -4a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= -4y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= -4y - \left(-\frac{-1+\sqrt{4y\,x^4+1}}{x^5}\right)(x) \\
&= \frac{-4y\,x^4+\sqrt{4y\,x^4+1}-1}{x^4} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-4y\,x^4+\sqrt{4y\,x^4+1}-1}{x^4}} dy
\end{aligned}$$

Which results in

$$S = -\frac{\ln(y)}{4} + \frac{\operatorname{arctanh}\left(\sqrt{4y\,x^4+1}\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-1+\sqrt{4y\,x^4+1}}{x^5}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{1}{x\sqrt{4y\,x^4+1}} \\
S_y &= \frac{-1-\frac{1}{\sqrt{4y\,x^4+1}}}{4y}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(y)}{4} + \frac{\operatorname{arctanh}\left(\sqrt{1+4yx^4}\right)}{2} = c_1$$

Which simplifies to

$$-\frac{\ln(y)}{4} + \frac{\operatorname{arctanh}\left(\sqrt{1+4yx^4}\right)}{2} = c_1$$

Summary

The solution(s) found are the following

$$-\frac{\ln(y)}{4} + \frac{\operatorname{arctanh}\left(\sqrt{1+4yx^4}\right)}{2} = c_1 \tag{1}$$

Verification of solutions

$$-\frac{\ln(y)}{4} + \frac{\operatorname{arctanh}\left(\sqrt{1+4yx^4}\right)}{2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      <- homogeneous successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      <- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 89

```
dsolve(x^6*diff(y(x),x)^2-2*x*diff(y(x),x)-4*y(x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{1}{4x^4}$$
$$y(x) = \frac{-ic_1 - x^2}{x^2 c_1^2}$$
$$y(x) = \frac{ic_1 - x^2}{x^2 c_1^2}$$
$$y(x) = \frac{ic_1 - x^2}{x^2 c_1^2}$$
$$y(x) = \frac{-ic_1 - x^2}{x^2 c_1^2}$$

✓ Solution by Mathematica

Time used: 0.532 (sec). Leaf size: 128

```
DSolve[x^6*(y'[x])^2-2*x*y'[x]-4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[-\frac{x\sqrt{4x^4y(x)+1}\operatorname{arctanh}\left(\sqrt{4x^4y(x)+1}\right)}{2\sqrt{4x^6y(x)+x^2}} - \frac{1}{4}\log(y(x)) = c_1, y(x)\right]$$
$$\text{Solve}\left[\frac{x\sqrt{4x^4y(x)+1}\operatorname{arctanh}\left(\sqrt{4x^4y(x)+1}\right)}{2\sqrt{4x^6y(x)+x^2}} - \frac{1}{4}\log(y(x)) = c_1, y(x)\right]$$
$$y(x) \to 0$$

2.6 problem 6

Internal problem ID [6870]
Internal file name [`OUTPUT/6113_Friday_August_05_2022_02_20_03_AM_76649283/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 16. Nonlinear equations. Miscellaneous Exercises. Page 340
Problem number: 6.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _dAlembert]
```

$$\boxed{5y'^2 + 6xy' - 2y = 0}$$

2.6.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$5p^2 + 6xp - 2y = 0$$

Solving for y from the above results in

$$y = \frac{5}{2}p^2 + 3xp \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = 3p$$
$$g = \frac{5p^2}{2}$$

Hence (2) becomes

$$-2p = (3x + 5p)\, p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$-2p = 0$$

Solving for p from the above gives

$$p = 0$$

Substituting these in (1A) gives

$$y = 0$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = -\frac{2p(x)}{3x + 5p\,(x)} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = -\frac{3x(p) + 5p}{2p} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = \frac{3}{2p}$$
$$q(p) = -\frac{5}{2}$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{3x(p)}{2p} = -\frac{5}{2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{3}{2p}dp} \\ &= p^{\frac{3}{2}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}(\mu x) &= (\mu)\left(-\frac{5}{2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}p}\left(p^{\frac{3}{2}}x\right) &= \left(p^{\frac{3}{2}}\right)\left(-\frac{5}{2}\right) \\ \mathrm{d}\left(p^{\frac{3}{2}}x\right) &= \left(-\frac{5p^{\frac{3}{2}}}{2}\right)\mathrm{d}p\end{aligned}$$

Integrating gives

$$\begin{aligned}p^{\frac{3}{2}}x &= \int -\frac{5p^{\frac{3}{2}}}{2}\,\mathrm{d}p \\ p^{\frac{3}{2}}x &= -p^{\frac{5}{2}} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = p^{\frac{3}{2}}$ results in

$$x(p) = -p + \frac{c_1}{p^{\frac{3}{2}}}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$\begin{aligned}p &= -\frac{3x}{5} + \frac{\sqrt{9x^2+10y}}{5} \\ p &= -\frac{3x}{5} - \frac{\sqrt{9x^2+10y}}{5}\end{aligned}$$

Substituting the above in the solution for x found above gives

$$\begin{aligned}x &= \frac{3x}{5} - \frac{\sqrt{9x^2+10y}}{5} + \frac{125c_1}{\left(-15x+5\sqrt{9x^2+10y}\right)^{\frac{3}{2}}} \\ x &= \frac{3x}{5} + \frac{\sqrt{9x^2+10y}}{5} + \frac{125c_1}{\left(-15x-5\sqrt{9x^2+10y}\right)^{\frac{3}{2}}}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$

$$x = \frac{3x}{5} - \frac{\sqrt{9x^2+10y}}{5} + \frac{125c_1}{\left(-15x+5\sqrt{9x^2+10y}\right)^{\frac{3}{2}}} \tag{2}$$

$$x = \frac{3x}{5} + \frac{\sqrt{9x^2+10y}}{5} + \frac{125c_1}{\left(-15x-5\sqrt{9x^2+10y}\right)^{\frac{3}{2}}} \tag{3}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$x = \frac{3x}{5} - \frac{\sqrt{9x^2+10y}}{5} + \frac{125c_1}{\left(-15x+5\sqrt{9x^2+10y}\right)^{\frac{3}{2}}}$$

Verified OK.

$$x = \frac{3x}{5} + \frac{\sqrt{9x^2+10y}}{5} + \frac{125c_1}{\left(-15x-5\sqrt{9x^2+10y}\right)^{\frac{3}{2}}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 85

```
dsolve(5*diff(y(x),x)^2+6*x*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$\frac{c_1}{\left(-15x-5\sqrt{9x^2+10y(x)}\right)^{\frac{3}{2}}}+\frac{2x}{5}-\frac{\sqrt{9x^2+10y(x)}}{5}=0$$

$$\frac{c_1}{\left(-15x+5\sqrt{9x^2+10y(x)}\right)^{\frac{3}{2}}}+\frac{2x}{5}+\frac{\sqrt{9x^2+10y(x)}}{5}=0$$

✓ Solution by Mathematica

Time used: 14.054 (sec). Leaf size: 771

```
DSolve[5*(y'[x])^2+6*x*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{Root}\left[4\#1^5 + 4\#1^4x^2 + \#1^3x^4 + 1000\#1^2e^{5c_1}x + 900\#1e^{5c_1}x^3 + 216e^{5c_1}x^5 - 25000e^{10c_1}\&, 1\right]$$

$$y(x) \to \text{Root}\left[4\#1^5 + 4\#1^4x^2 + \#1^3x^4 + 1000\#1^2e^{5c_1}x + 900\#1e^{5c_1}x^3 + 216e^{5c_1}x^5 - 25000e^{10c_1}\&, 2\right]$$

$$y(x) \to \text{Root}\left[4\#1^5 + 4\#1^4x^2 + \#1^3x^4 + 1000\#1^2e^{5c_1}x + 900\#1e^{5c_1}x^3 + 216e^{5c_1}x^5 - 25000e^{10c_1}\&, 3\right]$$

$$y(x) \to \text{Root}\left[4\#1^5 + 4\#1^4x^2 + \#1^3x^4 + 1000\#1^2e^{5c_1}x + 900\#1e^{5c_1}x^3 + 216e^{5c_1}x^5 - 25000e^{10c_1}\&, 4\right]$$

$$y(x) \to \text{Root}\left[4\#1^5 + 4\#1^4x^2 + \#1^3x^4 + 1000\#1^2e^{5c_1}x + 900\#1e^{5c_1}x^3 + 216e^{5c_1}x^5 - 25000e^{10c_1}\&, 5\right]$$

$$y(x) \to \text{Root}\left[100000\#1^5 + 100000\#1^4x^2 + 25000\#1^3x^4 - 1000\#1^2e^{5c_1}x - 900\#1e^{5c_1}x^3 - 216e^{5c_1}x^5 - e^{10c_1}\&, 1\right]$$

$$y(x) \to \text{Root}\left[100000\#1^5 + 100000\#1^4x^2 + 25000\#1^3x^4 - 1000\#1^2e^{5c_1}x - 900\#1e^{5c_1}x^3 - 216e^{5c_1}x^5 - e^{10c_1}\&, 2\right]$$

$$y(x) \to \text{Root}\left[100000\#1^5 + 100000\#1^4x^2 + 25000\#1^3x^4 - 1000\#1^2e^{5c_1}x - 900\#1e^{5c_1}x^3 - 216e^{5c_1}x^5 - e^{10c_1}\&, 3\right]$$

$$y(x) \to \text{Root}\left[100000\#1^5 + 100000\#1^4x^2 + 25000\#1^3x^4 - 1000\#1^2e^{5c_1}x - 900\#1e^{5c_1}x^3 - 216e^{5c_1}x^5 - e^{10c_1}\&, 4\right]$$

$$y(x) \to \text{Root}\left[100000\#1^5 + 100000\#1^4x^2 + 25000\#1^3x^4 - 1000\#1^2e^{5c_1}x - 900\#1e^{5c_1}x^3 - 216e^{5c_1}x^5 - e^{10c_1}\&, 5\right]$$

$$y(x) \to 0$$

2.7 problem 8

Internal problem ID [6871]
Internal file name [`OUTPUT/6114_Friday_August_05_2022_02_20_04_AM_46650619/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 16. Nonlinear equations. Miscellaneous Exercises. Page 340
Problem number: 8.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"exact", "quadrature", "separable", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y^2 {y'}^2 - y(x+1)\, y' = -x}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{1}{y} \tag{1}$$
$$y' = \frac{x}{y} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int y dy = x + c_1$$
$$\frac{y^2}{2} = x + c_1$$

Solving for y gives these solutions

$$
\begin{aligned}
y_1 &= \sqrt{2c_1 + 2x} \\
y_2 &= -\sqrt{2c_1 + 2x}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = \sqrt{2c_1 + 2x} \tag{1}$$
$$y = -\sqrt{2c_1 + 2x} \tag{2}$$

Verification of solutions

$$y = \sqrt{2c_1 + 2x}$$

Verified OK.

$$y = -\sqrt{2c_1 + 2x}$$

Verified OK.
Solving equation (2)

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x, y) \\
&= f(x)g(y) \\
&= \frac{x}{y}
\end{aligned}
$$

Where $f(x) = x$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{1}{y}}\, dy &= x\, dx \\
\int \frac{1}{\frac{1}{y}}\, dy &= \int x\, dx \\
\frac{y^2}{2} &= \frac{x^2}{2} + c_2
\end{aligned}
$$

Which results in

$$
\begin{aligned}
y &= \sqrt{x^2 + 2c_2} \\
y &= -\sqrt{x^2 + 2c_2}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = \sqrt{x^2 + 2c_2} \tag{1}$$
$$y = -\sqrt{x^2 + 2c_2} \tag{2}$$

Verification of solutions

$$y = \sqrt{x^2 + 2c_2}$$

Verified OK.

$$y = -\sqrt{x^2 + 2c_2}$$

Verified OK.

2.7.1 Maple step by step solution

Let's solve

$y^2 y'^2 - y(x+1)\, y' = -x$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $yy' = 1$

- Integrate both sides with respect to x

  $\int yy'dx = \int 1dx + c_1$

- Evaluate integral

  $\frac{y^2}{2} = x + c_1$

- Solve for y

  $\{y = \sqrt{2c_1 + 2x}, y = -\sqrt{2c_1 + 2x}\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
dsolve(y(x)^2*diff(y(x),x)^2-y(x)*(x+1)*diff(y(x),x)+x=0,y(x), singsol=all)
```

$$y(x) = \sqrt{c_1 + 2x}$$
$$y(x) = -\sqrt{c_1 + 2x}$$
$$y(x) = \sqrt{x^2 + c_1}$$
$$y(x) = -\sqrt{x^2 + c_1}$$

✓ Solution by Mathematica

Time used: 0.141 (sec). Leaf size: 72

```
DSolve[y[x]^2*(y'[x])^2-y[x]*(x+1)*y'[x]+x==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{2}\sqrt{x + c_1}$$
$$y(x) \to \sqrt{2}\sqrt{x + c_1}$$
$$y(x) \to -\sqrt{x^2 + 2c_1}$$
$$y(x) \to \sqrt{x^2 + 2c_1}$$

2.8 problem 9

Internal problem ID [6872]
Internal file name [`OUTPUT/6115_Friday_August_05_2022_02_20_05_AM_90610659/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 16. Nonlinear equations. Miscellaneous Exercises. Page 340
Problem number: 9.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`]]
```

$$\boxed{4x^5{y'}^2+12x^4yy'=-9}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y'=\frac{-\frac{3x^2y}{2}+\frac{3\sqrt{x^4y^2-x}}{2}}{x^3} \tag{1}$$

$$y'=-\frac{3\left(x^2y+\sqrt{x^4y^2-x}\right)}{2x^3} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y'=\frac{-\frac{3y\,x^2}{2}+\frac{3\sqrt{x^4y^2-x}}{2}}{x^3}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{3\left(-y\,x^2 + \sqrt{x^4y^2 - x}\right)(b_3 - a_2)}{2x^3} - \frac{9\left(-y\,x^2 + \sqrt{x^4y^2 - x}\right)^2 a_3}{4x^6} - \left(\frac{-3xy + \frac{3(4x^3y^2-1)}{4\sqrt{x^4y^2-x}}}{x^3} - \frac{9\left(-y\,x^2 + \sqrt{x^4y^2 - x}\right)}{2x^4}\right)(x a_2 + y a_3 + a_1) - \frac{3\left(-x^2 + \frac{y\,x^4}{\sqrt{x^4y^2-x}}\right)(x b_2 + y b_3 + b_1)}{2x^3} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{6x^8yb_2 - 24x^6y^3a_3 + 6x^7yb_1 - 6x^6y^2a_1 - 10b_2x^6\sqrt{x^4y^2 - x} + 15\sqrt{x^4y^2 - x}\,x^4y^2a_3 - 6\sqrt{x^4y^2 - x}\,x^5b_1 +}{4x^6\sqrt{x^4y^2 - x}} = 0$$

Setting the numerator to zero gives

$$-6x^8yb_2 + 24x^6y^3a_3 - 6x^7yb_1 + 6x^6y^2a_1 + 10b_2x^6\sqrt{x^4y^2 - x} - 15\sqrt{x^4y^2 - x}\,x^4y^2a_3 + 6\sqrt{x^4y^2 - x}\,x^5b_1 - 6\sqrt{x^4y^2 - x}\,x^4ya_1 - 9x^4a_2 - 6x^4b_3 - 33x^3ya_3 - 9\left(x^4y^2 - x\right)^{\frac{3}{2}} a_3 - 15x^3a_1 = 0 \tag{6E}$$

Simplifying the above gives

$$-6x^8yb_2 - 12x^7y^2a_2 - 6x^7y^2b_3 - 12x^6y^3a_3 - 6x^7yb_1 - 12x^6y^2a_1 + 10b_2x^6\sqrt{x^4y^2 - x} - 15\sqrt{x^4y^2 - x}\,x^4y^2a_3 + 6\sqrt{x^4y^2 - x}\,x^5b_1 - 6\sqrt{x^4y^2 - x}\,x^4ya_1 + 12\left(x^4y^2 - x\right)x^3a_2 + 6\left(x^4y^2 - x\right)x^3b_3 + 36\left(x^4y^2 - x\right)x^2ya_3 + 18\left(x^4y^2 - x\right)x^2a_1 + 3x^4a_2 + 3x^3ya_3 - 9\left(x^4y^2 - x\right)^{\frac{3}{2}} a_3 + 3x^3a_1 = 0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
x\Big(&-6x^7 y b_2+24x^5 y^3 a_3-6x^6 y b_1+6x^5 y^2 a_1+10\sqrt{x\left(x^3 y^2-1\right)}\, x^5 b_2 \\
&-24\sqrt{x\left(x^3 y^2-1\right)}\, x^3 y^2 a_3+6\sqrt{x\left(x^3 y^2-1\right)}\, x^4 b_1-6\sqrt{x\left(x^3 y^2-1\right)}\, x^3 y a_1 \\
&-9x^3 a_2-6x^3 b_3-33x^2 y a_3-15x^2 a_1+9\sqrt{x\left(x^3 y^2-1\right)}\, a_3\Big)=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{x\left(x^3 y^2-1\right)}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x=v_1, y=v_2, \sqrt{x\left(x^3 y^2-1\right)}=v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
v_1\big(&24v_1^5 v_2^3 a_3-6v_1^7 v_2 b_2+6v_1^5 v_2^2 a_1-6v_1^6 v_2 b_1-24v_3 v_1^3 v_2^2 a_3+10v_3 v_1^5 b_2 \\
&-6v_3 v_1^3 v_2 a_1+6v_3 v_1^4 b_1-9v_1^3 a_2-33v_1^2 v_2 a_3-6v_1^3 b_3-15v_1^2 a_1+9v_3 a_3\big)=0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&-6b_2 v_2 v_1^8-6b_1 v_2 v_1^7+24a_3 v_2^3 v_1^6+6a_1 v_2^2 v_1^6+10b_2 v_3 v_1^6+6b_1 v_3 v_1^5-24a_3 v_2^2 v_3 v_1^4 \\
&-6a_1 v_2 v_3 v_1^4+\left(-9a_2-6b_3\right) v_1^4-33a_3 v_2 v_1^3-15a_1 v_1^3+9v_3 a_3 v_1=0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-15a_1 &= 0 \\
-6a_1 &= 0 \\
6a_1 &= 0 \\
-33a_3 &= 0 \\
-24a_3 &= 0 \\
9a_3 &= 0 \\
24a_3 &= 0 \\
-6b_1 &= 0 \\
6b_1 &= 0 \\
-6b_2 &= 0 \\
10b_2 &= 0 \\
-9a_2 - 6b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= -\frac{3a_2}{2}
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= -\frac{3y}{2}
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= -\frac{3y}{2} - \left(\frac{-\frac{3y\,x^2}{2} + \frac{3\sqrt{x^4y^2 - x}}{2}}{x^3}\right)(x) \\
&= -\frac{3\sqrt{x^4y^2 - x}}{2x^2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\frac{3\sqrt{x^4y^2-x}}{2x^2}} dy \end{aligned}$$

Which results in

$$S = -\frac{2x^2 \ln\left(\frac{y\,x^4}{\sqrt{x^4}} + \sqrt{x^4y^2 - x}\right)}{3\sqrt{x^4}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-\frac{3y\,x^2}{2} + \frac{3\sqrt{x^4y^2-x}}{2}}{x^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{4x^3y^2 + 4x^{\frac{3}{2}}y\sqrt{x^3y^2 - 1} - 1}{\sqrt{x}\,\sqrt{x^3y^2 - 1}\,\left(3y\,x^2 + 3\sqrt{x}\,\sqrt{x^3y^2 - 1}\right)} \\ S_y &= -\frac{2x^2\left(y\,x^{\frac{3}{2}} + \sqrt{x^3y^2 - 1}\right)}{\sqrt{x^3y^2 - 1}\,\left(3y\,x^2 + 3\sqrt{x}\,\sqrt{x^3y^2 - 1}\right)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{\left(3y\,x^{2}+3\sqrt{x}\,\sqrt{x^{3}y^{2}-1}\right)\sqrt{x\left(x^{3}y^{2}-1\right)}+x^{4}y^{2}+x^{\frac{5}{2}}\sqrt{x^{3}y^{2}-1}\,y-x}{\sqrt{x^{3}y^{2}-1}\,x^{\frac{3}{2}}\left(3y\,x^{2}+3\sqrt{x}\,\sqrt{x^{3}y^{2}-1}\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{4}{3R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{4\ln(R)}{3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{2\ln\left(x^{2}y+\sqrt{x}\,\sqrt{y^{2}x^{3}-1}\right)}{3} = -\frac{4\ln(x)}{3} + c_1$$

Which simplifies to

$$-\frac{2\ln\left(x^{2}y+\sqrt{x}\,\sqrt{y^{2}x^{3}-1}\right)}{3} = -\frac{4\ln(x)}{3} + c_1$$

Which gives

$$y = \frac{\left(e^{3c_1}+x^{3}\right)e^{-\frac{3c_1}{2}}}{2x^{3}}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(e^{3c_1}+x^{3}\right)e^{-\frac{3c_1}{2}}}{2x^{3}} \tag{1}$$

Verification of solutions

$$y = \frac{\left(e^{3c_1}+x^{3}\right)e^{-\frac{3c_1}{2}}}{2x^{3}}$$

Verified OK.

Solving equation (2)

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{3\left(y\,x^{2}+\sqrt{x^{4}y^{2}-x}\right)}{2x^{3}} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = xa_2+ya_3+a_1 \tag{1E}
$$

$$
\eta = xb_2+yb_3+b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1,a_2,a_3,b_1,b_2,b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2-\frac{3\left(y\,x^{2}+\sqrt{x^{4}y^{2}-x}\right)(b_3-a_2)}{2x^{3}}-\frac{9\left(y\,x^{2}+\sqrt{x^{4}y^{2}-x}\right)^{2}a_3}{4x^{6}} \\
&-\left(-\frac{3\left(2xy+\frac{4x^{3}y^{2}-1}{2\sqrt{x^{4}y^{2}-x}}\right)}{2x^{3}}+\frac{\frac{9y\,x^{2}}{2}+\frac{9\sqrt{x^{4}y^{2}-x}}{2}}{x^{4}}\right)(xa_2+ya_3+a_1) \\
&+\frac{3\left(x^{2}+\frac{y\,x^{4}}{\sqrt{x^{4}y^{2}-x}}\right)(xb_2+yb_3+b_1)}{2x^{3}}=0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{-6x^{8}yb_2+24x^{6}y^{3}a_3-6x^{7}yb_1+6x^{6}y^{2}a_1-10b_2x^{6}\sqrt{x^{4}y^{2}-x}+15\sqrt{x^{4}y^{2}-x}\,x^{4}y^{2}a_3-6\sqrt{x^{4}y^{2}-x}\,x^{5}b_1}{4x^{6}\sqrt{x^{4}y^{2}-x}}
$$

$$
=0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&6x^{8}yb_2-24x^{6}y^{3}a_3+6x^{7}yb_1-6x^{6}y^{2}a_1+10b_2x^{6}\sqrt{x^{4}y^{2}-x} \\
&-15\sqrt{x^{4}y^{2}-x}\,x^{4}y^{2}a_3+6\sqrt{x^{4}y^{2}-x}\,x^{5}b_1-6\sqrt{x^{4}y^{2}-x}\,x^{4}ya_1 \\
&+9x^{4}a_2+6x^{4}b_3+33x^{3}ya_3-9\left(x^{4}y^{2}-x\right)^{\frac{3}{2}}a_3+15x^{3}a_1=0
\end{aligned} \tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
&6x^8yb_2+12x^7y^2a_2+6x^7y^2b_3+12x^6y^3a_3+6x^7yb_1\\
&+12x^6y^2a_1+10b_2x^6\sqrt{x^4y^2-x}-15\sqrt{x^4y^2-x}\,x^4y^2a_3\\
&+6\sqrt{x^4y^2-x}\,x^5b_1-6\sqrt{x^4y^2-x}\,x^4ya_1-12(x^4y^2-x)\,x^3a_2\\
&-6(x^4y^2-x)\,x^3b_3-36(x^4y^2-x)\,x^2ya_3-18(x^4y^2-x)\,x^2a_1\\
&-3x^4a_2-3x^3ya_3-9(x^4y^2-x)^{\frac{3}{2}}a_3-3x^3a_1=0
\end{aligned}
\tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&x\Big(6x^7yb_2-24x^5y^3a_3+6x^6yb_1-6x^5y^2a_1+10\sqrt{x\,(x^3y^2-1)}\,x^5b_2\\
&-24\sqrt{x\,(x^3y^2-1)}\,x^3y^2a_3+6\sqrt{x\,(x^3y^2-1)}\,x^4b_1-6\sqrt{x\,(x^3y^2-1)}\,x^3ya_1\\
&+9x^3a_2+6x^3b_3+33x^2ya_3+15x^2a_1+9\sqrt{x\,(x^3y^2-1)}\,a_3\Big)=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$
\left\{x,y,\sqrt{x\,(x^3y^2-1)}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$
\left\{x=v_1,y=v_2,\sqrt{x\,(x^3y^2-1)}=v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&v_1\big(-24v_1^5v_2^3a_3+6v_1^7v_2b_2-6v_1^5v_2^2a_1+6v_1^6v_2b_1-24v_3v_1^3v_2^2a_3+10v_3v_1^5b_2\\
&-6v_3v_1^3v_2a_1+6v_3v_1^4b_1+9v_1^3a_2+33v_1^2v_2a_3+6v_1^3b_3+15v_1^2a_1+9v_3a_3\big)=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1,v_2,v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&6b_2v_2v_1^8+6b_1v_2v_1^7-24a_3v_2^3v_1^6-6a_1v_2^2v_1^6+10b_2v_3v_1^6+6b_1v_3v_1^5-24a_3v_2^2v_3v_1^4\\
&-6a_1v_2v_3v_1^4+(9a_2+6b_3)\,v_1^4+33a_3v_2v_1^3+15a_1v_1^3+9v_3a_3v_1=0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -6a_1 &= 0 \\ 15a_1 &= 0 \\ -24a_3 &= 0 \\ 9a_3 &= 0 \\ 33a_3 &= 0 \\ 6b_1 &= 0 \\ 6b_2 &= 0 \\ 10b_2 &= 0 \\ 9a_2 + 6b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= a_2 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= -\frac{3a_2}{2} \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= -\frac{3y}{2} \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x,y)\,\xi \\ &= -\frac{3y}{2} - \left(-\frac{3\left(y\,x^2 + \sqrt{x^4y^2 - x}\right)}{2x^3}\right)(x) \\ &= \frac{3\sqrt{x^4y^2 - x}}{2x^2} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{3\sqrt{x^4y^2-x}}{2x^2}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{2x^2 \ln\left(\frac{y\,x^4}{\sqrt{x^4}} + \sqrt{x^4y^2 - x}\right)}{3\sqrt{x^4}}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = -\frac{3\left(y\,x^2 + \sqrt{x^4y^2 - x}\right)}{2x^3}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{4x^3y^2 + 4x^{\frac{3}{2}}y\sqrt{x^3y^2 - 1} - 1}{\sqrt{x}\,\sqrt{x^3y^2 - 1}\,\left(3y\,x^2 + 3\sqrt{x}\,\sqrt{x^3y^2 - 1}\right)} \\
S_y &= \frac{2x^2\left(y\,x^{\frac{3}{2}} + \sqrt{x^3y^2 - 1}\right)}{\sqrt{x^3y^2 - 1}\,\left(3y\,x^2 + 3\sqrt{x}\,\sqrt{x^3y^2 - 1}\right)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\left(-3y\,x^2 - 3\sqrt{x}\,\sqrt{x^3y^2-1}\right)\sqrt{x\,(x^3y^2-1)} + x^4y^2 + x^{\frac{5}{2}}\sqrt{x^3y^2-1}\,y - x}{\sqrt{x^3y^2-1}\,x^{\frac{3}{2}}\left(3y\,x^2 + 3\sqrt{x}\,\sqrt{x^3y^2-1}\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{2}{3R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{2\ln(R)}{3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{2\ln\left(x^2y + \sqrt{x}\,\sqrt{y^2x^3-1}\right)}{3} = -\frac{2\ln(x)}{3} + c_1$$

Which simplifies to

$$\frac{2\ln\left(x^2y + \sqrt{x}\,\sqrt{y^2x^3-1}\right)}{3} = -\frac{2\ln(x)}{3} + c_1$$

Which gives

$$y = \frac{(e^{3c_1} + x^3)\,e^{-\frac{3c_1}{2}}}{2x^3}$$

Summary

The solution(s) found are the following

$$y = \frac{(e^{3c_1} + x^3)\,e^{-\frac{3c_1}{2}}}{2x^3} \tag{1}$$

Verification of solutions

$$y = \frac{(e^{3c_1} + x^3)\,e^{-\frac{3c_1}{2}}}{2x^3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      1st order, trying the canonical coordinates of the invariance group
      <- 1st order, canonical coordinates successful
      <- homogeneous successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      1st order, trying the canonical coordinates of the invariance group
      <- 1st order, canonical coordinates successful
      <- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.234 (sec). Leaf size: 53

```
dsolve(4*x^5*diff(y(x),x)^2+12*x^4*y(x)*diff(y(x),x)+9=0,y(x), singsol=all)
```

$$y(x) = \frac{1}{x^{\frac{3}{2}}}$$
$$y(x) = -\frac{1}{x^{\frac{3}{2}}}$$
$$y(x) = \frac{c_1^2 x^3 + 1}{2c_1 x^3}$$
$$y(x) = \frac{x^3 + c_1^2}{2c_1 x^3}$$

✓ Solution by Mathematica

Time used: 6.994 (sec). Leaf size: 75

```
DSolve[4*x^5*(y'[x])^2+12*x^4*y[x]*y'[x]+9==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{\sqrt{x^3 \operatorname{sech}^2\left(\frac{3}{2}(-\log(x) + c_1)\right)}}$$
$$y(x) \to \frac{1}{\sqrt{x^3 \operatorname{sech}^2\left(\frac{3}{2}(-\log(x) + c_1)\right)}}$$
$$y(x) \to -\frac{1}{x^{3/2}}$$
$$y(x) \to \frac{1}{x^{3/2}}$$

2.9 problem 10

Internal problem ID [6873]
Internal file name [`OUTPUT/6116_Friday_August_05_2022_02_20_07_AM_96203312/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 16. Nonlinear equations. Miscellaneous Exercises. Page 340
Problem number: 10.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries]]
```

$$\boxed{4y^2y'^3 - 2xy' + y = 0}$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\left(-27y^2 + 3\sqrt{3}\,\sqrt{27y^4 - 8x^3}\right)^{\frac{1}{3}}}{6y} + \frac{x}{y\left(-27y^2 + 3\sqrt{3}\,\sqrt{27y^4 - 8x^3}\right)^{\frac{1}{3}}} \tag{1}$$

$$y' = -\frac{\left(-27y^2 + 3\sqrt{3}\,\sqrt{27y^4 - 8x^3}\right)^{\frac{1}{3}}}{12y} - \frac{x}{2y\left(-27y^2 + 3\sqrt{3}\,\sqrt{27y^4 - 8x^3}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{1}{3}}}{6y}\right.}{} \tag{2}$$

$$y' = -\frac{\left(-27y^2 + 3\sqrt{3}\,\sqrt{27y^4 - 8x^3}\right)^{\frac{1}{3}}}{12y} - \frac{x}{2y\left(-27y^2 + 3\sqrt{3}\,\sqrt{27y^4 - 8x^3}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{1}{3}}}{6y}\right.}{} \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$
\begin{aligned}
y' &= \frac{\left(-27y^2 + 3\sqrt{3}\,\sqrt{27y^4 - 8x^3}\right)^{\frac{2}{3}} + 6x}{6y\left(-27y^2 + 3\sqrt{3}\,\sqrt{27y^4 - 8x^3}\right)^{\frac{1}{3}}} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2+\frac{\left(\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}+6x\right)(b_3-a_2)}{6y\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{1}{3}}} - \frac{\left(\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}+6x\right)^2 a_3}{36y^2\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}} - \left(\frac{-\frac{24\sqrt{3}\,x^2}{\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{1}{3}}\sqrt{27y^4-8x^3}}+6}{6y\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{1}{3}}} + \frac{2\left(\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}+6x\right)\sqrt{3}\,x^2}{y\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{4}{3}}\sqrt{27y^4-8x^3}}\right)(xa_2+ya_3+a_1) - \left(\frac{-54y+\frac{162\sqrt{3}\,y^3}{\sqrt{27y^4-8x^3}}}{9y\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}} - \frac{\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}+6x}{6y^2\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{1}{3}}} - \frac{\left(\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}+6x\right)\left(-54y+\frac{162\sqrt{3}\,y^3}{\sqrt{27y^4-8x^3}}\right)}{18y\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{4}{3}}}\right)(xb_2+yb_3+b_1)=0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{1944\sqrt{3}\,x\,y^5a_2+9(27y^4-8x^3)^{\frac{3}{2}}a_3+108\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}\sqrt{3}\,x\,y^4b_2-24\left(-27y^2+3\sqrt{3}\,\sqrt{\right.}}{}$$

$$=0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-1944\sqrt{3}\,x\,y^5 a_2 - 9\left(27y^4-8x^3\right)^{\frac{3}{2}} a_3 \\
&- 108\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}} \sqrt{3}\,x\,y^4 b_2 \\
&+ 24\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}} \sqrt{3}\,x^3 y a_2 \\
&+ 24\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}} \sqrt{3}\,x^2 y^2 a_3 \\
&+ 36\sqrt{27y^4-8x^3}\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}} x\,y^2 b_2 \\
&+ 24\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}} \sqrt{3}\,x^2 y a_1 \\
&- 108\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}} \sqrt{3}\,y^5 b_3 \\
&+ 2\sqrt{27y^4-8x^3}\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{5}{3}} x b_2 \\
&- 2\sqrt{27y^4-8x^3}\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{5}{3}} y a_2 \\
&+ 4\sqrt{27y^4-8x^3}\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{5}{3}} y b_3 \\
&+ 12 b_2 y^2\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{4}{3}} \sqrt{27y^4-8x^3} \\
&- 108\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}} \sqrt{3}\,y^4 b_1 \\
&+ 1620\sqrt{3}\,x^2 y^4 b_2 + 2592\sqrt{3}\,x\,y^5 b_3 \\
&- 4\sqrt{27y^4-8x^3}\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{4}{3}} x a_3 \\
&+ 36\sqrt{27y^4-8x^3}\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}} y^3 b_3 \\
&+ 432\sqrt{3}\,x^4 y a_2 - 288\sqrt{3}\,x^3 y^2 a_3 + 1620\sqrt{3}\,x\,y^4 b_1 \\
&- 12\sqrt{27y^4-8x^3}\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}} x^2 a_3 \\
&+ 36\sqrt{27y^4-8x^3}\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}} y^2 b_1 \\
&- 540\sqrt{27y^4-8x^3}\,x^2 y^2 b_2 - 864\sqrt{27y^4-8x^3}\,x\,y^3 b_3 \\
&+ 144\sqrt{3}\,x^3 y a_1 + 486\sqrt{3}\,y^6 a_3 - 288\sqrt{3}\,x^5 b_2 \\
&- 972\sqrt{3}\,y^5 a_1 - 288\sqrt{3}\,x^4 b_1 - 576\sqrt{3}\,x^4 y b_3 \\
&+ 2\sqrt{27y^4-8x^3}\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{5}{3}} b_1 \\
&+ 81\sqrt{27y^4-8x^3}\,y^4 a_3 + 324\sqrt{27y^4-8x^3}\,y^3 a_1 \\
&- 540\sqrt{27y^4-8x^3}\,x\,y^2 b_1 + 648\sqrt{27y^4-8x^3}\,x\,y^3 a_2 = 0
\end{aligned}
\tag{6E}
$$

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{1}{3}}, \left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}, \sqrt{27y^4-8x^3}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{1}{3}} = v_3, \left(-27y^2 + 3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}} = v_4, \sqrt{27y^4-8x^3} = v_5\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-5832\sqrt{3}\,v_1v_2^5a_2 + 72v_4\sqrt{3}\,v_1^2v_2^2a_3 - 54v_5v_4v_1v_2^2b_2 + 72v_4\sqrt{3}\,v_1^2v_2a_1 \\
&- 864\sqrt{3}\,v_1^3v_3v_2^2b_2 - 972\sqrt{3}\,v_1v_3v_2^4a_3 - 288\sqrt{3}\,v_1^3v_4v_2b_3 \\
&+ 324v_1v_5v_3v_2^2a_3 - 1620v_5v_1^2v_2^2b_2 - 2592v_5v_1v_2^3b_3 + 432\sqrt{3}\,v_1^3v_2a_1 \\
&- 972v_5v_3v_2^4b_2 - 1728\sqrt{3}\,v_1^4v_2b_3 - 486\sqrt{3}\,v_4v_2^5a_2 + 2916\sqrt{3}\,v_3v_2^6b_2 \\
&- 144\sqrt{3}\,v_1^4v_4b_2 + 288\sqrt{3}\,v_1^4v_3a_3 - 144\sqrt{3}\,v_1^3v_4b_1 - 1620v_5v_1v_2^2b_1 \\
&+ 1944v_5v_1v_2^3a_2 + 216v_1^3v_5a_3 + 1458\sqrt{3}\,v_2^6a_3 - 864\sqrt{3}\,v_1^5b_2 \\
&- 2916\sqrt{3}\,v_2^5a_1 - 864\sqrt{3}\,v_1^4b_1 - 486v_5v_2^4a_3 + 972v_5v_2^3a_1 + 162v_5v_4v_2^3a_2 \\
&+ 648v_4\sqrt{3}\,v_2^5b_3 + 162v_4\sqrt{3}\,v_2^4b_1 + 4860\sqrt{3}\,v_1^2v_2^4b_2 + 7776\sqrt{3}\,v_1v_2^5b_3 \\
&- 216v_5v_4v_2^3b_3 + 1296\sqrt{3}\,v_1^4v_2a_2 - 864\sqrt{3}\,v_1^3v_2^2a_3 + 4860\sqrt{3}\,v_1v_2^4b_1 \\
&- 36v_5v_4v_1^2a_3 - 54v_5v_4v_2^2b_1 + 162v_4\sqrt{3}\,v_1v_2^4b_2 + 216v_4\sqrt{3}\,v_1^3v_2a_2 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$\begin{aligned}
&\left(1296\sqrt{3}\,a_2 - 1728\sqrt{3}\,b_3\right) v_1^4 v_2 + \left(-5832\sqrt{3}\,a_2 + 7776\sqrt{3}\,b_3\right) v_1 v_2^5 \\
&+ \left(-486\sqrt{3}\,a_2 + 648\sqrt{3}\,b_3\right) v_2^5 v_4 + (162a_2 - 216b_3)\, v_2^3 v_4 v_5 \\
&+ 72v_4\sqrt{3}\,v_1^2 v_2^2 a_3 - 54v_5 v_4 v_1 v_2^2 b_2 + 72v_4\sqrt{3}\,v_1^2 v_2 a_1 - 864\sqrt{3}\,v_1^3 v_3 v_2^2 b_2 \\
&- 972\sqrt{3}\,v_1 v_3 v_2^4 a_3 + 324 v_1 v_5 v_3 v_2^2 a_3 - 1620 v_5 v_1^2 v_2^2 b_2 + 432\sqrt{3}\,v_1^3 v_2 a_1 \\
&- 972 v_5 v_3 v_2^4 b_2 + 2916\sqrt{3}\,v_3 v_2^6 b_2 - 144\sqrt{3}\,v_1^4 v_4 b_2 + 288\sqrt{3}\,v_1^4 v_3 a_3 \\
&- 144\sqrt{3}\,v_1^3 v_4 b_1 - 1620 v_5 v_1 v_2^2 b_1 + (1944a_2 - 2592b_3)\, v_1 v_2^3 v_5 \\
&+ \left(216\sqrt{3}\,a_2 - 288\sqrt{3}\,b_3\right) v_1^3 v_2 v_4 + 216 v_1^3 v_5 a_3 + 1458\sqrt{3}\,v_2^6 a_3 \\
&- 864\sqrt{3}\,v_1^5 b_2 - 2916\sqrt{3}\,v_2^5 a_1 - 864\sqrt{3}\,v_1^4 b_1 - 486 v_5 v_2^4 a_3 \\
&+ 972 v_5 v_2^3 a_1 + 162 v_4\sqrt{3}\,v_2^4 b_1 + 4860\sqrt{3}\,v_1^2 v_2^4 b_2 - 864\sqrt{3}\,v_1^3 v_2^2 a_3 \\
&+ 4860\sqrt{3}\,v_1 v_2^4 b_1 - 36 v_5 v_4 v_1^2 a_3 - 54 v_5 v_4 v_2^2 b_1 + 162 v_4\sqrt{3}\,v_1 v_2^4 b_2 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
972a_1 &= 0 \\
-486a_3 &= 0 \\
-36a_3 &= 0 \\
216a_3 &= 0 \\
324a_3 &= 0 \\
-1620b_1 &= 0 \\
-54b_1 &= 0 \\
-1620b_2 &= 0 \\
-972b_2 &= 0 \\
-54b_2 &= 0 \\
-2916\sqrt{3}\,a_1 &= 0 \\
72\sqrt{3}\,a_1 &= 0 \\
432\sqrt{3}\,a_1 &= 0 \\
-972\sqrt{3}\,a_3 &= 0 \\
-864\sqrt{3}\,a_3 &= 0 \\
72\sqrt{3}\,a_3 &= 0 \\
288\sqrt{3}\,a_3 &= 0 \\
1458\sqrt{3}\,a_3 &= 0 \\
-864\sqrt{3}\,b_1 &= 0 \\
-144\sqrt{3}\,b_1 &= 0 \\
162\sqrt{3}\,b_1 &= 0 \\
4860\sqrt{3}\,b_1 &= 0 \\
-864\sqrt{3}\,b_2 &= 0 \\
-144\sqrt{3}\,b_2 &= 0 \\
162\sqrt{3}\,b_2 &= 0 \\
2916\sqrt{3}\,b_2 &= 0 \\
4860\sqrt{3}\,b_2 &= 0 \\
162a_2 - 216b_3 &= 0 \\
1944a_2 - 2592b_3 &= 0 \\
-5832\sqrt{3}\,a_2 + 7776\sqrt{3}\,b_3 &= 0 \\
-486\sqrt{3}\,a_2 + 648\sqrt{3}\,b_3 &= 0 \\
216\sqrt{3}\,a_2 - 288\sqrt{3}\,b_3 &= 0 \\
1296\sqrt{3}\,a_2 - 1728\sqrt{3}\,b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0\\
a_2 &= \frac{4b_3}{3}\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= \frac{4x}{3}\\
\eta &= y
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi}\\
&= \frac{y}{\frac{4x}{3}}\\
&= \frac{3y}{4x}
\end{aligned}$$

This is easily solved to give

$$y = c_1 x^{\frac{3}{4}}$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{y}{x^{\frac{3}{4}}}$$

And S is found from

$$
\begin{aligned}
dS &= \frac{dx}{\xi} \\
&= \frac{dx}{\frac{4x}{3}}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
S &= \int \frac{dx}{T} \\
&= \frac{3\ln(x)}{4}
\end{aligned}
$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{\left(-27y^2 + 3\sqrt{3}\,\sqrt{27y^4 - 8x^3}\right)^{\frac{2}{3}} + 6x}{6y\left(-27y^2 + 3\sqrt{3}\,\sqrt{27y^4 - 8x^3}\right)^{\frac{1}{3}}}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= -\frac{3y}{4x^{\frac{7}{4}}} \\
R_y &= \frac{1}{x^{\frac{3}{4}}} \\
S_x &= \frac{3}{4x} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{9x^{\frac{3}{4}}y\left(-27y^2 + 3\sqrt{3}\,\sqrt{27y^4 - 8x^3}\right)^{\frac{1}{3}}}{2\left(-27y^2 + 3\sqrt{3}\,\sqrt{27y^4 - 8x^3}\right)^{\frac{2}{3}} x - 9y^2\left(-27y^2 + 3\sqrt{3}\,\sqrt{27y^4 - 8x^3}\right)^{\frac{1}{3}} + 12x^2} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{9R3^{\frac{1}{3}}\left(\sqrt{3}\,\sqrt{27R^4-8}-9R^2\right)^{\frac{1}{3}}}{2\left(\sqrt{3}\,\sqrt{27R^4-8}-9R^2\right)^{\frac{2}{3}}3^{\frac{2}{3}}-9\left(\sqrt{3}\,\sqrt{27R^4-8}-9R^2\right)^{\frac{1}{3}}3^{\frac{1}{3}}R^2+12}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int \frac{9R\left(3\sqrt{81R^4-24}-27R^2\right)^{\frac{1}{3}}}{2\,9^{\frac{1}{3}}\left(\left(\sqrt{81R^4-24}-9R^2\right)^2\right)^{\frac{1}{3}}-9R^2\left(3\sqrt{81R^4-24}-27R^2\right)^{\frac{1}{3}}+12}dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{3\ln(x)}{4} = \int^{\frac{y}{x^{\frac{3}{4}}}} \frac{9\_a\left(3\sqrt{81\_a^4-24}-27\_a^2\right)^{\frac{1}{3}}}{2\,9^{\frac{1}{3}}\left(\left(\sqrt{81\_a^4-24}-9\_a^2\right)^2\right)^{\frac{1}{3}}-9\_a^2\left(3\sqrt{81\_a^4-24}-27\_a^2\right)^{\frac{1}{3}}+12}d\_a + c_1$$

Which simplifies to

$$\frac{3\ln(x)}{4} = \int^{\frac{y}{x^{\frac{3}{4}}}} \frac{9\_a\left(3\sqrt{81\_a^4-24}-27\_a^2\right)^{\frac{1}{3}}}{2\,9^{\frac{1}{3}}\left(\left(\sqrt{81\_a^4-24}-9\_a^2\right)^2\right)^{\frac{1}{3}}-9\_a^2\left(3\sqrt{81\_a^4-24}-27\_a^2\right)^{\frac{1}{3}}+12}d\_a + c_1$$

Summary

The solution(s) found are the following

$$\frac{3\ln(x)}{4} = \int^{\frac{y}{x^{\frac{3}{4}}}} \frac{9\_a\left(3\sqrt{81\_a^4-24}-27\_a^2\right)^{\frac{1}{3}}}{2\,9^{\frac{1}{3}}\left(\left(\sqrt{81\_a^4-24}-9\_a^2\right)^2\right)^{\frac{1}{3}}-9\_a^2\left(3\sqrt{81\_a^4-24}-27\_a^2\right)^{\frac{1}{3}}+12}d\_a + c_1 \tag{1}$$

Verification of solutions

$$\frac{3\ln(x)}{4} = \int^{\frac{y}{x^{\frac{3}{4}}}} \frac{9\_a\left(3\sqrt{81\_a^4-24}-27\_a^2\right)^{\frac{1}{3}}}{2\,9^{\frac{1}{3}}\left(\left(\sqrt{81\_a^4-24}-9\_a^2\right)^2\right)^{\frac{1}{3}}-9\_a^2\left(3\sqrt{81\_a^4-24}-27\_a^2\right)^{\frac{1}{3}}+12} d\_a + c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = \frac{i\sqrt{3}\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}-6i\sqrt{3}\,x-\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}-6x}{12y\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{1}{3}}}$$

$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1,a_2,a_3,b_1,b_2,b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 \\
&+\frac{\left(i\sqrt{3}\left(-27y^2+3\sqrt{3}\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}-6i\sqrt{3}\,x-\left(-27y^2+3\sqrt{3}\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}-6x\right)(b_3-a_2)}{12y\left(-27y^2+3\sqrt{3}\sqrt{27y^4-8x^3}\right)^{\frac{1}{3}}} \\
&-\frac{\left(i\sqrt{3}\left(-27y^2+3\sqrt{3}\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}-6i\sqrt{3}\,x-\left(-27y^2+3\sqrt{3}\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}-6x\right)^2 a_3}{144y^2\left(-27y^2+3\sqrt{3}\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}} \\
&-\left(\frac{-\frac{72ix^2}{\left(-27y^2+3\sqrt{3}\sqrt{27y^4-8x^3}\right)^{\frac{1}{3}}\sqrt{27y^4-8x^3}}-6i\sqrt{3}+\frac{24\sqrt{3}\,x^2}{\left(-27y^2+3\sqrt{3}\sqrt{27y^4-8x^3}\right)^{\frac{1}{3}}\sqrt{27y^4-8x^3}}-6}{12y\left(-27y^2+3\sqrt{3}\sqrt{27y^4-8x^3}\right)^{\frac{1}{3}}}\right. \\
&\left.+\frac{\left(i\sqrt{3}\left(-27y^2+3\sqrt{3}\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}-6i\sqrt{3}\,x-\left(-27y^2+3\sqrt{3}\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}-6x\right)\sqrt{3}\,x^2}{y\left(-27y^2+3\sqrt{3}\sqrt{27y^4-8x^3}\right)^{\frac{4}{3}}\sqrt{27y^4-8x^3}}\right)(xa_2 \\
&+ya_3+a_1)-\left(\frac{\frac{2i\sqrt{3}\left(-54y+\frac{162\sqrt{3}\,y^3}{\sqrt{27y^4-8x^3}}\right)}{3\left(-27y^2+3\sqrt{3}\sqrt{27y^4-8x^3}\right)^{\frac{1}{3}}}-\frac{2\left(-54y+\frac{162\sqrt{3}\,y^3}{\sqrt{27y^4-8x^3}}\right)}{3\left(-27y^2+3\sqrt{3}\sqrt{27y^4-8x^3}\right)^{\frac{1}{3}}}}{12y\left(-27y^2+3\sqrt{3}\sqrt{27y^4-8x^3}\right)^{\frac{1}{3}}}\right. \\
&-\frac{i\sqrt{3}\left(-27y^2+3\sqrt{3}\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}-6i\sqrt{3}\,x-\left(-27y^2+3\sqrt{3}\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}-6x}{12y^2\left(-27y^2+3\sqrt{3}\sqrt{27y^4-8x^3}\right)^{\frac{1}{3}}} \\
&\left.-\frac{\left(i\sqrt{3}\left(-27y^2+3\sqrt{3}\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}-6i\sqrt{3}\,x-\left(-27y^2+3\sqrt{3}\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}-6x\right)\left(-54y+\frac{162\sqrt{3}\,y^3}{\sqrt{27y^4-8x^3}}\right)}{36y\left(-27y^2+3\sqrt{3}\sqrt{27y^4-8x^3}\right)^{\frac{4}{3}}}\right) \\
&+yb_3+b_1)=0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{1}{3}}, \left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}, \sqrt{27y^4-8x^3}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{1}{3}} = v_3, \left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}} = v_4, \sqrt{27y^4-8x^3} = v_5\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-108i\sqrt{3}\, v_5 v_4 v_1 v_2^2 b_2 - 3888 v_5 v_1 v_2^3 a_2 + 1296 v_1 v_3 v_5 v_2^2 a_3 \\
&- 324\sqrt{3}\, v_4 v_1 v_2^4 b_2 - 432\sqrt{3}\, v_4 v_1^3 v_2 a_2 - 144\sqrt{3}\, v_4 v_1^2 v_2^2 a_3 \\
&- 144\sqrt{3}\, v_4 v_1^2 v_2 a_1 + 108 v_5 v_4 v_1 v_2^2 b_2 + 576\sqrt{3}\, v_1^3 v_4 v_2 b_3 \\
&- 3456\sqrt{3}\, v_1^3 v_3 v_2^2 b_2 - 3888\sqrt{3}\, v_1 v_3 v_2^4 a_3 + 972i\sqrt{3}\, v_5 v_2^4 a_3 \\
&- 432i\sqrt{3}\, v_1^3 v_5 a_3 - 1944i\sqrt{3}\, v_5 v_2^3 a_1 + 432i v_4 v_1^2 v_2^2 a_3 \\
&+ 432i v_4 v_1^2 v_2 a_1 + 972i v_4 v_1 v_2^4 b_2 + 1296i v_4 v_1^3 v_2 a_2 \\
&- 1728i v_1^3 v_4 v_2 b_3 - 2592\sqrt{3}\, v_1^4 v_2 a_2 + 1728\sqrt{3}\, v_1^3 v_2^2 a_3 \\
&- 9720\sqrt{3}\, v_1 v_2^4 b_1 + 432 v_5 v_4 v_2^3 b_3 - 864\sqrt{3}\, v_1^3 v_2 a_1 \\
&+ 108 v_5 v_4 v_2^2 b_1 + 3240 v_5 v_1^2 v_2^2 b_2 + 5184 v_5 v_1 v_2^3 b_3 \\
&+ 3240 v_5 v_1 v_2^2 b_1 + 11664\sqrt{3}\, v_1 v_2^5 a_2 + 3456\sqrt{3}\, v_1^4 v_2 b_3 \\
&+ 72 v_5 v_4 v_1^2 a_3 - 1296\sqrt{3}\, v_4 v_2^5 b_3 - 324\sqrt{3}\, v_4 v_2^4 b_1 \\
&- 9720\sqrt{3}\, v_1^2 v_2^4 b_2 - 15552\sqrt{3}\, v_1 v_2^5 b_3 - 324 v_4 v_5 v_2^3 a_2 \\
&- 29160i v_1 v_2^4 b_1 - 2592i v_1^3 v_2 a_1 - 2916i v_4 v_2^5 a_2 \\
&+ 3888i v_4 v_2^5 b_3 - 864i v_1^4 v_4 b_2 + 972i v_4 v_2^4 b_1 - 864i v_1^3 v_4 b_1 \\
&- 29160i v_1^2 v_2^4 b_2 + 34992i v_1 v_2^5 a_2 - 46656i v_1 v_2^5 b_3 \\
&- 7776i v_1^4 v_2 a_2 + 10368i v_1^4 v_2 b_3 + 5184i v_1^3 v_2^2 a_3 \\
&+ 3240i\sqrt{3}\, v_5 v_1 v_2^2 b_1 + 324i\sqrt{3}\, v_4 v_5 v_2^3 a_2 - 432i\sqrt{3}\, v_5 v_4 v_2^3 b_3 \\
&- 108i\sqrt{3}\, v_5 v_4 v_2^2 b_1 - 72i\sqrt{3}\, v_5 v_4 v_1^2 a_3 + 3240i\sqrt{3}\, v_5 v_1^2 v_2^2 b_2 \\
&- 3888i\sqrt{3}\, v_5 v_1 v_2^3 a_2 + 288\sqrt{3}\, v_1^3 v_4 b_1 + 972\sqrt{3}\, v_4 v_2^5 a_2 \\
&+ 11664\sqrt{3}\, v_3 v_2^6 b_2 + 288\sqrt{3}\, v_1^4 v_4 b_2 + 1152\sqrt{3}\, v_1^4 v_3 a_3 \\
&- 3888 v_3 v_5 v_2^4 b_2 + 972 v_5 v_2^4 a_3 - 1944 v_5 v_2^3 a_1 \\
&- 2916\sqrt{3}\, v_2^6 a_3 + 1728\sqrt{3}\, v_1^5 b_2 + 5832\sqrt{3}\, v_2^5 a_1 \\
&+ 1728\sqrt{3}\, v_1^4 b_1 + 5184i v_1^4 b_1 - 8748i v_2^6 a_3 + 5184i v_1^5 b_2 \\
&+ 17496i v_2^5 a_1 - 432 v_1^3 v_5 a_3 + 5184i\sqrt{3}\, v_5 v_1 v_2^3 b_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&\left(-432\sqrt{3}\,a_2 + 576\sqrt{3}\,b_3 + 1296ia_2 - 1728ib_3\right) v_1^3 v_2 v_4 \\
&+ \left(-144\sqrt{3}\,a_3 + 432ia_3\right) v_1^2 v_2^2 v_4 \\
&+ \left(3240i\sqrt{3}\,b_2 + 3240b_2\right) v_1^2 v_2^2 v_5 \\
&+ \left(-144\sqrt{3}\,a_1 + 432ia_1\right) v_1^2 v_2 v_4 \\
&+ \left(-72i\sqrt{3}\,a_3 + 72a_3\right) v_1^2 v_4 v_5 \\
&+ \left(-324\sqrt{3}\,b_2 + 972ib_2\right) v_1 v_2^4 v_4 \\
&+ \left(-3888i\sqrt{3}\,a_2 + 5184i\sqrt{3}\,b_3 - 3888a_2 + 5184b_3\right) v_1 v_2^3 v_5 \\
&+ \left(3240i\sqrt{3}\,b_1 + 3240b_1\right) v_1 v_2^2 v_5 \\
&+ \left(324i\sqrt{3}\,a_2 - 432i\sqrt{3}\,b_3 - 324a_2 + 432b_3\right) v_2^3 v_4 v_5 \\
&+ \left(-108i\sqrt{3}\,b_1 + 108b_1\right) v_2^2 v_4 v_5 + 1296 v_1 v_3 v_5 v_2^2 a_3 \\
&- 3456\sqrt{3}\,v_1^3 v_3 v_2^2 b_2 - 3888\sqrt{3}\,v_1 v_3 v_2^4 a_3 \\
&+ \left(-108i\sqrt{3}\,b_2 + 108b_2\right) v_1 v_2^2 v_4 v_5 \\
&+ \left(1728\sqrt{3}\,a_3 + 5184ia_3\right) v_1^3 v_2^2 \\
&+ \left(-864\sqrt{3}\,a_1 - 2592ia_1\right) v_1^3 v_2 \\
&+ \left(288\sqrt{3}\,b_1 - 864ib_1\right) v_1^3 v_4 + \left(-432i\sqrt{3}\,a_3 - 432a_3\right) v_1^3 v_5 \\
&+ \left(-9720\sqrt{3}\,b_2 - 29160ib_2\right) v_1^2 v_2^4 \\
&+ \left(11664\sqrt{3}\,a_2 - 15552\sqrt{3}\,b_3 + 34992ia_2 - 46656ib_3\right) v_1 v_2^5 \\
&+ \left(-9720\sqrt{3}\,b_1 - 29160ib_1\right) v_1 v_2^4 \\
&+ \left(972\sqrt{3}\,a_2 - 1296\sqrt{3}\,b_3 - 2916ia_2 + 3888ib_3\right) v_2^5 v_4 \\
&+ 11664\sqrt{3}\,v_3 v_2^6 b_2 + 1152\sqrt{3}\,v_1^4 v_3 a_3 - 3888 v_3 v_5 v_2^4 b_2 \\
&+ \left(1728\sqrt{3}\,b_2 + 5184ib_2\right) v_1^5 + \left(1728\sqrt{3}\,b_1 + 5184ib_1\right) v_1^4 \\
&+ \left(-2916\sqrt{3}\,a_3 - 8748ia_3\right) v_2^6 \\
&+ \left(5832\sqrt{3}\,a_1 + 17496ia_1\right) v_2^5 \\
&+ \left(-324\sqrt{3}\,b_1 + 972ib_1\right) v_2^4 v_4 + \left(972i\sqrt{3}\,a_3 + 972a_3\right) v_2^4 v_5 \\
&+ \left(-1944i\sqrt{3}\,a_1 - 1944a_1\right) v_2^3 v_5 \\
&+ \left(-2592\sqrt{3}\,a_2 + 3456\sqrt{3}\,b_3 - 7776ia_2 + 10368ib_3\right) v_1^4 v_2 \\
&+ \left(288\sqrt{3}\,b_2 - 864ib_2\right) v_1^4 v_4 = 0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
1296a_3 &= 0\\
-3888b_2 &= 0\\
-3888\sqrt{3}\,a_3 &= 0\\
1152\sqrt{3}\,a_3 &= 0\\
-3456\sqrt{3}\,b_2 &= 0\\
11664\sqrt{3}\,b_2 &= 0\\
-864\sqrt{3}\,a_1 - 2592ia_1 &= 0\\
-144\sqrt{3}\,a_1 + 432ia_1 &= 0\\
5832\sqrt{3}\,a_1 + 17496ia_1 &= 0\\
-2916\sqrt{3}\,a_3 - 8748ia_3 &= 0\\
-144\sqrt{3}\,a_3 + 432ia_3 &= 0\\
1728\sqrt{3}\,a_3 + 5184ia_3 &= 0\\
-9720\sqrt{3}\,b_1 - 29160ib_1 &= 0\\
-324\sqrt{3}\,b_1 + 972ib_1 &= 0\\
288\sqrt{3}\,b_1 - 864ib_1 &= 0\\
1728\sqrt{3}\,b_1 + 5184ib_1 &= 0\\
-9720\sqrt{3}\,b_2 - 29160ib_2 &= 0\\
-324\sqrt{3}\,b_2 + 972ib_2 &= 0\\
288\sqrt{3}\,b_2 - 864ib_2 &= 0\\
1728\sqrt{3}\,b_2 + 5184ib_2 &= 0\\
-1944i\sqrt{3}\,a_1 - 1944a_1 &= 0\\
-432i\sqrt{3}\,a_3 - 432a_3 &= 0\\
-108i\sqrt{3}\,b_1 + 108b_1 &= 0\\
-108i\sqrt{3}\,b_2 + 108b_2 &= 0\\
-72i\sqrt{3}\,a_3 + 72a_3 &= 0\\
972i\sqrt{3}\,a_3 + 972a_3 &= 0\\
3240i\sqrt{3}\,b_1 + 3240b_1 &= 0\\
3240i\sqrt{3}\,b_2 + 3240b_2 &= 0\\
-2592\sqrt{3}\,a_2 + 3456\sqrt{3}\,b_3 - 7776ia_2 + 10368ib_3 &= 0\\
-432\sqrt{3}\,a_2 + 576\sqrt{3}\,b_3 + 1296ia_2 - 1728ib_3 &= 0\\
972\sqrt{3}\,a_2 - 1296\sqrt{3}\,b_3 - 2916ia_2 + 3888ib_3 &= 0\\
11664\sqrt{3}\,a_2 - 15552\sqrt{3}\,b_3 + 34992ia_2 - 46656ib_3 &= 0\\
-3888i\sqrt{3}\,a_2 + 5184i\sqrt{3}\,b_3 - 3888a_2 + 5184b_3 &= 0\\
324i\sqrt{3}\,a_2 - 432i\sqrt{3}\,b_3 - 324a_2 + 432b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{4b_3}{3} \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= \frac{4x}{3} \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Solving equation (3)

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{i\sqrt{3}\left(-27y^2 + 3\sqrt{3}\,\sqrt{27y^4 - 8x^3}\right)^{\frac{2}{3}} - 6i\sqrt{3}\,x + \left(-27y^2 + 3\sqrt{3}\,\sqrt{27y^4 - 8x^3}\right)^{\frac{2}{3}} + 6x}{12y\left(-27y^2 + 3\sqrt{3}\,\sqrt{27y^4 - 8x^3}\right)^{\frac{1}{3}}} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 \qquad\qquad (5E)\\
&-\frac{\left(i\sqrt{3}\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}-6i\sqrt{3}\,x+\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}+6x\right)(b_3-a_2)}{12y\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{1}{3}}}\\
&-\frac{\left(i\sqrt{3}\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}-6i\sqrt{3}\,x+\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}+6x\right)^2 a_3}{144y^2\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}}\\
&-\left(-\frac{-\frac{72ix^2}{\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{1}{3}}\sqrt{27y^4-8x^3}}-6i\sqrt{3}-\frac{24\sqrt{3}\,x^2}{\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{1}{3}}\sqrt{27y^4-8x^3}}+6}{12y\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{1}{3}}}\right.\\
&\left.-\frac{\left(i\sqrt{3}\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}-6i\sqrt{3}\,x+\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}+6x\right)\sqrt{3}\,x^2}{y\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{4}{3}}\sqrt{27y^4-8x^3}}\right)(xa_2\\
&+ya_3+a_1)-\left(-\frac{\frac{2i\sqrt{3}\left(-54y+\frac{162\sqrt{3}\,y^3}{\sqrt{27y^4-8x^3}}\right)}{3\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{1}{3}}}+\frac{-36y+\frac{108\sqrt{3}\,y^3}{\sqrt{27y^4-8x^3}}}{\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{1}{3}}}}{12y\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{1}{3}}}\right.\\
&+\frac{i\sqrt{3}\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}-6i\sqrt{3}\,x+\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}+6x}{12y^2\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{1}{3}}}\\
&+\frac{\left(i\sqrt{3}\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}-6i\sqrt{3}\,x+\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}+6x\right)\left(-54y+\frac{162\sqrt{3}\,y}{\sqrt{27y^4-8}}\right.}{36y\left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{4}{3}}}\\
&+yb_3+b_1)=0
\end{aligned}
$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{1}{3}}, \left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}}, \sqrt{27y^4-8x^3}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{1}{3}} = v_3, \left(-27y^2+3\sqrt{3}\,\sqrt{27y^4-8x^3}\right)^{\frac{2}{3}} = v_4, \sqrt{27y^4-8x^3} = v_5\right\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-972iv_4v_1v_2^4b_2 - 1296iv_4v_1^3v_2a_2 + 1728iv_1^3v_4v_2b_3 \\
&- 432iv_4v_1^2v_2^2a_3 - 432iv_4v_1^2v_2a_1 - 972iv_5\sqrt{3}\,v_2^4a_3 \\
&+ 432i\sqrt{3}\,v_1^3v_5a_3 + 1944iv_5\sqrt{3}\,v_2^3a_1 + 8748iv_2^6a_3 \\
&- 5184iv_1^5b_2 - 17496iv_2^5a_1 - 5184iv_1^4b_1 + 7776iv_1^4v_2a_2 \\
&- 10368iv_1^4v_2b_3 - 5184iv_1^3v_2^2a_3 + 29160iv_1v_2^4b_1 \\
&- 432v_1^3v_5a_3 - 2916\sqrt{3}\,v_2^6a_3 + 1728\sqrt{3}\,v_1^5b_2 \\
&+ 5832\sqrt{3}\,v_2^5a_1 + 1728\sqrt{3}\,v_1^4b_1 + 972v_5v_2^4a_3 - 1944v_5v_2^3a_1 \\
&+ 576\sqrt{3}\,v_1^3v_4v_2b_3 - 3456\sqrt{3}\,v_1^3v_3v_2^2b_2 - 3888\sqrt{3}\,v_1v_3v_2^4a_3 \\
&- 144v_4\sqrt{3}\,v_1^2v_2a_1 + 1296v_1v_3v_5v_2^2a_3 - 324v_4\sqrt{3}\,v_1v_2^4b_2 \\
&- 432v_4\sqrt{3}\,v_1^3v_2a_2 - 144v_4\sqrt{3}\,v_1^2v_2^2a_3 + 108v_4v_5v_1v_2^2b_2 \\
&- 2592\sqrt{3}\,v_1^4v_2a_2 + 1728\sqrt{3}\,v_1^3v_2^2a_3 - 9720\sqrt{3}\,v_1v_2^4b_1 \\
&+ 108v_4v_5v_2^2b_1 + 3240v_5v_1^2v_2^2b_2 + 5184v_5v_1v_2^3b_3 \\
&- 864\sqrt{3}\,v_1^3v_2a_1 + 3240v_5v_1v_2^2b_1 + 11664\sqrt{3}\,v_1v_2^5a_2 \\
&+ 3456\sqrt{3}\,v_1^4v_2b_3 - 324v_4v_5v_2^3a_2 - 3888v_3v_5v_2^4b_2 \\
&+ 972\sqrt{3}\,v_4v_2^5a_2 + 11664\sqrt{3}\,v_3v_2^6b_2 + 288\sqrt{3}\,v_1^4v_4b_2 \\
&+ 1152\sqrt{3}\,v_1^4v_3a_3 + 288\sqrt{3}\,v_1^3v_4b_1 + 29160iv_1^2v_2^4b_2 \\
&+ 2592iv_1^3v_2a_1 - 972iv_4v_2^4b_1 + 864iv_1^3v_4b_1 + 2916iv_4v_2^5a_2 \\
&- 3888iv_4v_2^5b_3 + 864iv_1^4v_4b_2 - 34992iv_1v_2^5a_2 \\
&+ 46656iv_1v_2^5b_3 - 3240iv_5\sqrt{3}\,v_1^2v_2^2b_2 + 3888iv_5\sqrt{3}\,v_1v_2^3a_2 \\
&- 5184iv_5\sqrt{3}\,v_1v_2^3b_3 - 3240iv_5\sqrt{3}\,v_1v_2^2b_1 \\
&+ 72iv_4v_5\sqrt{3}\,v_1^2a_3 - 324i\sqrt{3}\,v_4v_5v_2^3a_2 + 432iv_4v_5\sqrt{3}\,v_2^3b_3 \\
&+ 108iv_4v_5\sqrt{3}\,v_2^2b_1 - 3888v_5v_1v_2^3a_2 + 72v_4v_5v_1^2a_3 \\
&- 1296v_4\sqrt{3}\,v_2^5b_3 - 324v_4\sqrt{3}\,v_2^4b_1 - 9720\sqrt{3}\,v_1^2v_2^4b_2 \\
&- 15552\sqrt{3}\,v_1v_2^5b_3 + 432v_4v_5v_2^3b_3 + 108iv_4v_5\sqrt{3}\,v_1v_2^2b_2 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&-3456\sqrt{3}\,v_1^3v_3v_2^2b_2 - 3888\sqrt{3}\,v_1v_3v_2^4a_3 + 1296v_1v_3v_5v_2^2a_3 \\
&+ \left(7776ia_2 - 10368ib_3 - 2592\sqrt{3}\,a_2 + 3456\sqrt{3}\,b_3\right) v_1^4v_2 \\
&+ \left(864ib_2 + 288\sqrt{3}\,b_2\right) v_1^4v_4 \\
&+ \left(-5184ia_3 + 1728\sqrt{3}\,a_3\right) v_1^3v_2^2 \\
&+ \left(2592ia_1 - 864\sqrt{3}\,a_1\right) v_1^3v_2 \\
&+ \left(864ib_1 + 288\sqrt{3}\,b_1\right) v_1^3v_4 + \left(432i\sqrt{3}\,a_3 - 432a_3\right) v_1^3v_5 \\
&+ \left(29160ib_2 - 9720\sqrt{3}\,b_2\right) v_1^2v_2^4 + \Big(-34992ia_2 \\
&+ 46656ib_3 + 11664\sqrt{3}\,a_2 - 15552\sqrt{3}\,b_3\Big) v_1v_2^5 \\
&+ \left(29160ib_1 - 9720\sqrt{3}\,b_1\right) v_1v_2^4 \\
&+ \left(2916ia_2 - 3888ib_3 + 972\sqrt{3}\,a_2 - 1296\sqrt{3}\,b_3\right) v_2^5v_4 \\
&+ \left(-972ib_1 - 324\sqrt{3}\,b_1\right) v_2^4v_4 \\
&+ \left(-972i\sqrt{3}\,a_3 + 972a_3\right) v_2^4v_5 \\
&+ \left(1944i\sqrt{3}\,a_1 - 1944a_1\right) v_2^3v_5 \\
&- 3888v_3v_5v_2^4b_2 + 11664\sqrt{3}\,v_3v_2^6b_2 + 1152\sqrt{3}\,v_1^4v_3a_3 \\
&+ \left(-5184ib_1 + 1728\sqrt{3}\,b_1\right) v_1^4 + \left(8748ia_3 - 2916\sqrt{3}\,a_3\right) v_2^6 \\
&+ \left(-17496ia_1 + 5832\sqrt{3}\,a_1\right) v_2^5 \\
&+ \left(-5184ib_2 + 1728\sqrt{3}\,b_2\right) v_1^5 \\
&+ \left(108i\sqrt{3}\,b_2 + 108b_2\right) v_1v_2^2v_4v_5 \\
&+ \left(-1296ia_2 + 1728ib_3 - 432\sqrt{3}\,a_2 + 576\sqrt{3}\,b_3\right) v_1^3v_2v_4 \\
&+ \left(-432ia_3 - 144\sqrt{3}\,a_3\right) v_1^2v_2^2v_4 \\
&+ \left(-3240i\sqrt{3}\,b_2 + 3240b_2\right) v_1^2v_2^2v_5 \\
&+ \left(-432ia_1 - 144\sqrt{3}\,a_1\right) v_1^2v_2v_4 \\
&+ \left(72i\sqrt{3}\,a_3 + 72a_3\right) v_1^2v_4v_5 \\
&+ \left(-972ib_2 - 324\sqrt{3}\,b_2\right) v_1v_2^4v_4 \\
&+ \left(3888i\sqrt{3}\,a_2 - 5184i\sqrt{3}\,b_3 - 3888a_2 + 5184b_3\right) v_1v_2^3v_5 \\
&+ \left(-3240i\sqrt{3}\,b_1 + 3240b_1\right) v_1v_2^2v_5 \\
&+ \left(-324i\sqrt{3}\,a_2 + 432i\sqrt{3}\,b_3 - 324a_2 + 432b_3\right) v_2^3v_4v_5 \\
&+ \left(108i\sqrt{3}\,b_1 + 108b_1\right) v_2^2v_4v_5 = 0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$1296a_3 = 0$$
$$-3888b_2 = 0$$
$$-3888\sqrt{3}\, a_3 = 0$$
$$1152\sqrt{3}\, a_3 = 0$$
$$-3456\sqrt{3}\, b_2 = 0$$
$$11664\sqrt{3}\, b_2 = 0$$
$$-17496ia_1 + 5832\sqrt{3}\, a_1 = 0$$
$$-5184ia_3 + 1728\sqrt{3}\, a_3 = 0$$
$$-5184ib_1 + 1728\sqrt{3}\, b_1 = 0$$
$$-5184ib_2 + 1728\sqrt{3}\, b_2 = 0$$
$$-972ib_1 - 324\sqrt{3}\, b_1 = 0$$
$$-972ib_2 - 324\sqrt{3}\, b_2 = 0$$
$$-432ia_1 - 144\sqrt{3}\, a_1 = 0$$
$$-432ia_3 - 144\sqrt{3}\, a_3 = 0$$
$$864ib_1 + 288\sqrt{3}\, b_1 = 0$$
$$864ib_2 + 288\sqrt{3}\, b_2 = 0$$
$$2592ia_1 - 864\sqrt{3}\, a_1 = 0$$
$$8748ia_3 - 2916\sqrt{3}\, a_3 = 0$$
$$29160ib_1 - 9720\sqrt{3}\, b_1 = 0$$
$$29160ib_2 - 9720\sqrt{3}\, b_2 = 0$$
$$-3240i\sqrt{3}\, b_1 + 3240b_1 = 0$$
$$-3240i\sqrt{3}\, b_2 + 3240b_2 = 0$$
$$-972i\sqrt{3}\, a_3 + 972a_3 = 0$$
$$72i\sqrt{3}\, a_3 + 72a_3 = 0$$
$$108i\sqrt{3}\, b_1 + 108b_1 = 0$$
$$108i\sqrt{3}\, b_2 + 108b_2 = 0$$
$$432i\sqrt{3}\, a_3 - 432a_3 = 0$$
$$1944i\sqrt{3}\, a_1 - 1944a_1 = 0$$
$$-34992ia_2 + 46656ib_3 + 11664\sqrt{3}\, a_2 - 15552\sqrt{3}\, b_3 = 0$$
$$-1296ia_2 + 1728ib_3 - 432\sqrt{3}\, a_2 + 576\sqrt{3}\, b_3 = 0$$
$$2916ia_2 - 3888ib_3 + 972\sqrt{3}\, a_2 - 1296\sqrt{3}\, b_3 = 0$$
$$7776ia_2 - 10368ib_3 - 2592\sqrt{3}\, a_2 + 3456\sqrt{3}\, b_3 = 0$$
$$-324i\sqrt{3}\, a_2 + 432i\sqrt{3}\, b_3 - 324a_2 + 432b_3 = 0$$
$$3888i\sqrt{3}\, a_2 - 5184i\sqrt{3}\, b_3 - 3888a_2 + 5184b_3 = 0$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{4b_3}{3} \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= \frac{4x}{3} \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 3 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
   -> Solving 1st order ODE of high degree, Lie methods, 1st trial
   `, `-> Computing symmetries using: way = 2
   `, `-> Computing symmetries using: way = 2
   -> Solving 1st order ODE of high degree, 2nd attempt. Trying parametric methods
   trying dAlembert
   -> Calling odsolve with the ODE`, diff(y(x), x) = (-8*y(x)^2*x^3+y(x))/(8*y(x)*x^4-x), y(
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   <- 1st order, parametric methods successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 100

```
dsolve(4*y(x)^2*diff(y(x),x)^3-2*x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{2^{\frac{3}{4}} 3^{\frac{1}{4}} x^{\frac{3}{4}}}{3}$$
$$y(x) = \frac{2^{\frac{3}{4}} 3^{\frac{1}{4}} x^{\frac{3}{4}}}{3}$$
$$y(x) = -\frac{i 2^{\frac{3}{4}} 3^{\frac{1}{4}} x^{\frac{3}{4}}}{3}$$
$$y(x) = \frac{i 2^{\frac{3}{4}} 3^{\frac{1}{4}} x^{\frac{3}{4}}}{3}$$
$$y(x) = 0$$
$$y(x) = \sqrt{2}\, \sqrt{c_1 \left(-2 c_1^2 + x\right)}$$
$$y(x) = -\sqrt{2}\, \sqrt{c_1 \left(-2 c_1^2 + x\right)}$$

✓ Solution by Mathematica

Time used: 83.967 (sec). Leaf size: 11250

```
DSolve[4*y[x]^2*(y'[x])^3-2*x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

2.10 problem 11

Internal problem ID [6874]
Internal file name [`OUTPUT/6117_Friday_August_05_2022_02_20_09_AM_27870328/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 16. Nonlinear equations. Miscellaneous Exercises. Page 340
Problem number: 11.
ODE order: 1.
ODE degree: 4.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _dAlembert]
```

$$\boxed{y'^4 + xy' - 3y = 0}$$

2.10.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$p^4 + xp - 3y = 0$$

Solving for y from the above results in

$$y = \frac{1}{3}p^4 + \frac{1}{3}xp \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = \frac{p}{3}$$
$$g = \frac{p^4}{3}$$

Hence (2) becomes

$$\frac{2p}{3} = \left(\frac{x}{3} + \frac{4p^3}{3}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$\frac{2p}{3} = 0$$

Solving for p from the above gives

$$p = 0$$

Substituting these in (1A) gives

$$y = 0$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{2p(x)}{3\left(\frac{x}{3} + \frac{4p(x)^3}{3}\right)} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{\frac{x(p)}{2} + 2p^3}{p} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = -\frac{1}{2p}$$
$$q(p) = 2p^2$$

Hence the ode is

$$\frac{d}{dp}x(p) - \frac{x(p)}{2p} = 2p^2$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{2p}dp} \\ &= \frac{1}{\sqrt{p}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}(\mu x) &= (\mu)\left(2p^2\right) \\ \frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{x}{\sqrt{p}}\right) &= \left(\frac{1}{\sqrt{p}}\right)\left(2p^2\right) \\ \mathrm{d}\left(\frac{x}{\sqrt{p}}\right) &= \left(2p^{\frac{3}{2}}\right)\,\mathrm{d}p\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{x}{\sqrt{p}} &= \int 2p^{\frac{3}{2}}\,\mathrm{d}p \\ \frac{x}{\sqrt{p}} &= \frac{4p^{\frac{5}{2}}}{5} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{p}}$ results in

$$x(p) = \frac{4p^3}{5} + c_1\sqrt{p}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = \mathrm{RootOf}\left(\_Z^4 + x\_Z - 3y\right)$$

Substituting the above in the solution for x found above gives

$$x = \frac{4\,\mathrm{RootOf}\left(\_Z^4 + x\_Z - 3y\right)^3}{5} + c_1\sqrt{\mathrm{RootOf}\left(\_Z^4 + x\_Z - 3y\right)}$$

<u>Summary</u>

The solution(s) found are the following

$$y = 0 \tag{1}$$

$$x = \frac{4\,\mathrm{RootOf}\left(\_Z^4 + x\_Z - 3y\right)^3}{5} + c_1\sqrt{\mathrm{RootOf}\left(\_Z^4 + x\_Z - 3y\right)} \tag{2}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$x = \frac{4 \operatorname{RootOf}\left(\_Z^4 + x\_Z - 3y\right)^3}{5} + c_1 \sqrt{\operatorname{RootOf}\left(\_Z^4 + x\_Z - 3y\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 34

```
dsolve(diff(y(x),x)^4+x*diff(y(x),x)-3*y(x)=0,y(x), singsol=all)
```

$$\left[x(\_T) = \frac{\sqrt{\_T}\left(4\_T^{\frac{5}{2}} + 5c_1\right)}{5}, y(\_T) = \frac{3\_T^4}{5} + \frac{\_T^{\frac{3}{2}} c_1}{3}\right]$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(y'[x])^4+x*y'[x]-3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Timed out

2.11 problem 13

Internal problem ID [6875]
Internal file name [`OUTPUT/6118_Friday_August_05_2022_02_20_50_AM_29999539/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 16. Nonlinear equations. Miscellaneous Exercises. Page 340
Problem number: 13.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _Clairaut]`

$$\boxed{x^2{y'}^3-2xy{y'}^2+y^2y'=-1}$$

2.11.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$x^2p^3 - 2xy\,p^2 + y^2p = -1$$

Solving for y from the above results in

$$y = \frac{px\sqrt{-p}-1}{\sqrt{-p}} \tag{1A}$$

$$y = \frac{px\sqrt{-p}+1}{\sqrt{-p}} \tag{2A}$$

Each of the above ode's is a Clairaut ode which is now solved. Solving ode 1A We start by replacing y' by p which gives

$$\begin{aligned} y &= px - \frac{1}{\sqrt{-p}} \\ &= px - \frac{1}{\sqrt{-p}} \end{aligned}$$

Writing the ode as

$$y = px + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = px + g \tag{1}$$

Then we see that

$$g = -\frac{1}{\sqrt{-p}}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\,\frac{dp}{dx} \\
0 &= (x + g')\,\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1 x - \frac{1}{\sqrt{-c_1}}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = -\frac{1}{\sqrt{-p}}$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x - \frac{1}{2\,(-p)^{\frac{3}{2}}} \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$
\begin{aligned}
p_1 &= -\frac{4^{\frac{2}{3}} (x^2)^{\frac{2}{3}}}{4x^2} \\
p_2 &= -\left(-\frac{4^{\frac{1}{3}} (x^2)^{\frac{1}{3}}}{4x} + \frac{i\sqrt{3}\, 4^{\frac{1}{3}} (x^2)^{\frac{1}{3}}}{4x}\right)^2 \\
p_3 &= -\left(-\frac{4^{\frac{1}{3}} (x^2)^{\frac{1}{3}}}{4x} - \frac{i\sqrt{3}\, 4^{\frac{1}{3}} (x^2)^{\frac{1}{3}}}{4x}\right)^2
\end{aligned}
$$

Substituting the above back in (1) results in

$$
\begin{aligned}
y_1 &= -\frac{\left(\frac{(x^2)^{\frac{2}{3}} \sqrt{\frac{(x^2)^{\frac{2}{3}}}{x^2}}}{2} + x\right) 2^{\frac{1}{3}}}{\sqrt{\frac{(x^2)^{\frac{2}{3}}}{x^2}}\, x} \\
y_2 &= \frac{\left(1 + i\sqrt{3}\right) (x^2)^{\frac{2}{3}}\, 2^{\frac{1}{3}} \sqrt{-\frac{(x^2)^{\frac{2}{3}} \left(1+i\sqrt{3}\right)}{x^2}} - 4\, 2^{\frac{5}{6}} x}{4\sqrt{-\frac{(x^2)^{\frac{2}{3}} \left(1+i\sqrt{3}\right)}{x^2}}\, x} \\
y_3 &= -\frac{\left(i\sqrt{3} - 1\right) (x^2)^{\frac{2}{3}}\, 2^{\frac{1}{3}} \sqrt{\frac{(x^2)^{\frac{2}{3}} \left(i\sqrt{3}-1\right)}{x^2}} + 4\, 2^{\frac{5}{6}} x}{4\sqrt{\frac{(x^2)^{\frac{2}{3}} \left(i\sqrt{3}-1\right)}{x^2}}\, x}
\end{aligned}
$$

Solving ode 2A We start by replacing y' by p which gives

$$
\begin{aligned}
y &= px + \frac{1}{\sqrt{-p}} \\
&= px + \frac{1}{\sqrt{-p}}
\end{aligned}
$$

Writing the ode as

$$
y = px + g(p)
$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$
y = px + g \tag{1}
$$

Then we see that

$$g = \frac{1}{\sqrt{-p}}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_2 x + \frac{1}{\sqrt{-c_2}}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = \frac{1}{\sqrt{-p}}$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x + \frac{1}{2(-p)^{\frac{3}{2}}} \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$\begin{aligned}
p_1 &= -\frac{(-4x^2)^{\frac{2}{3}}}{4x^2} \\
p_2 &= -\left(-\frac{(-4x^2)^{\frac{1}{3}}}{4x} - \frac{i\sqrt{3}\,(-4x^2)^{\frac{1}{3}}}{4x}\right)^2 \\
p_3 &= -\left(-\frac{(-4x^2)^{\frac{1}{3}}}{4x} + \frac{i\sqrt{3}\,(-4x^2)^{\frac{1}{3}}}{4x}\right)^2
\end{aligned}$$

Substituting the above back in (1) results in

$$
\begin{aligned}
y_1 &= -\frac{2^{\frac{1}{3}} \left((-x^2)^{\frac{2}{3}} \sqrt{\frac{(-x^2)^{\frac{2}{3}}}{x^2}} - 2x \right)}{2x \sqrt{\frac{(-x^2)^{\frac{2}{3}}}{x^2}}} \\
y_2 &= -\frac{\left((-x^2)^{\frac{2}{3}} \left(i\sqrt{6} - \sqrt{2}\right) \sqrt{\frac{(-x^2)^{\frac{2}{3}} \left(i\sqrt{3}-1\right)}{x^2}} - 8x \right) 2^{\frac{5}{6}}}{8\sqrt{\frac{(-x^2)^{\frac{2}{3}} \left(i\sqrt{3}-1\right)}{x^2}}\, x} \\
y_3 &= \frac{\left((-x^2)^{\frac{2}{3}} \left(i\sqrt{6} + \sqrt{2}\right) \sqrt{-\frac{(-x^2)^{\frac{2}{3}} \left(1+i\sqrt{3}\right)}{x^2}} + 8x \right) 2^{\frac{5}{6}}}{8\sqrt{-\frac{(-x^2)^{\frac{2}{3}} \left(1+i\sqrt{3}\right)}{x^2}}\, x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 x - \frac{1}{\sqrt{-c_1}} \tag{1}$$

$$y = -\frac{\left(\frac{(x^2)^{\frac{2}{3}} \sqrt{\frac{(x^2)^{\frac{2}{3}}}{x^2}}}{2} + x\right) 2^{\frac{1}{3}}}{\sqrt{\frac{(x^2)^{\frac{2}{3}}}{x^2}}\, x} \tag{2}$$

$$y = \frac{\left(1 + i\sqrt{3}\right) (x^2)^{\frac{2}{3}}\, 2^{\frac{1}{3}} \sqrt{-\frac{(x^2)^{\frac{2}{3}} \left(1+i\sqrt{3}\right)}{x^2}} - 4\, 2^{\frac{5}{6}} x}{4\sqrt{-\frac{(x^2)^{\frac{2}{3}} \left(1+i\sqrt{3}\right)}{x^2}}\, x} \tag{3}$$

$$y = -\frac{\left(i\sqrt{3} - 1\right) (x^2)^{\frac{2}{3}}\, 2^{\frac{1}{3}} \sqrt{\frac{(x^2)^{\frac{2}{3}} \left(i\sqrt{3}-1\right)}{x^2}} + 4\, 2^{\frac{5}{6}} x}{4\sqrt{\frac{(x^2)^{\frac{2}{3}} \left(i\sqrt{3}-1\right)}{x^2}}\, x} \tag{4}$$

$$y = c_2 x + \frac{1}{\sqrt{-c_2}} \tag{5}$$

$$y = -\frac{2^{\frac{1}{3}} \left((-x^2)^{\frac{2}{3}} \sqrt{\frac{(-x^2)^{\frac{2}{3}}}{x^2}} - 2x\right)}{2x\sqrt{\frac{(-x^2)^{\frac{2}{3}}}{x^2}}} \tag{6}$$

$$y = -\frac{\left((-x^2)^{\frac{2}{3}} \left(i\sqrt{6} - \sqrt{2}\right) \sqrt{\frac{(-x^2)^{\frac{2}{3}} \left(i\sqrt{3}-1\right)}{x^2}} - 8x\right) 2^{\frac{5}{6}}}{8\sqrt{\frac{(-x^2)^{\frac{2}{3}} \left(i\sqrt{3}-1\right)}{x^2}}\, x} \tag{7}$$

$$y = \frac{\left((-x^2)^{\frac{2}{3}} \left(i\sqrt{6} + \sqrt{2}\right) \sqrt{-\frac{(-x^2)^{\frac{2}{3}} \left(1+i\sqrt{3}\right)}{x^2}} + 8x\right) 2^{\frac{5}{6}}}{8\sqrt{-\frac{(-x^2)^{\frac{2}{3}} \left(1+i\sqrt{3}\right)}{x^2}}\, x} \tag{8}$$

Verification of solutions

$$y = c_1 x - \frac{1}{\sqrt{-c_1}}$$

Verified OK.

$$y = -\frac{\left(\frac{(x^2)^{\frac{2}{3}}\sqrt{\frac{(x^2)^{\frac{2}{3}}}{x^2}}}{2} + x\right) 2^{\frac{1}{3}}}{\sqrt{\frac{(x^2)^{\frac{2}{3}}}{x^2}}\, x}$$

Verified OK.

$$y = \frac{\left(1 + i\sqrt{3}\right) (x^2)^{\frac{2}{3}}\, 2^{\frac{1}{3}} \sqrt{-\frac{(x^2)^{\frac{2}{3}}\left(1+i\sqrt{3}\right)}{x^2}} - 4\, 2^{\frac{5}{6}} x}{4\sqrt{-\frac{(x^2)^{\frac{2}{3}}\left(1+i\sqrt{3}\right)}{x^2}}\, x}$$

Verified OK.

$$y = -\frac{\left(i\sqrt{3} - 1\right) (x^2)^{\frac{2}{3}}\, 2^{\frac{1}{3}} \sqrt{\frac{(x^2)^{\frac{2}{3}}\left(i\sqrt{3}-1\right)}{x^2}} + 4\, 2^{\frac{5}{6}} x}{4\sqrt{\frac{(x^2)^{\frac{2}{3}}\left(i\sqrt{3}-1\right)}{x^2}}\, x}$$

Verified OK.

$$y = c_2 x + \frac{1}{\sqrt{-c_2}}$$

Verified OK.

$$y = -\frac{2^{\frac{1}{3}}\left((-x^2)^{\frac{2}{3}} \sqrt{\frac{(-x^2)^{\frac{2}{3}}}{x^2}} - 2x\right)}{2x\sqrt{\frac{(-x^2)^{\frac{2}{3}}}{x^2}}}$$

Verified OK.

$$y = -\frac{\left((-x^2)^{\frac{2}{3}} \left(i\sqrt{6} - \sqrt{2}\right) \sqrt{\frac{(-x^2)^{\frac{2}{3}}\left(i\sqrt{3}-1\right)}{x^2}} - 8x\right) 2^{\frac{5}{6}}}{8\sqrt{\frac{(-x^2)^{\frac{2}{3}}\left(i\sqrt{3}-1\right)}{x^2}}\, x}$$

Verified OK.

$$y = \frac{\left((-x^2)^{\frac{2}{3}} \left(i\sqrt{6} + \sqrt{2}\right) \sqrt{-\frac{(-x^2)^{\frac{2}{3}}\left(1+i\sqrt{3}\right)}{x^2}} + 8x\right) 2^{\frac{5}{6}}}{8\sqrt{-\frac{(-x^2)^{\frac{2}{3}}\left(1+i\sqrt{3}\right)}{x^2}}\, x}$$

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 82

```
dsolve(x^2*diff(y(x),x)^3-2*x*y(x)*diff(y(x),x)^2+y(x)^2*diff(y(x),x)+1=0,y(x), singsol=all)
```

$$y(x) = \frac{3\,2^{\frac{1}{3}}(-x)^{\frac{1}{3}}}{2}$$
$$y(x) = -\frac{3\,2^{\frac{1}{3}}(-x)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)}{4}$$
$$y(x) = \frac{3\,2^{\frac{1}{3}}(-x)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)}{4}$$
$$y(x) = c_1x - \frac{1}{\sqrt{-c_1}}$$
$$y(x) = c_1x + \frac{1}{\sqrt{-c_1}}$$

✓ Solution by Mathematica

Time used: 66.431 (sec). Leaf size: 33909

```
DSolve[x^2*(y'[x])^3-2*x*y[x]*(y'[x])^2+y[x]^2*y'[x]+1==0,y[x],x,IncludeSingularSolutions ->
```

Too large to display

2.12 problem 14

Internal problem ID [6876]
Internal file name [`OUTPUT/6119_Friday_August_05_2022_02_21_13_AM_45924686/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 16. Nonlinear equations. Miscellaneous Exercises. Page 340
Problem number: 14.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`]]
```

$$\boxed{16x{y'}^2 + 8yy' + y^6 = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\left(-1 + \sqrt{1 - y^4 x}\right) y}{4x} \tag{1}$$

$$y' = -\frac{\left(1 + \sqrt{1 - y^4 x}\right) y}{4x} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = \frac{\left(-1 + \sqrt{-y^4 x + 1}\right) y}{4x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{\left(-1+\sqrt{-y^4x+1}\right) y(b_3 - a_2)}{4x} - \frac{\left(-1+\sqrt{-y^4x+1}\right)^2 y^2 a_3}{16x^2} \\ &- \left(-\frac{y^5}{8\sqrt{-y^4x+1}\, x} - \frac{\left(-1+\sqrt{-y^4x+1}\right) y}{4x^2}\right)(xa_2 + ya_3 + a_1) \\ &- \left(-\frac{y^4}{2\sqrt{-y^4x+1}} + \frac{-1+\sqrt{-y^4x+1}}{4x}\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-12x^3y^4b_2 - 2x^2y^5a_2 - 8x^2y^5b_3 + 4x\,y^6a_3 - 12x^2y^4b_1 + 2x\,y^5a_1 + (-y^4x+1)^{\frac{3}{2}}\, y^2a_3 - 20b_2\sqrt{-y^4x+1}}{16\sqrt{-y^4x+1}}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &12x^3y^4b_2 + 2x^2y^5a_2 + 8x^2y^5b_3 - 4x\,y^6a_3 + 12x^2y^4b_1 - 2x\,y^5a_1 \\ &- (-y^4x+1)^{\frac{3}{2}}\, y^2a_3 + 20b_2\sqrt{-y^4x+1}\, x^2 - 5y^2a_3\sqrt{-y^4x+1} \\ &+ 4\sqrt{-y^4x+1}\, xb_1 - 4\sqrt{-y^4x+1}\, ya_1 - 4x^2b_2 + 6y^2a_3 - 4xb_1 + 4ya_1 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &8x^3y^4b_2 + 2x^2y^5a_2 + 8x^2y^5b_3 + 2x\,y^6a_3 + 8x^2y^4b_1 + 2x\,y^5a_1 \\ &- (-y^4x+1)^{\frac{3}{2}}\, y^2a_3 - 4(-y^4x+1)\, x^2b_2 + 6(-y^4x+1)\, y^2a_3 \\ &- 4(-y^4x+1)\, xb_1 + 4(-y^4x+1)\, ya_1 + 20b_2\sqrt{-y^4x+1}\, x^2 \\ &- 5y^2a_3\sqrt{-y^4x+1} + 4\sqrt{-y^4x+1}\, xb_1 - 4\sqrt{-y^4x+1}\, ya_1 = 0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&x\sqrt{-y^4x+1}\,y^6a_3+12x^3y^4b_2+2x^2y^5a_2+8x^2y^5b_3-4x\,y^6a_3+12x^2y^4b_1 \\
&-2x\,y^5a_1+20b_2\sqrt{-y^4x+1}\,x^2-6y^2a_3\sqrt{-y^4x+1}-4x^2b_2 \\
&+4\sqrt{-y^4x+1}\,xb_1-4\sqrt{-y^4x+1}\,ya_1+6y^2a_3-4xb_1+4ya_1=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{-y^4x+1}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x=v_1, y=v_2, \sqrt{-y^4x+1}=v_3\right\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&v_1v_3v_2^6a_3+2v_1^2v_2^5a_2-4v_1v_2^6a_3+12v_1^3v_2^4b_2+8v_1^2v_2^5b_3-2v_1v_2^5a_1+12v_1^2v_2^4b_1 \\
&-6v_2^2a_3v_3+20b_2v_3v_1^2-4v_3v_2a_1+6v_2^2a_3+4v_3v_1b_1-4v_1^2b_2+4v_2a_1-4v_1b_1=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&12v_1^3v_2^4b_2+(2a_2+8b_3)\,v_1^2v_2^5+12v_1^2v_2^4b_1+20b_2v_3v_1^2-4v_1^2b_2+v_1v_3v_2^6a_3-4v_1v_2^6a_3 \\
&-2v_1v_2^5a_1+4v_3v_1b_1-4v_1b_1-6v_2^2a_3v_3+6v_2^2a_3-4v_3v_2a_1+4v_2a_1=0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_3 &= 0 \\
-4a_1 &= 0 \\
-2a_1 &= 0 \\
4a_1 &= 0 \\
-6a_3 &= 0 \\
-4a_3 &= 0 \\
6a_3 &= 0 \\
-4b_1 &= 0 \\
4b_1 &= 0 \\
12b_1 &= 0 \\
-4b_2 &= 0 \\
12b_2 &= 0 \\
20b_2 &= 0 \\
2a_2 + 8b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -4b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -4x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{\left(-1+\sqrt{-y^4x+1}\right) y}{4x}\right)(-4x) \\
&= y\sqrt{-y^4x+1} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y\sqrt{-y^4x+1}} dy \end{aligned}$$

Which results in

$$S = -\frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{-y^4x+1}}\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{\left(-1 + \sqrt{-y^4x+1}\right) y}{4x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{4x\sqrt{-y^4x+1}} \\ S_y &= \frac{1}{y\sqrt{-y^4x+1}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{4x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{4R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R)}{4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{1-y^4x}}\right)}{2} = \frac{\ln(x)}{4} + c_1$$

Which simplifies to

$$-\frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{1-y^4x}}\right)}{2} = \frac{\ln(x)}{4} + c_1$$

Summary

The solution(s) found are the following

$$-\frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{1-y^4x}}\right)}{2} = \frac{\ln(x)}{4} + c_1 \tag{1}$$

Verification of solutions

$$-\frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{1-y^4x}}\right)}{2} = \frac{\ln(x)}{4} + c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = -\frac{\left(1+\sqrt{-y^4x+1}\right)y}{4x}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{\left(1+\sqrt{-y^4x+1}\right) y(b_3-a_2)}{4x} - \frac{\left(1+\sqrt{-y^4x+1}\right)^2 y^2a_3}{16x^2} \\ &- \left(\frac{y^5}{8\sqrt{-y^4x+1}\,x} + \frac{\left(1+\sqrt{-y^4x+1}\right) y}{4x^2}\right)(xa_2+ya_3+a_1) \\ &- \left(\frac{y^4}{2\sqrt{-y^4x+1}} - \frac{1+\sqrt{-y^4x+1}}{4x}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{12x^3y^4b_2 + 2x^2y^5a_2 + 8x^2y^5b_3 - 4x\,y^6a_3 + 12x^2y^4b_1 - 2x\,y^5a_1 + (-y^4x+1)^{\frac{3}{2}}\,y^2a_3 - 20b_2\sqrt{-y^4x+1}\,x^2}{16\sqrt{-y^4x+1}}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-12x^3y^4b_2 - 2x^2y^5a_2 - 8x^2y^5b_3 + 4x\,y^6a_3 - 12x^2y^4b_1 + 2x\,y^5a_1 \\ &- \left(-y^4x+1\right)^{\frac{3}{2}} y^2a_3 + 20b_2\sqrt{-y^4x+1}\,x^2 - 5y^2a_3\sqrt{-y^4x+1} \\ &+ 4\sqrt{-y^4x+1}\,xb_1 - 4\sqrt{-y^4x+1}\,ya_1 + 4x^2b_2 - 6y^2a_3 + 4xb_1 - 4ya_1 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$
\begin{aligned}
&-8x^3y^4b_2 - 2x^2y^5a_2 - 8x^2y^5b_3 - 2x\,y^6a_3 - 8x^2y^4b_1 - 2x\,y^5a_1 \\
&- \left(-y^4x+1\right)^{\frac{3}{2}} y^2a_3 + 4\left(-y^4x+1\right)x^2b_2 - 6\left(-y^4x+1\right)y^2a_3 \\
&+ 4\left(-y^4x+1\right)xb_1 - 4\left(-y^4x+1\right)ya_1 + 20b_2\sqrt{-y^4x+1}\,x^2 \\
&- 5y^2a_3\sqrt{-y^4x+1} + 4\sqrt{-y^4x+1}\,xb_1 - 4\sqrt{-y^4x+1}\,ya_1 = 0
\end{aligned} \tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&x\sqrt{-y^4x+1}\,y^6a_3 - 12x^3y^4b_2 - 2x^2y^5a_2 - 8x^2y^5b_3 + 4x\,y^6a_3 \\
&- 12x^2y^4b_1 + 2x\,y^5a_1 + 20b_2\sqrt{-y^4x+1}\,x^2 - 6y^2a_3\sqrt{-y^4x+1} + 4x^2b_2 \\
&+ 4\sqrt{-y^4x+1}\,xb_1 - 4\sqrt{-y^4x+1}\,ya_1 - 6y^2a_3 + 4xb_1 - 4ya_1 = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{-y^4x+1}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x = v_1, y = v_2, \sqrt{-y^4x+1} = v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&v_1v_3v_2^6a_3 - 2v_1^2v_2^5a_2 + 4v_1v_2^6a_3 - 12v_1^3v_2^4b_2 - 8v_1^2v_2^5b_3 + 2v_1v_2^5a_1 - 12v_1^2v_2^4b_1 \\
&- 6v_2^2a_3v_3 + 20b_2v_3v_1^2 - 4v_3v_2a_1 - 6v_2^2a_3 + 4v_3v_1b_1 + 4v_1^2b_2 - 4v_2a_1 + 4v_1b_1 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&-12v_1^3v_2^4b_2 + (-2a_2 - 8b_3)\,v_1^2v_2^5 - 12v_1^2v_2^4b_1 + 20b_2v_3v_1^2 + 4v_1^2b_2 + v_1v_3v_2^6a_3 \\
&+ 4v_1v_2^6a_3 + 2v_1v_2^5a_1 + 4v_3v_1b_1 + 4v_1b_1 - 6v_2^2a_3v_3 - 6v_2^2a_3 - 4v_3v_2a_1 - 4v_2a_1 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_3 &= 0 \\
-4a_1 &= 0 \\
2a_1 &= 0 \\
-6a_3 &= 0 \\
4a_3 &= 0 \\
-12b_1 &= 0 \\
4b_1 &= 0 \\
-12b_2 &= 0 \\
4b_2 &= 0 \\
20b_2 &= 0 \\
-2a_2 - 8b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -4b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -4x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(-\frac{\left(1 + \sqrt{-y^4 x + 1}\right) y}{4x}\right)(-4x) \\
&= -y\sqrt{-y^4 x + 1} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-y\sqrt{-y^4x+1}} dy \end{aligned}$$

Which results in

$$S = \frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{-y^4x+1}}\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{\left(1 + \sqrt{-y^4x+1}\right) y}{4x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{1}{4x\sqrt{-y^4x+1}} \\ S_y &= -\frac{1}{y\sqrt{-y^4x+1}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{4x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{4R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R)}{4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{1-y^4x}}\right)}{2} = \frac{\ln(x)}{4} + c_1$$

Which simplifies to

$$\frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{1-y^4x}}\right)}{2} = \frac{\ln(x)}{4} + c_1$$

Summary

The solution(s) found are the following

$$\frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{1-y^4x}}\right)}{2} = \frac{\ln(x)}{4} + c_1 \tag{1}$$

Verification of solutions

$$\frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{1-y^4x}}\right)}{2} = \frac{\ln(x)}{4} + c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      <- homogeneous successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      <- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 103

```
dsolve(16*x*diff(y(x),x)^2+8*y(x)*diff(y(x),x)+y(x)^6=0,y(x), singsol=all)
```

$$y(x) = \frac{1}{x^{\frac{1}{4}}}$$
$$y(x) = -\frac{1}{x^{\frac{1}{4}}}$$
$$y(x) = -\frac{i}{x^{\frac{1}{4}}}$$
$$y(x) = \frac{i}{x^{\frac{1}{4}}}$$
$$y(x) = 0$$
$$y(x) = \frac{\mathrm{RootOf}\left(-\ln(x) + c_1 + 4\left(\int^{\_Z} \frac{1}{\_a\sqrt{-\_a^4+1}}d\_a\right)\right)}{x^{\frac{1}{4}}}$$
$$y(x) = \frac{\mathrm{RootOf}\left(-\ln(x) + c_1 - 4\left(\int^{\_Z} \frac{1}{\_a\sqrt{-\_a^4+1}}d\_a\right)\right)}{x^{\frac{1}{4}}}$$

✓ Solution by Mathematica

Time used: 0.684 (sec). Leaf size: 171

```
DSolve[16*x*(y'[x])^2+8*y[x]*y'[x]+y[x]^6==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{2}e^{\frac{c_1}{4}}}{\sqrt{x+e^{c_1}}}$$
$$y(x) \to -\frac{i\sqrt{2}e^{\frac{c_1}{4}}}{\sqrt{x+e^{c_1}}}$$
$$y(x) \to \frac{i\sqrt{2}e^{\frac{c_1}{4}}}{\sqrt{x+e^{c_1}}}$$
$$y(x) \to \frac{\sqrt{2}e^{\frac{c_1}{4}}}{\sqrt{x+e^{c_1}}}$$
$$y(x) \to 0$$
$$y(x) \to -\frac{1}{\sqrt[4]{x}}$$
$$y(x) \to -\frac{i}{\sqrt[4]{x}}$$
$$y(x) \to \frac{i}{\sqrt[4]{x}}$$
$$y(x) \to \frac{1}{\sqrt[4]{x}}$$

2.13 problem 15

Internal problem ID [6877]
Internal file name [`OUTPUT/6120_Friday_August_05_2022_02_21_16_AM_18104085/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 16. Nonlinear equations. Miscellaneous Exercises. Page 340
Problem number: 15.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{xy'^2 - (x^2 + 1)\, y' = -x}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{1}{x} \tag{1}$$
$$y' = x \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{1}{x}\, \mathrm{d}x \\ &= \ln(x) + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \ln(x) + c_1 \tag{1}$$

Verification of solutions

$$y = \ln(x) + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\begin{aligned} y &= \int x \, \mathrm{d}x \\ &= \frac{x^2}{2} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2}{2} + c_2 \tag{1}$$

Verification of solutions

$$y = \frac{x^2}{2} + c_2$$

Verified OK.

2.13.1 Maple step by step solution

Let's solve

$xy'^2 - (x^2 + 1)\, y' = -x$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \left(xy'^2 - (x^2 + 1)\, y'\right) dx = \int -x dx + c_1$

- Cannot compute integral

  $\int \left(xy'^2 - (x^2 + 1)\, y'\right) dx = -\frac{x^2}{2} + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(x*diff(y(x),x)^2-(x^2+1)*diff(y(x),x)+x=0,y(x), singsol=all)
```

$$y(x) = \frac{x^2}{2} + c_1$$
$$y(x) = \ln(x) + c_1$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 24

```
DSolve[x*(y'[x])^2-(x^2+1)*y'[x]+x==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2}{2} + c_1$$
$$y(x) \to \log(x) + c_1$$

2.14 problem 16

Internal problem ID [6878]
Internal file name [`OUTPUT/6121_Friday_August_05_2022_02_21_17_AM_29121151/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 16. Nonlinear equations. Miscellaneous Exercises. Page 340
Problem number: 16.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _dAlembert]`

$$\boxed{y'^3 - 2xy' - y = 0}$$

2.14.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$p^3 - 2xp - y = 0$$

Solving for y from the above results in

$$y = p^3 - 2xp \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = -2p$$
$$g = p^3$$

Hence (2) becomes

$$3p = \left(3p^2 - 2x\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$3p = 0$$

Solving for p from the above gives

$$p = 0$$

Substituting these in (1A) gives

$$y = 0$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{3p(x)}{3p\left(x\right)^2 - 2x} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{3p^2 - 2x(p)}{3p} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = \frac{2}{3p}$$
$$q(p) = p$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{2x(p)}{3p} = p$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{3p}dp} \\ &= p^{\frac{2}{3}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dp}(\mu x) &= (\mu)(p) \\ \frac{d}{dp}\left(p^{\frac{2}{3}}x\right) &= \left(p^{\frac{2}{3}}\right)(p) \\ d\left(p^{\frac{2}{3}}x\right) &= p^{\frac{5}{3}}\, dp\end{aligned}$$

Integrating gives

$$\begin{aligned}p^{\frac{2}{3}}x &= \int p^{\frac{5}{3}}\, dp \\ p^{\frac{2}{3}}x &= \frac{3p^{\frac{8}{3}}}{8} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = p^{\frac{2}{3}}$ results in

$$x(p) = \frac{3p^2}{8} + \frac{c_1}{p^{\frac{2}{3}}}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = \frac{\left(108y + 12\sqrt{-96x^3 + 81y^2}\right)^{\frac{1}{3}}}{6} + \frac{4x}{\left(108y + 12\sqrt{-96x^3 + 81y^2}\right)^{\frac{1}{3}}}$$

$$p = -\frac{\left(108y + 12\sqrt{-96x^3 + 81y^2}\right)^{\frac{1}{3}}}{12} - \frac{2x}{\left(108y + 12\sqrt{-96x^3 + 81y^2}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{1}{3}}}{6} - \frac{\cdots}{(10\cdots}\right.}{2\cdots}$$

$$p = -\frac{\left(108y + 12\sqrt{-96x^3 + 81y^2}\right)^{\frac{1}{3}}}{12} - \frac{2x}{\left(108y + 12\sqrt{-96x^3 + 81y^2}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{1}{3}}}{6} - \frac{\cdots}{(10\cdots}\right.}{2\cdots}$$

Substituting the above in the solution for x found above gives

$$x = \frac{\left(\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}+24x\right)^2}{96\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}}+\frac{c_1 6^{\frac{2}{3}}}{\left(\frac{\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}+24x}{\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{1}{3}}}\right)^{\frac{2}{3}}}$$

$$x = \frac{\left(-i\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}\sqrt{3}+24i\sqrt{3}\,x+\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}+24x\right)^2}{384\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}} + \frac{c_1 12^{\frac{2}{3}}}{\left(\frac{i\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}\sqrt{3}-24i\sqrt{3}\,x-\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}-24x}{\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{1}{3}}}\right)^{\frac{2}{3}}}$$

$$x = \frac{\left(i\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}\sqrt{3}-24i\sqrt{3}\,x+\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}+24x\right)^2}{384\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}} + \frac{2c_1 18^{\frac{1}{3}}}{\left(\frac{-i\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}\sqrt{3}+24i\sqrt{3}\,x-\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}-24x}{\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{1}{3}}}\right)^{\frac{2}{3}}}$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$

$$x = \frac{\left(\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}+24x\right)^2}{96\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}}+\frac{c_1 6^{\frac{2}{3}}}{\left(\frac{\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}+24x}{\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{1}{3}}}\right)^{\frac{2}{3}}} \tag{2}$$

$$x = \frac{\left(-i\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}\sqrt{3}+24i\sqrt{3}\,x+\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}+24x\right)^2}{384\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}}+\frac{c_1 12^{\frac{2}{3}}}{\left(\frac{i\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}\sqrt{3}-24i\sqrt{3}\,x-\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}-24x}{\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{1}{3}}}\right)^{\frac{2}{3}}} \tag{3}$$

$$x = \frac{\left(i\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}\sqrt{3}-24i\sqrt{3}\,x+\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}+24x\right)^2}{384\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}}+\frac{2c_1 18^{\frac{1}{3}}}{\left(\frac{-i\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}\sqrt{3}+24i\sqrt{3}\,x-\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}-24x}{\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{1}{3}}}\right)^{\frac{2}{3}}} \tag{4}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$x = \frac{\left(\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}+24x\right)^2}{96\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}} + \frac{c_1 6^{\frac{2}{3}}}{\left(\frac{\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}+24x}{\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{1}{3}}}\right)^{\frac{2}{3}}}$$

Verified OK.

$$x = \frac{\left(-i\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}\sqrt{3}+24i\sqrt{3}\,x+\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}+24x\right)^2}{384\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}} + \frac{c_1 12^{\frac{2}{3}}}{\left(\frac{i\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}\sqrt{3}-24i\sqrt{3}\,x-\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}-24x}{\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{1}{3}}}\right)^{\frac{2}{3}}}$$

Verified OK.

$$x = \frac{\left(i\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}\sqrt{3}-24i\sqrt{3}\,x+\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}+24x\right)^2}{384\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}} + \frac{2c_1 18^{\frac{1}{3}}}{\left(\frac{-i\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}\sqrt{3}+24i\sqrt{3}\,x-\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{2}{3}}-24x}{\left(108y+12\sqrt{-96x^3+81y^2}\right)^{\frac{1}{3}}}\right)^{\frac{2}{3}}}$$

Warning, solution could not be verified

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 450

```
dsolve(diff(y(x),x)^3-2*x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$-\frac{c_1}{\left(\frac{\left(108y(x)+12\sqrt{-96x^3+81y(x)^2}\right)^{\frac{2}{3}}+24x}{\left(108y(x)+12\sqrt{-96x^3+81y(x)^2}\right)^{\frac{1}{3}}}\right)^{\frac{2}{3}}}+x-\frac{\left(\left(108y(x)+12\sqrt{-96x^3+81y\left(x\right)^2}\right)^{\frac{2}{3}}+24x\right)^2}{96\left(108y\left(x\right)+12\sqrt{-96x^3+81y\left(x\right)^2}\right)^{\frac{2}{3}}}=0$$

$$-\frac{c_1}{\left(\frac{i\sqrt{3}\left(108y(x)+12\sqrt{-96x^3+81y(x)^2}\right)^{\frac{2}{3}}-24i\sqrt{3}\,x-\left(108y(x)+12\sqrt{-96x^3+81y(x)^2}\right)^{\frac{2}{3}}-24x}{\left(108y(x)+12\sqrt{-96x^3+81y(x)^2}\right)^{\frac{1}{3}}}\right)^{\frac{2}{3}}}+x+\frac{3\left(-\frac{\left(\sqrt{3}+i\right)\left(108y(x)+12\sqrt{-96x^3+81y(x)^2}\right)^{\frac{2}{3}}}{24}+x\left(-i+\sqrt{3}\right)\right)^2}{2\left(108y\left(x\right)+12\sqrt{-96x^3+81y\left(x\right)^2}\right)^{\frac{2}{3}}}=0$$

$$-\frac{12^{\frac{2}{3}}c_1}{\left(\frac{-i\sqrt{3}\left(108y(x)+12\sqrt{-96x^3+81y(x)^2}\right)^{\frac{2}{3}}+24i\sqrt{3}\,x-\left(108y(x)+12\sqrt{-96x^3+81y(x)^2}\right)^{\frac{2}{3}}-24x}{\left(108y(x)+12\sqrt{-96x^3+81y(x)^2}\right)^{\frac{1}{3}}}\right)^{\frac{2}{3}}}+x+\frac{3\left(\frac{\left(i-\sqrt{3}\right)\left(108y(x)+12\sqrt{-96x^3+81y(x)^2}\right)^{\frac{2}{3}}}{24}+x\left(\sqrt{3}+i\right)\right)^2}{2\left(108y\left(x\right)+12\sqrt{-96x^3+81y\left(x\right)^2}\right)^{\frac{2}{3}}}=0$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(y'[x])^3-2*x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Timed out

2.15 problem 18

Internal problem ID [6879]
Internal file name [`OUTPUT/6122_Friday_August_05_2022_02_21_22_AM_23605644/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 16. Nonlinear equations. Miscellaneous Exercises. Page 340
Problem number: 18.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{9xy^4y'^2 - 3y^5y' = 1}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{y^3 + \sqrt{y^6 + 4x}}{6xy^2} \tag{1}$$

$$y' = -\frac{-y^3 + \sqrt{y^6 + 4x}}{6xy^2} \tag{2}$$

Now each one of the above ODE is solved.

<u>Solving equation (1)</u>

Writing the ode as

$$y' = \frac{y^3 + \sqrt{y^6 + 4x}}{6x\,y^2}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{\left(y^3+\sqrt{y^6+4x}\right)(b_3-a_2)}{6x\,y^2} - \frac{\left(y^3+\sqrt{y^6+4x}\right)^2 a_3}{36x^2y^4} - \left(-\frac{y^3+\sqrt{y^6+4x}}{6x^2y^2} + \frac{1}{3x\sqrt{y^6+4x}\,y^2}\right)(x a_2 + y a_3 + a_1) - \left(\frac{3y^2+\frac{3y^5}{\sqrt{y^6+4x}}}{6x\,y^2} - \frac{y^3+\sqrt{y^6+4x}}{3x\,y^3}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{6x^2y^7b_2 - 4y^9a_3 + 6x\,y^7b_1 - 6y^8a_1 - 30b_2x^2y^4\sqrt{y^6+4x} - 5\sqrt{y^6+4x}\,y^6a_3 + 6\sqrt{y^6+4x}\,x\,y^4b_1 - 6\sqrt{y^6 +}}{36x^2y^4\sqrt{y^6+}} = 0$$

Setting the numerator to zero gives

$$-6x^2y^7b_2 + 4y^9a_3 - 6x\,y^7b_1 + 6y^8a_1 + 30b_2x^2y^4\sqrt{y^6+4x} + 5\sqrt{y^6+4x}\,y^6a_3 - 6\sqrt{y^6+4x}\,x\,y^4b_1 + 6\sqrt{y^6+4x}\,y^5a_1 + 48x^3yb_2 - 12x^2y^2a_2 + 72x^2y^2b_3 + 4x\,y^3a_3 - \left(y^6+4x\right)^{\frac{3}{2}}a_3 + 48x^2yb_1 + 12x\,y^2a_1 = 0 \tag{6E}$$

Simplifying the above gives

$$-18x^2y^7b_2 - 18x\,y^8b_3 - 18x\,y^7b_1 + 30b_2x^2y^4\sqrt{y^6+4x} + 5\sqrt{y^6+4x}\,y^6a_3 - 6\sqrt{y^6+4x}\,x\,y^4b_1 + 6\sqrt{y^6+4x}\,y^5a_1 + 12\left(y^6+4x\right)x^2yb_2 + 18\left(y^6+4x\right)x\,y^2b_3 + 4\left(y^6+4x\right)y^3a_3 + 12\left(y^6+4x\right)xyb_1 + 6\left(y^6+4x\right)y^2a_1 - 12x^2y^2a_2 - 12x\,y^3a_3 - \left(y^6+4x\right)^{\frac{3}{2}}a_3 - 12x\,y^2a_1 = 0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$-6x^2y^7b_2 + 4y^9a_3 - 6x\,y^7b_1 + 6y^8a_1 + 30b_2x^2y^4\sqrt{y^6+4x} + 4\sqrt{y^6+4x}\,y^6a_3 - 6\sqrt{y^6+4x}\,x\,y^4b_1 + 6\sqrt{y^6+4x}\,y^5a_1 + 48x^3yb_2 - 12x^2y^2a_2 + 72x^2y^2b_3 + 4x\,y^3a_3 + 48x^2yb_1 + 12x\,y^2a_1 - 4x\sqrt{y^6+4x}\,a_3 = 0$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{y^6+4x}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{y^6+4x} = v_3\right\}$$

The above PDE (6E) now becomes

$$4v_2^9a_3 - 6v_1^2v_2^7b_2 + 6v_2^8a_1 - 6v_1v_2^7b_1 + 4v_3v_2^6a_3 + 30b_2v_1^2v_2^4v_3 + 6v_3v_2^5a_1 - 6v_3v_1v_2^4b_1 - 12v_1^2v_2^2a_2 + 4v_1v_2^3a_3 + 48v_1^3v_2b_2 + 72v_1^2v_2^2b_3 + 12v_1v_2^2a_1 + 48v_1^2v_2b_1 - 4v_1v_3a_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$48v_1^3v_2b_2 - 6v_1^2v_2^7b_2 + 30b_2v_1^2v_2^4v_3 + (-12a_2 + 72b_3)\,v_1^2v_2^2 + 48v_1^2v_2b_1 - 6v_1v_2^7b_1 - 6v_3v_1v_2^4b_1 + 4v_1v_2^3a_3 + 12v_1v_2^2a_1 - 4v_1v_3a_3 + 4v_2^9a_3 + 6v_2^8a_1 + 4v_3v_2^6a_3 + 6v_3v_2^5a_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
6a_1 &= 0 \\
12a_1 &= 0 \\
-4a_3 &= 0 \\
4a_3 &= 0 \\
-6b_1 &= 0 \\
48b_1 &= 0 \\
-6b_2 &= 0 \\
30b_2 &= 0 \\
48b_2 &= 0 \\
-12a_2 + 72b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 6b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 6x \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi}\\ &= \frac{y}{6x}\\ &= \frac{y}{6x}\end{aligned}$$

This is easily solved to give

$$y = c_1 x^{\frac{1}{6}}$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{y}{x^{\frac{1}{6}}}$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi}\\ &= \frac{dx}{6x}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T}\\ &= \frac{\ln(x)}{6}\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y^3 + \sqrt{y^6 + 4x}}{6x\,y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= -\frac{y}{6x^{\frac{7}{6}}}\\ R_y &= \frac{1}{x^{\frac{1}{6}}}\\ S_x &= \frac{1}{6x}\\ S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{x^{\frac{1}{6}} y^2}{\sqrt{y^6 + 4x}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R^2}{\sqrt{R^6 + 4}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int \frac{R^2}{\sqrt{R^6 + 4}} dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(x)}{6} = \int^{\frac{y}{x^{\frac{1}{6}}}} \frac{\_a^2}{\sqrt{\_a^6 + 4}} d\_a + c_1$$

Which simplifies to

$$\frac{\ln(x)}{6} = \int^{\frac{y}{x^{\frac{1}{6}}}} \frac{\_a^2}{\sqrt{\_a^6 + 4}} d\_a + c_1$$

Summary

The solution(s) found are the following

$$\frac{\ln(x)}{6} = \int^{\frac{y}{x^{\frac{1}{6}}}} \frac{\_a^2}{\sqrt{\_a^6 + 4}} d\_a + c_1 \tag{1}$$

Verification of solutions

$$\frac{\ln(x)}{6} = \int^{\frac{y}{x^{\frac{1}{6}}}} \frac{\_a^2}{\sqrt{\_a^6 + 4}} d\_a + c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = -\frac{-y^3 + \sqrt{y^6 + 4x}}{6x\, y^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 &- \frac{\left(-y^3 + \sqrt{y^6 + 4x}\right)(b_3 - a_2)}{6x\,y^2} - \frac{\left(-y^3 + \sqrt{y^6 + 4x}\right)^2 a_3}{36x^2 y^4} \\ &- \left(-\frac{1}{3x\sqrt{y^6 + 4x}\,y^2} + \frac{-y^3 + \sqrt{y^6 + 4x}}{6x^2 y^2}\right)(x a_2 + y a_3 + a_1) \\ &- \left(-\frac{-3y^2 + \frac{3y^5}{\sqrt{y^6+4x}}}{6x\,y^2} + \frac{-y^3 + \sqrt{y^6 + 4x}}{3x\,y^3}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-6x^2y^7b_2 + 4y^9a_3 - 6x\,y^7b_1 + 6y^8a_1 - 30b_2x^2y^4\sqrt{y^6+4x} - 5\sqrt{y^6+4x}\,y^6a_3 + 6\sqrt{y^6+4x}\,x\,y^4b_1 - 6\sqrt{y}}{36x^2y^4\sqrt{y^6+}}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &6x^2y^7b_2 - 4y^9a_3 + 6x\,y^7b_1 - 6y^8a_1 + 30b_2x^2y^4\sqrt{y^6+4x} + 5\sqrt{y^6+4x}\,y^6a_3 \\ &- 6\sqrt{y^6+4x}\,x\,y^4b_1 + 6\sqrt{y^6+4x}\,y^5a_1 - 48x^3yb_2 + 12x^2y^2a_2 \\ &- 72x^2y^2b_3 - 4x\,y^3a_3 - \left(y^6+4x\right)^{\frac{3}{2}} a_3 - 48x^2yb_1 - 12x\,y^2a_1 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$
\begin{aligned}
&18x^2y^7b_2+18x\,y^8b_3+18x\,y^7b_1+30b_2x^2y^4\sqrt{y^6+4x}+5\sqrt{y^6+4x}\,y^6a_3\\
&-6\sqrt{y^6+4x}\,x\,y^4b_1+6\sqrt{y^6+4x}\,y^5a_1-12\left(y^6+4x\right)x^2yb_2\\
&-18\left(y^6+4x\right)x\,y^2b_3-4\left(y^6+4x\right)y^3a_3-12\left(y^6+4x\right)xyb_1\\
&-6\left(y^6+4x\right)y^2a_1+12x^2y^2a_2+12x\,y^3a_3-\left(y^6+4x\right)^{\frac{3}{2}}a_3+12x\,y^2a_1=0
\end{aligned}
\tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&6x^2y^7b_2-4y^9a_3+6x\,y^7b_1-6y^8a_1+30b_2x^2y^4\sqrt{y^6+4x}+4\sqrt{y^6+4x}\,y^6a_3\\
&-6\sqrt{y^6+4x}\,x\,y^4b_1+6\sqrt{y^6+4x}\,y^5a_1-48x^3yb_2+12x^2y^2a_2\\
&-72x^2y^2b_3-4x\,y^3a_3-48x^2yb_1-12x\,y^2a_1-4x\sqrt{y^6+4x}\,a_3=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$
\left\{x,y,\sqrt{y^6+4x}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$
\left\{x=v_1,y=v_2,\sqrt{y^6+4x}=v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-4v_2^9a_3+6v_1^2v_2^7b_2-6v_2^8a_1+6v_1v_2^7b_1+4v_3v_2^6a_3+30b_2v_1^2v_2^4v_3+6v_3v_2^5a_1\\
&-6v_3v_1v_2^4b_1+12v_1^2v_2^2a_2-4v_1v_2^3a_3-48v_1^3v_2b_2-72v_1^2v_2^2b_3-12v_1v_2^2a_1-48v_1^2v_2b_1\\
&-4v_1v_3a_3=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1,v_2,v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&-48v_1^3v_2b_2+6v_1^2v_2^7b_2+30b_2v_1^2v_2^4v_3+(12a_2-72b_3)\,v_1^2v_2^2\\
&-48v_1^2v_2b_1+6v_1v_2^7b_1-6v_3v_1v_2^4b_1-4v_1v_2^3a_3-12v_1v_2^2a_1\\
&-4v_1v_3a_3-4v_2^9a_3-6v_2^8a_1+4v_3v_2^6a_3+6v_3v_2^5a_1=0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-12a_1 &= 0 \\
-6a_1 &= 0 \\
6a_1 &= 0 \\
-4a_3 &= 0 \\
4a_3 &= 0 \\
-48b_1 &= 0 \\
-6b_1 &= 0 \\
6b_1 &= 0 \\
-48b_2 &= 0 \\
6b_2 &= 0 \\
30b_2 &= 0 \\
12a_2 - 72b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 6b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 6x \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      trying an integrating factor from the invariance group
      <- integrating factor successful
      <- homogeneous successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      trying an integrating factor from the invariance group
      <- integrating factor successful
      <- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 267

```
dsolve(9*x*y(x)^4*diff(y(x),x)^2-3*y(x)^5*diff(y(x),x)-1=0,y(x), singsol=all)
```

$$y(x) = 2^{\frac{1}{3}}(-x)^{\frac{1}{6}}$$
$$y(x) = -2^{\frac{1}{3}}(-x)^{\frac{1}{6}}$$
$$y(x) = -\frac{\left(1+i\sqrt{3}\right)2^{\frac{1}{3}}(-x)^{\frac{1}{6}}}{2}$$
$$y(x) = \frac{\left(i\sqrt{3}-1\right)2^{\frac{1}{3}}(-x)^{\frac{1}{6}}}{2}$$
$$y(x) = -\frac{\left(i\sqrt{3}-1\right)2^{\frac{1}{3}}(-x)^{\frac{1}{6}}}{2}$$
$$y(x) = \frac{\left(1+i\sqrt{3}\right)2^{\frac{1}{3}}(-x)^{\frac{1}{6}}}{2}$$
$$y(x) = \frac{\left((-x+c_1)^2 c_1^5\right)^{\frac{1}{6}}}{c_1}$$
$$y(x) = -\frac{\left((-x+c_1)^2 c_1^5\right)^{\frac{1}{6}}}{c_1}$$
$$y(x) = -\frac{\left(1+i\sqrt{3}\right)\left((-x+c_1)^2 c_1^5\right)^{\frac{1}{6}}}{2c_1}$$
$$y(x) = \frac{\left(i\sqrt{3}-1\right)\left((-x+c_1)^2 c_1^5\right)^{\frac{1}{6}}}{2c_1}$$
$$y(x) = -\frac{\left(i\sqrt{3}-1\right)\left((-x+c_1)^2 c_1^5\right)^{\frac{1}{6}}}{2c_1}$$
$$y(x) = \frac{\left(1+i\sqrt{3}\right)\left((-x+c_1)^2 c_1^5\right)^{\frac{1}{6}}}{2c_1}$$

✓ Solution by Mathematica

Time used: 3.005 (sec). Leaf size: 322

```
DSolve[9*x*y[x]^4*(y'[x])^2-3*y[x]^5*y'[x]-1==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt[3]{-\frac{1}{2}}e^{-\frac{c_1}{6}}\sqrt[3]{-4x+e^{c_1}}$$
$$y(x) \to \frac{e^{-\frac{c_1}{6}}\sqrt[3]{-4x+e^{c_1}}}{\sqrt[3]{2}}$$
$$y(x) \to \frac{(-1)^{2/3}e^{-\frac{c_1}{6}}\sqrt[3]{-4x+e^{c_1}}}{\sqrt[3]{2}}$$
$$y(x) \to -\sqrt[3]{-\frac{1}{2}}\sqrt[3]{-e^{-\frac{c_1}{2}}\left(-4x+e^{c_1}\right)}$$
$$y(x) \to \frac{\sqrt[3]{e^{-\frac{c_1}{2}}\left(4x-e^{c_1}\right)}}{\sqrt[3]{2}}$$
$$y(x) \to \frac{(-1)^{2/3}\sqrt[3]{-e^{-\frac{c_1}{2}}\left(-4x+e^{c_1}\right)}}{\sqrt[3]{2}}$$
$$y(x) \to -i\sqrt[3]{2}\sqrt[6]{x}$$
$$y(x) \to i\sqrt[3]{2}\sqrt[6]{x}$$
$$y(x) \to -\sqrt[6]{-1}\sqrt[3]{2}\sqrt[6]{x}$$
$$y(x) \to \sqrt[6]{-1}\sqrt[3]{2}\sqrt[6]{x}$$
$$y(x) \to -(-1)^{5/6}\sqrt[3]{2}\sqrt[6]{x}$$
$$y(x) \to (-1)^{5/6}\sqrt[3]{2}\sqrt[6]{x}$$

2.16 problem 19

Internal problem ID [6880]
Internal file name [`OUTPUT/6123_Friday_August_05_2022_02_21_26_AM_11129659/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 16. Nonlinear equations. Miscellaneous Exercises. Page 340
Problem number: 19.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _rational, _Clairaut]
```

$$\boxed{x^2y'^2 - (2yx + 1)\, y' + y^2 = -1}$$

2.16.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$x^2p^2 - (2xy + 1)\, p + y^2 = -1$$

Solving for y from the above results in

$$y = px + \sqrt{p-1} \tag{1A}$$
$$y = px - \sqrt{p-1} \tag{2A}$$

Each of the above ode's is a Clairaut ode which is now solved. <u>Solving ode 1A</u> We start by replacing y' by p which gives

$$\begin{aligned} y &= px + \sqrt{p-1} \\ &= px + \sqrt{p-1} \end{aligned}$$

Writing the ode as

$$y = px + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = px + g \tag{1}$$

Then we see that

$$g = \sqrt{p-1}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1 x + \sqrt{c_1 - 1}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = \sqrt{p-1}$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x + \frac{1}{2\sqrt{p-1}} \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$p_1 = \frac{4x^2+1}{4x^2}$$

Substituting the above back in (1) results in

$$y_1 = \frac{2\,\mathrm{csgn}\left(\frac{1}{x}\right)+4x^2+1}{4x}$$

Solving ode 2A We start by replacing y' by p which gives

$$\begin{aligned} y &= px-\sqrt{p-1} \\ &= px-\sqrt{p-1} \end{aligned}$$

Writing the ode as

$$y = px + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = px + g \tag{1}$$

Then we see that

$$g = -\sqrt{p-1}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned} p &= \frac{d}{dx}(xp+g) \\ p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\ p &= p + (x+g')\frac{dp}{dx} \\ 0 &= (x+g')\frac{dp}{dx} \end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned} \frac{dp}{dx} &= 0 \\ p &= c_1 \end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_2 x - \sqrt{c_2 - 1}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = -\sqrt{p-1}$, then the above equation becomes

$$\begin{aligned} x + g'(p) &= x - \frac{1}{2\sqrt{p-1}} \\ &= 0 \end{aligned}$$

Solving the above for p results in

$$p_1 = \frac{4x^2 + 1}{4x^2}$$

Substituting the above back in (1) results in

$$y_1 = \frac{4x^2 - 2\,\mathrm{csgn}\left(\frac{1}{x}\right) + 1}{4x}$$

Simplifying the solution $y = \frac{2\,\mathrm{csgn}\left(\frac{1}{x}\right) + 4x^2 + 1}{4x}$ to $y = \frac{4x^2+3}{4x}$ Simplifying the solution $y = \frac{4x^2 - 2\,\mathrm{csgn}\left(\frac{1}{x}\right) + 1}{4x}$ to $y = \frac{4x^2-1}{4x}$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= c_1 x + \sqrt{c_1 - 1} \\ y &= \frac{4x^2 + 3}{4x} \\ y &= c_2 x - \sqrt{c_2 - 1} \\ y &= \frac{4x^2 - 1}{4x} \end{aligned}$$

Verification of solutions

$$y = c_1 x + \sqrt{c_1 - 1}$$

Verified OK.

$$y = \frac{4x^2 + 3}{4x}$$

Verified OK.

$$y = c_2 x - \sqrt{c_2 - 1}$$

Verified OK.

$$y = \frac{4x^2 - 1}{4x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 42

```
dsolve(x^2*diff(y(x),x)^2-(2*x*y(x)+1)*diff(y(x),x)+y(x)^2+1=0,y(x), singsol=all)
```

$$y(x) = \frac{4x^2 - 1}{4x}$$
$$y(x) = c_1 x - \sqrt{c_1 - 1}$$
$$y(x) = c_1 x + \sqrt{c_1 - 1}$$

✓ Solution by Mathematica

Time used: 1.527 (sec). Leaf size: 66

```
DSolve[x^2*y'[x]^2-(2*x*y[x]+1)*y'[x]+y[x]^2+1==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to x + e^{-2c_1}x + e^{-c_1} \\
y(x) &\to x + \frac{1}{4}e^{-2c_1}x + \frac{e^{-c_1}}{2} \\
y(x) &\to x \\
y(x) &\to x - \frac{1}{4x}
\end{aligned}
$$

2.17 problem 20

Internal problem ID [6881]
Internal file name [`OUTPUT/6124_Friday_August_05_2022_02_21_29_AM_97548117/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 16. Nonlinear equations. Miscellaneous Exercises. Page 340
Problem number: 20.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`]]
```

$$\boxed{x^6 {y'}^2 - 16y - 8xy' = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{4 + 4\sqrt{y x^4 + 1}}{x^5} \tag{1}$$

$$y' = -\frac{4\left(-1 + \sqrt{y x^4 + 1}\right)}{x^5} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$\begin{aligned} y' &= \frac{4 + 4\sqrt{y\, x^4 + 1}}{x^5} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{4\left(1+\sqrt{y\,x^4+1}\right)(b_3-a_2)}{x^5} - \frac{16\left(1+\sqrt{y\,x^4+1}\right)^2 a_3}{x^{10}} \\ &- \left(\frac{8y}{\sqrt{y\,x^4+1}\,x^2} - \frac{20\left(1+\sqrt{y\,x^4+1}\right)}{x^6}\right)(xa_2+ya_3+a_1) \\ &- \frac{2(xb_2+yb_3+b_1)}{\sqrt{y\,x^4+1}\,x} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-b_2x^{10}\sqrt{y\,x^4+1} + 2x^{10}b_2 - 8x^9ya_2 - 2x^9yb_3 - 12x^8y^2a_3 + 2x^9b_1 - 12x^8ya_1 - 16\sqrt{y\,x^4+1}\,x^5a_2 - 4\sqrt{y \cdots}{\cdots}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &b_2x^{10}\sqrt{y\,x^4+1} - 2x^{10}b_2 + 8x^9ya_2 + 2x^9yb_3 + 12x^8y^2a_3 \\ &- 2x^9b_1 + 12x^8ya_1 + 16\sqrt{y\,x^4+1}\,x^5a_2 + 4\sqrt{y\,x^4+1}\,x^5b_3 \\ &+ 20\sqrt{y\,x^4+1}\,x^4ya_3 + 20\sqrt{y\,x^4+1}\,x^4a_1 + 16x^5a_2 + 4x^5b_3 \\ &- 12x^4ya_3 + 20x^4a_1 - 16\left(y\,x^4+1\right)^{\frac{3}{2}} a_3 - 16a_3\sqrt{y\,x^4+1} - 32a_3 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &b_2x^{10}\sqrt{y\,x^4+1} - 2x^{10}b_2 - 8x^9ya_2 - 2x^9yb_3 - 8x^8y^2a_3 \\ &- 2x^9b_1 - 8x^8ya_1 + 16\left(y\,x^4+1\right)x^5a_2 + 4\left(y\,x^4+1\right)x^5b_3 \\ &+ 20\left(y\,x^4+1\right)x^4ya_3 + 20\left(y\,x^4+1\right)x^4a_1 + 16\sqrt{y\,x^4+1}\,x^5a_2 \\ &+ 4\sqrt{y\,x^4+1}\,x^5b_3 + 20\sqrt{y\,x^4+1}\,x^4ya_3 + 20\sqrt{y\,x^4+1}\,x^4a_1 \\ &- 16\left(y\,x^4+1\right)^{\frac{3}{2}} a_3 - 32\left(y\,x^4+1\right)a_3 - 16a_3\sqrt{y\,x^4+1} = 0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&b_2x^{10}\sqrt{y\,x^4+1}-2x^{10}b_2+8x^9ya_2+2x^9yb_3+12x^8y^2a_3-2x^9b_1+12x^8ya_1 \\
&+16\sqrt{y\,x^4+1}\,x^5a_2+4\sqrt{y\,x^4+1}\,x^5b_3+4\sqrt{y\,x^4+1}\,x^4ya_3+16x^5a_2+4x^5b_3 \\
&+20\sqrt{y\,x^4+1}\,x^4a_1-12x^4ya_3+20x^4a_1-32a_3\sqrt{y\,x^4+1}-32a_3=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{y\,x^4+1}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x=v_1, y=v_2, \sqrt{y\,x^4+1}=v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&b_2v_1^{10}v_3+8v_1^9v_2a_2+12v_1^8v_2^2a_3-2v_1^{10}b_2+2v_1^9v_2b_3+12v_1^8v_2a_1 \\
&-2v_1^9b_1+16v_3v_1^5a_2+4v_3v_1^4v_2a_3+4v_3v_1^5b_3+20v_3v_1^4a_1 \\
&+16v_1^5a_2-12v_1^4v_2a_3+4v_1^5b_3+20v_1^4a_1-32a_3v_3-32a_3=0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&b_2v_1^{10}v_3-2v_1^{10}b_2+(8a_2+2b_3)\,v_1^9v_2-2v_1^9b_1+12v_1^8v_2^2a_3 \\
&+12v_1^8v_2a_1+(16a_2+4b_3)\,v_1^5v_3+(16a_2+4b_3)\,v_1^5+4v_3v_1^4v_2a_3 \\
&-12v_1^4v_2a_3+20v_3v_1^4a_1+20v_1^4a_1-32a_3v_3-32a_3=0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
b_2 &= 0\\
12a_1 &= 0\\
20a_1 &= 0\\
-32a_3 &= 0\\
-12a_3 &= 0\\
4a_3 &= 0\\
12a_3 &= 0\\
-2b_1 &= 0\\
-2b_2 &= 0\\
8a_2 + 2b_3 &= 0\\
16a_2 + 4b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0\\
a_2 &= a_2\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= -4a_2
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x\\
\eta &= -4y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi\\
&= -4y - \left(\frac{4 + 4\sqrt{y\,x^4 + 1}}{x^5}\right)(x)\\
&= \frac{-4y\,x^4 - 4\sqrt{y\,x^4 + 1} - 4}{x^4}\\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-4y\,x^4 - 4\sqrt{y\,x^4+1} - 4}{x^4}} dy \end{aligned}$$

Which results in

$$S = -\frac{\ln(y)}{4} - \frac{\operatorname{arctanh}\left(\sqrt{y\,x^4+1}\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{4 + 4\sqrt{y\,x^4+1}}{x^5}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{\sqrt{y\,x^4+1}\,x} \\ S_y &= \frac{-1 + \frac{1}{\sqrt{y\,x^4+1}}}{4y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(y)}{4} - \frac{\operatorname{arctanh}\left(\sqrt{y x^4 + 1}\right)}{2} = c_1$$

Which simplifies to

$$-\frac{\ln(y)}{4} - \frac{\operatorname{arctanh}\left(\sqrt{y x^4 + 1}\right)}{2} = c_1$$

Summary
The solution(s) found are the following

$$-\frac{\ln(y)}{4} - \frac{\operatorname{arctanh}\left(\sqrt{y x^4 + 1}\right)}{2} = c_1 \tag{1}$$

Verification of solutions

$$-\frac{\ln(y)}{4} - \frac{\operatorname{arctanh}\left(\sqrt{y x^4 + 1}\right)}{2} = c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = -\frac{4\left(-1 + \sqrt{y\, x^4 + 1}\right)}{x^5}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{4\left(-1+\sqrt{y\,x^4+1}\right)(b_3-a_2)}{x^5} - \frac{16\left(-1+\sqrt{y\,x^4+1}\right)^2 a_3}{x^{10}} \\ &- \left(-\frac{8y}{\sqrt{y\,x^4+1}\,x^2} + \frac{-20+20\sqrt{y\,x^4+1}}{x^6}\right)(xa_2+ya_3+a_1) \\ &+ \frac{2xb_2+2yb_3+2b_1}{\sqrt{y\,x^4+1}\,x} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-b_2x^{10}\sqrt{y\,x^4+1} - 2x^{10}b_2 + 8x^9ya_2 + 2x^9yb_3 + 12x^8y^2a_3 - 2x^9b_1 + 12x^8ya_1 - 16\sqrt{y\,x^4+1}\,x^5a_2 - 4\sqrt{y}}{}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &b_2x^{10}\sqrt{y\,x^4+1} + 2x^{10}b_2 - 8x^9ya_2 - 2x^9yb_3 - 12x^8y^2a_3 \\ &+ 2x^9b_1 - 12x^8ya_1 + 16\sqrt{y\,x^4+1}\,x^5a_2 + 4\sqrt{y\,x^4+1}\,x^5b_3 \\ &+ 20\sqrt{y\,x^4+1}\,x^4ya_3 + 20\sqrt{y\,x^4+1}\,x^4a_1 - 16x^5a_2 - 4x^5b_3 \\ &+ 12x^4ya_3 - 20x^4a_1 - 16\left(y\,x^4+1\right)^{\frac{3}{2}} a_3 - 16a_3\sqrt{y\,x^4+1} + 32a_3 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$
\begin{aligned}
& b_2 x^{10}\sqrt{y\,x^4+1}+2x^{10}b_2+8x^9 y a_2+2x^9 y b_3+8x^8 y^2 a_3 \\
& +2x^9 b_1+8x^8 y a_1-16\left(y\,x^4+1\right)x^5 a_2-4\left(y\,x^4+1\right)x^5 b_3 \\
& -20\left(y\,x^4+1\right)x^4 y a_3-20\left(y\,x^4+1\right)x^4 a_1+16\sqrt{y\,x^4+1}\,x^5 a_2 \\
& +4\sqrt{y\,x^4+1}\,x^5 b_3+20\sqrt{y\,x^4+1}\,x^4 y a_3+20\sqrt{y\,x^4+1}\,x^4 a_1 \\
& -16\left(y\,x^4+1\right)^{\frac{3}{2}}a_3+32\left(y\,x^4+1\right)a_3-16a_3\sqrt{y\,x^4+1}=0
\end{aligned} \tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
& b_2 x^{10}\sqrt{y\,x^4+1}+2x^{10}b_2-8x^9 y a_2-2x^9 y b_3-12x^8 y^2 a_3+2x^9 b_1-12x^8 y a_1 \\
& +16\sqrt{y\,x^4+1}\,x^5 a_2+4\sqrt{y\,x^4+1}\,x^5 b_3+4\sqrt{y\,x^4+1}\,x^4 y a_3-16x^5 a_2-4x^5 b_3 \\
& +20\sqrt{y\,x^4+1}\,x^4 a_1+12x^4 y a_3-20x^4 a_1-32a_3\sqrt{y\,x^4+1}+32a_3=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{y\,x^4+1}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x=v_1, y=v_2, \sqrt{y\,x^4+1}=v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
& b_2 v_1^{10} v_3-8v_1^9 v_2 a_2-12v_1^8 v_2^2 a_3+2v_1^{10} b_2-2v_1^9 v_2 b_3-12v_1^8 v_2 a_1 \\
& +2v_1^9 b_1+16v_3 v_1^5 a_2+4v_3 v_1^4 v_2 a_3+4v_3 v_1^5 b_3+20v_3 v_1^4 a_1 \\
& -16v_1^5 a_2+12v_1^4 v_2 a_3-4v_1^5 b_3-20v_1^4 a_1-32a_3 v_3+32a_3=0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
& b_2 v_1^{10} v_3+2v_1^{10} b_2+(-8a_2-2b_3)\,v_1^9 v_2+2v_1^9 b_1-12v_1^8 v_2^2 a_3 \\
& -12v_1^8 v_2 a_1+(16a_2+4b_3)\,v_1^5 v_3+(-16a_2-4b_3)\,v_1^5+4v_3 v_1^4 v_2 a_3 \\
& +12v_1^4 v_2 a_3+20v_3 v_1^4 a_1-20v_1^4 a_1-32a_3 v_3+32a_3=0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_2 &= 0 \\
-20a_1 &= 0 \\
-12a_1 &= 0 \\
20a_1 &= 0 \\
-32a_3 &= 0 \\
-12a_3 &= 0 \\
4a_3 &= 0 \\
12a_3 &= 0 \\
32a_3 &= 0 \\
2b_1 &= 0 \\
2b_2 &= 0 \\
-16a_2 - 4b_3 &= 0 \\
-8a_2 - 2b_3 &= 0 \\
16a_2 + 4b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= -4a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= -4y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= -4y - \left(-\frac{4\left(-1+\sqrt{y\,x^4+1}\right)}{x^5}\right)(x) \\ &= \frac{-4y\,x^4 + 4\sqrt{y\,x^4+1} - 4}{x^4} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-4y\,x^4+4\sqrt{y\,x^4+1}-4}{x^4}} dy\end{aligned}$$

Which results in

$$S = -\frac{\ln(y)}{4} + \frac{\operatorname{arctanh}\left(\sqrt{y\,x^4+1}\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{4\left(-1+\sqrt{y\,x^4+1}\right)}{x^5}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{1}{\sqrt{y\,x^4+1}\,x} \\
S_y &= \frac{-1-\frac{1}{\sqrt{y\,x^4+1}}}{4y}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(y)}{4} + \frac{\operatorname{arctanh}\left(\sqrt{yx^4+1}\right)}{2} = c_1$$

Which simplifies to

$$-\frac{\ln(y)}{4} + \frac{\operatorname{arctanh}\left(\sqrt{yx^4+1}\right)}{2} = c_1$$

Summary

The solution(s) found are the following

$$-\frac{\ln(y)}{4} + \frac{\operatorname{arctanh}\left(\sqrt{yx^4+1}\right)}{2} = c_1 \tag{1}$$

Verification of solutions

$$-\frac{\ln(y)}{4} + \frac{\operatorname{arctanh}\left(\sqrt{yx^4+1}\right)}{2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      <- homogeneous successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      <- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.844 (sec). Leaf size: 89

```
dsolve(x^6*diff(y(x),x)^2=8*(2*y(x)+x*diff(y(x),x)),y(x), singsol=all)
```

$$y(x) = -\frac{1}{x^4}$$
$$y(x) = \frac{2c_1 i - x^2}{x^2 c_1^2}$$
$$y(x) = \frac{-2c_1 i - x^2}{x^2 c_1^2}$$
$$y(x) = \frac{-2c_1 i - x^2}{x^2 c_1^2}$$
$$y(x) = \frac{2c_1 i - x^2}{x^2 c_1^2}$$

✓ Solution by Mathematica

Time used: 0.535 (sec). Leaf size: 122

```
DSolve[x^6*y'[x]^2==8*(2*y[x]+x*y'[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[-\frac{x\sqrt{x^4 y(x)+1}\operatorname{arctanh}\left(\sqrt{x^4 y(x)+1}\right)}{2\sqrt{x^6 y(x)+x^2}} - \frac{1}{4}\log(y(x)) = c_1, y(x)\right]$$
$$\text{Solve}\left[\frac{x\sqrt{x^4 y(x)+1}\operatorname{arctanh}\left(\sqrt{x^4 y(x)+1}\right)}{2\sqrt{x^6 y(x)+x^2}} - \frac{1}{4}\log(y(x)) = c_1, y(x)\right]$$
$$y(x) \to 0$$

2.18 problem 21

Internal problem ID [6882]
Internal file name [`OUTPUT/6125_Friday_August_05_2022_02_21_32_AM_77399903/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 16. Nonlinear equations. Miscellaneous Exercises. Page 340
Problem number: 21.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"exact", "linear", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{x^2 {y'}^2 - (x-y)^2 = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{y-x}{x} \tag{1}$$
$$y' = -\frac{y-x}{x} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = -1$$

Hence the ode is

$$y' - \frac{y}{x} = -1$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(-1) \\ \frac{d}{dx}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)(-1) \\ d\left(\frac{y}{x}\right) &= \left(-\frac{1}{x}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x} &= \int -\frac{1}{x}\, dx \\ \frac{y}{x} &= -\ln(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = -x\ln(x) + c_1 x$$

which simplifies to

$$y = x(-\ln(x) + c_1)$$

Summary
The solution(s) found are the following

$$y = x(-\ln(x) + c_1) \tag{1}$$

Verification of solutions

$$y = x(-\ln(x) + c_1)$$

Verified OK.

Solving equation (2)

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= 1 \end{aligned}$$

Hence the ode is

$$y' + \frac{y}{x} = 1$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int \frac{1}{x} dx} \\ &= x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu \\ \frac{\mathrm{d}}{\mathrm{d}x}(xy) &= x \\ \mathrm{d}(xy) &= x\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} xy &= \int x \,\mathrm{d}x \\ xy &= \frac{x^2}{2} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$y = \frac{x}{2} + \frac{c_2}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{x}{2} + \frac{c_2}{x} \tag{1}$$

Verification of solutions

$$y = \frac{x}{2} + \frac{c_2}{x}$$

Verified OK.

2.18.1 Maple step by step solution

Let's solve

$x^2 y'^2 - (x-y)^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -1 + \frac{y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x} = -1$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = -\mu(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int -\mu(x)\,dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int -\mu(x)\,dx + c_1$

- Solve for y

  $y = \frac{\int -\mu(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x}$

  $y = x\left(\int -\frac{1}{x}dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x(-\ln(x) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(x^2*diff(y(x),x)^2=(x-y(x))^2,y(x), singsol=all)
```

$$y(x) = (-\ln(x) + c_1) x$$
$$y(x) = \frac{x}{2} + \frac{c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.073 (sec). Leaf size: 30

```
DSolve[x^2*y'[x]^2==(x-y[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x}{2} + \frac{c_1}{x}$$
$$y(x) \to x(-\log(x) + c_1)$$

2.19 problem 22

Internal problem ID [6883]
Internal file name [OUTPUT/6126_Friday_August_05_2022_02_21_34_AM_73766698/index.tex]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 16. Nonlinear equations. Miscellaneous Exercises. Page 340
Problem number: 22.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _dAlembert]
```

$$\boxed{(y'+1)^2 (y-xy') = 1}$$

2.19.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$(p+1)^2 (-xp+y) = 1$$

Solving for y from the above results in

$$y = \frac{p^3x + 2p^2x + xp + 1}{(p+1)^2} \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= xp + \frac{1}{(p+1)^2} \\ &= xp + \frac{1}{(p+1)^2} \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = \frac{1}{(p+1)^2}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1 x + \frac{1}{(c_1+1)^2}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = \frac{1}{(p+1)^2}$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x - \frac{2}{(p+1)^3} \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$p_1 = \frac{2^{\frac{1}{3}} (x^2)^{\frac{1}{3}}}{x} - 1$$

$$p_2 = -\frac{2^{\frac{1}{3}} (x^2)^{\frac{1}{3}}}{2x} + \frac{i\sqrt{3}\, 2^{\frac{1}{3}} (x^2)^{\frac{1}{3}}}{2x} - 1$$

$$p_3 = -\frac{2^{\frac{1}{3}} (x^2)^{\frac{1}{3}}}{2x} - \frac{i\sqrt{3}\, 2^{\frac{1}{3}} (x^2)^{\frac{1}{3}}}{2x} - 1$$

Substituting the above back in (1) results in

$$y_1 = \frac{x\left(3\, 2^{\frac{1}{3}} x - 2(x^2)^{\frac{2}{3}}\right)}{2\,(x^2)^{\frac{2}{3}}}$$

$$y_2 = \frac{x\left(i(x^2)^{\frac{2}{3}} \sqrt{3} + 3\, 2^{\frac{1}{3}} x + (x^2)^{\frac{2}{3}}\right)}{\left(-i\sqrt{3} - 1\right) (x^2)^{\frac{2}{3}}}$$

$$y_3 = -\frac{x\left((x^2)^{\frac{2}{3}} \left(\sqrt{3} + i\right) + 3i2^{\frac{1}{3}} x\right)}{(x^2)^{\frac{2}{3}} \left(\sqrt{3} + i\right)}$$

Summary

The solution(s) found are the following

$$y = c_1 x + \frac{1}{(c_1 + 1)^2} \tag{1}$$

$$y = \frac{x\left(3\, 2^{\frac{1}{3}} x - 2(x^2)^{\frac{2}{3}}\right)}{2\,(x^2)^{\frac{2}{3}}} \tag{2}$$

$$y = \frac{x\left(i(x^2)^{\frac{2}{3}} \sqrt{3} + 3\, 2^{\frac{1}{3}} x + (x^2)^{\frac{2}{3}}\right)}{\left(-i\sqrt{3} - 1\right) (x^2)^{\frac{2}{3}}} \tag{3}$$

$$y = -\frac{x\left((x^2)^{\frac{2}{3}} \left(\sqrt{3} + i\right) + 3i2^{\frac{1}{3}} x\right)}{(x^2)^{\frac{2}{3}} \left(\sqrt{3} + i\right)} \tag{4}$$

Verification of solutions

$$y = c_1 x + \frac{1}{(c_1+1)^2}$$

Verified OK.

$$y = \frac{x\left(3\,2^{\frac{1}{3}}x - 2(x^2)^{\frac{2}{3}}\right)}{2\,(x^2)^{\frac{2}{3}}}$$

Verified OK.

$$y = \frac{x\left(i(x^2)^{\frac{2}{3}}\,\sqrt{3} + 3\,2^{\frac{1}{3}}x + (x^2)^{\frac{2}{3}}\right)}{\left(-i\sqrt{3}-1\right)(x^2)^{\frac{2}{3}}}$$

Verified OK.

$$y = -\frac{x\left((x^2)^{\frac{2}{3}}\left(\sqrt{3}+i\right) + 3i2^{\frac{1}{3}}x\right)}{(x^2)^{\frac{2}{3}}\left(\sqrt{3}+i\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 93

```
dsolve((diff(y(x),x)+1)^2*(y(x)-diff(y(x),x)*x)=1,y(x), singsol=all)
```

$$y(x) = \frac{3\,2^{\frac{1}{3}}(x^2)^{\frac{1}{3}}}{2} - x$$
$$y(x) = \frac{\left(-3i\sqrt{3}-3\right)2^{\frac{1}{3}}(x^2)^{\frac{1}{3}}}{4} - x$$
$$y(x) = \frac{\left(3i\sqrt{3}-3\right)2^{\frac{1}{3}}(x^2)^{\frac{1}{3}}}{4} - x$$
$$y(x) = \frac{c_1^3x + 2c_1^2x + c_1x + 1}{(c_1+1)^2}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 102

```
DSolve[(y'[x]+1)^2*(y[x]-y'[x]*x)==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1x + \frac{1}{(1+c_1)^2}$$
$$y(x) \to \frac{3x^{2/3}}{2^{2/3}} - x$$
$$y(x) \to -x + \frac{3i\left(\sqrt{3}+i\right)x^{2/3}}{2\ 2^{2/3}}$$
$$y(x) \to -x - \frac{3\left(1+i\sqrt{3}\right)x^{2/3}}{2\ 2^{2/3}}$$

2.20 problem 23

Internal problem ID [6884]
Internal file name [`OUTPUT/6127_Friday_August_05_2022_02_21_37_AM_75839393/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 16. Nonlinear equations. Miscellaneous Exercises. Page 340
Problem number: 23.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _Clairaut]
```

$$\boxed{y'^3 - y'^2 + xy' - y = 0}$$

2.20.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$p^3 - p^2 + xp - y = 0$$

Solving for y from the above results in

$$y = p^3 - p^2 + xp \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= p^3 - p^2 + xp \\ &= p^3 - p^2 + xp \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = p^3 - p^2$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned} p &= \frac{d}{dx}(xp + g) \\ p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\ p &= p + (x + g')\frac{dp}{dx} \\ 0 &= (x + g')\frac{dp}{dx} \end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned} \frac{dp}{dx} &= 0 \\ p &= c_1 \end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1^3 - c_1^2 + c_1 x$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = p^3 - p^2$, then the above equation becomes

$$\begin{aligned} x + g'(p) &= 3p^2 - 2p + x \\ &= 0 \end{aligned}$$

Solving the above for p results in

$$
\begin{aligned}
p_1 &= \frac{1}{3} + \frac{\sqrt{-3x+1}}{3} \\
p_2 &= \frac{1}{3} - \frac{\sqrt{-3x+1}}{3}
\end{aligned}
$$

Substituting the above back in (1) results in

$$
\begin{aligned}
y_1 &= \frac{(6x-2)\sqrt{-3x+1}}{27} + \frac{x}{3} - \frac{2}{27} \\
y_2 &= \frac{(-6x+2)\sqrt{-3x+1}}{27} + \frac{x}{3} - \frac{2}{27}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1^3 - c_1^2 + c_1 x \tag{1}$$

$$y = \frac{(6x-2)\sqrt{-3x+1}}{27} + \frac{x}{3} - \frac{2}{27} \tag{2}$$

$$y = \frac{(-6x+2)\sqrt{-3x+1}}{27} + \frac{x}{3} - \frac{2}{27} \tag{3}$$

Verification of solutions

$$y = c_1^3 - c_1^2 + c_1 x$$

Verified OK.

$$y = \frac{(6x-2)\sqrt{-3x+1}}{27} + \frac{x}{3} - \frac{2}{27}$$

Verified OK.

$$y = \frac{(-6x+2)\sqrt{-3x+1}}{27} + \frac{x}{3} - \frac{2}{27}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 56

```
dsolve(diff(y(x),x)^3-diff(y(x),x)^2+x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x}{3} - \frac{2}{27} - \frac{2\sqrt{-(3x-1)^3}}{27}$$
$$y(x) = \frac{x}{3} - \frac{2}{27} + \frac{2\sqrt{-(3x-1)^3}}{27}$$
$$y(x) = c_1(c_1^2 - c_1 + x)$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 74

```
DSolve[y'[x]^3-y'[x]^2+x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1(x + (-1 + c_1)c_1)$$
$$y(x) \to \frac{1}{27}\left(9x - 2\left(\sqrt{-(3x-1)^3} + 1\right)\right)$$
$$y(x) \to \frac{1}{27}\left(9x + 2\sqrt{-(3x-1)^3} - 2\right)$$

2.21 problem 24

Internal problem ID [6885]
Internal file name [`OUTPUT/6128_Friday_August_05_2022_02_21_42_AM_33139359/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 16. Nonlinear equations. Miscellaneous Exercises. Page 340
Problem number: 24.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"exact", "linear", "quadrature", "separable", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{xy'^2 + y(1-x)\, y' - y^2 = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = y \tag{1}$$
$$y' = -\frac{y}{x} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned}
\int \frac{1}{y} dy &= x + c_1 \\
\ln(y) &= x + c_1 \\
y &= e^{x+c_1} \\
y &= c_1 e^x
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{x} \tag{1}$$

Verification of solutions

$$y = c_1 e^{x}$$

Verified OK.

Solving equation (2)

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= -\frac{y}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y}\, dy &= -\frac{1}{x}\, dx \\ \int \frac{1}{y}\, dy &= \int -\frac{1}{x}\, dx \\ \ln(y) &= -\ln(x) + c_2 \\ y &= e^{-\ln(x)+c_2} \\ &= \frac{c_2}{x} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_2}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_2}{x}$$

Verified OK.

2.21.1 Maple step by step solution

Let's solve

$xy'^2 + y(1-x)\,y' - y^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int 1 dx + c_1$

- Evaluate integral

  $\ln(y) = x + c_1$

- Solve for y

  $y = \mathrm{e}^{x+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(x*diff(y(x),x)^2+y(x)*(1-x)*diff(y(x),x)-y(x)^2=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{x}$$
$$y(x) = \mathrm{e}^{x} c_1$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 26

```
DSolve[x*y'[x]^2+y[x]*(1-x)*y'[x]-y[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x$$
$$y(x) \to \frac{c_1}{x}$$
$$y(x) \to 0$$

2.22 problem 25

Internal problem ID [6886]
Internal file name [`OUTPUT/6129_Friday_August_05_2022_02_21_43_AM_19595941/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 16. Nonlinear equations. Miscellaneous Exercises. Page 340
Problem number: 25.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{yy'^2 - (x+y)\,y' + y = 0}$$

2.22.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$y\,p^2 - (x+y)\,p + y = 0$$

Solving for y from the above results in

$$y = \frac{xp}{p^2 - p + 1} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= \frac{p}{p^2 - p + 1} \\ g &= 0 \end{aligned}$$

Hence (2) becomes

$$p - \frac{p}{p^2 - p + 1} = x \left(\frac{1}{p^2 - p + 1} - \frac{p(2p - 1)}{(p^2 - p + 1)^2} \right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{p}{p^2 - p + 1} = 0$$

Solving for p from the above gives

$$\begin{aligned} p &= 1 \\ p &= 0 \\ p &= 0 \end{aligned}$$

Removing solutions for p which leads to undefined results and substituting these in (1A) gives

$$\begin{aligned} y &= 0 \\ y &= x \end{aligned}$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - \frac{p(x)}{p(x)^2 - p(x) + 1}}{x \left(\frac{1}{p(x)^2 - p(x) + 1} - \frac{p(x)(2p(x) - 1)}{\left(p(x)^2 - p(x) + 1\right)^2} \right)} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp} x(p) = \frac{x(p) \left(\frac{1}{p^2 - p + 1} - \frac{p(2p - 1)}{(p^2 - p + 1)^2} \right)}{p - \frac{p}{p^2 - p + 1}} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = -\frac{-p-1}{(p^2-p+1)\,p^2}$$
$$q(p) = 0$$

Hence the ode is

$$\frac{d}{dp}x(p) - \frac{(-p-1)\,x(p)}{(p^2-p+1)\,p^2} = 0$$

The integrating factor μ is

$$\mu = e^{\int -\frac{-p-1}{(p^2-p+1)p^2}dp}$$
$$= e^{-\frac{1}{p}+2\ln(p)-\ln(p^2-p+1)}$$

Which simplifies to

$$\mu = \frac{p^2 e^{-\frac{1}{p}}}{p^2-p+1}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}p}\mu x = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{p^2 e^{-\frac{1}{p}} x}{p^2-p+1}\right) = 0$$

Integrating gives

$$\frac{p^2 e^{-\frac{1}{p}} x}{p^2-p+1} = c_2$$

Dividing both sides by the integrating factor $\mu = \frac{p^2 e^{-\frac{1}{p}}}{p^2-p+1}$ results in

$$x(p) = \frac{c_2(p^2-p+1)\,e^{\frac{1}{p}}}{p^2}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = \frac{x + y + \sqrt{-3y^2 + 2yx + x^2}}{2y}$$
$$p = -\frac{-x - y + \sqrt{-3y^2 + 2yx + x^2}}{2y}$$

Substituting the above in the solution for x found above gives

$$x = \frac{2c_2 x \,\mathrm{e}^{\frac{2y}{x+y+\sqrt{(x+3y)(x-y)}}}}{x + y + \sqrt{(x+3y)(x-y)}}$$

$$x = \frac{2c_2 x \,\mathrm{e}^{\frac{2y}{x+y-\sqrt{(x+3y)(x-y)}}}}{x + y - \sqrt{(x+3y)(x-y)}}$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$
$$y = x \tag{2}$$
$$x = \frac{2c_2 x \,\mathrm{e}^{\frac{2y}{x+y+\sqrt{(x+3y)(x-y)}}}}{x + y + \sqrt{(x+3y)(x-y)}} \tag{3}$$
$$x = \frac{2c_2 x \,\mathrm{e}^{\frac{2y}{x+y-\sqrt{(x+3y)(x-y)}}}}{x + y - \sqrt{(x+3y)(x-y)}} \tag{4}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$y = x$$

Verified OK.

$$x = \frac{2c_2 x \,\mathrm{e}^{\frac{2y}{x+y+\sqrt{(x+3y)(x-y)}}}}{x + y + \sqrt{(x+3y)(x-y)}}$$

Verified OK.

$$x = \frac{2c_2 x \,\mathrm{e}^{\frac{2y}{x+y-\sqrt{(x+3y)(x-y)}}}}{x + y - \sqrt{(x+3y)(x-y)}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   <- symmetries for implicit equations successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 176

```
dsolve(y(x)*diff(y(x),x)^2-(x+y(x))*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = x$$
$$y(x) = 0$$
$$\frac{-x\sqrt{\frac{(3y(x)+x)(x-y(x))}{x^2}} + 2y(x)\ln\left(\frac{y(x)}{x}\right) + \left(-2\,\mathrm{arctanh}\left(\frac{x+y(x)}{x\sqrt{\frac{(3y(x)+x)(x-y(x))}{x^2}}}\right) - 2c_1 + 2\ln(x)\right)y(x) - x}{2y(x)} = 0$$
$$\frac{x\sqrt{\frac{(3y(x)+x)(x-y(x))}{x^2}} + 2y(x)\ln\left(\frac{y(x)}{x}\right) + \left(2\,\mathrm{arctanh}\left(\frac{x+y(x)}{x\sqrt{\frac{(3y(x)+x)(x-y(x))}{x^2}}}\right) - 2c_1 + 2\ln(x)\right)y(x) - x}{2y(x)} = 0$$

✓ Solution by Mathematica

Time used: 4.268 (sec). Leaf size: 320

```
DSolve[y[x]*y'[x]^2-(x+y[x])*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[-\frac{x\left(-i\sqrt{\frac{y(x)}{x}-1}\sqrt{\frac{3y(x)}{x}+1}+\frac{4y(x)\log\left(\sqrt{\frac{3y(x)}{x}-3}-\sqrt{\frac{3y(x)}{x}+1}\right)}{x}-\frac{4y(x)\log\left(-i\left(\frac{3y(x)}{x}+1\right)+i\sqrt{\frac{3y(x)}{x}-3}\sqrt{\frac{3y(x)}{x}+1}+\right.}{x}\right.}{4y(x)}\right.$$

$$\left.-\frac{\log(x)}{2}+c_1,y(x)\right]$$

$$\text{Solve}\left[-\frac{x\left(i\sqrt{\frac{y(x)}{x}-1}\sqrt{\frac{3y(x)}{x}+1}+\frac{4y(x)\log\left(\sqrt{\frac{3y(x)}{x}-3}-\sqrt{\frac{3y(x)}{x}+1}\right)}{x}-\frac{4y(x)\log\left(i\left(\frac{3y(x)}{x}+1\right)-i\sqrt{\frac{3y(x)}{x}-3}\sqrt{\frac{3y(x)}{x}+1}+\sqrt{2-}\right.}{x}\right.}{4y(x)}\right.$$

$$\left.-\frac{\log(x)}{2}+c_1,y(x)\right]$$

$$y(x)\to 0$$

2.23 problem 26

Internal problem ID [6887]
Internal file name [OUTPUT/6130_Friday_August_05_2022_02_21_46_AM_99865228/index.tex]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 16. Nonlinear equations. Miscellaneous Exercises. Page 340
Problem number: 26.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _rational, _dAlembert]
```

$$\boxed{xy'^2 + (k - x - y)\, y' + y = 0}$$

2.23.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$x\, p^2 + (k - x - y)\, p + y = 0$$

Solving for y from the above results in

$$y = \frac{p(px + k - x)}{p - 1} \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= px + \frac{pk}{p-1} \\ &= px + \frac{pk}{p-1} \end{aligned}$$

Writing the ode as

$$y = px + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = px + g \tag{1}$$

Then we see that

$$g = \frac{pk}{p-1}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1 x + \frac{c_1 k}{c_1 - 1}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = \frac{pk}{p-1}$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x + \frac{k}{p-1} - \frac{pk}{(p-1)^2} \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$p_1 = \frac{x+\sqrt{kx}}{x}$$
$$p_2 = -\frac{-x+\sqrt{kx}}{x}$$

Substituting the above back in (1) results in

$$y_1 = \frac{(k+x)\sqrt{kx}+2kx}{\sqrt{kx}}$$
$$y_2 = \frac{\sqrt{kx}\,k+x\sqrt{kx}-2kx}{\sqrt{kx}}$$

Summary

The solution(s) found are the following

$$y = c_1x+\frac{c_1k}{c_1-1} \tag{1}$$
$$y = \frac{(k+x)\sqrt{kx}+2kx}{\sqrt{kx}} \tag{2}$$
$$y = \frac{\sqrt{kx}\,k+x\sqrt{kx}-2kx}{\sqrt{kx}} \tag{3}$$

Verification of solutions

$$y = c_1x+\frac{c_1k}{c_1-1}$$

Verified OK.

$$y = \frac{(k+x)\sqrt{kx}+2kx}{\sqrt{kx}}$$

Verified OK.

$$y = \frac{\sqrt{kx}\,k+x\sqrt{kx}-2kx}{\sqrt{kx}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 45

```
dsolve(x*diff(y(x),x)^2+(k-x-y(x))*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = k + x - 2\sqrt{kx}$$
$$y(x) = k + x + 2\sqrt{kx}$$
$$y(x) = \frac{c_1(c_1x + k - x)}{c_1 - 1}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 54

```
DSolve[x*y'[x]^2+(k-x-y[x])*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1\left(x + \frac{k}{-1 + c_1}\right)$$
$$y(x) \to -2\sqrt{k}\sqrt{x} + k + x$$
$$y(x) \to \left(\sqrt{k} + \sqrt{x}\right)^2$$

2.24 problem 27

Internal problem ID [6888]
Internal file name [`OUTPUT/6131_Friday_August_05_2022_02_21_49_AM_68346215/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 16. Nonlinear equations. Miscellaneous Exercises. Page 340
Problem number: 27.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries]]
```

$$\boxed{xy'^3 - 2yy'^2 = -4x^2}$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\left(-54x^4 + 8y^3 + 6\sqrt{81x^4 - 24y^3}\, x^2\right)^{\frac{1}{3}}}{3x} + \frac{4y^2}{3x\left(-54x^4 + 8y^3 + 6\sqrt{81x^4 - 24y^3}\, x^2\right)^{\frac{1}{3}}} + \frac{2y}{3x} \tag{1}$$

$$y' = -\frac{\left(-54x^4 + 8y^3 + 6\sqrt{81x^4 - 24y^3}\, x^2\right)^{\frac{1}{3}}}{6x} - \frac{2y^2}{3x\left(-54x^4 + 8y^3 + 6\sqrt{81x^4 - 24y^3}\, x^2\right)^{\frac{1}{3}}} + \frac{2y}{3x} + \frac{i\sqrt{3}\left(\frac{(-}{}\right.}{} \tag{2}$$

$$y' = -\frac{\left(-54x^4 + 8y^3 + 6\sqrt{81x^4 - 24y^3}\, x^2\right)^{\frac{1}{3}}}{6x} - \frac{2y^2}{3x\left(-54x^4 + 8y^3 + 6\sqrt{81x^4 - 24y^3}\, x^2\right)^{\frac{1}{3}}} + \frac{2y}{3x} - \frac{i\sqrt{3}\left(\frac{(-}{}\right.}{} \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = \frac{\left(-54x^4+8y^3+6\sqrt{81x^4-24y^3}\,x^2\right)^{\frac{2}{3}}+2y\left(-54x^4+8y^3+6\sqrt{81x^4-24y^3}\,x^2\right)^{\frac{1}{3}}+4y^2}{3x\left(-54x^4+8y^3+6\sqrt{81x^4-24y^3}\,x^2\right)^{\frac{1}{3}}}$$

$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1,a_2,a_3,b_1,b_2,b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

Expression too large to display (5E)

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Simplifying the above gives

Expression too large to display (6E)

Since the PDE has radicals, simplifying gives

Expression too large to display

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(-54x^4+8y^3+6\sqrt{81x^4-24y^3}\,x^2\right)^{\frac{1}{3}}, \left(-54x^4+8y^3+6\sqrt{81x^4-24y^3}\,x^2\right)^{\frac{2}{3}}, \sqrt{81x^4-24y^3}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \left(-54x^4+8y^3+6\sqrt{81x^4-24y^3}\,x^2\right)^{\frac{1}{3}} = v_3, \left(-54x^4+8y^3+6\sqrt{81x^4-24y^3}\,x^2\right)^{\frac{2}{3}} = v_4, \sqrt{81x^4-24y^3} = v_5\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&52488v_1^{10}a_3+7776v_1^7v_2^2a_2-5832v_5v_1^8a_3-17496v_1^6v_2^3a_3\\
&-2916v_1^6v_3v_2^2a_3-11664v_1^8v_2b_2+1458v_1^8v_3b_2-5832v_1^7v_2^2b_3\\
&+13608v_1^6v_2^2a_1+2916v_1^6v_3v_2a_1-1944v_4v_1^7a_2-864v_5v_1^5v_2^2a_2\\
&-1152v_1^3v_2^5a_2-2430v_4v_1^6v_2a_3+1080v_5v_1^4v_2^3a_3\\
&+324v_1^4v_5v_3v_2^2a_3+1728v_1^2v_2^6a_3+864v_1^2v_3v_2^5a_3-11664v_1^7v_2b_1\\
&-2916v_1^7v_3b_1+1296v_5v_1^6v_2b_2-162v_1^6v_5v_3b_2+2592v_1^4v_2^4b_2\\
&-432v_1^4v_3v_2^3b_2+1458v_1^7v_4b_3+648v_5v_1^5v_2^2b_3+864v_1^3v_2^5b_3\\
&-486v_4v_1^6a_1-1512v_5v_1^4v_2^2a_1-324v_1^4v_5v_3v_2a_1-2880v_1^2v_2^5a_1\\
&-864v_1^2v_3v_2^4a_1+216v_5v_4v_1^5a_2+288v_4v_1^3v_2^3a_2+270v_5v_4v_1^4v_2a_3\\
&+432v_4v_1^2v_2^4a_3-96v_5v_2^6a_3-48v_5v_3v_2^5a_3+1296v_5v_1^5v_2b_1\\
&+324v_1^5v_5v_3b_1+2592v_1^3v_2^4b_1+864v_1^3v_3v_2^3b_1+216v_4v_1^4v_2^2b_2\\
&-96v_5v_1^2v_2^4b_2+24v_1^2v_5v_3v_2^3b_2-162v_1^5v_5v_4b_3-216v_4v_1^3v_2^3b_3\\
&+54v_5v_4v_1^4a_1-144v_4v_1^2v_2^3a_1+96v_5v_2^5a_1+48v_5v_3v_2^4a_1\\
&-24v_5v_4v_2^4a_3+216v_4v_1^3v_2^2b_1-96v_5v_1v_2^4b_1-48v_1v_5v_3v_2^3b_1\\
&-24v_5v_4v_1^2v_2^2b_2+24v_5v_4v_2^3a_1-24v_5v_4v_1v_2^2b_1=0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&1296 v_5 v_1^6 v_2 b_2 + (-864 a_2 + 648 b_3)\, v_1^5 v_2^2 v_5 + (216 a_2 - 162 b_3)\, v_1^5 v_4 v_5 \\
&+ (288 a_2 - 216 b_3)\, v_1^3 v_2^3 v_4 + (7776 a_2 - 5832 b_3)\, v_1^7 v_2^2 \\
&+ (-1944 a_2 + 1458 b_3)\, v_1^7 v_4 + (-1152 a_2 + 864 b_3)\, v_1^3 v_2^5 \\
&+ 432 v_4 v_1^2 v_2^4 a_3 + 1296 v_5 v_1^5 v_2 b_1 + 324 v_1^5 v_5 v_3 b_1 \\
&- 48 v_5 v_3 v_2^5 a_3 + 24 v_5 v_4 v_2^3 a_1 + 48 v_5 v_3 v_2^4 a_1 - 162 v_1^6 v_5 v_3 b_2 \\
&+ 54 v_5 v_4 v_1^4 a_1 - 24 v_5 v_4 v_2^4 a_3 - 1512 v_5 v_1^4 v_2^2 a_1 - 96 v_5 v_1^2 v_2^4 b_2 \\
&+ 216 v_4 v_1^3 v_2^2 b_1 - 96 v_5 v_1 v_2^4 b_1 + 864 v_1^2 v_3 v_2^5 a_3 + 864 v_1^3 v_3 v_2^3 b_1 \\
&- 864 v_1^2 v_3 v_2^4 a_1 - 144 v_4 v_1^2 v_2^3 a_1 - 2430 v_4 v_1^6 v_2 a_3 - 486 v_4 v_1^6 a_1 \\
&+ 1728 v_1^2 v_2^6 a_3 - 2880 v_1^2 v_2^5 a_1 - 5832 v_5 v_1^8 a_3 - 96 v_5 v_2^6 a_3 \\
&- 2916 v_1^7 v_3 b_1 + 96 v_5 v_2^5 a_1 - 11664 v_1^8 v_2 b_2 - 11664 v_1^7 v_2 b_1 \\
&- 17496 v_1^6 v_2^3 a_3 + 52488 v_1^{10} a_3 + 1080 v_5 v_1^4 v_2^3 a_3 + 216 v_4 v_1^4 v_2^2 b_2 \\
&- 2916 v_1^6 v_3 v_2^2 a_3 + 2916 v_1^6 v_3 v_2 a_1 - 432 v_1^4 v_3 v_2^3 b_2 + 13608 v_1^6 v_2^2 a_1 \\
&+ 2592 v_1^4 v_2^4 b_2 + 2592 v_1^3 v_2^4 b_1 + 1458 v_1^8 v_3 b_2 + 270 v_5 v_4 v_1^4 v_2 a_3 \\
&- 24 v_5 v_4 v_1^2 v_2^2 b_2 - 24 v_5 v_4 v_1 v_2^2 b_1 + 324 v_1^4 v_5 v_3 v_2^2 a_3 \\
&- 48 v_1 v_5 v_3 v_2^3 b_1 - 324 v_1^4 v_5 v_3 v_2 a_1 + 24 v_1^2 v_5 v_3 v_2^3 b_2 = 0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$-2880a_1 = 0$$
$$-1512a_1 = 0$$
$$-864a_1 = 0$$
$$-486a_1 = 0$$
$$-324a_1 = 0$$
$$-144a_1 = 0$$
$$24a_1 = 0$$
$$48a_1 = 0$$
$$54a_1 = 0$$
$$96a_1 = 0$$
$$2916a_1 = 0$$
$$13608a_1 = 0$$
$$-17496a_3 = 0$$
$$-5832a_3 = 0$$
$$-2916a_3 = 0$$
$$-2430a_3 = 0$$
$$-96a_3 = 0$$
$$-48a_3 = 0$$
$$-24a_3 = 0$$
$$270a_3 = 0$$
$$324a_3 = 0$$
$$432a_3 = 0$$
$$864a_3 = 0$$
$$1080a_3 = 0$$
$$1728a_3 = 0$$
$$52488a_3 = 0$$
$$-11664b_1 = 0$$
$$-2916b_1 = 0$$
$$-96b_1 = 0$$
$$-48b_1 = 0$$
$$-24b_1 = 0$$
$$216b_1 = 0$$
$$324b_1 = 0$$
$$864b_1 = 0$$
$$1296b_1 = 0$$
$$2592b_1 = 0$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= \frac{4a_2}{3}
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= \frac{4y}{3}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{\frac{4y}{3}}{x} \\
&= \frac{4y}{3x}
\end{aligned}
$$

This is easily solved to give

$$
y = c_1 x^{\frac{4}{3}}
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = \frac{y}{x^{\frac{4}{3}}}
$$

And S is found from

$$
\begin{aligned}
dS &= \frac{dx}{\xi} \\
&= \frac{dx}{x}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
S &= \int \frac{dx}{T} \\
&= \ln(x)
\end{aligned}
$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{\left(-54x^4+8y^3+6\sqrt{81x^4-24y^3}\,x^2\right)^{\frac{2}{3}} + 2y\left(-54x^4+8y^3+6\sqrt{81x^4-24y^3}\,x^2\right)^{\frac{1}{3}} + 4y^2}{3x\left(-54x^4+8y^3+6\sqrt{81x^4-24y^3}\,x^2\right)^{\frac{1}{3}}}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= -\frac{4y}{3x^{\frac{7}{3}}} \\
R_y &= \frac{1}{x^{\frac{4}{3}}} \\
S_x &= \frac{1}{x} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{3x^{\frac{4}{3}}\left(-54x^4+8y^3+6\sqrt{81x^4-24y^3}\,x^2\right)^{\frac{1}{3}}}{\left(-54x^4+8y^3+6\sqrt{81x^4-24y^3}\,x^2\right)^{\frac{2}{3}} - 2y\left(-54x^4+8y^3+6\sqrt{81x^4-24y^3}\,x^2\right)^{\frac{1}{3}} + 4y^2} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{3\,2^{\frac{1}{3}}\left(4R^3+3\sqrt{-24R^3+81}-27\right)^{\frac{1}{3}}}{\left(4R^3+3\sqrt{-24R^3+81}-27\right)^{\frac{2}{3}}2^{\frac{2}{3}} - 2\left(4R^3+3\sqrt{-24R^3+81}-27\right)^{\frac{1}{3}}2^{\frac{1}{3}}R + 4R^2}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int \frac{3\left(8R^3+6\sqrt{-24R^3+81}-54\right)^{\frac{1}{3}}}{4^{\frac{1}{3}}\left(\left(4R^3+3\sqrt{-24R^3+81}-27\right)^2\right)^{\frac{1}{3}}-2R\left(8R^3+6\sqrt{-24R^3+81}-54\right)^{\frac{1}{3}}+4R^2}dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(x) = \int^{\frac{y}{x^{\frac{4}{3}}}} \frac{3\left(8\_a^3+6\sqrt{-24\_a^3+81}-54\right)^{\frac{1}{3}}}{4^{\frac{1}{3}}\left(\left(4\_a^3+3\sqrt{-24\_a^3+81}-27\right)^2\right)^{\frac{1}{3}}-2\_a\left(8\_a^3+6\sqrt{-24\_a^3+81}-54\right)^{\frac{1}{3}}+4\_a^2}d\_$$

Which simplifies to

$$\ln(x) = \int^{\frac{y}{x^{\frac{4}{3}}}} \frac{3\left(8\_a^3+6\sqrt{-24\_a^3+81}-54\right)^{\frac{1}{3}}}{4^{\frac{1}{3}}\left(\left(4\_a^3+3\sqrt{-24\_a^3+81}-27\right)^2\right)^{\frac{1}{3}}-2\_a\left(8\_a^3+6\sqrt{-24\_a^3+81}-54\right)^{\frac{1}{3}}+4\_a^2}d\_$$

Summary

The solution(s) found are the following

$$\ln(x) = \int^{\frac{y}{x^{\frac{4}{3}}}} \frac{3\left(8\_a^3+6\sqrt{-24\_a^3+81}-54\right)^{\frac{1}{3}}}{4^{\frac{1}{3}}\left(\left(4\_a^3+3\sqrt{-24\_a^3+81}-27\right)^2\right)^{\frac{1}{3}}-2\_a\left(8\_a^3+6\sqrt{-24\_a^3+81}-54\right)^{\frac{1}{3}}+4\_a^2}d\_a + c_1 \tag{1}$$

Verification of solutions

$$\ln(x) = \int^{\frac{y}{x^{\frac{4}{3}}}} \frac{3\left(8\_a^3+6\sqrt{-24\_a^3+81}-54\right)^{\frac{1}{3}}}{4^{\frac{1}{3}}\left(\left(4\_a^3+3\sqrt{-24\_a^3+81}-27\right)^2\right)^{\frac{1}{3}}-2\_a\left(8\_a^3+6\sqrt{-24\_a^3+81}-54\right)^{\frac{1}{3}}+4\_a^2}d\_a + c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = \frac{i\left(-54x^4+8y^3+6\sqrt{81x^4-24y^3}\,x^2\right)^{\frac{2}{3}}\sqrt{3}-\left(-54x^4+8y^3+6\sqrt{81x^4-24y^3}\,x^2\right)^{\frac{2}{3}}+4y\left(-54x^4+8y^3\right.}{6x\left(-54x^4+8y^3+6\sqrt{81x^4-24y^3}\,x^2\right)^{\frac{1}{3}}}$$

$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1,a_2,a_3,b_1,b_2,b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

Expression too large to display (5E)

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Simplifying the above gives

Expression too large to display (6E)

Since the PDE has radicals, simplifying gives

Expression too large to display

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(-54x^4+8y^3+6\sqrt{81x^4-24y^3}\,x^2\right)^{\frac{1}{3}}, \left(-54x^4+8y^3+6\sqrt{81x^4-24y^3}\,x^2\right)^{\frac{2}{3}}, \sqrt{81x^4-24y^3}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x=v_1, y=v_2, \left(-54x^4+8y^3+6\sqrt{81x^4-24y^3}\,x^2\right)^{\frac{1}{3}}=v_3, \left(-54x^4+8y^3+6\sqrt{81x^4-24y^3}\,x^2\right)^{\frac{2}{3}}=v_4, \sqrt{81x^4-24y^3}=v_5\right\}$$

The above PDE (6E) now becomes

$$\text{Expression too large to display} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$\text{Expression too large to display} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$-3456a_1 = 0$$
$$-1296a_1 = 0$$
$$192a_1 = 0$$
$$11664a_1 = 0$$
$$-11664a_3 = 0$$
$$-192a_3 = 0$$
$$1296a_3 = 0$$
$$3456a_3 = 0$$
$$-11664b_1 = 0$$
$$-192b_1 = 0$$
$$1296b_1 = 0$$
$$3456b_1 = 0$$
$$-1728b_2 = 0$$
$$-648b_2 = 0$$
$$96b_2 = 0$$
$$5832b_2 = 0$$
$$-104976i\sqrt{3}\,a_3 - 104976a_3 = 0$$
$$-27216i\sqrt{3}\,a_1 - 27216a_1 = 0$$
$$-5184i\sqrt{3}\,b_1 - 5184b_1 = 0$$
$$-5184i\sqrt{3}\,b_2 - 5184b_2 = 0$$
$$-4860i\sqrt{3}\,a_3 + 4860a_3 = 0$$
$$-3456i\sqrt{3}\,a_3 - 3456a_3 = 0$$
$$-2592i\sqrt{3}\,b_1 - 2592b_1 = 0$$
$$-2592i\sqrt{3}\,b_2 - 2592b_2 = 0$$
$$-2160i\sqrt{3}\,a_3 - 2160a_3 = 0$$
$$-972i\sqrt{3}\,a_1 + 972a_1 = 0$$
$$-288i\sqrt{3}\,a_1 + 288a_1 = 0$$
$$-192i\sqrt{3}\,a_1 - 192a_1 = 0$$
$$-48i\sqrt{3}\,a_3 + 48a_3 = 0$$
$$-48i\sqrt{3}\,b_1 + 48b_1 = 0$$
$$-48i\sqrt{3}\,b_2 + 48b_2 = 0$$
$$48i\sqrt{3}\,a_1 - 48a_1 = 0$$
$$108i\sqrt{3}\,a_1 - 108a_1 = 0$$
$$192i\sqrt{3}\,a_3 + 192a_3 = 0$$
$$192i\sqrt{3}\,b_1 + 192b_1 = 0$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{3b_3}{4} \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= \frac{3x}{4} \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Solving equation (3)

Writing the ode as

$$
y' = -\frac{i\left(-54x^4 + 8y^3 + 6\sqrt{81x^4 - 24y^3}\, x^2\right)^{\frac{2}{3}} \sqrt{3} + \left(-54x^4 + 8y^3 + 6\sqrt{81x^4 - 24y^3}\, x^2\right)^{\frac{2}{3}} - 4y\left(-54x^4 + 8\right.}{6x\left(-54x^4 + 8y^3 + 6\sqrt{81x^4 - 24y^3}\, x^2\right)^{\frac{1}{3}}}
$$

$$
y' = \omega(x, y)
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\text{Expression too large to display} \tag{5E}$$

Putting the above in normal form gives

$$\text{Expression too large to display}$$

Setting the numerator to zero gives

$$\text{Expression too large to display} \tag{6E}$$

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(-54x^4 + 8y^3 + 6\sqrt{81x^4 - 24y^3}\, x^2\right)^{\frac{1}{3}}, \left(-54x^4 + 8y^3 + 6\sqrt{81x^4 - 24y^3}\, x^2\right)^{\frac{2}{3}}, \sqrt{81x^4 - 24y^3}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \left(-54x^4 + 8y^3 + 6\sqrt{81x^4 - 24y^3}\, x^2\right)^{\frac{1}{3}} = v_3, \left(-54x^4 + 8y^3 + 6\sqrt{81x^4 - 24y^3}\, x^2\right)^{\frac{2}{3}} = v_4, \sqrt{81x^4 - 24y^3} = v_5\right\}$$

The above PDE (6E) now becomes

$$\text{Expression too large to display} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$\text{Expression too large to display} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-3456a_1 &= 0 \\
-1296a_1 &= 0 \\
192a_1 &= 0 \\
11664a_1 &= 0 \\
-11664a_3 &= 0 \\
-192a_3 &= 0 \\
1296a_3 &= 0 \\
3456a_3 &= 0 \\
-11664b_1 &= 0 \\
-192b_1 &= 0 \\
1296b_1 &= 0 \\
3456b_1 &= 0 \\
-1728b_2 &= 0 \\
-648b_2 &= 0 \\
96b_2 &= 0 \\
5832b_2 &= 0 \\
-34992i\sqrt{3}\,a_3 + 34992a_3 &= 0 \\
-23328i\sqrt{3}\,b_1 + 23328b_1 &= 0 \\
-23328i\sqrt{3}\,b_2 + 23328b_2 &= 0 \\
-11664i\sqrt{3}\,a_3 + 11664a_3 &= 0 \\
-5760i\sqrt{3}\,a_1 + 5760a_1 &= 0 \\
-3024i\sqrt{3}\,a_1 + 3024a_1 &= 0 \\
-864i\sqrt{3}\,a_3 - 864a_3 &= 0 \\
-540i\sqrt{3}\,a_3 - 540a_3 &= 0 \\
-432i\sqrt{3}\,b_1 - 432b_1 &= 0 \\
-432i\sqrt{3}\,b_2 - 432b_2 &= 0 \\
-192i\sqrt{3}\,a_3 + 192a_3 &= 0 \\
-192i\sqrt{3}\,b_1 + 192b_1 &= 0 \\
-192i\sqrt{3}\,b_2 + 192b_2 &= 0 \\
-108i\sqrt{3}\,a_1 - 108a_1 &= 0 \\
-48i\sqrt{3}\,a_1 - 48a_1 &= 0 \\
48i\sqrt{3}\,a_3 + 48a_3 &= 0 \\
48i\sqrt{3}\,b_1 + 48b_1 &= 0 \\
48i\sqrt{3}\,b_2 + 48b_2 &= 0 \\
192i\sqrt{3}\,a_1 - 192a_1 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0\\
a_2 &= \frac{3b_3}{4}\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= \frac{3x}{4}\\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 3 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
   -> Solving 1st order ODE of high degree, Lie methods, 1st trial
   `, `-> Computing symmetries using: way = 2
   `, `-> Computing symmetries using: way = 2
   -> Solving 1st order ODE of high degree, 2nd attempt. Trying parametric methods
   -> Calling odsolve with the ODE`, diff(y(x), x) = (-3*x^5-3*(x^4+32*y(x))^(1/2)*x^3-32*y(
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      1st order, trying the canonical coordinates of the invariance group
      <- 1st order, canonical coordinates successful
      <- homogeneous successful
   -> Calling odsolve with the ODE`, diff(y(x), x) = (3*x^5-3*(x^4+32*y(x))^(1/2)*x^3+32*y(x
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      1st order, trying the canonical coordinates of the invariance group
      <- 1st order, canonical coordinates successful
      <- homogeneous successful
   <- 1st order, parametric methods successful`
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 800

```
dsolve(x*diff(y(x),x)^3-2*y(x)*diff(y(x),x)^2+4*x^2=0,y(x), singsol=all)
```

$$y(x) = \frac{3x^{\frac{4}{3}}}{2}$$

$$y(x) = -\frac{3x^{\frac{4}{3}}\left(1+i\sqrt{3}\right)}{4}$$

$$y(x) = \frac{3x^{\frac{4}{3}}\left(i\sqrt{3}-1\right)}{4}$$

$$y(x) = \frac{c_1^3 - 128x^2}{32c_1}$$

$$y(x) = \frac{c_1^3 + 128x^2}{32c_1}$$

$$y(x) = \frac{c_1\left(c_1^3 - 1728x^2 + 24\sqrt{6}\,\sqrt{-x^2\left(c_1^3 - 864x^2\right)}\right)^{\frac{1}{3}}}{96} + \frac{c_1^3}{96\left(c_1^3 - 1728x^2 + 24\sqrt{6}\,\sqrt{-x^2\left(c_1^3 - 864x^2\right)}\right)^{\frac{1}{3}}} + \frac{c_1^2}{96}$$

$$y(x) = \frac{c_1\left(c_1^3 + 24\sqrt{6}\,\sqrt{x^2\left(c_1^3 + 864x^2\right)} + 1728x^2\right)^{\frac{1}{3}}}{96} + \frac{c_1^3}{96\left(c_1^3 + 24\sqrt{6}\,\sqrt{x^2\left(c_1^3 + 864x^2\right)} + 1728x^2\right)^{\frac{1}{3}}} + \frac{c_1^2}{96}$$

$$y(x) = \frac{\left(c_1 - \left(c_1^3 - 1728x^2 + 24\sqrt{6}\,\sqrt{-c_1^3x^2 + 864x^4}\right)^{\frac{1}{3}}\right)c_1\left(i\left(\left(c_1^3 - 1728x^2 + 24\sqrt{6}\,\sqrt{-c_1^3x^2 + 864x^4}\right)^{\frac{1}{3}} + c\right.\right.}{192\left(c_1^3 - 1728x^2 + 24\sqrt{6}\,\sqrt{-c_1^3x^2 + 864x^4}\right)^{\frac{1}{3}}}$$

$$y(x) = \frac{\left(c_1^3 - 1728x^2 + 24\sqrt{2}\,\sqrt{3}\,\sqrt{-c_1^3x^2 + 864x^4}\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)c_1}{192} - \frac{\left(i\sqrt{3}\,c_1 + c_1 - 2\left(c_1^3 - 1728x^2 + 24\sqrt{2}\,\sqrt{3}\,\sqrt{-c_1^3x^2 + 864x^4}\right)^{\frac{1}{3}}\right)c_1^2}{192\left(c_1^3 - 1728x^2 + 24\sqrt{2}\,\sqrt{3}\,\sqrt{-c_1^3x^2 + 864x^4}\right)^{\frac{1}{3}}}$$

$$y(x) = \frac{\left(c_1 - \left(c_1^3 + 24\sqrt{6}\,\sqrt{c_1^3x^2 + 864x^4} + 1728x^2\right)^{\frac{1}{3}}\right)c_1\left(i\left(\left(c_1^3 + 24\sqrt{6}\,\sqrt{c_1^3x^2 + 864x^4} + 1728x^2\right)^{\frac{1}{3}} + c_1\right)\right.}{192\left(c_1^3 + 24\sqrt{6}\,\sqrt{c_1^3x^2 + 864x^4} + 1728x^2\right)^{\frac{1}{3}}}$$

$$y(x) = \frac{\left(c_1^3 + 24\sqrt{2}\,\sqrt{3}\,\sqrt{c_1^3x^2 + 864x^4} + 1728x^2\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)c_1}{192}$$

✓ Solution by Mathematica

Time used: 171.698 (sec). Leaf size: 15120

```
DSolve[x*y'[x]^3-2*y[x]*y'[x]^2+4*x^2==0,y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

3 CHAPTER 17. Power series solutions. 17.5. Solutions Near an Ordinary Point. Exercises page 355

3.1 problem 1

Internal problem ID [6889]
Internal file name [`OUTPUT/6132_Tuesday_August_09_2022_05_22_57_AM_9550685/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 17. Power series solutions. 17.5. Solutions Near an Ordinary Point. Exercises page 355
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2}\, f|_{x_0, y_0, y_0'} + \frac{x^3}{3!}\, f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (46) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (47) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
&= -y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1 \\
&= y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2 \\
&= y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3 \\
&= -y \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y}y' + \frac{\partial F_4}{\partial y'}F_4 \\
&= -y' \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y}y' + \frac{\partial F_5}{\partial y'}F_5 \\
&= y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0) \\
F_1 &= -y'(0) \\
F_2 &= y(0) \\
F_3 &= y'(0) \\
F_4 &= -y(0) \\
F_5 &= -y'(0) \\
F_6 &= y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{720}x^6+\frac{1}{40320}x^8\right) y(0) \\
&+ \left(x-\frac{1}{6}x^3+\frac{1}{120}x^5-\frac{1}{5040}x^7\right) y'(0)+O(x^8)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{3}$$

For $0 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n}{(n+2)\,(n+1)} \tag{5}$$

For $n = 0$ the recurrence equation gives

$$2a_2 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = -\frac{a_0}{2}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{720}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{5040}$$

For $n = 6$ the recurrence equation gives

$$56a_8 + a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{a_0}{40320}$$

For $n = 7$ the recurrence equation gives

$$72a_9 + a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{a_1}{362880}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{2} a_0 x^2 - \frac{1}{6} a_1 x^3 + \frac{1}{24} a_0 x^4 + \frac{1}{120} a_1 x^5 - \frac{1}{720} a_0 x^6 - \frac{1}{5040} a_1 x^7 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6\right) a_0 + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7\right) a_1 + O(x^8) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6\right) c_1 + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7\right) c_2 + O(x^8)$$

Summary
The solution(s) found are the following

$$\begin{aligned} y = &\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8\right) y(0) \\ &+ \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7\right) y'(0) + O(x^8) \end{aligned} \quad (1)$$

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6\right) c_1 + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7\right) c_2 + O(x^8) \quad (2)$$

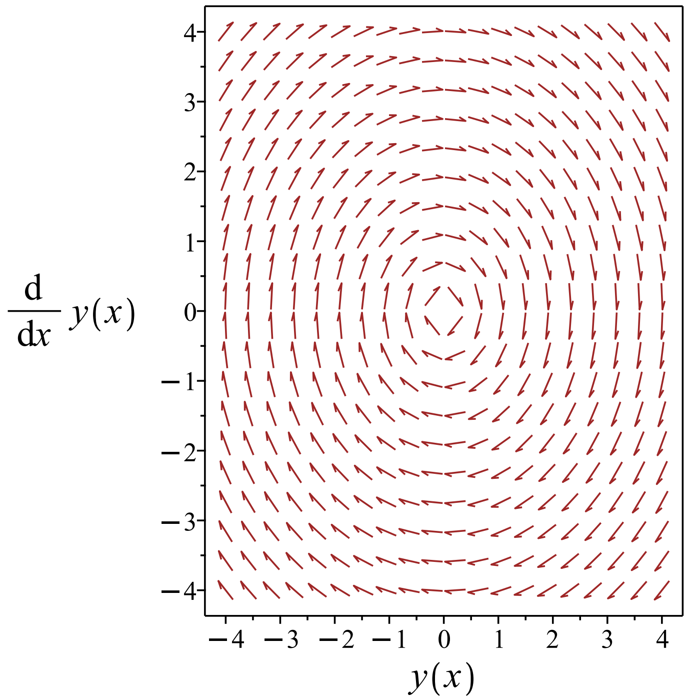

Figure 10: Slope field plot

Verification of solutions

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8\right) y(0) \\ + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7\right) y'(0) + O(x^8)$$

Verified OK.

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6\right) c_1 + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7\right) c_2 + O(x^8)$$

Verified OK.

3.1.1 Maple step by step solution

Let's solve

$y'' = -y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y = 0$

- Characteristic polynomial of ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \cos(x) + c_2 \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
Order:=8;
dsolve(diff(y(x),x$2)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6\right) y(0) + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7\right) D(y)(0) + O(x^8)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 56

```
AsymptoticDSolveValue[y''[x]+y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(-\frac{x^7}{5040}+\frac{x^5}{120}-\frac{x^3}{6}+x\right)+c_1\left(-\frac{x^6}{720}+\frac{x^4}{24}-\frac{x^2}{2}+1\right)$$

3.2 problem 2

Internal problem ID [6890]
Internal file name [`OUTPUT/6133_Tuesday_August_09_2022_05_22_59_AM_44998391/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 17. Power series solutions. 17.5. Solutions Near an Ordinary Point. Exercises page 355
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 9y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2}\, f|_{x_0,y_0,y_0'} + \frac{x^3}{3!}\, f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (49) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (50) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\ \vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y'_0} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= 9y\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
&= 9y'\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1\\
&= 81y\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2\\
&= 81y'\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3\\
&= 729y\\
F_5 &= \frac{dF_4}{dx}\\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y}y' + \frac{\partial F_4}{\partial y'}F_4\\
&= 729y'\\
F_6 &= \frac{dF_5}{dx}\\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y}y' + \frac{\partial F_5}{\partial y'}F_5\\
&= 6561y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 9y(0) \\
F_1 &= 9y'(0) \\
F_2 &= 81y(0) \\
F_3 &= 81y'(0) \\
F_4 &= 729y(0) \\
F_5 &= 729y'(0) \\
F_6 &= 6561y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1+\frac{9}{2}x^2+\frac{27}{8}x^4+\frac{81}{80}x^6+\frac{729}{4480}x^8\right)y(0)+\left(x+\frac{3}{2}x^3+\frac{27}{40}x^5+\frac{81}{560}x^7\right)y'(0)+O(x^8)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = 9\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=0}^{\infty} (-9a_n x^n) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=0}^{\infty} (-9 a_n x^n) = 0 \tag{3}$$

For $0 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - 9a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{9a_n}{(n+2)\,(n+1)} \tag{5}$$

For $n = 0$ the recurrence equation gives

$$2a_2 - 9a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = \frac{9a_0}{2}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - 9a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{3a_1}{2}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 9a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{27a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 9a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{27a_1}{40}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 9a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{81a_0}{80}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 9a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{81a_1}{560}$$

For $n = 6$ the recurrence equation gives

$$56a_8 - 9a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{729a_0}{4480}$$

For $n = 7$ the recurrence equation gives

$$72a_9 - 9a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{81a_1}{4480}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{9}{2} a_0 x^2 + \frac{3}{2} a_1 x^3 + \frac{27}{8} a_0 x^4 + \frac{27}{40} a_1 x^5 + \frac{81}{80} a_0 x^6 + \frac{81}{560} a_1 x^7 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{9}{2}x^2 + \frac{27}{8}x^4 + \frac{81}{80}x^6\right) a_0 + \left(x + \frac{3}{2}x^3 + \frac{27}{40}x^5 + \frac{81}{560}x^7\right) a_1 + O(x^8) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{9}{2}x^2 + \frac{27}{8}x^4 + \frac{81}{80}x^6\right) c_1 + \left(x + \frac{3}{2}x^3 + \frac{27}{40}x^5 + \frac{81}{560}x^7\right) c_2 + O(x^8)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & \left(1 + \frac{9}{2}x^2 + \frac{27}{8}x^4 + \frac{81}{80}x^6 + \frac{729}{4480}x^8\right) y(0) \\ & + \left(x + \frac{3}{2}x^3 + \frac{27}{40}x^5 + \frac{81}{560}x^7\right) y'(0) + O(x^8) \end{aligned} \quad (1)$$

$$y = \left(1 + \frac{9}{2}x^2 + \frac{27}{8}x^4 + \frac{81}{80}x^6\right) c_1 + \left(x + \frac{3}{2}x^3 + \frac{27}{40}x^5 + \frac{81}{560}x^7\right) c_2 + O(x^8) \quad (2)$$

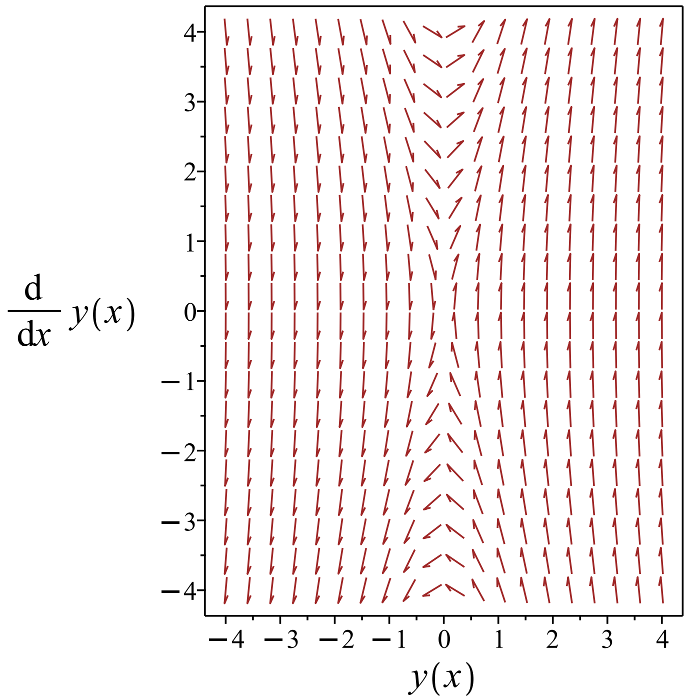

Figure 11: Slope field plot

Verification of solutions

$$y = \left(1+\frac{9}{2}x^2+\frac{27}{8}x^4+\frac{81}{80}x^6+\frac{729}{4480}x^8\right)y(0)+\left(x+\frac{3}{2}x^3+\frac{27}{40}x^5+\frac{81}{560}x^7\right)y'(0)+O(x^8)$$

Verified OK.

$$y = \left(1+\frac{9}{2}x^2+\frac{27}{8}x^4+\frac{81}{80}x^6\right)c_1+\left(x+\frac{3}{2}x^3+\frac{27}{40}x^5+\frac{81}{560}x^7\right)c_2+O(x^8)$$

Verified OK.

3.2.1 Maple step by step solution

Let's solve

$y'' = 9y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - 9y = 0$

- Characteristic polynomial of ODE

  $r^2 - 9 = 0$
- Factor the characteristic polynomial

  $(r - 3)(r + 3) = 0$
- Roots of the characteristic polynomial

  $r = (-3, 3)$
- 1st solution of the ODE

  $y_1(x) = e^{-3x}$
- 2nd solution of the ODE

  $y_2(x) = e^{3x}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$
- Substitute in solutions

  $y = c_1 e^{-3x} + c_2 e^{3x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
Order:=8;
dsolve(diff(y(x),x$2)-9*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + \frac{9}{2}x^2 + \frac{27}{8}x^4 + \frac{81}{80}x^6\right) y(0) + \left(x + \frac{3}{2}x^3 + \frac{27}{40}x^5 + \frac{81}{560}x^7\right) D(y)(0) + O(x^8)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 56

```
AsymptoticDSolveValue[y''[x]-9*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(\frac{81x^7}{560} + \frac{27x^5}{40} + \frac{3x^3}{2} + x\right) + c_1\left(\frac{81x^6}{80} + \frac{27x^4}{8} + \frac{9x^2}{2} + 1\right)$$

3.3 problem 3

Internal problem ID [6891]
Internal file name [`OUTPUT/6134_Tuesday_August_09_2022_05_23_00_AM_96673658/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 17. Power series solutions. 17.5. Solutions Near an Ordinary Point. Exercises page 355
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{y'' + 3xy' + 3y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2}\, f|_{x_0,y_0,y_0'} + \frac{x^3}{3!}\, f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (52) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (53) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -3xy' - 3y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= 9x^2y' + 9yx - 6y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -27x^3y' - 27x^2y + 45xy' + 27y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(81x^4 - 243x^2 + 72\right) y' + \left(81x^3 - 189x\right) y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-243x^5 + 1134x^3 - 891x\right) y' - 243y\left(x^4 - 4x^2 + \frac{5}{3}\right) \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \left(729x^6 - 4860x^4 + 7047x^2 - 1296\right) y' + 729x\left(x^4 - 6x^2 + \frac{19}{3}\right) y \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= \left(-2187x^7 + 19683x^5 - 44955x^3 + 22599x\right) y' - 2187\left(x^2 - \frac{7}{3}\right)\left(x^4 - 6x^2 + \frac{5}{3}\right) y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -3y(0) \\
F_1 &= -6y'(0) \\
F_2 &= 27y(0) \\
F_3 &= 72y'(0) \\
F_4 &= -405y(0) \\
F_5 &= -1296y'(0) \\
F_6 &= 8505y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{3}{2}x^2 + \frac{9}{8}x^4 - \frac{9}{16}x^6 + \frac{27}{128}x^8\right) y(0) + \left(x - x^3 + \frac{3}{5}x^5 - \frac{9}{35}x^7\right) y'(0) + O(x^8)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -3x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 3\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 3n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 3a_n x^n\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} 3n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 3a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 3a_0 = 0$$

$$a_2 = -\frac{3a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + 3na_n + 3a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{3a_n}{n+2} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 6a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -a_1$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 9a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{9a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 12a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{3a_1}{5}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 15a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{9a_0}{16}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 18a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{9a_1}{35}$$

For $n = 6$ the recurrence equation gives

$$56a_8 + 21a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{27a_0}{128}$$

For $n = 7$ the recurrence equation gives

$$72a_9 + 24a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{3a_1}{35}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{3}{2} a_0 x^2 - a_1 x^3 + \frac{9}{8} a_0 x^4 + \frac{3}{5} a_1 x^5 - \frac{9}{16} a_0 x^6 - \frac{9}{35} a_1 x^7 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{3}{2}x^2 + \frac{9}{8}x^4 - \frac{9}{16}x^6\right) a_0 + \left(x - x^3 + \frac{3}{5}x^5 - \frac{9}{35}x^7\right) a_1 + O(x^8) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{3}{2}x^2 + \frac{9}{8}x^4 - \frac{9}{16}x^6\right) c_1 + \left(x - x^3 + \frac{3}{5}x^5 - \frac{9}{35}x^7\right) c_2 + O(x^8)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{3}{2}x^2 + \frac{9}{8}x^4 - \frac{9}{16}x^6 + \frac{27}{128}x^8\right) y(0) + \left(x - x^3 + \frac{3}{5}x^5 - \frac{9}{35}x^7\right) y'(0) + O(x^8) \tag{1}$$

$$y = \left(1 - \frac{3}{2}x^2 + \frac{9}{8}x^4 - \frac{9}{16}x^6\right) c_1 + \left(x - x^3 + \frac{3}{5}x^5 - \frac{9}{35}x^7\right) c_2 + O(x^8) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{3}{2}x^2 + \frac{9}{8}x^4 - \frac{9}{16}x^6 + \frac{27}{128}x^8\right) y(0) + \left(x - x^3 + \frac{3}{5}x^5 - \frac{9}{35}x^7\right) y'(0) + O(x^8)$$

Verified OK.

$$y = \left(1 - \frac{3}{2}x^2 + \frac{9}{8}x^4 - \frac{9}{16}x^6\right) c_1 + \left(x - x^3 + \frac{3}{5}x^5 - \frac{9}{35}x^7\right) c_2 + O(x^8)$$

Verified OK.

3.3.1 Maple step by step solution

Let's solve

$y'' = -3xy' - 3y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 3xy' + 3y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ○ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ○ Shift index using $k- >k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 3a_k(k+1)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+1)\,(a_{k+2}(k+2) + 3a_k) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{3a_k}{k+2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
Order:=8;
dsolve(diff(y(x),x$2)+3*x*diff(y(x),x)+3*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{3}{2}x^2 + \frac{9}{8}x^4 - \frac{9}{16}x^6\right) y(0) + \left(x - x^3 + \frac{3}{5}x^5 - \frac{9}{35}x^7\right) D(y)(0) + O(x^8)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 54

```
AsymptoticDSolveValue[y''[x]+3*x*y'[x]+3*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(-\frac{9x^7}{35} + \frac{3x^5}{5} - x^3 + x\right) + c_1\left(-\frac{9x^6}{16} + \frac{9x^4}{8} - \frac{3x^2}{2} + 1\right)$$

3.4 problem 4

Internal problem ID [6892]
Internal file name [`OUTPUT/6135_Tuesday_August_09_2022_05_23_02_AM_31607739/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 17. Power series solutions. 17.5. Solutions Near an Ordinary Point. Exercises page 355
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(4x^2+1)\,y''-8y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy'_0+\frac{x^2}{2}\,f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\,f'|_{x_0,y_0,y'_0}+\cdots \\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (55)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (56)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4) \\
F_1 &= \frac{df}{dx} \\
&= \frac{dF_0}{dx} \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5) \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right) \\
&= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0 \\
&\vdots \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n \Big|_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= \frac{8y}{4x^2+1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{32x^2 y' - 64yx + 8y'}{(4x^2+1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -\frac{128(4x^2 y' - 8yx + y')\, x}{(4x^2+1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{10240\left(x^2 - \frac{1}{20}\right)\left(\left(x^2+\frac{1}{4}\right) y' - 2yx\right)}{(4x^2+1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -\frac{245760\left(\left(x^2+\frac{1}{4}\right) y' - 2yx\right)\left(x^2 - \frac{3}{20}\right) x}{(4x^2+1)^5} \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \frac{6881280\left(\left(x^2+\frac{1}{4}\right) y' - 2yx\right)\left(x^4 - \frac{3}{10}x^2 + \frac{3}{560}\right)}{(4x^2+1)^6} \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= -\frac{220200960\left(\left(x^2+\frac{1}{4}\right) y' - 2yx\right)\left(x^4 - \frac{1}{2}x^2 + \frac{3}{112}\right) x}{(4x^2+1)^7}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 8y(0) \\
F_1 &= 8y'(0) \\
F_2 &= 0 \\
F_3 &= -128y'(0) \\
F_4 &= 0 \\
F_5 &= 9216y'(0) \\
F_6 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = (4x^2 + 1)\, y(0) + \left(x + \frac{4}{3}x^3 - \frac{16}{15}x^5 + \frac{64}{35}x^7\right) y'(0) + O(x^8)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(4x^2 + 1)\, y'' - 8y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
(4x^2 + 1) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - 8\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} 4x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=0}^{\infty} (-8a_n x^n) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} 4x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=0}^{\infty} (-8a_n x^n) = 0 \quad (3)$$

$n = 0$ gives

$$2a_2 - 8a_0 = 0$$

$$a_2 = 4a_0$$

$n = 1$ gives

$$6a_3 - 8a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{4a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$4na_n(n-1) + (n+2)\, a_{n+2}(n+1) - 8a_n = 0 \quad (4)$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{4(n-2)\, a_n}{n+2} \quad (5)$$

For $n = 2$ the recurrence equation gives

$$12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 0$$

For $n = 3$ the recurrence equation gives

$$16a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{16a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$40a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$72a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{64a_1}{35}$$

For $n = 6$ the recurrence equation gives

$$112a_6 + 56a_8 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = 0$$

For $n = 7$ the recurrence equation gives

$$160a_7 + 72a_9 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = -\frac{256a_1}{63}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + 4a_0 x^2 + \frac{4}{3}a_1 x^3 - \frac{16}{15}a_1 x^5 + \frac{64}{35}a_1 x^7 + \dots$$

Collecting terms, the solution becomes

$$y = (4x^2 + 1)\, a_0 + \left(x + \frac{4}{3}x^3 - \frac{16}{15}x^5 + \frac{64}{35}x^7\right) a_1 + O(x^8) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = (4x^2 + 1)\, c_1 + \left(x + \frac{4}{3}x^3 - \frac{16}{15}x^5 + \frac{64}{35}x^7\right) c_2 + O(x^8)$$

Summary
The solution(s) found are the following

$$y = (4x^2 + 1)\, y(0) + \left(x + \frac{4}{3}x^3 - \frac{16}{15}x^5 + \frac{64}{35}x^7\right) y'(0) + O(x^8) \tag{1}$$

$$y = (4x^2 + 1)\, c_1 + \left(x + \frac{4}{3}x^3 - \frac{16}{15}x^5 + \frac{64}{35}x^7\right) c_2 + O(x^8) \tag{2}$$

Verification of solutions

$$y = (4x^2 + 1)\, y(0) + \left(x + \frac{4}{3}x^3 - \frac{16}{15}x^5 + \frac{64}{35}x^7\right) y'(0) + O(x^8)$$

Verified OK.

$$y = (4x^2 + 1)\, c_1 + \left(x + \frac{4}{3}x^3 - \frac{16}{15}x^5 + \frac{64}{35}x^7\right) c_2 + O(x^8)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=8;
dsolve((1+4*x^2)*diff(y(x),x$2)-8*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (4x^2 + 1)\, y(0) + \left(x + \frac{4}{3}x^3 - \frac{16}{15}x^5 + \frac{64}{35}x^7\right) D(y)\,(0) + O(x^8)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 40

```
AsymptoticDSolveValue[(1+4*x^2)*y''[x]-8*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1 (4x^2 + 1) + c_2 \left(\frac{64x^7}{35} - \frac{16x^5}{15} + \frac{4x^3}{3} + x\right)$$

3.5 problem 5

Internal problem ID [6893]
Internal file name [`OUTPUT/6136_Tuesday_August_09_2022_05_23_04_AM_85928680/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 17. Power series solutions. 17.5. Solutions Near an Ordinary Point. Exercises page 355
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{\left(-4x^2+1\right)y''+8y=0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (58) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (59) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{8y}{4x^2-1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{32x^2 y' - 64yx - 8y'}{(4x^2-1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{128(-4x^3+x)\, y' + 1024x^2 y}{(4x^2-1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{10240\left(x^2+\frac{1}{20}\right)\left(\left(x^2-\frac{1}{4}\right) y' - 2yx\right)}{(4x^2-1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -\frac{245760\left(x^2+\frac{3}{20}\right) x\left(\left(x^2-\frac{1}{4}\right) y' - 2yx\right)}{(4x^2-1)^5} \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \frac{6881280\left(x^4+\frac{3}{10}x^2+\frac{3}{560}\right)\left(\left(x^2-\frac{1}{4}\right) y' - 2yx\right)}{(4x^2-1)^6} \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= -\frac{220200960\left(x^4+\frac{1}{2}x^2+\frac{3}{112}\right) x\left(\left(x^2-\frac{1}{4}\right) y' - 2yx\right)}{(4x^2-1)^7}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -8y(0) \\
F_1 &= -8y'(0) \\
F_2 &= 0 \\
F_3 &= -128y'(0) \\
F_4 &= 0 \\
F_5 &= -9216y'(0) \\
F_6 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(-4x^2 + 1\right) y(0) + \left(x - \frac{4}{3}x^3 - \frac{16}{15}x^5 - \frac{64}{35}x^7\right) y'(0) + O\left(x^8\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
\left(-4x^2 + 1\right) y'' + 8y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(-4x^2 + 1\right) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + 8\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}
$$

Which simplifies to

$$
\sum_{n=2}^{\infty} \left(-4x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} 8a_n x^n\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\sum_{n=2}^{\infty} \left(-4x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=0}^{\infty} 8a_n x^n\right) = 0 \quad (3)$$

$n = 0$ gives

$$2a_2 + 8a_0 = 0$$

$$a_2 = -4a_0$$

$n = 1$ gives

$$6a_3 + 8a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{4a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$-4na_n(n-1) + (n+2)\, a_{n+2}(n+1) + 8a_n = 0 \quad (4)$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{4(n-2)\, a_n}{n+2} \quad (5)$$

For $n = 2$ the recurrence equation gives

$$12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 0$$

For $n = 3$ the recurrence equation gives

$$-16a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{16a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$-40a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$-72a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{64a_1}{35}$$

For $n = 6$ the recurrence equation gives

$$-112a_6 + 56a_8 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = 0$$

For $n = 7$ the recurrence equation gives

$$-160a_7 + 72a_9 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = -\frac{256a_1}{63}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x - 4a_0x^2 - \frac{4}{3}a_1x^3 - \frac{16}{15}a_1x^5 - \frac{64}{35}a_1x^7 + \dots$$

Collecting terms, the solution becomes

$$y = (-4x^2 + 1)\, a_0 + \left(x - \frac{4}{3}x^3 - \frac{16}{15}x^5 - \frac{64}{35}x^7\right) a_1 + O(x^8) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = (-4x^2 + 1)\, c_1 + \left(x - \frac{4}{3}x^3 - \frac{16}{15}x^5 - \frac{64}{35}x^7\right) c_2 + O(x^8)$$

Summary

The solution(s) found are the following

$$y = (-4x^2 + 1)\, y(0) + \left(x - \frac{4}{3}x^3 - \frac{16}{15}x^5 - \frac{64}{35}x^7\right) y'(0) + O(x^8) \tag{1}$$

$$y = (-4x^2 + 1)\, c_1 + \left(x - \frac{4}{3}x^3 - \frac{16}{15}x^5 - \frac{64}{35}x^7\right) c_2 + O(x^8) \tag{2}$$

Verification of solutions

$$y = (-4x^2 + 1)\, y(0) + \left(x - \frac{4}{3}x^3 - \frac{16}{15}x^5 - \frac{64}{35}x^7\right) y'(0) + O(x^8)$$

Verified OK.

$$y = (-4x^2 + 1)\, c_1 + \left(x - \frac{4}{3}x^3 - \frac{16}{15}x^5 - \frac{64}{35}x^7\right) c_2 + O(x^8)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=8;
dsolve((1-4*x^2)*diff(y(x),x$2)+8*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(-4x^2+1\right) y(0) + \left(x - \frac{4}{3}x^3 - \frac{16}{15}x^5 - \frac{64}{35}x^7\right) D(y)(0) + O\left(x^8\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 40

```
AsymptoticDSolveValue[(1-4*x^2)*y''[x]+8*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(1-4x^2\right) + c_2\left(-\frac{64x^7}{35} - \frac{16x^5}{15} - \frac{4x^3}{3} + x\right)$$

3.6 problem 6

Internal problem ID [6894]
Internal file name [`OUTPUT/6137_Tuesday_August_09_2022_05_23_05_AM_69199443/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 17. Power series solutions. 17.5. Solutions Near an Ordinary Point. Exercises page 355
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+1)\, y''-4xy'+6y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy'_0+\frac{x^2}{2}\,f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\,f'|_{x_0,y_0,y'_0}+\cdots \\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{61}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{62}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{align*}
F_0 &= f(x, y, y') \tag{4}\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \tag{6}
\end{align*}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{4xy' - 6y}{x^2 + 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{6x^2y' - 12yx - 2y'}{(x^2 + 1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= 0 \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= 0 \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= 0 \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= 0 \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= 0
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -6y(0) \\
F_1 &= -2y'(0) \\
F_2 &= 0 \\
F_3 &= 0 \\
F_4 &= 0 \\
F_5 &= 0 \\
F_6 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(-3x^2 + 1\right) y(0) + \left(x - \frac{1}{3}x^3\right) y'(0) + O\left(x^8\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
\left(x^2 + 1\right) y'' - 4xy' + 6y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(x^2 + 1\right) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - 4x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 6\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} \left(-4n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 6a_n x^n\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned} &\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) \\ &+ \sum_{n=1}^{\infty} (-4n a_n x^n) + \left(\sum_{n=0}^{\infty} 6a_n x^n\right) = 0 \end{aligned} \tag{3}$$

$n = 0$ gives

$$2a_2 + 6a_0 = 0$$

$$a_2 = -3a_0$$

$n = 1$ gives

$$6a_3 + 2a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$n a_n(n-1) + (n+2) a_{n+2}(n+1) - 4n a_n + 6a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(n^2 - 5n + 6)}{(n+2)(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 0$$

For $n = 3$ the recurrence equation gives

$$20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$2a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$6a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

For $n = 6$ the recurrence equation gives

$$12a_6 + 56a_8 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = 0$$

For $n = 7$ the recurrence equation gives

$$20a_7 + 72a_9 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - 3a_0 x^2 - \frac{1}{3} a_1 x^3 + \dots$$

Collecting terms, the solution becomes

$$y = (-3x^2 + 1)\, a_0 + \left(x - \frac{1}{3}x^3\right) a_1 + O(x^8) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = (-3x^2 + 1)\, c_1 + \left(x - \frac{1}{3}x^3\right) c_2 + O(x^8)$$

Summary

The solution(s) found are the following

$$y = (-3x^2 + 1)\, y(0) + \left(x - \frac{1}{3}x^3\right) y'(0) + O(x^8) \tag{1}$$

$$y = (-3x^2 + 1)\, c_1 + \left(x - \frac{1}{3}x^3\right) c_2 + O(x^8) \tag{2}$$

Verification of solutions

$$y = (-3x^2 + 1)\, y(0) + \left(x - \frac{1}{3}x^3\right) y'(0) + O(x^8)$$

Verified OK.

$$y = (-3x^2 + 1)\, c_1 + \left(x - \frac{1}{3}x^3\right) c_2 + O(x^8)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
Order:=8;
dsolve((1+x^2)*diff(y(x),x$2)-4*x*diff(y(x),x)+6*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = y(0) + D(y)(0)\,x - 3x^2y(0) - \frac{D(y)(0)\,x^3}{3}$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 26

```
AsymptoticDSolveValue[(1+x^2)*y''[x]-4*x*y'[x]+6*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(x - \frac{x^3}{3}\right) + c_1(1 - 3x^2)$$

3.7 problem 7

Internal problem ID [6895]
Internal file name [`OUTPUT/6138_Tuesday_August_09_2022_05_23_07_AM_96573020/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 17. Power series solutions. 17.5. Solutions Near an Ordinary Point. Exercises page 355
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+1)\,y''+10xy'+20y=0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\,y'(x_0) + \frac{(x - x_0)^2}{2}y''(x_0) + \frac{(x - x_0)^3}{3!}y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2}\,f|_{x_0,y_0,y'_0} + \frac{x^3}{3!}\,f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!}\,\frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{64}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{65}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n\big|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{10(xy'+2y)}{x^2+1}\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x}+\frac{\partial F_0}{\partial y}y'+\frac{\partial F_0}{\partial y'}F_0\\
&= \frac{90x^2y'+240yx-30y'}{(x^2+1)^2}\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x}+\frac{\partial F_1}{\partial y}y'+\frac{\partial F_1}{\partial y'}F_1\\
&= \frac{840(-x^3+x)\,y'+840(-3x^2+1)\,y}{(x^2+1)^3}\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x}+\frac{\partial F_2}{\partial y}y'+\frac{\partial F_2}{\partial y'}F_2\\
&= \frac{1680(5x^4-10x^2+1)\,y'+26880(x^3-x)\,y}{(x^2+1)^4}\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x}+\frac{\partial F_3}{\partial y}y'+\frac{\partial F_3}{\partial y'}F_3\\
&= \frac{(-90720x^5+302400x^3-90720x)\,y'-302400\left(x^4-2x^2+\frac{1}{5}\right)y}{(x^2+1)^5}\\
F_5 &= \frac{dF_4}{dx}\\
&= \frac{\partial F_4}{\partial x}+\frac{\partial F_4}{\partial y}y'+\frac{\partial F_4}{\partial y'}F_4\\
&= \frac{(1058400x^6-5292000x^4+3175200x^2-151200)\,y'+3628800\left(x^2-\frac{1}{3}\right)(x^2-3)\,xy}{(x^2+1)^6}\\
F_6 &= \frac{dF_5}{dx}\\
&= \frac{\partial F_5}{\partial x}+\frac{\partial F_5}{\partial y}y'+\frac{\partial F_5}{\partial y'}F_5\\
&= \frac{13305600(-x^7+7x^5-7x^3+x)\,y'-46569600\left(x^6-5x^4+3x^2-\frac{1}{7}\right)y}{(x^2+1)^7}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$\begin{aligned}
F_0 &= -20y(0)\\
F_1 &= -30y'(0)\\
F_2 &= 840y(0)\\
F_3 &= 1680y'(0)\\
F_4 &= -60480y(0)\\
F_5 &= -151200y'(0)\\
F_6 &= 6652800y(0)
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(165x^8 - 84x^6 + 35x^4 - 10x^2 + 1\right) y(0) + \left(-30x^7 + 14x^5 - 5x^3 + x\right) y'(0) + O\left(x^8\right)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(x^2 + 1\right) y'' + 10xy' + 20y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1}\\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}$$

Substituting the above back into the ode gives

$$\left(x^2 + 1\right) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + 10x \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 20 \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 10 n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 20 a_n x^n\right) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} 10 n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 20 a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 20a_0 = 0$$

$$a_2 = -10a_0$$

$n = 1$ gives

$$6a_3 + 30a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -5a_1$$

For $2 \leq n$, the recurrence equation is

$$n a_n(n-1) + (n+2)\, a_{n+2}(n+1) + 10 n a_n + 20 a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(n^2 + 9n + 20)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$42a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 35a_0$$

For $n = 3$ the recurrence equation gives

$$56a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 14a_1$$

For $n = 4$ the recurrence equation gives

$$72a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -84a_0$$

For $n = 5$ the recurrence equation gives

$$90a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -30a_1$$

For $n = 6$ the recurrence equation gives

$$110a_6 + 56a_8 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = 165a_0$$

For $n = 7$ the recurrence equation gives

$$132a_7 + 72a_9 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = 55a_1$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = -30a_1x^7 - 84a_0x^6 + 14a_1x^5 + 35a_0x^4 - 5a_1x^3 - 10a_0x^2 + a_1x + a_0 + \dots$$

Collecting terms, the solution becomes

$$y = \left(-84x^6 + 35x^4 - 10x^2 + 1\right) a_0 + \left(-30x^7 + 14x^5 - 5x^3 + x\right) a_1 + O\left(x^8\right) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(-84x^6 + 35x^4 - 10x^2 + 1\right) c_1 + \left(-30x^7 + 14x^5 - 5x^3 + x\right) c_2 + O\left(x^8\right)$$

Summary

The solution(s) found are the following

$$y = \left(165x^8 - 84x^6 + 35x^4 - 10x^2 + 1\right) y(0) + \left(-30x^7 + 14x^5 - 5x^3 + x\right) y'(0) + O\left(x^8\right) \quad (1)$$

$$y = \left(-84x^6 + 35x^4 - 10x^2 + 1\right) c_1 + \left(-30x^7 + 14x^5 - 5x^3 + x\right) c_2 + O\left(x^8\right) \quad (2)$$

Verification of solutions

$$y = \left(165x^8 - 84x^6 + 35x^4 - 10x^2 + 1\right) y(0) + \left(-30x^7 + 14x^5 - 5x^3 + x\right) y'(0) + O\left(x^8\right)$$

Verified OK.

$$y = \left(-84x^6 + 35x^4 - 10x^2 + 1\right) c_1 + \left(-30x^7 + 14x^5 - 5x^3 + x\right) c_2 + O\left(x^8\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 44

```
Order:=8;
dsolve((1+x^2)*diff(y(x),x$2)+10*x*diff(y(x),x)+20*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (-84x^6 + 35x^4 - 10x^2 + 1)\, y(0) + (-30x^7 + 14x^5 - 5x^3 + x)\, D(y)(0) + O(x^8)$$

✓ Solution by Mathematica
Time used: 0.001 (sec). Leaf size: 44

```
AsymptoticDSolveValue[(1+x^2)*y''[x]+10*x*y'[x]+20*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2(-30x^7 + 14x^5 - 5x^3 + x) + c_1(-84x^6 + 35x^4 - 10x^2 + 1)$$

3.8 problem 8

Internal problem ID [6896]
Internal file name [`OUTPUT/6139_Tuesday_August_09_2022_05_23_09_AM_85755146/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 17. Power series solutions. 17.5. Solutions Near an Ordinary Point. Exercises page 355
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+4)\,y''+2xy'-12y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy'_0+\frac{x^2}{2}\,f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\,f'|_{x_0,y_0,y'_0}+\cdots \\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{67}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{68}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2(xy'-6y)}{x^2+4}\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
&= \frac{18x^2y' - 48yx + 40y'}{\left(x^2+4\right)^2}\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1\\
&= \frac{-120x^3y' + 360x^2y - 288xy' + 288y}{\left(x^2+4\right)^3}\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2\\
&= \frac{192(5x^3y' - 15x^2y + 12xy' - 12y)\,x}{\left(x^2+4\right)^4}\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3\\
&= -\frac{8640\left(x+\frac{2}{3}\right)\left(x-\frac{2}{3}\right)\left(\left(x^3+\frac{12}{5}x\right)y' + y\left(-3x^2-\frac{12}{5}\right)\right)}{\left(x^2+4\right)^5}\\
F_5 &= \frac{dF_4}{dx}\\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y}y' + \frac{\partial F_4}{\partial y'}F_4\\
&= \frac{86400x\left(\left(x^3+\frac{12}{5}x\right)y' + y\left(-3x^2-\frac{12}{5}\right)\right)\left(x^2-\frac{4}{3}\right)}{\left(x^2+4\right)^6}\\
F_6 &= \frac{dF_5}{dx}\\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y}y' + \frac{\partial F_5}{\partial y'}F_5\\
&= -\frac{950400\left(\left(x^3+\frac{12}{5}x\right)y' + y\left(-3x^2-\frac{12}{5}\right)\right)\left(x^4-\frac{8}{3}x^2+\frac{16}{33}\right)}{\left(x^2+4\right)^7}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$\begin{aligned}
F_0 &= 3y(0) \\
F_1 &= \frac{5y'(0)}{2} \\
F_2 &= \frac{9y(0)}{2} \\
F_3 &= 0 \\
F_4 &= -9y(0) \\
F_5 &= 0 \\
F_6 &= \frac{135y(0)}{2}
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1 + \frac{3}{2}x^2 + \frac{3}{16}x^4 - \frac{1}{80}x^6 + \frac{3}{1792}x^8\right) y(0) + \left(x + \frac{5}{12}x^3\right) y'(0) + O(x^8)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2 + 4)\, y'' + 2xy' - 12y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}$$

Substituting the above back into the ode gives

$$(x^2 + 4)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + 2x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 12\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} 4n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 2n a_n x^n\right) + \sum_{n=0}^{\infty} (-12 a_n x^n) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 4n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} 4(n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned}&\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} 4(n+2)\, a_{n+2}(n+1)\, x^n\right)\\ &+ \left(\sum_{n=1}^{\infty} 2n a_n x^n\right) + \sum_{n=0}^{\infty} (-12 a_n x^n) = 0\end{aligned} \tag{3}$$

$n = 0$ gives

$$8a_2 - 12a_0 = 0$$

$$a_2 = \frac{3a_0}{2}$$

$n = 1$ gives

$$24a_3 - 10a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{5a_1}{12}$$

For $2 \leq n$, the recurrence equation is

$$n a_n(n-1) + 4(n+2)\, a_{n+2}(n+1) + 2n a_n - 12a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(n^2+n-12)}{4(n+2)(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$-6a_2 + 48a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{3a_0}{16}$$

For $n = 3$ the recurrence equation gives

$$80a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$8a_4 + 120a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{80}$$

For $n = 5$ the recurrence equation gives

$$18a_5 + 168a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

For $n = 6$ the recurrence equation gives

$$30a_6 + 224a_8 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{3a_0}{1792}$$

For $n = 7$ the recurrence equation gives

$$44a_7 + 288a_9 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x + \frac{3}{2}a_0x^2 + \frac{5}{12}a_1x^3 + \frac{3}{16}a_0x^4 - \frac{1}{80}a_0x^6 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{3}{2}x^2 + \frac{3}{16}x^4 - \frac{1}{80}x^6\right) a_0 + \left(x + \frac{5}{12}x^3\right) a_1 + O(x^8) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{3}{2}x^2 + \frac{3}{16}x^4 - \frac{1}{80}x^6\right) c_1 + \left(x + \frac{5}{12}x^3\right) c_2 + O(x^8)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \left(1 + \frac{3}{2}x^2 + \frac{3}{16}x^4 - \frac{1}{80}x^6 + \frac{3}{1792}x^8\right) y(0) + \left(x + \frac{5}{12}x^3\right) y'(0) + O(x^8) \tag{1}$$

$$y = \left(1 + \frac{3}{2}x^2 + \frac{3}{16}x^4 - \frac{1}{80}x^6\right) c_1 + \left(x + \frac{5}{12}x^3\right) c_2 + O(x^8) \tag{2}$$

Verification of solutions

$$y = \left(1 + \frac{3}{2}x^2 + \frac{3}{16}x^4 - \frac{1}{80}x^6 + \frac{3}{1792}x^8\right) y(0) + \left(x + \frac{5}{12}x^3\right) y'(0) + O(x^8)$$

Verified OK.

$$y = \left(1 + \frac{3}{2}x^2 + \frac{3}{16}x^4 - \frac{1}{80}x^6\right) c_1 + \left(x + \frac{5}{12}x^3\right) c_2 + O(x^8)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=8;
dsolve((x^2+4)*diff(y(x),x$2)+2*x*diff(y(x),x)-12*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + \frac{3}{2}x^2 + \frac{3}{16}x^4 - \frac{1}{80}x^6\right) y(0) + \left(x + \frac{5}{12}x^3\right) D(y)(0) + O(x^8)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[(x^2+4)*y''[x]+2*x*y'[x]-12*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(\frac{5x^3}{12}+x\right)+c_1\left(-\frac{x^6}{80}+\frac{3x^4}{16}+\frac{3x^2}{2}+1\right)$$

3.9 problem 9

Internal problem ID [6897]
Internal file name [`OUTPUT/6140_Tuesday_August_09_2022_05_23_10_AM_95537058/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 17. Power series solutions. 17.5. Solutions Near an Ordinary Point. Exercises page 355
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_2", "second order series method. Taylor series method", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2-9)\,y''+3xy'-3y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2}\, f|_{x_0, y_0, y_0'} + \frac{x^3}{3!}\, f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (70) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (71) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{3(xy'-y)}{x^2-9}\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
&= \frac{15(xy'-y)\,x}{\left(x^2-9\right)^2}\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1\\
&= -\frac{90(xy'-y)\left(x^2+\frac{3}{2}\right)}{\left(x^2-9\right)^3}\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2\\
&= \frac{630(xy'-y)\,x\left(x^2+\frac{9}{2}\right)}{\left(x^2-9\right)^4}\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3\\
&= -\frac{5040(xy'-y)\left(x^4+9x^2+\frac{81}{16}\right)}{\left(x^2-9\right)^5}\\
F_5 &= \frac{dF_4}{dx}\\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y}y' + \frac{\partial F_4}{\partial y'}F_4\\
&= \frac{45360(xy'-y)\left(x^4+15x^2+\frac{405}{16}\right)x}{\left(x^2-9\right)^6}\\
F_6 &= \frac{dF_5}{dx}\\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y}y' + \frac{\partial F_5}{\partial y'}F_5\\
&= -\frac{14175(xy'-y)\left(32x^6+720x^4+2430x^2+729\right)}{\left(x^2-9\right)^7}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -\frac{y(0)}{3} \\
F_1 &= 0 \\
F_2 &= -\frac{5y(0)}{27} \\
F_3 &= 0 \\
F_4 &= -\frac{35y(0)}{81} \\
F_5 &= 0 \\
F_6 &= -\frac{175y(0)}{81}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{6}x^2 - \frac{5}{648}x^4 - \frac{7}{11664}x^6 - \frac{5}{93312}x^8\right) y(0) + xy'(0) + O(x^8)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(x^2 - 9)\, y'' + 3xy' - 3y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
(x^2 - 9) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + 3x \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 3\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}
$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right)+\sum_{n=2}^{\infty}\left(-9n(n-1)\, a_n x^{n-2}\right)+\left(\sum_{n=1}^{\infty} 3n a_n x^n\right)+\sum_{n=0}^{\infty}\left(-3a_n x^n\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty}\left(-9n(n-1)\, a_n x^{n-2}\right)=\sum_{n=0}^{\infty}\left(-9(n+2)\, a_{n+2}(n+1)\, x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned}\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right)&+\sum_{n=0}^{\infty}\left(-9(n+2)\, a_{n+2}(n+1)\, x^n\right)\\ &+\left(\sum_{n=1}^{\infty} 3n a_n x^n\right)+\sum_{n=0}^{\infty}\left(-3a_n x^n\right)=0\end{aligned} \tag{3}$$

$n=0$ gives

$$-18a_2-3a_0=0$$

$$a_2=-\frac{a_0}{6}$$

For $2\leq n$, the recurrence equation is

$$n a_n(n-1)-9(n+2)\, a_{n+2}(n+1)+3n a_n-3a_n=0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2}=\frac{a_n(n^2+2n-3)}{9\,(n+2)\,(n+1)} \tag{5}$$

For $n=2$ the recurrence equation gives

$$5a_2-108a_4=0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{5a_0}{648}$$

For $n = 3$ the recurrence equation gives

$$12a_3 - 180a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$21a_4 - 270a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{7a_0}{11664}$$

For $n = 5$ the recurrence equation gives

$$32a_5 - 378a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

For $n = 6$ the recurrence equation gives

$$45a_6 - 504a_8 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = -\frac{5a_0}{93312}$$

For $n = 7$ the recurrence equation gives

$$60a_7 - 648a_9 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{6} a_0 x^2 - \frac{5}{648} a_0 x^4 - \frac{7}{11664} a_0 x^6 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{6} x^2 - \frac{5}{648} x^4 - \frac{7}{11664} x^6\right) a_0 + a_1 x + O(x^8) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{6} x^2 - \frac{5}{648} x^4 - \frac{7}{11664} x^6\right) c_1 + c_2 x + O(x^8)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{6} x^2 - \frac{5}{648} x^4 - \frac{7}{11664} x^6 - \frac{5}{93312} x^8\right) y(0) + x y'(0) + O(x^8) \tag{1}$$

$$y = \left(1 - \frac{1}{6} x^2 - \frac{5}{648} x^4 - \frac{7}{11664} x^6\right) c_1 + c_2 x + O(x^8) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{1}{6} x^2 - \frac{5}{648} x^4 - \frac{7}{11664} x^6 - \frac{5}{93312} x^8\right) y(0) + x y'(0) + O(x^8)$$

Verified OK.

$$y = \left(1 - \frac{1}{6} x^2 - \frac{5}{648} x^4 - \frac{7}{11664} x^6\right) c_1 + c_2 x + O(x^8)$$

Verified OK.

3.9.1 Maple step by step solution

Let's solve

$(x^2-9)\,y''+3xy'-3y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{3xy'}{x^2-9}+\frac{3y}{x^2-9}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{3xy'}{x^2-9}-\frac{3y}{x^2-9}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{3x}{x^2-9}, P_3(x)=-\frac{3}{x^2-9}\right]$

  ◦ $(x+3)\cdot P_2(x)$ is analytic at $x=-3$

  $\left((x+3)\cdot P_2(x)\right)\bigg|_{x=-3}=\frac{3}{2}$

  ◦ $(x+3)^2\cdot P_3(x)$ is analytic at $x=-3$

  $\left((x+3)^2\cdot P_3(x)\right)\bigg|_{x=-3}=0$

  ◦ $x=-3$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-3$

- Multiply by denominators

  $(x^2-9)\,y''+3xy'-3y=0$

- Change variables using $x=u-3$ so that the regular singular point is at $u=0$

  $(u^2-6u)\left(\frac{d^2}{du^2}y(u)\right)+(3u-9)\left(\frac{d}{du}y(u)\right)-3y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-3a_0r(1+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-3a_{k+1}(k+1+r)\,(2k+3+2r) + a_k(k+r+3)\,(k+r-1)\right) u^{k+r}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-3r(1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-6\left(k+\frac{3}{2}+r\right)(k+1+r)\, a_{k+1} + a_k(k+r+3)\,(k+r-1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+r+3)(k+r-1)}{3(2k+3+2r)(k+1+r)}$$

- Recursion relation for $r = 0$; series terminates at $k = 1$

$$a_{k+1} = \frac{a_k(k+3)(k-1)}{3(2k+3)(k+1)}$$

- Apply recursion relation for $k = 0$

$$a_1 = -\frac{a_0}{3}$$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

$$y(u) = a_0 \cdot \left(1 - \frac{u}{3}\right)$$

- Revert the change of variables $u = x + 3$

$$\left[y = -\frac{a_0 x}{3}\right]$$

- Recursion relation for $r = -\frac{1}{2}$

$a_{k+1} = \frac{a_k \left(k+\frac{5}{2}\right) \left(k-\frac{3}{2}\right)}{3(2k+2) \left(k+\frac{1}{2}\right)}$

- Solution for $r = -\frac{1}{2}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{1}{2}}, a_{k+1} = \frac{a_k \left(k+\frac{5}{2}\right) \left(k-\frac{3}{2}\right)}{3(2k+2) \left(k+\frac{1}{2}\right)}\right]$

- Revert the change of variables $u = x + 3$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+3)^{k-\frac{1}{2}}, a_{k+1} = \frac{a_k \left(k+\frac{5}{2}\right) \left(k-\frac{3}{2}\right)}{3(2k+2) \left(k+\frac{1}{2}\right)}\right]$

- Combine solutions and rename parameters

  $\left[y = -\frac{a_0 x}{3} + \left(\sum_{k=0}^{\infty} b_k (x+3)^{k-\frac{1}{2}}\right), b_{k+1} = \frac{b_k \left(k+\frac{5}{2}\right) \left(k-\frac{3}{2}\right)}{3(2k+2) \left(k+\frac{1}{2}\right)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 29

```
Order:=8;
dsolve((x^2-9)*diff(y(x),x$2)+3*x*diff(y(x),x)-3*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{6}x^2 - \frac{5}{648}x^4 - \frac{7}{11664}x^6\right) y(0) + D(y)(0)\, x + O(x^8)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 34

```
AsymptoticDSolveValue[(x^2-9)*y''[x]+3*x*y'[x]-3*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1 \left(-\frac{7x^6}{11664} - \frac{5x^4}{648} - \frac{x^2}{6} + 1 \right) + c_2 x$$

3.10 problem 10

Internal problem ID [6898]
Internal file name [OUTPUT/6141_Tuesday_August_09_2022_05_23_12_AM_4787677/index.tex]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 17. Power series solutions. 17.5. Solutions Near an Ordinary Point. Exercises page 355
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 2xy' + 5y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (73) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (74) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}F_n|_{x_0,y_0,y_0'} \tag{7}$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -2xy' - 5y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= 4x^2y' + 10yx - 7y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -8x^3y' - 20x^2y + 32xy' + 45y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(16x^4 - 108x^2 + 77\right) y' + 40yx\left(x^2 - 5\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-32x^5 + 320x^3 - 570x\right) y' + \left(-80x^4 + 660x^2 - 585\right) y \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \left(64x^6 - 880x^4 + 2760x^2 - 1155\right) y' + 160x\left(x^4 - 12x^2 + \frac{417}{16}\right) y \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= \left(-128x^7 + 2304x^5 - 10960x^3 + 12000x\right) y' - 320\left(x^6 - \frac{65}{4}x^4 + \frac{489}{8}x^2 - \frac{1989}{64}\right) y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -5y(0) \\
F_1 &= -7y'(0) \\
F_2 &= 45y(0) \\
F_3 &= 77y'(0) \\
F_4 &= -585y(0) \\
F_5 &= -1155y'(0) \\
F_6 &= 9945y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{5}{2}x^2 + \frac{15}{8}x^4 - \frac{13}{16}x^6 + \frac{221}{896}x^8\right) y(0) + \left(x - \frac{7}{6}x^3 + \frac{77}{120}x^5 - \frac{11}{48}x^7\right) y'(0) + O(x^8)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -2x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 5\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 2n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 5 a_n x^n\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} 2n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 5a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 5a_0 = 0$$

$$a_2 = -\frac{5a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + 2na_n + 5a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(2n+5)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 7a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{7a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 9a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{15a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 11a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{77a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 13a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{13a_0}{16}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 15a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{11a_1}{48}$$

For $n = 6$ the recurrence equation gives

$$56a_8 + 17a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{221a_0}{896}$$

For $n = 7$ the recurrence equation gives

$$72a_9 + 19a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{209a_1}{3456}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x - \frac{5}{2}a_0x^2 - \frac{7}{6}a_1x^3 + \frac{15}{8}a_0x^4 + \frac{77}{120}a_1x^5 - \frac{13}{16}a_0x^6 - \frac{11}{48}a_1x^7 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{5}{2}x^2 + \frac{15}{8}x^4 - \frac{13}{16}x^6\right)a_0 + \left(x - \frac{7}{6}x^3 + \frac{77}{120}x^5 - \frac{11}{48}x^7\right)a_1 + O(x^8) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{5}{2}x^2 + \frac{15}{8}x^4 - \frac{13}{16}x^6\right)c_1 + \left(x - \frac{7}{6}x^3 + \frac{77}{120}x^5 - \frac{11}{48}x^7\right)c_2 + O(x^8)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & \left(1 - \frac{5}{2}x^2 + \frac{15}{8}x^4 - \frac{13}{16}x^6 + \frac{221}{896}x^8\right)y(0) \\ & + \left(x - \frac{7}{6}x^3 + \frac{77}{120}x^5 - \frac{11}{48}x^7\right)y'(0) + O(x^8) \end{aligned} \quad (1)$$

$$y = \left(1 - \frac{5}{2}x^2 + \frac{15}{8}x^4 - \frac{13}{16}x^6\right)c_1 + \left(x - \frac{7}{6}x^3 + \frac{77}{120}x^5 - \frac{11}{48}x^7\right)c_2 + O(x^8) \quad (2)$$

Verification of solutions

$$y = \left(1-\frac{5}{2}x^2+\frac{15}{8}x^4-\frac{13}{16}x^6+\frac{221}{896}x^8\right)y(0)+\left(x-\frac{7}{6}x^3+\frac{77}{120}x^5-\frac{11}{48}x^7\right)y'(0)+O(x^8)$$

Verified OK.

$$y = \left(1-\frac{5}{2}x^2+\frac{15}{8}x^4-\frac{13}{16}x^6\right)c_1+\left(x-\frac{7}{6}x^3+\frac{77}{120}x^5-\frac{11}{48}x^7\right)c_2+O(x^8)$$

Verified OK.

3.10.1 Maple step by step solution

Let's solve

$y'' = -2xy' - 5y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 2xy' + 5y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ○ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ○ Shift index using $k- >k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} (a_{k+2}(k+2)(k+1) + a_k(2k+5))\, x^k = 0$

- Each term in the series must be 0, giving the recursion relation

$$\left(k^{2}+3k+2\right)a_{k+2}+2a_{k}k+5a_{k}=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y=\sum_{k=0}^{\infty}a_{k}x^{k},a_{k+2}=-\frac{a_{k}(2k+5)}{k^{2}+3k+2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 44

```
Order:=8;
dsolve(diff(y(x),x$2)+2*x*diff(y(x),x)+5*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\left(1-\frac{5}{2}x^{2}+\frac{15}{8}x^{4}-\frac{13}{16}x^{6}\right)y(0)+\left(x-\frac{7}{6}x^{3}+\frac{77}{120}x^{5}-\frac{11}{48}x^{7}\right)D(y)\left(0\right)+O\left(x^{8}\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 56

```
AsymptoticDSolveValue[y''[x]+2*x*y'[x]+5*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(-\frac{11x^7}{48} + \frac{77x^5}{120} - \frac{7x^3}{6} + x\right) + c_1\left(-\frac{13x^6}{16} + \frac{15x^4}{8} - \frac{5x^2}{2} + 1\right)$$

3.11 problem 11

Internal problem ID [6899]
Internal file name [`OUTPUT/6142_Tuesday_August_09_2022_05_23_13_AM_4655115/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 17. Power series solutions. 17.5. Solutions Near an Ordinary Point. Exercises page 355
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(x^2+4)\,y''+6xy'+4y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\ &= y_0+xy'_0+\frac{x^2}{2}f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}f'|_{x_0,y_0,y'_0}+\cdots \\ &= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (76) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (77) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2(3xy' + 2y)}{x^2 + 4} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{38x^2 y' + 32yx - 40y'}{\left(x^2 + 4\right)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{-272x^3 y' - 248x^2 y + 832xy' + 288y}{\left(x^2 + 4\right)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{\left(2200x^4 - 13120x^2 + 4480\right) y' + \left(2080x^3 - 7040x\right) y}{\left(x^2 + 4\right)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{\left(-19920x^5 + 193920x^3 - 195840x\right) y' - 19200\left(x^4 - \frac{33}{5}x^2 + \frac{12}{5}\right) y}{\left(x^2 + 4\right)^5} \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \frac{\left(199920x^6 - 2869440x^4 + 5725440x^2 - 967680\right) y' + 194880x\left(x^4 - \frac{312}{29}x^2 + \frac{336}{29}\right) y}{\left(x^2 + 4\right)^6} \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= \frac{\left(-2204160x^7 + 43653120x^5 - 143646720x^3 + 72253440x\right) y' - 2163840\left(x^6 - \frac{364}{23}x^4 + \frac{5424}{161}x^2 - \frac{960}{161}\right)}{\left(x^2 + 4\right)^7}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0) \\
F_1 &= -\frac{5y'(0)}{2} \\
F_2 &= \frac{9y(0)}{2} \\
F_3 &= \frac{35y'(0)}{2} \\
F_4 &= -45y(0) \\
F_5 &= -\frac{945y'(0)}{4} \\
F_6 &= \frac{1575y(0)}{2}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1-\frac{1}{2}x^2+\frac{3}{16}x^4-\frac{1}{16}x^6+\frac{5}{256}x^8\right)y(0)+\left(x-\frac{5}{12}x^3+\frac{7}{48}x^5-\frac{3}{64}x^7\right)y'(0)+O(x^8)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(x^2+4)\,y''+6xy'+4y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
(x^2+4)\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right)+6x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)+4\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}
$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} 4n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 6n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 4a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 4n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} 4(n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned}&\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} 4(n+2)\, a_{n+2}(n+1)\, x^n\right) \\ &+ \left(\sum_{n=1}^{\infty} 6n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 4a_n x^n\right) = 0\end{aligned} \tag{3}$$

$n = 0$ gives

$$8a_2 + 4a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

$n = 1$ gives

$$24a_3 + 10a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{5a_1}{12}$$

For $2 \leq n$, the recurrence equation is

$$n a_n(n-1) + 4(n+2)\, a_{n+2}(n+1) + 6n a_n + 4a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{(n+4)\, a_n}{4\,(n+2)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$18a_2 + 48a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{3a_0}{16}$$

For $n = 3$ the recurrence equation gives

$$28a_3 + 80a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{7a_1}{48}$$

For $n = 4$ the recurrence equation gives

$$40a_4 + 120a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{16}$$

For $n = 5$ the recurrence equation gives

$$54a_5 + 168a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{3a_1}{64}$$

For $n = 6$ the recurrence equation gives

$$70a_6 + 224a_8 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{5a_0}{256}$$

For $n = 7$ the recurrence equation gives

$$88a_7 + 288a_9 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{11a_1}{768}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{2} a_0 x^2 - \frac{5}{12} a_1 x^3 + \frac{3}{16} a_0 x^4 + \frac{7}{48} a_1 x^5 - \frac{1}{16} a_0 x^6 - \frac{3}{64} a_1 x^7 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{3}{16}x^4 - \frac{1}{16}x^6\right) a_0 + \left(x - \frac{5}{12}x^3 + \frac{7}{48}x^5 - \frac{3}{64}x^7\right) a_1 + O(x^8) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{3}{16}x^4 - \frac{1}{16}x^6\right) c_1 + \left(x - \frac{5}{12}x^3 + \frac{7}{48}x^5 - \frac{3}{64}x^7\right) c_2 + O(x^8)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & \left(1 - \frac{1}{2}x^2 + \frac{3}{16}x^4 - \frac{1}{16}x^6 + \frac{5}{256}x^8\right) y(0) \\ & + \left(x - \frac{5}{12}x^3 + \frac{7}{48}x^5 - \frac{3}{64}x^7\right) y'(0) + O(x^8) \end{aligned} \quad (1)$$

$$y = \left(1 - \frac{1}{2}x^2 + \frac{3}{16}x^4 - \frac{1}{16}x^6\right) c_1 + \left(x - \frac{5}{12}x^3 + \frac{7}{48}x^5 - \frac{3}{64}x^7\right) c_2 + O(x^8) \quad (2)$$

Verification of solutions

$$y = \left(1-\frac{1}{2}x^2+\frac{3}{16}x^4-\frac{1}{16}x^6+\frac{5}{256}x^8\right)y(0)+\left(x-\frac{5}{12}x^3+\frac{7}{48}x^5-\frac{3}{64}x^7\right)y'(0)+O(x^8)$$

Verified OK.

$$y = \left(1-\frac{1}{2}x^2+\frac{3}{16}x^4-\frac{1}{16}x^6\right)c_1+\left(x-\frac{5}{12}x^3+\frac{7}{48}x^5-\frac{3}{64}x^7\right)c_2+O(x^8)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
Order:=8;
dsolve((x^2+4)*diff(y(x),x$2)+6*x*diff(y(x),x)+4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1-\frac{1}{2}x^2+\frac{3}{16}x^4-\frac{1}{16}x^6\right)y(0)+\left(x-\frac{5}{12}x^3+\frac{7}{48}x^5-\frac{3}{64}x^7\right)D(y)(0)+O(x^8)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 56

```
AsymptoticDSolveValue[(x^2+4)*y''[x]+6*x*y'[x]+4*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(-\frac{3x^7}{64}+\frac{7x^5}{48}-\frac{5x^3}{12}+x\right)+c_1\left(-\frac{x^6}{16}+\frac{3x^4}{16}-\frac{x^2}{2}+1\right)$$

3.12 problem 12

Internal problem ID [6900]
Internal file name [`OUTPUT/6143_Tuesday_August_09_2022_05_23_15_AM_97496273/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 17. Power series solutions. 17.5. Solutions Near an Ordinary Point. Exercises page 355
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(2x^2+1)\,y''-5xy'+3y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots\\
&= y_0+xy'_0+\frac{x^2}{2}f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}f'|_{x_0,y_0,y'_0}+\cdots\\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{79}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{80}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= \frac{5xy' - 3y}{2x^2+1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
&= \frac{9x^2y' - 3yx + 2y'}{\left(2x^2+1\right)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1 \\
&= \frac{3x^3y' - 9x^2y + 9xy' - 9y}{\left(2x^2+1\right)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2 \\
&= -\frac{21\left(\left(x^3+3x\right)y' + \left(-3x^2-3\right)y\right)x}{\left(2x^2+1\right)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3 \\
&= \frac{189\left(\left(x^3+3x\right)y' + \left(-3x^2-3\right)y\right)\left(x+\frac{1}{3}\right)\left(x-\frac{1}{3}\right)}{\left(2x^2+1\right)^5} \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y}y' + \frac{\partial F_4}{\partial y'}F_4 \\
&= -\frac{2079\left(\left(x^3+3x\right)y' + \left(-3x^2-3\right)y\right)\left(x^2-\frac{1}{3}\right)x}{\left(2x^2+1\right)^6} \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y}y' + \frac{\partial F_5}{\partial y'}F_5 \\
&= \frac{27027\left(x^4 - \frac{2}{3}x^2 + \frac{1}{39}\right)\left(\left(x^3+3x\right)y' + \left(-3x^2-3\right)y\right)}{\left(2x^2+1\right)^7}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -3y(0)\\
F_1 &= 2y'(0)\\
F_2 &= -9y(0)\\
F_3 &= 0\\
F_4 &= 63y(0)\\
F_5 &= 0\\
F_6 &= -2079y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{3}{2}x^2 - \frac{3}{8}x^4 + \frac{7}{80}x^6 - \frac{33}{640}x^8\right) y(0) + \left(x + \frac{1}{3}x^3\right) y'(0) + O(x^8)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(2x^2 + 1) y'' - 5xy' + 3y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1}\\
y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
(2x^2 + 1) \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) - 5x \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 3 \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} 2x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-5n a_n x^n) + \left(\sum_{n=0}^{\infty} 3a_n x^n\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} 2x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=1}^{\infty} \left(-5n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 3a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 3a_0 = 0$$

$$a_2 = -\frac{3a_0}{2}$$

$n = 1$ gives

$$6a_3 - 2a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$2n a_n(n-1) + (n+2)\, a_{n+2}(n+1) - 5n a_n + 3a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(2n^2 - 7n + 3)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$-3a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{3a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$7a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{7a_0}{80}$$

For $n = 5$ the recurrence equation gives

$$18a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

For $n = 6$ the recurrence equation gives

$$33a_6 + 56a_8 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = -\frac{33a_0}{640}$$

For $n = 7$ the recurrence equation gives

$$52a_7 + 72a_9 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{3}{2} a_0 x^2 + \frac{1}{3} a_1 x^3 - \frac{3}{8} a_0 x^4 + \frac{7}{80} a_0 x^6 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{3}{2}x^2 - \frac{3}{8}x^4 + \frac{7}{80}x^6\right) a_0 + \left(x + \frac{1}{3}x^3\right) a_1 + O(x^8) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{3}{2}x^2 - \frac{3}{8}x^4 + \frac{7}{80}x^6\right) c_1 + \left(x + \frac{1}{3}x^3\right) c_2 + O(x^8)$$

<u>Summary</u>

The solution(s) found are the following

$$y = \left(1 - \frac{3}{2}x^2 - \frac{3}{8}x^4 + \frac{7}{80}x^6 - \frac{33}{640}x^8\right) y(0) + \left(x + \frac{1}{3}x^3\right) y'(0) + O(x^8) \tag{1}$$

$$y = \left(1 - \frac{3}{2}x^2 - \frac{3}{8}x^4 + \frac{7}{80}x^6\right) c_1 + \left(x + \frac{1}{3}x^3\right) c_2 + O(x^8) \tag{2}$$

<u>Verification of solutions</u>

$$y = \left(1 - \frac{3}{2}x^2 - \frac{3}{8}x^4 + \frac{7}{80}x^6 - \frac{33}{640}x^8\right) y(0) + \left(x + \frac{1}{3}x^3\right) y'(0) + O(x^8)$$

Verified OK.

$$y = \left(1 - \frac{3}{2}x^2 - \frac{3}{8}x^4 + \frac{7}{80}x^6\right) c_1 + \left(x + \frac{1}{3}x^3\right) c_2 + O(x^8)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Legendre successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=8;
dsolve((1+2*x^2)*diff(y(x),x$2)-5*x*diff(y(x),x)+3*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{3}{2}x^2 - \frac{3}{8}x^4 + \frac{7}{80}x^6\right)y(0) + \left(x + \frac{1}{3}x^3\right)D(y)(0) + O(x^8)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[(1+2*x^2)*y''[x]-5*x*y'[x]+3*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(\frac{x^3}{3} + x\right) + c_1\left(\frac{7x^6}{80} - \frac{3x^4}{8} - \frac{3x^2}{2} + 1\right)$$

3.13 problem 13

Internal problem ID [6901]
Internal file name [`OUTPUT/6144_Tuesday_August_09_2022_05_23_16_AM_81295945/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 17. Power series solutions. 17.5. Solutions Near an Ordinary Point. Exercises page 355
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' + x^2y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2}\, f\big|_{x_0, y_0, y_0'} + \frac{x^3}{3!}\, f'\big|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (82) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (83) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -x^2 y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -(xy' + 2y)\, x \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= yx^4 - 4xy' - 2y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= y'x^4 + 8yx^3 - 6y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= 12x^3 y' - x^2 y\left(x^4 - 30\right) \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= -\left(\left(x^5 - 66x\right) y' + \left(18x^4 - 60\right) y\right) x \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= \left(-24x^5 + 192x\right) y' + y\left(x^8 - 156x^4 + 60\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 0 \\
F_2 &= -2y(0) \\
F_3 &= -6y'(0) \\
F_4 &= 0 \\
F_5 &= 0 \\
F_6 &= 60y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{12}x^4 + \frac{1}{672}x^8\right) y(0) + \left(x - \frac{1}{20}x^5\right) y'(0) + O(x^8)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} x^{n+2} a_n\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

$$\sum_{n=0}^{\infty} x^{n+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^n\right) = 0 \tag{3}$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + a_{n-2} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_{n-2}}{(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_0}{12}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1}{20}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$42a_7 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

For $n = 6$ the recurrence equation gives

$$56a_8 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{a_0}{672}$$

For $n = 7$ the recurrence equation gives

$$72a_9 + a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{a_1}{1440}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{12} a_0 x^4 - \frac{1}{20} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{x^4}{12}\right) a_0 + \left(x - \frac{1}{20} x^5\right) a_1 + O\left(x^8\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{x^4}{12}\right) c_1 + \left(x - \frac{1}{20} x^5\right) c_2 + O\left(x^8\right)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{12} x^4 + \frac{1}{672} x^8\right) y(0) + \left(x - \frac{1}{20} x^5\right) y'(0) + O\left(x^8\right) \tag{1}$$

$$y = \left(1 - \frac{x^4}{12}\right) c_1 + \left(x - \frac{1}{20} x^5\right) c_2 + O\left(x^8\right) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{1}{12} x^4 + \frac{1}{672} x^8\right) y(0) + \left(x - \frac{1}{20} x^5\right) y'(0) + O\left(x^8\right)$$

Verified OK.

$$y = \left(1 - \frac{x^4}{12}\right) c_1 + \left(x - \frac{1}{20} x^5\right) c_2 + O\left(x^8\right)$$

Verified OK.

3.13.1 Maple step by step solution

Let's solve

$y'' = -x^2 y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + x^2 y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^k$$

□ Rewrite ODE with series expansions

- ○ Convert $x^2 \cdot y$ to series expansion

$$x^2 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+2}$$

- ○ Shift index using $k- >k-2$

$$x^2 \cdot y = \sum_{k=2}^{\infty} a_{k-2} x^k$$

- ○ Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- ○ Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$6a_3 x + 2a_2 + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 = 0, 6a_3 = 0]$$

- Solve for the dependent coefficient(s)

$$\{a_2 = 0, a_3 = 0\}$$

- Each term in the series must be 0, giving the recursion relation

$$\left(k^2 + 3k + 2\right) a_{k+2} + a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} + a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{a_k}{k^2+7k+12}, a_2 = 0, a_3 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=8;
dsolve(diff(y(x),x$2)+x^2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{x^4}{12}\right) y(0) + \left(x - \frac{1}{20}x^5\right) D(y)(0) + O(x^8)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 28

```
AsymptoticDSolveValue[y''[x]+x^2*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(x - \frac{x^5}{20}\right) + c_1\left(1 - \frac{x^4}{12}\right)$$

3.14 problem 14

Internal problem ID [6902]
Internal file name [`OUTPUT/6145_Tuesday_August_09_2022_05_23_18_AM_40367264/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 17. Power series solutions. 17.5. Solutions Near an Ordinary Point. Exercises page 355
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{\left(-4x^2+1\right) y''+6xy'-4y=0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (85)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (86)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n\big|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= \frac{6xy' - 4y}{4x^2-1}\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
&= \frac{-4x^2y' + 8yx - 2y'}{\left(4x^2-1\right)^2}\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1\\
&= \frac{20(2x^2y' - 4yx + y')\,x}{\left(4x^2-1\right)^3}\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2\\
&= -\frac{560\left(x^2+\frac{1}{14}\right)\left(\left(x^2+\frac{1}{2}\right)y' - 2yx\right)}{\left(4x^2-1\right)^4}\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3\\
&= \frac{10080\left(x^2+\frac{3}{14}\right)x\left(\left(x^2+\frac{1}{2}\right)y' - 2yx\right)}{\left(4x^2-1\right)^5}\\
F_5 &= \frac{dF_4}{dx}\\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4\\
&= -\frac{221760\left(x^4+\frac{3}{7}x^2+\frac{3}{308}\right)\left(\left(x^2+\frac{1}{2}\right)y' - 2yx\right)}{\left(4x^2-1\right)^6}\\
F_6 &= \frac{dF_5}{dx}\\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5\\
&= \frac{5765760x\left(x^4+\frac{5}{7}x^2+\frac{15}{308}\right)\left(\left(x^2+\frac{1}{2}\right)y' - 2yx\right)}{\left(4x^2-1\right)^7}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 4y(0) \\
F_1 &= -2y'(0) \\
F_2 &= 0 \\
F_3 &= -20y'(0) \\
F_4 &= 0 \\
F_5 &= -1080y'(0) \\
F_6 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(2x^2 + 1\right) y(0) + \left(x - \frac{1}{3}x^3 - \frac{1}{6}x^5 - \frac{3}{14}x^7\right) y'(0) + O\left(x^8\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
\left(-4x^2 + 1\right) y'' + 6xy' - 4y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(-4x^2 + 1\right) \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + 6x \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 4\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \quad (1)
$$

Which simplifies to

$$
\sum_{n=2}^{\infty} \left(-4x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 6n a_n x^n\right) + \sum_{n=0}^{\infty} \left(-4a_n x^n\right) = 0 \quad (2)
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\sum_{n=2}^{\infty} (-4x^n a_n n(n-1)) + \left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \left(\sum_{n=1}^{\infty} 6n a_n x^n\right) + \sum_{n=0}^{\infty} (-4a_n x^n) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - 4a_0 = 0$$

$$a_2 = 2a_0$$

$n = 1$ gives

$$6a_3 + 2a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$-4n a_n(n-1) + (n+2) a_{n+2}(n+1) + 6n a_n - 4a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{2a_n(2n^2 - 5n + 2)}{(n+2)(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 0$$

For $n = 3$ the recurrence equation gives

$$-10a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1}{6}$$

For $n = 4$ the recurrence equation gives

$$-28a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$-54a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{3a_1}{14}$$

For $n = 6$ the recurrence equation gives

$$-88a_6 + 56a_8 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = 0$$

For $n = 7$ the recurrence equation gives

$$-130a_7 + 72a_9 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = -\frac{65a_1}{168}$$

And so on. Therefore the solution is

$$y = \sum_{n=0}^{\infty} a_n x^n$$
$$= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x + 2a_0x^2 - \frac{1}{3}a_1x^3 - \frac{1}{6}a_1x^5 - \frac{3}{14}a_1x^7 + \dots$$

Collecting terms, the solution becomes

$$y = (2x^2 + 1)\, a_0 + \left(x - \frac{1}{3}x^3 - \frac{1}{6}x^5 - \frac{3}{14}x^7\right) a_1 + O(x^8) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = (2x^2 + 1)\, c_1 + \left(x - \frac{1}{3}x^3 - \frac{1}{6}x^5 - \frac{3}{14}x^7\right) c_2 + O(x^8)$$

Summary

The solution(s) found are the following

$$y = (2x^2 + 1)\, y(0) + \left(x - \frac{1}{3}x^3 - \frac{1}{6}x^5 - \frac{3}{14}x^7\right) y'(0) + O(x^8) \tag{1}$$

$$y = (2x^2 + 1)\, c_1 + \left(x - \frac{1}{3}x^3 - \frac{1}{6}x^5 - \frac{3}{14}x^7\right) c_2 + O(x^8) \tag{2}$$

Verification of solutions

$$y = (2x^2 + 1)\, y(0) + \left(x - \frac{1}{3}x^3 - \frac{1}{6}x^5 - \frac{3}{14}x^7\right) y'(0) + O(x^8)$$

Verified OK.

$$y = (2x^2 + 1)\, c_1 + \left(x - \frac{1}{3}x^3 - \frac{1}{6}x^5 - \frac{3}{14}x^7\right) c_2 + O(x^8)$$

Verified OK.

3.14.1 Maple step by step solution

Let's solve

$(-4x^2+1)\,y''+6xy'-4y=0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y''=\frac{6xy'}{4x^2-1}-\frac{4y}{4x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''-\frac{6xy'}{4x^2-1}+\frac{4y}{4x^2-1}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=-\frac{6x}{4x^2-1}, P_3(x)=\frac{4}{4x^2-1}\right]$

- ◦ $\left(x+\frac{1}{2}\right)\cdot P_2(x)$ is analytic at $x=-\frac{1}{2}$

$\left(\left(x+\frac{1}{2}\right)\cdot P_2(x)\right)\Bigg|_{x=-\frac{1}{2}}=-\frac{3}{4}$

- ◦ $\left(x+\frac{1}{2}\right)^2\cdot P_3(x)$ is analytic at $x=-\frac{1}{2}$

$\left(\left(x+\frac{1}{2}\right)^2\cdot P_3(x)\right)\Bigg|_{x=-\frac{1}{2}}=0$

- ◦ $x=-\frac{1}{2}$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-\frac{1}{2}$

- Multiply by denominators

$y''(4x^2-1)-6xy'+4y=0$

- Change variables using $x=u-\frac{1}{2}$ so that the regular singular point is at $u=0$

$(4u^2-4u)\left(\frac{d^2}{du^2}y(u)\right)+(-6u+3)\left(\frac{d}{du}y(u)\right)+4y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0 r(-7+4r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(4k-3+4r) + 2a_k(2k+2r-1)\,(k+r-2)\right) u$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-r(-7+4r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{7}{4}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$-4(k+1+r)\left(k-\frac{3}{4}+r\right) a_{k+1} + 4(k+r-2)\, a_k\left(k+r-\frac{1}{2}\right) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{2(k+r-2)a_k(2k+2r-1)}{(k+1+r)(4k-3+4r)}$$

- Recursion relation for $r = 0$; series terminates at $k = 2$

  $$a_{k+1} = \frac{2(k-2)a_k(2k-1)}{(k+1)(4k-3)}$$

- Apply recursion relation for $k = 0$

  $$a_1 = -\frac{4a_0}{3}$$

- Apply recursion relation for $k = 1$

  $$a_2 = -a_1$$

- Express in terms of a_0

  $$a_2 = \frac{4a_0}{3}$$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - \frac{4}{3}u + \frac{4}{3}u^2\right)$
- Revert the change of variables $u = x + \frac{1}{2}$

  $\left[y = a_0 \left(\frac{2}{3} + \frac{4x^2}{3}\right)\right]$
- Recursion relation for $r = \frac{7}{4}$

  $a_{k+1} = \frac{2\left(k-\frac{1}{4}\right)a_k\left(2k+\frac{5}{2}\right)}{\left(k+\frac{11}{4}\right)(4k+4)}$
- Solution for $r = \frac{7}{4}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{7}{4}}, a_{k+1} = \frac{2\left(k-\frac{1}{4}\right)a_k\left(2k+\frac{5}{2}\right)}{\left(k+\frac{11}{4}\right)(4k+4)}\right]$
- Revert the change of variables $u = x + \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k \left(x + \frac{1}{2}\right)^{k+\frac{7}{4}}, a_{k+1} = \frac{2\left(k-\frac{1}{4}\right)a_k\left(2k+\frac{5}{2}\right)}{\left(k+\frac{11}{4}\right)(4k+4)}\right]$
- Combine solutions and rename parameters

  $\left[y = a_0 \left(\frac{2}{3} + \frac{4x^2}{3}\right) + \left(\sum_{k=0}^{\infty} b_k \left(x + \frac{1}{2}\right)^{k+\frac{7}{4}}\right), b_{k+1} = \frac{2\left(k-\frac{1}{4}\right)b_k\left(2k+\frac{5}{2}\right)}{\left(k+\frac{11}{4}\right)(4k+4)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Legendre successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=8;
dsolve((1-4*x^2)*diff(y(x),x$2)+6*x*diff(y(x),x)-4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (2x^2+1)\,y(0) + \left(x - \frac{1}{3}x^3 - \frac{1}{6}x^5 - \frac{3}{14}x^7\right) D(y)\,(0) + O(x^8)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 40

```
AsymptoticDSolveValue[(1-4*x^2)*y''[x]+6*x*y'[x]-4*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1(2x^2+1) + c_2\left(-\frac{3x^7}{14} - \frac{x^5}{6} - \frac{x^3}{3} + x\right)$$

3.15 problem 15

Internal problem ID [6903]
Internal file name [`OUTPUT/6146_Tuesday_August_09_2022_05_23_20_AM_47998857/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 17. Power series solutions. 17.5. Solutions Near an Ordinary Point. Exercises page 355
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_2", "second order series method. Taylor series method", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(2x^2+1)\,y''+3xy'-3y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (88) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (89) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\ \vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y'_0} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{3(xy'-y)}{2x^2+1}\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
&= \frac{21(xy'-y)\,x}{\left(2x^2+1\right)^2}\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1\\
&= -\frac{21(xy'-y)\left(9x^2-1\right)}{\left(2x^2+1\right)^3}\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2\\
&= \frac{2079(xy'-y)\left(x^2-\frac{1}{3}\right)x}{\left(2x^2+1\right)^4}\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3\\
&= -\frac{693(xy'-y)\left(39x^4-26x^2+1\right)}{\left(2x^2+1\right)^5}\\
F_5 &= \frac{dF_4}{dx}\\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y}y' + \frac{\partial F_4}{\partial y'}F_4\\
&= \frac{405405(xy'-y)\left(x^4-\frac{10}{9}x^2+\frac{5}{39}\right)x}{\left(2x^2+1\right)^6}\\
F_6 &= \frac{dF_5}{dx}\\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y}y' + \frac{\partial F_5}{\partial y'}F_5\\
&= -\frac{10395(xy'-y)\left(663x^6-1105x^4+255x^2-5\right)}{\left(2x^2+1\right)^7}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 3y(0) \\
F_1 &= 0 \\
F_2 &= -21y(0) \\
F_3 &= 0 \\
F_4 &= 693y(0) \\
F_5 &= 0 \\
F_6 &= -51975y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + \frac{3}{2}x^2 - \frac{7}{8}x^4 + \frac{77}{80}x^6 - \frac{165}{128}x^8\right) y(0) + xy'(0) + O(x^8)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(2x^2 + 1) y'' + 3xy' - 3y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
(2x^2 + 1) \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + 3x \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 3\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} 2x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 3n a_n x^n\right) + \sum_{n=0}^{\infty} (-3a_n x^n) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} 2x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} 3n a_n x^n\right) + \sum_{n=0}^{\infty} (-3a_n x^n) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - 3a_0 = 0$$

$$a_2 = \frac{3a_0}{2}$$

For $2 \leq n$, the recurrence equation is

$$2na_n(n-1) + (n+2)\, a_{n+2}(n+1) + 3na_n - 3a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(2n^2+n-3)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$7a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{7a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$18a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$33a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{77a_0}{80}$$

For $n = 5$ the recurrence equation gives

$$52a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

For $n = 6$ the recurrence equation gives

$$75a_6 + 56a_8 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = -\frac{165a_0}{128}$$

For $n = 7$ the recurrence equation gives

$$102a_7 + 72a_9 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = 0$$

And so on. Therefore the solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
y = a_0 + a_1 x + \frac{3}{2} a_0 x^2 - \frac{7}{8} a_0 x^4 + \frac{77}{80} a_0 x^6 + \dots
$$

Collecting terms, the solution becomes

$$
y = \left(1 + \frac{3}{2} x^2 - \frac{7}{8} x^4 + \frac{77}{80} x^6\right) a_0 + a_1 x + O\left(x^8\right) \tag{3}
$$

At $x = 0$ the solution above becomes

$$
y = \left(1 + \frac{3}{2} x^2 - \frac{7}{8} x^4 + \frac{77}{80} x^6\right) c_1 + c_2 x + O\left(x^8\right)
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \left(1 + \frac{3}{2} x^2 - \frac{7}{8} x^4 + \frac{77}{80} x^6 - \frac{165}{128} x^8\right) y(0) + x y'(0) + O\left(x^8\right) \tag{1}
$$

$$
y = \left(1 + \frac{3}{2} x^2 - \frac{7}{8} x^4 + \frac{77}{80} x^6\right) c_1 + c_2 x + O\left(x^8\right) \tag{2}
$$

<u>Verification of solutions</u>

$$
y = \left(1 + \frac{3}{2} x^2 - \frac{7}{8} x^4 + \frac{77}{80} x^6 - \frac{165}{128} x^8\right) y(0) + x y'(0) + O\left(x^8\right)
$$

Verified OK.

$$
y = \left(1 + \frac{3}{2} x^2 - \frac{7}{8} x^4 + \frac{77}{80} x^6\right) c_1 + c_2 x + O\left(x^8\right)
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Legendre successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 29

```
Order:=8;
dsolve((1+2*x^2)*diff(y(x),x$2)+3*x*diff(y(x),x)-3*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + \frac{3}{2}x^2 - \frac{7}{8}x^4 + \frac{77}{80}x^6\right) y(0) + D(y)\,(0)\,x + O\left(x^8\right)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 34

```
AsymptoticDSolveValue[(1+2*x^2)*y''[x]+3*x*y'[x]-3*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(\frac{77x^6}{80} - \frac{7x^4}{8} + \frac{3x^2}{2} + 1\right) + c_2x$$

3.16 problem 16

Internal problem ID [6904]
Internal file name [`OUTPUT/6147_Tuesday_August_09_2022_05_23_21_AM_91064150/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 17. Power series solutions. 17.5. Solutions Near an Ordinary Point. Exercises page 355
Problem number: 16.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_3rd_order, _exact, _linear, _homogeneous]]
```

Unable to solve or complete the solution.

$$\boxed{y''' + x^2 y'' + 5xy' + 3y = 0}$$

Unable to solve this ODE.

3.16.1 Maple step by step solution

Let's solve

$y''' + x^2 y'' + 5xy' + 3y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k k \, x^k$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k k(k-1) \, x^k$$

- Convert y''' to series expansion

$$y''' = \sum_{k=3}^{\infty} a_k k(k-1)(k-2) \, x^{k-3}$$

- Shift index using $k- > k+3$

$$y''' = \sum_{k=0}^{\infty} a_{k+3}(k+3)(k+2)(k+1) \, x^k$$

Rewrite DE with series expansions

$$\sum_{k=0}^{\infty} \left(a_{k+3}(k+3)(k+2)(k+1) + a_k(k+3)(k+1)\right) x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k+3)(k+1)(ka_{k+3} + a_k + 2a_{k+3}) = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = -\frac{a_k}{k+2}\right]$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying high order exact linear fully integrable
trying to convert to a linear ODE with constant coefficients
trying differential order: 3; missing the dependent variable
trying Louvillian solutions for 3rd order ODEs, imprimitive case
Louvillian solutions for 3rd order ODEs, imprimitive case: input is reducible, switching to
checking if the LODE is of Euler type
expon. solutions partially successful. Result(s) =`, [exp(-(1/3)*x^3)*x]
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 98

```
dsolve(diff(y(x),x$3)+x^2*diff(y(x),x$2)+5*x*diff(y(x),x)+3*y(x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\left(-c_1 e^{-\frac{x^3}{3}} x + c_3 3^{\frac{1}{3}}\right) \left(-x^3\right)^{\frac{2}{3}} + x^2 e^{-\frac{x^3}{3}} \left(3 c_2 \left(-x^3\right)^{\frac{1}{3}} \Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right) \Gamma\left(\frac{2}{3}\right) - 2 c_2 \left(-x^3\right)^{\frac{1}{3}} \sqrt{3}\, \pi + x \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right) c}{\left(-x^3\right)^{\frac{2}{3}}}$$

✓ Solution by Mathematica

Time used: 0.093 (sec). Leaf size: 88

```
DSolve[y'''[x]+x^2*y''[x]+5*x*y'[x]+3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{e^{-\frac{x^3}{3}} \left(-2\ 3^{2/3} c_3 \sqrt[3]{-x^3} x \Gamma\left(-\frac{1}{3}, -\frac{x^3}{3}\right) + 3 \sqrt[3]{3} c_1 \left(-x^3\right)^{2/3} \Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right) + 18 c_2 x^2\right)}{18 x}$$

3.17 problem 17

Internal problem ID [6905]
Internal file name [`OUTPUT/6148_Tuesday_August_09_2022_05_23_22_AM_43400438/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 17. Power series solutions. 17.5. Solutions Near an Ordinary Point. Exercises page 355
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + xy' + 3y = x^2}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{91}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{92}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -xy' - 3y + x^2 \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= x^2 y' - x^3 + 3yx - 4y' + 2x \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(-x^3 + 9x\right) y' + x^4 - 3x^2 y - 7x^2 + 15y + 2 \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(x^4 - 15x^2 + 24\right) y' + \left(3x^3 - 33x\right) y - x^5 + 13x^3 - 14x \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-x^5 + 22x^3 - 87x\right) y' + \left(-3x^4 + 54x^2 - 105\right) y + x^6 - 20x^4 + 63x^2 - 14 \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \left(x^6 - 30x^4 + 207x^2 - 192\right) y' - x\left(\left(-3x^4 + 78x^2 - 369\right) y + x^6 - 28x^4 + 167x^2 - 126\right) \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= \left(-x^7 + 39x^5 - 405x^3 + 975x\right) y' + \left(-3x^6 + 105x^4 - 855x^2 + 945\right) y + x^8 - 37x^6 + 347x^4 - 693x^2 -
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -3y(0) \\
F_1 &= -4y'(0) \\
F_2 &= 2 + 15y(0) \\
F_3 &= 24y'(0) \\
F_4 &= -14 - 105y(0) \\
F_5 &= -192y'(0) \\
F_6 &= 126 + 945y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1 - \frac{3}{2}x^2 + \frac{5}{8}x^4 - \frac{7}{48}x^6 + \frac{3}{128}x^8\right) y(0) \\
&+ \left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5 - \frac{4}{105}x^7\right) y'(0) + \frac{x^4}{12} - \frac{7x^6}{360} + \frac{x^8}{320} + O(x^8)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 3\left(\sum_{n=0}^{\infty} a_n x^n\right) + x^2 \tag{1}
$$

Expanding x^2 as Taylor series around $x = 0$ and keeping only the first 8 terms gives

$$
\begin{aligned}
x^2 &= x^2 + \dots \\
&= x^2
\end{aligned}
$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 3\left(\sum_{n=0}^{\infty} a_n x^n\right) = x^2$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 3 a_n x^n\right) = x^2 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 3 a_n x^n\right) = x^2 \tag{3}$$

$n = 0$ gives

$$2a_2 + 3a_0 = 0$$

$$a_2 = -\frac{3a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$((n+2)\, a_{n+2}(n+1) + n a_n + 3a_n)\, x^n = x^2 \tag{4}$$

For $n = 1$ the recurrence equation gives

$$(6a_3 + 4a_1)\, x = 0$$

$$6a_3 + 4a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{2a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$\begin{aligned}(12a_4 + 5a_2)\,x^2 &= x^2\\ 12a_4 + 5a_2 &= 1\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{1}{12} + \frac{5a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$\begin{aligned}(20a_5 + 6a_3)\,x^3 &= 0\\ 20a_5 + 6a_3 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{5}$$

For $n = 4$ the recurrence equation gives

$$\begin{aligned}(30a_6 + 7a_4)\,x^4 &= 0\\ 30a_6 + 7a_4 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{7}{360} - \frac{7a_0}{48}$$

For $n = 5$ the recurrence equation gives

$$\begin{aligned}(42a_7 + 8a_5)\,x^5 &= 0\\ 42a_7 + 8a_5 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{4a_1}{105}$$

For $n = 6$ the recurrence equation gives

$$(56a_8 + 9a_6)\,x^6 = 0$$
$$56a_8 + 9a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{1}{320} + \frac{3a_0}{128}$$

For $n = 7$ the recurrence equation gives

$$(72a_9 + 10a_7)\,x^7 = 0$$
$$72a_9 + 10a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{a_1}{189}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x - \frac{3a_0x^2}{2} - \frac{2a_1x^3}{3} + \left(\frac{1}{12} + \frac{5a_0}{8}\right)x^4 + \frac{a_1x^5}{5} + \left(-\frac{7}{360} - \frac{7a_0}{48}\right)x^6 - \frac{4a_1x^7}{105} + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{3}{2}x^2 + \frac{5}{8}x^4 - \frac{7}{48}x^6\right)a_0 + \left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5 - \frac{4}{105}x^7\right)a_1 + \frac{x^4}{12} - \frac{7x^6}{360} + O(x^8) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{3}{2}x^2 + \frac{5}{8}x^4 - \frac{7}{48}x^6\right)c_1 + \left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5 - \frac{4}{105}x^7\right)c_2 + \frac{x^4}{12} - \frac{7x^6}{360} + O(x^8)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{3}{2}x^2 + \frac{5}{8}x^4 - \frac{7}{48}x^6 + \frac{3}{128}x^8\right)y(0) + \left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5 - \frac{4}{105}x^7\right)y'(0) + \frac{x^4}{12} - \frac{7x^6}{360} + \frac{x^8}{320} + O(x^8) \tag{1}$$

$$y = \left(1 - \frac{3}{2}x^2 + \frac{5}{8}x^4 - \frac{7}{48}x^6\right)c_1 + \left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5 - \frac{4}{105}x^7\right)c_2 + \frac{x^4}{12} - \frac{7x^6}{360} + O(x^8) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{3}{2}x^2 + \frac{5}{8}x^4 - \frac{7}{48}x^6 + \frac{3}{128}x^8\right)y(0) + \left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5 - \frac{4}{105}x^7\right)y'(0) + \frac{x^4}{12} - \frac{7x^6}{360} + \frac{x^8}{320} + O(x^8)$$

Verified OK.

$$y = \left(1 - \frac{3}{2}x^2 + \frac{5}{8}x^4 - \frac{7}{48}x^6\right)c_1 + \left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5 - \frac{4}{105}x^7\right)c_2 + \frac{x^4}{12} - \frac{7x^6}{360} + O(x^8)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
      Solution has integrals. Trying a special function solution free of integrals...
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         <- Kummer successful
      <- special function solution successful
         -> Trying to convert hypergeometric functions to elementary form...
         <- elementary form could result into a too large expression - returning special fun
      <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 48

```
Order:=8;
dsolve(diff(y(x),x$2)+x*diff(y(x),x)+3*y(x)=x^2,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{3}{2}x^2 + \frac{5}{8}x^4 - \frac{7}{48}x^6\right) y(0) + \left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5 - \frac{4}{105}x^7\right) D(y)(0) + \frac{x^4}{12} - \frac{7x^6}{360} + O(x^8)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 70

```
AsymptoticDSolveValue[y''[x]+x*y'[x]+3*y[x]==x^2,y[x],{x,0,7}]
```

$$y(x) \to -\frac{7x^6}{360} + \frac{x^4}{12} + c_2\left(-\frac{4x^7}{105} + \frac{x^5}{5} - \frac{2x^3}{3} + x\right) + c_1\left(-\frac{7x^6}{48} + \frac{5x^4}{8} - \frac{3x^2}{2} + 1\right)$$

3.18 problem 18

Internal problem ID [6906]
Internal file name [`OUTPUT/6149_Tuesday_August_09_2022_05_23_24_AM_31298264/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 17. Power series solutions. 17.5. Solutions Near an Ordinary Point. Exercises page 355
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{y'' + 2xy' + 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (94) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (95) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -2xy' - 2y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= 4x^2 y' + 4yx - 4y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -8x^3 y' - 8x^2 y + 20xy' + 12y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(16x^4 - 72x^2 + 32\right) y' + \left(16x^3 - 56x\right) y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-32x^5 + 224x^3 - 264x\right) y' - 32\left(x^4 - 6x^2 + \frac{15}{4}\right) y \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \left(64x^6 - 640x^4 + 1392x^2 - 384\right) y' + 64\left(x^4 - 9x^2 + \frac{57}{4}\right) xy \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= \left(-128x^7 + 1728x^5 - 5920x^3 + 4464x\right) y' - 128\left(x^4 - 9x^2 + \frac{15}{4}\right)\left(x^2 - \frac{7}{2}\right) y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -2y(0) \\
F_1 &= -4y'(0) \\
F_2 &= 12y(0) \\
F_3 &= 32y'(0) \\
F_4 &= -120y(0) \\
F_5 &= -384y'(0) \\
F_6 &= 1680y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8\right) y(0) + \left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{105}x^7\right) y'(0) + O\left(x^8\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = -2x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 2n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} 2n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 2a_0 = 0$$

$$a_2 = -a_0$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + 2na_n + 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{2a_n}{n+2} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 4a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{2a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 6a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{2}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 8a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{4a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 10a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{6}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 12a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{8a_1}{105}$$

For $n = 6$ the recurrence equation gives

$$56a_8 + 14a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{a_0}{24}$$

For $n = 7$ the recurrence equation gives

$$72a_9 + 16a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{16a_1}{945}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - a_0 x^2 - \frac{2}{3} a_1 x^3 + \frac{1}{2} a_0 x^4 + \frac{4}{15} a_1 x^5 - \frac{1}{6} a_0 x^6 - \frac{8}{105} a_1 x^7 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6\right) a_0 + \left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{105}x^7\right) a_1 + O(x^8) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6\right) c_1 + \left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{105}x^7\right) c_2 + O(x^8)$$

Summary

The solution(s) found are the following

$$y = \left(1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8\right) y(0) + \left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{105}x^7\right) y'(0) + O(x^8) \tag{1}$$

$$y = \left(1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6\right) c_1 + \left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{105}x^7\right) c_2 + O(x^8) \tag{2}$$

Verification of solutions

$$y = \left(1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8\right) y(0) + \left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{105}x^7\right) y'(0) + O(x^8)$$

Verified OK.

$$y = \left(1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6\right) c_1 + \left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{105}x^7\right) c_2 + O(x^8)$$

Verified OK.

3.18.1 Maple step by step solution

Let's solve

$y'' = -2xy' - 2y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 2xy' + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} (a_{k+2}(k+2)\,(k+1) + 2a_k(k+1))\, x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+1)\,(a_{k+2}(k+2) + 2a_k) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{2a_k}{k+2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
Order:=8;
dsolve(diff(y(x),x$2)+2*x*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6\right) y(0) + \left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{105}x^7\right) D(y)(0) + O(x^8)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 54

```
AsymptoticDSolveValue[y''[x]+2*x*y'[x]+2*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(-\frac{8x^7}{105} + \frac{4x^5}{15} - \frac{2x^3}{3} + x\right) + c_1\left(-\frac{x^6}{6} + \frac{x^4}{2} - x^2 + 1\right)$$

3.19 problem 19

Internal problem ID [6907]
Internal file name [`OUTPUT/6150_Tuesday_August_09_2022_05_23_25_AM_88058161/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 17. Power series solutions. 17.5. Solutions Near an Ordinary Point. Exercises page 355
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 3xy' + 7y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (97)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (98)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -3xy' - 7y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= 9x^2 y' + 21yx - 10y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -27x^3 y' - 63x^2 y + 69xy' + 91y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(81x^4 - 351x^2 + 160\right) y' + \left(189x^3 - 609x\right) y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-243x^5 + 1566x^3 - 1791x\right) y' + \left(-567x^4 + 3024x^2 - 1729\right) y \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \left(729x^6 - 6480x^4 + 13095x^2 - 3520\right) y' + 1701x\left(x^4 - \frac{70}{9}x^2 + \frac{295}{27}\right) y \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= \left(-2187x^7 + 25515x^5 - 78435x^3 + 55335x\right) y' - 5103\left(x^6 - \frac{95}{9}x^4 + \frac{695}{27}x^2 - \frac{6175}{729}\right) y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -7y(0) \\
F_1 &= -10y'(0) \\
F_2 &= 91y(0) \\
F_3 &= 160y'(0) \\
F_4 &= -1729y(0) \\
F_5 &= -3520y'(0) \\
F_6 &= 43225y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1-\frac{7}{2}x^2+\frac{91}{24}x^4-\frac{1729}{720}x^6+\frac{1235}{1152}x^8\right)y(0)+\left(x-\frac{5}{3}x^3+\frac{4}{3}x^5-\frac{44}{63}x^7\right)y'(0)+O\left(x^8\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -3x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 7\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 3n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 7a_n x^n\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} 3n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 7a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 7a_0 = 0$$

$$a_2 = -\frac{7a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + 3na_n + 7a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(3n+7)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 10a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{5a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 13a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{91a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 16a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{4a_1}{3}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 19a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{1729a_0}{720}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 22a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{44a_1}{63}$$

For $n = 6$ the recurrence equation gives

$$56a_8 + 25a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{1235a_0}{1152}$$

For $n = 7$ the recurrence equation gives

$$72a_9 + 28a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{22a_1}{81}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{7}{2} a_0 x^2 - \frac{5}{3} a_1 x^3 + \frac{91}{24} a_0 x^4 + \frac{4}{3} a_1 x^5 - \frac{1729}{720} a_0 x^6 - \frac{44}{63} a_1 x^7 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{7}{2}x^2 + \frac{91}{24}x^4 - \frac{1729}{720}x^6\right) a_0 + \left(x - \frac{5}{3}x^3 + \frac{4}{3}x^5 - \frac{44}{63}x^7\right) a_1 + O(x^8) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{7}{2}x^2 + \frac{91}{24}x^4 - \frac{1729}{720}x^6\right) c_1 + \left(x - \frac{5}{3}x^3 + \frac{4}{3}x^5 - \frac{44}{63}x^7\right) c_2 + O(x^8)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = &\left(1 - \frac{7}{2}x^2 + \frac{91}{24}x^4 - \frac{1729}{720}x^6 + \frac{1235}{1152}x^8\right) y(0) \\ &+ \left(x - \frac{5}{3}x^3 + \frac{4}{3}x^5 - \frac{44}{63}x^7\right) y'(0) + O(x^8) \end{aligned} \quad (1)$$

$$y = \left(1 - \frac{7}{2}x^2 + \frac{91}{24}x^4 - \frac{1729}{720}x^6\right) c_1 + \left(x - \frac{5}{3}x^3 + \frac{4}{3}x^5 - \frac{44}{63}x^7\right) c_2 + O(x^8) \quad (2)$$

Verification of solutions

$$y = \left(1-\frac{7}{2}x^2+\frac{91}{24}x^4-\frac{1729}{720}x^6+\frac{1235}{1152}x^8\right)y(0)+\left(x-\frac{5}{3}x^3+\frac{4}{3}x^5-\frac{44}{63}x^7\right)y'(0)+O(x^8)$$

Verified OK.

$$y = \left(1-\frac{7}{2}x^2+\frac{91}{24}x^4-\frac{1729}{720}x^6\right)c_1+\left(x-\frac{5}{3}x^3+\frac{4}{3}x^5-\frac{44}{63}x^7\right)c_2+O(x^8)$$

Verified OK.

3.19.1 Maple step by step solution

Let's solve

$y'' = -3xy' - 7y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 3xy' + 7y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)(k+1) + a_k(3k+7)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

$$\left(k^2+3k+2\right) a_{k+2}+3a_k k+7a_k=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=-\frac{a_k(3k+7)}{k^2+3k+2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
Order:=8;
dsolve(diff(y(x),x$2)+3*x*diff(y(x),x)+7*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\left(1-\frac{7}{2}x^2+\frac{91}{24}x^4-\frac{1729}{720}x^6\right)y(0)+\left(x-\frac{5}{3}x^3+\frac{4}{3}x^5-\frac{44}{63}x^7\right)D(y)(0)+O\left(x^8\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 56

```
AsymptoticDSolveValue[y''[x]+3*x*y'[x]+7*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(-\frac{44x^7}{63} + \frac{4x^5}{3} - \frac{5x^3}{3} + x\right) + c_1\left(-\frac{1729x^6}{720} + \frac{91x^4}{24} - \frac{7x^2}{2} + 1\right)$$

3.20 problem 20

Internal problem ID [6908]
Internal file name [`OUTPUT/6151_Tuesday_August_09_2022_05_23_27_AM_54658325/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 17. Power series solutions. 17.5. Solutions Near an Ordinary Point. Exercises page 355
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2y'' + 9xy' - 36y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{100}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{101}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{9xy'}{2} + 18y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{81x^2y'}{4} - 81xy + \frac{27y'}{2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{81(-9x^3 - 10x)\, y'}{8} + \frac{81y(9x^2 + 4)}{2} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{243(27x^4 + 36x^2 + 4)\, y'}{16} - \frac{6561yx\left(x^2 + \frac{2}{3}\right)}{4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{2187(-27x^5 - 36x^3 - 4x)\, y'}{32} + \frac{59049yx^2\left(x^2 + \frac{2}{3}\right)}{8} \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \frac{531441\left(x^2 - \frac{2}{9}\right)\left(\left(x^4 + \frac{4}{3}x^2 + \frac{4}{27}\right) y' - 4yx\left(x^2 + \frac{2}{3}\right)\right)}{64} \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= \frac{59049(-81x^7 - 54x^5 + 60x^3 + 8x)\, y'}{128} + \frac{531441(9x^6 - 4x^2)\, y}{32}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 18y(0) \\
F_1 &= \frac{27y'(0)}{2} \\
F_2 &= 162y(0) \\
F_3 &= \frac{243y'(0)}{4} \\
F_4 &= 0 \\
F_5 &= -\frac{2187y'(0)}{8} \\
F_6 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + 9x^2 + \frac{27}{4}x^4\right) y(0) + \left(x + \frac{9}{4}x^3 + \frac{81}{160}x^5 - \frac{243}{4480}x^7\right) y'(0) + O(x^8)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = -\frac{9x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)}{2} + 18\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} 2n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 9n\, x^n a_n\right) + \sum_{n=0}^{\infty} (-36 a_n x^n) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 2n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} 2(n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} 2(n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} 9n\, x^n a_n\right) + \sum_{n=0}^{\infty} \left(-36 a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$4a_2 - 36a_0 = 0$$

$$a_2 = 9a_0$$

For $1 \leq n$, the recurrence equation is

$$2(n+2)\, a_{n+2}(n+1) + 9n a_n - 36a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{9a_n(n-4)}{2\,(n+2)\,(n+1)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$12a_3 - 27a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{9a_1}{4}$$

For $n = 2$ the recurrence equation gives

$$24a_4 - 18a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{27a_0}{4}$$

For $n = 3$ the recurrence equation gives

$$40a_5 - 9a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{81a_1}{160}$$

For $n = 4$ the recurrence equation gives

$$60a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$84a_7 + 9a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{243a_1}{4480}$$

For $n = 6$ the recurrence equation gives

$$112a_8 + 18a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = 0$$

For $n = 7$ the recurrence equation gives

$$144a_9 + 27a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{729a_1}{71680}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x + 9a_0x^2 + \frac{9}{4}a_1x^3 + \frac{27}{4}a_0x^4 + \frac{81}{160}a_1x^5 - \frac{243}{4480}a_1x^7 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + 9x^2 + \frac{27}{4}x^4\right)a_0 + \left(x + \frac{9}{4}x^3 + \frac{81}{160}x^5 - \frac{243}{4480}x^7\right)a_1 + O(x^8) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + 9x^2 + \frac{27}{4}x^4\right)c_1 + \left(x + \frac{9}{4}x^3 + \frac{81}{160}x^5 - \frac{243}{4480}x^7\right)c_2 + O(x^8)$$

Summary

The solution(s) found are the following

$$y = \left(1 + 9x^2 + \frac{27}{4}x^4\right)y(0) + \left(x + \frac{9}{4}x^3 + \frac{81}{160}x^5 - \frac{243}{4480}x^7\right)y'(0) + O(x^8) \tag{1}$$

$$y = \left(1 + 9x^2 + \frac{27}{4}x^4\right)c_1 + \left(x + \frac{9}{4}x^3 + \frac{81}{160}x^5 - \frac{243}{4480}x^7\right)c_2 + O(x^8) \tag{2}$$

Verification of solutions

$$y = \left(1 + 9x^2 + \frac{27}{4}x^4\right)y(0) + \left(x + \frac{9}{4}x^3 + \frac{81}{160}x^5 - \frac{243}{4480}x^7\right)y'(0) + O(x^8)$$

Verified OK.

$$y = \left(1 + 9x^2 + \frac{27}{4}x^4\right)c_1 + \left(x + \frac{9}{4}x^3 + \frac{81}{160}x^5 - \frac{243}{4480}x^7\right)c_2 + O(x^8)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 39

```
Order:=8;
dsolve(2*diff(y(x),x$2)+9*x*diff(y(x),x)-36*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(\frac{27}{4}x^4 + 9x^2 + 1\right) y(0) + \left(x + \frac{9}{4}x^3 + \frac{81}{160}x^5 - \frac{243}{4480}x^7\right) D(y)(0) + O(x^8)$$

✓ Solution by Mathematica
Time used: 0.001 (sec). Leaf size: 47

```
AsymptoticDSolveValue[2*y''[x]+9*x*y'[x]-36*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(\frac{27x^4}{4} + 9x^2 + 1\right) + c_2\left(-\frac{243x^7}{4480} + \frac{81x^5}{160} + \frac{9x^3}{4} + x\right)$$

3.21 problem 21

Internal problem ID [6909]
Internal file name [`OUTPUT/6152_Tuesday_August_09_2022_05_23_28_AM_8669685/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 17. Power series solutions. 17.5. Solutions Near an Ordinary Point. Exercises page 355
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{(x^2+4)\, y''+xy'-9y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (103) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (104) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{xy' - 9y}{x^2+4} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{11x^2y' - 27yx + 32y'}{(x^2+4)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{60(-x^3-3x)\,y' + 180y(x^2+1)}{(x^2+4)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{420((x^3+3x)\,y' + (-3x^2-3)\,y)\,x}{(x^2+4)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -\frac{3360\left(x^2-\frac{1}{2}\right)\left((x^3+3x)\,y' + (-3x^2-3)\,y\right)}{(x^2+4)^5} \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \frac{30240((x^3+3x)\,y' + (-3x^2-3)\,y)\left(x^2-\frac{3}{2}\right)x}{(x^2+4)^6} \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= -\frac{302400((x^3+3x)\,y' + (-3x^2-3)\,y)\left(x^4-3x^2+\frac{3}{5}\right)}{(x^2+4)^7}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= \frac{9y(0)}{4} \\
F_1 &= 2y'(0) \\
F_2 &= \frac{45y(0)}{16} \\
F_3 &= 0 \\
F_4 &= -\frac{315y(0)}{64} \\
F_5 &= 0 \\
F_6 &= \frac{8505y(0)}{256}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + \frac{9}{8}x^2 + \frac{15}{128}x^4 - \frac{7}{1024}x^6 + \frac{27}{32768}x^8\right) y(0) + \left(x + \frac{1}{3}x^3\right) y'(0) + O\left(x^8\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
\left(x^2 + 4\right) y'' + xy' - 9y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(x^2 + 4\right) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + x \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 9 \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}
$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} 4n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \sum_{n=0}^{\infty} (-9 a_n x^n) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 4n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} 4(n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned} &\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} 4(n+2)\, a_{n+2}(n+1)\, x^n\right) \\ &+ \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \sum_{n=0}^{\infty} (-9 a_n x^n) = 0 \end{aligned} \quad (3)$$

$n = 0$ gives

$$8a_2 - 9a_0 = 0$$

$$a_2 = \frac{9a_0}{8}$$

$n = 1$ gives

$$24a_3 - 8a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$n a_n(n-1) + 4(n+2)\, a_{n+2}(n+1) + n a_n - 9a_n = 0 \quad (4)$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(n^2-9)}{4(n+2)(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$-5a_2 + 48a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{15a_0}{128}$$

For $n = 3$ the recurrence equation gives

$$80a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$7a_4 + 120a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{7a_0}{1024}$$

For $n = 5$ the recurrence equation gives

$$16a_5 + 168a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

For $n = 6$ the recurrence equation gives

$$27a_6 + 224a_8 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{27a_0}{32768}$$

For $n = 7$ the recurrence equation gives

$$40a_7 + 288a_9 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{9}{8} a_0 x^2 + \frac{1}{3} a_1 x^3 + \frac{15}{128} a_0 x^4 - \frac{7}{1024} a_0 x^6 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{9}{8}x^2 + \frac{15}{128}x^4 - \frac{7}{1024}x^6\right) a_0 + \left(x + \frac{1}{3}x^3\right) a_1 + O(x^8) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{9}{8}x^2 + \frac{15}{128}x^4 - \frac{7}{1024}x^6\right) c_1 + \left(x + \frac{1}{3}x^3\right) c_2 + O(x^8)$$

Summary

The solution(s) found are the following

$$y = \left(1 + \frac{9}{8}x^2 + \frac{15}{128}x^4 - \frac{7}{1024}x^6 + \frac{27}{32768}x^8\right) y(0) + \left(x + \frac{1}{3}x^3\right) y'(0) + O(x^8) \tag{1}$$

$$y = \left(1 + \frac{9}{8}x^2 + \frac{15}{128}x^4 - \frac{7}{1024}x^6\right) c_1 + \left(x + \frac{1}{3}x^3\right) c_2 + O(x^8) \tag{2}$$

Verification of solutions

$$y = \left(1+\frac{9}{8}x^2+\frac{15}{128}x^4-\frac{7}{1024}x^6+\frac{27}{32768}x^8\right)y(0)+\left(x+\frac{1}{3}x^3\right)y'(0)+O(x^8)$$

Verified OK.

$$y = \left(1+\frac{9}{8}x^2+\frac{15}{128}x^4-\frac{7}{1024}x^6\right)c_1+\left(x+\frac{1}{3}x^3\right)c_2+O(x^8)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 34

```
Order:=8;
dsolve((x^2+4)*diff(y(x),x$2)+x*diff(y(x),x)-9*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1+\frac{9}{8}x^2+\frac{15}{128}x^4-\frac{7}{1024}x^6\right)y(0)+\left(x+\frac{1}{3}x^3\right)D(y)(0)+O(x^8)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[(x^2+4)*y''[x]+x*y'[x]-9*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(\frac{x^3}{3}+x\right)+c_1\left(-\frac{7x^6}{1024}+\frac{15x^4}{128}+\frac{9x^2}{8}+1\right)$$

3.22 problem 22

Internal problem ID [6910]
Internal file name [`OUTPUT/6153_Tuesday_August_09_2022_05_23_30_AM_29950073/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 17. Power series solutions. 17.5. Solutions Near an Ordinary Point. Exercises page 355
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+4)\,y''+3xy'-8y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\ &= y_0+xy'_0+\frac{x^2}{2}\,f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\,f'|_{x_0,y_0,y'_0}+\cdots \\ &= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{106}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{107}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{3xy' - 8y}{x^2+4} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{20x^2y' - 40yx + 20y'}{\left(x^2+4\right)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -\frac{140(x^2y' - 2yx + y')\, x}{\left(x^2+4\right)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{1120\left(x^2 - \frac{1}{2}\right)\left(\left(x^2+1\right) y' - 2yx\right)}{\left(x^2+4\right)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -\frac{10080(\left(x^2+1\right) y' - 2yx)\left(x^2 - \frac{3}{2}\right) x}{\left(x^2+4\right)^5} \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \frac{100800(\left(x^2+1\right) y' - 2yx)\left(x^4 - 3x^2 + \frac{3}{5}\right)}{\left(x^2+4\right)^6} \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= -\frac{1108800(\left(x^2+1\right) y' - 2yx)\, x(x^4 - 5x^2 + 3)}{\left(x^2+4\right)^7}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 2y(0) \\
F_1 &= \frac{5y'(0)}{4} \\
F_2 &= 0 \\
F_3 &= -\frac{35y'(0)}{16} \\
F_4 &= 0 \\
F_5 &= \frac{945y'(0)}{64} \\
F_6 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = (x^2 + 1)\, y(0) + \left(x + \frac{5}{24}x^3 - \frac{7}{384}x^5 + \frac{3}{1024}x^7\right) y'(0) + O(x^8)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(x^2 + 4)\, y'' + 3xy' - 8y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
(x^2 + 4) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + 3x \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 8 \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}
$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} 4n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 3n a_n x^n\right) + \sum_{n=0}^{\infty} (-8 a_n x^n) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 4n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} 4(n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned} \left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} 4(n+2)\, a_{n+2}(n+1)\, x^n\right) \\ + \left(\sum_{n=1}^{\infty} 3n a_n x^n\right) + \sum_{n=0}^{\infty} (-8 a_n x^n) = 0 \end{aligned} \tag{3}$$

$n = 0$ gives

$$8a_2 - 8a_0 = 0$$

$$a_2 = a_0$$

$n = 1$ gives

$$24a_3 - 5a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{5a_1}{24}$$

For $2 \leq n$, the recurrence equation is

$$n a_n(n-1) + 4(n+2)\, a_{n+2}(n+1) + 3n a_n - 8a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(n^2+2n-8)}{4(n+2)(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$48a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 0$$

For $n = 3$ the recurrence equation gives

$$7a_3 + 80a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{7a_1}{384}$$

For $n = 4$ the recurrence equation gives

$$16a_4 + 120a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$27a_5 + 168a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{3a_1}{1024}$$

For $n = 6$ the recurrence equation gives

$$40a_6 + 224a_8 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = 0$$

For $n = 7$ the recurrence equation gives

$$55a_7 + 288a_9 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = -\frac{55a_1}{98304}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + a_0 x^2 + \frac{5}{24} a_1 x^3 - \frac{7}{384} a_1 x^5 + \frac{3}{1024} a_1 x^7 + \dots$$

Collecting terms, the solution becomes

$$y = (x^2 + 1)\, a_0 + \left(x + \frac{5}{24} x^3 - \frac{7}{384} x^5 + \frac{3}{1024} x^7\right) a_1 + O(x^8) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = (x^2 + 1)\, c_1 + \left(x + \frac{5}{24} x^3 - \frac{7}{384} x^5 + \frac{3}{1024} x^7\right) c_2 + O(x^8)$$

Summary

The solution(s) found are the following

$$y = (x^2 + 1)\, y(0) + \left(x + \frac{5}{24} x^3 - \frac{7}{384} x^5 + \frac{3}{1024} x^7\right) y'(0) + O(x^8) \tag{1}$$

$$y = (x^2 + 1)\, c_1 + \left(x + \frac{5}{24} x^3 - \frac{7}{384} x^5 + \frac{3}{1024} x^7\right) c_2 + O(x^8) \tag{2}$$

Verification of solutions

$$y = (x^2+1)\,y(0) + \left(x + \frac{5}{24}x^3 - \frac{7}{384}x^5 + \frac{3}{1024}x^7\right) y'(0) + O(x^8)$$

Verified OK.

$$y = (x^2+1)\,c_1 + \left(x + \frac{5}{24}x^3 - \frac{7}{384}x^5 + \frac{3}{1024}x^7\right) c_2 + O(x^8)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
Order:=8;
dsolve((x^2+4)*diff(y(x),x$2)+3*x*diff(y(x),x)-8*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (x^2+1)\,y(0) + \left(x + \frac{5}{24}x^3 - \frac{7}{384}x^5 + \frac{3}{1024}x^7\right) D(y)(0) + O(x^8)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 38

```
AsymptoticDSolveValue[(x^2+4)*y''[x]+3*x*y'[x]-8*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1(x^2+1) + c_2\left(\frac{3x^7}{1024} - \frac{7x^5}{384} + \frac{5x^3}{24} + x\right)$$

3.23 problem 23

Internal problem ID [6911]
Internal file name [`OUTPUT/6154_Tuesday_August_09_2022_05_23_31_AM_15034228/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 17. Power series solutions. 17.5. Solutions Near an Ordinary Point. Exercises page 355
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(9x^2+1)\,y''-18y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy_0'+\frac{x^2}{2}f|_{x_0,y_0,y_0'}+\frac{x^3}{3!}f'|_{x_0,y_0,y_0'}+\cdots \\
&= y_0+xy_0'+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}\left.\frac{d^nf}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (109)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (110)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n\big|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= \frac{18y}{9x^2+1}\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
&= \frac{162x^2y' - 324yx + 18y'}{\left(9x^2+1\right)^2}\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1\\
&= -\frac{648(9x^2y' - 18yx + y')\,x}{\left(9x^2+1\right)^3}\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2\\
&= \frac{262440\left(\left(x^2+\frac{1}{9}\right)y' - 2yx\right)\left(x^2-\frac{1}{45}\right)}{\left(9x^2+1\right)^4}\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3\\
&= -\frac{14171760\left(\left(x^2+\frac{1}{9}\right)y' - 2yx\right)x\left(x^2-\frac{1}{15}\right)}{\left(9x^2+1\right)^5}\\
F_5 &= \frac{dF_4}{dx}\\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y}y' + \frac{\partial F_4}{\partial y'}F_4\\
&= \frac{892820880\left(\left(x^2+\frac{1}{9}\right)y' - 2yx\right)\left(x^4-\frac{2}{15}x^2+\frac{1}{945}\right)}{\left(9x^2+1\right)^6}\\
F_6 &= \frac{dF_5}{dx}\\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y}y' + \frac{\partial F_5}{\partial y'}F_5\\
&= -\frac{64283103360\left(\left(x^2+\frac{1}{9}\right)y' - 2yx\right)x\left(x^4-\frac{2}{9}x^2+\frac{1}{189}\right)}{\left(9x^2+1\right)^7}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 18y(0) \\
F_1 &= 18y'(0) \\
F_2 &= 0 \\
F_3 &= -648y'(0) \\
F_4 &= 0 \\
F_5 &= 104976y'(0) \\
F_6 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(9x^2 + 1\right) y(0) + \left(x + 3x^3 - \frac{27}{5}x^5 + \frac{729}{35}x^7\right) y'(0) + O\left(x^8\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
\left(9x^2 + 1\right) y'' - 18y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(9x^2 + 1\right) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - 18\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} 9x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=0}^{\infty} \left(-18a_n x^n\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} 9x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=0}^{\infty} (-18 a_n x^n) = 0 \quad (3)$$

$n = 0$ gives

$$2a_2 - 18a_0 = 0$$

$$a_2 = 9a_0$$

$n = 1$ gives

$$6a_3 - 18a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = 3a_1$$

For $2 \leq n$, the recurrence equation is

$$9na_n(n-1) + (n+2)\, a_{n+2}(n+1) - 18a_n = 0 \quad (4)$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{9(n-2)\, a_n}{n+2} \quad (5)$$

For $n = 2$ the recurrence equation gives

$$12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 0$$

For $n = 3$ the recurrence equation gives

$$36a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{27a_1}{5}$$

For $n = 4$ the recurrence equation gives

$$90a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$162a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{729a_1}{35}$$

For $n = 6$ the recurrence equation gives

$$252a_6 + 56a_8 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = 0$$

For $n = 7$ the recurrence equation gives

$$360a_7 + 72a_9 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = -\frac{729a_1}{7}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + 9a_0 x^2 + 3a_1 x^3 - \frac{27}{5} a_1 x^5 + \frac{729}{35} a_1 x^7 + \dots$$

Collecting terms, the solution becomes

$$y = (9x^2 + 1)\, a_0 + \left(x + 3x^3 - \frac{27}{5}x^5 + \frac{729}{35}x^7\right) a_1 + O(x^8) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = (9x^2 + 1)\, c_1 + \left(x + 3x^3 - \frac{27}{5}x^5 + \frac{729}{35}x^7\right) c_2 + O(x^8)$$

Summary

The solution(s) found are the following

$$y = (9x^2 + 1)\, y(0) + \left(x + 3x^3 - \frac{27}{5}x^5 + \frac{729}{35}x^7\right) y'(0) + O(x^8) \tag{1}$$

$$y = (9x^2 + 1)\, c_1 + \left(x + 3x^3 - \frac{27}{5}x^5 + \frac{729}{35}x^7\right) c_2 + O(x^8) \tag{2}$$

Verification of solutions

$$y = (9x^2 + 1)\, y(0) + \left(x + 3x^3 - \frac{27}{5}x^5 + \frac{729}{35}x^7\right) y'(0) + O(x^8)$$

Verified OK.

$$y = (9x^2 + 1)\, c_1 + \left(x + 3x^3 - \frac{27}{5}x^5 + \frac{729}{35}x^7\right) c_2 + O(x^8)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 34

```
Order:=8;
dsolve((1+9*x^2)*diff(y(x),x$2)-18*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (9x^2 + 1)\, y(0) + \left(x + 3x^3 - \frac{27}{5}x^5 + \frac{729}{35}x^7\right) D(y)\,(0) + O(x^8)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 38

```
AsymptoticDSolveValue[(1+9*x^2)*y''[x]-18*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1(9x^2 + 1) + c_2\left(\frac{729x^7}{35} - \frac{27x^5}{5} + 3x^3 + x\right)$$

3.24 problem 24

Internal problem ID [6912]
Internal file name [`OUTPUT/6155_Tuesday_August_09_2022_05_23_33_AM_6249229/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 17. Power series solutions. 17.5. Solutions Near an Ordinary Point. Exercises page 355
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(3x^2+1)\, y''+13xy'+7y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy'_0+\frac{x^2}{2}f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}f'|_{x_0,y_0,y'_0}+\cdots \\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (112) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (113) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n \big|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{13xy'+7y}{3x^2+1}\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x}+\frac{\partial F_0}{\partial y}y'+\frac{\partial F_0}{\partial y'}F_0\\
&= \frac{187x^2y'+133yx-20y'}{\left(3x^2+1\right)^2}\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x}+\frac{\partial F_1}{\partial y}y'+\frac{\partial F_1}{\partial y'}F_1\\
&= \frac{-3154x^3y'-2506x^2y+1007xy'+273y}{\left(3x^2+1\right)^3}\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x}+\frac{\partial F_2}{\partial y}y'+\frac{\partial F_2}{\partial y'}F_2\\
&= \frac{\left(61870x^4-39345x^2+1280\right)y'+\left(52150x^3-16975x\right)y}{\left(3x^2+1\right)^4}\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x}+\frac{\partial F_3}{\partial y}y'+\frac{\partial F_3}{\partial y'}F_3\\
&= \frac{\left(-1390300x^5+1468400x^3-143025x\right)y'+\left(-1215340x^4+788340x^2-25935\right)y}{\left(3x^2+1\right)^5}\\
F_5 &= \frac{dF_4}{dx}\\
&= \frac{\partial F_4}{\partial x}+\frac{\partial F_4}{\partial y}y'+\frac{\partial F_4}{\partial y'}F_4\\
&= \frac{\left(35282380x^6-55727420x^4+10836735x^2-168960\right)y'+31608220\left(x^4-\frac{7848}{7283}x^2+\frac{3093}{29132}\right)xy}{\left(3x^2+1\right)^6}\\
F_6 &= \frac{dF_5}{dx}\\
&= \frac{\partial F_5}{\partial x}+\frac{\partial F_5}{\partial y}y'+\frac{\partial F_5}{\partial y'}F_5\\
&= \frac{\left(-998929120x^7+2203036080x^5-712881890x^3+33308415x\right)y'+\left(-910749280x^6+1467761680x\right.}{\left(3x^2+1\right)^7}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -7y(0) \\
F_1 &= -20y'(0) \\
F_2 &= 273y(0) \\
F_3 &= 1280y'(0) \\
F_4 &= -25935y(0) \\
F_5 &= -168960y'(0) \\
F_6 &= 4538625y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1-\frac{7}{2}x^2+\frac{91}{8}x^4-\frac{1729}{48}x^6+\frac{43225}{384}x^8\right)y(0) \\
&+\left(x-\frac{10}{3}x^3+\frac{32}{3}x^5-\frac{704}{21}x^7\right)y'(0)+O(x^8)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(3x^2+1)\,y''+13xy'+7y=0
$$

Let the solution be represented as power series of the form

$$
y=\sum_{n=0}^{\infty}a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
(3x^2+1)\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right)+13x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)+7\left(\sum_{n=0}^{\infty} a_n x^n\right)=0 \quad (1)
$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} 3x^{n} a_{n} n(n-1)\right)+\left(\sum_{n=2}^{\infty} n(n-1) a_{n} x^{n-2}\right)+\left(\sum_{n=1}^{\infty} 13 n a_{n} x^{n}\right)+\left(\sum_{n=0}^{\infty} 7 a_{n} x^{n}\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_{n} x^{n-2}=\sum_{n=0}^{\infty}(n+2) a_{n+2}(n+1) x^{n}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned}&\left(\sum_{n=2}^{\infty} 3x^{n} a_{n} n(n-1)\right)+\left(\sum_{n=0}^{\infty}(n+2) a_{n+2}(n+1) x^{n}\right) \\ &+\left(\sum_{n=1}^{\infty} 13 n a_{n} x^{n}\right)+\left(\sum_{n=0}^{\infty} 7 a_{n} x^{n}\right)=0\end{aligned} \tag{3}$$

$n=0$ gives

$$2a_{2}+7a_{0}=0$$

$$a_{2}=-\frac{7a_{0}}{2}$$

$n=1$ gives

$$6a_{3}+20a_{1}=0$$

Which after substituting earlier equations, simplifies to

$$a_{3}=-\frac{10a_{1}}{3}$$

For $2 \leq n$, the recurrence equation is

$$3n a_{n}(n-1)+(n+2) a_{n+2}(n+1)+13 n a_{n}+7a_{n}=0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{(3n+7)\,a_n}{n+2} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$39a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{91a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$64a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{32a_1}{3}$$

For $n = 4$ the recurrence equation gives

$$95a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{1729a_0}{48}$$

For $n = 5$ the recurrence equation gives

$$132a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{704a_1}{21}$$

For $n = 6$ the recurrence equation gives

$$175a_6 + 56a_8 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{43225 a_0}{384}$$

For $n = 7$ the recurrence equation gives

$$224 a_7 + 72 a_9 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{2816 a_1}{27}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{7}{2} a_0 x^2 - \frac{10}{3} a_1 x^3 + \frac{91}{8} a_0 x^4 + \frac{32}{3} a_1 x^5 - \frac{1729}{48} a_0 x^6 - \frac{704}{21} a_1 x^7 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{7}{2} x^2 + \frac{91}{8} x^4 - \frac{1729}{48} x^6\right) a_0 + \left(x - \frac{10}{3} x^3 + \frac{32}{3} x^5 - \frac{704}{21} x^7\right) a_1 + O\left(x^8\right) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{7}{2} x^2 + \frac{91}{8} x^4 - \frac{1729}{48} x^6\right) c_1 + \left(x - \frac{10}{3} x^3 + \frac{32}{3} x^5 - \frac{704}{21} x^7\right) c_2 + O\left(x^8\right)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = &\left(1 - \frac{7}{2} x^2 + \frac{91}{8} x^4 - \frac{1729}{48} x^6 + \frac{43225}{384} x^8\right) y(0) \\ &+ \left(x - \frac{10}{3} x^3 + \frac{32}{3} x^5 - \frac{704}{21} x^7\right) y'(0) + O\left(x^8\right) \end{aligned} \quad (1)$$

$$y = \left(1 - \frac{7}{2} x^2 + \frac{91}{8} x^4 - \frac{1729}{48} x^6\right) c_1 + \left(x - \frac{10}{3} x^3 + \frac{32}{3} x^5 - \frac{704}{21} x^7\right) c_2 + O\left(x^8\right) \quad (2)$$

Verification of solutions

$$y = \left(1 - \frac{7}{2}x^2 + \frac{91}{8}x^4 - \frac{1729}{48}x^6 + \frac{43225}{384}x^8\right) y(0) + \left(x - \frac{10}{3}x^3 + \frac{32}{3}x^5 - \frac{704}{21}x^7\right) y'(0) + O(x^8)$$

Verified OK.

$$y = \left(1 - \frac{7}{2}x^2 + \frac{91}{8}x^4 - \frac{1729}{48}x^6\right) c_1 + \left(x - \frac{10}{3}x^3 + \frac{32}{3}x^5 - \frac{704}{21}x^7\right) c_2 + O(x^8)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
Order:=8;
dsolve((1+3*x^2)*diff(y(x),x$2)+13*x*diff(y(x),x)+7*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{7}{2}x^2 + \frac{91}{8}x^4 - \frac{1729}{48}x^6\right) y(0) + \left(x - \frac{10}{3}x^3 + \frac{32}{3}x^5 - \frac{704}{21}x^7\right) D(y)(0) + O(x^8)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 56

```
AsymptoticDSolveValue[(1+3*x^2)*y''[x]+13*x*y'[x]+7*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(-\frac{704x^7}{21} + \frac{32x^5}{3} - \frac{10x^3}{3} + x\right) + c_1\left(-\frac{1729x^6}{48} + \frac{91x^4}{8} - \frac{7x^2}{2} + 1\right)$$

3.25 problem 25

Internal problem ID [6913]
Internal file name [`OUTPUT/6156_Tuesday_August_09_2022_05_23_34_AM_90181001/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 17. Power series solutions. 17.5. Solutions Near an Ordinary Point. Exercises page 355
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(2x^2+1)\,y''+11xy'+9y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy'_0+\frac{x^2}{2}f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}f'|_{x_0,y_0,y'_0}+\cdots \\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{115}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{116}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{11xy' + 9y}{2x^2 + 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{125x^2 y' + 135yx - 20y'}{(2x^2 + 1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{-1605x^3 y' - 1935x^2 y + 765xy' + 315y}{(2x^2 + 1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(23415x^4 - 22185x^2 + 1080)\, y' + 29925x\left(x^2 - \frac{17}{35}\right) y}{(2x^2 + 1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-385035x^5 + 604770x^3 - 88065x)\, y' + (-509985x^4 + 492930x^2 - 24255)\, y}{(2x^2 + 1)^5} \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \frac{(7065765x^6 - 16568550x^4 + 4812615x^2 - 112320)\, y' + 9585135xy\left(x^4 - \frac{550}{343}x^2 + \frac{81}{343}\right)}{(2x^2 + 1)^6} \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= \frac{(-143342325x^7 + 468591075x^5 - 226307925x^3 + 15819975x)\, y' - 197783775\left(x^6 - \frac{77975}{32557}x^4 + \frac{22917}{32557}x\right.}{(2x^2 + 1)^7}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -9y(0) \\
F_1 &= -20y'(0) \\
F_2 &= 315y(0) \\
F_3 &= 1080y'(0) \\
F_4 &= -24255y(0) \\
F_5 &= -112320y'(0) \\
F_6 &= 3274425y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{9}{2}x^2 + \frac{105}{8}x^4 - \frac{539}{16}x^6 + \frac{10395}{128}x^8\right) y(0) + \left(x - \frac{10}{3}x^3 + 9x^5 - \frac{156}{7}x^7\right) y'(0) + O(x^8)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(2x^2 + 1)\, y'' + 11xy' + 9y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
(2x^2 + 1)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + 11x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 9\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \quad (1)
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} 2x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 11 n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 9 a_n x^n\right) = 0 \quad (2)
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} 2x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \left(\sum_{n=1}^{\infty} 11 n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 9 a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 9a_0 = 0$$

$$a_2 = -\frac{9a_0}{2}$$

$n = 1$ gives

$$6a_3 + 20a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{10a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$2na_n(n-1) + (n+2) a_{n+2}(n+1) + 11na_n + 9a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(2n^2 + 9n + 9)}{(n+2)(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$35a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{105a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$54a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 9a_1$$

For $n = 4$ the recurrence equation gives

$$77a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{539a_0}{16}$$

For $n = 5$ the recurrence equation gives

$$104a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{156a_1}{7}$$

For $n = 6$ the recurrence equation gives

$$135a_6 + 56a_8 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{10395a_0}{128}$$

For $n = 7$ the recurrence equation gives

$$170a_7 + 72a_9 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{1105a_1}{21}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{9}{2}a_0 x^2 - \frac{10}{3}a_1 x^3 + \frac{105}{8}a_0 x^4 + 9a_1 x^5 - \frac{539}{16}a_0 x^6 - \frac{156}{7}a_1 x^7 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{9}{2}x^2 + \frac{105}{8}x^4 - \frac{539}{16}x^6\right)a_0 + \left(x - \frac{10}{3}x^3 + 9x^5 - \frac{156}{7}x^7\right)a_1 + O(x^8) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{9}{2}x^2 + \frac{105}{8}x^4 - \frac{539}{16}x^6\right)c_1 + \left(x - \frac{10}{3}x^3 + 9x^5 - \frac{156}{7}x^7\right)c_2 + O(x^8)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= \left(1 - \frac{9}{2}x^2 + \frac{105}{8}x^4 - \frac{539}{16}x^6 + \frac{10395}{128}x^8\right)y(0) \\ &\quad + \left(x - \frac{10}{3}x^3 + 9x^5 - \frac{156}{7}x^7\right)y'(0) + O(x^8) \end{aligned} \quad (1)$$

$$y = \left(1 - \frac{9}{2}x^2 + \frac{105}{8}x^4 - \frac{539}{16}x^6\right)c_1 + \left(x - \frac{10}{3}x^3 + 9x^5 - \frac{156}{7}x^7\right)c_2 + O(x^8) \quad (2)$$

Verification of solutions

$$y = \left(1-\frac{9}{2}x^2+\frac{105}{8}x^4-\frac{539}{16}x^6+\frac{10395}{128}x^8\right)y(0) + \left(x-\frac{10}{3}x^3+9x^5-\frac{156}{7}x^7\right)y'(0)+O(x^8)$$

Verified OK.

$$y = \left(1-\frac{9}{2}x^2+\frac{105}{8}x^4-\frac{539}{16}x^6\right)c_1+\left(x-\frac{10}{3}x^3+9x^5-\frac{156}{7}x^7\right)c_2+O(x^8)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Legendre successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
Order:=8;
dsolve((1+2*x^2)*diff(y(x),x$2)+11*x*diff(y(x),x)+9*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1-\frac{9}{2}x^{2}+\frac{105}{8}x^{4}-\frac{539}{16}x^{6}\right)y(0)+\left(x-\frac{10}{3}x^{3}+9x^{5}-\frac{156}{7}x^{7}\right)D(y)(0)+O\left(x^{8}\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 54

```
AsymptoticDSolveValue[(1+2*x^2)*y''[x]+11*x*y'[x]+9*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(-\frac{156x^7}{7}+9x^5-\frac{10x^3}{3}+x\right)+c_1\left(-\frac{539x^6}{16}+\frac{105x^4}{8}-\frac{9x^2}{2}+1\right)$$

3.26 problem 26

Internal problem ID [6914]
Internal file name [`OUTPUT/6157_Tuesday_August_09_2022_05_23_36_AM_15502206/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 17. Power series solutions. 17.5. Solutions Near an Ordinary Point. Exercises page 355
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 2(x+3)\, y' - 3y = 0}$$

With the expansion point for the power series method at $x = -3$.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x + 3$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\frac{d^2}{dt^2}y(t) - 2t\left(\frac{d}{dt}y(t)\right) - 3y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (118)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (119)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= 2t\left(\frac{d}{dt}y(t)\right) + 3y(t) \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y(t)}F_0 \\
&= 4\left(\frac{d}{dt}y(t)\right)t^2 + 6y(t)\,t + 5\frac{d}{dt}y(t) \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y(t)}F_1 \\
&= 8\left(t^3 + 3t\right)\left(\frac{d}{dt}y(t)\right) + 3y(t)\left(4t^2 + 7\right) \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y(t)}F_2 \\
&= \left(16t^4 + 84t^2 + 45\right)\left(\frac{d}{dt}y(t)\right) + 24y(t)\,t\left(t^2 + 4\right) \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y(t)}F_3 \\
&= \left(32t^5 + 256t^3 + 354t\right)\left(\frac{d}{dt}y(t)\right) + \left(48t^4 + 324t^2 + 231\right)y(t) \\
F_5 &= \frac{dF_4}{dt} \\
&= \frac{\partial F_4}{\partial t} + \frac{\partial F_4}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_4}{\partial \frac{d}{dt}y(t)}F_4 \\
&= \left(64t^6 + 720t^4 + 1800t^2 + 585\right)\left(\frac{d}{dt}y(t)\right) + 96\left(t^4 + 10t^2 + \frac{285}{16}\right)ty(t) \\
F_6 &= \frac{dF_5}{dt} \\
&= \frac{\partial F_5}{\partial t} + \frac{\partial F_5}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_5}{\partial \frac{d}{dt}y(t)}F_5 \\
&= \left(128t^7 + 1920t^5 + 7440t^3 + 6480t\right)\left(\frac{d}{dt}y(t)\right) + 192\left(t^6 + \frac{55}{4}t^4 + \frac{345}{8}t^2 + \frac{1155}{64}\right)y(t)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 3y(0) \\
F_1 &= 5y'(0) \\
F_2 &= 21y(0) \\
F_3 &= 45y'(0) \\
F_4 &= 231y(0) \\
F_5 &= 585y'(0) \\
F_6 &= 3465y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y(t) = \left(1 + \frac{3}{2}t^2 + \frac{7}{8}t^4 + \frac{77}{240}t^6 + \frac{11}{128}t^8\right) y(0) + \left(t + \frac{5}{6}t^3 + \frac{3}{8}t^5 + \frac{13}{112}t^7\right) y'(0) + O(t^8)
$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y(t) = \sum_{n=0}^{\infty} a_n t^n
$$

Then

$$
\begin{aligned}
\frac{d}{dt}y(t) &= \sum_{n=1}^{\infty} n a_n t^{n-1} \\
\frac{d^2}{dt^2}y(t) &= \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = 2t\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + 3\left(\sum_{n=0}^{\infty} a_n t^n\right) \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right) + \sum_{n=1}^{\infty} \left(-2n\, t^n a_n\right) + \sum_{n=0}^{\infty} \left(-3a_n t^n\right) = 0 \tag{2}
$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, t^n\right) + \sum_{n=1}^{\infty} (-2n\, t^n a_n) + \sum_{n=0}^{\infty} (-3a_n t^n) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - 3a_0 = 0$$

$$a_2 = \frac{3a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - 2na_n - 3a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_n(2n+3)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - 5a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{5a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 7a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{7a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 9a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{3a_1}{8}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 11a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{77a_0}{240}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 13a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{13a_1}{112}$$

For $n = 6$ the recurrence equation gives

$$56a_8 - 15a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{11a_0}{128}$$

For $n = 7$ the recurrence equation gives

$$72a_9 - 17a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{221a_1}{8064}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t + \frac{3}{2} a_0 t^2 + \frac{5}{6} a_1 t^3 + \frac{7}{8} a_0 t^4 + \frac{3}{8} a_1 t^5 + \frac{77}{240} a_0 t^6 + \frac{13}{112} a_1 t^7 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 + \frac{3}{2} t^2 + \frac{7}{8} t^4 + \frac{77}{240} t^6\right) a_0 + \left(t + \frac{5}{6} t^3 + \frac{3}{8} t^5 + \frac{13}{112} t^7\right) a_1 + O(t^8) \quad (3)$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 + \frac{3}{2} t^2 + \frac{7}{8} t^4 + \frac{77}{240} t^6\right) c_1 + \left(t + \frac{5}{6} t^3 + \frac{3}{8} t^5 + \frac{13}{112} t^7\right) c_2 + O(t^8)$$

Replacing t in the above with the original independent variable $x$susing $t = x + 3$ results in

$$\begin{aligned} y = &\left(1 + \frac{3(x+3)^2}{2} + \frac{7(x+3)^4}{8} + \frac{77(x+3)^6}{240} + \frac{11(x+3)^8}{128}\right) y(-3) \\ &+ \left(x + 3 + \frac{5(x+3)^3}{6} + \frac{3(x+3)^5}{8} + \frac{13(x+3)^7}{112}\right) y'(-3) + O\left((x+3)^8\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \left(1+\frac{3(x+3)^{2}}{2}+\frac{7(x+3)^{4}}{8}+\frac{77(x+3)^{6}}{240}+\frac{11(x+3)^{8}}{128}\right) y(-3) \\
& + \left(x+3+\frac{5(x+3)^{3}}{6}+\frac{3(x+3)^{5}}{8}+\frac{13(x+3)^{7}}{112}\right) y'(-3)+O\left((x+3)^{8}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & \left(1+\frac{3(x+3)^{2}}{2}+\frac{7(x+3)^{4}}{8}+\frac{77(x+3)^{6}}{240}+\frac{11(x+3)^{8}}{128}\right) y(-3) \\
& + \left(x+3+\frac{5(x+3)^{3}}{6}+\frac{3(x+3)^{5}}{8}+\frac{13(x+3)^{7}}{112}\right) y'(-3)+O\left((x+3)^{8}\right)
\end{aligned}
$$

Verified OK.

3.26.1 Maple step by step solution

Let's solve

$y''+(-2x-6)\,y'-3y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m=0..1$

    $x^m \cdot y' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$

  ◦ Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, x^k$

  ◦ Convert y'' to series expansion

    $y''=\sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- Shift index using $k->k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$$

  Rewrite DE with series expansions

  $$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)(k+1) - 6a_{k+1}(k+1) - a_k(2k+3)\right)x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$k^2 a_{k+2} + (-2a_k - 6a_{k+1} + 3a_{k+2})k - 3a_k - 6a_{k+1} + 2a_{k+2} = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2a_k k + 6a_{k+1}k + 3a_k + 6a_{k+1}}{k^2 + 3k + 2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
Order:=8;
dsolve(diff(y(x),x$2)-2*(x+3)*diff(y(x),x)-3*y(x)=0,y(x),type='series',x=-3);
```

$$y(x) = \left(1 + \frac{3(x+3)^2}{2} + \frac{7(x+3)^4}{8} + \frac{77(x+3)^6}{240}\right) y(-3) + \left(x + 3 + \frac{5(x+3)^3}{6} + \frac{3(x+3)^5}{8} + \frac{13(x+3)^7}{112}\right) D(y)(-3) + O(x^8)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 69

```
AsymptoticDSolveValue[y''[x]-2*(x+3)*y'[x]-3*y[x]==0,y[x],{x,-3,7}]
```

$$y(x) \to c_1\left(\frac{77}{240}(x+3)^6 + \frac{7}{8}(x+3)^4 + \frac{3}{2}(x+3)^2 + 1\right) + c_2\left(\frac{13}{112}(x+3)^7 + \frac{3}{8}(x+3)^5 + \frac{5}{6}(x+3)^3 + x + 3\right)$$

3.27 problem 27

Internal problem ID [6915]
Internal file name [`OUTPUT/6158_Tuesday_August_09_2022_05_23_38_AM_91740308/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 17. Power series solutions. 17.5. Solutions Near an Ordinary Point. Exercises page 355
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_airy", "second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + (x-2)\,y = 0}$$

With the expansion point for the power series method at $x = 2$.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 2$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\frac{d^2}{dt^2}y(t) + y(t)\,t = 0$$

With its expansion point and initial conditions now at $t = 0$. The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (121)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (122)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -y(t)\,t \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y(t)} F_0 \\
&= -t\left(\frac{d}{dt}y(t)\right) - y(t) \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y(t)} F_1 \\
&= -2\frac{d}{dt}y(t) + y(t)\,t^2 \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y(t)} F_2 \\
&= t\left(t\left(\frac{d}{dt}y(t)\right) + 4y(t)\right) \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y(t)} F_3 \\
&= -y(t)\,t^3 + 6t\left(\frac{d}{dt}y(t)\right) + 4y(t) \\
F_5 &= \frac{dF_4}{dt} \\
&= \frac{\partial F_4}{\partial t} + \frac{\partial F_4}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_4}{\partial \frac{d}{dt}y(t)} F_4 \\
&= -\left(\frac{d}{dt}y(t)\right)t^3 - 9y(t)\,t^2 + 10\frac{d}{dt}y(t) \\
F_6 &= \frac{dF_5}{dt} \\
&= \frac{\partial F_5}{\partial t} + \frac{\partial F_5}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_5}{\partial \frac{d}{dt}y(t)} F_5 \\
&= -12\left(\frac{d}{dt}y(t)\right)t^2 + ty(t)\left(t^3 - 28\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -y(0) \\
F_2 &= -2y'(0) \\
F_3 &= 0 \\
F_4 &= 4y(0) \\
F_5 &= 10y'(0) \\
F_6 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y(t) = \left(1 - \frac{1}{6}t^3 + \frac{1}{180}t^6\right) y(0) + \left(t - \frac{1}{12}t^4 + \frac{1}{504}t^7\right) y'(0) + O(t^8)
$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y(t) = \sum_{n=0}^{\infty} a_n t^n
$$

Then

$$
\begin{aligned}
\frac{d}{dt}y(t) &= \sum_{n=1}^{\infty} n a_n t^{n-1} \\
\frac{d^2}{dt^2}y(t) &= \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = -\left(\sum_{n=0}^{\infty} a_n t^n\right) t \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right) + \left(\sum_{n=0}^{\infty} t^{1+n} a_n\right) = 0 \tag{2}
$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, t^n$$

$$\sum_{n=0}^{\infty} t^{1+n} a_n = \sum_{n=1}^{\infty} a_{n-1} t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, t^n\right) + \left(\sum_{n=1}^{\infty} a_{n-1} t^n\right) = 0 \tag{3}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) + a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_{n-1}}{(n+2)\,(1+n)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_0}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_1}{12}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$30a_6 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{180}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{504}$$

For $n = 6$ the recurrence equation gives

$$56a_8 + a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = 0$$

For $n = 7$ the recurrence equation gives

$$72a_9 + a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = -\frac{a_0}{12960}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t - \frac{1}{6} a_0 t^3 - \frac{1}{12} a_1 t^4 + \frac{1}{180} a_0 t^6 + \frac{1}{504} a_1 t^7 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 - \frac{1}{6} t^3 + \frac{1}{180} t^6\right) a_0 + \left(t - \frac{1}{12} t^4 + \frac{1}{504} t^7\right) a_1 + O(t^8) \tag{3}$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 - \frac{1}{6} t^3 + \frac{1}{180} t^6\right) c_1 + \left(t - \frac{1}{12} t^4 + \frac{1}{504} t^7\right) c_2 + O(t^8)$$

Replacing t in the above with the original independent variable $x$susing $t = x - 2$ results in

$$y = \left(1 - \frac{(x-2)^3}{6} + \frac{(x-2)^6}{180}\right) y(2) + \left(x - 2 - \frac{(x-2)^4}{12} + \frac{(x-2)^7}{504}\right) y'(2) + O\left((x-2)^8\right)$$

<u>Summary</u>

The solution(s) found are the following

$$\begin{aligned} y = {} & \left(1 - \frac{(x-2)^3}{6} + \frac{(x-2)^6}{180}\right) y(2) \\ & + \left(x - 2 - \frac{(x-2)^4}{12} + \frac{(x-2)^7}{504}\right) y'(2) + O\left((x-2)^8\right) \end{aligned} \tag{1}$$

<u>Verification of solutions</u>

$$y = \left(1 - \frac{(x-2)^3}{6} + \frac{(x-2)^6}{180}\right) y(2) + \left(x - 2 - \frac{(x-2)^4}{12} + \frac{(x-2)^7}{504}\right) y'(2) + O\left((x-2)^8\right)$$

Verified OK.

3.27.1 Maple step by step solution

Let's solve

$y'' + (x-2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 - 2a_0 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - 2a_k + a_{k-1}\right) x^k\right) = 0$

- Each term must be 0

  $2a_2 - 2a_0 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} - 2a_k + a_{k-1} = 0$

- Shift index using $k- >k+1$

  $\left((k+1)^2 + 3k + 5\right) a_{k+3} - 2a_{k+1} + a_k = 0$

- Recursion relation that defines the series solution to the ODE

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+3}=-\frac{-2a_{k+1}+a_k}{k^2+5k+6}, 2a_2-2a_0=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=8;
dsolve(diff(y(x),x$2)+(x-2)*y(x)=0,y(x),type='series',x=2);
```

$$y(x)=\left(1-\frac{(-2+x)^3}{6}+\frac{(-2+x)^6}{180}\right)y(2)+\left(-2+x-\frac{(-2+x)^4}{12}+\frac{(-2+x)^7}{504}\right)D(y)(2)+O(x^8)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 51

```
AsymptoticDSolveValue[y''[x]+(x-2)*y[x]==0,y[x],{x,2,7}]
```

$$y(x)\to c_1\left(\frac{1}{180}(x-2)^6-\frac{1}{6}(x-2)^3+1\right)+c_2\left(\frac{1}{504}(x-2)^7-\frac{1}{12}(x-2)^4+x-2\right)$$

3.28 problem 28

Internal problem ID [6916]
Internal file name [`OUTPUT/6159_Tuesday_August_09_2022_05_23_40_AM_6449168/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 17. Power series solutions. 17.5. Solutions Near an Ordinary Point. Exercises page 355
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x^2 - 2x + 2)\, y'' - 4(x - 1)\, y' + 6y = 0}$$

With the expansion point for the power series method at $x = 1$.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left((t+1)^2 - 2t\right)\left(\frac{d^2}{dt^2}y(t)\right) - 4t\left(\frac{d}{dt}y(t)\right) + 6y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f\big|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'\big|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (124) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (125) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}F_n|_{x_0,y_0,y_0'} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{4t\left(\frac{d}{dt}y(t)\right) - 6y(t)}{t^2+1} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y\left(t\right)}F_0 \\
&= \frac{6\left(\frac{d}{dt}y(t)\right)t^2 - 12y(t)\,t - 2\frac{d}{dt}y(t)}{\left(t^2+1\right)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y\left(t\right)}F_1 \\
&= 0 \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y\left(t\right)}F_2 \\
&= 0 \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y\left(t\right)}F_3 \\
&= 0 \\
F_5 &= \frac{dF_4}{dt} \\
&= \frac{\partial F_4}{\partial t} + \frac{\partial F_4}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_4}{\partial \frac{d}{dt}y\left(t\right)}F_4 \\
&= 0 \\
F_6 &= \frac{dF_5}{dt} \\
&= \frac{\partial F_5}{\partial t} + \frac{\partial F_5}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_5}{\partial \frac{d}{dt}y\left(t\right)}F_5 \\
&= 0
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -6y(0) \\
F_1 &= -2y'(0) \\
F_2 &= 0 \\
F_3 &= 0 \\
F_4 &= 0 \\
F_5 &= 0 \\
F_6 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y(t) = \left(-3t^2 + 1\right) y(0) + \left(t - \frac{1}{3}t^3\right) y'(0) + O\left(t^8\right)
$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
\left(\frac{d^2}{dt^2}y(t)\right)\left(t^2 + 1\right) - 4t\left(\frac{d}{dt}y(t)\right) + 6y(t) = 0
$$

Let the solution be represented as power series of the form

$$
y(t) = \sum_{n=0}^{\infty} a_n t^n
$$

Then

$$
\begin{aligned}
\frac{d}{dt}y(t) &= \sum_{n=1}^{\infty} n a_n t^{n-1} \\
\frac{d^2}{dt^2}y(t) &= \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)\left(t^2 + 1\right) - 4t\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + 6\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} t^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right) + \sum_{n=1}^{\infty} \left(-4n a_n t^n\right) + \left(\sum_{n=0}^{\infty} 6a_n t^n\right) = 0 \tag{2}
$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} t^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, t^n\right) + \sum_{n=1}^{\infty} (-4n a_n t^n) + \left(\sum_{n=0}^{\infty} 6 a_n t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 6a_0 = 0$$

$$a_2 = -3a_0$$

$n = 1$ gives

$$6a_3 + 2a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$n a_n(n-1) + (n+2)\, a_{n+2}(n+1) - 4n a_n + 6a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(n^2 - 5n + 6)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 0$$

For $n = 3$ the recurrence equation gives

$$20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$2a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$6a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

For $n = 6$ the recurrence equation gives

$$12a_6 + 56a_8 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = 0$$

For $n = 7$ the recurrence equation gives

$$20a_7 + 72a_9 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t - 3a_0 t^2 - \frac{1}{3} a_1 t^3 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(-3t^2 + 1\right) a_0 + \left(t - \frac{1}{3} t^3\right) a_1 + O\left(t^8\right) \tag{3}$$

At $t = 0$ the solution above becomes

$$y(t) = \left(-3t^2 + 1\right) c_1 + \left(t - \frac{1}{3} t^3\right) c_2 + O\left(t^8\right)$$

Replacing t in the above with the original independent variable $x$susing $t = x - 1$ results in

$$y = \left(-3(x-1)^2 + 1\right) y(1) + \left(x - 1 - \frac{(x-1)^3}{3}\right) y'(1) + O\left((x-1)^8\right)$$

Summary

The solution(s) found are the following

$$y = \left(-3(x-1)^2 + 1\right) y(1) + \left(x - 1 - \frac{(x-1)^3}{3}\right) y'(1) + O\left((x-1)^8\right) \tag{1}$$

Verification of solutions

$$y = \left(-3(x-1)^2 + 1\right) y(1) + \left(x - 1 - \frac{(x-1)^3}{3}\right) y'(1) + O\left((x-1)^8\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
Order:=8;
dsolve((x^2-2*x+2)*diff(y(x),x$2)-4*(x-1)*diff(y(x),x)+6*y(x)=0,y(x),type='series',x=1);
```

$$y(x) = \frac{(-x^3 + 3x^2 - 2)\, D(y)(1)}{3} - 3y(1)\left(x^2 - 2x + \frac{2}{3}\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 31

```
AsymptoticDSolveValue[(x^2-2*x+2)*y''[x]-4*(x-1)*y'[x]+6*y[x]==0,y[x],{x,1,7}]
```

$$y(x) \to c_1\left(1 - 3(x-1)^2\right) + c_2\left(-\frac{1}{3}(x-1)^3 + x - 1\right)$$

4 CHAPTER 18. Power series solutions. 18.4 Indicial Equation with Difference of Roots Nonintegral. Exercises page 365

4.1 problem 1

Internal problem ID [6917]
Internal file name [`OUTPUT/6160_Tuesday_August_09_2022_05_23_42_AM_74600233/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.4 Indicial Equation with Difference of Roots Nonintegral. Exercises page 365
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{2x(x+1)\,y''+3(x+1)\,y'-y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(2x^2+2x\right)y''+(3x+3)\,y'-y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{3}{2x}$$
$$q(x)=-\frac{1}{2x\,(x+1)}$$

Table 26: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{2x(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -1, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x(x+1)\, y'' + (3x+3)\, y' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &2x(x+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) \\ &+ (3x+3)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r) = \sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1} a_n (n+r)(n+r-1) + 3(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$2x^{-1+r} a_0 r(-1+r) + 3r a_0 x^{-1+r} = 0$$

Or

$$\left(2x^{-1+r} r(-1+r) + 3r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 + r\right) x^{-1+r} = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 + r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = -\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 + r\right) x^{-1+r} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_{n-1}(n+r-1)(n+r-2) + 2a_n(n+r)(n+r-1) + 3a_{n-1}(n+r-1) + 3a_n(n+r) - a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(2n+2r-3)\, a_{n-1}}{2n+1+2r} \tag{4}$$

Which for the root $r=0$ becomes

$$a_n = \frac{a_{n-1}(3-2n)}{2n+1} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r=0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n=1$, using the above recursive equation gives

$$a_1 = \frac{1-2r}{3+2r}$$

Which for the root $r=0$ becomes

$$a_1 = \frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-2r}{3+2r}$	$\frac{1}{3}$

For $n=2$, using the above recursive equation gives

$$a_2 = \frac{4r^2-1}{4r^2+16r+15}$$

Which for the root $r=0$ becomes

$$a_2 = -\frac{1}{15}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-2r}{3+2r}$	$\frac{1}{3}$
a_2	$\frac{4r^2-1}{4r^2+16r+15}$	$-\frac{1}{15}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-4r^2 + 1}{4r^2 + 24r + 35}$$

Which for the root $r = 0$ becomes

$$a_3 = \frac{1}{35}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-2r}{3+2r}$	$\frac{1}{3}$
a_2	$\frac{4r^2-1}{4r^2+16r+15}$	$-\frac{1}{15}$
a_3	$\frac{-4r^2+1}{4r^2+24r+35}$	$\frac{1}{35}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4r^2 - 1}{4r^2 + 32r + 63}$$

Which for the root $r = 0$ becomes

$$a_4 = -\frac{1}{63}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-2r}{3+2r}$	$\frac{1}{3}$
a_2	$\frac{4r^2-1}{4r^2+16r+15}$	$-\frac{1}{15}$
a_3	$\frac{-4r^2+1}{4r^2+24r+35}$	$\frac{1}{35}$
a_4	$\frac{4r^2-1}{4r^2+32r+63}$	$-\frac{1}{63}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-4r^2 + 1}{4r^2 + 40r + 99}$$

Which for the root $r = 0$ becomes

$$a_5 = \frac{1}{99}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-2r}{3+2r}$	$\frac{1}{3}$
a_2	$\frac{4r^2-1}{4r^2+16r+15}$	$-\frac{1}{15}$
a_3	$\frac{-4r^2+1}{4r^2+24r+35}$	$\frac{1}{35}$
a_4	$\frac{4r^2-1}{4r^2+32r+63}$	$-\frac{1}{63}$
a_5	$\frac{-4r^2+1}{4r^2+40r+99}$	$\frac{1}{99}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{4r^2 - 1}{4r^2 + 48r + 143}$$

Which for the root $r = 0$ becomes

$$a_6 = -\frac{1}{143}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-2r}{3+2r}$	$\frac{1}{3}$
a_2	$\frac{4r^2-1}{4r^2+16r+15}$	$-\frac{1}{15}$
a_3	$\frac{-4r^2+1}{4r^2+24r+35}$	$\frac{1}{35}$
a_4	$\frac{4r^2-1}{4r^2+32r+63}$	$-\frac{1}{63}$
a_5	$\frac{-4r^2+1}{4r^2+40r+99}$	$\frac{1}{99}$
a_6	$\frac{4r^2-1}{4r^2+48r+143}$	$-\frac{1}{143}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{-4r^2 + 1}{4r^2 + 56r + 195}$$

Which for the root $r = 0$ becomes

$$a_7 = \frac{1}{195}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-2r}{3+2r}$	$\frac{1}{3}$
a_2	$\frac{4r^2-1}{4r^2+16r+15}$	$-\frac{1}{15}$
a_3	$\frac{-4r^2+1}{4r^2+24r+35}$	$\frac{1}{35}$
a_4	$\frac{4r^2-1}{4r^2+32r+63}$	$-\frac{1}{63}$
a_5	$\frac{-4r^2+1}{4r^2+40r+99}$	$\frac{1}{99}$
a_6	$\frac{4r^2-1}{4r^2+48r+143}$	$-\frac{1}{143}$
a_7	$\frac{-4r^2+1}{4r^2+56r+195}$	$\frac{1}{195}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \dots \\ &= 1 + \frac{x}{3} - \frac{x^2}{15} + \frac{x^3}{35} - \frac{x^4}{63} + \frac{x^5}{99} - \frac{x^6}{143} + \frac{x^7}{195} + O(x^8) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &2b_{n-1}(n+r-1)(n+r-2) + 2b_n(n+r)(n+r-1) \\ &+ 3b_{n-1}(n+r-1) + 3(n+r)b_n - b_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{(2n+2r-3)b_{n-1}}{2n+1+2r} \tag{4}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_n = -\frac{(n-2)b_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{1 - 2r}{3 + 2r}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_1 = 1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1-2r}{3+2r}$	1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4r^2 - 1}{4r^2 + 16r + 15}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1-2r}{3+2r}$	1
b_2	$\frac{4r^2-1}{4r^2+16r+15}$	0

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{-4r^2 + 1}{4r^2 + 24r + 35}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1-2r}{3+2r}$	1
b_2	$\frac{4r^2-1}{4r^2+16r+15}$	0
b_3	$\frac{-4r^2+1}{4r^2+24r+35}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{4r^2 - 1}{4r^2 + 32r + 63}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1-2r}{3+2r}$	1
b_2	$\frac{4r^2-1}{4r^2+16r+15}$	0
b_3	$\frac{-4r^2+1}{4r^2+24r+35}$	0
b_4	$\frac{4r^2-1}{4r^2+32r+63}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{-4r^2 + 1}{4r^2 + 40r + 99}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1-2r}{3+2r}$	1
b_2	$\frac{4r^2-1}{4r^2+16r+15}$	0
b_3	$\frac{-4r^2+1}{4r^2+24r+35}$	0
b_4	$\frac{4r^2-1}{4r^2+32r+63}$	0
b_5	$\frac{-4r^2+1}{4r^2+40r+99}$	0

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{4r^2 - 1}{4r^2 + 48r + 143}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_6 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1-2r}{3+2r}$	1
b_2	$\frac{4r^2-1}{4r^2+16r+15}$	0
b_3	$\frac{-4r^2+1}{4r^2+24r+35}$	0
b_4	$\frac{4r^2-1}{4r^2+32r+63}$	0
b_5	$\frac{-4r^2+1}{4r^2+40r+99}$	0
b_6	$\frac{4r^2-1}{4r^2+48r+143}$	0

For $n = 7$, using the above recursive equation gives

$$b_7 = \frac{-4r^2 + 1}{4r^2 + 56r + 195}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_7 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1-2r}{3+2r}$	1
b_2	$\frac{4r^2-1}{4r^2+16r+15}$	0
b_3	$\frac{-4r^2+1}{4r^2+24r+35}$	0
b_4	$\frac{4r^2-1}{4r^2+32r+63}$	0
b_5	$\frac{-4r^2+1}{4r^2+40r+99}$	0
b_6	$\frac{4r^2-1}{4r^2+48r+143}$	0
b_7	$\frac{-4r^2+1}{4r^2+56r+195}$	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots\right) \\
&= \frac{1 + x + O(x^8)}{\sqrt{x}}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \left(1 + \frac{x}{3} - \frac{x^2}{15} + \frac{x^3}{35} - \frac{x^4}{63} + \frac{x^5}{99} - \frac{x^6}{143} + \frac{x^7}{195} + O(x^8)\right) + \frac{c_2(1 + x + O(x^8))}{\sqrt{x}}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \left(1 + \frac{x}{3} - \frac{x^2}{15} + \frac{x^3}{35} - \frac{x^4}{63} + \frac{x^5}{99} - \frac{x^6}{143} + \frac{x^7}{195} + O(x^8)\right) + \frac{c_2(1 + x + O(x^8))}{\sqrt{x}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \left(1 + \frac{x}{3} - \frac{x^2}{15} + \frac{x^3}{35} - \frac{x^4}{63} + \frac{x^5}{99} - \frac{x^6}{143} + \frac{x^7}{195} + O(x^8)\right) + \frac{c_2(1 + x + O(x^8))}{\sqrt{x}} \tag{1}
$$

Verification of solutions

$$y = c_1\left(1+\frac{x}{3}-\frac{x^2}{15}+\frac{x^3}{35}-\frac{x^4}{63}+\frac{x^5}{99}-\frac{x^6}{143}+\frac{x^7}{195}+O(x^8)\right)+\frac{c_2(1+x+O(x^8))}{\sqrt{x}}$$

Verified OK.

4.1.1 Maple step by step solution

Let's solve

$2x(1+x)\,y''+(3x+3)\,y'-y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{y}{2x(1+x)}-\frac{3y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{3y'}{2x}-\frac{y}{2x(1+x)}=0$

□ Check to see if x_0 is a regular singular point

  ○ Define functions

  $\left[P_2(x)=\frac{3}{2x}, P_3(x)=-\frac{1}{2x(1+x)}\right]$

  ○ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1}=0$

  ○ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  ○ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $2x(1+x)\,y''+(3x+3)\,y'-y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(2u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+3u\left(\frac{d}{du}y(u)\right)-y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  ○ Convert $u\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion

  $u\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r}$

  ○ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  ○ Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r(-1+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}(-2a_{k+1}(k+r+1)\,(k+r)+a_k(k+r+1)\,(2k+2r-1))\,u^{k+r}\right)=$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-1+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{0,1\}$

- Each term in the series must be 0, giving the recursion relation

  $2(k+r+1)\left(\left(k+r-\frac{1}{2}\right)a_k-a_{k+1}(k+r)\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{(2k+2r-1)a_k}{2(k+r)}$

- Recursion relation for $r=0$

  $a_{k+1}=\frac{(2k-1)a_k}{2k}$

- Solution for $r=0$

  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+1}=\frac{(2k-1)a_k}{2k}\right]$

- Revert the change of variables $u=1+x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^k\ , a_{k+1} = \frac{(2k-1)a_k}{2k}\right]$$

- Recursion relation for $r = 1$

$$a_{k+1} = \frac{(2k+1)a_k}{2(k+1)}$$

- Solution for $r = 1$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+1}, a_{k+1} = \frac{(2k+1)a_k}{2(k+1)}\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+1}\ , a_{k+1} = \frac{(2k+1)a_k}{2(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (1+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k+1}\right), a_{k+1} = \frac{(2k-1)a_k}{2k}, b_{k+1} = \frac{(2k+1)b_k}{2(k+1)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 40

```
Order:=8;
dsolve(2*x*(x+1)*diff(y(x),x$2)+3*(x+1)*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1(1 + x + O(x^8))}{\sqrt{x}} + c_2\left(1 + \frac{1}{3}x - \frac{1}{15}x^2 + \frac{1}{35}x^3 - \frac{1}{63}x^4 + \frac{1}{99}x^5 - \frac{1}{143}x^6 + \frac{1}{195}x^7 + O(x^8)\right)$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 67

```
AsymptoticDSolveValue[2*x*(x+1)*y''[x]+3*(x+1)*y'[x]-y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(\frac{x^7}{195}-\frac{x^6}{143}+\frac{x^5}{99}-\frac{x^4}{63}+\frac{x^3}{35}-\frac{x^2}{15}+\frac{x}{3}+1\right)+\frac{c_2(x+1)}{\sqrt{x}}$$

4.2 problem 2

Internal problem ID [6918]
Internal file name [`OUTPUT/6161_Tuesday_August_09_2022_05_23_45_AM_84888398/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.4 Indicial Equation with Difference of Roots Nonintegral. Exercises page 365
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' + 4xy' + \left(4x^2 - 1\right) y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4x^2y'' + 4xy' + \left(4x^2 - 1\right) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{4x^2 - 1}{4x^2}$$

Table 28: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{4x^2-1}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2y'' + 4xy' + \left(4x^2 - 1\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + 4x \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(4x^2 - 1\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n(n+r)\,(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n(n+r)\right) \\ +\left(\sum_{n=0}^{\infty} 4x^{n+r+2} a_n\right)+\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 4x^{n+r+2} a_n=\sum_{n=2}^{\infty} 4a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n(n+r)\,(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n(n+r)\right) \\ +\left(\sum_{n=2}^{\infty} 4a_{n-2} x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r} a_n(n+r)\,(n+r-1)+4x^{n+r} a_n(n+r)-a_n x^{n+r}=0$$

When $n=0$ the above becomes

$$4x^r a_0 r(-1+r)+4x^r a_0 r-a_0 x^r=0$$

Or

$$\left(4x^r r(-1+r)+4x^r r-x^r\right) a_0=0$$

Since $a_0\neq 0$ then the above simplifies to

$$\left(4r^2-1\right) x^r=0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2-1=0$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= \frac{1}{2} \\
r_2 &= -\frac{1}{2}
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
\left(4r^2 - 1\right) x^r = 0
$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sqrt{x} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\
y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{\sqrt{x}}
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}} \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}} \right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
4a_n(n+r)(n+r-1) + 4a_n(n+r) + 4a_{n-2} - a_n = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{4a_{n-2}}{4n^2 + 8nr + 4r^2 - 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{a_{n-2}}{n (n+1)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{4}{4r^2 + 16r + 15}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{6}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{(4r^2 + 16r + 15)(4r^2 + 32r + 63)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{120}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{120}$
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = -\frac{64}{(4r^2 + 16r + 15)(4r^2 + 32r + 63)(4r^2 + 48r + 143)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_6 = -\frac{1}{5040}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{120}$
a_5	0	0
a_6	$-\frac{64}{(4r^2+16r+15)(4r^2+32r+63)(4r^2+48r+143)}$	$-\frac{1}{5040}$

For $n = 7$, using the above recursive equation gives

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{120}$
a_5	0	0
a_6	$-\frac{64}{(4r^2+16r+15)(4r^2+32r+63)(4r^2+48r+143)}$	$-\frac{1}{5040}$
a_7	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\ &= \sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040} + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= 0 \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} 0 &= \lim_{r \to -\frac{1}{2}} 0 \\ &= 0 \end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$4b_n(n+r)(n+r-1) + 4b_n(n+r) + 4b_{n-2} - b_n = 0 \tag{4}$$

Which for for the root $r = -\frac{1}{2}$ becomes

$$4b_n\left(n - \frac{1}{2}\right)\left(n - \frac{3}{2}\right) + 4b_n\left(n - \frac{1}{2}\right) + 4b_{n-2} - b_n = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{4b_{n-2}}{4n^2 + 8nr + 4r^2 - 1} \tag{5}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_n = -\frac{4b_{n-2}}{4n^2 - 4n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{4}{4r^2 + 16r + 15}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_2 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16}{(4r^2 + 16r + 15)(4r^2 + 32r + 63)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = \frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{24}$
b_5	0	0

For $n = 6$, using the above recursive equation gives

$$b_6 = -\frac{64}{(4r^2 + 16r + 15)(4r^2 + 32r + 63)(4r^2 + 48r + 143)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_6 = -\frac{1}{720}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{24}$
b_5	0	0
b_6	$-\frac{64}{(4r^2+16r+15)(4r^2+32r+63)(4r^2+48r+143)}$	$-\frac{1}{720}$

For $n = 7$, using the above recursive equation gives

$$b_7 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{24}$
b_5	0	0
b_6	$-\frac{64}{(4r^2+16r+15)(4r^2+32r+63)(4r^2+48r+143)}$	$-\frac{1}{720}$
b_7	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= \sqrt{x}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots\right) \\ &= \frac{1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + O(x^8)}{\sqrt{x}} \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\sqrt{x}\left(1-\frac{x^2}{6}+\frac{x^4}{120}-\frac{x^6}{5040}+O(x^8)\right)+\frac{c_2\left(1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+O(x^8)\right)}{\sqrt{x}}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\sqrt{x}\left(1-\frac{x^2}{6}+\frac{x^4}{120}-\frac{x^6}{5040}+O(x^8)\right)+\frac{c_2\left(1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+O(x^8)\right)}{\sqrt{x}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\sqrt{x}\left(1-\frac{x^2}{6}+\frac{x^4}{120}-\frac{x^6}{5040}+O(x^8)\right)+\frac{c_2\left(1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+O(x^8)\right)}{\sqrt{x}} \tag{1}
$$

Verification of solutions

$$
y = c_1\sqrt{x}\left(1-\frac{x^2}{6}+\frac{x^4}{120}-\frac{x^6}{5040}+O(x^8)\right)+\frac{c_2\left(1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+O(x^8)\right)}{\sqrt{x}}
$$

Verified OK.

4.2.1 Maple step by step solution

Let's solve

$4x^2y''+4xy'+(4x^2-1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(4x^2-1)y}{4x^2}-\frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y'}{x}+\frac{(4x^2-1)y}{4x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (4x^2 - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)(-1+2r)=0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0

  $a_1(3+2r)(1+2r)=0$
- Solve for the dependent coefficient(s)

  $a_1=0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2+8kr+4r^2-1)+4a_{k-2}=0$
- Shift index using $k->k+2$

  $a_{k+2}\left(4(k+2)^2+8(k+2)r+4r^2-1\right)+4a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r=-\frac{1}{2}$

  $a_{k+2}=-\frac{4a_k}{4k^2+12k+8}$
- Solution for $r=-\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2}=-\frac{4a_k}{4k^2+12k+8}, a_1=0\right]$
- Recursion relation for $r=\frac{1}{2}$

  $a_{k+2}=-\frac{4a_k}{4k^2+20k+24}$
- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2}=-\frac{4a_k}{4k^2+20k+24}, a_1=0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2}=-\frac{4a_k}{4k^2+12k+8}, a_1=0, b_{k+2}=-\frac{4b_k}{4k^2+20k+24}, b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 39

```
Order:=8;
dsolve(4*x^2*diff(y(x),x$2)+4*x*diff(y(x),x)+(4*x^2-1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x\left(1-\frac{1}{6}x^2+\frac{1}{120}x^4-\frac{1}{5040}x^6+\mathrm{O}\left(x^8\right)\right)+c_2\left(1-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{720}x^6+\mathrm{O}\left(x^8\right)\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 76

```
AsymptoticDSolveValue[4*x^2*y''[x]+4*x*y'[x]+(4*x^2-1)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(-\frac{x^{11/2}}{720}+\frac{x^{7/2}}{24}-\frac{x^{3/2}}{2}+\frac{1}{\sqrt{x}}\right)+c_2\left(-\frac{x^{13/2}}{5040}+\frac{x^{9/2}}{120}-\frac{x^{5/2}}{6}+\sqrt{x}\right)$$

4.3 problem 3

Internal problem ID [6919]
Internal file name [`OUTPUT/6162_Tuesday_August_09_2022_05_23_47_AM_91692522/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.4 Indicial Equation with Difference of Roots Nonintegral. Exercises page 365
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' + 4xy' - \left(4x^2 + 1\right) y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4x^2y'' + 4xy' + \left(-4x^2 - 1\right) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -\frac{4x^2 + 1}{4x^2}$$

Table 30: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{4x^2+1}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2y'' + 4xy' + \left(-4x^2 - 1\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2 \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) + 4x \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + \left(-4x^2 - 1\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-4x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-4x^{n+r+2} a_n\right) = \sum_{n=2}^{\infty} \left(-4a_{n-2} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)\right) + \sum_{n=2}^{\infty} \left(-4a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$4x^{n+r} a_n (n+r)(n+r-1) + 4x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$4x^r a_0 r(-1+r) + 4x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(4x^r r(-1+r) + 4x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2 - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = -\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(4r^2 - 1)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sqrt{x} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{\sqrt{x}}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$4a_n(n + r)(n + r - 1) + 4a_n(n + r) - 4a_{n-2} - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{4a_{n-2}}{4n^2 + 8nr + 4r^2 - 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{a_{n-2}}{n(n+1)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4}{4r^2 + 16r + 15}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{4}{4r^2+16r+15}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{4}{4r^2+16r+15}$	$\frac{1}{6}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{(4r^2 + 16r + 15)(4r^2 + 32r + 63)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{4}{4r^2+16r+15}$	$\frac{1}{6}$
a_3	0	0
a_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{120}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{4}{4r^2+16r+15}$	$\frac{1}{6}$
a_3	0	0
a_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{120}$
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{64}{(4r^2 + 16r + 15)(4r^2 + 32r + 63)(4r^2 + 48r + 143)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_6 = \frac{1}{5040}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{4}{4r^2+16r+15}$	$\frac{1}{6}$
a_3	0	0
a_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{120}$
a_5	0	0
a_6	$\frac{64}{(4r^2+16r+15)(4r^2+32r+63)(4r^2+48r+143)}$	$\frac{1}{5040}$

For $n = 7$, using the above recursive equation gives

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{4}{4r^2+16r+15}$	$\frac{1}{6}$
a_3	0	0
a_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{120}$
a_5	0	0
a_6	$\frac{64}{(4r^2+16r+15)(4r^2+32r+63)(4r^2+48r+143)}$	$\frac{1}{5040}$
a_7	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\big(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\big) \\ &= \sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{120} + \frac{x^6}{5040} + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= 0 \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} 0 &= \lim_{r\to -\frac{1}{2}} 0 \\ &= 0 \end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$4b_n(n+r)(n+r-1) + 4b_n(n+r) - 4b_{n-2} - b_n = 0 \tag{4}$$

Which for for the root $r = -\frac{1}{2}$ becomes

$$4b_n\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right) + 4b_n\left(n-\frac{1}{2}\right) - 4b_{n-2} - b_n = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = \frac{4b_{n-2}}{4n^2 + 8nr + 4r^2 - 1} \tag{5}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_n = \frac{4b_{n-2}}{4n^2 - 4n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4}{4r^2 + 16r + 15}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_2 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{4}{4r^2+16r+15}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{4}{4r^2+16r+15}$	$\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16}{(4r^2 + 16r + 15)(4r^2 + 32r + 63)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = \frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{4}{4r^2+16r+15}$	$\frac{1}{2}$
b_3	0	0
b_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{4}{4r^2+16r+15}$	$\frac{1}{2}$
b_3	0	0
b_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{24}$
b_5	0	0

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{64}{(4r^2 + 16r + 15)(4r^2 + 32r + 63)(4r^2 + 48r + 143)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_6 = \frac{1}{720}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{4}{4r^2+16r+15}$	$\frac{1}{2}$
b_3	0	0
b_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{24}$
b_5	0	0
b_6	$\frac{64}{(4r^2+16r+15)(4r^2+32r+63)(4r^2+48r+143)}$	$\frac{1}{720}$

For $n = 7$, using the above recursive equation gives

$$b_7 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{4}{4r^2+16r+15}$	$\frac{1}{2}$
b_3	0	0
b_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{24}$
b_5	0	0
b_6	$\frac{64}{(4r^2+16r+15)(4r^2+32r+63)(4r^2+48r+143)}$	$\frac{1}{720}$
b_7	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= \sqrt{x}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \ldots\right) \\ &= \frac{1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + O(x^8)}{\sqrt{x}} \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \sqrt{x}\left(1+\frac{x^2}{6}+\frac{x^4}{120}+\frac{x^6}{5040}+O(x^8)\right)+\frac{c_2\left(1+\frac{x^2}{2}+\frac{x^4}{24}+\frac{x^6}{720}+O(x^8)\right)}{\sqrt{x}}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \sqrt{x}\left(1+\frac{x^2}{6}+\frac{x^4}{120}+\frac{x^6}{5040}+O(x^8)\right)+\frac{c_2\left(1+\frac{x^2}{2}+\frac{x^4}{24}+\frac{x^6}{720}+O(x^8)\right)}{\sqrt{x}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \sqrt{x}\left(1+\frac{x^2}{6}+\frac{x^4}{120}+\frac{x^6}{5040}+O(x^8)\right)+\frac{c_2\left(1+\frac{x^2}{2}+\frac{x^4}{24}+\frac{x^6}{720}+O(x^8)\right)}{\sqrt{x}} \tag{1}
$$

Verification of solutions

$$
y = c_1 \sqrt{x}\left(1+\frac{x^2}{6}+\frac{x^4}{120}+\frac{x^6}{5040}+O(x^8)\right)+\frac{c_2\left(1+\frac{x^2}{2}+\frac{x^4}{24}+\frac{x^6}{720}+O(x^8)\right)}{\sqrt{x}}
$$

Verified OK.

4.3.1 Maple step by step solution

Let's solve

$4x^2y'' + 4xy' + (-4x^2 - 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(4x^2+1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} - \frac{(4x^2+1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = -\frac{4x^2+1}{4x^2}\right]$
- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 1$
- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{1}{4}$
- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (-4x^2 - 1)\, y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$
- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1 + 2r)\,(-1 + 2r)\, x^r + a_1(3 + 2r)\,(1 + 2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k + 2r + 1)\,(2k + 2r - 1) - 4a_{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)(-1+2r)=0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0

  $a_1(3+2r)(1+2r)=0$
- Solve for the dependent coefficient(s)

  $a_1=0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2+8kr+4r^2-1)-4a_{k-2}=0$
- Shift index using $k->k+2$

  $a_{k+2}\left(4(k+2)^2+8(k+2)r+4r^2-1\right)-4a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r=-\frac{1}{2}$

  $a_{k+2}=\frac{4a_k}{4k^2+12k+8}$
- Solution for $r=-\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2}=\frac{4a_k}{4k^2+12k+8}, a_1=0\right]$
- Recursion relation for $r=\frac{1}{2}$

  $a_{k+2}=\frac{4a_k}{4k^2+20k+24}$
- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2}=\frac{4a_k}{4k^2+20k+24}, a_1=0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2}=\frac{4a_k}{4k^2+12k+8}, a_1=0, b_{k+2}=\frac{4b_k}{4k^2+20k+24}, b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 39

```
Order:=8;
dsolve(4*x^2*diff(y(x),x$2)+4*x*diff(y(x),x)-(4*x^2+1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x \left(1+\frac{1}{6}x^2+\frac{1}{120}x^4+\frac{1}{5040}x^6+\mathrm{O}\left(x^8\right)\right)+c_2\left(1+\frac{1}{2}x^2+\frac{1}{24}x^4+\frac{1}{720}x^6+\mathrm{O}\left(x^8\right)\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 76

```
AsymptoticDSolveValue[4*x^2*y''[x]+4*x*y'[x]-(4*x^2+1)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(\frac{x^{11/2}}{720}+\frac{x^{7/2}}{24}+\frac{x^{3/2}}{2}+\frac{1}{\sqrt{x}}\right)+c_2\left(\frac{x^{13/2}}{5040}+\frac{x^{9/2}}{120}+\frac{x^{5/2}}{6}+\sqrt{x}\right)$$

4.4 problem 4

Internal problem ID [6920]
Internal file name [`OUTPUT/6163_Tuesday_August_09_2022_05_23_49_AM_59709833/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.4 Indicial Equation with Difference of Roots Nonintegral. Exercises page 365
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{4xy'' + 3y' + 3y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4xy'' + 3y' + 3y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{3}{4x}$$
$$q(x) = \frac{3}{4x}$$

Table 32: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{4x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{3}{4x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4xy'' + 3y' + 3y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + 3\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + 3\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 3a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 3a_n x^{n+r} = \sum_{n=1}^{\infty} 3a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 4x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r-1} a_n (n+r)(n+r-1) + 3(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$4x^{-1+r} a_0 r(-1+r) + 3r a_0 x^{-1+r} = 0$$

Or

$$\left(4x^{-1+r} r(-1+r) + 3r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-1+4r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{4}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(-1+4r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{4}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{4}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$4a_n(n+r)(n+r-1) + 3a_n(n+r) + 3a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{3a_{n-1}}{4n^2 + 8nr + 4r^2 - n - r} \tag{4}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_n = -\frac{3a_{n-1}}{4n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{4}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{3}{4r^2 + 7r + 3}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_1 = -\frac{3}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{4r^2+7r+3}$	$-\frac{3}{5}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{9}{(4r^2 + 7r + 3)(4r^2 + 15r + 14)}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_2 = \frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{4r^2+7r+3}$	$-\frac{3}{5}$
a_2	$\frac{9}{(4r^2+7r+3)(4r^2+15r+14)}$	$\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{27}{(4r^2 + 7r + 3)(4r^2 + 15r + 14)(4r^2 + 23r + 33)}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_3 = -\frac{1}{130}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{4r^2+7r+3}$	$-\frac{3}{5}$
a_2	$\frac{9}{(4r^2+7r+3)(4r^2+15r+14)}$	$\frac{1}{10}$
a_3	$-\frac{27}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)}$	$-\frac{1}{130}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{81}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)(4r^2+31r+60)}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_4 = \frac{3}{8840}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{4r^2+7r+3}$	$-\frac{3}{5}$
a_2	$\frac{9}{(4r^2+7r+3)(4r^2+15r+14)}$	$\frac{1}{10}$
a_3	$-\frac{27}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)}$	$-\frac{1}{130}$
a_4	$\frac{81}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)(4r^2+31r+60)}$	$\frac{3}{8840}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{243}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)(4r^2+31r+60)(4r^2+39r+95)}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_5 = -\frac{3}{309400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{4r^2+7r+3}$	$-\frac{3}{5}$
a_2	$\frac{9}{(4r^2+7r+3)(4r^2+15r+14)}$	$\frac{1}{10}$
a_3	$-\frac{27}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)}$	$-\frac{1}{130}$
a_4	$\frac{81}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)(4r^2+31r+60)}$	$\frac{3}{8840}$
a_5	$-\frac{243}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)(4r^2+31r+60)(4r^2+39r+95)}$	$-\frac{3}{309400}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{729}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)(4r^2+31r+60)(4r^2+39r+95)(4r^2+47r+138}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_6 = \frac{3}{15470000}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{4r^2+7r+3}$	$-\frac{3}{5}$
a_2	$\frac{9}{(4r^2+7r+3)(4r^2+15r+14)}$	$\frac{1}{10}$
a_3	$-\frac{27}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)}$	$-\frac{1}{130}$
a_4	$\frac{81}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)(4r^2+31r+60)}$	$\frac{3}{8840}$
a_5	$-\frac{243}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)(4r^2+31r+60)(4r^2+39r+95)}$	$-\frac{3}{309400}$
a_6	$\frac{729}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)(4r^2+31r+60)(4r^2+39r+95)(4r^2+47r+138)}$	$\frac{3}{15470000}$

For $n = 7$, using the above recursive equation gives

$$a_7 = -\frac{2187}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)(4r^2+31r+60)(4r^2+39r+95)(4r^2+47r+1}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_7 = -\frac{9}{3140410000}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{4r^2+7r+3}$	$-\frac{3}{5}$
a_2	$\frac{9}{(4r^2+7r+3)(4r^2+15r+14)}$	$\frac{1}{10}$
a_3	$-\frac{27}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)}$	$-\frac{1}{130}$
a_4	$\frac{81}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)(4r^2+31r+60)}$	$\frac{3}{8840}$
a_5	$-\frac{243}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)(4r^2+31r+60)(4r^2+39r+95)}$	$-\frac{3}{309400}$
a_6	$\frac{729}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)(4r^2+31r+60)(4r^2+39r+95)(4r^2+47r+138)}$	$\frac{3}{15470000}$
a_7	$-\frac{2187}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)(4r^2+31r+60)(4r^2+39r+95)(4r^2+47r+138)(4r^2+55r+189)}$	$-\frac{9}{3140410000}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^{\frac{1}{4}}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \dots\right) \\
&= x^{\frac{1}{4}}\left(1 - \frac{3x}{5} + \frac{x^2}{10} - \frac{x^3}{130} + \frac{3x^4}{8840} - \frac{3x^5}{309400} + \frac{3x^6}{15470000} - \frac{9x^7}{3140410000} + O(x^8)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
4b_n(n+r)(n+r-1) + 3(n+r)b_n + 3b_{n-1} = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = -\frac{3b_{n-1}}{4n^2 + 8nr + 4r^2 - n - r} \tag{4}
$$

Which for the root $r = 0$ becomes

$$
b_n = -\frac{3b_{n-1}}{n(4n-1)} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{3}{4r^2 + 7r + 3}$$

Which for the root $r = 0$ becomes

$$b_1 = -1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{4r^2+7r+3}$	-1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{9}{(4r^2 + 7r + 3)(4r^2 + 15r + 14)}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{3}{14}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{4r^2+7r+3}$	-1
b_2	$\frac{9}{(4r^2+7r+3)(4r^2+15r+14)}$	$\frac{3}{14}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{27}{(4r^2 + 7r + 3)(4r^2 + 15r + 14)(4r^2 + 23r + 33)}$$

Which for the root $r = 0$ becomes

$$b_3 = -\frac{3}{154}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{4r^2+7r+3}$	-1
b_2	$\frac{9}{(4r^2+7r+3)(4r^2+15r+14)}$	$\frac{3}{14}$
b_3	$-\frac{27}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)}$	$-\frac{3}{154}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{81}{(4r^2 + 7r + 3)(4r^2 + 15r + 14)(4r^2 + 23r + 33)(4r^2 + 31r + 60)}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{3}{3080}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{4r^2+7r+3}$	-1
b_2	$\frac{9}{(4r^2+7r+3)(4r^2+15r+14)}$	$\frac{3}{14}$
b_3	$-\frac{27}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)}$	$-\frac{3}{154}$
b_4	$\frac{81}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)(4r^2+31r+60)}$	$\frac{3}{3080}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{243}{(4r^2 + 7r + 3)(4r^2 + 15r + 14)(4r^2 + 23r + 33)(4r^2 + 31r + 60)(4r^2 + 39r + 95)}$$

Which for the root $r = 0$ becomes

$$b_5 = -\frac{9}{292600}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{4r^2+7r+3}$	-1
b_2	$\frac{9}{(4r^2+7r+3)(4r^2+15r+14)}$	$\frac{3}{14}$
b_3	$-\frac{27}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)}$	$-\frac{3}{154}$
b_4	$\frac{81}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)(4r^2+31r+60)}$	$\frac{3}{3080}$
b_5	$-\frac{243}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)(4r^2+31r+60)(4r^2+39r+95)}$	$-\frac{9}{292600}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{729}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)(4r^2+31r+60)(4r^2+39r+95)(4r^2+47r+138}$$

Which for the root $r = 0$ becomes

$$b_6 = \frac{9}{13459600}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{4r^2+7r+3}$	-1
b_2	$\frac{9}{(4r^2+7r+3)(4r^2+15r+14)}$	$\frac{3}{14}$
b_3	$-\frac{27}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)}$	$-\frac{3}{154}$
b_4	$\frac{81}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)(4r^2+31r+60)}$	$\frac{3}{3080}$
b_5	$-\frac{243}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)(4r^2+31r+60)(4r^2+39r+95)}$	$-\frac{9}{292600}$
b_6	$\frac{729}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)(4r^2+31r+60)(4r^2+39r+95)(4r^2+47r+138)}$	$\frac{9}{13459600}$

For $n = 7$, using the above recursive equation gives

$$b_7 = -\frac{2187}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)(4r^2+31r+60)(4r^2+39r+95)(4r^2+47r+13}$$

Which for the root $r = 0$ becomes

$$b_7 = -\frac{1}{94217200}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{4r^2+7r+3}$	-1
b_2	$\frac{9}{(4r^2+7r+3)(4r^2+15r+14)}$	$\frac{3}{14}$
b_3	$-\frac{27}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)}$	$-\frac{3}{154}$
b_4	$\frac{81}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)(4r^2+31r+60)}$	$\frac{3}{3080}$
b_5	$-\frac{243}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)(4r^2+31r+60)(4r^2+39r+95)}$	$-\frac{9}{292600}$
b_6	$\frac{729}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)(4r^2+31r+60)(4r^2+39r+95)(4r^2+47r+138)}$	$\frac{9}{13459600}$
b_7	$-\frac{2187}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)(4r^2+31r+60)(4r^2+39r+95)(4r^2+47r+138)(4r^2+55r+189)}$	$-\frac{1}{94217200}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\
&= 1 - x + \frac{3x^2}{14} - \frac{3x^3}{154} + \frac{3x^4}{3080} - \frac{9x^5}{292600} + \frac{9x^6}{13459600} - \frac{x^7}{94217200} + O(x^8)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{1}{4}} \left(1 - \frac{3x}{5} + \frac{x^2}{10} - \frac{x^3}{130} + \frac{3x^4}{8840} - \frac{3x^5}{309400} + \frac{3x^6}{15470000} - \frac{9x^7}{3140410000} + O(x^8)\right) \\
&\quad + c_2 \left(1 - x + \frac{3x^2}{14} - \frac{3x^3}{154} + \frac{3x^4}{3080} - \frac{9x^5}{292600} + \frac{9x^6}{13459600} - \frac{x^7}{94217200} + O(x^8)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{1}{4}} \left(1 - \frac{3x}{5} + \frac{x^2}{10} - \frac{x^3}{130} + \frac{3x^4}{8840} - \frac{3x^5}{309400} + \frac{3x^6}{15470000} - \frac{9x^7}{3140410000} + O(x^8)\right) \\
&\quad + c_2 \left(1 - x + \frac{3x^2}{14} - \frac{3x^3}{154} + \frac{3x^4}{3080} - \frac{9x^5}{292600} + \frac{9x^6}{13459600} - \frac{x^7}{94217200} + O(x^8)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1 x^{\frac{1}{4}}\left(1-\frac{3x}{5}+\frac{x^2}{10}-\frac{x^3}{130}+\frac{3x^4}{8840}-\frac{3x^5}{309400}+\frac{3x^6}{15470000}-\frac{9x^7}{3140410000}+O(x^8)\right) + c_2\left(1-x+\frac{3x^2}{14}-\frac{3x^3}{154}+\frac{3x^4}{3080}-\frac{9x^5}{292600}+\frac{9x^6}{13459600}-\frac{x^7}{94217200}+O(x^8)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{4}}\left(1-\frac{3x}{5}+\frac{x^2}{10}-\frac{x^3}{130}+\frac{3x^4}{8840}-\frac{3x^5}{309400}+\frac{3x^6}{15470000}-\frac{9x^7}{3140410000}+O(x^8)\right) + c_2\left(1-x+\frac{3x^2}{14}-\frac{3x^3}{154}+\frac{3x^4}{3080}-\frac{9x^5}{292600}+\frac{9x^6}{13459600}-\frac{x^7}{94217200}+O(x^8)\right)$$

Verified OK.

4.4.1 Maple step by step solution

Let's solve

$4xy''+3y'+3y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{3y}{4x}-\frac{3y'}{4x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{3y'}{4x}+\frac{3y}{4x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{3}{4x}, P_3(x)=\frac{3}{4x}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{3}{4}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

  ◦ $x=0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$4xy'' + 3y' + 3y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert y' to series expansion

$y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

- ○ Shift index using $k->k+1$

$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$

- ○ Convert $x \cdot y''$ to series expansion

$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

- ○ Shift index using $k->k+1$

$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

Rewrite ODE with series expansions

$$a_0 r(-1+4r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(4k+3+4r) + 3a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-1+4r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{0, \frac{1}{4}\right\}$

- Each term in the series must be 0, giving the recursion relation

$4(k+1+r)\left(k+\frac{3}{4}+r\right) a_{k+1} + 3a_k = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = -\frac{3a_k}{(k+1+r)(4k+3+4r)}$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{3a_k}{(k+1)(4k+3)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{3a_k}{(k+1)(4k+3)}\right]$$

- Recursion relation for $r = \frac{1}{4}$

$$a_{k+1} = -\frac{3a_k}{\left(k+\frac{5}{4}\right)(4k+4)}$$

- Solution for $r = \frac{1}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+1} = -\frac{3a_k}{\left(k+\frac{5}{4}\right)(4k+4)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{4}}\right), a_{k+1} = -\frac{3a_k}{(k+1)(4k+3)}, b_{k+1} = -\frac{3b_k}{\left(k+\frac{5}{4}\right)(4k+4)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 52

```
Order:=8;
dsolve(4*x*diff(y(x),x$2)+3*diff(y(x),x)+3*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{\frac{1}{4}} \left(1 - \frac{3}{5}x + \frac{1}{10}x^2 - \frac{1}{130}x^3 + \frac{3}{8840}x^4 - \frac{3}{309400}x^5 + \frac{3}{15470000}x^6 - \frac{9}{3140410000}x^7 + \mathrm{O}\left(x^8\right)\right) + c_2 \left(1 - x + \frac{3}{14}x^2 - \frac{3}{154}x^3 + \frac{3}{3080}x^4 - \frac{9}{292600}x^5 + \frac{9}{13459600}x^6 - \frac{1}{94217200}x^7 + \mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 111

```
AsymptoticDSolveValue[4*x*y''[x]+3*y'[x]+3*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1 \sqrt[4]{x} \left(-\frac{9x^7}{3140410000} + \frac{3x^6}{15470000} - \frac{3x^5}{309400} + \frac{3x^4}{8840} - \frac{x^3}{130} + \frac{x^2}{10} - \frac{3x}{5} + 1\right) + c_2 \left(-\frac{x^7}{94217200} + \frac{9x^6}{13459600} - \frac{9x^5}{292600} + \frac{3x^4}{3080} - \frac{3x^3}{154} + \frac{3x^2}{14} - x + 1\right)$$

4.5 problem 5

Internal problem ID [6921]
Internal file name [`OUTPUT/6164_Tuesday_August_09_2022_05_23_52_AM_82600808/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.4 Indicial Equation with Difference of Roots Nonintegral. Exercises page 365
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2(1-x)\,y''-x(1+7x)\,y'+y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-2x^3+2x^2\right)y''+\left(-7x^2-x\right)y'+y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=\frac{1+7x}{2x\,(x-1)}\\ q(x)&=-\frac{1}{2x^2\,(x-1)}\end{aligned}$$

Table 34: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1+7x}{2x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

$q(x) = -\frac{1}{2x^2(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 1, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-2y''x^2(x-1) + \left(-7x^2 - x\right) y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x^2(x-1) + \left(-7x^2 - x\right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty}\left(-2x^{1+n+r}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}2x^{n+r}a_n(n+r)(n+r-1)\right) + \sum_{n=0}^{\infty}\left(-7x^{1+n+r}a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-x^{n+r}a_n(n+r)\right)+\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-2x^{1+n+r}a_n(n+r)(n+r-1)\right)=\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)(n+r-2)\,x^{n+r}\right)$$

$$\sum_{n=0}^{\infty}\left(-7x^{1+n+r}a_n(n+r)\right)=\sum_{n=1}^{\infty}\left(-7a_{n-1}(n+r-1)\,x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)(n+r-2)\,x^{n+r}\right)+\left(\sum_{n=0}^{\infty}2x^{n+r}a_n(n+r)(n+r-1)\right)+\sum_{n=1}^{\infty}\left(-7a_{n-1}(n+r-1)\,x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-x^{n+r}a_n(n+r)\right)+\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r}a_n(n+r)(n+r-1)-x^{n+r}a_n(n+r)+a_n x^{n+r}=0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r)-x^r a_0 r+a_0 x^r=0$$

Or

$$\left(2x^r r(-1+r)-x^r r+x^r\right)a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 3r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = \frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} -2a_{n-1}(n+r-1)(n+r-2) + 2a_n(n+r)(n+r-1) \\ - 7a_{n-1}(n+r-1) - a_n(n+r) + a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{(2n+2r+3)\, a_{n-1}}{2n+2r-1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{(2n+5)\, a_{n-1}}{2n+1} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{5+2r}{1+2r}$$

Which for the root $r = 1$ becomes

$$a_1 = \frac{7}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{5+2r}{1+2r}$	$\frac{7}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4r^2+24r+35}{4r^2+8r+3}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{21}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{5+2r}{1+2r}$	$\frac{7}{3}$
a_2	$\frac{4r^2+24r+35}{4r^2+8r+3}$	$\frac{21}{5}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{4r^2 + 32r + 63}{4r^2 + 8r + 3}$$

Which for the root $r = 1$ becomes

$$a_3 = \frac{33}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{5+2r}{1+2r}$	$\frac{7}{3}$
a_2	$\frac{4r^2+24r+35}{4r^2+8r+3}$	$\frac{21}{5}$
a_3	$\frac{4r^2+32r+63}{4r^2+8r+3}$	$\frac{33}{5}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4r^2 + 40r + 99}{4r^2 + 8r + 3}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{143}{15}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{5+2r}{1+2r}$	$\frac{7}{3}$
a_2	$\frac{4r^2+24r+35}{4r^2+8r+3}$	$\frac{21}{5}$
a_3	$\frac{4r^2+32r+63}{4r^2+8r+3}$	$\frac{33}{5}$
a_4	$\frac{4r^2+40r+99}{4r^2+8r+3}$	$\frac{143}{15}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{4r^2 + 48r + 143}{4r^2 + 8r + 3}$$

Which for the root $r = 1$ becomes

$$a_5 = 13$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{5+2r}{1+2r}$	$\frac{7}{3}$
a_2	$\frac{4r^2+24r+35}{4r^2+8r+3}$	$\frac{21}{5}$
a_3	$\frac{4r^2+32r+63}{4r^2+8r+3}$	$\frac{33}{5}$
a_4	$\frac{4r^2+40r+99}{4r^2+8r+3}$	$\frac{143}{15}$
a_5	$\frac{4r^2+48r+143}{4r^2+8r+3}$	13

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{4r^2 + 56r + 195}{4r^2 + 8r + 3}$$

Which for the root $r = 1$ becomes

$$a_6 = 17$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{5+2r}{1+2r}$	$\frac{7}{3}$
a_2	$\frac{4r^2+24r+35}{4r^2+8r+3}$	$\frac{21}{5}$
a_3	$\frac{4r^2+32r+63}{4r^2+8r+3}$	$\frac{33}{5}$
a_4	$\frac{4r^2+40r+99}{4r^2+8r+3}$	$\frac{143}{15}$
a_5	$\frac{4r^2+48r+143}{4r^2+8r+3}$	13
a_6	$\frac{4r^2+56r+195}{4r^2+8r+3}$	17

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{4r^2 + 64r + 255}{4r^2 + 8r + 3}$$

Which for the root $r = 1$ becomes

$$a_7 = \frac{323}{15}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{5+2r}{1+2r}$	$\frac{7}{3}$
a_2	$\frac{4r^2+24r+35}{4r^2+8r+3}$	$\frac{21}{5}$
a_3	$\frac{4r^2+32r+63}{4r^2+8r+3}$	$\frac{33}{5}$
a_4	$\frac{4r^2+40r+99}{4r^2+8r+3}$	$\frac{143}{15}$
a_5	$\frac{4r^2+48r+143}{4r^2+8r+3}$	13
a_6	$\frac{4r^2+56r+195}{4r^2+8r+3}$	17
a_7	$\frac{4r^2+64r+255}{4r^2+8r+3}$	$\frac{323}{15}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \ldots\right) \\ &= x\left(1 + \frac{7x}{3} + \frac{21x^2}{5} + \frac{33x^3}{5} + \frac{143x^4}{15} + 13x^5 + 17x^6 + \frac{323x^7}{15} + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \le n$ the recursive equation is

$$\begin{aligned} &-2b_{n-1}(n+r-1)(n+r-2) + 2b_n(n+r)(n+r-1) \\ &- 7b_{n-1}(n+r-1) - b_n(n+r) + b_n = 0 \end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{(2n+2r+3)\, b_{n-1}}{2n+2r-1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = \frac{(n+2)\, b_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{5 + 2r}{1 + 2r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_1 = 3$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{5+2r}{1+2r}$	3

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4r^2 + 24r + 35}{4r^2 + 8r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = 6$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{5+2r}{1+2r}$	3
b_2	$\frac{4r^2+24r+35}{4r^2+8r+3}$	6

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{4r^2 + 32r + 63}{4r^2 + 8r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_3 = 10$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{5+2r}{1+2r}$	3
b_2	$\frac{4r^2+24r+35}{4r^2+8r+3}$	6
b_3	$\frac{4r^2+32r+63}{4r^2+8r+3}$	10

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{4r^2 + 40r + 99}{4r^2 + 8r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = 15$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{5+2r}{1+2r}$	3
b_2	$\frac{4r^2+24r+35}{4r^2+8r+3}$	6
b_3	$\frac{4r^2+32r+63}{4r^2+8r+3}$	10
b_4	$\frac{4r^2+40r+99}{4r^2+8r+3}$	15

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{4r^2 + 48r + 143}{4r^2 + 8r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_5 = 21$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{5+2r}{1+2r}$	3
b_2	$\frac{4r^2+24r+35}{4r^2+8r+3}$	6
b_3	$\frac{4r^2+32r+63}{4r^2+8r+3}$	10
b_4	$\frac{4r^2+40r+99}{4r^2+8r+3}$	15
b_5	$\frac{4r^2+48r+143}{4r^2+8r+3}$	21

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{4r^2 + 56r + 195}{4r^2 + 8r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_6 = 28$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{5+2r}{1+2r}$	3
b_2	$\frac{4r^2+24r+35}{4r^2+8r+3}$	6
b_3	$\frac{4r^2+32r+63}{4r^2+8r+3}$	10
b_4	$\frac{4r^2+40r+99}{4r^2+8r+3}$	15
b_5	$\frac{4r^2+48r+143}{4r^2+8r+3}$	21
b_6	$\frac{4r^2+56r+195}{4r^2+8r+3}$	28

For $n = 7$, using the above recursive equation gives

$$b_7 = \frac{4r^2 + 64r + 255}{4r^2 + 8r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_7 = 36$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{5+2r}{1+2r}$	3
b_2	$\frac{4r^2+24r+35}{4r^2+8r+3}$	6
b_3	$\frac{4r^2+32r+63}{4r^2+8r+3}$	10
b_4	$\frac{4r^2+40r+99}{4r^2+8r+3}$	15
b_5	$\frac{4r^2+48r+143}{4r^2+8r+3}$	21
b_6	$\frac{4r^2+56r+195}{4r^2+8r+3}$	28
b_7	$\frac{4r^2+64r+255}{4r^2+8r+3}$	36

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots\right) \\
&= \sqrt{x}\left(1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5 + 28x^6 + 36x^7 + O(x^8)\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1 + \frac{7x}{3} + \frac{21x^2}{5} + \frac{33x^3}{5} + \frac{143x^4}{15} + 13x^5 + 17x^6 + \frac{323x^7}{15} + O(x^8)\right) \\
&\quad + c_2\sqrt{x}\left(1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5 + 28x^6 + 36x^7 + O(x^8)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x\left(1 + \frac{7x}{3} + \frac{21x^2}{5} + \frac{33x^3}{5} + \frac{143x^4}{15} + 13x^5 + 17x^6 + \frac{323x^7}{15} + O(x^8)\right) \\
&\quad + c_2\sqrt{x}\left(1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5 + 28x^6 + 36x^7 + O(x^8)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y = c_1 x\left(1 + \frac{7x}{3} + \frac{21x^2}{5} + \frac{33x^3}{5} + \frac{143x^4}{15} + 13x^5 + 17x^6 + \frac{323x^7}{15} + O(x^8)\right) \\
+ c_2\sqrt{x}\left(1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5 + 28x^6 + 36x^7 + O(x^8)\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1 x \left(1 + \frac{7x}{3} + \frac{21x^2}{5} + \frac{33x^3}{5} + \frac{143x^4}{15} + 13x^5 + 17x^6 + \frac{323x^7}{15} + O(x^8)\right)$$
$$+ c_2 \sqrt{x}\,(1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5 + 28x^6 + 36x^7 + O(x^8))$$

Verified OK.

4.5.1 Maple step by step solution

Let's solve

$-2y''x^2(x-1) + (-7x^2 - x)\,y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{2x^2(x-1)} - \frac{(1+7x)y'}{2x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(1+7x)y'}{2x(x-1)} - \frac{y}{2x^2(x-1)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1+7x}{2x(x-1)}, P_3(x) = -\frac{1}{2x^2(x-1)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -\frac{1}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{1}{2}$

  - $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2y''x^2(x-1) + x(1+7x)\,y' - y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$
  - ○ Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - ○ Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

    $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$
  - ○ Shift index using $k->k+2-m$

    $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0(-1+2r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} (-a_k(2k+2r-1)\,(k+r-1) + a_{k-1}(k+r-1)\,(2k+3+2r\right.$

- • $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(-1+2r)\,(-1+r) = 0$
- • Values of r that satisfy the indicial equation

  $r \in \left\{1, \frac{1}{2}\right\}$
- • Each term in the series must be 0, giving the recursion relation

  $-2\left(\left(-k-r-\frac{3}{2}\right) a_{k-1} + a_k\left(k+r-\frac{1}{2}\right)\right)(k+r-1) = 0$
- • Shift index using $k->k+1$

  $-2\left(\left(-k-\frac{5}{2}-r\right) a_k + a_{k+1}\left(k+\frac{1}{2}+r\right)\right)(k+r) = 0$
- • Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{(2k+2r+5)a_k}{2k+1+2r}$
- • Recursion relation for $r = 1$

  $a_{k+1} = \frac{(2k+7)a_k}{2k+3}$
- • Solution for $r = 1$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1}=\frac{(2k+7)a_k}{2k+3}\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+1}=\frac{(2k+6)a_k}{2k+2}$$

- Solution for $r=\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1}=\frac{(2k+6)a_k}{2k+2}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1}=\frac{(2k+7)a_k}{2k+3}, b_{k+1}=\frac{(2k+6)b_k}{2k+2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 53

```
Order:=8;
dsolve(2*x^2*(1-x)*diff(y(x),x$2)-x*(1+7*x)*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x)=c_1\sqrt{x}\left(1+3x+6x^2+10x^3+15x^4+21x^5+28x^6+36x^7+\mathrm{O}\left(x^8\right)\right)$$
$$+c_2x\left(1+\frac{7}{3}x+\frac{21}{5}x^2+\frac{33}{5}x^3+\frac{143}{15}x^4+13x^5+17x^6+\frac{323}{15}x^7+\mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 96

```
AsymptoticDSolveValue[2*x^2*(1-x)*y''[x]-x*(1+7*x)*y'[x]+y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1 x \left(\frac{323x^7}{15} + 17x^6 + 13x^5 + \frac{143x^4}{15} + \frac{33x^3}{5} + \frac{21x^2}{5} + \frac{7x}{3} + 1\right) + c_2\sqrt{x}\left(36x^7 + 28x^6 + 21x^5 + 15x^4 + 10x^3 + 6x^2 + 3x + 1\right)$$

4.6 problem 6

Internal problem ID [6922]
Internal file name [`OUTPUT/6165_Tuesday_August_09_2022_05_23_54_AM_9883834/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.4 Indicial Equation with Difference of Roots Nonintegral. Exercises page 365
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2xy'' + 5(1 - 2x)\, y' - 5y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2xy'' + (-10x + 5)\, y' - 5y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{5(2x - 1)}{2x}$$
$$q(x) = -\frac{5}{2x}$$

Table 36: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{5(2x-1)}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{5}{2x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2xy'' + (-10x + 5) y' - 5y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2} \right) + (-10x + 5) \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1} \right) - 5 \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-10x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 5(n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-5a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-10x^{n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-10a_{n-1}(n+r-1)\, x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-5a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-5a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-10a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 5(n+r) a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-5a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1} a_n (n+r)(n+r-1) + 5(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$2x^{-1+r} a_0 r(-1+r) + 5r a_0 x^{-1+r} = 0$$

Or

$$\left(2x^{-1+r} r(-1+r) + 5r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(3+2r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 + 3r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = -\frac{3}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(3 + 2r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{3}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{3}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n + r)(n + r - 1) - 10a_{n-1}(n + r - 1) + 5a_n(n + r) - 5a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{5a_{n-1}(2n + 2r - 1)}{2n^2 + 4nr + 2r^2 + 3n + 3r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{(10n - 5)\, a_{n-1}}{n\, (2n + 3)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{10r + 5}{2r^2 + 7r + 5}$$

Which for the root $r = 0$ becomes

$$a_1 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{10r+5}{2r^2+7r+5}$	1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{100r^2 + 200r + 75}{(2r^2 + 7r + 5)(2r^2 + 11r + 14)}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{15}{14}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{10r+5}{2r^2+7r+5}$	1
a_2	$\frac{100r^2+200r+75}{(2r^2+7r+5)(2r^2+11r+14)}$	$\frac{15}{14}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{500r^2 + 1000r + 375}{4r^5 + 56r^4 + 299r^3 + 754r^2 + 885r + 378}$$

Which for the root $r = 0$ becomes

$$a_3 = \frac{125}{126}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{10r+5}{2r^2+7r+5}$	1
a_2	$\frac{100r^2+200r+75}{(2r^2+7r+5)(2r^2+11r+14)}$	$\frac{15}{14}$
a_3	$\frac{500r^2+1000r+375}{4r^5+56r^4+299r^3+754r^2+885r+378}$	$\frac{125}{126}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{2500r^2 + 5000r + 1875}{4r^6 + 80r^5 + 639r^4 + 2590r^3 + 5561r^2 + 5910r + 2376}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{625}{792}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{10r+5}{2r^2+7r+5}$	1
a_2	$\frac{100r^2+200r+75}{(2r^2+7r+5)(2r^2+11r+14)}$	$\frac{15}{14}$
a_3	$\frac{500r^2+1000r+375}{4r^5+56r^4+299r^3+754r^2+885r+378}$	$\frac{125}{126}$
a_4	$\frac{2500r^2+5000r+1875}{4r^6+80r^5+639r^4+2590r^3+5561r^2+5910r+2376}$	$\frac{625}{792}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{12500r^2 + 25000r + 9375}{4r^7 + 108r^6 + 1203r^5 + 7125r^4 + 24051r^3 + 45807r^2 + 44942r + 17160}$$

Which for the root $r = 0$ becomes

$$a_5 = \frac{625}{1144}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{10r+5}{2r^2+7r+5}$	1
a_2	$\frac{100r^2+200r+75}{(2r^2+7r+5)(2r^2+11r+14)}$	$\frac{15}{14}$
a_3	$\frac{500r^2+1000r+375}{4r^5+56r^4+299r^3+754r^2+885r+378}$	$\frac{125}{126}$
a_4	$\frac{2500r^2+5000r+1875}{4r^6+80r^5+639r^4+2590r^3+5561r^2+5910r+2376}$	$\frac{625}{792}$
a_5	$\frac{12500r^2+25000r+9375}{4r^7+108r^6+1203r^5+7125r^4+24051r^3+45807r^2+44942r+17160}$	$\frac{625}{1144}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{62500r^2 + 125000r + 46875}{4r^8 + 140r^7 + 2071r^6 + 16835r^5 + 81781r^4 + 241325r^3 + 418344r^2 + 384300r + 140400}$$

Which for the root $r = 0$ becomes

$$a_6 = \frac{625}{1872}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{10r+5}{2r^2+7r+5}$	1
a_2	$\frac{100r^2+200r+75}{(2r^2+7r+5)(2r^2+11r+14)}$	$\frac{15}{14}$
a_3	$\frac{500r^2+1000r+375}{4r^5+56r^4+299r^3+754r^2+885r+378}$	$\frac{125}{126}$
a_4	$\frac{2500r^2+5000r+1875}{4r^6+80r^5+639r^4+2590r^3+5561r^2+5910r+2376}$	$\frac{625}{792}$
a_5	$\frac{12500r^2+25000r+9375}{4r^7+108r^6+1203r^5+7125r^4+24051r^3+45807r^2+44942r+17160}$	$\frac{625}{1144}$
a_6	$\frac{62500r^2+125000r+46875}{4r^8+140r^7+2071r^6+16835r^5+81781r^4+241325r^3+418344r^2+384300r+140400}$	$\frac{625}{1872}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{312500r^2 + 625000r + 234375}{4r^9 + 176r^8 + 3335r^7 + 35588r^6 + 234626r^5 + 985544r^4 + 2618815r^3 + 4205172r^2 + 3654900r + 12}$$

Which for the root $r = 0$ becomes

$$a_7 = \frac{3125}{17136}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{10r+5}{2r^2+7r+5}$	1
a_2	$\frac{100r^2+200r+75}{(2r^2+7r+5)(2r^2+11r+14)}$	$\frac{15}{14}$
a_3	$\frac{500r^2+1000r+375}{4r^5+56r^4+299r^3+754r^2+885r+378}$	$\frac{125}{126}$
a_4	$\frac{2500r^2+5000r+1875}{4r^6+80r^5+639r^4+2590r^3+5561r^2+5910r+2376}$	$\frac{625}{792}$
a_5	$\frac{12500r^2+25000r+9375}{4r^7+108r^6+1203r^5+7125r^4+24051r^3+45807r^2+44942r+17160}$	$\frac{625}{1144}$
a_6	$\frac{62500r^2+125000r+46875}{4r^8+140r^7+2071r^6+16835r^5+81781r^4+241325r^3+418344r^2+384300r+140400}$	$\frac{625}{1872}$
a_7	$\frac{312500r^2+625000r+234375}{4r^9+176r^8+3335r^7+35588r^6+234626r^5+985544r^4+2618815r^3+4205172r^2+3654900r+1285200}$	$\frac{3125}{17136}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots \\ &= 1 + x + \frac{15x^2}{14} + \frac{125x^3}{126} + \frac{625x^4}{792} + \frac{625x^5}{1144} + \frac{625x^6}{1872} + \frac{3125x^7}{17136} + O(x^8) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - 10b_{n-1}(n+r-1) + 5(n+r)\, b_n - 5b_{n-1} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{5b_{n-1}(2n+2r-1)}{2n^2+4nr+2r^2+3n+3r} \tag{4}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_n = \frac{10b_{n-1}(n-2)}{n\,(2n-3)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{10r + 5}{2r^2 + 7r + 5}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_1 = 10$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{10r+5}{2r^2+7r+5}$	10

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{100r^2 + 200r + 75}{(2r^2 + 7r + 5)(2r^2 + 11r + 14)}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{10r+5}{2r^2+7r+5}$	10
b_2	$\frac{100r^2+200r+75}{(2r^2+7r+5)(2r^2+11r+14)}$	0

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{500r^2 + 1000r + 375}{4r^5 + 56r^4 + 299r^3 + 754r^2 + 885r + 378}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{10r+5}{2r^2+7r+5}$	10
b_2	$\frac{100r^2+200r+75}{(2r^2+7r+5)(2r^2+11r+14)}$	0
b_3	$\frac{500r^2+1000r+375}{4r^5+56r^4+299r^3+754r^2+885r+378}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{2500r^2 + 5000r + 1875}{4r^6 + 80r^5 + 639r^4 + 2590r^3 + 5561r^2 + 5910r + 2376}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{10r+5}{2r^2+7r+5}$	10
b_2	$\frac{100r^2+200r+75}{(2r^2+7r+5)(2r^2+11r+14)}$	0
b_3	$\frac{500r^2+1000r+375}{4r^5+56r^4+299r^3+754r^2+885r+378}$	0
b_4	$\frac{2500r^2+5000r+1875}{4r^6+80r^5+639r^4+2590r^3+5561r^2+5910r+2376}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{12500r^2 + 25000r + 9375}{4r^7 + 108r^6 + 1203r^5 + 7125r^4 + 24051r^3 + 45807r^2 + 44942r + 17160}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{10r+5}{2r^2+7r+5}$	10
b_2	$\frac{100r^2+200r+75}{(2r^2+7r+5)(2r^2+11r+14)}$	0
b_3	$\frac{500r^2+1000r+375}{4r^5+56r^4+299r^3+754r^2+885r+378}$	0
b_4	$\frac{2500r^2+5000r+1875}{4r^6+80r^5+639r^4+2590r^3+5561r^2+5910r+2376}$	0
b_5	$\frac{12500r^2+25000r+9375}{4r^7+108r^6+1203r^5+7125r^4+24051r^3+45807r^2+44942r+17160}$	0

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{62500r^2 + 125000r + 46875}{4r^8 + 140r^7 + 2071r^6 + 16835r^5 + 81781r^4 + 241325r^3 + 418344r^2 + 384300r + 140400}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_6 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{10r+5}{2r^2+7r+5}$	10
b_2	$\frac{100r^2+200r+75}{(2r^2+7r+5)(2r^2+11r+14)}$	0
b_3	$\frac{500r^2+1000r+375}{4r^5+56r^4+299r^3+754r^2+885r+378}$	0
b_4	$\frac{2500r^2+5000r+1875}{4r^6+80r^5+639r^4+2590r^3+5561r^2+5910r+2376}$	0
b_5	$\frac{12500r^2+25000r+9375}{4r^7+108r^6+1203r^5+7125r^4+24051r^3+45807r^2+44942r+17160}$	0
b_6	$\frac{62500r^2+125000r+46875}{4r^8+140r^7+2071r^6+16835r^5+81781r^4+241325r^3+418344r^2+384300r+140400}$	0

For $n = 7$, using the above recursive equation gives

$$b_7 = \frac{312500r^2 + 625000r + 234375}{4r^9 + 176r^8 + 3335r^7 + 35588r^6 + 234626r^5 + 985544r^4 + 2618815r^3 + 4205172r^2 + 3654900r + 12}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_7 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{10r+5}{2r^2+7r+5}$	10
b_2	$\frac{100r^2+200r+75}{(2r^2+7r+5)(2r^2+11r+14)}$	0
b_3	$\frac{500r^2+1000r+375}{4r^5+56r^4+299r^3+754r^2+885r+378}$	0
b_4	$\frac{2500r^2+5000r+1875}{4r^6+80r^5+639r^4+2590r^3+5561r^2+5910r+2376}$	0
b_5	$\frac{12500r^2+25000r+9375}{4r^7+108r^6+1203r^5+7125r^4+24051r^3+45807r^2+44942r+17160}$	0
b_6	$\frac{62500r^2+125000r+46875}{4r^8+140r^7+2071r^6+16835r^5+81781r^4+241325r^3+418344r^2+384300r+140400}$	0
b_7	$\frac{312500r^2+625000r+234375}{4r^9+176r^8+3335r^7+35588r^6+234626r^5+985544r^4+2618815r^3+4205172r^2+3654900r+1285200}$	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots\right) \\ &= \frac{1 + 10x + O(x^8)}{x^{\frac{3}{2}}} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1\left(1 + x + \frac{15x^2}{14} + \frac{125x^3}{126} + \frac{625x^4}{792} + \frac{625x^5}{1144} + \frac{625x^6}{1872} + \frac{3125x^7}{17136} + O(x^8)\right) \\ &+ \frac{c_2(1 + 10x + O(x^8))}{x^{\frac{3}{2}}} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1\left(1 + x + \frac{15x^2}{14} + \frac{125x^3}{126} + \frac{625x^4}{792} + \frac{625x^5}{1144} + \frac{625x^6}{1872} + \frac{3125x^7}{17136} + O(x^8)\right) \\ &+ \frac{c_2(1 + 10x + O(x^8))}{x^{\frac{3}{2}}} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1\left(1+x+\frac{15x^2}{14}+\frac{125x^3}{126}+\frac{625x^4}{792}+\frac{625x^5}{1144}+\frac{625x^6}{1872}+\frac{3125x^7}{17136}+O(x^8)\right) + \frac{c_2(1+10x+O(x^8))}{x^{\frac{3}{2}}} \tag{1}$$

Verification of solutions

$$y = c_1\left(1+x+\frac{15x^2}{14}+\frac{125x^3}{126}+\frac{625x^4}{792}+\frac{625x^5}{1144}+\frac{625x^6}{1872}+\frac{3125x^7}{17136}+O(x^8)\right) + \frac{c_2(1+10x+O(x^8))}{x^{\frac{3}{2}}}$$

Verified OK.

4.6.1 Maple step by step solution

Let's solve

$2xy''+(-10x+5)\,y'-5y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{5y}{2x}+\frac{5(2x-1)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{5(2x-1)y'}{2x}-\frac{5y}{2x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-\frac{5(2x-1)}{2x}, P_3(x)=-\frac{5}{2x}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{5}{2}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2xy'' + (-10x+5)\,y' - 5y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$

- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r (3+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1} (k+1+r)\,(2k+5+2r) - 5a_k (2k+2r+1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, -\frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2(k+1+r)\left(k+\frac{5}{2}+r\right) a_{k+1} - 10a_k\left(k+r+\frac{1}{2}\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{5a_k (2k+2r+1)}{(k+1+r)(2k+5+2r)}$

- Recursion relation for $r=0$

  $a_{k+1}=\frac{5a_k(2k+1)}{(k+1)(2k+5)}$

- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+1}=\frac{5a_k(2k+1)}{(k+1)(2k+5)}\right]$

- Recursion relation for $r=-\frac{3}{2}$; series terminates at $k=1$

  $a_{k+1}=\frac{5a_k(2k-2)}{\left(k-\frac{1}{2}\right)(2k+2)}$

- Apply recursion relation for $k=0$

  $a_1=10a_0$

- Terminating series solution of the ODE for $r=-\frac{3}{2}$. Use reduction of order to find the second

  $y=a_0\cdot(1+10x)$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+b_0\cdot(1+10x), a_{k+1}=\frac{5a_k(2k+1)}{(k+1)(2k+5)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacic algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 40

```
Order:=8;
dsolve(2*x*diff(y(x),x$2)+5*(1-2*x)*diff(y(x),x)-5*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1(1 + 10x + O(x^8))}{x^{\frac{3}{2}}} + c_2\left(1 + x + \frac{15}{14}x^2 + \frac{125}{126}x^3 + \frac{625}{792}x^4 + \frac{625}{1144}x^5 + \frac{625}{1872}x^6 + \frac{3125}{17136}x^7 + O(x^8)\right)$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 65

```
AsymptoticDSolveValue[2*x*y''[x]+5*(1-2*x)*y'[x]-5*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to \frac{c_2(10x+1)}{x^{3/2}} + c_1\left(\frac{3125x^7}{17136} + \frac{625x^6}{1872} + \frac{625x^5}{1144} + \frac{625x^4}{792} + \frac{125x^3}{126} + \frac{15x^2}{14} + x + 1\right)$$

4.7 problem 7

Internal problem ID [6923]
Internal file name [`OUTPUT/6166_Tuesday_August_09_2022_05_23_57_AM_62176316/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.4 Indicial Equation with Difference of Roots Nonintegral. Exercises page 365
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{8x^2y'' + 10xy' - (1 + x) y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$8x^2y'' + 10xy' + (-x - 1) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{5}{4x}$$
$$q(x) = -\frac{1 + x}{8x^2}$$

Table 38: Table $p(x), q(x)$ singularites.

$p(x) = \frac{5}{4x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1+x}{8x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$8x^2y'' + 10xy' + (-x - 1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$8x^2 \left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2} \right) + 10x \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1} \right) + (-x - 1) \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 8x^{n+r}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 10x^{n+r}a_n(n+r)\right) \tag{2A}$$
$$+\sum_{n=0}^{\infty}\left(-x^{1+n+r}a_n\right)+\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)=0$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-x^{1+n+r}a_n\right)=\sum_{n=1}^{\infty}\left(-a_{n-1}x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 8x^{n+r}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 10x^{n+r}a_n(n+r)\right) \tag{2B}$$
$$+\sum_{n=1}^{\infty}\left(-a_{n-1}x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)=0$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$8x^{n+r}a_n(n+r)(n+r-1)+10x^{n+r}a_n(n+r)-a_n x^{n+r}=0$$

When $n=0$ the above becomes

$$8x^r a_0 r(-1+r)+10x^r a_0 r-a_0 x^r=0$$

Or

$$\left(8x^r r(-1+r)+10x^r r-x^r\right)a_0=0$$

Since $a_0\neq 0$ then the above simplifies to

$$\left(8r^2+2r-1\right)x^r=0$$

Since the above is true for all x then the indicial equation becomes

$$8r^2+2r-1=0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{4}$$
$$r_2 = -\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(8r^2 + 2r - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{3}{4}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{4}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$8a_n(n+r)(n+r-1) + 10a_n(n+r) - a_{n-1} - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{8n^2 + 16nr + 8r^2 + 2n + 2r - 1} \tag{4}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_n = \frac{a_{n-1}}{8n^2 + 6n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{4}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{8r^2 + 18r + 9}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_1 = \frac{1}{14}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{8r^2+18r+9}$	$\frac{1}{14}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{64r^4 + 416r^3 + 964r^2 + 936r + 315}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_2 = \frac{1}{616}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{8r^2+18r+9}$	$\frac{1}{14}$
a_2	$\frac{1}{64r^4+416r^3+964r^2+936r+315}$	$\frac{1}{616}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{512r^6 + 6528r^5 + 33440r^4 + 87720r^3 + 123548r^2 + 87822r + 24255}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_3 = \frac{1}{55440}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{8r^2+18r+9}$	$\frac{1}{14}$
a_2	$\frac{1}{64r^4+416r^3+964r^2+936r+315}$	$\frac{1}{616}$
a_3	$\frac{1}{512r^6+6528r^5+33440r^4+87720r^3+123548r^2+87822r+24255}$	$\frac{1}{55440}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(512r^6 + 6528r^5 + 33440r^4 + 87720r^3 + 123548r^2 + 87822r + 24255)(8r^2 + 66r + 135)}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_4 = \frac{1}{8426880}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{8r^2+18r+9}$	$\frac{1}{14}$
a_2	$\frac{1}{64r^4+416r^3+964r^2+936r+315}$	$\frac{1}{616}$
a_3	$\frac{1}{512r^6+6528r^5+33440r^4+87720r^3+123548r^2+87822r+24255}$	$\frac{1}{55440}$
a_4	$\frac{1}{(512r^6+6528r^5+33440r^4+87720r^3+123548r^2+87822r+24255)(8r^2+66r+135)}$	$\frac{1}{8426880}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{(512r^6 + 6528r^5 + 33440r^4 + 87720r^3 + 123548r^2 + 87822r + 24255)(8r^2 + 66r + 135)(8r^2 + 82r -}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_5 = \frac{1}{1938182400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{8r^2+18r+9}$	$\frac{1}{14}$
a_2	$\frac{1}{64r^4+416r^3+964r^2+936r+315}$	$\frac{1}{616}$
a_3	$\frac{1}{512r^6+6528r^5+33440r^4+87720r^3+123548r^2+87822r+24255}$	$\frac{1}{55440}$
a_4	$\frac{1}{(512r^6+6528r^5+33440r^4+87720r^3+123548r^2+87822r+24255)(8r^2+66r+135)}$	$\frac{1}{8426880}$
a_5	$\frac{1}{(512r^6+6528r^5+33440r^4+87720r^3+123548r^2+87822r+24255)(8r^2+66r+135)(8r^2+82r+209)}$	$\frac{1}{1938182400}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{1}{(512r^6 + 6528r^5 + 33440r^4 + 87720r^3 + 123548r^2 + 87822r + 24255)(8r^2 + 66r + 135)(8r^2 + 82r}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_6 = \frac{1}{627971097600}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{8r^2+18r+9}$	$\frac{1}{14}$
a_2	$\frac{1}{64r^4+416r^3+964r^2+936r+315}$	$\frac{1}{616}$
a_3	$\frac{1}{512r^6+6528r^5+33440r^4+87720r^3+123548r^2+87822r+24255}$	$\frac{1}{55440}$
a_4	$\frac{1}{(512r^6+6528r^5+33440r^4+87720r^3+123548r^2+87822r+24255)(8r^2+66r+135)}$	$\frac{1}{8426880}$
a_5	$\frac{1}{(512r^6+6528r^5+33440r^4+87720r^3+123548r^2+87822r+24255)(8r^2+66r+135)(8r^2+82r+209)}$	$\frac{1}{1938182400}$
a_6	$\frac{1}{(512r^6+6528r^5+33440r^4+87720r^3+123548r^2+87822r+24255)(8r^2+66r+135)(8r^2+82r+209)(8r^2+98r+299)}$	$\frac{1}{627971097600}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{1}{(512r^6 + 6528r^5 + 33440r^4 + 87720r^3 + 123548r^2 + 87822r + 24255)(8r^2 + 66r + 135)(8r^2 + 82r}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_7 = \frac{1}{272539456358400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{8r^2+18r+9}$	$\frac{1}{14}$
a_2	$\frac{1}{64r^4+416r^3+964r^2+936r+315}$	$\frac{1}{616}$
a_3	$\frac{1}{512r^6+6528r^5+33440r^4+87720r^3+123548r^2+87822r+24255}$	$\frac{1}{5544}$
a_4	$\frac{1}{(512r^6+6528r^5+33440r^4+87720r^3+123548r^2+87822r+24255)(8r^2+66r+135)}$	$\frac{1}{8426}$
a_5	$\frac{1}{(512r^6+6528r^5+33440r^4+87720r^3+123548r^2+87822r+24255)(8r^2+66r+135)(8r^2+82r+209)}$	1938
a_6	$\frac{1}{(512r^6+6528r^5+33440r^4+87720r^3+123548r^2+87822r+24255)(8r^2+66r+135)(8r^2+82r+209)(8r^2+98r+299)}$	6279
a_7	$\frac{1}{(512r^6+6528r^5+33440r^4+87720r^3+123548r^2+87822r+24255)(8r^2+66r+135)(8r^2+82r+209)(8r^2+98r+299)(8r^2+114r+405)}$	2725

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^{\frac{1}{4}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\
&= x^{\frac{1}{4}}\left(1 + \frac{x}{14} + \frac{x^2}{616} + \frac{x^3}{55440} + \frac{x^4}{8426880} + \frac{x^5}{1938182400} + \frac{x^6}{627971097600} + \frac{x^7}{272539456358400} + O
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
8b_n(n+r)(n+r-1) + 10b_n(n+r) - b_{n-1} - b_n = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = \frac{b_{n-1}}{8n^2 + 16nr + 8r^2 + 2n + 2r - 1} \tag{4}
$$

Which for the root $r = -\frac{1}{2}$ becomes

$$
b_n = \frac{b_{n-1}}{8n^2 - 6n} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{1}{8r^2 + 18r + 9}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_1 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{8r^2+18r+9}$	$\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{64r^4 + 416r^3 + 964r^2 + 936r + 315}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_2 = \frac{1}{40}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{8r^2+18r+9}$	$\frac{1}{2}$
b_2	$\frac{1}{64r^4+416r^3+964r^2+936r+315}$	$\frac{1}{40}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{1}{512r^6 + 6528r^5 + 33440r^4 + 87720r^3 + 123548r^2 + 87822r + 24255}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_3 = \frac{1}{2160}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{8r^2+18r+9}$	$\frac{1}{2}$
b_2	$\frac{1}{64r^4+416r^3+964r^2+936r+315}$	$\frac{1}{40}$
b_3	$\frac{1}{512r^6+6528r^5+33440r^4+87720r^3+123548r^2+87822r+24255}$	$\frac{1}{2160}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(512r^6 + 6528r^5 + 33440r^4 + 87720r^3 + 123548r^2 + 87822r + 24255)(8r^2 + 66r + 135)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = \frac{1}{224640}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{8r^2+18r+9}$	$\frac{1}{2}$
b_2	$\frac{1}{64r^4+416r^3+964r^2+936r+315}$	$\frac{1}{40}$
b_3	$\frac{1}{512r^6+6528r^5+33440r^4+87720r^3+123548r^2+87822r+24255}$	$\frac{1}{2160}$
b_4	$\frac{1}{(512r^6+6528r^5+33440r^4+87720r^3+123548r^2+87822r+24255)(8r^2+66r+135)}$	$\frac{1}{224640}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{1}{(512r^6 + 6528r^5 + 33440r^4 + 87720r^3 + 123548r^2 + 87822r + 24255)(8r^2 + 66r + 135)(8r^2 + 82r +}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_5 = \frac{1}{38188800}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{8r^2+18r+9}$	$\frac{1}{2}$
b_2	$\frac{1}{64r^4+416r^3+964r^2+936r+315}$	$\frac{1}{40}$
b_3	$\frac{1}{512r^6+6528r^5+33440r^4+87720r^3+123548r^2+87822r+24255}$	$\frac{1}{2160}$
b_4	$\frac{1}{(512r^6+6528r^5+33440r^4+87720r^3+123548r^2+87822r+24255)(8r^2+66r+135)}$	$\frac{1}{224640}$
b_5	$\frac{1}{(512r^6+6528r^5+33440r^4+87720r^3+123548r^2+87822r+24255)(8r^2+66r+135)(8r^2+82r+209)}$	$\frac{1}{38188800}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{1}{(512r^6 + 6528r^5 + 33440r^4 + 87720r^3 + 123548r^2 + 87822r + 24255)(8r^2 + 66r + 135)(8r^2 + 82r +}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_6 = \frac{1}{9623577600}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{8r^2+18r+9}$	$\frac{1}{2}$
b_2	$\frac{1}{64r^4+416r^3+964r^2+936r+315}$	$\frac{1}{40}$
b_3	$\frac{1}{512r^6+6528r^5+33440r^4+87720r^3+123548r^2+87822r+24255}$	$\frac{1}{2160}$
b_4	$\frac{1}{(512r^6+6528r^5+33440r^4+87720r^3+123548r^2+87822r+24255)(8r^2+66r+135)}$	$\frac{1}{224640}$
b_5	$\frac{1}{(512r^6+6528r^5+33440r^4+87720r^3+123548r^2+87822r+24255)(8r^2+66r+135)(8r^2+82r+209)}$	$\frac{1}{38188800}$
b_6	$\frac{1}{(512r^6+6528r^5+33440r^4+87720r^3+123548r^2+87822r+24255)(8r^2+66r+135)(8r^2+82r+209)(8r^2+98r+299)}$	$\frac{1}{9623577600}$

For $n = 7$, using the above recursive equation gives

$$b_7 = \frac{1}{(512r^6 + 6528r^5 + 33440r^4 + 87720r^3 + 123548r^2 + 87822r + 24255)(8r^2 + 66r + 135)(8r^2 + 82r +}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_7 = \frac{1}{3368252160000}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{8r^2+18r+9}$	$\frac{1}{2}$
b_2	$\frac{1}{64r^4+416r^3+964r^2+936r+315}$	$\frac{1}{40}$
b_3	$\frac{1}{512r^6+6528r^5+33440r^4+87720r^3+123548r^2+87822r+24255}$	$\frac{1}{2160}$
b_4	$\frac{1}{(512r^6+6528r^5+33440r^4+87720r^3+123548r^2+87822r+24255)(8r^2+66r+135)}$	$\frac{1}{2246\cdots}$
b_5	$\frac{1}{(512r^6+6528r^5+33440r^4+87720r^3+123548r^2+87822r+24255)(8r^2+66r+135)(8r^2+82r+209)}$	$\frac{\cdots}{3818\cdots}$
b_6	$\frac{1}{(512r^6+6528r^5+33440r^4+87720r^3+123548r^2+87822r+24255)(8r^2+66r+135)(8r^2+82r+209)(8r^2+98r+299)}$	$\frac{\cdots}{9623\cdots}$
b_7	$\frac{1}{(512r^6+6528r^5+33440r^4+87720r^3+123548r^2+87822r+24255)(8r^2+66r+135)(8r^2+82r+209)(8r^2+98r+299)(8r^2+114r+405)}$	$\frac{\cdots}{3368\cdots}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^{\frac{1}{4}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots\right) \\
&= \frac{1 + \frac{x}{2} + \frac{x^2}{40} + \frac{x^3}{2160} + \frac{x^4}{224640} + \frac{x^5}{38188800} + \frac{x^6}{9623577600} + \frac{x^7}{3368252160000} + O(x^8)}{\sqrt{x}}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{1}{4}}\left(1 + \frac{x}{14} + \frac{x^2}{616} + \frac{x^3}{55440} + \frac{x^4}{8426880} + \frac{x^5}{1938182400} + \frac{x^6}{627971097600} + \frac{x^7}{272539456358400} + O(x^8)\right) \\
&+ \frac{c_2\left(1 + \frac{x}{2} + \frac{x^2}{40} + \frac{x^3}{2160} + \frac{x^4}{224640} + \frac{x^5}{38188800} + \frac{x^6}{9623577600} + \frac{x^7}{3368252160000} + O(x^8)\right)}{\sqrt{x}}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{1}{4}}\left(1 + \frac{x}{14} + \frac{x^2}{616} + \frac{x^3}{55440} + \frac{x^4}{8426880} + \frac{x^5}{1938182400} + \frac{x^6}{627971097600} + \frac{x^7}{272539456358400} + O(x^8)\right) \\
&+ \frac{c_2\left(1 + \frac{x}{2} + \frac{x^2}{40} + \frac{x^3}{2160} + \frac{x^4}{224640} + \frac{x^5}{38188800} + \frac{x^6}{9623577600} + \frac{x^7}{3368252160000} + O(x^8)\right)}{\sqrt{x}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{1}{4}} \left(1 + \frac{x}{14} + \frac{x^2}{616} + \frac{x^3}{55440} + \frac{x^4}{8426880} + \frac{x^5}{1938182400} + \frac{x^6}{627971097600} + \frac{x^7}{272539456358400} + O(x^8)\right) + \frac{c_2 \left(1 + \frac{x}{2} + \frac{x^2}{40} + \frac{x^3}{2160} + \frac{x^4}{224640} + \frac{x^5}{38188800} + \frac{x^6}{9623577600} + \frac{x^7}{3368252160000} + O(x^8)\right)}{\sqrt{x}}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{4}} \left(1 + \frac{x}{14} + \frac{x^2}{616} + \frac{x^3}{55440} + \frac{x^4}{8426880} + \frac{x^5}{1938182400} + \frac{x^6}{627971097600} + \frac{x^7}{272539456358400} + O(x^8)\right) + \frac{c_2 \left(1 + \frac{x}{2} + \frac{x^2}{40} + \frac{x^3}{2160} + \frac{x^4}{224640} + \frac{x^5}{38188800} + \frac{x^6}{9623577600} + \frac{x^7}{3368252160000} + O(x^8)\right)}{\sqrt{x}}$$

Verified OK.

4.7.1 Maple step by step solution

Let's solve

$8x^2y'' + 10xy' + (-x-1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{5y'}{4x} + \frac{(1+x)y}{8x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{5y'}{4x} - \frac{(1+x)y}{8x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{5}{4x}, P_3(x) = -\frac{1+x}{8x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{5}{4}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{8}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $8x^2y'' + 10xy' + (-x-1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+4r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)\,(4k+4r-1) - a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+4r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{4}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $8\left(k+r-\frac{1}{4}\right)\left(k+r+\frac{1}{2}\right)a_k-a_{k-1}=0$

- Shift index using $k->k+1$

  $8\left(k+\frac{3}{4}+r\right)\left(k+\frac{3}{2}+r\right)a_{k+1}-a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k}{(4k+3+4r)(2k+3+2r)}$

- Recursion relation for $r=-\frac{1}{2}$

  $a_{k+1}=\frac{a_k}{(4k+1)(2k+2)}$

- Solution for $r=-\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{2}}, a_{k+1}=\frac{a_k}{(4k+1)(2k+2)}\right]$

- Recursion relation for $r=\frac{1}{4}$

  $a_{k+1}=\frac{a_k}{(4k+4)\left(2k+\frac{7}{2}\right)}$

- Solution for $r=\frac{1}{4}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{4}}, a_{k+1}=\frac{a_k}{(4k+4)\left(2k+\frac{7}{2}\right)}\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{4}}\right), a_{k+1}=\frac{a_k}{(4k+1)(2k+2)}, b_{k+1}=\frac{b_k}{(4k+4)\left(2k+\frac{7}{2}\right)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 55

```
Order:=8;
dsolve(8*x^2*diff(y(x),x$2)+10*x*diff(y(x),x)-(1+x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_2 x^{\frac{3}{4}}\left(1+\frac{1}{14}x+\frac{1}{616}x^2+\frac{1}{55440}x^3+\frac{1}{8426880}x^4+\frac{1}{1938182400}x^5+\frac{1}{627971097600}x^6+\frac{1}{272539456358400}x^7+\mathrm{O}\left(x^8\right)\right) -}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 118

```
AsymptoticDSolveValue[8*x^2*y''[x]+10*x*y'[x]-(1+x)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\sqrt[4]{x}\left(\frac{x^7}{272539456358400}+\frac{x^6}{627971097600}+\frac{x^5}{1938182400}+\frac{x^4}{8426880}+\frac{x^3}{55440}+\frac{x^2}{616}+\frac{x}{14}+1\right) + \frac{c_2\left(\frac{x^7}{3368252160000}+\frac{x^6}{9623577600}+\frac{x^5}{38188800}+\frac{x^4}{224640}+\frac{x^3}{2160}+\frac{x^2}{40}+\frac{x}{2}+1\right)}{\sqrt{x}}$$

4.8 problem 8

Internal problem ID [6924]
Internal file name [`OUTPUT/6167_Tuesday_August_09_2022_05_23_59_AM_67397934/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.4 Indicial Equation with Difference of Roots Nonintegral. Exercises page 365
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2xy'' + (-x+2)\, y' - 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2xy'' + (-x+2)\, y' - 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x-2}{2x}$$
$$q(x) = -\frac{1}{x}$$

Table 40: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x-2}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2xy'' + (-x + 2) y' - 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2} \right) + (-x + 2) \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1} \right) - 2 \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1) x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1) x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1} a_n (n+r)(n+r-1) + 2(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$2x^{-1+r} a_0 r(-1+r) + 2r a_0 x^{-1+r} = 0$$

Or

$$\left(2x^{-1+r} r(-1+r) + 2r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$2x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$2x^{-1+r}r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) + 2a_n(n+r) - 2a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(n+r+1)}{2n^2 + 4nr + 2r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-1}(n+1)}{2n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{2+r}{2(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_1 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2+r}{2(r+1)^2}$	1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{3+r}{4(2+r)(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{3}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2+r}{2(r+1)^2}$	1
a_2	$\frac{3+r}{4(2+r)(r+1)^2}$	$\frac{3}{8}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{4 + r}{8 (3 + r) (2 + r) (r + 1)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = \frac{1}{12}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2+r}{2(r+1)^2}$	1
a_2	$\frac{3+r}{4(2+r)(r+1)^2}$	$\frac{3}{8}$
a_3	$\frac{4+r}{8(3+r)(2+r)(r+1)^2}$	$\frac{1}{12}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{5 + r}{16 (4 + r) (3 + r) (2 + r) (r + 1)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{5}{384}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2+r}{2(r+1)^2}$	1
a_2	$\frac{3+r}{4(2+r)(r+1)^2}$	$\frac{3}{8}$
a_3	$\frac{4+r}{8(3+r)(2+r)(r+1)^2}$	$\frac{1}{12}$
a_4	$\frac{5+r}{16(4+r)(3+r)(2+r)(r+1)^2}$	$\frac{5}{384}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{6 + r}{32 (5 + r) (4 + r) (3 + r) (2 + r) (r + 1)^2}$$

Which for the root $r = 0$ becomes

$$a_5 = \frac{1}{640}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2+r}{2(r+1)^2}$	1
a_2	$\frac{3+r}{4(2+r)(r+1)^2}$	$\frac{3}{8}$
a_3	$\frac{4+r}{8(3+r)(2+r)(r+1)^2}$	$\frac{1}{12}$
a_4	$\frac{5+r}{16(4+r)(3+r)(2+r)(r+1)^2}$	$\frac{5}{384}$
a_5	$\frac{6+r}{32(5+r)(4+r)(3+r)(2+r)(r+1)^2}$	$\frac{1}{640}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{7+r}{64(6+r)(5+r)(4+r)(3+r)(2+r)(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_6 = \frac{7}{46080}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2+r}{2(r+1)^2}$	1
a_2	$\frac{3+r}{4(2+r)(r+1)^2}$	$\frac{3}{8}$
a_3	$\frac{4+r}{8(3+r)(2+r)(r+1)^2}$	$\frac{1}{12}$
a_4	$\frac{5+r}{16(4+r)(3+r)(2+r)(r+1)^2}$	$\frac{5}{384}$
a_5	$\frac{6+r}{32(5+r)(4+r)(3+r)(2+r)(r+1)^2}$	$\frac{1}{640}$
a_6	$\frac{7+r}{64(6+r)(5+r)(4+r)(3+r)(2+r)(r+1)^2}$	$\frac{7}{46080}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{8+r}{128\,(7+r)\,(6+r)\,(5+r)\,(4+r)\,(3+r)\,(2+r)\,(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_7 = \frac{1}{80640}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2+r}{2(r+1)^2}$	1
a_2	$\frac{3+r}{4(2+r)(r+1)^2}$	$\frac{3}{8}$
a_3	$\frac{4+r}{8(3+r)(2+r)(r+1)^2}$	$\frac{1}{12}$
a_4	$\frac{5+r}{16(4+r)(3+r)(2+r)(r+1)^2}$	$\frac{5}{384}$
a_5	$\frac{6+r}{32(5+r)(4+r)(3+r)(2+r)(r+1)^2}$	$\frac{1}{640}$
a_6	$\frac{7+r}{64(6+r)(5+r)(4+r)(3+r)(2+r)(r+1)^2}$	$\frac{7}{46080}$
a_7	$\frac{8+r}{128(7+r)(6+r)(5+r)(4+r)(3+r)(2+r)(r+1)^2}$	$\frac{1}{80640}$

Using the above table, then the first solution $y_1(x)$ becomes

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots \\ &= 1 + x + \frac{3x^2}{8} + \frac{x^3}{12} + \frac{5x^4}{384} + \frac{x^5}{640} + \frac{7x^6}{46080} + \frac{x^7}{80640} + O(x^8) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_{}$
b_0	1	1	N/A since b_n starts from 1	N
b_1	$\frac{2+r}{2(r+1)^2}$	1	$\frac{-3-r}{2(r+1)^3}$	−
b_2	$\frac{3+r}{4(2+r)(r+1)^2}$	$\frac{3}{8}$	$\frac{-2r^2-11r-13}{4(2+r)^2(r+1)^3}$	−
b_3	$\frac{4+r}{8(3+r)(2+r)(r+1)^2}$	$\frac{1}{12}$	$\frac{-3r^3-27r^2-74r-62}{8(3+r)^2(2+r)^2(r+1)^3}$	−
b_4	$\frac{5+r}{16(4+r)(3+r)(2+r)(r+1)^2}$	$\frac{5}{384}$	$\frac{-2r^4-27r^3-128r^2-252r-173}{8(4+r)^2(3+r)^2(2+r)^2(r+1)^3}$	−
b_5	$\frac{6+r}{32(5+r)(4+r)(3+r)(2+r)(r+1)^2}$	$\frac{1}{640}$	$\frac{-5r^5-95r^4-685r^3-2335r^2-3744r-2244}{32(5+r)^2(4+r)^2(3+r)^2(2+r)^2(r+1)^3}$	−
b_6	$\frac{7+r}{64(6+r)(5+r)(4+r)(3+r)(2+r)(r+1)^2}$	$\frac{7}{46080}$	$\frac{-6r^6-153r^5-1555r^4-8035r^3-22163r^2-30764r-16668}{64(6+r)^2(5+r)^2(4+r)^2(3+r)^2(2+r)^2(r+1)^3}$	−
b_7	$\frac{8+r}{128(7+r)(6+r)(5+r)(4+r)(3+r)(2+r)(r+1)^2}$	$\frac{1}{80640}$	$\frac{-7r^7-231r^6-3143r^5-22785r^4-94682r^3-224448r^2-279376r-139824}{128(7+r)^2(6+r)^2(5+r)^2(4+r)^2(3+r)^2(2+r)^2(r+1)^3}$	−

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 + b_7x^7 + b_8x^8\dots \\
&= \left(1 + x + \frac{3x^2}{8} + \frac{x^3}{12} + \frac{5x^4}{384} + \frac{x^5}{640} + \frac{7x^6}{46080} + \frac{x^7}{80640} + O(x^8)\right)\ln(x) \\
&\quad - \frac{3x}{2} - \frac{13x^2}{16} - \frac{31x^3}{144} - \frac{173x^4}{4608} - \frac{187x^5}{38400} - \frac{463x^6}{921600} - \frac{971x^7}{22579200} + O(x^8)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1\left(1 + x + \frac{3x^2}{8} + \frac{x^3}{12} + \frac{5x^4}{384} + \frac{x^5}{640} + \frac{7x^6}{46080} + \frac{x^7}{80640} + O(x^8)\right) \\
&\quad + c_2\left(\left(1 + x + \frac{3x^2}{8} + \frac{x^3}{12} + \frac{5x^4}{384} + \frac{x^5}{640} + \frac{7x^6}{46080} + \frac{x^7}{80640} + O(x^8)\right)\ln(x) - \frac{3x}{2}\right. \\
&\quad \left. - \frac{13x^2}{16} - \frac{31x^3}{144} - \frac{173x^4}{4608} - \frac{187x^5}{38400} - \frac{463x^6}{921600} - \frac{971x^7}{22579200} + O(x^8)\right)
\end{aligned}
$$

Hence the final solution is

$$
y = y_h
$$

$$
\begin{aligned}
&= c_1\left(1+x+\frac{3x^2}{8}+\frac{x^3}{12}+\frac{5x^4}{384}+\frac{x^5}{640}+\frac{7x^6}{46080}+\frac{x^7}{80640}+O(x^8)\right)\\
&+ c_2\left(\left(1+x+\frac{3x^2}{8}+\frac{x^3}{12}+\frac{5x^4}{384}+\frac{x^5}{640}+\frac{7x^6}{46080}+\frac{x^7}{80640}+O(x^8)\right)\ln(x)-\frac{3x}{2}\right.\\
&\left.-\frac{13x^2}{16}-\frac{31x^3}{144}-\frac{173x^4}{4608}-\frac{187x^5}{38400}-\frac{463x^6}{921600}-\frac{971x^7}{22579200}+O(x^8)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\left(1+x+\frac{3x^2}{8}+\frac{x^3}{12}+\frac{5x^4}{384}+\frac{x^5}{640}+\frac{7x^6}{46080}+\frac{x^7}{80640}+O(x^8)\right)\\
&+ c_2\left(\left(1+x+\frac{3x^2}{8}+\frac{x^3}{12}+\frac{5x^4}{384}+\frac{x^5}{640}+\frac{7x^6}{46080}+\frac{x^7}{80640}+O(x^8)\right)\ln(x)-\frac{3x}{2}\right.\\
&\left.-\frac{13x^2}{16}-\frac{31x^3}{144}-\frac{173x^4}{4608}-\frac{187x^5}{38400}-\frac{463x^6}{921600}-\frac{971x^7}{22579200}+O(x^8)\right) \quad (1)
\end{aligned}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1\left(1+x+\frac{3x^2}{8}+\frac{x^3}{12}+\frac{5x^4}{384}+\frac{x^5}{640}+\frac{7x^6}{46080}+\frac{x^7}{80640}+O(x^8)\right)\\
&+ c_2\left(\left(1+x+\frac{3x^2}{8}+\frac{x^3}{12}+\frac{5x^4}{384}+\frac{x^5}{640}+\frac{7x^6}{46080}+\frac{x^7}{80640}+O(x^8)\right)\ln(x)-\frac{3x}{2}\right.\\
&\left.-\frac{13x^2}{16}-\frac{31x^3}{144}-\frac{173x^4}{4608}-\frac{187x^5}{38400}-\frac{463x^6}{921600}-\frac{971x^7}{22579200}+O(x^8)\right)
\end{aligned}
$$

Verified OK.

4.8.1 Maple step by step solution

Let's solve

$2xy''+(-x+2)\,y'-2y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{y}{x}+\frac{(x-2)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(x-2)y'}{2x}-\frac{y}{x}=0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = -\frac{x-2}{2x}, P_3(x) = -\frac{1}{x}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2xy'' + (-x+2)\,y' - 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ◦ Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$

- ◦ Shift index using $k- >k+1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$2a_0r^2x^{-1+r}+\left(\sum_{k=0}^{\infty}\left(2a_{k+1}(k+1+r)^2-a_k(k+r+2)\right)x^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $2r^2=0$
- Values of r that satisfy the indicial equation

  $r=0$
- Each term in the series must be 0, giving the recursion relation

  $2a_{k+1}(k+1)^2-a_k(k+2)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k(k+2)}{2(k+1)^2}$
- Recursion relation for $r=0$

  $a_{k+1}=\frac{a_k(k+2)}{2(k+1)^2}$
- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty}a_kx^k, a_{k+1}=\frac{a_k(k+2)}{2(k+1)^2}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 71

```
Order:=8;
dsolve(2*x*diff(y(x),x$2)+(2-x)*diff(y(x),x)-2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln(x) + c_1)\left(1 + x + \frac{3}{8}x^2 + \frac{1}{12}x^3 + \frac{5}{384}x^4 + \frac{1}{640}x^5 + \frac{7}{46080}x^6 + \frac{1}{80640}x^7 + \mathrm{O}\left(x^8\right)\right) + \left(-\frac{3}{2}x - \frac{13}{16}x^2 - \frac{31}{144}x^3 - \frac{173}{4608}x^4 - \frac{187}{38400}x^5 - \frac{463}{921600}x^6 - \frac{971}{22579200}x^7 + \mathrm{O}\left(x^8\right)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 151

```
AsymptoticDSolveValue[2*x*y''[x]+(2-x)*y'[x]-2*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(\frac{x^7}{80640} + \frac{7x^6}{46080} + \frac{x^5}{640} + \frac{5x^4}{384} + \frac{x^3}{12} + \frac{3x^2}{8} + x + 1\right) + c_2\left(-\frac{971x^7}{22579200} - \frac{463x^6}{921600} - \frac{187x^5}{38400} - \frac{173x^4}{4608} - \frac{31x^3}{144} - \frac{13x^2}{16} + \left(\frac{x^7}{80640} + \frac{7x^6}{46080} + \frac{x^5}{640} + \frac{5x^4}{384} + \frac{x^3}{12} + \frac{3x^2}{8} + x + 1\right)\log(x) - \frac{3x}{2}\right)$$

4.9 problem 9

Internal problem ID [6925]
Internal file name [`OUTPUT/6168_Tuesday_August_09_2022_05_24_01_AM_88069339/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.4 Indicial Equation with Difference of Roots Nonintegral. Exercises page 365
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x(x+3)\,y'' - 3(1+x)\,y' + 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(2x^2 + 6x)\,y'' + (-3x - 3)\,y' + 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{3(1+x)}{2x\,(x+3)}$$
$$q(x) = \frac{1}{x\,(x+3)}$$

Table 42: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{3(1+x)}{2x(x+3)}$	
singularity	type
$x = -3$	"regular"
$x = 0$	"regular"

$q(x) = \frac{1}{x(x+3)}$	
singularity	type
$x = -3$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-3, 0, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x(x+3)\, y'' + (-3x-3)\, y' + 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x(x+3) \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (-3x-3) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 6x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-3(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-3a_{n-1}(n+r-1)\, x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} 2a_n x^{n+r} = \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 6x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-3a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-3(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$6x^{n+r-1} a_n (n+r)(n+r-1) - 3(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$6x^{-1+r} a_0 r(-1+r) - 3r a_0 x^{-1+r} = 0$$

Or

$$\left(6x^{-1+r}r(-1+r)-3r\,x^{-1+r}\right)a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(6r^2-9r\right)x^{-1+r}=0$$

Since the above is true for all x then the indicial equation becomes

$$6r^2-9r=0$$

Solving for r gives the roots of the indicial equation as

$$r_1=\frac{3}{2}$$
$$r_2=0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(6r^2-9r\right)x^{-1+r}=0$$

Solving for r gives the roots of the indicial equation as Since $r_1-r_2=\frac{3}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x)=x^{r_1}\left(\sum_{n=0}^{\infty}a_nx^n\right)$$
$$y_2(x)=x^{r_2}\left(\sum_{n=0}^{\infty}b_nx^n\right)$$

Or

$$y_1(x)=\sum_{n=0}^{\infty}a_nx^{n+\frac{3}{2}}$$
$$y_2(x)=\sum_{n=0}^{\infty}b_nx^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n=0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0=1$. For $1\leq n$ the recursive equation is

$$\begin{aligned}2a_{n-1}(n+r-1)\,(n+r-2)+6a_n(n+r)\,(n+r-1)\\-3a_{n-1}(n+r-1)-3a_n(n+r)+2a_{n-1}=0\end{aligned}\tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(n+r-3)\, a_{n-1}}{3\,(n+r)} \tag{4}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_n = \frac{a_{n-1}(-2n+3)}{6n+9} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{2-r}{3+3r}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_1 = \frac{1}{15}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2-r}{3+3r}$	$\frac{1}{15}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^2-3r+2}{9\,(1+r)\,(2+r)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_2 = -\frac{1}{315}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2-r}{3+3r}$	$\frac{1}{15}$
a_2	$\frac{r^2-3r+2}{9(1+r)(2+r)}$	$-\frac{1}{315}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{(r^2 - 3r + 2)\, r}{27\,(1 + r)\,(2 + r)\,(3 + r)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_3 = \frac{1}{2835}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2-r}{3+3r}$	$\frac{1}{15}$
a_2	$\frac{r^2-3r+2}{9(1+r)(2+r)}$	$-\frac{1}{315}$
a_3	$-\frac{(r^2-3r+2)r}{27(1+r)(2+r)(3+r)}$	$\frac{1}{2835}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r(r^2 - 3r + 2)}{81\,(3 + r)\,(2 + r)\,(4 + r)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_4 = -\frac{1}{18711}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2-r}{3+3r}$	$\frac{1}{15}$
a_2	$\frac{r^2-3r+2}{9(1+r)(2+r)}$	$-\frac{1}{315}$
a_3	$-\frac{(r^2-3r+2)r}{27(1+r)(2+r)(3+r)}$	$\frac{1}{2835}$
a_4	$\frac{r(r^2-3r+2)}{81(3+r)(2+r)(4+r)}$	$-\frac{1}{18711}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{r(r^2 - 3r + 2)}{243 (4 + r) (3 + r) (5 + r)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_5 = \frac{1}{104247}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2-r}{3+3r}$	$\frac{1}{15}$
a_2	$\frac{r^2-3r+2}{9(1+r)(2+r)}$	$-\frac{1}{315}$
a_3	$-\frac{(r^2-3r+2)r}{27(1+r)(2+r)(3+r)}$	$\frac{1}{2835}$
a_4	$\frac{r(r^2-3r+2)}{81(3+r)(2+r)(4+r)}$	$-\frac{1}{18711}$
a_5	$-\frac{r(r^2-3r+2)}{243(4+r)(3+r)(5+r)}$	$\frac{1}{104247}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{(r^2 - 3r + 2) r}{729 (5 + r) (4 + r) (6 + r)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_6 = -\frac{1}{521235}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2-r}{3+3r}$	$\frac{1}{15}$
a_2	$\frac{r^2-3r+2}{9(1+r)(2+r)}$	$-\frac{1}{315}$
a_3	$-\frac{(r^2-3r+2)r}{27(1+r)(2+r)(3+r)}$	$\frac{1}{2835}$
a_4	$\frac{r(r^2-3r+2)}{81(3+r)(2+r)(4+r)}$	$-\frac{1}{18711}$
a_5	$-\frac{r(r^2-3r+2)}{243(4+r)(3+r)(5+r)}$	$\frac{1}{104247}$
a_6	$\frac{(r^2-3r+2)r}{729(5+r)(4+r)(6+r)}$	$-\frac{1}{521235}$

For $n = 7$, using the above recursive equation gives

$$a_7 = -\frac{r(r^2 - 3r + 2)}{2187\,(6 + r)\,(5 + r)\,(7 + r)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_7 = \frac{1}{2416635}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2-r}{3+3r}$	$\frac{1}{15}$
a_2	$\frac{r^2-3r+2}{9(1+r)(2+r)}$	$-\frac{1}{315}$
a_3	$-\frac{(r^2-3r+2)r}{27(1+r)(2+r)(3+r)}$	$\frac{1}{2835}$
a_4	$\frac{r(r^2-3r+2)}{81(3+r)(2+r)(4+r)}$	$-\frac{1}{18711}$
a_5	$-\frac{r(r^2-3r+2)}{243(4+r)(3+r)(5+r)}$	$\frac{1}{104247}$
a_6	$\frac{(r^2-3r+2)r}{729(5+r)(4+r)(6+r)}$	$-\frac{1}{521235}$
a_7	$-\frac{r(r^2-3r+2)}{2187(6+r)(5+r)(7+r)}$	$\frac{1}{2416635}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^{\frac{3}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\
&= x^{\frac{3}{2}}\left(1 + \frac{x}{15} - \frac{x^2}{315} + \frac{x^3}{2835} - \frac{x^4}{18711} + \frac{x^5}{104247} - \frac{x^6}{521235} + \frac{x^7}{2416635} + O(x^8)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
\begin{aligned}
&2b_{n-1}(n+r-1)(n+r-2) + 6b_n(n+r)(n+r-1) \\
&- 3b_{n-1}(n+r-1) - 3(n+r)\,b_n + 2b_{n-1} = 0
\end{aligned} \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = -\frac{(n+r-3)\,b_{n-1}}{3\,(n+r)} \tag{4}
$$

Which for the root $r = 0$ becomes

$$
b_n = -\frac{(n-3)\,b_{n-1}}{3n} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$
b_1 = \frac{2-r}{3+3r}
$$

Which for the root $r = 0$ becomes

$$
b_1 = \frac{2}{3}
$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2-r}{3+3r}$	$\frac{2}{3}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{r^2 - 3r + 2}{9(1+r)(2+r)}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{9}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2-r}{3+3r}$	$\frac{2}{3}$
b_2	$\frac{r^2-3r+2}{9(1+r)(2+r)}$	$\frac{1}{9}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{(r^2 - 3r + 2)\, r}{27(1+r)(2+r)(3+r)}$$

Which for the root $r = 0$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2-r}{3+3r}$	$\frac{2}{3}$
b_2	$\frac{r^2-3r+2}{9(1+r)(2+r)}$	$\frac{1}{9}$
b_3	$-\frac{(r^2-3r+2)r}{27(1+r)(2+r)(3+r)}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r(r^2 - 3r + 2)}{81(3+r)(2+r)(4+r)}$$

Which for the root $r = 0$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2-r}{3+3r}$	$\frac{2}{3}$
b_2	$\frac{r^2-3r+2}{9(1+r)(2+r)}$	$\frac{1}{9}$
b_3	$-\frac{(r^2-3r+2)r}{27(1+r)(2+r)(3+r)}$	0
b_4	$\frac{r(r^2-3r+2)}{81(3+r)(2+r)(4+r)}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{r(r^2 - 3r + 2)}{243\,(4 + r)\,(3 + r)\,(5 + r)}$$

Which for the root $r = 0$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2-r}{3+3r}$	$\frac{2}{3}$
b_2	$\frac{r^2-3r+2}{9(1+r)(2+r)}$	$\frac{1}{9}$
b_3	$-\frac{(r^2-3r+2)r}{27(1+r)(2+r)(3+r)}$	0
b_4	$\frac{r(r^2-3r+2)}{81(3+r)(2+r)(4+r)}$	0
b_5	$-\frac{r(r^2-3r+2)}{243(4+r)(3+r)(5+r)}$	0

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{(r^2 - 3r + 2)\,r}{729\,(5 + r)\,(4 + r)\,(6 + r)}$$

Which for the root $r = 0$ becomes

$$b_6 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2-r}{3+3r}$	$\frac{2}{3}$
b_2	$\frac{r^2-3r+2}{9(1+r)(2+r)}$	$\frac{1}{9}$
b_3	$-\frac{(r^2-3r+2)r}{27(1+r)(2+r)(3+r)}$	0
b_4	$\frac{r(r^2-3r+2)}{81(3+r)(2+r)(4+r)}$	0
b_5	$-\frac{r(r^2-3r+2)}{243(4+r)(3+r)(5+r)}$	0
b_6	$\frac{(r^2-3r+2)r}{729(5+r)(4+r)(6+r)}$	0

For $n = 7$, using the above recursive equation gives

$$b_7 = -\frac{r(r^2-3r+2)}{2187\,(6+r)\,(5+r)\,(7+r)}$$

Which for the root $r = 0$ becomes

$$b_7 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2-r}{3+3r}$	$\frac{2}{3}$
b_2	$\frac{r^2-3r+2}{9(1+r)(2+r)}$	$\frac{1}{9}$
b_3	$-\frac{(r^2-3r+2)r}{27(1+r)(2+r)(3+r)}$	0
b_4	$\frac{r(r^2-3r+2)}{81(3+r)(2+r)(4+r)}$	0
b_5	$-\frac{r(r^2-3r+2)}{243(4+r)(3+r)(5+r)}$	0
b_6	$\frac{(r^2-3r+2)r}{729(5+r)(4+r)(6+r)}$	0
b_7	$-\frac{r(r^2-3r+2)}{2187(6+r)(5+r)(7+r)}$	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\ &= 1 + \frac{2x}{3} + \frac{x^2}{9} + O(x^8) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{3}{2}} \left(1 + \frac{x}{15} - \frac{x^2}{315} + \frac{x^3}{2835} - \frac{x^4}{18711} + \frac{x^5}{104247} - \frac{x^6}{521235} + \frac{x^7}{2416635} + O(x^8)\right) \\ &\quad + c_2 \left(1 + \frac{2x}{3} + \frac{x^2}{9} + O(x^8)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^{\frac{3}{2}} \left(1 + \frac{x}{15} - \frac{x^2}{315} + \frac{x^3}{2835} - \frac{x^4}{18711} + \frac{x^5}{104247} - \frac{x^6}{521235} + \frac{x^7}{2416635} + O(x^8)\right) \\ &\quad + c_2 \left(1 + \frac{2x}{3} + \frac{x^2}{9} + O(x^8)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = c_1 x^{\frac{3}{2}} &\left(1 + \frac{x}{15} - \frac{x^2}{315} + \frac{x^3}{2835} - \frac{x^4}{18711} + \frac{x^5}{104247} - \frac{x^6}{521235} + \frac{x^7}{2416635} + O(x^8)\right) \\ &+ c_2 \left(1 + \frac{2x}{3} + \frac{x^2}{9} + O(x^8)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y = c_1 x^{\frac{3}{2}} &\left(1 + \frac{x}{15} - \frac{x^2}{315} + \frac{x^3}{2835} - \frac{x^4}{18711} + \frac{x^5}{104247} - \frac{x^6}{521235} + \frac{x^7}{2416635} + O(x^8)\right) \\ &+ c_2 \left(1 + \frac{2x}{3} + \frac{x^2}{9} + O(x^8)\right) \end{aligned}$$

Verified OK.

4.9.1 Maple step by step solution

Let's solve

$2x(x+3)\, y'' + (-3x-3)\, y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3(1+x)y'}{2x(x+3)} - \frac{y}{x(x+3)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{3(1+x)y'}{2x(x+3)} + \frac{y}{x(x+3)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{3(1+x)}{2x(x+3)}, P_3(x) = \frac{1}{x(x+3)}\right]$

  ◦ $(x+3)\cdot P_2(x)$ is analytic at $x=-3$

  $\left((x+3)\cdot P_2(x)\right)\bigg|_{x=-3} = -1$

  ◦ $(x+3)^2\cdot P_3(x)$ is analytic at $x=-3$

  $\left((x+3)^2\cdot P_3(x)\right)\bigg|_{x=-3} = 0$

  ◦ $x=-3$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -3$

- Multiply by denominators

  $2x(x+3)\, y'' + (-3x-3)\, y' + 2y = 0$

- Change variables using $x = u-3$ so that the regular singular point is at $u=0$

  $(2u^2-6u)\left(\frac{d^2}{du^2}y(u)\right) + (-3u+6)\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$
- Shift index using $k- > k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$
- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$
- Shift index using $k- > k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-6a_0r(-2+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-6a_{k+1}(k+1+r)\,(k+r-1) + a_k(2k+2r-1)\,(k+r-2)\right) u^{k+r}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-6r(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $2(k+r-2)\left(k+r-\frac{1}{2}\right)a_k - 6a_{k+1}(k+1+r)\,(k+r-1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{(k+r-2)(2k+2r-1)a_k}{6(k+1+r)(k+r-1)}$
- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{(k-2)(2k-1)a_k}{6(k+1)(k-1)}$
- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 1$

  $a_{k+1} = \frac{(k-2)(2k-1)a_k}{6(k+1)(k-1)}$
- Recursion relation for $r = 2$

  $a_{k+1} = \frac{k(2k+3)a_k}{6(k+3)(k+1)}$
- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{k(2k+3)a_k}{6(k+3)(k+1)}\right]$$

- Revert the change of variables $u = x+3$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+3)^{k+2}, a_{k+1} = \frac{k(2k+3)a_k}{6(k+3)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 42

```
Order:=8;
dsolve(2*x*(x+3)*diff(y(x),x$2)-3*(x+1)*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{\frac{3}{2}} \left(1 + \frac{1}{15}x - \frac{1}{315}x^2 + \frac{1}{2835}x^3 - \frac{1}{18711}x^4 + \frac{1}{104247}x^5 - \frac{1}{521235}x^6 + \frac{1}{2416635}x^7 + \mathrm{O}\left(x^8\right)\right) + c_2\left(1 + \frac{2}{3}x + \frac{1}{9}x^2 + \mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 78

```
AsymptoticDSolveValue[2*x*(x+3)*y''[x]-3*(x+1)*y'[x]+2*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(\frac{x^2}{9}+\frac{2x}{3}+1\right) + c_1\left(\frac{x^7}{2416635}-\frac{x^6}{521235}+\frac{x^5}{104247}-\frac{x^4}{18711}+\frac{x^3}{2835}-\frac{x^2}{315}+\frac{x}{15}+1\right)x^{3/2}$$

4.10 problem 10

Internal problem ID [6926]
Internal file name [OUTPUT/6169_Tuesday_August_09_2022_05_24_04_AM_21677104/index.tex]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.4 Indicial Equation with Difference of Roots Nonintegral. Exercises page 365
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{2xy'' + \left(-2x^2+1\right)y' - 4yx = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2xy'' + \left(-2x^2+1\right)y' - 4yx = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{2x^2-1}{2x}$$
$$q(x) = -2$$

Table 44: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{2x^2-1}{2x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = -2$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2xy'' + (-2x^2 + 1)\, y' - 4yx = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (-2x^2+1) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - 4\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) x = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n (n+r)\right) \\ + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-4x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n (n+r)\right) = \sum_{n=2}^{\infty} \left(-2a_{n-2}(n+r-2)\, x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-4x^{1+n+r} a_n\right) = \sum_{n=2}^{\infty} \left(-4a_{n-2} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=2}^{\infty} \left(-2a_{n-2}(n+r-2)\, x^{n+r-1}\right) \\ + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=2}^{\infty} \left(-4a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$2x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(2x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-1+2r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(-1 + 2r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2a_n(n + r)(n + r - 1) - 2a_{n-2}(n + r - 2) + a_n(n + r) - 4a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2a_{n-2}}{2n - 1 + 2r} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{a_{n-2}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{2}{3 + 2r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2}{3+2r}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2}{3+2r}$	$\frac{1}{2}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4}{4r^2 + 20r + 21}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{1}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2}{3+2r}$	$\frac{1}{2}$
a_3	0	0
a_4	$\frac{4}{4r^2+20r+21}$	$\frac{1}{8}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2}{3+2r}$	$\frac{1}{2}$
a_3	0	0
a_4	$\frac{4}{4r^2+20r+21}$	$\frac{1}{8}$
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{8}{8r^3 + 84r^2 + 262r + 231}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_6 = \frac{1}{48}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2}{3+2r}$	$\frac{1}{2}$
a_3	0	0
a_4	$\frac{4}{4r^2+20r+21}$	$\frac{1}{8}$
a_5	0	0
a_6	$\frac{8}{8r^3+84r^2+262r+231}$	$\frac{1}{48}$

For $n = 7$, using the above recursive equation gives

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2}{3+2r}$	$\frac{1}{2}$
a_3	0	0
a_4	$\frac{4}{4r^2+20r+21}$	$\frac{1}{8}$
a_5	0	0
a_6	$\frac{8}{8r^3+84r^2+262r+231}$	$\frac{1}{48}$
a_7	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\ &= \sqrt{x}\left(1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - 2b_{n-2}(n+r-2) + (n+r)b_n - 4b_{n-2} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{2b_{n-2}}{2n-1+2r} \tag{4}$$

Which for the root $r = 0$ becomes

$$b_n = \frac{2b_{n-2}}{2n-1} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{2}{3+2r}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{2}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{2}{3+2r}$	$\frac{2}{3}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{2}{3+2r}$	$\frac{2}{3}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{4}{4r^2 + 20r + 21}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{4}{21}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{2}{3+2r}$	$\frac{2}{3}$
b_3	0	0
b_4	$\frac{4}{4r^2+20r+21}$	$\frac{4}{21}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{2}{3+2r}$	$\frac{2}{3}$
b_3	0	0
b_4	$\frac{4}{4r^2+20r+21}$	$\frac{4}{21}$
b_5	0	0

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{8}{8r^3 + 84r^2 + 262r + 231}$$

Which for the root $r = 0$ becomes

$$b_6 = \frac{8}{231}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{2}{3+2r}$	$\frac{2}{3}$
b_3	0	0
b_4	$\frac{4}{4r^2+20r+21}$	$\frac{4}{21}$
b_5	0	0
b_6	$\frac{8}{8r^3+84r^2+262r+231}$	$\frac{8}{231}$

For $n = 7$, using the above recursive equation gives

$$b_7 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{2}{3+2r}$	$\frac{2}{3}$
b_3	0	0
b_4	$\frac{4}{4r^2+20r+21}$	$\frac{4}{21}$
b_5	0	0
b_6	$\frac{8}{8r^3+84r^2+262r+231}$	$\frac{8}{231}$
b_7	0	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\
&= 1 + \frac{2x^2}{3} + \frac{4x^4}{21} + \frac{8x^6}{231} + O(x^8)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \sqrt{x} \left(1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + O(x^8)\right) + c_2 \left(1 + \frac{2x^2}{3} + \frac{4x^4}{21} + \frac{8x^6}{231} + O(x^8)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \sqrt{x} \left(1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + O(x^8)\right) + c_2 \left(1 + \frac{2x^2}{3} + \frac{4x^4}{21} + \frac{8x^6}{231} + O(x^8)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \sqrt{x} \left(1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + O(x^8)\right) + c_2 \left(1 + \frac{2x^2}{3} + \frac{4x^4}{21} + \frac{8x^6}{231} + O(x^8)\right) \quad (1)
$$

Verification of solutions

$$
y = c_1 \sqrt{x} \left(1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + O(x^8)\right) + c_2 \left(1 + \frac{2x^2}{3} + \frac{4x^4}{21} + \frac{8x^6}{231} + O(x^8)\right)
$$

Verified OK.

4.10.1 Maple step by step solution

Let's solve

$2xy'' + (-2x^2 + 1) y' - 4yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = 2y + \frac{(2x^2-1)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x^2-1)y'}{2x} - 2y = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2x^2-1}{2x}, P_3(x) = -2\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{1}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $2xy'' + (-2x^2 + 1)\, y' - 4yx = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

  ◦ Shift index using $k- >k - 1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$

  ◦ Shift index using $k- >k + 1 - m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k + 1 - m + r)\, x^{k+r}$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$$

- Shift index using $k->k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0r(-1+2r)\,x^{-1+r} + a_1(1+r)\,(1+2r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(2k+1+2r) - 2a_{k-1}(k+r\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term must be 0

  $$a_1(1+r)\,(1+2r) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$2(k+r+1)\left(\left(k+r+\frac{1}{2}\right)a_{k+1} - a_{k-1}\right) = 0$$

- Shift index using $k->k+1$

  $$2(k+r+2)\left(\left(k+\frac{3}{2}+r\right)a_{k+2} - a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = \frac{2a_k}{2k+3+2r}$$

- Recursion relation for $r = 0$

  $$a_{k+2} = \frac{2a_k}{2k+3}$$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2a_k}{2k+3}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{1}{2}$

  $$a_{k+2} = \frac{2a_k}{2k+4}$$

- Solution for $r = \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{2a_k}{2k+4}, 3a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^k\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{2}}\right), a_{k+2}=\frac{2a_k}{2k+3}, a_1=0, b_{k+2}=\frac{2b_k}{2k+4}, 3b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 36

```
Order:=8;
dsolve(2*x*diff(y(x),x$2)+(1-2*x^2)*diff(y(x),x)-4*x*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=c_1\sqrt{x}\left(1+\frac{1}{2}x^2+\frac{1}{8}x^4+\frac{1}{48}x^6+\mathrm{O}\left(x^8\right)\right)+c_2\left(1+\frac{2}{3}x^2+\frac{4}{21}x^4+\frac{8}{231}x^6+\mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 61

```
AsymptoticDSolveValue[2*x*y''[x]+(1-2*x^2)*y'[x]-4*x*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\sqrt{x}\left(\frac{x^6}{48}+\frac{x^4}{8}+\frac{x^2}{2}+1\right)+c_2\left(\frac{8x^6}{231}+\frac{4x^4}{21}+\frac{2x^2}{3}+1\right)$$

4.11 problem 11

Internal problem ID [6927]
Internal file name [`OUTPUT/6170_Friday_August_12_2022_11_04_09_PM_9550685/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.4 Indicial Equation with Difference of Roots Nonintegral. Exercises page 365
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{x(-x+4)\,y''+(-x+2)\,y'+4y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-x^2+4x\right)y''+(-x+2)\,y'+4y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{x-2}{x\,(x-4)}$$
$$q(x)=-\frac{4}{x\,(x-4)}$$

Table 46: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x-2}{x(x-4)}$	
singularity	type
$x = 0$	"regular"
$x = 4$	"regular"

$q(x) = -\frac{4}{x(x-4)}$	
singularity	type
$x = 0$	"regular"
$x = 4$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 4, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''x(x-4) + (-x+2)\, y' + 4y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x(x-4) + (-x+2) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 4\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty}\left(-x^{n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 4 x^{n+r-1} a_n(n+r)(n+r-1)\right) + \sum_{n=0}^{\infty}\left(-x^{n+r} a_n(n+r)\right)+\left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1}\right)+\left(\sum_{n=0}^{\infty} 4 a_n x^{n+r}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-x^{n+r} a_n(n+r)(n+r-1)\right)=\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)(n+r-2) x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty}\left(-x^{n+r} a_n(n+r)\right)=\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1) x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} 4 a_n x^{n+r}=\sum_{n=1}^{\infty} 4 a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)(n+r-2) x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 4 x^{n+r-1} a_n(n+r)(n+r-1)\right)+\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1) x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1}\right)+\left(\sum_{n=1}^{\infty} 4 a_{n-1} x^{n+r-1}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4 x^{n+r-1} a_n(n+r)(n+r-1)+2(n+r) a_n x^{n+r-1}=0$$

When $n=0$ the above becomes

$$4 x^{-1+r} a_0 r(-1+r)+2 r a_0 x^{-1+r}=0$$

Or

$$\left(4x^{-1+r}r(-1+r)+2r\,x^{-1+r}\right)a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2-2r\right)x^{-1+r}=0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2-2r=0$$

Solving for r gives the roots of the indicial equation as

$$r_1=\frac{1}{2}$$
$$r_2=0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(4r^2-2r\right)x^{-1+r}=0$$

Solving for r gives the roots of the indicial equation as Since $r_1-r_2=\frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x)=x^{r_1}\left(\sum_{n=0}^{\infty}a_n x^n\right)$$
$$y_2(x)=x^{r_2}\left(\sum_{n=0}^{\infty}b_n x^n\right)$$

Or

$$y_1(x)=\sum_{n=0}^{\infty}a_n x^{n+\frac{1}{2}}$$
$$y_2(x)=\sum_{n=0}^{\infty}b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n=0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0=1$. For $1\leq n$ the recursive equation is

$$-a_{n-1}(n+r-1)(n+r-2)+4a_n(n+r)(n+r-1)-a_{n-1}(n+r-1)+2a_n(n+r)+4a_{n-1}=0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(n^2 + 2nr + r^2 - 2n - 2r - 3)}{4n^2 + 8nr + 4r^2 - 2n - 2r} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{a_{n-1}(4n^2 - 4n - 15)}{16n^2 + 8n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{r^2 - 4}{4r^2 + 6r + 2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = -\frac{5}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2-4}{4r^2+6r+2}$	$-\frac{5}{8}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^3 - 7r + 6}{16r^3 + 48r^2 + 44r + 12}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{7}{128}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2-4}{4r^2+6r+2}$	$-\frac{5}{8}$
a_2	$\frac{r^3-7r+6}{16r^3+48r^2+44r+12}$	$\frac{7}{128}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{(r+4)\, r(-1+r)\,(r-2)}{64r^4 + 352r^3 + 656r^2 + 488r + 120}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = \frac{3}{1024}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2-4}{4r^2+6r+2}$	$-\frac{5}{8}$
a_2	$\frac{r^3-7r+6}{16r^3+48r^2+44r+12}$	$\frac{7}{128}$
a_3	$\frac{(r+4)r(-1+r)(r-2)}{64r^4+352r^3+656r^2+488r+120}$	$\frac{3}{1024}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(r-2)\,(-1+r)\, r(r+5)}{256r^4 + 2048r^3 + 5504r^2 + 5632r + 1680}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{11}{32768}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2-4}{4r^2+6r+2}$	$-\frac{5}{8}$
a_2	$\frac{r^3-7r+6}{16r^3+48r^2+44r+12}$	$\frac{7}{128}$
a_3	$\frac{(r+4)r(-1+r)(r-2)}{64r^4+352r^3+656r^2+488r+120}$	$\frac{3}{1024}$
a_4	$\frac{(r-2)(-1+r)r(r+5)}{256r^4+2048r^3+5504r^2+5632r+1680}$	$\frac{11}{32768}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{(r+6)(r+2)\, r(r-2)(-1+r)}{1024r^5 + 12800r^4 + 58880r^3 + 121600r^2 + 108096r + 30240}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = \frac{13}{262144}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2-4}{4r^2+6r+2}$	$-\frac{5}{8}$
a_2	$\frac{r^3-7r+6}{16r^3+48r^2+44r+12}$	$\frac{7}{128}$
a_3	$\frac{(r+4)r(-1+r)(r-2)}{64r^4+352r^3+656r^2+488r+120}$	$\frac{3}{1024}$
a_4	$\frac{(r-2)(-1+r)r(r+5)}{256r^4+2048r^3+5504r^2+5632r+1680}$	$\frac{11}{32768}$
a_5	$\frac{(r+6)(r+2)r(r-2)(-1+r)}{1024r^5+12800r^4+58880r^3+121600r^2+108096r+30240}$	$\frac{13}{262144}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{(r^2+10r+21)(-1+r)\, r(r^2-4)}{64(2r+11)(32r^5+400r^4+1840r^3+3800r^2+3378r+945)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_6 = \frac{35}{4194304}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2-4}{4r^2+6r+2}$	$-\frac{5}{8}$
a_2	$\frac{r^3-7r+6}{16r^3+48r^2+44r+12}$	$\frac{7}{128}$
a_3	$\frac{(r+4)r(-1+r)(r-2)}{64r^4+352r^3+656r^2+488r+120}$	$\frac{3}{1024}$
a_4	$\frac{(r-2)(-1+r)r(r+5)}{256r^4+2048r^3+5504r^2+5632r+1680}$	$\frac{11}{32768}$
a_5	$\frac{(r+6)(r+2)r(r-2)(-1+r)}{1024r^5+12800r^4+58880r^3+121600r^2+108096r+30240}$	$\frac{13}{262144}$
a_6	$\frac{(r^2+10r+21)(-1+r)r(r^2-4)}{64(2r+11)(32r^5+400r^4+1840r^3+3800r^2+3378r+945)}$	$\frac{35}{4194304}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{(r^2+12r+32)(r^2-4)r(-1+r)(r+3)}{128(2r+13)(2r+11)(32r^5+400r^4+1840r^3+3800r^2+3378r+945)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_7 = \frac{51}{33554432}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2-4}{4r^2+6r+2}$	$-\frac{5}{8}$
a_2	$\frac{r^3-7r+6}{16r^3+48r^2+44r+12}$	$\frac{7}{128}$
a_3	$\frac{(r+4)r(-1+r)(r-2)}{64r^4+352r^3+656r^2+488r+120}$	$\frac{3}{1024}$
a_4	$\frac{(r-2)(-1+r)r(r+5)}{256r^4+2048r^3+5504r^2+5632r+1680}$	$\frac{11}{32768}$
a_5	$\frac{(r+6)(r+2)r(r-2)(-1+r)}{1024r^5+12800r^4+58880r^3+121600r^2+108096r+30240}$	$\frac{13}{262144}$
a_6	$\frac{(r^2+10r+21)(-1+r)r(r^2-4)}{64(2r+11)(32r^5+400r^4+1840r^3+3800r^2+3378r+945)}$	$\frac{35}{4194304}$
a_7	$\frac{(r^2+12r+32)(r^2-4)r(-1+r)(r+3)}{128(2r+13)(2r+11)(32r^5+400r^4+1840r^3+3800r^2+3378r+945)}$	$\frac{51}{33554432}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \dots) \\ &= \sqrt{x}\left(1 - \frac{5x}{8} + \frac{7x^2}{128} + \frac{3x^3}{1024} + \frac{11x^4}{32768} + \frac{13x^5}{262144} + \frac{35x^6}{4194304} + \frac{51x^7}{33554432} + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \le n$ the recursive equation is

$$\begin{aligned} -b_{n-1}(n+r-1)(n+r-2) + 4b_n(n+r)(n+r-1) \\ - b_{n-1}(n+r-1) + 2(n+r)b_n + 4b_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-1}(n^2+2nr+r^2-2n-2r-3)}{4n^2+8nr+4r^2-2n-2r} \tag{4}$$

Which for the root $r = 0$ becomes

$$b_n = \frac{b_{n-1}(n^2 - 2n - 3)}{4n^2 - 2n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{r^2 - 4}{4r^2 + 6r + 2}$$

Which for the root $r = 0$ becomes

$$b_1 = -2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r^2-4}{4r^2+6r+2}$	-2

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{r^3 - 7r + 6}{16r^3 + 48r^2 + 44r + 12}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r^2-4}{4r^2+6r+2}$	-2
b_2	$\frac{r^3-7r+6}{16r^3+48r^2+44r+12}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{(r+4)\, r(-1+r)\,(r-2)}{64r^4 + 352r^3 + 656r^2 + 488r + 120}$$

Which for the root $r = 0$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r^2-4}{4r^2+6r+2}$	-2
b_2	$\frac{r^3-7r+6}{16r^3+48r^2+44r+12}$	$\frac{1}{2}$
b_3	$\frac{(r+4)r(-1+r)(r-2)}{64r^4+352r^3+656r^2+488r+120}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{(r-2)\,(-1+r)\, r(r+5)}{256r^4 + 2048r^3 + 5504r^2 + 5632r + 1680}$$

Which for the root $r = 0$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r^2-4}{4r^2+6r+2}$	-2
b_2	$\frac{r^3-7r+6}{16r^3+48r^2+44r+12}$	$\frac{1}{2}$
b_3	$\frac{(r+4)r(-1+r)(r-2)}{64r^4+352r^3+656r^2+488r+120}$	0
b_4	$\frac{(r-2)(-1+r)r(r+5)}{256r^4+2048r^3+5504r^2+5632r+1680}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{(r+6)\,(r+2)\, r(r-2)\,(-1+r)}{1024r^5 + 12800r^4 + 58880r^3 + 121600r^2 + 108096r + 30240}$$

Which for the root $r = 0$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r^2-4}{4r^2+6r+2}$	-2
b_2	$\frac{r^3-7r+6}{16r^3+48r^2+44r+12}$	$\frac{1}{2}$
b_3	$\frac{(r+4)r(-1+r)(r-2)}{64r^4+352r^3+656r^2+488r+120}$	0
b_4	$\frac{(r-2)(-1+r)r(r+5)}{256r^4+2048r^3+5504r^2+5632r+1680}$	0
b_5	$\frac{(r+6)(r+2)r(r-2)(-1+r)}{1024r^5+12800r^4+58880r^3+121600r^2+108096r+30240}$	0

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{(r^2+10r+21)(-1+r)\,r(r^2-4)}{64(2r+11)(32r^5+400r^4+1840r^3+3800r^2+3378r+945)}$$

Which for the root $r = 0$ becomes

$$b_6 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r^2-4}{4r^2+6r+2}$	-2
b_2	$\frac{r^3-7r+6}{16r^3+48r^2+44r+12}$	$\frac{1}{2}$
b_3	$\frac{(r+4)r(-1+r)(r-2)}{64r^4+352r^3+656r^2+488r+120}$	0
b_4	$\frac{(r-2)(-1+r)r(r+5)}{256r^4+2048r^3+5504r^2+5632r+1680}$	0
b_5	$\frac{(r+6)(r+2)r(r-2)(-1+r)}{1024r^5+12800r^4+58880r^3+121600r^2+108096r+30240}$	0
b_6	$\frac{(r^2+10r+21)(-1+r)r(r^2-4)}{64(2r+11)(32r^5+400r^4+1840r^3+3800r^2+3378r+945)}$	0

For $n = 7$, using the above recursive equation gives

$$b_7 = \frac{(r^2+12r+32)(r^2-4)\,r(-1+r)(r+3)}{128(2r+13)(2r+11)(32r^5+400r^4+1840r^3+3800r^2+3378r+945)}$$

Which for the root $r = 0$ becomes

$$b_7 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r^2-4}{4r^2+6r+2}$	-2
b_2	$\frac{r^3-7r+6}{16r^3+48r^2+44r+12}$	$\frac{1}{2}$
b_3	$\frac{(r+4)r(-1+r)(r-2)}{64r^4+352r^3+656r^2+488r+120}$	0
b_4	$\frac{(r-2)(-1+r)r(r+5)}{256r^4+2048r^3+5504r^2+5632r+1680}$	0
b_5	$\frac{(r+6)(r+2)r(r-2)(-1+r)}{1024r^5+12800r^4+58880r^3+121600r^2+108096r+30240}$	0
b_6	$\frac{(r^2+10r+21)(-1+r)r(r^2-4)}{64(2r+11)(32r^5+400r^4+1840r^3+3800r^2+3378r+945)}$	0
b_7	$\frac{(r^2+12r+32)(r^2-4)r(-1+r)(r+3)}{128(2r+13)(2r+11)(32r^5+400r^4+1840r^3+3800r^2+3378r+945)}$	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\
&= 1 - 2x + \frac{x^2}{2} + O(x^8)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \sqrt{x}\left(1 - \frac{5x}{8} + \frac{7x^2}{128} + \frac{3x^3}{1024} + \frac{11x^4}{32768} + \frac{13x^5}{262144} + \frac{35x^6}{4194304} + \frac{51x^7}{33554432} + O(x^8)\right) \\
&\quad + c_2\left(1 - 2x + \frac{x^2}{2} + O(x^8)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \sqrt{x}\left(1 - \frac{5x}{8} + \frac{7x^2}{128} + \frac{3x^3}{1024} + \frac{11x^4}{32768} + \frac{13x^5}{262144} + \frac{35x^6}{4194304} + \frac{51x^7}{33554432} + O(x^8)\right) \\
&\quad + c_2\left(1 - 2x + \frac{x^2}{2} + O(x^8)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1\sqrt{x}\left(1-\frac{5x}{8}+\frac{7x^2}{128}+\frac{3x^3}{1024}+\frac{11x^4}{32768}+\frac{13x^5}{262144}+\frac{35x^6}{4194304}+\frac{51x^7}{33554432}+O(x^8)\right) + c_2\left(1-2x+\frac{x^2}{2}+O(x^8)\right) \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\left(1-\frac{5x}{8}+\frac{7x^2}{128}+\frac{3x^3}{1024}+\frac{11x^4}{32768}+\frac{13x^5}{262144}+\frac{35x^6}{4194304}+\frac{51x^7}{33554432}+O(x^8)\right) + c_2\left(1-2x+\frac{x^2}{2}+O(x^8)\right)$$

Verified OK.

4.11.1 Maple step by step solution

Let's solve

$-y''x(x-4)+(-x+2)\,y'+4y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{4y}{x(x-4)}-\frac{(x-2)y'}{x(x-4)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(x-2)y'}{x(x-4)}-\frac{4y}{x(x-4)}=0$

□ Check to see if x_0 is a regular singular point

  ∘ Define functions

    $\left[P_2(x)=\frac{x-2}{x(x-4)}, P_3(x)=-\frac{4}{x(x-4)}\right]$

  ∘ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{1}{2}$

  ∘ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

- $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''x(x-4)+(x-2)\,y'-4y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m=0..1$

  $x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m=1..2$

  $x^m \cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m \cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r(-1+2r)\,x^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)\,(2k+1+2r)+a_k(k+r+2)\,(k+r-2)\right)x^{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-1+2r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{0,\frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $-4(k+1+r)\left(k+r+\frac{1}{2}\right)a_{k+1}+a_k(k+r+2)\,(k+r-2)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k(k+r+2)(k+r-2)}{2(k+1+r)(2k+1+2r)}$

- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{a_k(k+2)(k-2)}{2(k+1)(2k+1)}$
- Apply recursion relation for $k = 0$

  $a_1 = -2a_0$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{4}$
- Express in terms of a_0

  $a_2 = \frac{a_0}{2}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y = a_0 \cdot \left(1 - 2x + \frac{1}{2}x^2\right)$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{2\left(k+\frac{3}{2}\right)(2k+2)}$
- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{2\left(k+\frac{3}{2}\right)(2k+2)}\right]$
- Combine solutions and rename parameters

  $\left[y = a_0 \cdot \left(1 - 2x + \frac{1}{2}x^2\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), b_{k+1} = \frac{b_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{2\left(k+\frac{3}{2}\right)(2k+2)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
   Solution is available but has compositions of trig with ln functions of radicals. Attempt
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 42

```
Order:=8;
dsolve(x*(4-x)*diff(y(x),x$2)+(2-x)*diff(y(x),x)+4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\sqrt{x}\left(1-\frac{5}{8}x+\frac{7}{128}x^2+\frac{3}{1024}x^3+\frac{11}{32768}x^4+\frac{13}{262144}x^5+\frac{35}{4194304}x^6+\frac{51}{33554432}x^7+\mathrm{O}\left(x^8\right)\right)+c_2\left(1-2x+\frac{1}{2}x^2+\mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 76

```
AsymptoticDSolveValue[x*(4-x)*y''[x]+(2-x)*y'[x]+4*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(\frac{x^2}{2}-2x+1\right)+c_1\sqrt{x}\left(\frac{51x^7}{33554432}+\frac{35x^6}{4194304}+\frac{13x^5}{262144}+\frac{11x^4}{32768}+\frac{3x^3}{1024}+\frac{7x^2}{128}-\frac{5x}{8}+1\right)$$

4.12 problem 12

Internal problem ID [6928]
Internal file name [`OUTPUT/6171_Friday_August_12_2022_11_04_13_PM_5854370/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.4 Indicial Equation with Difference of Roots Nonintegral. Exercises page 365
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{3x^2y'' + xy' - (1+x)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$3x^2y'' + xy' + (-x-1)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{3x}$$
$$q(x) = -\frac{1+x}{3x^2}$$

Table 48: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{3x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1+x}{3x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$3x^2y'' + xy' + (-x - 1)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$3x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + x\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (-x-1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$3x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$3x^r a_0 r(-1+r) + x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(3x^r r(-1+r) + x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(3r^2 - 2r - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$3r^2 - 2r - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = -\frac{1}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(3r^2 - 2r - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{4}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{3}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$3a_n(n+r)(n+r-1) + a_n(n+r) - a_{n-1} - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{3n^2 + 6nr + 3r^2 - 2n - 2r - 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-1}}{n(3n+4)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{r(3r+4)}$$

Which for the root $r = 1$ becomes

$$a_1 = \frac{1}{7}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r(3r+4)}$	$\frac{1}{7}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{9r^4 + 42r^3 + 61r^2 + 28r}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{140}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r(3r+4)}$	$\frac{1}{7}$
a_2	$\frac{1}{9r^4+42r^3+61r^2+28r}$	$\frac{1}{140}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{27r^6 + 270r^5 + 1035r^4 + 1900r^3 + 1668r^2 + 560r}$$

Which for the root $r = 1$ becomes

$$a_3 = \frac{1}{5460}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r(3r+4)}$	$\frac{1}{7}$
a_2	$\frac{1}{9r^4+42r^3+61r^2+28r}$	$\frac{1}{140}$
a_3	$\frac{1}{27r^6+270r^5+1035r^4+1900r^3+1668r^2+560r}$	$\frac{1}{5460}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{81r^8 + 1404r^7 + 10098r^6 + 39000r^5 + 87169r^4 + 112476r^3 + 77372r^2 + 21840r}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{349440}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r(3r+4)}$	$\frac{1}{7}$
a_2	$\frac{1}{9r^4+42r^3+61r^2+28r}$	$\frac{1}{140}$
a_3	$\frac{1}{27r^6+270r^5+1035r^4+1900r^3+1668r^2+560r}$	$\frac{1}{5460}$
a_4	$\frac{1}{81r^8+1404r^7+10098r^6+39000r^5+87169r^4+112476r^3+77372r^2+21840r}$	$\frac{1}{349440}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{243r^{10} + 6480r^9 + 74790r^8 + 489600r^7 + 1999779r^6 + 5274160r^5 + 8960260r^4 + 9430400r^3 + 55633}$$

Which for the root $r = 1$ becomes

$$a_5 = \frac{1}{33196800}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r(3r+4)}$	$\frac{1}{7}$
a_2	$\frac{1}{9r^4+42r^3+61r^2+28r}$	$\frac{1}{140}$
a_3	$\frac{1}{27r^6+270r^5+1035r^4+1900r^3+1668r^2+560r}$	$\frac{1}{5460}$
a_4	$\frac{1}{81r^8+1404r^7+10098r^6+39000r^5+87169r^4+112476r^3+77372r^2+21840r}$	$\frac{1}{349440}$
a_5	$\frac{1}{243r^{10}+6480r^9+74790r^8+489600r^7+1999779r^6+5274160r^5+8960260r^4+9430400r^3+5563328r^2+1397760r}$	$\frac{1}{33196800}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{1}{r\left(243r^9 + 6480r^8 + 74790r^7 + 489600r^6 + 1999779r^5 + 5274160r^4 + 8960260r^3 + 9430400r^2 + 556\right.}$$

Which for the root $r = 1$ becomes

$$a_6 = \frac{1}{4381977600}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r(3r+4)}$	$\frac{1}{7}$
a_2	$\frac{1}{9r^4+42r^3+61r^2+28r}$	$\frac{1}{140}$
a_3	$\frac{1}{27r^6+270r^5+1035r^4+1900r^3+1668r^2+560r}$	$\frac{1}{5460}$
a_4	$\frac{1}{81r^8+1404r^7+10098r^6+39000r^5+87169r^4+112476r^3+77372r^2+21840r}$	$\frac{1}{349440}$
a_5	$\frac{1}{243r^{10}+6480r^9+74790r^8+489600r^7+1999779r^6+5274160r^5+8960260r^4+9430400r^3+5563328r^2+1397760r}$	$\frac{1}{33196800}$
a_6	$\frac{1}{r(243r^9+6480r^8+74790r^7+489600r^6+1999779r^5+5274160r^4+8960260r^3+9430400r^2+5563328r+1397760)(3r^2+34r+95)}$	$\frac{1}{43819776}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{1}{r\left(243r^9 + 6480r^8 + 74790r^7 + 489600r^6 + 1999779r^5 + 5274160r^4 + 8960260r^3 + 9430400r^2 + 556\right.}$$

Which for the root $r = 1$ becomes

$$a_7 = \frac{1}{766846080000}$$

And the table now becomes

n	$a_{n,r}$
a_0	1
a_1	$\frac{1}{r(3r+4)}$
a_2	$\frac{1}{9r^4+42r^3+61r^2+28r}$
a_3	$\frac{1}{27r^6+270r^5+1035r^4+1900r^3+1668r^2+560r}$
a_4	$\frac{1}{81r^8+1404r^7+10098r^6+39000r^5+87169r^4+112476r^3+77372r^2+21840r}$
a_5	$\frac{1}{243r^{10}+6480r^9+74790r^8+489600r^7+1999779r^6+5274160r^5+8960260r^4+9430400r^3+5563328r^2+1397760r}$
a_6	$\frac{1}{r(243r^9+6480r^8+74790r^7+489600r^6+1999779r^5+5274160r^4+8960260r^3+9430400r^2+5563328r+1397760)(3r^2+34r+95)}$
a_7	$\frac{1}{r(243r^9+6480r^8+74790r^7+489600r^6+1999779r^5+5274160r^4+8960260r^3+9430400r^2+5563328r+1397760)(3r^2+34r+95)(3r^2+40r+13}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \dots\right) \\
&= x\left(1 + \frac{x}{7} + \frac{x^2}{140} + \frac{x^3}{5460} + \frac{x^4}{349440} + \frac{x^5}{33196800} + \frac{x^6}{4381977600} + \frac{x^7}{766846080000} + O\left(x^8\right)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
3b_n(n+r)(n+r-1) + b_n(n+r) - b_{n-1} - b_n = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = \frac{b_{n-1}}{3n^2 + 6nr + 3r^2 - 2n - 2r - 1} \tag{4}
$$

Which for the root $r = -\frac{1}{3}$ becomes

$$
b_n = \frac{b_{n-1}}{n(3n-4)} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{1}{r(3r+4)}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_1 = -1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{r(3r+4)}$	-1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{9r^4 + 42r^3 + 61r^2 + 28r}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_2 = -\frac{1}{4}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{r(3r+4)}$	-1
b_2	$\frac{1}{9r^4+42r^3+61r^2+28r}$	$-\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{1}{27r^6 + 270r^5 + 1035r^4 + 1900r^3 + 1668r^2 + 560r}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_3 = -\frac{1}{60}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{r(3r+4)}$	-1
b_2	$\frac{1}{9r^4+42r^3+61r^2+28r}$	$-\frac{1}{4}$
b_3	$\frac{1}{27r^6+270r^5+1035r^4+1900r^3+1668r^2+560r}$	$-\frac{1}{60}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{81r^8 + 1404r^7 + 10098r^6 + 39000r^5 + 87169r^4 + 112476r^3 + 77372r^2 + 21840r}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_4 = -\frac{1}{1920}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{r(3r+4)}$	-1
b_2	$\frac{1}{9r^4+42r^3+61r^2+28r}$	$-\frac{1}{4}$
b_3	$\frac{1}{27r^6+270r^5+1035r^4+1900r^3+1668r^2+560r}$	$-\frac{1}{60}$
b_4	$\frac{1}{81r^8+1404r^7+10098r^6+39000r^5+87169r^4+112476r^3+77372r^2+21840r}$	$-\frac{1}{1920}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{1}{243r^{10} + 6480r^9 + 74790r^8 + 489600r^7 + 1999779r^6 + 5274160r^5 + 8960260r^4 + 9430400r^3 + 55633}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_5 = -\frac{1}{105600}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{r(3r+4)}$	-1
b_2	$\frac{1}{9r^4+42r^3+61r^2+28r}$	$-\frac{1}{4}$
b_3	$\frac{1}{27r^6+270r^5+1035r^4+1900r^3+1668r^2+560r}$	$-\frac{1}{60}$
b_4	$\frac{1}{81r^8+1404r^7+10098r^6+39000r^5+87169r^4+112476r^3+77372r^2+21840r}$	$-\frac{1}{1920}$
b_5	$\frac{1}{243r^{10}+6480r^9+74790r^8+489600r^7+1999779r^6+5274160r^5+8960260r^4+9430400r^3+5563328r^2+1397760r}$	$-\frac{1}{105600}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{1}{r\left(243r^9 + 6480r^8 + 74790r^7 + 489600r^6 + 1999779r^5 + 5274160r^4 + 8960260r^3 + 9430400r^2 + 556\right.}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_6 = -\frac{1}{8870400}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{r(3r+4)}$	-1
b_2	$\frac{1}{9r^4+42r^3+61r^2+28r}$	$-\frac{1}{4}$
b_3	$\frac{1}{27r^6+270r^5+1035r^4+1900r^3+1668r^2+560r}$	$-\frac{1}{60}$
b_4	$\frac{1}{81r^8+1404r^7+10098r^6+39000r^5+87169r^4+112476r^3+77372r^2+21840r}$	$-\frac{1}{1920}$
b_5	$\frac{1}{243r^{10}+6480r^9+74790r^8+489600r^7+1999779r^6+5274160r^5+8960260r^4+9430400r^3+5563328r^2+1397760r}$	$-\frac{1}{105600}$
b_6	$\frac{1}{r(243r^9+6480r^8+74790r^7+489600r^6+1999779r^5+5274160r^4+8960260r^3+9430400r^2+5563328r+1397760)(3r^2+34r+95)}$	$-\frac{1}{887040}$

For $n = 7$, using the above recursive equation gives

$$b_7 = \frac{1}{r\left(243r^9 + 6480r^8 + 74790r^7 + 489600r^6 + 1999779r^5 + 5274160r^4 + 8960260r^3 + 9430400r^2 + 556\right.}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_7 = -\frac{1}{1055577600}$$

And the table now becomes

n	$b_{n,r}$
b_0	1
b_1	$\frac{1}{r(3r+4)}$
b_2	$\frac{1}{9r^4+42r^3+61r^2+28r}$
b_3	$\frac{1}{27r^6+270r^5+1035r^4+1900r^3+1668r^2+560r}$
b_4	$\frac{1}{81r^8+1404r^7+10098r^6+39000r^5+87169r^4+112476r^3+77372r^2+21840r}$
b_5	$\frac{1}{243r^{10}+6480r^9+74790r^8+489600r^7+1999779r^6+5274160r^5+8960260r^4+9430400r^3+5563328r^2+1397760r}$
b_6	$\frac{1}{r(243r^9+6480r^8+74790r^7+489600r^6+1999779r^5+5274160r^4+8960260r^3+9430400r^2+5563328r+1397760)(3r^2+34r+95)}$
b_7	$\frac{1}{r(243r^9+6480r^8+74790r^7+489600r^6+1999779r^5+5274160r^4+8960260r^3+9430400r^2+5563328r+1397760)(3r^2+34r+95)(3r^2+40r+13}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots\right) \\
&= \frac{1 - x - \frac{x^2}{4} - \frac{x^3}{60} - \frac{x^4}{1920} - \frac{x^5}{105600} - \frac{x^6}{8870400} - \frac{x^7}{1055577600} + O(x^8)}{x^{\frac{1}{3}}}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x \left(1 + \frac{x}{7} + \frac{x^2}{140} + \frac{x^3}{5460} + \frac{x^4}{349440} + \frac{x^5}{33196800} + \frac{x^6}{4381977600} + \frac{x^7}{766846080000} \right. \\
&\quad \left. + O(x^8)\right) + \frac{c_2 \left(1 - x - \frac{x^2}{4} - \frac{x^3}{60} - \frac{x^4}{1920} - \frac{x^5}{105600} - \frac{x^6}{8870400} - \frac{x^7}{1055577600} + O(x^8)\right)}{x^{\frac{1}{3}}}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x \left(1 + \frac{x}{7} + \frac{x^2}{140} + \frac{x^3}{5460} + \frac{x^4}{349440} + \frac{x^5}{33196800} + \frac{x^6}{4381977600} + \frac{x^7}{766846080000} \right. \\
&\quad \left. + O(x^8)\right) + \frac{c_2 \left(1 - x - \frac{x^2}{4} - \frac{x^3}{60} - \frac{x^4}{1920} - \frac{x^5}{105600} - \frac{x^6}{8870400} - \frac{x^7}{1055577600} + O(x^8)\right)}{x^{\frac{1}{3}}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x \left(1+\frac{x}{7}+\frac{x^2}{140}+\frac{x^3}{5460}+\frac{x^4}{349440}+\frac{x^5}{33196800}+\frac{x^6}{4381977600}+\frac{x^7}{766846080000}+O\left(x^8\right)\right) + \frac{c_2 \left(1-x-\frac{x^2}{4}-\frac{x^3}{60}-\frac{x^4}{1920}-\frac{x^5}{105600}-\frac{x^6}{8870400}-\frac{x^7}{1055577600}+O\left(x^8\right)\right)}{x^{\frac{1}{3}}}
$$

Verification of solutions

$$
y = c_1 x \left(1+\frac{x}{7}+\frac{x^2}{140}+\frac{x^3}{5460}+\frac{x^4}{349440}+\frac{x^5}{33196800}+\frac{x^6}{4381977600}+\frac{x^7}{766846080000}+O\left(x^8\right)\right) + \frac{c_2 \left(1-x-\frac{x^2}{4}-\frac{x^3}{60}-\frac{x^4}{1920}-\frac{x^5}{105600}-\frac{x^6}{8870400}-\frac{x^7}{1055577600}+O\left(x^8\right)\right)}{x^{\frac{1}{3}}}
$$

Verified OK.

4.12.1 Maple step by step solution

Let's solve

$3x^2 y'' + x y' + (-x-1) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{3x} + \frac{(1+x)y}{3x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{3x} - \frac{(1+x)y}{3x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{3x}, P_3(x) = -\frac{1+x}{3x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{1}{3}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^{2}\cdot P_{3}(x)\right)\bigg|_{x=0}=-\frac{1}{3}$

- $x=0$is a regular singular point

  Check to see if $x_{0}=0$ is a regular singular point

  $x_{0}=0$

- Multiply by denominators

  $3x^{2}y''+xy'+\left(-x-1\right)y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_{k}x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $x^{m}\cdot y$ to series expansion for $m=0..1$

    $x^{m}\cdot y=\sum_{k=0}^{\infty}a_{k}x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^{m}\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  - Convert $x\cdot y'$ to series expansion

    $x\cdot y'=\sum_{k=0}^{\infty}a_{k}(k+r)\,x^{k+r}$

  - Convert $x^{2}\cdot y''$ to series expansion

    $x^{2}\cdot y''=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(1+3r)\,(-1+r)\,x^{r}+\left(\sum_{k=1}^{\infty}\left(a_{k}(3k+3r+1)\,(k+r-1)-a_{k-1}\right)x^{k+r}\right)=0$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(1+3r)\,(-1+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{1,-\frac{1}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $3\left(k+r+\frac{1}{3}\right)(k+r-1)\,a_{k}-a_{k-1}=0$

- Shift index using $k->k+1$

  $3\left(k+\frac{4}{3}+r\right)(k+r)\, a_{k+1}-a_{k}=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_{k}}{(3k+4+3r)(k+r)}$

- Recursion relation for $r=1$

  $a_{k+1}=\frac{a_{k}}{(3k+7)(k+1)}$

- Solution for $r=1$

  $\left[y=\sum_{k=0}^{\infty} a_{k} x^{k+1}, a_{k+1}=\frac{a_{k}}{(3k+7)(k+1)}\right]$

- Recursion relation for $r=-\frac{1}{3}$

  $a_{k+1}=\frac{a_{k}}{(3k+3)\left(k-\frac{1}{3}\right)}$

- Solution for $r=-\frac{1}{3}$

  $\left[y=\sum_{k=0}^{\infty} a_{k} x^{k-\frac{1}{3}}, a_{k+1}=\frac{a_{k}}{(3k+3)\left(k-\frac{1}{3}\right)}\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_{k} x^{k+1}\right)+\left(\sum_{k=0}^{\infty} b_{k} x^{k-\frac{1}{3}}\right), a_{k+1}=\frac{a_{k}}{(3k+7)(k+1)}, b_{k+1}=\frac{b_{k}}{(3k+3)\left(k-\frac{1}{3}\right)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 53

```
Order:=8;
dsolve(3*x^2*diff(y(x),x$2)+x*diff(y(x),x)-(1+x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1\left(1 - x - \frac{1}{4}x^2 - \frac{1}{60}x^3 - \frac{1}{1920}x^4 - \frac{1}{105600}x^5 - \frac{1}{8870400}x^6 - \frac{1}{1055577600}x^7 + \mathrm{O}\left(x^8\right)\right)}{x^{\frac{1}{3}}} + c_2 x\left(1 + \frac{1}{7}x + \frac{1}{140}x^2 + \frac{1}{5460}x^3 + \frac{1}{349440}x^4 + \frac{1}{33196800}x^5 + \frac{1}{4381977600}x^6 + \frac{1}{766846080000}x^7 + \mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 112

```
AsymptoticDSolveValue[3*x^2*y''[x]+x*y'[x]-(1+x)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1 x\left(\frac{x^7}{766846080000} + \frac{x^6}{4381977600} + \frac{x^5}{33196800} + \frac{x^4}{349440} + \frac{x^3}{5460} + \frac{x^2}{140} + \frac{x}{7} + 1\right) + \frac{c_2\left(-\frac{x^7}{1055577600} - \frac{x^6}{8870400} - \frac{x^5}{105600} - \frac{x^4}{1920} - \frac{x^3}{60} - \frac{x^2}{4} - x + 1\right)}{\sqrt[3]{x}}$$

4.13 problem 13

Internal problem ID [6929]
Internal file name [`OUTPUT/6172_Friday_August_12_2022_11_04_15_PM_72395354/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.4 Indicial Equation with Difference of Roots Nonintegral. Exercises page 365
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2xy'' + (2x+1)\,y' + 4y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2xy'' + (2x+1)\,y' + 4y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{2x+1}{2x}$$
$$q(x) = \frac{2}{x}$$

Table 50: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2x+1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{2}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2xy'' + (2x+1)\,y' + 4y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + (2x+1)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + 4\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 2x^{n+r}a_n(n+r)\right)+\left(\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}\right)+\left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{n+r}a_n(n+r)=\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\,x^{n+r-1}$$

$$\sum_{n=0}^{\infty} 4a_n x^{n+r}=\sum_{n=1}^{\infty} 4a_{n-1}x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\,x^{n+r-1}\right)+\left(\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}\right)+\left(\sum_{n=1}^{\infty} 4a_{n-1}x^{n+r-1}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1}a_n(n+r)(n+r-1)+(n+r)\,a_n x^{n+r-1}=0$$

When $n=0$ the above becomes

$$2x^{-1+r}a_0r(-1+r)+ra_0x^{-1+r}=0$$

Or

$$\left(2x^{-1+r}r(-1+r)+r\,x^{-1+r}\right)a_0=0$$

Since $a_0\neq 0$ then the above simplifies to

$$r\,x^{-1+r}(-1+2r)=0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(-1 + 2r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) + 2a_{n-1}(n+r-1) + a_n(n+r) + 4a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2a_{n-1}(n+r+1)}{2n^2 + 4nr + 2r^2 - n - r} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{a_{n-1}(-3 - 2n)}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-4 - 2r}{2r^2 + 3r + 1}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = -\frac{5}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4-2r}{2r^2+3r+1}$	$-\frac{5}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{12 + 4r}{4r^3 + 12r^2 + 11r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{7}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4-2r}{2r^2+3r+1}$	$-\frac{5}{3}$
a_2	$\frac{12+4r}{4r^3+12r^2+11r+3}$	$\frac{7}{6}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-32 - 8r}{8r^4 + 44r^3 + 82r^2 + 61r + 15}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = -\frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4-2r}{2r^2+3r+1}$	$-\frac{5}{3}$
a_2	$\frac{12+4r}{4r^3+12r^2+11r+3}$	$\frac{7}{6}$
a_3	$\frac{-32-8r}{8r^4+44r^3+82r^2+61r+15}$	$-\frac{1}{2}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{80 + 16r}{16r^5 + 144r^4 + 472r^3 + 696r^2 + 457r + 105}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{11}{72}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4-2r}{2r^2+3r+1}$	$-\frac{5}{3}$
a_2	$\frac{12+4r}{4r^3+12r^2+11r+3}$	$\frac{7}{6}$
a_3	$\frac{-32-8r}{8r^4+44r^3+82r^2+61r+15}$	$-\frac{1}{2}$
a_4	$\frac{80+16r}{16r^5+144r^4+472r^3+696r^2+457r+105}$	$\frac{11}{72}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-192 - 32r}{32r^6 + 432r^5 + 2240r^4 + 5640r^3 + 7178r^2 + 4323r + 945}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = -\frac{13}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4-2r}{2r^2+3r+1}$	$-\frac{5}{3}$
a_2	$\frac{12+4r}{4r^3+12r^2+11r+3}$	$\frac{7}{6}$
a_3	$\frac{-32-8r}{8r^4+44r^3+82r^2+61r+15}$	$-\frac{1}{2}$
a_4	$\frac{80+16r}{16r^5+144r^4+472r^3+696r^2+457r+105}$	$\frac{11}{72}$
a_5	$\frac{-192-32r}{32r^6+432r^5+2240r^4+5640r^3+7178r^2+4323r+945}$	$-\frac{13}{360}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{448 + 64r}{64r^7 + 1216r^6 + 9232r^5 + 35920r^4 + 76396r^3 + 87604r^2 + 49443r + 10395}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_6 = \frac{1}{144}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4-2r}{2r^2+3r+1}$	$-\frac{5}{3}$
a_2	$\frac{12+4r}{4r^3+12r^2+11r+3}$	$\frac{7}{6}$
a_3	$\frac{-32-8r}{8r^4+44r^3+82r^2+61r+15}$	$-\frac{1}{2}$
a_4	$\frac{80+16r}{16r^5+144r^4+472r^3+696r^2+457r+105}$	$\frac{11}{72}$
a_5	$\frac{-192-32r}{32r^6+432r^5+2240r^4+5640r^3+7178r^2+4323r+945}$	$-\frac{13}{360}$
a_6	$\frac{448+64r}{64r^7+1216r^6+9232r^5+35920r^4+76396r^3+87604r^2+49443r+10395}$	$\frac{1}{144}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{-1024 - 128r}{128r^8 + 3264r^7 + 34272r^6 + 191856r^5 + 619752r^4 + 1168356r^3 + 1237738r^2 + 663549r + 135135}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_7 = -\frac{17}{15120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4-2r}{2r^2+3r+1}$	$-\frac{5}{3}$
a_2	$\frac{12+4r}{4r^3+12r^2+11r+3}$	$\frac{7}{6}$
a_3	$\frac{-32-8r}{8r^4+44r^3+82r^2+61r+15}$	$-\frac{1}{2}$
a_4	$\frac{80+16r}{16r^5+144r^4+472r^3+696r^2+457r+105}$	$\frac{11}{72}$
a_5	$\frac{-192-32r}{32r^6+432r^5+2240r^4+5640r^3+7178r^2+4323r+945}$	$-\frac{13}{360}$
a_6	$\frac{448+64r}{64r^7+1216r^6+9232r^5+35920r^4+76396r^3+87604r^2+49443r+10395}$	$\frac{1}{144}$
a_7	$\frac{-1024-128r}{128r^8+3264r^7+34272r^6+191856r^5+619752r^4+1168356r^3+1237738r^2+663549r+135135}$	$-\frac{17}{15120}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\ &= \sqrt{x}\left(1 - \frac{5x}{3} + \frac{7x^2}{6} - \frac{x^3}{2} + \frac{11x^4}{72} - \frac{13x^5}{360} + \frac{x^6}{144} - \frac{17x^7}{15120} + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) + 2b_{n-1}(n+r-1) + (n+r)b_n + 4b_{n-1} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{2b_{n-1}(n+r+1)}{2n^2 + 4nr + 2r^2 - n - r} \tag{4}$$

Which for the root $r = 0$ becomes

$$b_n = -\frac{2b_{n-1}(n+1)}{n(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{-4 - 2r}{2r^2 + 3r + 1}$$

Which for the root $r = 0$ becomes

$$b_1 = -4$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4-2r}{2r^2+3r+1}$	-4

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{12 + 4r}{4r^3 + 12r^2 + 11r + 3}$$

Which for the root $r = 0$ becomes

$$b_2 = 4$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4-2r}{2r^2+3r+1}$	-4
b_2	$\frac{12+4r}{4r^3+12r^2+11r+3}$	4

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{-32 - 8r}{8r^4 + 44r^3 + 82r^2 + 61r + 15}$$

Which for the root $r = 0$ becomes

$$b_3 = -\frac{32}{15}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4-2r}{2r^2+3r+1}$	-4
b_2	$\frac{12+4r}{4r^3+12r^2+11r+3}$	4
b_3	$\frac{-32-8r}{8r^4+44r^3+82r^2+61r+15}$	$-\frac{32}{15}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{80 + 16r}{16r^5 + 144r^4 + 472r^3 + 696r^2 + 457r + 105}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{16}{21}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4-2r}{2r^2+3r+1}$	-4
b_2	$\frac{12+4r}{4r^3+12r^2+11r+3}$	4
b_3	$\frac{-32-8r}{8r^4+44r^3+82r^2+61r+15}$	$-\frac{32}{15}$
b_4	$\frac{80+16r}{16r^5+144r^4+472r^3+696r^2+457r+105}$	$\frac{16}{21}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{-192 - 32r}{32r^6 + 432r^5 + 2240r^4 + 5640r^3 + 7178r^2 + 4323r + 945}$$

Which for the root $r = 0$ becomes

$$b_5 = -\frac{64}{315}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4-2r}{2r^2+3r+1}$	-4
b_2	$\frac{12+4r}{4r^3+12r^2+11r+3}$	4
b_3	$\frac{-32-8r}{8r^4+44r^3+82r^2+61r+15}$	$-\frac{32}{15}$
b_4	$\frac{80+16r}{16r^5+144r^4+472r^3+696r^2+457r+105}$	$\frac{16}{21}$
b_5	$\frac{-192-32r}{32r^6+432r^5+2240r^4+5640r^3+7178r^2+4323r+945}$	$-\frac{64}{315}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{448 + 64r}{64r^7 + 1216r^6 + 9232r^5 + 35920r^4 + 76396r^3 + 87604r^2 + 49443r + 10395}$$

Which for the root $r = 0$ becomes

$$b_6 = \frac{64}{1485}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4-2r}{2r^2+3r+1}$	-4
b_2	$\frac{12+4r}{4r^3+12r^2+11r+3}$	4
b_3	$\frac{-32-8r}{8r^4+44r^3+82r^2+61r+15}$	$-\frac{32}{15}$
b_4	$\frac{80+16r}{16r^5+144r^4+472r^3+696r^2+457r+105}$	$\frac{16}{21}$
b_5	$\frac{-192-32r}{32r^6+432r^5+2240r^4+5640r^3+7178r^2+4323r+945}$	$-\frac{64}{315}$
b_6	$\frac{448+64r}{64r^7+1216r^6+9232r^5+35920r^4+76396r^3+87604r^2+49443r+10395}$	$\frac{64}{1485}$

For $n = 7$, using the above recursive equation gives

$$b_7 = \frac{-1024 - 128r}{128r^8 + 3264r^7 + 34272r^6 + 191856r^5 + 619752r^4 + 1168356r^3 + 1237738r^2 + 663549r + 135135}$$

Which for the root $r = 0$ becomes

$$b_7 = -\frac{1024}{135135}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4-2r}{2r^2+3r+1}$	-4
b_2	$\frac{12+4r}{4r^3+12r^2+11r+3}$	4
b_3	$\frac{-32-8r}{8r^4+44r^3+82r^2+61r+15}$	$-\frac{32}{15}$
b_4	$\frac{80+16r}{16r^5+144r^4+472r^3+696r^2+457r+105}$	$\frac{16}{21}$
b_5	$\frac{-192-32r}{32r^6+432r^5+2240r^4+5640r^3+7178r^2+4323r+945}$	$-\frac{64}{315}$
b_6	$\frac{448+64r}{64r^7+1216r^6+9232r^5+35920r^4+76396r^3+87604r^2+49443r+10395}$	$\frac{64}{1485}$
b_7	$\frac{-1024-128r}{128r^8+3264r^7+34272r^6+191856r^5+619752r^4+1168356r^3+1237738r^2+663549r+135135}$	$-\frac{1024}{135135}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\
&= 1 - 4x + 4x^2 - \frac{32x^3}{15} + \frac{16x^4}{21} - \frac{64x^5}{315} + \frac{64x^6}{1485} - \frac{1024x^7}{135135} + O(x^8)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \sqrt{x} \left(1 - \frac{5x}{3} + \frac{7x^2}{6} - \frac{x^3}{2} + \frac{11x^4}{72} - \frac{13x^5}{360} + \frac{x^6}{144} - \frac{17x^7}{15120} + O(x^8)\right) \\
&\quad + c_2 \left(1 - 4x + 4x^2 - \frac{32x^3}{15} + \frac{16x^4}{21} - \frac{64x^5}{315} + \frac{64x^6}{1485} - \frac{1024x^7}{135135} + O(x^8)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \sqrt{x} \left(1 - \frac{5x}{3} + \frac{7x^2}{6} - \frac{x^3}{2} + \frac{11x^4}{72} - \frac{13x^5}{360} + \frac{x^6}{144} - \frac{17x^7}{15120} + O(x^8)\right) \\
&\quad + c_2 \left(1 - 4x + 4x^2 - \frac{32x^3}{15} + \frac{16x^4}{21} - \frac{64x^5}{315} + \frac{64x^6}{1485} - \frac{1024x^7}{135135} + O(x^8)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x}\left(1-\frac{5x}{3}+\frac{7x^2}{6}-\frac{x^3}{2}+\frac{11x^4}{72}-\frac{13x^5}{360}+\frac{x^6}{144}-\frac{17x^7}{15120}+O(x^8)\right) + c_2\left(1-4x+4x^2-\frac{32x^3}{15}+\frac{16x^4}{21}-\frac{64x^5}{315}+\frac{64x^6}{1485}-\frac{1024x^7}{135135}+O(x^8)\right) \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\left(1-\frac{5x}{3}+\frac{7x^2}{6}-\frac{x^3}{2}+\frac{11x^4}{72}-\frac{13x^5}{360}+\frac{x^6}{144}-\frac{17x^7}{15120}+O(x^8)\right) + c_2\left(1-4x+4x^2-\frac{32x^3}{15}+\frac{16x^4}{21}-\frac{64x^5}{315}+\frac{64x^6}{1485}-\frac{1024x^7}{135135}+O(x^8)\right)$$

Verified OK.

4.13.1 Maple step by step solution

Let's solve

$2xy''+(2x+1)\,y'+4y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(2x+1)y'}{2x}-\frac{2y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(2x+1)y'}{2x}+\frac{2y}{x}=0$

☐ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{2x+1}{2x}, P_3(x)=\frac{2}{x}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{1}{2}$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

  - $x=0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$2xy'' + (2x+1)\,y' + 4y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$

  ◦ Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

  ◦ Shift index using $k- >k+1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, x^{k+r}$

Rewrite ODE with series expansions

$$a_0 r(-1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k+1+2r) + 2a_k(k+r+2)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-1+2r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{0, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

$2(k+1+r)\left(k+r+\frac{1}{2}\right) a_{k+1} + 2a_k(k+r+2) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = -\frac{2a_k(k+r+2)}{(k+1+r)(2k+1+2r)}$

- Recursion relation for $r = 0$

$a_{k+1} = -\frac{2a_k(k+2)}{(k+1)(2k+1)}$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{2a_k(k+2)}{(k+1)(2k+1)}\right]$$

- Recursion relation for $r = \frac{1}{2}$

$a_{k+1} = -\frac{2a_k\left(k+\frac{5}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{2a_k\left(k+\frac{5}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{2a_k(k+2)}{(k+1)(2k+1)}, b_{k+1} = -\frac{2b_k\left(k+\frac{5}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 52

```
Order:=8;
dsolve(2*x*diff(y(x),x$2)+(1+2*x)*diff(y(x),x)+4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\sqrt{x}\left(1-\frac{5}{3}x+\frac{7}{6}x^2-\frac{1}{2}x^3+\frac{11}{72}x^4-\frac{13}{360}x^5+\frac{1}{144}x^6-\frac{17}{15120}x^7+\mathrm{O}\left(x^8\right)\right)$$
$$+c_2\left(1-4x+4x^2-\frac{32}{15}x^3+\frac{16}{21}x^4-\frac{64}{315}x^5+\frac{64}{1485}x^6-\frac{1024}{135135}x^7+\mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 109

```
AsymptoticDSolveValue[2*x*y''[x]+(1+2*x)*y'[x]+4*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\sqrt{x}\left(-\frac{17x^7}{15120}+\frac{x^6}{144}-\frac{13x^5}{360}+\frac{11x^4}{72}-\frac{x^3}{2}+\frac{7x^2}{6}-\frac{5x}{3}+1\right) + c_2\left(-\frac{1024x^7}{135135}+\frac{64x^6}{1485}-\frac{64x^5}{315}+\frac{16x^4}{21}-\frac{32x^3}{15}+4x^2-4x+1\right)$$

4.14 problem 14

Internal problem ID [6930]
Internal file name [OUTPUT/6173_Friday_August_12_2022_11_04_18_PM_66620466/index.tex]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.4 Indicial Equation with Difference of Roots Nonintegral. Exercises page 365
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2xy'' + (2x+1)\,y' - 5y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2xy'' + (2x+1)\,y' - 5y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{2x+1}{2x}$$
$$q(x) = -\frac{5}{2x}$$

Table 52: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2x+1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{5}{2x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2xy'' + (2x + 1)\, y' - 5y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x \left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}\right) + (2x + 1) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) - 5 \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-5a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r) = \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} \left(-5a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-5a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-5a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$2x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(2x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-1+2r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(-1+2r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) + 2a_{n-1}(n+r-1) + a_n(n+r) - 5a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(2n+2r-7)}{2n^2+4nr+2r^2-n-r} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{2a_{n-1}(n-3)}{2n^2+n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{5 - 2r}{2r^2 + 3r + 1}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = \frac{4}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{5-2r}{2r^2+3r+1}$	$\frac{4}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4r^2 - 16r + 15}{4r^4 + 20r^3 + 35r^2 + 25r + 6}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{4}{15}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{5-2r}{2r^2+3r+1}$	$\frac{4}{3}$
a_2	$\frac{4r^2-16r+15}{4r^4+20r^3+35r^2+25r+6}$	$\frac{4}{15}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-8r^3 + 36r^2 - 46r + 15}{8r^6 + 84r^5 + 350r^4 + 735r^3 + 812r^2 + 441r + 90}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{5-2r}{2r^2+3r+1}$	$\frac{4}{3}$
a_2	$\frac{4r^2-16r+15}{4r^4+20r^3+35r^2+25r+6}$	$\frac{4}{15}$
a_3	$\frac{-8r^3+36r^2-46r+15}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{8r^3 - 36r^2 + 46r - 15}{8r^7 + 140r^6 + 1022r^5 + 4025r^4 + 9212r^3 + 12215r^2 + 8658r + 2520}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{5-2r}{2r^2+3r+1}$	$\frac{4}{3}$
a_2	$\frac{4r^2-16r+15}{4r^4+20r^3+35r^2+25r+6}$	$\frac{4}{15}$
a_3	$\frac{-8r^3+36r^2-46r+15}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	0
a_4	$\frac{8r^3-36r^2+46r-15}{8r^7+140r^6+1022r^5+4025r^4+9212r^3+12215r^2+8658r+2520}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-8r^3 + 36r^2 - 46r + 15}{8r^8 + 204r^7 + 2226r^6 + 13545r^5 + 50127r^4 + 115101r^3 + 159319r^2 + 120630r + 37800}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{5-2r}{2r^2+3r+1}$	$\frac{4}{3}$
a_2	$\frac{4r^2-16r+15}{4r^4+20r^3+35r^2+25r+6}$	$\frac{4}{15}$
a_3	$\frac{-8r^3+36r^2-46r+15}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	0
a_4	$\frac{8r^3-36r^2+46r-15}{8r^7+140r^6+1022r^5+4025r^4+9212r^3+12215r^2+8658r+2520}$	0
a_5	$\frac{-8r^3+36r^2-46r+15}{8r^8+204r^7+2226r^6+13545r^5+50127r^4+115101r^3+159319r^2+120630r+37800}$	0

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{8r^3 - 36r^2 + 46r - 15}{8r^9 + 276r^8 + 4146r^7 + 35511r^6 + 190575r^5 + 662109r^4 + 1481899r^3 + 2046384r^2 + 1566612r + 49}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_6 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{5-2r}{2r^2+3r+1}$	$\frac{4}{3}$
a_2	$\frac{4r^2-16r+15}{4r^4+20r^3+35r^2+25r+6}$	$\frac{4}{15}$
a_3	$\frac{-8r^3+36r^2-46r+15}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	0
a_4	$\frac{8r^3-36r^2+46r-15}{8r^7+140r^6+1022r^5+4025r^4+9212r^3+12215r^2+8658r+2520}$	0
a_5	$\frac{-8r^3+36r^2-46r+15}{8r^8+204r^7+2226r^6+13545r^5+50127r^4+115101r^3+159319r^2+120630r+37800}$	0
a_6	$\frac{8r^3-36r^2+46r-15}{8r^9+276r^8+4146r^7+35511r^6+190575r^5+662109r^4+1481899r^3+2046384r^2+1566612r+498960}$	0

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{-8r^3 + 36r^2 - 46r + 15}{8r^{10} + 356r^9 + 6990r^8 + 79575r^7 + 580104r^6 + 2820258r^5 + 9220630r^4 + 19905775r^3 + 26948988r^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{5-2r}{2r^2+3r+1}$	$\frac{4}{3}$
a_2	$\frac{4r^2-16r+15}{4r^4+20r^3+35r^2+25r+6}$	$\frac{4}{15}$
a_3	$\frac{-8r^3+36r^2-46r+15}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	0
a_4	$\frac{8r^3-36r^2+46r-15}{8r^7+140r^6+1022r^5+4025r^4+9212r^3+12215r^2+8658r+2520}$	0
a_5	$\frac{-8r^3+36r^2-46r+15}{8r^8+204r^7+2226r^6+13545r^5+50127r^4+115101r^3+159319r^2+120630r+37800}$	0
a_6	$\frac{8r^3-36r^2+46r-15}{8r^9+276r^8+4146r^7+35511r^6+190575r^5+662109r^4+1481899r^3+2046384r^2+1566612r+498960}$	0
a_7	$\frac{-8r^3+36r^2-46r+15}{8r^{10}+356r^9+6990r^8+79575r^7+580104r^6+2820258r^5+9220630r^4+19905775r^3+26948988r^2+20437236r+6486480}$	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\ &= \sqrt{x}\left(1 + \frac{4x}{3} + \frac{4x^2}{15} + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) + 2b_{n-1}(n+r-1) + (n+r)b_n - 5b_{n-1} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-1}(2n+2r-7)}{2n^2+4nr+2r^2-n-r} \tag{4}$$

Which for the root $r = 0$ becomes

$$b_n = \frac{b_{n-1}(7-2n)}{2n^2-n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{5 - 2r}{2r^2 + 3r + 1}$$

Which for the root $r = 0$ becomes

$$b_1 = 5$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{5-2r}{2r^2+3r+1}$	5

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4r^2 - 16r + 15}{4r^4 + 20r^3 + 35r^2 + 25r + 6}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{5}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{5-2r}{2r^2+3r+1}$	5
b_2	$\frac{4r^2-16r+15}{4r^4+20r^3+35r^2+25r+6}$	$\frac{5}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{-8r^3 + 36r^2 - 46r + 15}{8r^6 + 84r^5 + 350r^4 + 735r^3 + 812r^2 + 441r + 90}$$

Which for the root $r = 0$ becomes

$$b_3 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{5-2r}{2r^2+3r+1}$	5
b_2	$\frac{4r^2-16r+15}{4r^4+20r^3+35r^2+25r+6}$	$\frac{5}{2}$
b_3	$\frac{-8r^3+36r^2-46r+15}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{8r^3 - 36r^2 + 46r - 15}{8r^7 + 140r^6 + 1022r^5 + 4025r^4 + 9212r^3 + 12215r^2 + 8658r + 2520}$$

Which for the root $r = 0$ becomes

$$b_4 = -\frac{1}{168}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{5-2r}{2r^2+3r+1}$	5
b_2	$\frac{4r^2-16r+15}{4r^4+20r^3+35r^2+25r+6}$	$\frac{5}{2}$
b_3	$\frac{-8r^3+36r^2-46r+15}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$\frac{1}{6}$
b_4	$\frac{8r^3-36r^2+46r-15}{8r^7+140r^6+1022r^5+4025r^4+9212r^3+12215r^2+8658r+2520}$	$-\frac{1}{168}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{-8r^3 + 36r^2 - 46r + 15}{8r^8 + 204r^7 + 2226r^6 + 13545r^5 + 50127r^4 + 115101r^3 + 159319r^2 + 120630r + 37800}$$

Which for the root $r = 0$ becomes

$$b_5 = \frac{1}{2520}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{5-2r}{2r^2+3r+1}$	5
b_2	$\frac{4r^2-16r+15}{4r^4+20r^3+35r^2+25r+6}$	$\frac{5}{2}$
b_3	$\frac{-8r^3+36r^2-46r+15}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$\frac{1}{6}$
b_4	$\frac{8r^3-36r^2+46r-15}{8r^7+140r^6+1022r^5+4025r^4+9212r^3+12215r^2+8658r+2520}$	$-\frac{1}{168}$
b_5	$\frac{-8r^3+36r^2-46r+15}{8r^8+204r^7+2226r^6+13545r^5+50127r^4+115101r^3+159319r^2+120630r+37800}$	$\frac{1}{2520}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{8r^3 - 36r^2 + 46r - 15}{8r^9 + 276r^8 + 4146r^7 + 35511r^6 + 190575r^5 + 662109r^4 + 1481899r^3 + 2046384r^2 + 1566612r + 49}$$

Which for the root $r = 0$ becomes

$$b_6 = -\frac{1}{33264}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{5-2r}{2r^2+3r+1}$	5
b_2	$\frac{4r^2-16r+15}{4r^4+20r^3+35r^2+25r+6}$	$\frac{5}{2}$
b_3	$\frac{-8r^3+36r^2-46r+15}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$\frac{1}{6}$
b_4	$\frac{8r^3-36r^2+46r-15}{8r^7+140r^6+1022r^5+4025r^4+9212r^3+12215r^2+8658r+2520}$	$-\frac{1}{168}$
b_5	$\frac{-8r^3+36r^2-46r+15}{8r^8+204r^7+2226r^6+13545r^5+50127r^4+115101r^3+159319r^2+120630r+37800}$	$\frac{1}{2520}$
b_6	$\frac{8r^3-36r^2+46r-15}{8r^9+276r^8+4146r^7+35511r^6+190575r^5+662109r^4+1481899r^3+2046384r^2+1566612r+498960}$	$-\frac{1}{33264}$

For $n = 7$, using the above recursive equation gives

$$b_7 = \frac{-8r^3 + 36r^2 - 46r + 15}{8r^{10} + 356r^9 + 6990r^8 + 79575r^7 + 580104r^6 + 2820258r^5 + 9220630r^4 + 19905775r^3 + 26948988r^2}$$

Which for the root $r = 0$ becomes

$$b_7 = \frac{1}{432432}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{5-2r}{2r^2+3r+1}$	5
b_2	$\frac{4r^2-16r+15}{4r^4+20r^3+35r^2+25r+6}$	$\frac{5}{2}$
b_3	$\frac{-8r^3+36r^2-46r+15}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$\frac{1}{6}$
b_4	$\frac{8r^3-36r^2+46r-15}{8r^7+140r^6+1022r^5+4025r^4+9212r^3+12215r^2+8658r+2520}$	$-\frac{1}{168}$
b_5	$\frac{-8r^3+36r^2-46r+15}{8r^8+204r^7+2226r^6+13545r^5+50127r^4+115101r^3+159319r^2+120630r+37800}$	$\frac{1}{2520}$
b_6	$\frac{8r^3-36r^2+46r-15}{8r^9+276r^8+4146r^7+35511r^6+190575r^5+662109r^4+1481899r^3+2046384r^2+1566612r+498960}$	$-\frac{1}{33264}$
b_7	$\frac{-8r^3+36r^2-46r+15}{8r^{10}+356r^9+6990r^8+79575r^7+580104r^6+2820258r^5+9220630r^4+19905775r^3+26948988r^2+20437236r+6486480}$	$\frac{1}{432432}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\ &= 1 + 5x + \frac{5x^2}{2} + \frac{x^3}{6} - \frac{x^4}{168} + \frac{x^5}{2520} - \frac{x^6}{33264} + \frac{x^7}{432432} + O(x^8) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \sqrt{x} \left(1 + \frac{4x}{3} + \frac{4x^2}{15} + O(x^8)\right) \\ &\quad + c_2 \left(1 + 5x + \frac{5x^2}{2} + \frac{x^3}{6} - \frac{x^4}{168} + \frac{x^5}{2520} - \frac{x^6}{33264} + \frac{x^7}{432432} + O(x^8)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 \sqrt{x} \left(1 + \frac{4x}{3} + \frac{4x^2}{15} + O(x^8)\right) \\ &\quad + c_2 \left(1 + 5x + \frac{5x^2}{2} + \frac{x^3}{6} - \frac{x^4}{168} + \frac{x^5}{2520} - \frac{x^6}{33264} + \frac{x^7}{432432} + O(x^8)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1\sqrt{x}\left(1+\frac{4x}{3}+\frac{4x^2}{15}+O(x^8)\right) + c_2\left(1+5x+\frac{5x^2}{2}+\frac{x^3}{6}-\frac{x^4}{168}+\frac{x^5}{2520}-\frac{x^6}{33264}+\frac{x^7}{432432}+O(x^8)\right) \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\left(1+\frac{4x}{3}+\frac{4x^2}{15}+O(x^8)\right) + c_2\left(1+5x+\frac{5x^2}{2}+\frac{x^3}{6}-\frac{x^4}{168}+\frac{x^5}{2520}-\frac{x^6}{33264}+\frac{x^7}{432432}+O(x^8)\right)$$

Verified OK.

4.14.1 Maple step by step solution

Let's solve

$2xy'' + (2x+1)\,y' - 5y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(2x+1)y'}{2x} + \frac{5y}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2x+1)y'}{2x} - \frac{5y}{2x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2x+1}{2x}, P_3(x) = -\frac{5}{2x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{1}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$2xy'' + (2x+1)\,y' - 5y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$

  ◦ Shift index using $k- >k+1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

Rewrite ODE with series expansions

$a_0r(-1+2r)\,x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k+1+2r) + a_k(2k+2r-5)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-1+2r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{0, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

$2(k+1+r)\left(k+r+\frac{1}{2}\right)a_{k+1} + 2a_k\left(k+r-\frac{5}{2}\right) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = -\frac{a_k(2k+2r-5)}{(k+1+r)(2k+1+2r)}$

- Recursion relation for $r = 0$

$a_{k+1} = -\frac{a_k(2k-5)}{(k+1)(2k+1)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k(2k-5)}{(k+1)(2k+1)}\right]$

- Recursion relation for $r = \frac{1}{2}$; series terminates at $k = 2$

  $a_{k+1} = -\frac{a_k(2k-4)}{\left(k+\frac{3}{2}\right)(2k+2)}$

- Apply recursion relation for $k = 0$

  $a_1 = \frac{4a_0}{3}$

- Apply recursion relation for $k = 1$

  $a_2 = \frac{a_1}{5}$

- Express in terms of a_0

  $a_2 = \frac{4a_0}{15}$

- Terminating series solution of the ODE for $r = \frac{1}{2}$. Use reduction of order to find the second li

  $y = a_0 \cdot \left(1 + \frac{4}{3}x + \frac{4}{15}x^2\right)$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + b_0 \cdot \left(1 + \frac{4}{3}x + \frac{4}{15}x^2\right), a_{k+1} = -\frac{a_k(2k-5)}{(k+1)(2k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 42

```
Order:=8;
dsolve(2*x*diff(y(x),x$2)+(1+2*x)*diff(y(x),x)-5*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\sqrt{x}\left(1+\frac{4}{3}x+\frac{4}{15}x^2+\mathrm{O}\left(x^8\right)\right) + c_2\left(1+5x+\frac{5}{2}x^2+\frac{1}{6}x^3-\frac{1}{168}x^4+\frac{1}{2520}x^5-\frac{1}{33264}x^6+\frac{1}{432432}x^7+\mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 76

```
AsymptoticDSolveValue[2*x*y''[x]+(1+2*x)*y'[x]-5*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\sqrt{x}\left(\frac{4x^2}{15}+\frac{4x}{3}+1\right)+c_2\left(\frac{x^7}{432432}-\frac{x^6}{33264}+\frac{x^5}{2520}-\frac{x^4}{168}+\frac{x^3}{6}+\frac{5x^2}{2}+5x+1\right)$$

4.15 problem 15

Internal problem ID [6931]
Internal file name [`OUTPUT/6174_Friday_August_12_2022_11_04_20_PM_14734510/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.4 Indicial Equation with Difference of Roots Nonintegral. Exercises page 365
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2y'' - 3x(1-x)\,y' + 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' + \left(3x^2 - 3x\right)y' + 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{\frac{3x}{2} - \frac{3}{2}}{x}$$
$$q(x) = \frac{1}{x^2}$$

Table 54: Table $p(x), q(x)$ singularites.

$p(x) = \frac{\frac{3x}{2}-\frac{3}{2}}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' + \left(3x^2 - 3x\right) y' + 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(3x^2 - 3x\right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 3x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)\, x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n (n+r)(n+r-1) - 3x^{n+r} a_n (n+r) + 2a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r) - 3x^r a_0 r + 2a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) - 3x^r r + 2x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 - 5r + 2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 5r + 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned}r_1 &= 2\\ r_2 &= \frac{1}{2}\end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 - 5r + 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{3}{2}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned}y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)\\ y_2(x) &= x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)\end{aligned}$$

Or

$$\begin{aligned}y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+2}\\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}\end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) + 3a_{n-1}(n+r-1) - 3a_n(n+r) + 2a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{3a_{n-1}(n+r-1)}{2n^2 + 4nr + 2r^2 - 5n - 5r + 2} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = -\frac{3a_{n-1}(1+n)}{n(2n+3)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{3r}{2r^2 - r - 1}$$

Which for the root $r = 2$ becomes

$$a_1 = -\frac{6}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3r}{2r^2-r-1}$	$-\frac{6}{5}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{9 + 9r}{4r^3 + 4r^2 - 5r - 3}$$

Which for the root $r = 2$ becomes

$$a_2 = \frac{27}{35}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3r}{2r^2-r-1}$	$-\frac{6}{5}$
a_2	$\frac{9+9r}{4r^3+4r^2-5r-3}$	$\frac{27}{35}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-54 - 27r}{8r^4 + 28r^3 + 10r^2 - 31r - 15}$$

Which for the root $r = 2$ becomes

$$a_3 = -\frac{12}{35}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3r}{2r^2-r-1}$	$-\frac{6}{5}$
a_2	$\frac{9+9r}{4r^3+4r^2-5r-3}$	$\frac{27}{35}$
a_3	$\frac{-54-27r}{8r^4+28r^3+10r^2-31r-15}$	$-\frac{12}{35}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{243 + 81r}{16r^5 + 112r^4 + 216r^3 + 8r^2 - 247r - 105}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{9}{77}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3r}{2r^2-r-1}$	$-\frac{6}{5}$
a_2	$\frac{9+9r}{4r^3+4r^2-5r-3}$	$\frac{27}{35}$
a_3	$\frac{-54-27r}{8r^4+28r^3+10r^2-31r-15}$	$-\frac{12}{35}$
a_4	$\frac{243+81r}{16r^5+112r^4+216r^3+8r^2-247r-105}$	$\frac{9}{77}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-972 - 243r}{32r^6 + 368r^5 + 1440r^4 + 1960r^3 - 422r^2 - 2433r - 945}$$

Which for the root $r = 2$ becomes

$$a_5 = -\frac{162}{5005}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3r}{2r^2-r-1}$	$-\frac{6}{5}$
a_2	$\frac{9+9r}{4r^3+4r^2-5r-3}$	$\frac{27}{35}$
a_3	$\frac{-54-27r}{8r^4+28r^3+10r^2-31r-15}$	$-\frac{12}{35}$
a_4	$\frac{243+81r}{16r^5+112r^4+216r^3+8r^2-247r-105}$	$\frac{9}{77}$
a_5	$\frac{-972-243r}{32r^6+368r^5+1440r^4+1960r^3-422r^2-2433r-945}$	$-\frac{162}{5005}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{3645 + 729r}{64r^7 + 1088r^6 + 6928r^5 + 19760r^4 + 20716r^3 - 9508r^2 - 28653r - 10395}$$

Which for the root $r = 2$ becomes

$$a_6 = \frac{27}{3575}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3r}{2r^2-r-1}$	$-\frac{6}{5}$
a_2	$\frac{9+9r}{4r^3+4r^2-5r-3}$	$\frac{27}{35}$
a_3	$\frac{-54-27r}{8r^4+28r^3+10r^2-31r-15}$	$-\frac{12}{35}$
a_4	$\frac{243+81r}{16r^5+112r^4+216r^3+8r^2-247r-105}$	$\frac{9}{77}$
a_5	$\frac{-972-243r}{32r^6+368r^5+1440r^4+1960r^3-422r^2-2433r-945}$	$-\frac{162}{5005}$
a_6	$\frac{3645+729r}{64r^7+1088r^6+6928r^5+19760r^4+20716r^3-9508r^2-28653r-10395}$	$\frac{27}{3575}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{-13122 - 2187r}{128r^8 + 3008r^7 + 28000r^6 + 129584r^5 + 298312r^4 + 250292r^3 - 180910r^2 - 393279r - 135135}$$

Which for the root $r = 2$ becomes

$$a_7 = -\frac{648}{425425}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3r}{2r^2-r-1}$	$-\frac{6}{5}$
a_2	$\frac{9+9r}{4r^3+4r^2-5r-3}$	$\frac{27}{35}$
a_3	$\frac{-54-27r}{8r^4+28r^3+10r^2-31r-15}$	$-\frac{12}{35}$
a_4	$\frac{243+81r}{16r^5+112r^4+216r^3+8r^2-247r-105}$	$\frac{9}{77}$
a_5	$\frac{-972-243r}{32r^6+368r^5+1440r^4+1960r^3-422r^2-2433r-945}$	$-\frac{162}{5005}$
a_6	$\frac{3645+729r}{64r^7+1088r^6+6928r^5+19760r^4+20716r^3-9508r^2-28653r-10395}$	$\frac{27}{3575}$
a_7	$\frac{-13122-2187r}{128r^8+3008r^7+28000r^6+129584r^5+298312r^4+250292r^3-180910r^2-393279r-135135}$	$-\frac{648}{425425}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^2\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \dots\right) \\ &= x^2\left(1 - \frac{6x}{5} + \frac{27x^2}{35} - \frac{12x^3}{35} + \frac{9x^4}{77} - \frac{162x^5}{5005} + \frac{27x^6}{3575} - \frac{648x^7}{425425} + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) + 3b_{n-1}(n+r-1) - 3b_n(n+r) + 2b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{3b_{n-1}(n+r-1)}{2n^2+4nr+2r^2-5n-5r+2} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = \frac{-6nb_{n-1}+3b_{n-1}}{4n^2-6n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{3r}{2r^2 - r - 1}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_1 = \frac{3}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3r}{2r^2-r-1}$	$\frac{3}{2}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{9 + 9r}{4r^3 + 4r^2 - 5r - 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = -\frac{27}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3r}{2r^2-r-1}$	$\frac{3}{2}$
b_2	$\frac{9+9r}{4r^3+4r^2-5r-3}$	$-\frac{27}{8}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{-54 - 27r}{8r^4 + 28r^3 + 10r^2 - 31r - 15}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_3 = \frac{45}{16}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3r}{2r^2-r-1}$	$\frac{3}{2}$
b_2	$\frac{9+9r}{4r^3+4r^2-5r-3}$	$-\frac{27}{8}$
b_3	$\frac{-54-27r}{8r^4+28r^3+10r^2-31r-15}$	$\frac{45}{16}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{243 + 81r}{16r^5 + 112r^4 + 216r^3 + 8r^2 - 247r - 105}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = -\frac{189}{128}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3r}{2r^2-r-1}$	$\frac{3}{2}$
b_2	$\frac{9+9r}{4r^3+4r^2-5r-3}$	$-\frac{27}{8}$
b_3	$\frac{-54-27r}{8r^4+28r^3+10r^2-31r-15}$	$\frac{45}{16}$
b_4	$\frac{243+81r}{16r^5+112r^4+216r^3+8r^2-247r-105}$	$-\frac{189}{128}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{-972 - 243r}{32r^6 + 368r^5 + 1440r^4 + 1960r^3 - 422r^2 - 2433r - 945}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_5 = \frac{729}{1280}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3r}{2r^2-r-1}$	$\frac{3}{2}$
b_2	$\frac{9+9r}{4r^3+4r^2-5r-3}$	$-\frac{27}{8}$
b_3	$\frac{-54-27r}{8r^4+28r^3+10r^2-31r-15}$	$\frac{45}{16}$
b_4	$\frac{243+81r}{16r^5+112r^4+216r^3+8r^2-247r-105}$	$-\frac{189}{128}$
b_5	$\frac{-972-243r}{32r^6+368r^5+1440r^4+1960r^3-422r^2-2433r-945}$	$\frac{729}{1280}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{3645 + 729r}{64r^7 + 1088r^6 + 6928r^5 + 19760r^4 + 20716r^3 - 9508r^2 - 28653r - 10395}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_6 = -\frac{891}{5120}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3r}{2r^2-r-1}$	$\frac{3}{2}$
b_2	$\frac{9+9r}{4r^3+4r^2-5r-3}$	$-\frac{27}{8}$
b_3	$\frac{-54-27r}{8r^4+28r^3+10r^2-31r-15}$	$\frac{45}{16}$
b_4	$\frac{243+81r}{16r^5+112r^4+216r^3+8r^2-247r-105}$	$-\frac{189}{128}$
b_5	$\frac{-972-243r}{32r^6+368r^5+1440r^4+1960r^3-422r^2-2433r-945}$	$\frac{729}{1280}$
b_6	$\frac{3645+729r}{64r^7+1088r^6+6928r^5+19760r^4+20716r^3-9508r^2-28653r-10395}$	$-\frac{891}{5120}$

For $n = 7$, using the above recursive equation gives

$$b_7 = \frac{-13122 - 2187r}{128r^8 + 3008r^7 + 28000r^6 + 129584r^5 + 298312r^4 + 250292r^3 - 180910r^2 - 393279r - 135135}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_7 = \frac{3159}{71680}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3r}{2r^2-r-1}$	$\frac{3}{2}$
b_2	$\frac{9+9r}{4r^3+4r^2-5r-3}$	$-\frac{27}{8}$
b_3	$\frac{-54-27r}{8r^4+28r^3+10r^2-31r-15}$	$\frac{45}{16}$
b_4	$\frac{243+81r}{16r^5+112r^4+216r^3+8r^2-247r-105}$	$-\frac{189}{128}$
b_5	$\frac{-972-243r}{32r^6+368r^5+1440r^4+1960r^3-422r^2-2433r-945}$	$\frac{729}{1280}$
b_6	$\frac{3645+729r}{64r^7+1088r^6+6928r^5+19760r^4+20716r^3-9508r^2-28653r-10395}$	$-\frac{891}{5120}$
b_7	$\frac{-13122-2187r}{128r^8+3008r^7+28000r^6+129584r^5+298312r^4+250292r^3-180910r^2-393279r-135135}$	$\frac{3159}{71680}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^2 \left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots\right) \\
&= \sqrt{x}\left(1 + \frac{3x}{2} - \frac{27x^2}{8} + \frac{45x^3}{16} - \frac{189x^4}{128} + \frac{729x^5}{1280} - \frac{891x^6}{5120} + \frac{3159x^7}{71680} + O(x^8)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^2 \left(1 - \frac{6x}{5} + \frac{27x^2}{35} - \frac{12x^3}{35} + \frac{9x^4}{77} - \frac{162x^5}{5005} + \frac{27x^6}{3575} - \frac{648x^7}{425425} + O(x^8)\right) \\
&\quad + c_2 \sqrt{x}\left(1 + \frac{3x}{2} - \frac{27x^2}{8} + \frac{45x^3}{16} - \frac{189x^4}{128} + \frac{729x^5}{1280} - \frac{891x^6}{5120} + \frac{3159x^7}{71680} + O(x^8)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^2 \left(1 - \frac{6x}{5} + \frac{27x^2}{35} - \frac{12x^3}{35} + \frac{9x^4}{77} - \frac{162x^5}{5005} + \frac{27x^6}{3575} - \frac{648x^7}{425425} + O(x^8)\right) \\
&\quad + c_2 \sqrt{x}\left(1 + \frac{3x}{2} - \frac{27x^2}{8} + \frac{45x^3}{16} - \frac{189x^4}{128} + \frac{729x^5}{1280} - \frac{891x^6}{5120} + \frac{3159x^7}{71680} + O(x^8)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 x^2 \left(1 - \frac{6x}{5} + \frac{27x^2}{35} - \frac{12x^3}{35} + \frac{9x^4}{77} - \frac{162x^5}{5005} + \frac{27x^6}{3575} - \frac{648x^7}{425425} + O(x^8)\right) + c_2\sqrt{x}\left(1 + \frac{3x}{2} - \frac{27x^2}{8} + \frac{45x^3}{16} - \frac{189x^4}{128} + \frac{729x^5}{1280} - \frac{891x^6}{5120} + \frac{3159x^7}{71680} + O(x^8)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^2 \left(1 - \frac{6x}{5} + \frac{27x^2}{35} - \frac{12x^3}{35} + \frac{9x^4}{77} - \frac{162x^5}{5005} + \frac{27x^6}{3575} - \frac{648x^7}{425425} + O(x^8)\right) + c_2\sqrt{x}\left(1 + \frac{3x}{2} - \frac{27x^2}{8} + \frac{45x^3}{16} - \frac{189x^4}{128} + \frac{729x^5}{1280} - \frac{891x^6}{5120} + \frac{3159x^7}{71680} + O(x^8)\right)$$

Verified OK.

4.15.1 Maple step by step solution

Let's solve

$2x^2 y'' + (3x^2 - 3x) y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{x^2} - \frac{3(x-1)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3(x-1)y'}{2x} + \frac{y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{3(x-1)}{2x}, P_3(x) = \frac{1}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -\frac{3}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 1$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2y'' + 3x(x-1)\,y' + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - ◦ Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - ◦ Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)\,(-2+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)\,(k+r-2) + 3a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{2, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2(k+r-2)\left(k+r-\frac{1}{2}\right) a_k + 3a_{k-1}(k+r-1) = 0$

- Shift index using $k- >k+1$

  $2(k+r-1)\left(k+\frac{1}{2}+r\right) a_{k+1} + 3a_k(k+r) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{3a_k(k+r)}{(k+r-1)(2k+1+2r)}$

- Recursion relation for $r = 2$

  $a_{k+1} = -\frac{3a_k(k+2)}{(k+1)(2k+5)}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{3a_k(k+2)}{(k+1)(2k+5)}\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = -\frac{3a_k\left(k+\frac{1}{2}\right)}{\left(k-\frac{1}{2}\right)(2k+2)}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{3a_k\left(k+\frac{1}{2}\right)}{\left(k-\frac{1}{2}\right)(2k+2)}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{3a_k(k+2)}{(k+1)(2k+5)}, b_{k+1} = -\frac{3b_k\left(k+\frac{1}{2}\right)}{\left(k-\frac{1}{2}\right)(2k+2)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 55

```
Order:=8;
dsolve(2*x^2*diff(y(x),x$2)-3*x*(1-x)*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\sqrt{x}\left(1+\frac{3}{2}x-\frac{27}{8}x^2+\frac{45}{16}x^3-\frac{189}{128}x^4+\frac{729}{1280}x^5-\frac{891}{5120}x^6+\frac{3159}{71680}x^7+\mathrm{O}\left(x^8\right)\right) + c_2x^2\left(1-\frac{6}{5}x+\frac{27}{35}x^2-\frac{12}{35}x^3+\frac{9}{77}x^4-\frac{162}{5005}x^5+\frac{27}{3575}x^6-\frac{648}{425425}x^7+\mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 116

```
AsymptoticDSolveValue[2*x^2*y''[x]-3*x*(1-x)*y'[x]+2*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1 \left(-\frac{648x^7}{425425} + \frac{27x^6}{3575} - \frac{162x^5}{5005} + \frac{9x^4}{77} - \frac{12x^3}{35} + \frac{27x^2}{35} - \frac{6x}{5} + 1\right) x^2 + c_2 \left(\frac{3159x^7}{71680} - \frac{891x^6}{5120} + \frac{729x^5}{1280} - \frac{189x^4}{128} + \frac{45x^3}{16} - \frac{27x^2}{8} + \frac{3x}{2} + 1\right) \sqrt{x}$$

4.16 problem 16

Internal problem ID [6932]
Internal file name [`OUTPUT/6175_Friday_August_12_2022_11_04_23_PM_19162768/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.4 Indicial Equation with Difference of Roots Nonintegral. Exercises page 365
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2y'' + x(4x-1)\,y' + 2(3x-1)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' + \left(4x^2 - x\right)y' + (6x-2)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{4x-1}{2x}$$
$$q(x) = \frac{3x-1}{x^2}$$

Table 56: Table $p(x), q(x)$ singularites.

$p(x) = \frac{4x-1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{3x-1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' + (4x^2 - x)\, y' + (6x - 2)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} \right) + (4x^2 - x) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \right) + (6x - 2) \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n(n+r)\right) \\ + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n(n+r)\right) + \left(\sum_{n=0}^{\infty} 6x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 4x^{1+n+r} a_n(n+r) = \sum_{n=1}^{\infty} 4a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 6x^{1+n+r} a_n = \sum_{n=1}^{\infty} 6a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 4a_{n-1}(n+r-1)\, x^{n+r}\right) \\ + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n(n+r)\right) + \left(\sum_{n=1}^{\infty} 6a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n(n+r)(n+r-1) - x^{n+r} a_n(n+r) - 2a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r) - x^r a_0 r - 2a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) - x^r r - 2x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 - 3r - 2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 3r - 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = -\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(2r^2 - 3r - 2)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{5}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) + 4a_{n-1}(n+r-1) - a_n(n+r) + 6a_{n-1} - 2a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2a_{n-1}}{n+r-2} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = -\frac{2a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{2}{-1 + r}$$

Which for the root $r = 2$ becomes

$$a_1 = -2$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{-1+r}$	-2

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4}{(-1 + r) r}$$

Which for the root $r = 2$ becomes

$$a_2 = 2$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{-1+r}$	-2
a_2	$\frac{4}{(-1+r)r}$	2

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{8}{r^3 - r}$$

Which for the root $r = 2$ becomes

$$a_3 = -\frac{4}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{-1+r}$	-2
a_2	$\frac{4}{(-1+r)r}$	2
a_3	$-\frac{8}{r^3-r}$	$-\frac{4}{3}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{r\,(r^2-1)\,(2+r)}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{2}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{-1+r}$	-2
a_2	$\frac{4}{(-1+r)r}$	2
a_3	$-\frac{8}{r^3-r}$	$-\frac{4}{3}$
a_4	$\frac{16}{r(r^2-1)(2+r)}$	$\frac{2}{3}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{32}{r^5+5r^4+5r^3-5r^2-6r}$$

Which for the root $r = 2$ becomes

$$a_5 = -\frac{4}{15}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{-1+r}$	-2
a_2	$\frac{4}{(-1+r)r}$	2
a_3	$-\frac{8}{r^3-r}$	$-\frac{4}{3}$
a_4	$\frac{16}{r(r^2-1)(2+r)}$	$\frac{2}{3}$
a_5	$-\frac{32}{r^5+5r^4+5r^3-5r^2-6r}$	$-\frac{4}{15}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{64}{r\left(r^4 + 5r^3 + 5r^2 - 5r - 6\right)\left(4 + r\right)}$$

Which for the root $r = 2$ becomes

$$a_6 = \frac{4}{45}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{-1+r}$	-2
a_2	$\frac{4}{(-1+r)r}$	2
a_3	$-\frac{8}{r^3-r}$	$-\frac{4}{3}$
a_4	$\frac{16}{r(r^2-1)(2+r)}$	$\frac{2}{3}$
a_5	$-\frac{32}{r^5+5r^4+5r^3-5r^2-6r}$	$-\frac{4}{15}$
a_6	$\frac{64}{r(r^4+5r^3+5r^2-5r-6)(4+r)}$	$\frac{4}{45}$

For $n = 7$, using the above recursive equation gives

$$a_7 = -\frac{128}{r\left(r^4 + 5r^3 + 5r^2 - 5r - 6\right)\left(4 + r\right)\left(5 + r\right)}$$

Which for the root $r = 2$ becomes

$$a_7 = -\frac{8}{315}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{-1+r}$	-2
a_2	$\frac{4}{(-1+r)r}$	2
a_3	$-\frac{8}{r^3-r}$	$-\frac{4}{3}$
a_4	$\frac{16}{r(r^2-1)(2+r)}$	$\frac{2}{3}$
a_5	$-\frac{32}{r^5+5r^4+5r^3-5r^2-6r}$	$-\frac{4}{15}$
a_6	$\frac{64}{r(r^4+5r^3+5r^2-5r-6)(4+r)}$	$\frac{4}{45}$
a_7	$-\frac{128}{r(r^4+5r^3+5r^2-5r-6)(4+r)(5+r)}$	$-\frac{8}{315}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^2\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \dots\right) \\
&= x^2\left(1 - 2x + 2x^2 - \frac{4x^3}{3} + \frac{2x^4}{3} - \frac{4x^5}{15} + \frac{4x^6}{45} - \frac{8x^7}{315} + O\left(x^8\right)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
2b_n(n+r)(n+r-1) + 4b_{n-1}(n+r-1) - b_n(n+r) + 6b_{n-1} - 2b_n = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = -\frac{2b_{n-1}}{n+r-2} \tag{4}
$$

Which for the root $r = -\frac{1}{2}$ becomes

$$
b_n = -\frac{4b_{n-1}}{2n-5} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{2}{-1+r}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_1 = \frac{4}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{-1+r}$	$\frac{4}{3}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4}{(-1+r)\, r}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_2 = \frac{16}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{-1+r}$	$\frac{4}{3}$
b_2	$\frac{4}{(-1+r)r}$	$\frac{16}{3}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{8}{r^3 - r}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_3 = -\frac{64}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{-1+r}$	$\frac{4}{3}$
b_2	$\frac{4}{(-1+r)r}$	$\frac{16}{3}$
b_3	$-\frac{8}{r^3-r}$	$-\frac{64}{3}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16}{r(r^2 - 1)(2 + r)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = \frac{256}{9}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{-1+r}$	$\frac{4}{3}$
b_2	$\frac{4}{(-1+r)r}$	$\frac{16}{3}$
b_3	$-\frac{8}{r^3-r}$	$-\frac{64}{3}$
b_4	$\frac{16}{r(r^2-1)(2+r)}$	$\frac{256}{9}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{32}{r^5 + 5r^4 + 5r^3 - 5r^2 - 6r}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_5 = -\frac{1024}{45}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{-1+r}$	$\frac{4}{3}$
b_2	$\frac{4}{(-1+r)r}$	$\frac{16}{3}$
b_3	$-\frac{8}{r^3-r}$	$-\frac{64}{3}$
b_4	$\frac{16}{r(r^2-1)(2+r)}$	$\frac{256}{9}$
b_5	$-\frac{32}{r^5+5r^4+5r^3-5r^2-6r}$	$-\frac{1024}{45}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{64}{r\left(r^4 + 5r^3 + 5r^2 - 5r - 6\right)(4 + r)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_6 = \frac{4096}{315}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{-1+r}$	$\frac{4}{3}$
b_2	$\frac{4}{(-1+r)r}$	$\frac{16}{3}$
b_3	$-\frac{8}{r^3-r}$	$-\frac{64}{3}$
b_4	$\frac{16}{r(r^2-1)(2+r)}$	$\frac{256}{9}$
b_5	$-\frac{32}{r^5+5r^4+5r^3-5r^2-6r}$	$-\frac{1024}{45}$
b_6	$\frac{64}{r(r^4+5r^3+5r^2-5r-6)(4+r)}$	$\frac{4096}{315}$

For $n = 7$, using the above recursive equation gives

$$b_7 = -\frac{128}{r\left(r^4 + 5r^3 + 5r^2 - 5r - 6\right)(4 + r)(5 + r)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_7 = -\frac{16384}{2835}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{-1+r}$	$\frac{4}{3}$
b_2	$\frac{4}{(-1+r)r}$	$\frac{16}{3}$
b_3	$-\frac{8}{r^3-r}$	$-\frac{64}{3}$
b_4	$\frac{16}{r(r^2-1)(2+r)}$	$\frac{256}{9}$
b_5	$-\frac{32}{r^5+5r^4+5r^3-5r^2-6r}$	$-\frac{1024}{45}$
b_6	$\frac{64}{r(r^4+5r^3+5r^2-5r-6)(4+r)}$	$\frac{4096}{315}$
b_7	$-\frac{128}{r(r^4+5r^3+5r^2-5r-6)(4+r)(5+r)}$	$-\frac{16384}{2835}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^2\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots\right) \\
&= \frac{1 + \frac{4x}{3} + \frac{16x^2}{3} - \frac{64x^3}{3} + \frac{256x^4}{9} - \frac{1024x^5}{45} + \frac{4096x^6}{315} - \frac{16384x^7}{2835} + O(x^8)}{\sqrt{x}}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^2 \left(1 - 2x + 2x^2 - \frac{4x^3}{3} + \frac{2x^4}{3} - \frac{4x^5}{15} + \frac{4x^6}{45} - \frac{8x^7}{315} + O\left(x^8\right)\right) \\
&+ \frac{c_2\left(1 + \frac{4x}{3} + \frac{16x^2}{3} - \frac{64x^3}{3} + \frac{256x^4}{9} - \frac{1024x^5}{45} + \frac{4096x^6}{315} - \frac{16384x^7}{2835} + O(x^8)\right)}{\sqrt{x}}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^2 \left(1 - 2x + 2x^2 - \frac{4x^3}{3} + \frac{2x^4}{3} - \frac{4x^5}{15} + \frac{4x^6}{45} - \frac{8x^7}{315} + O\left(x^8\right)\right) \\
&+ \frac{c_2\left(1 + \frac{4x}{3} + \frac{16x^2}{3} - \frac{64x^3}{3} + \frac{256x^4}{9} - \frac{1024x^5}{45} + \frac{4096x^6}{315} - \frac{16384x^7}{2835} + O(x^8)\right)}{\sqrt{x}}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
\begin{align*}
y &= c_1 x^2 \left(1-2x+2x^2-\frac{4x^3}{3}+\frac{2x^4}{3}-\frac{4x^5}{15}+\frac{4x^6}{45}-\frac{8x^7}{315}+O\left(x^8\right)\right) \\
&+ \frac{c_2 \left(1+\frac{4x}{3}+\frac{16x^2}{3}-\frac{64x^3}{3}+\frac{256x^4}{9}-\frac{1024x^5}{45}+\frac{4096x^6}{315}-\frac{16384x^7}{2835}+O\left(x^8\right)\right)}{\sqrt{x}} \tag{1}
\end{align*}
$$

Verification of solutions

$$
\begin{align*}
y &= c_1 x^2 \left(1-2x+2x^2-\frac{4x^3}{3}+\frac{2x^4}{3}-\frac{4x^5}{15}+\frac{4x^6}{45}-\frac{8x^7}{315}+O\left(x^8\right)\right) \\
&+ \frac{c_2 \left(1+\frac{4x}{3}+\frac{16x^2}{3}-\frac{64x^3}{3}+\frac{256x^4}{9}-\frac{1024x^5}{45}+\frac{4096x^6}{315}-\frac{16384x^7}{2835}+O\left(x^8\right)\right)}{\sqrt{x}}
\end{align*}
$$

Verified OK.

4.16.1 Maple step by step solution

Let's solve

$2x^2y''+(4x^2-x)\,y'+(6x-2)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(3x-1)y}{x^2}-\frac{(4x-1)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(4x-1)y'}{2x}+\frac{(3x-1)y}{x^2}=0$

☐ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{4x-1}{2x}, P_3(x)=\frac{3x-1}{x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=-\frac{1}{2}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=-1$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2y'' + x(4x-1)\,y' + (6x-2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-2+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)\,(k+r-2) + 2a_{k-1}(2k+2r+1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{2, -\frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

$$2(a_k(k+r-2)+2a_{k-1})\left(k+r+\frac{1}{2}\right)=0$$

- Shift index using $k->k+1$

$$2(a_{k+1}(k+r-1)+2a_k)\left(k+\frac{3}{2}+r\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=-\frac{2a_k}{k+r-1}$$

- Recursion relation for $r=2$

$$a_{k+1}=-\frac{2a_k}{k+1}$$

- Solution for $r=2$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k+2}, a_{k+1}=-\frac{2a_k}{k+1}\right]$$

- Recursion relation for $r=-\frac{1}{2}$

$$a_{k+1}=-\frac{2a_k}{k-\frac{3}{2}}$$

- Solution for $r=-\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{2}}, a_{k+1}=-\frac{2a_k}{k-\frac{3}{2}}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k+2}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k-\frac{1}{2}}\right), a_{k+1}=-\frac{2a_k}{k+1}, b_{k+1}=-\frac{2b_k}{k-\frac{3}{2}}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 55

```
Order:=8;
dsolve(2*x^2*diff(y(x),x$2)+x*(4*x-1)*diff(y(x),x)+2*(3*x-1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1\left(1+\frac{4}{3}x+\frac{16}{3}x^2-\frac{64}{3}x^3+\frac{256}{9}x^4-\frac{1024}{45}x^5+\frac{4096}{315}x^6-\frac{16384}{2835}x^7+\mathrm{O}\left(x^8\right)\right)}{\sqrt{x}} + c_2x^2\left(1-2x+2x^2-\frac{4}{3}x^3+\frac{2}{3}x^4-\frac{4}{15}x^5+\frac{4}{45}x^6-\frac{8}{315}x^7+\mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 112

```
AsymptoticDSolveValue[2*x^2*y''[x]+x*(4*x-1)*y'[x]+2*(3*x-1)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(-\frac{8x^7}{315}+\frac{4x^6}{45}-\frac{4x^5}{15}+\frac{2x^4}{3}-\frac{4x^3}{3}+2x^2-2x+1\right)x^2 + \frac{c_2\left(-\frac{16384x^7}{2835}+\frac{4096x^6}{315}-\frac{1024x^5}{45}+\frac{256x^4}{9}-\frac{64x^3}{3}+\frac{16x^2}{3}+\frac{4x}{3}+1\right)}{\sqrt{x}}$$

4.17 problem 17

Internal problem ID [6933]
Internal file name [`OUTPUT/6176_Friday_August_12_2022_11_04_25_PM_96573020/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.4 Indicial Equation with Difference of Roots Nonintegral. Exercises page 365
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2xy'' - (2x^2+1)\,y' - yx = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2xy'' + (-2x^2-1)\,y' - yx = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{2x^2+1}{2x}$$
$$q(x) = -\frac{1}{2}$$

Table 58: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{2x^2+1}{2x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = -\frac{1}{2}$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2xy'' + (-2x^2 - 1)\, y' - yx = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (-2x^2 - 1)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) x = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1}a_n(n+r)(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-2x^{1+n+r}a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-(n+r)a_n x^{n+r-1}\right)+\sum_{n=0}^{\infty}\left(-x^{1+n+r}a_n\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-2x^{1+n+r}a_n(n+r)\right)=\sum_{n=2}^{\infty}\left(-2a_{n-2}(n+r-2)\,x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty}\left(-x^{1+n+r}a_n\right)=\sum_{n=2}^{\infty}\left(-a_{n-2}x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1}a_n(n+r)(n+r-1)\right)+\sum_{n=2}^{\infty}\left(-2a_{n-2}(n+r-2)\,x^{n+r-1}\right)+\sum_{n=0}^{\infty}\left(-(n+r)a_n x^{n+r-1}\right)+\sum_{n=2}^{\infty}\left(-a_{n-2}x^{n+r-1}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1}a_n(n+r)(n+r-1)-(n+r)\,a_n x^{n+r-1}=0$$

When $n=0$ the above becomes

$$2x^{-1+r}a_0 r(-1+r)-ra_0x^{-1+r}=0$$

Or

$$\left(2x^{-1+r}r(-1+r)-r\,x^{-1+r}\right)a_0=0$$

Since $a_0\neq 0$ then the above simplifies to

$$r\,x^{-1+r}(-3+2r)=0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 3r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{3}{2}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(-3 + 2r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{3}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2a_n(n + r)(n + r - 1) - 2a_{n-2}(n + r - 2) - a_n(n + r) - a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}}{n + r} \tag{4}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_n = \frac{2a_{n-2}}{2n+3} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{2+r}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_2 = \frac{2}{7}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2+r}$	$\frac{2}{7}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2+r}$	$\frac{2}{7}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(2+r)(4+r)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_4 = \frac{4}{77}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2+r}$	$\frac{2}{7}$
a_3	0	0
a_4	$\frac{1}{(2+r)(4+r)}$	$\frac{4}{77}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2+r}$	$\frac{2}{7}$
a_3	0	0
a_4	$\frac{1}{(2+r)(4+r)}$	$\frac{4}{77}$
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{1}{(2+r)(4+r)(6+r)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_6 = \frac{8}{1155}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2+r}$	$\frac{2}{7}$
a_3	0	0
a_4	$\frac{1}{(2+r)(4+r)}$	$\frac{4}{77}$
a_5	0	0
a_6	$\frac{1}{(2+r)(4+r)(6+r)}$	$\frac{8}{1155}$

For $n = 7$, using the above recursive equation gives

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2+r}$	$\frac{2}{7}$
a_3	0	0
a_4	$\frac{1}{(2+r)(4+r)}$	$\frac{4}{77}$
a_5	0	0
a_6	$\frac{1}{(2+r)(4+r)(6+r)}$	$\frac{8}{1155}$
a_7	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{3}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\ &= x^{\frac{3}{2}}\left(1 + \frac{2x^2}{7} + \frac{4x^4}{77} + \frac{8x^6}{1155} + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - 2b_{n-2}(n+r-2) - (n+r)b_n - b_{n-2} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-2}}{n+r} \tag{4}$$

Which for the root $r = 0$ becomes

$$b_n = \frac{b_{n-2}}{n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{2+r}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2+r}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2+r}$	$\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(2+r)(4+r)}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{1}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2+r}$	$\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{(2+r)(4+r)}$	$\frac{1}{8}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2+r}$	$\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{(2+r)(4+r)}$	$\frac{1}{8}$
b_5	0	0

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{1}{(2+r)(4+r)(6+r)}$$

Which for the root $r = 0$ becomes

$$b_6 = \frac{1}{48}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2+r}$	$\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{(2+r)(4+r)}$	$\frac{1}{8}$
b_5	0	0
b_6	$\frac{1}{(2+r)(4+r)(6+r)}$	$\frac{1}{48}$

For $n = 7$, using the above recursive equation gives

$$b_7 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2+r}$	$\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{(2+r)(4+r)}$	$\frac{1}{8}$
b_5	0	0
b_6	$\frac{1}{(2+r)(4+r)(6+r)}$	$\frac{1}{48}$
b_7	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\ &= 1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + O(x^8)\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{3}{2}} \left(1 + \frac{2x^2}{7} + \frac{4x^4}{77} + \frac{8x^6}{1155} + O(x^8)\right) + c_2 \left(1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + O(x^8)\right)\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y &= y_h \\ &= c_1 x^{\frac{3}{2}} \left(1 + \frac{2x^2}{7} + \frac{4x^4}{77} + \frac{8x^6}{1155} + O(x^8)\right) + c_2 \left(1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + O(x^8)\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{3}{2}} \left(1 + \frac{2x^2}{7} + \frac{4x^4}{77} + \frac{8x^6}{1155} + O(x^8)\right) + c_2 \left(1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + O(x^8)\right) \quad (1)$$

Verification of solutions

$$y = c_1 x^{\frac{3}{2}} \left(1 + \frac{2x^2}{7} + \frac{4x^4}{77} + \frac{8x^6}{1155} + O(x^8)\right) + c_2 \left(1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + O(x^8)\right)$$

Verified OK.

4.17.1 Maple step by step solution

Let's solve

$2xy'' + (-2x^2 - 1)\, y' - yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{2} + \frac{(2x^2+1)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x^2+1)y'}{2x} - \frac{y}{2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2x^2+1}{2x}, P_3(x) = -\frac{1}{2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -\frac{1}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $2xy'' + (-2x^2 - 1)\, y' - yx = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

  ◦ Shift index using $k- >k - 1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  ◦ Shift index using $k- >k + 1 - m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-1}$

- Shift index using $k->k+1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)(k+r) x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-3+2r) x^{-1+r} + a_1(1+r)(-1+2r) x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+r+1)(2k-1+2r) - a_{k-1}(2k -\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{3}{2}\right\}$

- Each term must be 0

  $a_1(1+r)(-1+2r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k - \frac{1}{2} + r\right)(a_{k+1}(k+r+1) - a_{k-1}) = 0$

- Shift index using $k->k+1$

  $2\left(k + \frac{1}{2} + r\right)(a_{k+2}(k+2+r) - a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k}{k+2+r}$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{a_k}{k+2}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a_k}{k+2}, -a_1 = 0\right]$

- Recursion relation for $r = \frac{3}{2}$

  $a_{k+2} = \frac{a_k}{k+\frac{7}{2}}$

- Solution for $r = \frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = \frac{a_k}{k+\frac{7}{2}}, 5a_1 = 0\right]$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^k\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{3}{2}}\right), a_{k+2}=\frac{a_k}{k+2}, -a_1=0, b_{k+2}=\frac{b_k}{k+\frac{7}{2}}, 5b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 36

```
Order:=8;
dsolve(2*x*diff(y(x),x$2)-(1+2*x^2)*diff(y(x),x)-x*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=c_1 x^{\frac{3}{2}}\left(1+\frac{2}{7}x^2+\frac{4}{77}x^4+\frac{8}{1155}x^6+\mathrm{O}\left(x^8\right)\right)+c_2\left(1+\frac{1}{2}x^2+\frac{1}{8}x^4+\frac{1}{48}x^6+\mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 61

```
AsymptoticDSolveValue[2*x*y''[x]-(1+2*x^2)*y'[x]-x*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(\frac{x^6}{48}+\frac{x^4}{8}+\frac{x^2}{2}+1\right)+c_1\left(\frac{8x^6}{1155}+\frac{4x^4}{77}+\frac{2x^2}{7}+1\right)x^{3/2}$$

4.18 problem 19

Internal problem ID [6934]
Internal file name [OUTPUT/6177_Friday_August_12_2022_11_04_28_PM_25807087/index.tex]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.4 Indicial Equation with Difference of Roots Nonintegral. Exercises page 365
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{2x^2y'' + xy' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' + xy' - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{2x}$$
$$q(x) = -\frac{1}{2x^2}$$

Table 60: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' + xy' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} \right) + x \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)\,(n+r-1) \right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r) \right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r} \right) = 0 \quad \text{(2A)}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r}a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)=0 \quad \text{(2B)}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r}a_n(n+r)(n+r-1)+x^{n+r}a_n(n+r)-a_n x^{n+r}=0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r)+x^r a_0 r-a_0 x^r=0$$

Or

$$\left(2x^r r(-1+r)+x^r r-x^r\right)a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2-r-1\right)x^r=0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2-r-1=0$$

Solving for r gives the roots of the indicial equation as

$$r_1=1$$
$$r_2=-\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2-r-1\right)x^r=0$$

Solving for r gives the roots of the indicial equation as Since $r_1-r_2=\frac{3}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x)=x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x)=x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $0 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) + a_n(n+r) - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = 0 \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = 0 \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0
a_5	0	0
a_6	0	0

For $n = 7$, using the above recursive equation gives

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0
a_5	0	0
a_6	0	0
a_7	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\ &= x\left(1 + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $0 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) + b_n(n+r) - b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = 0 \tag{4}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_n = 0 \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	0	0

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	0	0
b_5	0	0

For $n = 6$, using the above recursive equation gives

$$b_6 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	0	0
b_5	0	0
b_6	0	0

For $n = 7$, using the above recursive equation gives

$$b_7 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	0	0
b_5	0	0
b_6	0	0
b_7	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots\right) \\ &= \frac{1 + O(x^8)}{\sqrt{x}} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1 + O\left(x^8\right)\right) + \frac{c_2(1 + O(x^8))}{\sqrt{x}} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x\left(1 + O\left(x^8\right)\right) + \frac{c_2(1 + O(x^8))}{\sqrt{x}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x\left(1 + O\left(x^8\right)\right) + \frac{c_2(1 + O(x^8))}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = c_1 x\left(1 + O\left(x^8\right)\right) + \frac{c_2(1 + O(x^8))}{\sqrt{x}}$$

Verified OK.

4.18.1 Maple step by step solution

Let's solve

$2x^2 y'' + x y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{2x} + \frac{y}{2x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{2x} - \frac{y}{2x^2} = 0$

- Multiply by denominators of the ODE

  $2x^2 y'' + x y' - y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt} y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2} y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt} y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2} y(t)}{x^2} - \frac{\frac{d}{dt} y(t)}{x^2}$

  Substitute the change of variables back into the ODE

$$2x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}\right)+\frac{d}{dt}y(t)-y(t)=0$$

- Simplify
$$2\frac{d^2}{dt^2}y(t)-\frac{d}{dt}y(t)-y(t)=0$$
- Isolate 2nd derivative
$$\frac{d^2}{dt^2}y(t)=\frac{\frac{d}{dt}y(t)}{2}+\frac{y(t)}{2}$$
- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
$$\frac{d^2}{dt^2}y(t)-\frac{\frac{d}{dt}y(t)}{2}-\frac{y(t)}{2}=0$$
- Characteristic polynomial of ODE
$$r^2-\frac{1}{2}r-\frac{1}{2}=0$$
- Factor the characteristic polynomial
$$\frac{(2r+1)(r-1)}{2}=0$$
- Roots of the characteristic polynomial
$$r=\left(1,-\frac{1}{2}\right)$$
- 1st solution of the ODE
$$y_1(t)=\mathrm{e}^t$$
- 2nd solution of the ODE
$$y_2(t)=\mathrm{e}^{-\frac{t}{2}}$$
- General solution of the ODE
$$y(t)=c_1y_1(t)+c_2y_2(t)$$
- Substitute in solutions
$$y(t)=c_1\mathrm{e}^t+c_2\mathrm{e}^{-\frac{t}{2}}$$
- Change variables back using $t=\ln(x)$
$$y=c_1x+\frac{c_2}{\sqrt{x}}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 25

```
Order:=8;
dsolve(2*x^2*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{x^{\frac{3}{2}}c_2 + c_1}{\sqrt{x}} + O(x^8)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 16

```
AsymptoticDSolveValue[2*x^2*y''[x]+x*y'[x]-y[x]==0,y[x],{x,0,7}]
```

$$y(x) \rightarrow c_1 x + \frac{c_2}{\sqrt{x}}$$

4.19 problem 20

Internal problem ID [6935]
Internal file name [OUTPUT/6178_Friday_August_12_2022_11_04_30_PM_75656588/index.tex]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.4 Indicial Equation with Difference of Roots Nonintegral. Exercises page 365
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{2x^2y'' - 3xy' + 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' - 3xy' + 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{3}{2x}$$
$$q(x) = \frac{1}{x^2}$$

Table 62: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{3}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' - 3xy' + 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} \right) - 3x \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \right) + 2 \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)\,(n+r-1) \right) + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r} \right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \quad \text{(2B)}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n (n+r)(n+r-1) - 3x^{n+r} a_n (n+r) + 2a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r) - 3x^r a_0 r + 2a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) - 3x^r r + 2x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 - 5r + 2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 5r + 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = \frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 - 5r + 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{3}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+2} \\
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}
\end{aligned}
$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $0 \leq n$ the recursive equation is

$$
2a_n(n+r)(n+r-1) - 3a_n(n+r) + 2a_n = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = 0 \tag{4}
$$

Which for the root $r = 2$ becomes

$$
a_n = 0 \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$
a_1 = 0
$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$
a_2 = 0
$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0
a_5	0	0
a_6	0	0

For $n = 7$, using the above recursive equation gives

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0
a_5	0	0
a_6	0	0
a_7	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^2 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\
&= x^2 \left(1 + O(x^8)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $0 \leq n$ the recursive equation is

$$2b_n(n + r)(n + r - 1) - 3b_n(n + r) + 2b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = 0 \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = 0 \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	0	0

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	0	0
b_5	0	0

For $n = 6$, using the above recursive equation gives

$$b_6 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	0	0
b_5	0	0
b_6	0	0

For $n = 7$, using the above recursive equation gives

$$b_7 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	0	0
b_5	0	0
b_6	0	0
b_7	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^2 \left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots\right) \\ &= \sqrt{x}\,\left(1 + O(x^8)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^2 \left(1 + O(x^8)\right) + c_2 \sqrt{x}\,\left(1 + O(x^8)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^2 \left(1 + O(x^8)\right) + c_2 \sqrt{x}\,\left(1 + O(x^8)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^2 \left(1 + O(x^8)\right) + c_2 \sqrt{x}\,\left(1 + O(x^8)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^2 \left(1 + O(x^8)\right) + c_2 \sqrt{x}\,\left(1 + O(x^8)\right)$$

Verified OK.

4.19.1 Maple step by step solution

Let's solve

$2x^2y'' - 3xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3y'}{2x} - \frac{y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{3y'}{2x} + \frac{y}{x^2} = 0$

- Multiply by denominators of the ODE

  $2x^2y'' - 3xy' + 2y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $2x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 3\frac{d}{dt}y(t) + 2y(t) = 0$

- Simplify

  $2\frac{d^2}{dt^2}y(t) - 5\frac{d}{dt}y(t) + 2y(t) = 0$

- Isolate 2nd derivative

$\frac{d^2}{dt^2}y(t) = \frac{5\frac{d}{dt}y(t)}{2} - y(t)$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2}y(t) - \frac{5\frac{d}{dt}y(t)}{2} + y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - \frac{5}{2}r + 1 = 0$

- Factor the characteristic polynomial

  $\frac{(2r-1)(r-2)}{2} = 0$

- Roots of the characteristic polynomial

  $r = \left(2, \frac{1}{2}\right)$

- 1st solution of the ODE

  $y_1(t) = e^{2t}$

- 2nd solution of the ODE

  $y_2(t) = e^{\frac{t}{2}}$

- General solution of the ODE

  $y(t) = c_1y_1(t) + c_2y_2(t)$

- Substitute in solutions

  $y(t) = c_1e^{2t} + c_2e^{\frac{t}{2}}$

- Change variables back using $t = \ln(x)$

  $y = c_1x^2 + c_2\sqrt{x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 27

```
Order:=8;
dsolve(2*x^2*diff(y(x),x$2)-3*x*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\sqrt{x} + c_2x^2 + O(x^8)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 18

```
AsymptoticDSolveValue[2*x^2*y''[x]-3*x*y'[x]+2*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1x^2 + c_2\sqrt{x}$$

4.20 problem 21

Internal problem ID [6936]
Internal file name [`OUTPUT/6179_Friday_August_12_2022_11_04_32_PM_79654824/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.4 Indicial Equation with Difference of Roots Nonintegral. Exercises page 365
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{9x^2y'' + 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$9x^2y'' + 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 0$$
$$q(x) = \frac{2}{9x^2}$$

Table 64: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = \frac{2}{9x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$9x^2y'' + 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$9x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$9x^{n+r} a_n (n+r)(n+r-1) + 2a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$9x^r a_0 r(-1+r) + 2a_0 x^r = 0$$

Or

$$\left(9x^r r(-1+r) + 2x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(9r^2 - 9r + 2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$9r^2 - 9r + 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{2}{3}$$
$$r_2 = \frac{1}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(9r^2 - 9r + 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{2}{3}} \\
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{3}}
\end{aligned}
$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $0 \leq n$ the recursive equation is

$$
9a_n(n+r)(n+r-1) + 2a_n = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = 0 \tag{4}
$$

Which for the root $r = \frac{2}{3}$ becomes

$$
a_n = 0 \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{2}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$
a_1 = 0
$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$
a_2 = 0
$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0
a_5	0	0
a_6	0	0

For $n = 7$, using the above recursive equation gives

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0
a_5	0	0
a_6	0	0
a_7	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{2}{3}} \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\ &= x^{\frac{2}{3}} \left(1 + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $0 \leq n$ the recursive equation is

$$9b_n(n + r)(n + r - 1) + 2b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = 0 \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_n = 0 \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	0	0

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	0	0
b_5	0	0

For $n = 6$, using the above recursive equation gives

$$b_6 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	0	0
b_5	0	0
b_6	0	0

For $n = 7$, using the above recursive equation gives

$$b_7 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	0	0
b_5	0	0
b_6	0	0
b_7	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\frac{2}{3}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots\right) \\ &= x^{\frac{1}{3}}\left(1 + O(x^8)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{2}{3}}\left(1 + O(x^8)\right) + c_2 x^{\frac{1}{3}}\left(1 + O(x^8)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^{\frac{2}{3}}\left(1 + O(x^8)\right) + c_2 x^{\frac{1}{3}}\left(1 + O(x^8)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{2}{3}}\left(1 + O(x^8)\right) + c_2 x^{\frac{1}{3}}\left(1 + O(x^8)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{2}{3}}\left(1 + O(x^8)\right) + c_2 x^{\frac{1}{3}}\left(1 + O(x^8)\right)$$

Verified OK.

4.20.1 Maple step by step solution

Let's solve

$9x^2y'' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y}{9x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y}{9x^2} = 0$

- Multiply by denominators of the ODE

  $9x^2y'' + 2y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $9x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + 2y(t) = 0$

- Simplify

  $9\frac{d^2}{dt^2}y(t) - 9\frac{d}{dt}y(t) + 2y(t) = 0$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = \frac{d}{dt}y(t) - \frac{2y(t)}{9}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2}y(t)-\frac{d}{dt}y(t)+\frac{2y(t)}{9}=0$
- Characteristic polynomial of ODE

  $r^2-r+\frac{2}{9}=0$
- Factor the characteristic polynomial

  $\frac{(3r-1)(3r-2)}{9}=0$
- Roots of the characteristic polynomial

  $r=\left(\frac{1}{3},\frac{2}{3}\right)$
- 1st solution of the ODE

  $y_1(t)=\mathrm{e}^{\frac{t}{3}}$
- 2nd solution of the ODE

  $y_2(t)=\mathrm{e}^{\frac{2t}{3}}$
- General solution of the ODE

  $y(t)=c_1y_1(t)+c_2y_2(t)$
- Substitute in solutions

  $y(t)=c_1\mathrm{e}^{\frac{t}{3}}+c_2\mathrm{e}^{\frac{2t}{3}}$
- Change variables back using $t=\ln(x)$

  $y=c_1x^{\frac{1}{3}}+c_2x^{\frac{2}{3}}$
- Simplify

  $y=x^{\frac{1}{3}}\left(x^{\frac{1}{3}}c_2+c_1\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
Order:=8;
dsolve(9*x^2*diff(y(x),x$2)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = x^{\frac{1}{3}}\left(c_2 x^{\frac{1}{3}} + c_1\right) + O(x^8)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 20

```
AsymptoticDSolveValue[9*x^2*y''[x]+2*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1 x^{2/3} + c_2 \sqrt[3]{x}$$

4.21 problem 22

Internal problem ID [6937]
Internal file name [`OUTPUT/6180_Friday_August_12_2022_11_04_34_PM_17396529/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.4 Indicial Equation with Difference of Roots Nonintegral. Exercises page 365
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{2x^2y'' + 5xy' - 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' + 5xy' - 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{5}{2x}$$
$$q(x) = -\frac{1}{x^2}$$

Table 66: Table $p(x), q(x)$ singularites.

$p(x) = \frac{5}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' + 5xy' - 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + 5x\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \quad \text{(2A)}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r} a_n(n+r)\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \quad \text{(2B)}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n(n+r)(n+r-1) + 5x^{n+r} a_n(n+r) - 2a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r) + 5x^r a_0 r - 2a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) + 5x^r r - 2x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 + 3r - 2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 + 3r - 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = -2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 + 3r - 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{5}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-2}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $0 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) + 5a_n(n+r) - 2a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = 0 \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = 0 \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0
a_5	0	0
a_6	0	0

For $n = 7$, using the above recursive equation gives

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0
a_5	0	0
a_6	0	0
a_7	0	0

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\
&= \sqrt{x}\,\left(1 + O\left(x^8\right)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $0 \leq n$ the recursive equation is

$$
2b_n(n + r)\,(n + r - 1) + 5b_n(n + r) - 2b_n = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = 0 \tag{4}
$$

Which for the root $r = -2$ becomes

$$
b_n = 0 \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -2$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$
b_1 = 0
$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	0	0

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	0	0
b_5	0	0

For $n = 6$, using the above recursive equation gives

$$b_6 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	0	0
b_5	0	0
b_6	0	0

For $n = 7$, using the above recursive equation gives

$$b_7 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	0	0
b_5	0	0
b_6	0	0
b_7	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= \sqrt{x}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \ldots\right) \\ &= \frac{1 + O(x^8)}{x^2} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \sqrt{x}\left(1 + O(x^8)\right) + \frac{c_2(1 + O(x^8))}{x^2} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 \sqrt{x}\left(1 + O(x^8)\right) + \frac{c_2(1 + O(x^8))}{x^2} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \sqrt{x}\left(1 + O(x^8)\right) + \frac{c_2(1 + O(x^8))}{x^2} \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\left(1+O\left(x^8\right)\right)+\frac{c_2\left(1+O\left(x^8\right)\right)}{x^2}$$

Verified OK.

4.21.1 Maple step by step solution

Let's solve

$2x^2y''+5xy'-2y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{5y'}{2x}+\frac{y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{5y'}{2x}-\frac{y}{x^2}=0$

- Multiply by denominators of the ODE

  $2x^2y''+5xy'-2y=0$

- Make a change of variables

  $t=\ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y'=\left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

    $y'=\frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y''=\left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2+t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y''=\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $2x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}\right)+5\frac{d}{dt}y(t)-2y(t)=0$

- Simplify
  $2 \frac{d^{2}}{d t^{2}}y(t)+3 \frac{d}{d t}y(t)-2 y(t)=0$
- Isolate 2nd derivative
  $\frac{d^{2}}{d t^{2}}y(t)=-\frac{3 \frac{d}{d t}y(t)}{2}+y(t)$
- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^{2}}{d t^{2}}y(t)+\frac{3 \frac{d}{d t}y(t)}{2}-y(t)=0$
- Characteristic polynomial of ODE
  $r^{2}+\frac{3}{2}r-1=0$
- Factor the characteristic polynomial
  $\frac{(r+2)(2r-1)}{2}=0$
- Roots of the characteristic polynomial
  $r=\left(-2, \frac{1}{2}\right)$
- 1st solution of the ODE
  $y_{1}(t)=\mathrm{e}^{-2t}$
- 2nd solution of the ODE
  $y_{2}(t)=\mathrm{e}^{\frac{t}{2}}$
- General solution of the ODE
  $y(t)=c_{1}y_{1}(t)+c_{2}y_{2}(t)$
- Substitute in solutions
  $y(t)=c_{1}\mathrm{e}^{-2t}+c_{2}\mathrm{e}^{\frac{t}{2}}$
- Change variables back using $t=\ln(x)$
  $y=\frac{c_{1}}{x^{2}}+c_{2}\sqrt{x}$
- Simplify
  $y=\frac{c_{1}}{x^{2}}+c_{2}\sqrt{x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 27

```
Order:=8;
dsolve(2*x^2*diff(y(x),x$2)+5*x*diff(y(x),x)-2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{x^{\frac{5}{2}} c_2 + c_1}{x^2} + O(x^8)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 18

```
AsymptoticDSolveValue[2*x^2*y''[x]+5*x*y'[x]-2*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to \frac{c_2}{x^2} + c_1\sqrt{x}$$

4.22 problem 25

Internal problem ID [6938]
Internal file name [`OUTPUT/6181_Friday_August_12_2022_11_04_36_PM_64086885/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.4 Indicial Equation with Difference of Roots Nonintegral. Exercises page 365
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{2x^2y'' + xy' - y = 0}$$

4.22.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$2x^2(r(r-1))x^{r-2} + xrx^{r-1} - x^r = 0$$

Simplifying gives

$$2r(r-1)\,x^r + r\,x^r - x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$2r(r-1)+r-1=0$$

Or

$$2r^2-r-1=0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 1$$
$$r_2 = -\frac{1}{2}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_1 x + \frac{c_2}{\sqrt{x}}$$

Summary

The solution(s) found are the following

$$y = c_1 x + \frac{c_2}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = c_1 x + \frac{c_2}{\sqrt{x}}$$

Verified OK.

4.22.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$2x^2y''+xy'-y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$p(x) = \frac{1}{2x}$$
$$q(x) = -\frac{1}{2x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$
$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int \frac{1}{2x}dx\right)} dx \\
&= \int e^{-\frac{\ln(x)}{2}} dx \\
&= \int \frac{1}{\sqrt{x}} dx \\
&= 2\sqrt{x}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{1}{2x^2}}{\frac{1}{x}} \\
&= -\frac{1}{2x}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - \frac{y(\tau)}{2x} &= 0
\end{aligned}$$

But in terms of τ

$$-\frac{1}{2x} = -\frac{2}{\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) - \frac{2y(\tau)}{\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 - 2y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} - 2\tau^r = 0$$

Simplifying gives

$$r(r-1)\,\tau^r + 0\,\tau^r - 2\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$r(r-1) + 0 - 2 = 0$$

Or

$$r^2 - r - 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned}
r_1 &= -1 \\
r_2 &= 2
\end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = \frac{c_1}{\tau} + c_2\tau^2$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1}{2\sqrt{x}} + 4c_2x$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{2\sqrt{x}} + 4c_2x \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{2\sqrt{x}} + 4c_2x$$

Verified OK.

4.22.3 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$2x^2y'' + xy' - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{2x} \\ q(x) &= -\frac{1}{2x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{2x^2} - \frac{1}{2x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{5v'(x)}{2x} &= 0 \\ v''(x) + \frac{5v'(x)}{2x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{5u(x)}{2x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{5u}{2x} \end{aligned}$$

Where $f(x) = -\frac{5}{2x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{5}{2x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{5}{2x}\, dx \\ \ln(u) &= -\frac{5\ln(x)}{2} + c_1 \\ u &= e^{-\frac{5\ln(x)}{2} + c_1} \\ &= \frac{c_1}{x^{\frac{5}{2}}} \end{aligned}$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= -\frac{2c_1}{3x^{\frac{3}{2}}} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\,x^n \\
&= \left(-\frac{2c_1}{3x^{\frac{3}{2}}} + c_2\right)x \\
&= \left(-\frac{2c_1}{3x^{\frac{3}{2}}} + c_2\right)x
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \left(-\frac{2c_1}{3x^{\frac{3}{2}}} + c_2\right)x \tag{1}
$$

Verification of solutions

$$
y = \left(-\frac{2c_1}{3x^{\frac{3}{2}}} + c_2\right)x
$$

Verified OK.

4.22.4 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$
Ay'' + By' + Cy = F(x)
$$

This method reduces the order ode the ODE by one by applying the transformation

$$
y = Bv
$$

This results in

$$
\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}
$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \qquad (1)
\end{aligned}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$
ABv'' + \left(2AB' + B^2\right) v' = 0
$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$
ABu' + \left(2AB' + B^2\right) u = 0
$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A &= 2x^2 \\
B &= x \\
C &= -1 \\
F &= 0
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= \left(2x^2\right)(0) + (x)(1) + (-1)(x) \\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
2x^3 v'' + \left(5x^2\right) v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
2x^3 u'(x) + 5x^2 u(x) = 0
$$

Which is now solved for u. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{5u}{2x}
\end{aligned}
$$

Where $f(x) = -\frac{5}{2x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{5}{2x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{5}{2x}\,dx \\
\ln(u) &= -\frac{5\ln(x)}{2} + c_1 \\
u &= \mathrm{e}^{-\frac{5\ln(x)}{2}+c_1} \\
&= \frac{c_1}{x^{\frac{5}{2}}}
\end{aligned}$$

The ode for v now becomes

$$\begin{aligned}
v' &= u \\
&= \frac{c_1}{x^{\frac{5}{2}}}
\end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned}
v(x) &= \int \frac{c_1}{x^{\frac{5}{2}}}\,\mathrm{d}x \\
&= -\frac{2c_1}{3x^{\frac{3}{2}}} + c_2
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y(x) &= Bv \\
&= (x)\left(-\frac{2c_1}{3x^{\frac{3}{2}}} + c_2\right) \\
&= \left(-\frac{2c_1}{3x^{\frac{3}{2}}} + c_2\right)x
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \left(-\frac{2c_1}{3x^{\frac{3}{2}}} + c_2\right)x \tag{1}$$

Verification of solutions

$$y = \left(-\frac{2c_1}{3x^{\frac{3}{2}}} + c_2\right)x$$

Verified OK.

4.22.5 Solving using Kovacic algorithm

Writing the ode as

$$2x^2y'' + xy' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 5 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 68: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{5}{16x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{5}{16x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-}=-\frac{1}{4}$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-}-\left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{4}-\left(-\frac{1}{4}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_{\infty}
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^{-}}{x-c_1}\right)+(-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{4x}+(-)\,(0) \\
&= -\frac{1}{4x} \\
&= -\frac{1}{4x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}
$$

Let

$$
p(x)=1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0)+2\left(-\frac{1}{4x}\right)(0)+\left(\left(\frac{1}{4x^2}\right)+\left(-\frac{1}{4x}\right)^2-\left(\frac{5}{16x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\,dx} \\
&= \mathrm{e}^{\int -\frac{1}{4x}dx} \\
&= \frac{1}{x^{\frac{1}{4}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{2x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{4}} \\
&= z_1 \left(\frac{1}{x^{\frac{1}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{\sqrt{x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{2x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2x^{\frac{3}{2}}}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{\sqrt{x}}\right) + c_2 \left(\frac{1}{\sqrt{x}}\left(\frac{2x^{\frac{3}{2}}}{3}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1}{\sqrt{x}} + \frac{2c_2 x}{3} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1}{\sqrt{x}} + \frac{2c_2 x}{3}$$

Verified OK.

4.22.6 Maple step by step solution

Let's solve

$2x^2 y'' + x y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{2x} + \frac{y}{2x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{2x} - \frac{y}{2x^2} = 0$

- Multiply by denominators of the ODE

  $2x^2 y'' + x y' - y = 0$

- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt} y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2} y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt} y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2} y(t)}{x^2} - \frac{\frac{d}{dt} y(t)}{x^2}$

  Substitute the change of variables back into the ODE

$$2x^2 \left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + \frac{d}{dt}y(t) - y(t) = 0$$

- Simplify

$$2\frac{d^2}{dt^2}y(t) - \frac{d}{dt}y(t) - y(t) = 0$$

- Isolate 2nd derivative

$$\frac{d^2}{dt^2}y(t) = \frac{\frac{d}{dt}y(t)}{2} + \frac{y(t)}{2}$$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$$\frac{d^2}{dt^2}y(t) - \frac{\frac{d}{dt}y(t)}{2} - \frac{y(t)}{2} = 0$$

- Characteristic polynomial of ODE

$$r^2 - \frac{1}{2}r - \frac{1}{2} = 0$$

- Factor the characteristic polynomial

$$\frac{(2r+1)(r-1)}{2} = 0$$

- Roots of the characteristic polynomial

$$r = \left(1, -\frac{1}{2}\right)$$

- 1st solution of the ODE

$$y_1(t) = \mathrm{e}^t$$

- 2nd solution of the ODE

$$y_2(t) = \mathrm{e}^{-\frac{t}{2}}$$

- General solution of the ODE

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

- Substitute in solutions

$$y(t) = c_1 \mathrm{e}^t + c_2 \mathrm{e}^{-\frac{t}{2}}$$

- Change variables back using $t = \ln(x)$

$$y = c_1 x + \frac{c_2}{\sqrt{x}}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(2*x^2*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + \frac{c_2}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 18

```
DSolve[2*x^2*y''[x]+x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1}{\sqrt{x}} + c_2 x$$

4.23 problem 26

Internal problem ID [6939]
Internal file name [`OUTPUT/6182_Friday_August_12_2022_11_04_37_PM_47063498/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.4 Indicial Equation with Difference of Roots Nonintegral. Exercises page 365
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{2x^2y'' - 3xy' + 2y = 0}$$

4.23.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$2x^2(r(r-1))x^{r-2} - 3xrx^{r-1} + 2x^r = 0$$

Simplifying gives

$$2r(r-1)\,x^r - 3r\,x^r + 2x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$2r(r-1) - 3r + 2 = 0$$

Or

$$2r^2 - 5r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 2$$
$$r_2 = \frac{1}{2}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_1 x^2 + c_2 \sqrt{x}$$

Summary

The solution(s) found are the following

$$y = c_1 x^2 + c_2 \sqrt{x} \tag{1}$$

Verification of solutions

$$y = c_1 x^2 + c_2 \sqrt{x}$$

Verified OK.

4.23.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$2x^2 y'' - 3xy' + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{3}{2x}$$
$$q(x) = \frac{1}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int e^{-\left(\int p(x)dx\right)}dx \\
&= \int e^{-\left(\int -\frac{3}{2x}dx\right)}dx \\
&= \int e^{\frac{3\ln(x)}{2}}\,dx \\
&= \int x^{\frac{3}{2}}dx \\
&= \frac{2x^{\frac{5}{2}}}{5}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{1}{x^2}}{x^3} \\
&= \frac{1}{x^5}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{x^5} &= 0
\end{aligned}$$

But in terms of τ

$$\frac{1}{x^5} = \frac{4}{25\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{4y(\tau)}{25\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$25\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 4y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$25\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 4\tau^r = 0$$

Simplifying gives

$$25r(r-1)\,\tau^r + 0\,\tau^r + 4\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$25r(r-1) + 0 + 4 = 0$$

Or

$$25r^2 - 25r + 4 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{5}$$
$$r_2 = \frac{4}{5}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1\tau^{\frac{1}{5}} + c_2\tau^{\frac{4}{5}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 2^{\frac{1}{5}} 5^{\frac{4}{5}} \left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}}{5} + \frac{c_2 2^{\frac{4}{5}} 5^{\frac{1}{5}} \left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}}{5}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 2^{\frac{1}{5}} 5^{\frac{4}{5}} \left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}}{5} + \frac{c_2 2^{\frac{4}{5}} 5^{\frac{1}{5}} \left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}}{5} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 2^{\frac{1}{5}} 5^{\frac{4}{5}} \left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}}{5} + \frac{c_2 2^{\frac{4}{5}} 5^{\frac{1}{5}} \left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}}{5}$$

Verified OK.

4.23.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$2x^2y'' - 3xy' + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{3}{2x}$$
$$q(x) = \frac{1}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$
\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{\frac{1}{x^2}}}{c} \\
\tau'' &= -\frac{1}{c\sqrt{\frac{1}{x^2}}\, x^3}
\end{aligned} \tag{6}
$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{1}{c\sqrt{\frac{1}{x^2}}\, x^3} - \frac{3}{2x}\frac{\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= -\frac{5c}{2}
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - \frac{5c\left(\frac{d}{d\tau}y(\tau)\right)}{2} + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{\frac{5c\tau}{4}}\left(c_1 \cosh\left(\frac{3c\tau}{4}\right) + i c_2 \sinh\left(\frac{3c\tau}{4}\right)\right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\, dx \\
&= \frac{\int \sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{\frac{1}{x^2}}\, x \ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$y = x^{\frac{5}{4}}\left(c_1 \cosh\left(\frac{3\ln(x)}{4}\right) + ic_2 \sinh\left(\frac{3\ln(x)}{4}\right)\right)$$

Summary
The solution(s) found are the following

$$y = x^{\frac{5}{4}}\left(c_1 \cosh\left(\frac{3\ln(x)}{4}\right) + ic_2 \sinh\left(\frac{3\ln(x)}{4}\right)\right) \tag{1}$$

Verification of solutions

$$y = x^{\frac{5}{4}}\left(c_1 \cosh\left(\frac{3\ln(x)}{4}\right) + ic_2 \sinh\left(\frac{3\ln(x)}{4}\right)\right)$$

Verified OK.

4.23.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$2x^2y'' - 3xy' + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{2x} \\ q(x) &= \frac{1}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right)v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right)v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{3n}{2x^2} + \frac{1}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{5v'(x)}{2x} &= 0 \\ v''(x) + \frac{5v'(x)}{2x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{5u(x)}{2x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{5u}{2x} \end{aligned}$$

Where $f(x) = -\frac{5}{2x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{5}{2x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{5}{2x}\,dx \\ \ln(u) &= -\frac{5\ln(x)}{2} + c_1 \\ u &= e^{-\frac{5\ln(x)}{2}+c_1} \\ &= \frac{c_1}{x^{\frac{5}{2}}} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= -\frac{2c_1}{3x^{\frac{3}{2}}} + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{2c_1}{3x^{\frac{3}{2}}} + c_2\right) x^2 \\
&= c_2 x^2 - \frac{2\sqrt{x}\, c_1}{3}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \left(-\frac{2c_1}{3x^{\frac{3}{2}}} + c_2\right) x^2 \tag{1}$$

Verification of solutions

$$y = \left(-\frac{2c_1}{3x^{\frac{3}{2}}} + c_2\right) x^2$$

Verified OK.

4.23.5 Solving using Kovacic algorithm

Writing the ode as

$$2x^2y'' - 3xy' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}
A &= 2x^2 \\
B &= -3x \\
C &= 2
\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 5 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 70: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 0\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5}{16x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{5}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{4} - \left(-\frac{1}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{4x} + (-)\,(0) \\
&= -\frac{1}{4x} \\
&= -\frac{1}{4x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{4x}\right)(0) + \left(\left(\frac{1}{4x^2}\right) + \left(-\frac{1}{4x}\right)^2 - \left(\frac{5}{16x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{1}{4x} dx} \\ &= \frac{1}{x^{\frac{1}{4}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-3x}{2x^2} \, dx} \\ &= z_1 e^{\frac{3\ln(x)}{4}} \\ &= z_1 \left(x^{\frac{3}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3x}{2x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\frac{3\ln(x)}{2}}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{2x^{\frac{3}{2}}}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\sqrt{x}) + c_2\left(\sqrt{x}\left(\frac{2x^{\frac{3}{2}}}{3}\right)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \sqrt{x}\,c_1 + \frac{2c_2x^2}{3} \tag{1}$$

Verification of solutions

$$y = \sqrt{x}\,c_1 + \frac{2c_2x^2}{3}$$

Verified OK.

4.23.6 Maple step by step solution

Let's solve

$2x^2y'' - 3xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3y'}{2x} - \frac{y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{3y'}{2x} + \frac{y}{x^2} = 0$

- Multiply by denominators of the ODE

  $2x^2 y'' - 3xy' + 2y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $2x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 3\frac{d}{dt}y(t) + 2y(t) = 0$

- Simplify

  $2\frac{d^2}{dt^2}y(t) - 5\frac{d}{dt}y(t) + 2y(t) = 0$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = \frac{5\frac{d}{dt}y(t)}{2} - y(t)$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2}y(t) - \frac{5\frac{d}{dt}y(t)}{2} + y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - \frac{5}{2}r + 1 = 0$

- Factor the characteristic polynomial

  $\frac{(2r-1)(r-2)}{2} = 0$

- Roots of the characteristic polynomial

  $r = \left(2, \frac{1}{2}\right)$

- 1st solution of the ODE

  $y_1(t) = e^{2t}$

- 2nd solution of the ODE

  $y_2(t) = e^{\frac{t}{2}}$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 e^{2t} + c_2 e^{\frac{t}{2}}$

- Change variables back using $t = \ln(x)$

  $y = c_1 x^2 + c_2 \sqrt{x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(2*x^2*diff(y(x),x$2)-3*x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^2 + \sqrt{x}\, c_2$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 20

```
DSolve[2*x^2*y''[x]-3*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 x^2 + c_1 \sqrt{x}$$

4.24 problem 27

Internal problem ID [6940]
Internal file name [`OUTPUT/6183_Friday_August_12_2022_11_04_38_PM_40386715/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.4 Indicial Equation with Difference of Roots Nonintegral. Exercises page 365
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{9x^2y'' + 2y = 0}$$

4.24.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$9x^2(r(r-1))x^{r-2} + 0rx^{r-1} + 2x^r = 0$$

Simplifying gives

$$9r(r-1)\,x^r + 0\,x^r + 2x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$9r(r-1) + 0 + 2 = 0$$

Or

$$9r^2 - 9r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{3}$$
$$r_2 = \frac{2}{3}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_1 x^{\frac{1}{3}} + c_2 x^{\frac{2}{3}}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{1}{3}} + c_2 x^{\frac{2}{3}} \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{3}} + c_2 x^{\frac{2}{3}}$$

Verified OK.

4.24.2 Solving using Kovacic algorithm

Writing the ode as

$$9x^2 y'' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^2 \\ B &= 0 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{9x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -2 \\ t &= 9x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{2}{9x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 72: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 9x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{9x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{2}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{2}{3} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{3} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{2}{9x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{2}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{2}{3} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{3} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{2}{9x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{2}{3}$	$\frac{1}{3}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{2}{3}$	$\frac{1}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{3}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{3} - \left(\frac{1}{3}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{3x} + (-)\,(0) \\ &= \frac{1}{3x} \\ &= \frac{1}{3x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{3x}\right)(0) + \left(\left(-\frac{1}{3x^2}\right) + \left(\frac{1}{3x}\right)^2 - \left(-\frac{2}{9x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \frac{1}{3x} dx} \\ &= x^{\frac{1}{3}} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= x^{\frac{1}{3}} \end{aligned}$$

Which simplifies to

$$y_1 = x^{\frac{1}{3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\
&= x^{\frac{1}{3}} \int \frac{1}{x^{\frac{2}{3}}} \, dx \\
&= x^{\frac{1}{3}} \left(3x^{\frac{1}{3}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{\frac{1}{3}}\right) + c_2 \left(x^{\frac{1}{3}} \left(3x^{\frac{1}{3}}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^{\frac{1}{3}} + 3c_2 x^{\frac{2}{3}} \tag{1}
$$

Verification of solutions

$$
y = c_1 x^{\frac{1}{3}} + 3c_2 x^{\frac{2}{3}}
$$

Verified OK.

4.24.3 Maple step by step solution

Let's solve

$9x^2 y'' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y}{9x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y}{9x^2} = 0$

- Multiply by denominators of the ODE

$9x^2y'' + 2y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $9x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + 2y(t) = 0$

- Simplify

  $9\frac{d^2}{dt^2}y(t) - 9\frac{d}{dt}y(t) + 2y(t) = 0$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = \frac{d}{dt}y(t) - \frac{2y(t)}{9}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2}y(t) - \frac{d}{dt}y(t) + \frac{2y(t)}{9} = 0$

- Characteristic polynomial of ODE

  $r^2 - r + \frac{2}{9} = 0$

- Factor the characteristic polynomial

  $\frac{(3r-1)(3r-2)}{9} = 0$

- Roots of the characteristic polynomial

  $r = \left(\frac{1}{3}, \frac{2}{3}\right)$

- 1st solution of the ODE

  $y_1(t) = e^{\frac{t}{3}}$

- 2nd solution of the ODE

  $y_2(t) = e^{\frac{2t}{3}}$
- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$
- Substitute in solutions

  $y(t) = c_1 e^{\frac{t}{3}} + c_2 e^{\frac{2t}{3}}$
- Change variables back using $t = \ln(x)$

  $y = c_1 x^{\frac{1}{3}} + c_2 x^{\frac{2}{3}}$
- Simplify

  $y = x^{\frac{1}{3}}\left(x^{\frac{1}{3}} c_2 + c_1\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(9*x^2*diff(y(x),x$2)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = x^{\frac{1}{3}}\left(c_1 x^{\frac{1}{3}} + c_2\right)$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 22

```
DSolve[9*x^2*y''[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt[3]{x}\left(c_2 \sqrt[3]{x} + c_1\right)$$

4.25 problem 28

Internal problem ID [6941]
Internal file name [`OUTPUT/6184_Friday_August_12_2022_11_04_40_PM_31476194/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.4 Indicial Equation with Difference of Roots Nonintegral. Exercises page 365
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{2x^2y'' + 5xy' - 2y = 0}$$

4.25.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$2x^2(r(r-1))x^{r-2} + 5xrx^{r-1} - 2x^r = 0$$

Simplifying gives

$$2r(r-1)\,x^r + 5r\,x^r - 2x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$2r(r-1) + 5r - 2 = 0$$

Or

$$2r^2 + 3r - 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -2$$
$$r_2 = \frac{1}{2}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x^2} + c_2\sqrt{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^2} + c_2\sqrt{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x^2} + c_2\sqrt{x}$$

Verified OK.

4.25.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$2x^2y'' + 5xy' - 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{5}{2x}$$
$$q(x) = -\frac{1}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int \frac{5}{2x}dx\right)}dx \\ &= \int e^{-\frac{5\ln(x)}{2}}\,dx \\ &= \int \frac{1}{x^{\frac{5}{2}}}dx \\ &= -\frac{2}{3x^{\frac{3}{2}}}\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{-\frac{1}{x^2}}{\frac{1}{x^5}} \\ &= -x^3\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) - x^3 y(\tau) = 0$$

But in terms of τ

$$-x^3 = -\frac{4}{9\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) - \frac{4y(\tau)}{9\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$9\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 - 4y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$9\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} - 4\tau^r = 0$$

Simplifying gives

$$9r(r-1)\,\tau^r + 0\,\tau^r - 4\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$9r(r-1) + 0 - 4 = 0$$

Or

$$9r^2 - 9r - 4 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -\frac{1}{3}$$
$$r_2 = \frac{4}{3}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = \frac{c_1}{\tau^{\frac{1}{3}}} + c_2\tau^{\frac{4}{3}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\left(2c_2 18^{\frac{2}{3}} \left(-\frac{1}{x^{\frac{3}{2}}}\right)^{\frac{5}{3}} + 27c_1\right) 18^{\frac{2}{3}}}{162 \left(-\frac{1}{x^{\frac{3}{2}}}\right)^{\frac{1}{3}}}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(2c_2 18^{\frac{2}{3}} \left(-\frac{1}{x^{\frac{3}{2}}}\right)^{\frac{5}{3}} + 27c_1\right) 18^{\frac{2}{3}}}{162 \left(-\frac{1}{x^{\frac{3}{2}}}\right)^{\frac{1}{3}}} \tag{1}$$

Verification of solutions

$$y = \frac{\left(2c_2 18^{\frac{2}{3}} \left(-\frac{1}{x^{\frac{3}{2}}}\right)^{\frac{5}{3}} + 27c_1\right) 18^{\frac{2}{3}}}{162 \left(-\frac{1}{x^{\frac{3}{2}}}\right)^{\frac{1}{3}}}$$

Verified OK.

4.25.3 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$2x^2y'' + 5xy' - 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{5}{2x} \\ q(x) &= -\frac{1}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{5n}{2x^2} - \frac{1}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = \frac{1}{2} \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{7v'(x)}{2x} &= 0 \\ v''(x) + \frac{7v'(x)}{2x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{7u(x)}{2x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{7u}{2x} \end{aligned}$$

Where $f(x) = -\frac{7}{2x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{7}{2x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{7}{2x}\,dx \\ \ln(u) &= -\frac{7\ln(x)}{2} + c_1 \\ u &= e^{-\frac{7\ln(x)}{2} + c_1} \\ &= \frac{c_1}{x^{\frac{7}{2}}} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= -\frac{2c_1}{5x^{\frac{5}{2}}} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\, x^n \\ &= \left(-\frac{2c_1}{5x^{\frac{5}{2}}} + c_2\right)\sqrt{x} \\ &= \frac{5c_2 x^{\frac{5}{2}} - 2c_1}{5x^2} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \left(-\frac{2c_1}{5x^{\frac{5}{2}}} + c_2\right)\sqrt{x} \tag{1}$$

Verification of solutions

$$y = \left(-\frac{2c_1}{5x^{\frac{5}{2}}} + c_2\right)\sqrt{x}$$

Verified OK.

4.25.4 Solving using Kovacic algorithm

Writing the ode as

$$2x^2y'' + 5xy' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= 5x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{21}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 21 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{21}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 74: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{21}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{21}{16x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{21}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = -\frac{3}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{3}{4} - \left(-\frac{3}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{4x} + (-)\,(0) \\
&= -\frac{3}{4x} \\
&= -\frac{3}{4x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{3}{4x}\right)(0) + \left(\left(\frac{3}{4x^2}\right) + \left(-\frac{3}{4x}\right)^2 - \left(\frac{21}{16x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int -\frac{3}{4x} dx} \\
&= \frac{1}{x^{\frac{3}{4}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{5x}{2x^2}\,dx} \\
&= z_1 e^{-\frac{5\ln(x)}{4}} \\
&= z_1\left(\frac{1}{x^{\frac{5}{4}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5x}{2x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{2x^{\frac{5}{2}}}{5}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x^2}\right) + c_2\left(\frac{1}{x^2}\left(\frac{2x^{\frac{5}{2}}}{5}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^2} + \frac{2c_2\sqrt{x}}{5} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x^2} + \frac{2c_2\sqrt{x}}{5}$$

Verified OK.

4.25.5 Maple step by step solution

Let's solve

$2x^2y'' + 5xy' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{5y'}{2x} + \frac{y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{5y'}{2x} - \frac{y}{x^2} = 0$

- Multiply by denominators of the ODE

  $2x^2y'' + 5xy' - 2y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

$$2x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}\right)+5\frac{d}{dt}y(t)-2y(t)=0$$

- Simplify

$$2\frac{d^2}{dt^2}y(t)+3\frac{d}{dt}y(t)-2y(t)=0$$

- Isolate 2nd derivative

$$\frac{d^2}{dt^2}y(t)=-\frac{3\frac{d}{dt}y(t)}{2}+y(t)$$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$$\frac{d^2}{dt^2}y(t)+\frac{3\frac{d}{dt}y(t)}{2}-y(t)=0$$

- Characteristic polynomial of ODE

$$r^2+\frac{3}{2}r-1=0$$

- Factor the characteristic polynomial

$$\frac{(r+2)(2r-1)}{2}=0$$

- Roots of the characteristic polynomial

$$r=\left(-2,\frac{1}{2}\right)$$

- 1st solution of the ODE

$$y_1(t)=\mathrm{e}^{-2t}$$

- 2nd solution of the ODE

$$y_2(t)=\mathrm{e}^{\frac{t}{2}}$$

- General solution of the ODE

$$y(t)=c_1y_1(t)+c_2y_2(t)$$

- Substitute in solutions

$$y(t)=c_1\mathrm{e}^{-2t}+c_2\mathrm{e}^{\frac{t}{2}}$$

- Change variables back using $t=\ln(x)$

$$y=\frac{c_1}{x^2}+c_2\sqrt{x}$$

- Simplify

$$y=\frac{c_1}{x^2}+c_2\sqrt{x}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 15

```
dsolve(2*x^2*diff(y(x),x$2)+5*x*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x^{\frac{5}{2}} c_2 + c_1}{x^2}$$

✓ Solution by Mathematica
Time used: 0.012 (sec). Leaf size: 20

```
DSolve[2*x^2*y''[x]+5*x*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 x^{5/2} + c_1}{x^2}$$

4.26 problem 29

Internal problem ID [6942]
Internal file name [`OUTPUT/6185_Friday_August_12_2022_11_04_41_PM_784968/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.4 Indicial Equation with Difference of Roots Nonintegral. Exercises page 365
Problem number: 29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' + 2xy' - 12y = 0}$$

4.26.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 2xrx^{r-1} - 12x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 2r\,x^r - 12x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 2r - 12 = 0$$

Or

$$r^2 + r - 12 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -4$$
$$r_2 = 3$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x^4} + c_2 x^3$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^4} + c_2 x^3 \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x^4} + c_2 x^3$$

Verified OK.

4.26.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2 y'' + 2xy' - 12y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{2}{x}$$
$$q(x) = -\frac{12}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{2}{x}dx\right)}dx \\
&= \int e^{-2\ln(x)}\,dx \\
&= \int \frac{1}{x^2}dx \\
&= -\frac{1}{x}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{12}{x^2}}{\frac{1}{x^4}} \\
&= -12x^2
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) - 12x^2 y(\tau) = 0$$

But in terms of τ

$$-12x^2 = -\frac{12}{\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) - \frac{12y(\tau)}{\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 - 12y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} - 12\tau^r = 0$$

Simplifying gives

$$r(r-1)\,\tau^r + 0\,\tau^r - 12\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$r(r-1) + 0 - 12 = 0$$

Or

$$r^2 - r - 12 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -3$$
$$r_2 = 4$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = \frac{c_1}{\tau^3} + c_2\tau^4$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{-c_1x^7 + c_2}{x^4}$$

Summary
The solution(s) found are the following

$$y = \frac{-c_1 x^7 + c_2}{x^4} \tag{1}$$

Verification of solutions

$$y = \frac{-c_1 x^7 + c_2}{x^4}$$

Verified OK.

4.26.3 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2 y'' + 2xy' - 12y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{2}{x} \\ q(x) &= -\frac{12}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{2n}{x^2} - \frac{12}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{8v'(x)}{x} &= 0 \\ v''(x) + \frac{8v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{8u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{8u}{x} \end{aligned}$$

Where $f(x) = -\frac{8}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{8}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{8}{x}\,dx \\ \ln(u) &= -8\ln(x) + c_1 \\ u &= \mathrm{e}^{-8\ln(x)+c_1} \\ &= \frac{c_1}{x^8} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= -\frac{c_1}{7x^7} + c_2 \end{aligned}$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{c_1}{7x^7} + c_2\right) x^3 \\
&= \frac{7c_2x^7 - c_1}{7x^4}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \left(-\frac{c_1}{7x^7} + c_2\right) x^3 \tag{1}
$$

Verification of solutions

$$
y = \left(-\frac{c_1}{7x^7} + c_2\right) x^3
$$

Verified OK.

4.26.4 Solving using Kovacic algorithm

Writing the ode as

$$
x^2y'' + 2xy' - 12y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 \\
B &= 2x \\
C &= -12
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{12}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 12 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{12}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 76: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{12}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 12$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 4 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -3
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{12}{x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 12$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 4 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -3
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{12}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	4	-3

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	4	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -3$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -3 - (-3) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{x} + (-)\,(0) \\
&= -\frac{3}{x} \\
&= -\frac{3}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{x}\right)(0) + \left(\left(\frac{3}{x^2}\right) + \left(-\frac{3}{x}\right)^2 - \left(\frac{12}{x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{3}{x} dx} \\
&= \frac{1}{x^3}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{2x}{x^2} \, dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1 \left(\frac{1}{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{x^7}{7}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{1}{x^4}\right) + c_2\left(\frac{1}{x^4}\left(\frac{x^7}{7}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x^4} + \frac{c_2x^3}{7} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{x^4} + \frac{c_2x^3}{7}
$$

Verified OK.

4.26.5 Maple step by step solution

Let's solve

$x^2y'' + 2xy' - 12y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{x} + \frac{12y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x} - \frac{12y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2y'' + 2xy' - 12y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

- ∘ Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

- Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

- Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + 2\frac{d}{dt}y(t) - 12y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) + \frac{d}{dt}y(t) - 12y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 + r - 12 = 0$

- Factor the characteristic polynomial

  $(r+4)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (-4, 3)$

- 1st solution of the ODE

  $y_1(t) = e^{-4t}$

- 2nd solution of the ODE

  $y_2(t) = e^{3t}$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 e^{-4t} + c_2 e^{3t}$

- Change variables back using $t = \ln(x)$

  $y = \frac{c_1}{x^4} + c_2 x^3$

- Simplify

  $y = \frac{c_1}{x^4} + c_2 x^3$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)+2*x*diff(y(x),x)-12*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_2 x^7 + c_1}{x^4}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 18

```
DSolve[x^2*y''[x]+2*x*y'[x]-12*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 x^7 + c_1}{x^4}$$

4.27 problem 30

Internal problem ID [6943]
Internal file name [`OUTPUT/6186_Friday_August_12_2022_11_04_42_PM_19754612/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.4 Indicial Equation with Difference of Roots Nonintegral. Exercises page 365
Problem number: 30.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{x^2y'' + xy' - 9y = 0}$$

4.27.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} - 9x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r - 9x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1)+r-9=0$$

Or

$$r^2-9=0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -3 \\ r_2 &= 3 \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x^3} + c_2 x^3$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^3} + c_2 x^3 \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x^3} + c_2 x^3$$

Verified OK.

4.27.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2 y'' + x y' - 9y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{9}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int e^{-\left(\int p(x)dx\right)} dx \\
&= \int e^{-\left(\int \frac{1}{x}dx\right)} dx \\
&= \int e^{-\ln(x)}\, dx \\
&= \int \frac{1}{x} dx \\
&= \ln(x)
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{9}{x^2}}{\frac{1}{x^2}} \\
&= -9
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - 9y(\tau) &= 0
\end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = -9$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} - 9\, e^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 - 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-9)}\\ &= \pm 3\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3\\ \lambda_2 &= -3\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3\\ \lambda_2 &= -3\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y(\tau) &= c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau}\\ y(\tau) &= c_1 e^{(3)\tau} + c_2 e^{(-3)\tau}\end{aligned}$$

Or

$$y(\tau) = c_1 e^{3\tau} + c_2 e^{-3\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 x^6 + c_2}{x^3}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^6 + c_2}{x^3} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^6 + c_2}{x^3}$$

Verified OK.

4.27.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2 y'' + x y' - 9y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{9}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau) \right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{3\sqrt{-\frac{1}{x^2}}}{c} \\ \tau'' &= \frac{3}{c\sqrt{-\frac{1}{x^2}}\, x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{\frac{3}{c\sqrt{-\frac{1}{x^2}}\, x^3} + \frac{1}{x}\frac{3\sqrt{-\frac{1}{x^2}}}{c}}{\left(\frac{3\sqrt{-\frac{1}{x^2}}}{c}\right)^2} \\ &= 0 \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned} \tau &= \int \frac{1}{c}\sqrt{q}\, dx \\ &= \frac{\int 3\sqrt{-\frac{1}{x^2}}dx}{c} \\ &= \frac{3\sqrt{-\frac{1}{x^2}}\, x \ln(x)}{c} \end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cosh(3\ln(x)) + i c_2 \sinh(3\ln(x))$$

Summary
The solution(s) found are the following

$$y = c_1 \cosh(3\ln(x)) + ic_2 \sinh(3\ln(x)) \tag{1}$$

Verification of solutions

$$y = c_1 \cosh(3\ln(x)) + ic_2 \sinh(3\ln(x))$$

Verified OK.

4.27.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' + xy' - 9y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{9}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{x^2} - \frac{9}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{7v'(x)}{x} &= 0 \\ v''(x) + \frac{7v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{7u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{7u}{x} \end{aligned}$$

Where $f(x) = -\frac{7}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{7}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{7}{x}\, dx \\ \ln(u) &= -7\ln(x) + c_1 \\ u &= e^{-7\ln(x)+c_1} \\ &= \frac{c_1}{x^7} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= -\frac{c_1}{6x^6} + c_2 \end{aligned}$$

Hence

$$\begin{aligned}y &= v(x)\,x^n \\ &= \left(-\frac{c_1}{6x^6}+c_2\right)x^3 \\ &= \frac{6c_2x^6-c_1}{6x^3}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \left(-\frac{c_1}{6x^6}+c_2\right)x^3 \tag{1}$$

Verification of solutions

$$y = \left(-\frac{c_1}{6x^6}+c_2\right)x^3$$

Verified OK.

4.27.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2y''+xy'-9y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= x^2 \\ B &= x \\ C &= -9\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{35}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 35 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{35}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 78: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{35}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{35}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{5}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{5}{2} - \left(-\frac{5}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{5}{2x} + (-)\,(0) \\ &= -\frac{5}{2x} \\ &= -\frac{5}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{5}{2x}\right)(0) + \left(\left(\frac{5}{2x^2}\right) + \left(-\frac{5}{2x}\right)^2 - \left(\frac{35}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{5}{2x} dx} \\
&= \frac{1}{x^{\frac{5}{2}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{x^2} \, dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{1}{x^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{x^6}{6}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x^3}\right) + c_2 \left(\frac{1}{x^3} \left(\frac{x^6}{6}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x^3} + \frac{c_2 x^3}{6} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{x^3} + \frac{c_2 x^3}{6}
$$

Verified OK.

4.27.6 Maple step by step solution

Let's solve

$x^2 y'' + x y' - 9y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} + \frac{9y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} - \frac{9y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2 y'' + x y' - 9y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

- Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

- Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + \frac{d}{dt}y(t) - 9y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - 9y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - 9 = 0$

- Factor the characteristic polynomial

  $(r-3)(r+3) = 0$

- Roots of the characteristic polynomial

  $r = (-3, 3)$

- 1st solution of the ODE

  $y_1(t) = e^{-3t}$

- 2nd solution of the ODE

  $y_2(t) = e^{3t}$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 e^{-3t} + c_2 e^{3t}$

- Change variables back using $t = \ln(x)$

  $y = \frac{c_1}{x^3} + c_2 x^3$

- Simplify

  $y = \frac{c_1}{x^3} + c_2 x^3$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)-9*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_2 x^6 + c_1}{x^3}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 18

```
DSolve[x^2*y''[x]+x*y'[x]-9*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 x^6 + c_1}{x^3}$$

4.28 problem 31

Internal problem ID [6944]
Internal file name [`OUTPUT/6187_Friday_August_12_2022_11_04_44_PM_59055855/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.4 Indicial Equation with Difference of Roots Nonintegral. Exercises page 365
Problem number: 31.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{x^2y'' - 3xy' + 4y = 0}$$

4.28.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 3xrx^{r-1} + 4x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 3r\,x^r + 4x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 3r + 4 = 0$$

Or

$$r^2 - 4r + 4 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= 2 \\ r_2 &= 2 \end{aligned}$$

Since the roots are equal, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = c_1 x^2 + c_2 x^2 \ln(x)$$

Summary

The solution(s) found are the following

$$y = c_1 x^2 + c_2 x^2 \ln(x) \tag{1}$$

Verification of solutions

$$y = c_1 x^2 + c_2 x^2 \ln(x)$$

Verified OK.

4.28.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2 y'' - 3xy' + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{3}{x}dx\right)} dx \\
&= \int e^{3\ln(x)}\, dx \\
&= \int x^3 dx \\
&= \frac{x^4}{4}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{4}{x^2}}{x^6} \\
&= \frac{4}{x^8}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{4y(\tau)}{x^8} = 0$$

But in terms of τ

$$\frac{4}{x^8} = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{(c_2\ln(x^4) - 2c_2\ln(2) + c_1)\sqrt{x^4}}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{\left(c_2 \ln\left(x^4\right) - 2c_2 \ln\left(2\right) + c_1\right)\sqrt{x^4}}{2} \tag{1}$$

Verification of solutions

$$y = \frac{\left(c_2 \ln\left(x^4\right) - 2c_2 \ln\left(2\right) + c_1\right)\sqrt{x^4}}{2}$$

Verified OK.

4.28.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2y'' - 3xy' + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'\left(x\right)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'\left(x\right)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{\frac{1}{x^2}}}{c} \end{aligned} \tag{6}$$

$$\tau'' = -\frac{2}{c\sqrt{\frac{1}{x^2}}\, x^3}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{2}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{3}{x}\frac{2\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{2\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= -2c
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - 2c\left(\frac{d}{d\tau}y(\tau)\right) + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = e^{c\tau} c_1
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int 2\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{2\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 x^2
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^2 \qquad (1)
$$

Verification of solutions

$$
y = c_1 x^2
$$

Verified OK.

4.28.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' - 3xy' + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{3n}{x^2} + \frac{4}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{v'(x)}{x} &= 0 \\ v''(x) + \frac{v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{u}{x}
\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{1}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\
\ln(u) &= -\ln(x) + c_1 \\
u &= e^{-\ln(x)+c_1} \\
&= \frac{c_1}{x}
\end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= c_1 \ln(x) + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\,x^n \\
&= (c_1 \ln(x) + c_2)\,x^2 \\
&= (c_1 \ln(x) + c_2)\,x^2
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = (c_1 \ln(x) + c_2)\,x^2 \tag{1}$$

Verification of solutions

$$y = (c_1 \ln(x) + c_2)\,x^2$$

Verified OK.

4.28.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 3xy' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -3x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 80: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{1}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{1}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\,(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x}{x^2}\,dx} \\
&= z_1 e^{\frac{3\ln(x)}{2}} \\
&= z_1\left(x^{\frac{3}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^2
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^2\right) + c_2\left(x^2(\ln(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^2 + c_2 x^2 \ln(x) \tag{1}
$$

Verification of solutions

$$
y = c_1 x^2 + c_2 x^2 \ln(x)
$$

Verified OK.

4.28.6 Maple step by step solution

Let's solve

$x^2 y'' - 3 x y' + 4 y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3y'}{x} - \frac{4y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{3y'}{x} + \frac{4y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2 y'' - 3 x y' + 4 y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt} y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2} y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt} y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2} y(t)}{x^2} - \frac{\frac{d}{dt} y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2 \left(\frac{\frac{d^2}{dt^2} y(t)}{x^2} - \frac{\frac{d}{dt} y(t)}{x^2}\right) - 3 \frac{d}{dt} y(t) + 4 y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2} y(t) - 4 \frac{d}{dt} y(t) + 4 y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r-2)^2 = 0$
- Root of the characteristic polynomial

  $r = 2$
- 1st solution of the ODE

  $y_1(t) = e^{2t}$
- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\,e^{2t}$
- General solution of the ODE

  $y(t) = c_1y_1(t) + c_2y_2(t)$
- Substitute in solutions

  $y(t) = c_1e^{2t} + c_2t\,e^{2t}$
- Change variables back using $t = \ln(x)$

  $y = c_1x^2 + c_2x^2\ln(x)$
- Simplify

  $y = x^2(c_1 + c_2\ln(x))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(x^2*diff(y(x),x$2)-3*x*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = x^2(c_2\ln(x) + c_1)$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 18

```
DSolve[x^2*y''[x]-3*x*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2(2c_2 \log(x) + c_1)$$

4.29 problem 32

Internal problem ID [6945]
Internal file name [`OUTPUT/6188_Friday_August_12_2022_11_04_45_PM_54640770/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.4 Indicial Equation with Difference of Roots Nonintegral. Exercises page 365
Problem number: 32.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' - 5xy' + 9y = 0}$$

4.29.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 5xrx^{r-1} + 9x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 5r\,x^r + 9x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 5r + 9 = 0$$

Or

$$r^2 - 6r + 9 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 3$$
$$r_2 = 3$$

Since the roots are equal, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = c_1 x^3 + c_2 x^3 \ln(x)$$

Summary

The solution(s) found are the following

$$y = c_1 x^3 + c_2 x^3 \ln(x) \tag{1}$$

Verification of solutions

$$y = c_1 x^3 + c_2 x^3 \ln(x)$$

Verified OK.

4.29.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2 y'' - 5xy' + 9y = 0 \tag{1}$$

Becomes

$$y'' + p(x) y' + q(x) y = 0 \tag{2}$$

Where

$$p(x) = -\frac{5}{x}$$
$$q(x) = \frac{9}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{5}{x}dx\right)}dx \\
&= \int e^{5\ln(x)}\,dx \\
&= \int x^5 dx \\
&= \frac{x^6}{6}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{9}{x^2}}{x^{10}} \\
&= \frac{9}{x^{12}}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{9y(\tau)}{x^{12}} &= 0
\end{aligned}$$

But in terms of τ

$$\frac{9}{x^{12}} = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\sqrt{6}\,\sqrt{x^6}\,(c_1 + c_2\ln(x^6) - c_2\ln(3) - c_2\ln(2))}{6}$$

Summary
The solution(s) found are the following

$$y = \frac{\sqrt{6}\,\sqrt{x^6}\,(c_1 + c_2 \ln(x^6) - c_2 \ln(3) - c_2 \ln(2))}{6} \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{6}\,\sqrt{x^6}\,(c_1 + c_2 \ln(x^6) - c_2 \ln(3) - c_2 \ln(2))}{6}$$

Verified OK.

4.29.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2 y'' - 5xy' + 9y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{5}{x}$$
$$q(x) = \frac{9}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\tau' = \frac{1}{c}\sqrt{q}$$
$$= \frac{3\sqrt{\frac{1}{x^2}}}{c} \tag{6}$$
$$\tau'' = -\frac{3}{c\sqrt{\frac{1}{x^2}}\, x^3}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{3}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{5}{x}\frac{3\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{3\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= -2c
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - 2c\left(\frac{d}{d\tau}y(\tau)\right) + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{c\tau} c_1
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int 3\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{3\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 x^3
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^3 \tag{1}
$$

Verification of solutions

$$
y = c_1 x^3
$$

Verified OK.

4.29.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' - 5xy' + 9y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{5}{x} \\ q(x) &= \frac{9}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{5n}{x^2} + \frac{9}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{v'(x)}{x} &= 0 \\ v''(x) + \frac{v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{1}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{1}{x}\, dx \\ \ln(u) &= -\ln(x) + c_1 \\ u &= \mathrm{e}^{-\ln(x)+c_1} \\ &= \frac{c_1}{x} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= c_1 \ln(x) + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\, x^n \\ &= (c_1 \ln(x) + c_2)\, x^3 \\ &= (c_1 \ln(x) + c_2)\, x^3 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = (c_1 \ln(x) + c_2)\, x^3 \tag{1}$$

Verification of solutions

$$y = (c_1 \ln(x) + c_2)\, x^3$$

Verified OK.

4.29.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 5xy' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -5x \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 82: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{1}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{1}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\,(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \,dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-5x}{x^2}\, dx} \\
&= z_1 e^{\frac{5 \ln(x)}{2}} \\
&= z_1 \left(x^{\frac{5}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^3
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-5x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{5 \ln(x)}}{(y_1)^2}\, dx \\
&= y_1 (\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^3\right) + c_2 \left(x^3 (\ln(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^3 + c_2 x^3 \ln(x) \tag{1}
$$

Verification of solutions

$$
y = c_1 x^3 + c_2 x^3 \ln(x)
$$

Verified OK.

4.29.6 Maple step by step solution

Let's solve

$x^2 y'' - 5xy' + 9y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{5y'}{x} - \frac{9y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{5y'}{x} + \frac{9y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2 y'' - 5xy' + 9y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt} y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2} y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt} y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2} y(t)}{x^2} - \frac{\frac{d}{dt} y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2 \left(\frac{\frac{d^2}{dt^2} y(t)}{x^2} - \frac{\frac{d}{dt} y(t)}{x^2}\right) - 5 \frac{d}{dt} y(t) + 9y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2} y(t) - 6 \frac{d}{dt} y(t) + 9y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - 6r + 9 = 0$

- Factor the characteristic polynomial

  $(r-3)^2 = 0$
- Root of the characteristic polynomial

  $r = 3$
- 1st solution of the ODE

  $y_1(t) = e^{3t}$
- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\,e^{3t}$
- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$
- Substitute in solutions

  $y(t) = c_1 e^{3t} + c_2 t\,e^{3t}$
- Change variables back using $t = \ln(x)$

  $y = c_1 x^3 + c_2 x^3 \ln(x)$
- Simplify

  $y = x^3(c_1 + c_2 \ln(x))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(x^2*diff(y(x),x$2)-5*x*diff(y(x),x)+9*y(x)=0,y(x), singsol=all)
```

$$y(x) = x^3(c_2 \ln(x) + c_1)$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 18

```
DSolve[x^2*y''[x]-5*x*y'[x]+9*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^3(3c_2 \log(x) + c_1)$$

4.30 problem 33

Internal problem ID [6946]
Internal file name [`OUTPUT/6189_Friday_August_12_2022_11_04_47_PM_93797961/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.4 Indicial Equation with Difference of Roots Nonintegral. Exercises page 365
Problem number: 33.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' + 5xy' + 5y = 0}$$

4.30.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 5xrx^{r-1} + 5x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 5r\,x^r + 5x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1)+5r+5=0$$

Or

$$r^2+4r+5=0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -2-i \\ r_2 &= -2+i \end{aligned}$$

The roots are complex conjugate of each others. Let the roots be

$$\begin{aligned} r_1 &= \alpha+i\beta \\ r_2 &= \alpha-i\beta \end{aligned}$$

Where in this case $\alpha=-2$ and $\beta=-1$. Hence the solution becomes

$$\begin{aligned} y &= c_1x^{r_1}+c_2x^{r_2} \\ &= c_1x^{\alpha+i\beta}+c_2x^{\alpha-i\beta} \\ &= x^\alpha\left(c_1x^{i\beta}+c_2x^{-i\beta}\right) \\ &= x^\alpha\left(c_1e^{\ln\left(x^{i\beta}\right)}+c_2e^{\ln\left(x^{-i\beta}\right)}\right) \\ &= x^\alpha\left(c_1e^{i(\beta\ln x)}+c_2e^{-i(\beta\ln x)}\right) \end{aligned}$$

Using the values for $\alpha=-2, \beta=-1$, the above becomes

$$y=x^{-2}\left(c_1e^{-i\ln(x)}+c_2e^{i\ln(x)}\right)$$

Using Euler relation, the expression $c_1e^{iA}+c_2e^{-iA}$ is transformed to $c_1\cos A+c_1\sin A$ where the constants are free to change. Applying this to the above result gives

$$y=\frac{1}{x^2}\left(c_1\cos\left(\ln\left(x\right)\right)+c_2\sin\left(\ln\left(x\right)\right)\right)$$

Summary

The solution(s) found are the following

$$y=\frac{c_1\cos\left(\ln\left(x\right)\right)+c_2\sin\left(\ln\left(x\right)\right)}{x^2} \tag{1}$$

Verification of solutions

$$y=\frac{c_1\cos\left(\ln\left(x\right)\right)+c_2\sin\left(\ln\left(x\right)\right)}{x^2}$$

Verified OK.

4.30.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' + 5xy' + 5y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{5}{x} \\ q(x) &= \frac{5}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int e^{-\left(\int p(x)dx\right)} dx \\ &= \int e^{-\left(\int \frac{5}{x}dx\right)} dx \\ &= \int e^{-5\ln(x)}\, dx \\ &= \int \frac{1}{x^5} dx \\ &= -\frac{1}{4x^4} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{5}{x^2}}{\frac{1}{x^{10}}} \\
&= 5x^8
\end{aligned}
\tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + 5x^8 y(\tau) &= 0
\end{aligned}
$$

But in terms of τ

$$
5x^8 = \frac{5}{16\tau^2}
$$

Hence the above ode becomes

$$
\frac{d^2}{d\tau^2}y(\tau) + \frac{5y(\tau)}{16\tau^2} = 0
$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$
16\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 5y(\tau) = 0
$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$
16\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 5\tau^r = 0
$$

Simplifying gives

$$
16r(r-1)\,\tau^r + 0\,\tau^r + 5\tau^r = 0
$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$
16r(r-1) + 0 + 5 = 0
$$

Or

$$
16r^2 - 16r + 5 = 0 \tag{1}
$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$
\begin{aligned}
r_1 &= \frac{1}{2} - \frac{i}{4} \\
r_2 &= \frac{1}{2} + \frac{i}{4}
\end{aligned}
$$

The roots are complex conjugate of each others. Let the roots be

$$
\begin{aligned}
r_1 &= \alpha + i\beta \\
r_2 &= \alpha - i\beta
\end{aligned}
$$

Where in this case $\alpha = \frac{1}{2}$ and $\beta = -\frac{1}{4}$. Hence the solution becomes

$$
\begin{aligned}
y(\tau) &= c_1 \tau^{r_1} + c_2 \tau^{r_2} \\
&= c_1 \tau^{\alpha + i\beta} + c_2 \tau^{\alpha - i\beta} \\
&= \tau^{\alpha} \left(c_1 \tau^{i\beta} + c_2 \tau^{-i\beta}\right) \\
&= \tau^{\alpha} \left(c_1 e^{\ln\left(\tau^{i\beta}\right)} + c_2 e^{\ln\left(\tau^{-i\beta}\right)}\right) \\
&= \tau^{\alpha} \left(c_1 e^{i(\beta \ln \tau)} + c_2 e^{-i(\beta \ln \tau)}\right)
\end{aligned}
$$

Using the values for $\alpha = \frac{1}{2}, \beta = -\frac{1}{4}$, the above becomes

$$
y(\tau) = \tau^{\frac{1}{2}} \left(c_1 e^{-\frac{i \ln(\tau)}{4}} + c_2 e^{\frac{i \ln(\tau)}{4}}\right)
$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$
y(\tau) = \sqrt{\tau} \left(c_1 \cos\left(\frac{\ln(\tau)}{4}\right) + c_2 \sin\left(\frac{\ln(\tau)}{4}\right)\right)
$$

The above solution is now transformed back to y using (6) which results in

$$
y = \frac{\sqrt{-\frac{1}{x^4}} \left(c_1 \cos\left(-\frac{\ln(2)}{2} + \frac{\ln\left(-\frac{1}{x^4}\right)}{4}\right) + c_2 \sin\left(-\frac{\ln(2)}{2} + \frac{\ln\left(-\frac{1}{x^4}\right)}{4}\right)\right)}{2}
$$

Summary

The solution(s) found are the following

$$
y = \frac{\sqrt{-\frac{1}{x^4}} \left(c_1 \cos\left(-\frac{\ln(2)}{2} + \frac{\ln\left(-\frac{1}{x^4}\right)}{4}\right) + c_2 \sin\left(-\frac{\ln(2)}{2} + \frac{\ln\left(-\frac{1}{x^4}\right)}{4}\right)\right)}{2} \tag{1}
$$

Verification of solutions

$$y = \frac{\sqrt{-\frac{1}{x^4}}\left(c_1 \cos\left(-\frac{\ln(2)}{2} + \frac{\ln\left(-\frac{1}{x^4}\right)}{4}\right) + c_2 \sin\left(-\frac{\ln(2)}{2} + \frac{\ln\left(-\frac{1}{x^4}\right)}{4}\right)\right)}{2}$$

Verified OK.

4.30.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2 y'' + 5xy' + 5y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{5}{x}$$
$$q(x) = \frac{5}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1\left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{5}\,\sqrt{\frac{1}{x^2}}}{c} \end{aligned} \tag{6}$$

$$\tau'' = -\frac{\sqrt{5}}{c\sqrt{\frac{1}{x^2}}\, x^3}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{\sqrt{5}}{c\sqrt{\frac{1}{x^2}}\,x^3} + \frac{5}{x}\frac{\sqrt{5}\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{5}\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= \frac{4c\sqrt{5}}{5}
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{4c\sqrt{5}\left(\frac{d}{d\tau}y(\tau)\right)}{5} + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{-\frac{2\sqrt{5}\,c\tau}{5}}\left(c_1\cos\left(\frac{\sqrt{5}\,c\tau}{5}\right) + c_2\sin\left(\frac{\sqrt{5}\,c\tau}{5}\right)\right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{5}\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{5}\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = \frac{c_1\cos(\ln(x)) + c_2\sin(\ln(x))}{x^2}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \frac{c_1\cos(\ln(x)) + c_2\sin(\ln(x))}{x^2} \qquad (1)
$$

Verification of solutions

$$y = \frac{c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))}{x^2}$$

Verified OK.

4.30.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' + 5xy' + 5y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{5}{x} \\ q(x) &= \frac{5}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{5n}{x^2} + \frac{5}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -2 + i \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{-4+2i}{x} + \frac{5}{x}\right) v'(x) &= 0 \\ v''(x) + \frac{(1+2i)\,v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(1+2i)\, u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{(-1-2i)\, u}{x} \end{aligned}$$

Where $f(x) = \frac{-1-2i}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= \frac{-1-2i}{x}\, dx \\ \int \frac{1}{u}\, du &= \int \frac{-1-2i}{x}\, dx \\ \ln(u) &= (-1-2i)\ln(x) + c_1 \\ u &= e^{(-1-2i)\ln(x)+c_1} \\ &= c_1 e^{(-1-2i)\ln(x)} \end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1 x^{-2i}}{x}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= \frac{i c_1 x^{-2i}}{2} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\, x^n \\ &= \left(\frac{i c_1 x^{-2i}}{2} + c_2\right) x^{-2+i} \\ &= c_2 x^{-2+i} + \frac{i c_1 x^{-2-i}}{2} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \left(\frac{i c_1 x^{-2i}}{2} + c_2\right) x^{-2+i} \tag{1}$$

Verification of solutions

$$y = \left(\frac{i c_1 x^{-2i}}{2} + c_2\right) x^{-2+i}$$

Verified OK.

4.30.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' + 5xy' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 5x \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -5$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{5}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 84: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + i \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - i\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{5}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{5}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + i \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - i\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{5}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + i$	$\frac{1}{2} - i$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}+i$	$\frac{1}{2}-i$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2} - i$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{2} - i - \left(\frac{1}{2} - i\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{\frac{1}{2}-i}{x} + (-)(0) \\ &= \frac{\frac{1}{2}-i}{x} \\ &= \frac{\frac{1}{2}-i}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{\frac{1}{2}-i}{x}\right)(0)+\left(\left(\frac{-\frac{1}{2}+i}{x^2}\right)+\left(\frac{\frac{1}{2}-i}{x}\right)^2-\left(-\frac{5}{4x^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}z_1(x)&=pe^{\int \omega\,dx}\\&=\mathrm{e}^{\int \frac{\frac{1}{2}-i}{x}dx}\\&=x^{\frac{1}{2}-i}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1&=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\&=z_1e^{-\int \frac{1}{2}\frac{5x}{x^2}\,dx}\\&=z_1e^{-\frac{5\ln(x)}{2}}\\&=z_1\left(\frac{1}{x^{\frac{5}{2}}}\right)\end{aligned}$$

Which simplifies to

$$y_1=x^{-2-i}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2&=y_1\int \frac{e^{\int -\frac{5x}{x^2}\,dx}}{(y_1)^2}\,dx\\&=y_1\int \frac{e^{-5\ln(x)}}{(y_1)^2}\,dx\\&=y_1\left(-\frac{ix^{2i}}{2}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{-2-i}\right) + c_2 \left(x^{-2-i} \left(-\frac{i x^{2i}}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^{-2-i} - \frac{i c_2 x^{-2+i}}{2} \tag{1}
$$

Verification of solutions

$$
y = c_1 x^{-2-i} - \frac{i c_2 x^{-2+i}}{2}
$$

Verified OK.

4.30.6 Maple step by step solution

Let's solve

$x^2 y'' + 5 x y' + 5 y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{5y'}{x} - \frac{5y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{5y'}{x} + \frac{5y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2 y'' + 5 x y' + 5 y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

- Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

- Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + 5\frac{d}{dt}y(t) + 5y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) + 4\frac{d}{dt}y(t) + 5y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 + 4r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-4)\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-2 - \mathrm{I}, -2 + \mathrm{I})$

- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{-2t}\cos(t)$

- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{-2t}\sin(t)$

- General solution of the ODE

  $y(t) = c_1y_1(t) + c_2y_2(t)$

- Substitute in solutions

  $y(t) = c_1\mathrm{e}^{-2t}\cos(t) + c_2\mathrm{e}^{-2t}\sin(t)$

- Change variables back using $t = \ln(x)$

  $y = \frac{c_1\cos(\ln(x))}{x^2} + \frac{c_2\sin(\ln(x))}{x^2}$

- Simplify

  $y = \frac{c_1\cos(\ln(x))}{x^2} + \frac{c_2\sin(\ln(x))}{x^2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x$2)+5*x*diff(y(x),x)+5*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(\ln(x)) + c_2 \cos(\ln(x))}{x^2}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 22

```
DSolve[x^2*y''[x]+5*x*y'[x]+5*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 \cos(\log(x)) + c_1 \sin(\log(x))}{x^2}$$

4.31 problem 34

Internal problem ID [6947]
Internal file name [`OUTPUT/6190_Friday_August_12_2022_11_04_48_PM_13727061/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.4 Indicial Equation with Difference of Roots Nonintegral. Exercises page 365
Problem number: 34.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{x^3y''' + 4x^2y'' - 8xy' + 8y = 0}$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned} y' &= \lambda\, x^{\lambda-1} \\ y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\ y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \end{aligned}$$

Substituting these back into

$$x^3y''' + 4x^2y'' - 8xy' + 8y = 0$$

gives

$$-8x\lambda\, x^{\lambda-1} + 4x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} + 8x^\lambda = 0$$

Which simplifies to

$$-8\lambda\, x^\lambda + 4\lambda(\lambda-1)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\, x^\lambda + 8x^\lambda = 0$$

And since $x^\lambda \neq 0$ then dividing through by x^λ, the above becomes

$$-8\lambda + 4\lambda(\lambda - 1) + \lambda(\lambda - 1)(\lambda - 2) + 8 = 0$$

Simplifying gives the characteristic equation as

$$\lambda^3 + \lambda^2 - 10\lambda + 8 = 0$$

Solving the above gives the following roots

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = -4$$

This table summarises the result

root	multiplicity	type of root
1	1	real root
2	1	real root
-4	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a $c_1 x^\lambda$ basis solution. Each real root of multiplicty two, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ basis solutions. Each real root of multiplicty three, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ and $c_3 x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = c_2 x^2 + c_1 x + \frac{c_3}{x^4}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = x$$
$$y_2 = x^2$$
$$y_3 = \frac{1}{x^4}$$

Summary

The solution(s) found are the following

$$y = c_2 x^2 + c_1 x + \frac{c_3}{x^4} \tag{1}$$

Verification of solutions

$$y = c_2 x^2 + c_1 x + \frac{c_3}{x^4}$$

Verified OK.

4.31.1 Maple step by step solution

Let's solve

$x^3 y''' + 4x^2 y'' - 8xy' + 8y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = -\frac{8y}{x^3} - \frac{4(xy'' - 2y')}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' + \frac{4y''}{x} - \frac{8y'}{x^2} + \frac{8y}{x^3} = 0$

- Multiply by denominators of the ODE

  $x^3 y''' + 4x^2 y'' - 8xy' + 8y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt} y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2} y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt} y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2} y(t)}{x^2} - \frac{\frac{d}{dt} y(t)}{x^2}$

  - Calculate the 3rd derivative of y with respect to x , using the chain rule

    $y''' = \left(\frac{d^3}{dt^3} y(t)\right) t'(x)^3 + 3t'(x)\, t''(x) \left(\frac{d^2}{dt^2} y(t)\right) + t'''(x) \left(\frac{d}{dt} y(t)\right)$

- Compute derivative

  $y''' = \frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}$

  Substitute the change of variables back into the ODE

  $x^3\left(\frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}\right) + 4x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 8\frac{d}{dt}y(t) + 8y(t) = 0$

- Simplify

  $\frac{d^3}{dt^3}y(t) + \frac{d^2}{dt^2}y(t) - 10\frac{d}{dt}y(t) + 8y(t) = 0$

☐ Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(t)$

  $y_1(t) = y(t)$

- Define new variable $y_2(t)$

  $y_2(t) = \frac{d}{dt}y(t)$

- Define new variable $y_3(t)$

  $y_3(t) = \frac{d^2}{dt^2}y(t)$

- Isolate for $\frac{d}{dt}y_3(t)$ using original ODE

  $\frac{d}{dt}y_3(t) = -y_3(t) + 10y_2(t) - 8y_1(t)$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(t) = \frac{d}{dt}y_1(t), y_3(t) = \frac{d}{dt}y_2(t), \frac{d}{dt}y_3(t) = -y_3(t) + 10y_2(t) - 8y_1(t)\right]$

- Define vector

  $$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- System to solve

  $$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & 10 & -1 \end{bmatrix} \cdot \vec{y}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & 10 & -1 \end{bmatrix}$$

- Rewrite the system as

  $\frac{d}{dt}\vec{y}(t) = A \cdot \vec{y}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-4, \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{4} \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-4, \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{4} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-4t} \cdot \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{4} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \mathrm{e}^{t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \left[\begin{array}{c}\frac{1}{4} \\ \frac{1}{2} \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{2t} \cdot \left[\begin{array}{c}\frac{1}{4} \\ \frac{1}{2} \\ 1\end{array}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-4t} \cdot \left[\begin{array}{c}\frac{1}{16} \\ -\frac{1}{4} \\ 1\end{array}\right] + c_2 e^{t} \cdot \left[\begin{array}{c}1 \\ 1 \\ 1\end{array}\right] + c_3 e^{2t} \cdot \left[\begin{array}{c}\frac{1}{4} \\ \frac{1}{2} \\ 1\end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y(t) = \frac{(4c_3 e^{6t} + 16c_2 e^{5t} + c_1) e^{-4t}}{16}$$

- Change variables back using $t = \ln(x)$

$$y = \frac{4c_3 x^6 + 16c_2 x^5 + c_1}{16x^4}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve(x^3*diff(y(x),x$3)+4*x^2*diff(y(x),x$2)-8*x*diff(y(x),x)+8*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^6 + c_3 x^5 + c_2}{x^4}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 22

```
DSolve[x^3*y'''[x]+4*x^2*y''[x]-8*x*y'[x]+8*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1}{x^4} + c_3 x^2 + c_2 x$$

5 CHAPTER 18. Power series solutions. 18.6. Indicial Equation with Equal Roots. Exercises page 373

5.1 problem 1

Internal problem ID [6948]
Internal file name [`OUTPUT/6191_Friday_August_12_2022_11_04_49_PM_3851012/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.6. Indicial Equation with Equal Roots. Exercises page 373
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - x(1+x)\,y' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(-x^2 - x\right)y' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1+x}{x}$$
$$q(x) = \frac{1}{x^2}$$

Table 87: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1+x}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + \left(-x^2 - x\right) y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(-x^2 - x\right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(-1+r)^2\, x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(-1+r)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(-1 + r)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{1+n}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n + r)(n + r - 1) - a_{n-1}(n + r - 1) - a_n(n + r) + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{n + r - 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{r}$$

Which for the root $r = 1$ becomes

$$a_1 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r}$	1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(1+r)\, r}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r}$	1
a_2	$\frac{1}{(1+r)r}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{(1+r) r (2+r)}$$

Which for the root $r = 1$ becomes

$$a_3 = \frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r}$	1
a_2	$\frac{1}{(1+r)r}$	$\frac{1}{2}$
a_3	$\frac{1}{(1+r)r(2+r)}$	$\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(3+r)(1+r) r (2+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r}$	1
a_2	$\frac{1}{(1+r)r}$	$\frac{1}{2}$
a_3	$\frac{1}{(1+r)r(2+r)}$	$\frac{1}{6}$
a_4	$\frac{1}{(3+r)(1+r)r(2+r)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{(4+r)(3+r)(1+r) r (2+r)}$$

Which for the root $r = 1$ becomes

$$a_5 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r}$	1
a_2	$\frac{1}{(1+r)r}$	$\frac{1}{2}$
a_3	$\frac{1}{(1+r)r(2+r)}$	$\frac{1}{6}$
a_4	$\frac{1}{(3+r)(1+r)r(2+r)}$	$\frac{1}{24}$
a_5	$\frac{1}{(4+r)(3+r)(1+r)r(2+r)}$	$\frac{1}{120}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{1}{(4+r)(3+r)(1+r)r(2+r)(5+r)}$$

Which for the root $r = 1$ becomes

$$a_6 = \frac{1}{720}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r}$	1
a_2	$\frac{1}{(1+r)r}$	$\frac{1}{2}$
a_3	$\frac{1}{(1+r)r(2+r)}$	$\frac{1}{6}$
a_4	$\frac{1}{(3+r)(1+r)r(2+r)}$	$\frac{1}{24}$
a_5	$\frac{1}{(4+r)(3+r)(1+r)r(2+r)}$	$\frac{1}{120}$
a_6	$\frac{1}{(4+r)(3+r)(1+r)r(2+r)(5+r)}$	$\frac{1}{720}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{1}{(6+r)(4+r)(3+r)(1+r)r(2+r)(5+r)}$$

Which for the root $r = 1$ becomes

$$a_7 = \frac{1}{5040}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r}$	1
a_2	$\frac{1}{(1+r)r}$	$\frac{1}{2}$
a_3	$\frac{1}{(1+r)r(2+r)}$	$\frac{1}{6}$
a_4	$\frac{1}{(3+r)(1+r)r(2+r)}$	$\frac{1}{24}$
a_5	$\frac{1}{(4+r)(3+r)(1+r)r(2+r)}$	$\frac{1}{120}$
a_6	$\frac{1}{(4+r)(3+r)(1+r)r(2+r)(5+r)}$	$\frac{1}{720}$
a_7	$\frac{1}{(6+r)(4+r)(3+r)(1+r)r(2+r)(5+r)}$	$\frac{1}{5040}$

Using the above table, then the first solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\
&= x\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O\left(x^8\right)\right)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = 1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r=1)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{1}{r}$	1	$-\frac{1}{r^2}$	-1
b_2	$\frac{1}{(1+r)r}$	$\frac{1}{2}$	$\frac{-2r-1}{(1+r)^2 r^2}$	$-\frac{3}{4}$
b_3	$\frac{1}{(1+r)r(2+r)}$	$\frac{1}{6}$	$\frac{-3r^2-6r-2}{(1+r)^2 r^2 (2+r)^2}$	$-\frac{11}{36}$
b_4	$\frac{1}{(3+r)(1+r)r(2+r)}$	$\frac{1}{24}$	$\frac{-4r^3-18r^2-22r-6}{(3+r)^2(1+r)^2 r^2 (2+r)^2}$	$-\frac{25}{288}$
b_5	$\frac{1}{(4+r)(3+r)(1+r)r(2+r)}$	$\frac{1}{120}$	$\frac{-5r^4-40r^3-105r^2-100r-24}{(4+r)^2(3+r)^2(1+r)^2 r^2 (2+r)^2}$	$-\frac{137}{7200}$
b_6	$\frac{1}{(4+r)(3+r)(1+r)r(2+r)(5+r)}$	$\frac{1}{720}$	$\frac{-6r^5-75r^4-340r^3-675r^2-548r-120}{(4+r)^2(3+r)^2(1+r)^2 r^2 (2+r)^2 (5+r)^2}$	$-\frac{49}{14400}$
b_7	$\frac{1}{(6+r)(4+r)(3+r)(1+r)r(2+r)(5+r)}$	$\frac{1}{5040}$	$\frac{-7r^6-126r^5-875r^4-2940r^3-4872r^2-3528r-720}{(6+r)^2(4+r)^2(3+r)^2(1+r)^2 r^2 (2+r)^2 (5+r)^2}$	$-\frac{121}{235200}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\
&= x\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8)\right)\ln(x) \\
&\quad + x\left(-x - \frac{3x^2}{4} - \frac{11x^3}{36} - \frac{25x^4}{288} - \frac{137x^5}{7200} - \frac{49x^6}{14400} - \frac{121x^7}{235200} + O(x^8)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8)\right) \\
&\quad + c_2\left(x\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8)\right)\ln(x)\right. \\
&\quad \left. + x\left(-x - \frac{3x^2}{4} - \frac{11x^3}{36} - \frac{25x^4}{288} - \frac{137x^5}{7200} - \frac{49x^6}{14400} - \frac{121x^7}{235200} + O(x^8)\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
y = y_h
$$

$$
\begin{aligned}
&= c_1 x\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}+\frac{x^7}{5040}+O(x^8)\right)\\
&\quad + c_2\left(x\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}+\frac{x^7}{5040}+O(x^8)\right)\ln(x)\right.\\
&\qquad \left.+ x\left(-x-\frac{3x^2}{4}-\frac{11x^3}{36}-\frac{25x^4}{288}-\frac{137x^5}{7200}-\frac{49x^6}{14400}-\frac{121x^7}{235200}+O(x^8)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}+\frac{x^7}{5040}+O(x^8)\right)\\
&\quad + c_2\left(x\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}+\frac{x^7}{5040}+O(x^8)\right)\ln(x)\right.\\
&\qquad \left.+ x\left(-x-\frac{3x^2}{4}-\frac{11x^3}{36}-\frac{25x^4}{288}-\frac{137x^5}{7200}-\frac{49x^6}{14400}-\frac{121x^7}{235200}+O(x^8)\right)\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}+\frac{x^7}{5040}+O(x^8)\right)\\
&\quad + c_2\left(x\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}+\frac{x^7}{5040}+O(x^8)\right)\ln(x)\right.\\
&\qquad \left.+ x\left(-x-\frac{3x^2}{4}-\frac{11x^3}{36}-\frac{25x^4}{288}-\frac{137x^5}{7200}-\frac{49x^6}{14400}-\frac{121x^7}{235200}+O(x^8)\right)\right)
\end{aligned}
$$

Verified OK.

5.1.1 Maple step by step solution

Let's solve

$x^2y'' + (-x^2 - x)\, y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{x^2} + \frac{(1+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(1+x)y'}{x} + \frac{y}{x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{1+x}{x}, P_3(x) = \frac{1}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 1$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $x^2 y'' - x(1+x)\, y' + y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

    Rewrite ODE with series expansions

    $a_0(-1+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)^2 - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 1$
- Each term in the series must be 0, giving the recursion relation

  $(k + r - 1)(a_k(k + r - 1) - a_{k-1}) = 0$
- Shift index using $k- >k + 1$

  $(k + r)(a_{k+1}(k + r) - a_k) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+r}$
- Recursion relation for $r = 1$

  $a_{k+1} = \frac{a_k}{k+1}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = \frac{a_k}{k+1}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 75

```
Order:=8;
dsolve(x^2*diff(y(x),x$2)-x*(1+x)*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left((c_2 \ln(x) + c_1)\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 + \mathrm{O}\left(x^8\right)\right) + \left(-x - \frac{3}{4}x^2 - \frac{11}{36}x^3 - \frac{25}{288}x^4 - \frac{137}{7200}x^5 - \frac{49}{14400}x^6 - \frac{121}{235200}x^7 + \mathrm{O}\left(x^8\right)\right)c_2\right)x$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 154

```
AsymptoticDSolveValue[x^2*y''[x]-x*(1+x)*y'[x]+y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1 x\left(\frac{x^7}{5040} + \frac{x^6}{720} + \frac{x^5}{120} + \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + x + 1\right) + c_2\left(x\left(-\frac{121x^7}{235200} - \frac{49x^6}{14400} - \frac{137x^5}{7200} - \frac{25x^4}{288} - \frac{11x^3}{36} - \frac{3x^2}{4} - x\right) + x\left(\frac{x^7}{5040} + \frac{x^6}{720} + \frac{x^5}{120} + \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + x + 1\right)\log(x)\right)$$

5.2 problem 2

Internal problem ID [6949]
Internal file name [`OUTPUT/6192_Friday_August_12_2022_11_04_52_PM_59914383/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.6. Indicial Equation with Equal Roots. Exercises page 373
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' + (1-2x)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4x^2y'' + (1-2x)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= -\frac{2x-1}{4x^2} \end{aligned}$$

Table 89: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = -\frac{2x-1}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2y'' + (1-2x)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + (1-2x)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-2x^{1+n+r}a_n\right) = \sum_{n=1}^{\infty}\left(-2a_{n-1}x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 4x^{n+r}a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \sum_{n=1}^{\infty}\left(-2a_{n-1}x^{n+r}\right) = 0 \quad \text{(2B)}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r}a_n(n+r)(n+r-1) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$4x^r a_0 r(-1+r) + a_0 x^r = 0$$

Or

$$\left(4x^r r(-1+r) + x^r\right)a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^r(2r-1)^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$(2r-1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^r(2r-1)^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = \frac{1}{2}$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+\frac{1}{2}}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$4a_n(n+r)(n+r-1) + a_n - 2a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2a_{n-1}}{4n^2 + 8nr + 4r^2 - 4n - 4r + 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{a_{n-1}}{2n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{2}{(2r+1)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{(2r+1)^2}$	$\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4}{(2r+1)^2 (2r+3)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{1}{16}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{(2r+1)^2}$	$\frac{1}{2}$
a_2	$\frac{4}{(2r+1)^2(2r+3)^2}$	$\frac{1}{16}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{8}{(2r+1)^2 (2r+3)^2 (5+2r)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = \frac{1}{288}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{(2r+1)^2}$	$\frac{1}{2}$
a_2	$\frac{4}{(2r+1)^2(2r+3)^2}$	$\frac{1}{16}$
a_3	$\frac{8}{(2r+1)^2(2r+3)^2(5+2r)^2}$	$\frac{1}{288}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{(2r+1)^2 (2r+3)^2 (5+2r)^2 (7+2r)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{1}{9216}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{(2r+1)^2}$	$\frac{1}{2}$
a_2	$\frac{4}{(2r+1)^2(2r+3)^2}$	$\frac{1}{16}$
a_3	$\frac{8}{(2r+1)^2(2r+3)^2(5+2r)^2}$	$\frac{1}{288}$
a_4	$\frac{16}{(2r+1)^2(2r+3)^2(5+2r)^2(7+2r)^2}$	$\frac{1}{9216}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{32}{(2r+1)^2 (2r+3)^2 (5+2r)^2 (7+2r)^2 (9+2r)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = \frac{1}{460800}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{(2r+1)^2}$	$\frac{1}{2}$
a_2	$\frac{4}{(2r+1)^2(2r+3)^2}$	$\frac{1}{16}$
a_3	$\frac{8}{(2r+1)^2(2r+3)^2(5+2r)^2}$	$\frac{1}{288}$
a_4	$\frac{16}{(2r+1)^2(2r+3)^2(5+2r)^2(7+2r)^2}$	$\frac{1}{9216}$
a_5	$\frac{32}{(2r+1)^2(2r+3)^2(5+2r)^2(7+2r)^2(9+2r)^2}$	$\frac{1}{460800}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{64}{(2r+1)^2(2r+3)^2(5+2r)^2(7+2r)^2(9+2r)^2(11+2r)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_6 = \frac{1}{33177600}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{(2r+1)^2}$	$\frac{1}{2}$
a_2	$\frac{4}{(2r+1)^2(2r+3)^2}$	$\frac{1}{16}$
a_3	$\frac{8}{(2r+1)^2(2r+3)^2(5+2r)^2}$	$\frac{1}{288}$
a_4	$\frac{16}{(2r+1)^2(2r+3)^2(5+2r)^2(7+2r)^2}$	$\frac{1}{9216}$
a_5	$\frac{32}{(2r+1)^2(2r+3)^2(5+2r)^2(7+2r)^2(9+2r)^2}$	$\frac{1}{460800}$
a_6	$\frac{64}{(2r+1)^2(2r+3)^2(5+2r)^2(7+2r)^2(9+2r)^2(11+2r)^2}$	$\frac{1}{33177600}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{128}{(2r+1)^2(2r+3)^2(5+2r)^2(7+2r)^2(9+2r)^2(11+2r)^2(13+2r)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_7 = \frac{1}{3251404800}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{(2r+1)^2}$	$\frac{1}{2}$
a_2	$\frac{4}{(2r+1)^2(2r+3)^2}$	$\frac{1}{16}$
a_3	$\frac{8}{(2r+1)^2(2r+3)^2(5+2r)^2}$	$\frac{1}{288}$
a_4	$\frac{16}{(2r+1)^2(2r+3)^2(5+2r)^2(7+2r)^2}$	$\frac{1}{9216}$
a_5	$\frac{32}{(2r+1)^2(2r+3)^2(5+2r)^2(7+2r)^2(9+2r)^2}$	$\frac{1}{460800}$
a_6	$\frac{64}{(2r+1)^2(2r+3)^2(5+2r)^2(7+2r)^2(9+2r)^2(11+2r)^2}$	$\frac{1}{33177600}$
a_7	$\frac{128}{(2r+1)^2(2r+3)^2(5+2r)^2(7+2r)^2(9+2r)^2(11+2r)^2(13+2r)^2}$	$\frac{1}{3251404800}$

Using the above table, then the first solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\
&= \sqrt{x}\left(1 + \frac{x}{2} + \frac{x^2}{16} + \frac{x^3}{288} + \frac{x^4}{9216} + \frac{x^5}{460800} + \frac{x^6}{33177600} + \frac{x^7}{3251404800} + O(x^8)\right)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = \frac{1}{2}$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	$\frac{2}{(2r+1)^2}$	$\frac{1}{2}$	$-\frac{8}{(2r+1)^3}$
b_2	$\frac{4}{(2r+1)^2(2r+3)^2}$	$\frac{1}{16}$	$\frac{-64r-64}{(2r+1)^3(2r+3)^3}$
b_3	$\frac{8}{(2r+1)^2(2r+3)^2(5+2r)^2}$	$\frac{1}{288}$	$\frac{-384r^2-1152r-736}{(2r+1)^3(2r+3)^3(5+2r)^3}$
b_4	$\frac{16}{(2r+1)^2(2r+3)^2(5+2r)^2(7+2r)^2}$	$\frac{1}{9216}$	$-\frac{512(4r^3+24r^2+43r+22)}{(2r+1)^3(2r+3)^3(5+2r)^3(7+2r)^3}$
b_5	$\frac{32}{(2r+1)^2(2r+3)^2(5+2r)^2(7+2r)^2(9+2r)^2}$	$\frac{1}{460800}$	$-\frac{128(80r^4+800r^3+2760r^2+3800r+1689)}{(2r+1)^3(2r+3)^3(5+2r)^3(7+2r)^3(9+2r)^3}$
b_6	$\frac{64}{(2r+1)^2(2r+3)^2(5+2r)^2(7+2r)^2(9+2r)^2(11+2r)^2}$	$\frac{1}{33177600}$	$-\frac{49152(3+r)\left(r^4+12r^3+\frac{289}{6}r^2+73r+\frac{1627}{48}\right)}{(2r+1)^3(2r+3)^3(5+2r)^3(7+2r)^3(9+2r)^3(11+2r)^3}$
b_7	$\frac{128}{(2r+1)^2(2r+3)^2(5+2r)^2(7+2r)^2(9+2r)^2(11+2r)^2(13+2r)^2}$	$\frac{1}{3251404800}$	$-\frac{512(448r^6+9408r^5+77840r^4+321440r^3+688548r^2+7\ldots}{(2r+1)^3(2r+3)^3(5+2r)^3(7+2r)^3(9+2r)^3(11+2r\ldots}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 + b_7x^7 + b_8x^8\ldots \\
&= \sqrt{x}\left(1+\frac{x}{2}+\frac{x^2}{16}+\frac{x^3}{288}+\frac{x^4}{9216}+\frac{x^5}{460800}+\frac{x^6}{33177600}+\frac{x^7}{3251404800}+O(x^8)\right)\ln(x) \\
&\quad+\sqrt{x}\left(-x-\frac{3x^2}{16}-\frac{11x^3}{864}-\frac{25x^4}{55296}-\frac{137x^5}{13824000}-\frac{49x^6}{331776000}-\frac{121x^7}{75866112000}+O(x^8)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1\sqrt{x}\left(1+\frac{x}{2}+\frac{x^2}{16}+\frac{x^3}{288}+\frac{x^4}{9216}+\frac{x^5}{460800}+\frac{x^6}{33177600}+\frac{x^7}{3251404800}+O(x^8)\right) \\
&\quad+c_2\left(\sqrt{x}\left(1+\frac{x}{2}+\frac{x^2}{16}+\frac{x^3}{288}+\frac{x^4}{9216}+\frac{x^5}{460800}+\frac{x^6}{33177600}+\frac{x^7}{3251404800}\right.\right. \\
&\quad\left.+O(x^8)\right)\ln(x)+\sqrt{x}\left(-x-\frac{3x^2}{16}-\frac{11x^3}{864}-\frac{25x^4}{55296}-\frac{137x^5}{13824000}-\frac{49x^6}{331776000}\right. \\
&\quad\left.\left.-\frac{121x^7}{75866112000}+O(x^8)\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\sqrt{x}\left(1+\frac{x}{2}+\frac{x^2}{16}+\frac{x^3}{288}+\frac{x^4}{9216}+\frac{x^5}{460800}+\frac{x^6}{33177600}+\frac{x^7}{3251404800}+O(x^8)\right) \\
&\quad + c_2\left(\sqrt{x}\left(1+\frac{x}{2}+\frac{x^2}{16}+\frac{x^3}{288}+\frac{x^4}{9216}+\frac{x^5}{460800}+\frac{x^6}{33177600}+\frac{x^7}{3251404800}\right.\right. \\
&\quad \left. +O(x^8)\right)\ln(x)+\sqrt{x}\left(-x-\frac{3x^2}{16}-\frac{11x^3}{864}-\frac{25x^4}{55296}-\frac{137x^5}{13824000}-\frac{49x^6}{331776000}\right. \\
&\quad \left.\left. -\frac{121x^7}{75866112000}+O(x^8)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\sqrt{x}\left(1+\frac{x}{2}+\frac{x^2}{16}+\frac{x^3}{288}+\frac{x^4}{9216}+\frac{x^5}{460800}+\frac{x^6}{33177600}+\frac{x^7}{3251404800}+O(x^8)\right) \\
&\quad + c_2\left(\sqrt{x}\left(1+\frac{x}{2}+\frac{x^2}{16}+\frac{x^3}{288}+\frac{x^4}{9216}+\frac{x^5}{460800}+\frac{x^6}{33177600}+\frac{x^7}{3251404800}\right.\right. \\
&\quad \left. +O(x^8)\right)\ln(x)+\sqrt{x}\left(-x-\frac{3x^2}{16}-\frac{11x^3}{864}-\frac{25x^4}{55296}-\frac{137x^5}{13824000}-\frac{49x^6}{331776000}\right. \\
&\quad \left.\left. -\frac{121x^7}{75866112000}+O(x^8)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1\sqrt{x}\left(1+\frac{x}{2}+\frac{x^2}{16}+\frac{x^3}{288}+\frac{x^4}{9216}+\frac{x^5}{460800}+\frac{x^6}{33177600}+\frac{x^7}{3251404800}+O(x^8)\right) \\
&\quad + c_2\left(\sqrt{x}\left(1+\frac{x}{2}+\frac{x^2}{16}+\frac{x^3}{288}+\frac{x^4}{9216}+\frac{x^5}{460800}+\frac{x^6}{33177600}+\frac{x^7}{3251404800}\right.\right. \\
&\quad \left. +O(x^8)\right)\ln(x)+\sqrt{x}\left(-x-\frac{3x^2}{16}-\frac{11x^3}{864}-\frac{25x^4}{55296}-\frac{137x^5}{13824000}-\frac{49x^6}{331776000}\right. \\
&\quad \left.\left. -\frac{121x^7}{75866112000}+O(x^8)\right)\right)
\end{aligned}
$$

Verified OK.

5.2.1 Maple step by step solution

Let's solve

$4x^2y'' + (1-2x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(2x-1)y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

  $\left[P_2(x) = 0, P_3(x) = -\frac{2x-1}{4x^2}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 0$

  ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{1}{4}$

  ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + (1-2x)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ○ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ○ Shift index using $k- > k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+2r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k (2k+2r-1)^2 - 2a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+2r)^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = \frac{1}{2}$$

- Each term in the series must be 0, giving the recursion relation

  $$4\left(k+r-\frac{1}{2}\right)^2 a_k - 2a_{k-1} = 0$$

- Shift index using $k- > k+1$

  $$4\left(k+\frac{1}{2}+r\right)^2 a_{k+1} - 2a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{2a_k}{(2k+1+2r)^2}$$

- Recursion relation for $r = \frac{1}{2}$

  $$a_{k+1} = \frac{2a_k}{(2k+2)^2}$$

- Solution for $r = \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{2a_k}{(2k+2)^2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 81

```
Order:=8;
dsolve(4*x^2*diff(y(x),x$2)+(1-2*x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left((c_2 \ln(x) + c_1) \left(1 + \frac{1}{2}x + \frac{1}{16}x^2 + \frac{1}{288}x^3 + \frac{1}{9216}x^4 + \frac{1}{460800}x^5 + \frac{1}{33177600}x^6 + \frac{1}{3251404800}x^7 + \mathrm{O}\left(x^8\right) \right) + \left(-x - \frac{3}{16}x^2 - \frac{11}{864}x^3 - \frac{25}{55296}x^4 - \frac{137}{13824000}x^5 - \frac{49}{331776000}x^6 - \frac{121}{75866112000}x^7 + \mathrm{O}\left(x^8\right) \right) c_2 \right) \sqrt{x}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 174

```
AsymptoticDSolveValue[4*x^2*y''[x]+(1-2*x)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\sqrt{x}\left(\frac{x^7}{3251404800}+\frac{x^6}{33177600}+\frac{x^5}{460800}+\frac{x^4}{9216}+\frac{x^3}{288}+\frac{x^2}{16}+\frac{x}{2}+1\right) + c_2\left(\sqrt{x}\left(-\frac{121x^7}{75866112000}-\frac{49x^6}{331776000}-\frac{137x^5}{13824000}-\frac{25x^4}{55296}-\frac{11x^3}{864}-\frac{3x^2}{16}-x\right) + \sqrt{x}\left(\frac{x^7}{3251404800}+\frac{x^6}{33177600}+\frac{x^5}{460800}+\frac{x^4}{9216}+\frac{x^3}{288}+\frac{x^2}{16}+\frac{x}{2}+1\right)\log(x)\right)$$

5.3 problem 3

Internal problem ID [6950]
Internal file name [`OUTPUT/6193_Friday_August_12_2022_11_04_54_PM_10656506/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.6. Indicial Equation with Equal Roots. Exercises page 373
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x(-3+x)\,y' + 4y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(x^2 - 3x\right)y' + 4y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{-3+x}{x}$$
$$q(x) = \frac{4}{x^2}$$

Table 91: Table $p(x), q(x)$ singularites.

$p(x) = \frac{-3+x}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{4}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + (x^2 - 3x) y' + 4y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + (x^2 - 3x) \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + 4\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\right) \\ + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}\right) \\ + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 3x^{n+r} a_n (n+r) + 4a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - 3x^r a_0 r + 4a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 3x^r r + 4x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r-2)^2\, x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r-2)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= 2 \\
r_2 &= 2
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
(r-2)^2 x^r = 0
$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$
y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}
$$

Now the second solution y_2 is found using

$$
y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r} \right) \tag{1B}
$$

Then the general solution will be

$$
y = c_1 y_1(x) + c_2 y_2(x)
$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 2$, Eqs (1A,1B) become

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+2} \\
y_2(x) &= y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+2} \right)
\end{aligned}
$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
a_n(n+r)(n+r-1) + a_{n-1}(n+r-1) - 3a_n(n+r) + 4a_n = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{a_{n-1}(n+r-1)}{n^2 + 2nr + r^2 - 4n - 4r + 4} \tag{4}
$$

Which for the root $r = 2$ becomes

$$a_n = -\frac{a_{n-1}(1+n)}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{r}{(-1+r)^2}$$

Which for the root $r = 2$ becomes

$$a_1 = -2$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{(-1+r)^2}$	-2

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1+r}{r(-1+r)^2}$$

Which for the root $r = 2$ becomes

$$a_2 = \frac{3}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{(-1+r)^2}$	-2
a_2	$\frac{1+r}{r(-1+r)^2}$	$\frac{3}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-2-r}{(1+r)\, r\, (-1+r)^2}$$

Which for the root $r = 2$ becomes

$$a_3 = -\frac{2}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{(-1+r)^2}$	-2
a_2	$\frac{1+r}{r(-1+r)^2}$	$\frac{3}{2}$
a_3	$\frac{-2-r}{(1+r)r(-1+r)^2}$	$-\frac{2}{3}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{3+r}{(2+r)\,(1+r)\, r\, (-1+r)^2}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{5}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{(-1+r)^2}$	-2
a_2	$\frac{1+r}{r(-1+r)^2}$	$\frac{3}{2}$
a_3	$\frac{-2-r}{(1+r)r(-1+r)^2}$	$-\frac{2}{3}$
a_4	$\frac{3+r}{(2+r)(1+r)r(-1+r)^2}$	$\frac{5}{24}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-4-r}{(3+r)\,(2+r)\,(1+r)\, r\, (-1+r)^2}$$

Which for the root $r = 2$ becomes

$$a_5 = -\frac{1}{20}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{(-1+r)^2}$	-2
a_2	$\frac{1+r}{r(-1+r)^2}$	$\frac{3}{2}$
a_3	$\frac{-2-r}{(1+r)r(-1+r)^2}$	$-\frac{2}{3}$
a_4	$\frac{3+r}{(2+r)(1+r)r(-1+r)^2}$	$\frac{5}{24}$
a_5	$\frac{-4-r}{(3+r)(2+r)(1+r)r(-1+r)^2}$	$-\frac{1}{20}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{5+r}{(4+r)(3+r)(2+r)(1+r)r(-1+r)^2}$$

Which for the root $r = 2$ becomes

$$a_6 = \frac{7}{720}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{(-1+r)^2}$	-2
a_2	$\frac{1+r}{r(-1+r)^2}$	$\frac{3}{2}$
a_3	$\frac{-2-r}{(1+r)r(-1+r)^2}$	$-\frac{2}{3}$
a_4	$\frac{3+r}{(2+r)(1+r)r(-1+r)^2}$	$\frac{5}{24}$
a_5	$\frac{-4-r}{(3+r)(2+r)(1+r)r(-1+r)^2}$	$-\frac{1}{20}$
a_6	$\frac{5+r}{(4+r)(3+r)(2+r)(1+r)r(-1+r)^2}$	$\frac{7}{720}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{-6-r}{(5+r)(4+r)(3+r)(2+r)(1+r)\, r\, (-1+r)^2}$$

Which for the root $r = 2$ becomes

$$a_7 = -\frac{1}{630}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{(-1+r)^2}$	-2
a_2	$\frac{1+r}{r(-1+r)^2}$	$\frac{3}{2}$
a_3	$\frac{-2-r}{(1+r)r(-1+r)^2}$	$-\frac{2}{3}$
a_4	$\frac{3+r}{(2+r)(1+r)r(-1+r)^2}$	$\frac{5}{24}$
a_5	$\frac{-4-r}{(3+r)(2+r)(1+r)r(-1+r)^2}$	$-\frac{1}{20}$
a_6	$\frac{5+r}{(4+r)(3+r)(2+r)(1+r)r(-1+r)^2}$	$\frac{7}{720}$
a_7	$\frac{-6-r}{(5+r)(4+r)(3+r)(2+r)(1+r)r(-1+r)^2}$	$-\frac{1}{630}$

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^2 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \ldots\right) \\ &= x^2 \left(1 - 2x + \frac{3x^2}{2} - \frac{2x^3}{3} + \frac{5x^4}{24} - \frac{x^5}{20} + \frac{7x^6}{720} - \frac{x^7}{630} + O(x^8)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 2$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r=2)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$-\frac{r}{(-1+r)^2}$	-2	$\frac{1+r}{(-1+r)^3}$	3
b_2	$\frac{1+r}{r(-1+r)^2}$	$\frac{3}{2}$	$\frac{-2r^2-3r+1}{r^2(-1+r)^3}$	$-\frac{13}{4}$
b_3	$\frac{-2-r}{(1+r)r(-1+r)^2}$	$-\frac{2}{3}$	$\frac{3r^3+9r^2+2r-2}{(1+r)^2r^2(-1+r)^3}$	$\frac{31}{18}$
b_4	$\frac{3+r}{(2+r)(1+r)r(-1+r)^2}$	$\frac{5}{24}$	$\frac{-4r^4-22r^3-28r^2+6}{(2+r)^2(1+r)^2r^2(-1+r)^3}$	$-\frac{173}{288}$
b_5	$\frac{-4-r}{(3+r)(2+r)(1+r)r(-1+r)^2}$	$-\frac{1}{20}$	$\frac{5r^5+45r^4+125r^3+105r^2-16r-24}{(3+r)^2(2+r)^2(1+r)^2r^2(-1+r)^3}$	$\frac{187}{1200}$
b_6	$\frac{5+r}{(4+r)(3+r)(2+r)(1+r)r(-1+r)^2}$	$\frac{7}{720}$	$\frac{-6r^6-81r^5-385r^4-755r^3-473r^2+140r+120}{(4+r)^2(3+r)^2(2+r)^2(1+r)^2r^2(-1+r)^3}$	$-\frac{463}{14400}$
b_7	$\frac{-6-r}{(5+r)(4+r)(3+r)(2+r)(1+r)r(-1+r)^2}$	$-\frac{1}{630}$	$\frac{7r^7+133r^6+959r^5+3255r^4+5082r^3+2492r^2-1128r-720}{(5+r)^2(4+r)^2(3+r)^2(2+r)^2(1+r)^2r^2(-1+r)^3}$	$\frac{971}{176400}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 + b_7x^7 + b_8x^8\dots \\
&= x^2\left(1 - 2x + \frac{3x^2}{2} - \frac{2x^3}{3} + \frac{5x^4}{24} - \frac{x^5}{20} + \frac{7x^6}{720} - \frac{x^7}{630} + O(x^8)\right)\ln(x) \\
&\quad + x^2\left(3x - \frac{13x^2}{4} + \frac{31x^3}{18} - \frac{173x^4}{288} + \frac{187x^5}{1200} - \frac{463x^6}{14400} + \frac{971x^7}{176400} + O(x^8)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1x^2\left(1 - 2x + \frac{3x^2}{2} - \frac{2x^3}{3} + \frac{5x^4}{24} - \frac{x^5}{20} + \frac{7x^6}{720} - \frac{x^7}{630} + O(x^8)\right) \\
&\quad + c_2\left(x^2\left(1 - 2x + \frac{3x^2}{2} - \frac{2x^3}{3} + \frac{5x^4}{24} - \frac{x^5}{20} + \frac{7x^6}{720} - \frac{x^7}{630} + O(x^8)\right)\ln(x)\right. \\
&\quad \left. + x^2\left(3x - \frac{13x^2}{4} + \frac{31x^3}{18} - \frac{173x^4}{288} + \frac{187x^5}{1200} - \frac{463x^6}{14400} + \frac{971x^7}{176400} + O(x^8)\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
y = y_h
$$

$$
\begin{aligned}
&= c_1x^2\left(1-2x+\frac{3x^2}{2}-\frac{2x^3}{3}+\frac{5x^4}{24}-\frac{x^5}{20}+\frac{7x^6}{720}-\frac{x^7}{630}+O(x^8)\right) \\
&+ c_2\left(x^2\left(1-2x+\frac{3x^2}{2}-\frac{2x^3}{3}+\frac{5x^4}{24}-\frac{x^5}{20}+\frac{7x^6}{720}-\frac{x^7}{630}+O(x^8)\right)\ln(x)\right. \\
&\left.+ x^2\left(3x-\frac{13x^2}{4}+\frac{31x^3}{18}-\frac{173x^4}{288}+\frac{187x^5}{1200}-\frac{463x^6}{14400}+\frac{971x^7}{176400}+O(x^8)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1x^2\left(1-2x+\frac{3x^2}{2}-\frac{2x^3}{3}+\frac{5x^4}{24}-\frac{x^5}{20}+\frac{7x^6}{720}-\frac{x^7}{630}+O(x^8)\right) \\
&+ c_2\left(x^2\left(1-2x+\frac{3x^2}{2}-\frac{2x^3}{3}+\frac{5x^4}{24}-\frac{x^5}{20}+\frac{7x^6}{720}-\frac{x^7}{630}+O(x^8)\right)\ln(x)\right. \\
&\left.+ x^2\left(3x-\frac{13x^2}{4}+\frac{31x^3}{18}-\frac{173x^4}{288}+\frac{187x^5}{1200}-\frac{463x^6}{14400}+\frac{971x^7}{176400}+O(x^8)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1x^2\left(1-2x+\frac{3x^2}{2}-\frac{2x^3}{3}+\frac{5x^4}{24}-\frac{x^5}{20}+\frac{7x^6}{720}-\frac{x^7}{630}+O(x^8)\right) \\
&+ c_2\left(x^2\left(1-2x+\frac{3x^2}{2}-\frac{2x^3}{3}+\frac{5x^4}{24}-\frac{x^5}{20}+\frac{7x^6}{720}-\frac{x^7}{630}+O(x^8)\right)\ln(x)\right. \\
&\left.+ x^2\left(3x-\frac{13x^2}{4}+\frac{31x^3}{18}-\frac{173x^4}{288}+\frac{187x^5}{1200}-\frac{463x^6}{14400}+\frac{971x^7}{176400}+O(x^8)\right)\right)
\end{aligned}
$$

Verified OK.

5.3.1 Maple step by step solution

Let's solve

$x^2y''+(x^2-3x)\,y'+4y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{4y}{x^2}-\frac{(-3+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(-3+x)y'}{x}+\frac{4y}{x^2}=0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{-3+x}{x}, P_3(x) = \frac{4}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -3$$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 4$$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $x^2 y'' + x(-3+x)\, y' + 4y = 0$
- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$
  - Shift index using $k- >k+1-m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$
  - Convert $x^2 \cdot y''$ to series expansion

    $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

    Rewrite ODE with series expansions

    $$a_0(-2+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k (k+r-2)^2 + a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 2$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-2)^2 + a_{k-1}(k+r-1) = 0$
- Shift index using $k- >k+1$

  $a_{k+1}(k+r-1)^2 + a_k(k+r) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r)}{(k+r-1)^2}$
- Recursion relation for $r = 2$

  $a_{k+1} = -\frac{a_k(k+2)}{(k+1)^2}$
- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{a_k(k+2)}{(k+1)^2}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 81

```
Order:=8;
dsolve(x^2*diff(y(x),x$2)+x*(x-3)*diff(y(x),x)+4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left((c_2 \ln(x) + c_1)\left(1 - 2x + \frac{3}{2}x^2 - \frac{2}{3}x^3 + \frac{5}{24}x^4 - \frac{1}{20}x^5 + \frac{7}{720}x^6 - \frac{1}{630}x^7 + \mathrm{O}\left(x^8\right)\right) + \left(3x - \frac{13}{4}x^2 + \frac{31}{18}x^3 - \frac{173}{288}x^4 + \frac{187}{1200}x^5 - \frac{463}{14400}x^6 + \frac{971}{176400}x^7 + \mathrm{O}\left(x^8\right)\right)c_2\right)x^2$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 164

```
AsymptoticDSolveValue[x^2*y''[x]+x*(x-3)*y'[x]+4*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(-\frac{x^7}{630} + \frac{7x^6}{720} - \frac{x^5}{20} + \frac{5x^4}{24} - \frac{2x^3}{3} + \frac{3x^2}{2} - 2x + 1\right)x^2 + c_2\left(\left(\frac{971x^7}{176400} - \frac{463x^6}{14400} + \frac{187x^5}{1200} - \frac{173x^4}{288} + \frac{31x^3}{18} - \frac{13x^2}{4} + 3x\right)x^2 + \left(-\frac{x^7}{630} + \frac{7x^6}{720} - \frac{x^5}{20} + \frac{5x^4}{24} - \frac{2x^3}{3} + \frac{3x^2}{2} - 2x + 1\right)x^2\log(x)\right)$$

5.4 problem 4

Internal problem ID [6951]
Internal file name [`OUTPUT/6194_Friday_August_12_2022_11_04_56_PM_78487314/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.6. Indicial Equation with Equal Roots. Exercises page 373
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + 3xy' + \left(4x^2 + 1\right) y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + 3xy' + \left(4x^2 + 1\right) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{3}{x}$$
$$q(x) = \frac{4x^2 + 1}{x^2}$$

Table 93: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{4x^2+1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + 3xy' + \left(4x^2 + 1\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + 3x \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(4x^2 + 1\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 4x^{n+r+2} a_n = \sum_{n=2}^{\infty} 4a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 4a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + 3x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + 3x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + 3x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r+1)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r+1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -1$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r+1)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = -1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-1}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n-1}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 3a_n(n+r) + 4a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{4a_{n-2}}{n^2 + 2nr + r^2 + 2n + 2r + 1} \tag{4}$$

Which for the root $r = -1$ becomes

$$a_n = -\frac{4a_{n-2}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{4}{(r+3)^2}$$

Which for the root $r = -1$ becomes

$$a_2 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{(r+3)^2}$	-1

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{(r+3)^2}$	-1
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{(r+3)^2 (r+5)^2}$$

Which for the root $r = -1$ becomes

$$a_4 = \frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{(r+3)^2}$	-1
a_3	0	0
a_4	$\frac{16}{(r+3)^2(r+5)^2}$	$\frac{1}{4}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{(r+3)^2}$	-1
a_3	0	0
a_4	$\frac{16}{(r+3)^2(r+5)^2}$	$\frac{1}{4}$
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = -\frac{64}{(r+3)^2 (r+5)^2 (r+7)^2}$$

Which for the root $r = -1$ becomes

$$a_6 = -\frac{1}{36}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{(r+3)^2}$	-1
a_3	0	0
a_4	$\frac{16}{(r+3)^2(r+5)^2}$	$\frac{1}{4}$
a_5	0	0
a_6	$-\frac{64}{(r+3)^2(r+5)^2(r+7)^2}$	$-\frac{1}{36}$

For $n = 7$, using the above recursive equation gives

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{(r+3)^2}$	-1
a_3	0	0
a_4	$\frac{16}{(r+3)^2(r+5)^2}$	$\frac{1}{4}$
a_5	0	0
a_6	$-\frac{64}{(r+3)^2(r+5)^2(r+7)^2}$	$-\frac{1}{36}$
a_7	0	0

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= \frac{1}{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\
&= \frac{-x^2 + 1 + \frac{x^4}{4} - \frac{x^6}{36} + O(x^8)}{x}
\end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = -1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = -1)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$-\frac{4}{(r+3)^2}$	-1	$\frac{8}{(r+3)^3}$	1
b_3	0	0	0	0
b_4	$\frac{16}{(r+3)^2(r+5)^2}$	$\frac{1}{4}$	$\frac{-64r-256}{(r+3)^3(r+5)^3}$	$-\frac{3}{8}$
b_5	0	0	0	0
b_6	$-\frac{64}{(r+3)^2(r+5)^2(r+7)^2}$	$-\frac{1}{36}$	$\frac{384r^2+3840r+9088}{(r+3)^3(r+5)^3(r+7)^3}$	$\frac{11}{216}$
b_7	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \ldots \\
&= \frac{\left(-x^2 + 1 + \frac{x^4}{4} - \frac{x^6}{36} + O(x^8)\right)\ln(x)}{x} + \frac{x^2 - \frac{3x^4}{8} + \frac{11x^6}{216} + O(x^8)}{x}
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= \frac{c_1\left(-x^2 + 1 + \frac{x^4}{4} - \frac{x^6}{36} + O(x^8)\right)}{x} \\
&\quad + c_2\left(\frac{\left(-x^2 + 1 + \frac{x^4}{4} - \frac{x^6}{36} + O(x^8)\right)\ln(x)}{x} + \frac{x^2 - \frac{3x^4}{8} + \frac{11x^6}{216} + O(x^8)}{x}\right)
\end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= \frac{c_1 \left(-x^2+1+\frac{x^4}{4}-\frac{x^6}{36}+O(x^8)\right)}{x} \\
&+ c_2 \left(\frac{\left(-x^2+1+\frac{x^4}{4}-\frac{x^6}{36}+O(x^8)\right) \ln(x)}{x}+\frac{x^2-\frac{3x^4}{8}+\frac{11x^6}{216}+O(x^8)}{x}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = &\frac{c_1 \left(-x^2+1+\frac{x^4}{4}-\frac{x^6}{36}+O(x^8)\right)}{x} \\
&+ c_2 \left(\frac{\left(-x^2+1+\frac{x^4}{4}-\frac{x^6}{36}+O(x^8)\right) \ln(x)}{x}+\frac{x^2-\frac{3x^4}{8}+\frac{11x^6}{216}+O(x^8)}{x}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = &\frac{c_1 \left(-x^2+1+\frac{x^4}{4}-\frac{x^6}{36}+O(x^8)\right)}{x} \\
&+ c_2 \left(\frac{\left(-x^2+1+\frac{x^4}{4}-\frac{x^6}{36}+O(x^8)\right) \ln(x)}{x}+\frac{x^2-\frac{3x^4}{8}+\frac{11x^6}{216}+O(x^8)}{x}\right)
\end{aligned}
$$

Verified OK.

5.4.1 Maple step by step solution

Let's solve

$x^2 y'' + 3xy' + (4x^2+1) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2+1) y}{x^2} - \frac{3y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{3y'}{x} + \frac{(4x^2+1)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{3}{x}, P_3(x) = \frac{4x^2+1}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 3$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 1$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + 3xy' + (4x^2 + 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\, (k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(1+r)^2 x^r + a_1(2+r)^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+1)^2 + 4a_{k-2}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = -1$
- Each term must be 0

  $a_1(2+r)^2 = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)^2 + 4a_{k-2} = 0$
- Shift index using $k- >k+2$

  $a_{k+2}(k+3+r)^2 + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{(k+3+r)^2}$
- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{4a_k}{(k+2)^2}$
- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{4a_k}{(k+2)^2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 57

```
Order:=8;
dsolve(x^2*diff(y(x),x$2)+3*x*diff(y(x),x)+(1+4*x^2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{(c_2 \ln(x) + c_1)\left(1 - x^2 + \frac{1}{4}x^4 - \frac{1}{36}x^6 + \mathrm{O}(x^8)\right) + \left(x^2 - \frac{3}{8}x^4 + \frac{11}{216}x^6 + \mathrm{O}(x^8)\right)c_2}{x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 84

```
AsymptoticDSolveValue[x^2*y''[x]+3*x*y'[x]+(1+4*x^2)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to \frac{c_1\left(-\frac{x^6}{36} + \frac{x^4}{4} - x^2 + 1\right)}{x} + c_2\left(\frac{\frac{11x^6}{216} - \frac{3x^4}{8} + x^2}{x} + \frac{\left(-\frac{x^6}{36} + \frac{x^4}{4} - x^2 + 1\right)\log(x)}{x}\right)$$

5.5 problem 5

Internal problem ID [6952]
Internal file name [`OUTPUT/6195_Friday_August_12_2022_11_04_58_PM_95537058/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.6. Indicial Equation with Equal Roots. Exercises page 373
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x(1+x)\,y''+(1+5x)\,y'+3y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^2+x)\,y''+(1+5x)\,y'+3y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=\frac{1+5x}{x\,(1+x)}\\ q(x)&=\frac{3}{x\,(1+x)}\end{aligned}$$

Table 95: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1+5x}{x(1+x)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

$q(x) = \frac{3}{x(1+x)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x(1+x)\,y'' + (1+5x)\,y' + 3y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} x(1+x)&\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) \\ &+ (1+5x)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + 3\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$
\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 3a_n x^{n+r}\right) = 0 \tag{2A}
$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} a_{n-1} (n+r-1)(n+r-2) x^{n+r-1}
$$

$$
\sum_{n=0}^{\infty} 5x^{n+r} a_n (n+r) = \sum_{n=1}^{\infty} 5a_{n-1} (n+r-1) x^{n+r-1}
$$

$$
\sum_{n=0}^{\infty} 3a_n x^{n+r} = \sum_{n=1}^{\infty} 3a_{n-1} x^{n+r-1}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$
\left(\sum_{n=1}^{\infty} a_{n-1} (n+r-1)(n+r-2) x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 5a_{n-1} (n+r-1) x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0
$$

When $n=0$ the above becomes

$$
x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0
$$

Or

$$\left(x^{-1+r} r(-1+r)+r\, x^{-1+r}\right) a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2=0$$

Since the above is true for all x then the indicial equation becomes

$$r^2=0$$

Solving for r gives the roots of the indicial equation as

$$r_1=0$$
$$r_2=0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2=0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x)=\sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x)=y_1(x) \ln (x)+\left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y=c_1 y_1(x)+c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0=1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n=0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0=1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} & a_{n-1}(n+r-1)(n+r-2)+a_n(n+r)(n+r-1) \\ & \quad+5 a_{n-1}(n+r-1)+a_n(n+r)+3 a_{n-1}=0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(n+r+2)\, a_{n-1}}{n+r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{(n+2)\, a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-3-r}{1+r}$$

Which for the root $r = 0$ becomes

$$a_1 = -3$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-r}{1+r}$	-3

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^2+7r+12}{(r+2)\,(1+r)}$$

Which for the root $r = 0$ becomes

$$a_2 = 6$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-r}{1+r}$	-3
a_2	$\frac{r^2+7r+12}{(r+2)(1+r)}$	6

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-r^2 - 9r - 20}{(1+r)(r+2)}$$

Which for the root $r = 0$ becomes

$$a_3 = -10$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-r}{1+r}$	-3
a_2	$\frac{r^2+7r+12}{(r+2)(1+r)}$	6
a_3	$\frac{-r^2-9r-20}{(1+r)(r+2)}$	-10

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^2 + 11r + 30}{(1+r)(r+2)}$$

Which for the root $r = 0$ becomes

$$a_4 = 15$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-r}{1+r}$	-3
a_2	$\frac{r^2+7r+12}{(r+2)(1+r)}$	6
a_3	$\frac{-r^2-9r-20}{(1+r)(r+2)}$	-10
a_4	$\frac{r^2+11r+30}{(1+r)(r+2)}$	15

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-r^2 - 13r - 42}{(1+r)(r+2)}$$

Which for the root $r = 0$ becomes

$$a_5 = -21$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-r}{1+r}$	-3
a_2	$\frac{r^2+7r+12}{(r+2)(1+r)}$	6
a_3	$\frac{-r^2-9r-20}{(1+r)(r+2)}$	-10
a_4	$\frac{r^2+11r+30}{(1+r)(r+2)}$	15
a_5	$\frac{-r^2-13r-42}{(1+r)(r+2)}$	-21

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{r^2 + 15r + 56}{(1 + r)(r + 2)}$$

Which for the root $r = 0$ becomes

$$a_6 = 28$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-r}{1+r}$	-3
a_2	$\frac{r^2+7r+12}{(r+2)(1+r)}$	6
a_3	$\frac{-r^2-9r-20}{(1+r)(r+2)}$	-10
a_4	$\frac{r^2+11r+30}{(1+r)(r+2)}$	15
a_5	$\frac{-r^2-13r-42}{(1+r)(r+2)}$	-21
a_6	$\frac{r^2+15r+56}{(1+r)(r+2)}$	28

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{-r^2 - 17r - 72}{(1 + r)(r + 2)}$$

Which for the root $r = 0$ becomes

$$a_7 = -36$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-r}{1+r}$	-3
a_2	$\frac{r^2+7r+12}{(r+2)(1+r)}$	6
a_3	$\frac{-r^2-9r-20}{(1+r)(r+2)}$	-10
a_4	$\frac{r^2+11r+30}{(1+r)(r+2)}$	15
a_5	$\frac{-r^2-13r-42}{(1+r)(r+2)}$	-21
a_6	$\frac{r^2+15r+56}{(1+r)(r+2)}$	28
a_7	$\frac{-r^2-17r-72}{(1+r)(r+2)}$	-36

Using the above table, then the first solution $y_1(x)$ becomes

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots \\ &= -36x^7 + 28x^6 - 21x^5 + 15x^4 - 10x^3 + 6x^2 - 3x + 1 + O(x^8) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r} \right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r=0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{-3-r}{1+r}$	-3	$\frac{2}{(1+r)^2}$	2
b_2	$\frac{r^2+7r+12}{(r+2)(1+r)}$	6	$\frac{-4r^2-20r-22}{(1+r)^2(r+2)^2}$	$-\frac{11}{2}$
b_3	$\frac{-r^2-9r-20}{(1+r)(r+2)}$	-10	$\frac{6r^2+36r+42}{(1+r)^2(r+2)^2}$	$\frac{21}{2}$
b_4	$\frac{r^2+11r+30}{(1+r)(r+2)}$	15	$\frac{-8r^2-56r-68}{(1+r)^2(r+2)^2}$	-17
b_5	$\frac{-r^2-13r-42}{(1+r)(r+2)}$	-21	$\frac{10r^2+80r+100}{(1+r)^2(r+2)^2}$	25
b_6	$\frac{r^2+15r+56}{(1+r)(r+2)}$	28	$\frac{-12r^2-108r-138}{(1+r)^2(r+2)^2}$	$-\frac{69}{2}$
b_7	$\frac{-r^2-17r-72}{(1+r)(r+2)}$	-36	$\frac{14r^2+140r+182}{(1+r)^2(r+2)^2}$	$\frac{91}{2}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 + b_7x^7 + b_8x^8 \dots \\
&= \left(-36x^7 + 28x^6 - 21x^5 + 15x^4 - 10x^3 + 6x^2 - 3x + 1 + O(x^8)\right)\ln(x) \\
&\quad + 2x - \frac{11x^2}{2} + \frac{21x^3}{2} - 17x^4 + 25x^5 - \frac{69x^6}{2} + \frac{91x^7}{2} + O(x^8)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1\left(-36x^7 + 28x^6 - 21x^5 + 15x^4 - 10x^3 + 6x^2 - 3x + 1 + O(x^8)\right) \\
&\quad + c_2\Bigg(\left(-36x^7 + 28x^6 - 21x^5 + 15x^4 - 10x^3 + 6x^2 - 3x + 1 + O(x^8)\right)\ln(x) \\
&\quad + 2x - \frac{11x^2}{2} + \frac{21x^3}{2} - 17x^4 + 25x^5 - \frac{69x^6}{2} + \frac{91x^7}{2} + O(x^8)\Bigg)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(-36x^7 + 28x^6 - 21x^5 + 15x^4 - 10x^3 + 6x^2 - 3x + 1 + O(x^8)\right) \\
&\quad + c_2\Bigg(\left(-36x^7 + 28x^6 - 21x^5 + 15x^4 - 10x^3 + 6x^2 - 3x + 1 + O(x^8)\right)\ln(x) + 2x \\
&\quad - \frac{11x^2}{2} + \frac{21x^3}{2} - 17x^4 + 25x^5 - \frac{69x^6}{2} + \frac{91x^7}{2} + O(x^8)\Bigg)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1\left(-36x^7+28x^6-21x^5+15x^4-10x^3+6x^2-3x+1+O(x^8)\right) \\
& + c_2\left(\left(-36x^7+28x^6-21x^5+15x^4-10x^3+6x^2-3x+1+O(x^8)\right)\ln(x)+2x\right. \\
& \left. -\frac{11x^2}{2}+\frac{21x^3}{2}-17x^4+25x^5-\frac{69x^6}{2}+\frac{91x^7}{2}+O(x^8)\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1\left(-36x^7+28x^6-21x^5+15x^4-10x^3+6x^2-3x+1+O(x^8)\right) \\
& + c_2\left(\left(-36x^7+28x^6-21x^5+15x^4-10x^3+6x^2-3x+1+O(x^8)\right)\ln(x)+2x\right. \\
& \left. -\frac{11x^2}{2}+\frac{21x^3}{2}-17x^4+25x^5-\frac{69x^6}{2}+\frac{91x^7}{2}+O(x^8)\right)
\end{aligned}
$$

Verified OK.

5.5.1 Maple step by step solution

Let's solve

$x(1+x)\,y''+(1+5x)\,y'+3y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{3y}{x(1+x)}-\frac{(1+5x)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(1+5x)y'}{x(1+x)}+\frac{3y}{x(1+x)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{1+5x}{x(1+x)}, P_3(x)=\frac{3}{x(1+x)}\right]$

  ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left.((1+x)\cdot P_2(x))\right|_{x=-1}=4$

  ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

$$\left((1+x)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $x(1+x)\, y'' + (1+5x)\, y' + 3y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^2 - u)\left(\frac{d^2}{du^2}y(u)\right) + (-4+5u)\left(\frac{d}{du}y(u)\right) + 3y(u) = 0$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$$

- Shift index using $k- > k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- > k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0 r(3+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+r+1)(k+4+r) + a_k (k+r+3)(k+r+1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(3+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{-3, 0\}$

- Each term in the series must be 0, giving the recursion relation

  $((-k-r-4)\,a_{k+1} + a_k(k+r+3))\,(k+r+1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r+3)}{k+4+r}$

- Recursion relation for $r = -3$

  $a_{k+1} = \frac{a_k k}{k+1}$

- Solution for $r = -3$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-3}, a_{k+1} = \frac{a_k k}{k+1}\right]$

- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k-3}, a_{k+1} = \frac{a_k k}{k+1}\right]$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(k+3)}{k+4}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(k+3)}{k+4}\right]$

- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k (1+x)^k, a_{k+1} = \frac{a_k(k+3)}{k+4}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k (1+x)^{k-3}\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^k\right), a_{k+1} = \frac{a_k k}{k+1}, b_{k+1} = \frac{b_k(k+3)}{k+4}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 71

```
Order:=8;
dsolve(x*(1+x)*diff(y(x),x$2)+(1+5*x)*diff(y(x),x)+3*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln(x) + c_1)\left(1 - 3x + 6x^2 - 10x^3 + 15x^4 - 21x^5 + 28x^6 - 36x^7 + \mathrm{O}\left(x^8\right)\right) + \left(2x - \frac{11}{2}x^2 + \frac{21}{2}x^3 - 17x^4 + 25x^5 - \frac{69}{2}x^6 + \frac{91}{2}x^7 + \mathrm{O}\left(x^8\right)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 125

```
AsymptoticDSolveValue[x*(1+x)*y''[x]+(1+5*x)*y'[x]+3*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(-36x^7 + 28x^6 - 21x^5 + 15x^4 - 10x^3 + 6x^2 - 3x + 1\right) + c_2\left(\frac{91x^7}{2} - \frac{69x^6}{2} + 25x^5 - 17x^4 + \frac{21x^3}{2} - \frac{11x^2}{2} + \left(-36x^7 + 28x^6 - 21x^5 + 15x^4 - 10x^3 + 6x^2 - 3x + 1\right)\log(x) + 2x\right)$$

5.6 problem 6

Internal problem ID [6953]
Internal file name [`OUTPUT/6196_Friday_August_12_2022_11_05_00_PM_72649872/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.6. Indicial Equation with Equal Roots. Exercises page 373
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - x(3x+1)\,y' + (1-6x)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(-3x^2 - x\right)y' + (1-6x)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= -\frac{3x+1}{x} \\ q(x) &= -\frac{6x-1}{x^2} \end{aligned}$$

Table 97: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{3x+1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{6x-1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + (-3x^2 - x)\,y' + (1 - 6x)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + (-3x^2 - x) \left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (1-6x) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-3x^{1+n+r} a_n(n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n(n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-6x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-3x^{1+n+r} a_n(n+r)\right) = \sum_{n=1}^{\infty} \left(-3a_{n-1}(n+r-1)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} \left(-6x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-6a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-3a_{n-1}(n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n(n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \sum_{n=1}^{\infty} \left(-6a_{n-1} x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n(n+r)(n+r-1) - x^{n+r} a_n(n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(-1+r)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(-1+r)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(-1+r)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{1+n}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - 3a_{n-1}(n+r-1) - a_n(n+r) + a_n - 6a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{3a_{n-1}(1+n+r)}{n^2+2nr+r^2-2n-2r+1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{3a_{n-1}(2+n)}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{3r+6}{r^2}$$

Which for the root $r = 1$ becomes

$$a_1 = 9$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3r+6}{r^2}$	9

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{9(2+r)(3+r)}{r^2(1+r)^2}$$

Which for the root $r = 1$ becomes

$$a_2 = 27$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3r+6}{r^2}$	9
a_2	$\frac{9(2+r)(3+r)}{r^2(1+r)^2}$	27

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{27(4+r)(3+r)}{(2+r)\, r^2 (1+r)^2}$$

Which for the root $r = 1$ becomes

$$a_3 = 45$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3r+6}{r^2}$	9
a_2	$\frac{9(2+r)(3+r)}{r^2(1+r)^2}$	27
a_3	$\frac{27(4+r)(3+r)}{(2+r)r^2(1+r)^2}$	45

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{81(5+r)(4+r)}{(2+r)(3+r)\, r^2 (1+r)^2}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{405}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3r+6}{r^2}$	9
a_2	$\frac{9(2+r)(3+r)}{r^2(1+r)^2}$	27
a_3	$\frac{27(4+r)(3+r)}{(2+r)r^2(1+r)^2}$	45
a_4	$\frac{81(5+r)(4+r)}{(2+r)(3+r)r^2(1+r)^2}$	$\frac{405}{8}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{243(6+r)(5+r)}{(4+r)(2+r)(3+r)\, r^2 (1+r)^2}$$

Which for the root $r = 1$ becomes

$$a_5 = \frac{1701}{40}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3r+6}{r^2}$	9
a_2	$\frac{9(2+r)(3+r)}{r^2(1+r)^2}$	27
a_3	$\frac{27(4+r)(3+r)}{(2+r)r^2(1+r)^2}$	45
a_4	$\frac{81(5+r)(4+r)}{(2+r)(3+r)r^2(1+r)^2}$	$\frac{405}{8}$
a_5	$\frac{243(6+r)(5+r)}{(4+r)(2+r)(3+r)r^2(1+r)^2}$	$\frac{1701}{40}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{729(7 + r)(6 + r)}{(5 + r)(4 + r)(2 + r)(3 + r)r^2(1 + r)^2}$$

Which for the root $r = 1$ becomes

$$a_6 = \frac{567}{20}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3r+6}{r^2}$	9
a_2	$\frac{9(2+r)(3+r)}{r^2(1+r)^2}$	27
a_3	$\frac{27(4+r)(3+r)}{(2+r)r^2(1+r)^2}$	45
a_4	$\frac{81(5+r)(4+r)}{(2+r)(3+r)r^2(1+r)^2}$	$\frac{405}{8}$
a_5	$\frac{243(6+r)(5+r)}{(4+r)(2+r)(3+r)r^2(1+r)^2}$	$\frac{1701}{40}$
a_6	$\frac{729(7+r)(6+r)}{(5+r)(4+r)(2+r)(3+r)r^2(1+r)^2}$	$\frac{567}{20}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{2187(8+r)(7+r)}{(6+r)(5+r)(4+r)(2+r)(3+r)r^2(1+r)^2}$$

Which for the root $r = 1$ becomes

$$a_7 = \frac{2187}{140}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3r+6}{r^2}$	9
a_2	$\frac{9(2+r)(3+r)}{r^2(1+r)^2}$	27
a_3	$\frac{27(4+r)(3+r)}{(2+r)r^2(1+r)^2}$	45
a_4	$\frac{81(5+r)(4+r)}{(2+r)(3+r)r^2(1+r)^2}$	$\frac{405}{8}$
a_5	$\frac{243(6+r)(5+r)}{(4+r)(2+r)(3+r)r^2(1+r)^2}$	$\frac{1701}{40}$
a_6	$\frac{729(7+r)(6+r)}{(5+r)(4+r)(2+r)(3+r)r^2(1+r)^2}$	$\frac{567}{20}$
a_7	$\frac{2187(8+r)(7+r)}{(6+r)(5+r)(4+r)(2+r)(3+r)r^2(1+r)^2}$	$\frac{2187}{140}$

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\ &= x\left(45x^3 + 27x^2 + 9x + 1 + \frac{405x^4}{8} + \frac{1701x^5}{40} + \frac{567x^6}{20} + \frac{2187x^7}{140} + O\left(x^8\right)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	$\frac{3r+6}{r^2}$	9	$\frac{-3r-12}{r^3}$
b_2	$\frac{9(2+r)(3+r)}{r^2(1+r)^2}$	27	$\frac{-18r^3-135r^2-261r-108}{r^3(1+r)^3}$
b_3	$\frac{27(4+r)(3+r)}{(2+r)r^2(1+r)^2}$	45	$-\frac{27(3r^4+33r^3+116r^2+146r+48)}{(2+r)^2r^3(1+r)^3}$
b_4	$\frac{81(5+r)(4+r)}{(2+r)(3+r)r^2(1+r)^2}$	$\frac{405}{8}$	$-\frac{162(2r^5+31r^4+172r^3+416r^2+417r+120)}{(3+r)^2(2+r)^2r^3(1+r)^3}$
b_5	$\frac{243(6+r)(5+r)}{(4+r)(2+r)(3+r)r^2(1+r)^2}$	$\frac{1701}{40}$	$-\frac{243(5r^6+105r^5+845r^4+3285r^3+6344r^2+5484r+1440)}{(4+r)^2(3+r)^2(2+r)^2r^3(1+r)^3}$
b_6	$\frac{729(7+r)(6+r)}{(5+r)(4+r)(2+r)(3+r)r^2(1+r)^2}$	$\frac{567}{20}$	$-\frac{729(6r^7+165r^6+1819r^5+10315r^4+32003r^3+52952r^2+41124r+10080)}{(5+r)^2(4+r)^2(3+r)^2(2+r)^2r^3(1+r)^3}$
b_7	$\frac{2187(8+r)(7+r)}{(6+r)(5+r)(4+r)(2+r)(3+r)r^2(1+r)^2}$	$\frac{2187}{140}$	$-\frac{2187(7r^8+245r^7+3549r^6+27587r^5+124572r^4+329504r^3+485960r^2+347472r+80}{(6+r)^2(5+r)^2(4+r)^2(3+r)^2(2+r)^2r^3(1+r)^3}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 + b_7x^7 + b_8x^8 \dots \\
&= x\left(45x^3 + 27x^2 + 9x + 1 + \frac{405x^4}{8} + \frac{1701x^5}{40} + \frac{567x^6}{20} + \frac{2187x^7}{140} + O(x^8)\right)\ln(x) \\
&\quad + x\left(-15x - \frac{261x^2}{4} - \frac{519x^3}{4} - \frac{5211x^4}{32} - \frac{118179x^5}{800} - \frac{83511x^6}{800} - \frac{2361717x^7}{39200} + O(x^8)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1x\left(45x^3 + 27x^2 + 9x + 1 + \frac{405x^4}{8} + \frac{1701x^5}{40} + \frac{567x^6}{20} + \frac{2187x^7}{140} + O(x^8)\right) \\
&\quad + c_2\left(x\left(45x^3+27x^2+9x+1+\frac{405x^4}{8}+\frac{1701x^5}{40}+\frac{567x^6}{20}+\frac{2187x^7}{140}+O(x^8)\right)\ln(x)\right. \\
&\quad + x\left(-15x - \frac{261x^2}{4} - \frac{519x^3}{4} - \frac{5211x^4}{32} - \frac{118179x^5}{800} - \frac{83511x^6}{800} - \frac{2361717x^7}{39200}\right. \\
&\quad \left.\left. + O(x^8)\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x\left(45x^3 + 27x^2 + 9x + 1 + \frac{405x^4}{8} + \frac{1701x^5}{40} + \frac{567x^6}{20} + \frac{2187x^7}{140} + O(x^8)\right) \\
&\quad + c_2\left(x\left(45x^3 + 27x^2 + 9x + 1 + \frac{405x^4}{8} + \frac{1701x^5}{40} + \frac{567x^6}{20} + \frac{2187x^7}{140} + O(x^8)\right)\ln(x)\right. \\
&\quad \left. + x\left(-15x - \frac{261x^2}{4} - \frac{519x^3}{4} - \frac{5211x^4}{32} - \frac{118179x^5}{800} - \frac{83511x^6}{800} - \frac{2361717x^7}{39200} + O(x^8)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x\left(45x^3 + 27x^2 + 9x + 1 + \frac{405x^4}{8} + \frac{1701x^5}{40} + \frac{567x^6}{20} + \frac{2187x^7}{140} + O(x^8)\right) \\
&\quad + c_2\left(x\left(45x^3 + 27x^2 + 9x + 1 + \frac{405x^4}{8} + \frac{1701x^5}{40} + \frac{567x^6}{20} + \frac{2187x^7}{140} + O(x^8)\right)\ln(x)\right. \\
&\quad \left. + x\left(-15x - \frac{261x^2}{4} - \frac{519x^3}{4} - \frac{5211x^4}{32} - \frac{118179x^5}{800} - \frac{83511x^6}{800} - \frac{2361717x^7}{39200} + O(x^8)\right)\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x\left(45x^3 + 27x^2 + 9x + 1 + \frac{405x^4}{8} + \frac{1701x^5}{40} + \frac{567x^6}{20} + \frac{2187x^7}{140} + O(x^8)\right) \\
&\quad + c_2\left(x\left(45x^3 + 27x^2 + 9x + 1 + \frac{405x^4}{8} + \frac{1701x^5}{40} + \frac{567x^6}{20} + \frac{2187x^7}{140} + O(x^8)\right)\ln(x)\right. \\
&\quad \left. + x\left(-15x - \frac{261x^2}{4} - \frac{519x^3}{4} - \frac{5211x^4}{32} - \frac{118179x^5}{800} - \frac{83511x^6}{800} - \frac{2361717x^7}{39200} + O(x^8)\right)\right)
\end{aligned}
$$

Verified OK.

5.6.1 Maple step by step solution

Let's solve

$x^2 y'' + (-3x^2 - x)\, y' + (1 - 6x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(6x-1)y}{x^2} + \frac{(3x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(3x+1)y'}{x} - \frac{(6x-1)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{3x+1}{x}, P_3(x) = -\frac{6x-1}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 1$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' - x(3x+1)\, y' + (1 - 6x)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- > k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)^2 - 3a_{k-1}(k+1+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+r)^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = 1$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(k+r-1)^2 - 3a_{k-1}(k+1+r) = 0$$

- Shift index using $k- >k+1$

  $$a_{k+1}(k+r)^2 - 3a_k(k+r+2) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{3a_k(k+r+2)}{(k+r)^2}$$

- Recursion relation for $r = 1$

  $$a_{k+1} = \frac{3a_k(k+3)}{(k+1)^2}$$

- Solution for $r = 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = \frac{3a_k(k+3)}{(k+1)^2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 75

```
Order:=8;
dsolve(x^2*diff(y(x),x$2)-x*(1+3*x)*diff(y(x),x)+(1-6*x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left((c_2 \ln(x) + c_1)\left(1 + 9x + 27x^2 + 45x^3 + \frac{405}{8}x^4 + \frac{1701}{40}x^5 + \frac{567}{20}x^6 + \frac{2187}{140}x^7 + O(x^8)\right) + \left((-15)x - \frac{261}{4}x^2 - \frac{519}{4}x^3 - \frac{5211}{32}x^4 - \frac{118179}{800}x^5 - \frac{83511}{800}x^6 - \frac{2361717}{39200}x^7 + O(x^8)\right)c_2\right)x$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 150

```
AsymptoticDSolveValue[x^2*y''[x]-x*(1+3*x)*y'[x]+(1-6*x)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1 x\left(\frac{2187x^7}{140} + \frac{567x^6}{20} + \frac{1701x^5}{40} + \frac{405x^4}{8} + 45x^3 + 27x^2 + 9x + 1\right) + c_2\left(x\left(-\frac{2361717x^7}{39200} - \frac{83511x^6}{800} - \frac{118179x^5}{800} - \frac{5211x^4}{32} - \frac{519x^3}{4} - \frac{261x^2}{4} - 15x\right) + x\left(\frac{2187x^7}{140} + \frac{567x^6}{20} + \frac{1701x^5}{40} + \frac{405x^4}{8} + 45x^3 + 27x^2 + 9x + 1\right)\log(x)\right)$$

5.7 problem 7

Internal problem ID [6954]
Internal file name [`OUTPUT/6197_Friday_August_12_2022_11_05_02_PM_58516527/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.6. Indicial Equation with Equal Roots. Exercises page 373
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x(x-1)\,y' + (1-x)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(x^2 - x\right)y' + (1-x)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x-1}{x}$$
$$q(x) = -\frac{x-1}{x^2}$$

Table 99: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x-1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{x-1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + (x^2 - x)\, y' + (1 - x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}\right) + (x^2 - x) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (1 - x) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\right) \\ + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}\right) \\ + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(-1+r)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(-1+r)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(-1+r)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{1+n}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_{n-1}(n+r-1) - a_n(n+r) + a_n - a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n+r-2)}{n^2+2nr+r^2-2n-2r+1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-1}(n-1)}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1-r}{r^2}$$

Which for the root $r = 1$ becomes

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-r}{r^2}$	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-1+r}{r(r+1)^2}$$

Which for the root $r = 1$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-r}{r^2}$	0
a_2	$\frac{-1+r}{r(r+1)^2}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1-r}{r(r+1)(r+2)^2}$$

Which for the root $r = 1$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-r}{r^2}$	0
a_2	$\frac{-1+r}{r(r+1)^2}$	0
a_3	$\frac{1-r}{r(r+1)(r+2)^2}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{-1+r}{(r+2)(r+1)r(r+3)^2}$$

Which for the root $r = 1$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-r}{r^2}$	0
a_2	$\frac{-1+r}{r(r+1)^2}$	0
a_3	$\frac{1-r}{r(r+1)(r+2)^2}$	0
a_4	$\frac{-1+r}{(r+2)(r+1)r(r+3)^2}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1-r}{r(r+1)(r+2)(r+3)(r+4)^2}$$

Which for the root $r = 1$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-r}{r^2}$	0
a_2	$\frac{-1+r}{r(r+1)^2}$	0
a_3	$\frac{1-r}{r(r+1)(r+2)^2}$	0
a_4	$\frac{-1+r}{(r+2)(r+1)r(r+3)^2}$	0
a_5	$\frac{1-r}{r(r+1)(r+2)(r+3)(r+4)^2}$	0

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{-1+r}{(r+4)(r+3)(r+2)(r+1)r(r+5)^2}$$

Which for the root $r = 1$ becomes

$$a_6 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-r}{r^2}$	0
a_2	$\frac{-1+r}{r(r+1)^2}$	0
a_3	$\frac{1-r}{r(r+1)(r+2)^2}$	0
a_4	$\frac{-1+r}{(r+2)(r+1)r(r+3)^2}$	0
a_5	$\frac{1-r}{r(r+1)(r+2)(r+3)(r+4)^2}$	0
a_6	$\frac{-1+r}{(r+4)(r+3)(r+2)(r+1)r(r+5)^2}$	0

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{1-r}{r(r+1)(r+2)(r+3)(r+4)(r+5)(r+6)^2}$$

Which for the root $r = 1$ becomes

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-r}{r^2}$	0
a_2	$\frac{-1+r}{r(r+1)^2}$	0
a_3	$\frac{1-r}{r(r+1)(r+2)^2}$	0
a_4	$\frac{-1+r}{(r+2)(r+1)r(r+3)^2}$	0
a_5	$\frac{1-r}{r(r+1)(r+2)(r+3)(r+4)^2}$	0
a_6	$\frac{-1+r}{(r+4)(r+3)(r+2)(r+1)r(r+5)^2}$	0
a_7	$\frac{1-r}{r(r+1)(r+2)(r+3)(r+4)(r+5)(r+6)^2}$	0

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \ldots\right) \\ &= x\left(1 + O\left(x^8\right)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r=1)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{1-r}{r^2}$	0	$\frac{r-2}{r^3}$	-1
b_2	$\frac{-1+r}{r(r+1)^2}$	0	$\frac{-2r^2+3r+1}{r^2(r+1)^3}$	$\frac{1}{4}$
b_3	$\frac{1-r}{r(r+1)(r+2)^2}$	0	$\frac{3r^3-7r-2}{r^2(r+1)^2(r+2)^3}$	$-\frac{1}{18}$
b_4	$\frac{-1+r}{(r+2)(r+1)r(r+3)^2}$	0	$\frac{-4r^4-10r^3+8r^2+24r+6}{r^2(r+1)^2(r+2)^2(r+3)^3}$	$\frac{1}{96}$
b_5	$\frac{1-r}{r(r+1)(r+2)(r+3)(r+4)^2}$	0	$\frac{5r^5+30r^4+35r^3-60r^2-106r-24}{r^2(r+1)^2(r+2)^2(r+3)^2(r+4)^3}$	$-\frac{1}{600}$
b_6	$\frac{-1+r}{(r+4)(r+3)(r+2)(r+1)r(r+5)^2}$	0	$\frac{-6r^6-63r^5-205r^4-125r^3+427r^2+572r+120}{r^2(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^3}$	$\frac{1}{4320}$
b_7	$\frac{1-r}{r(r+1)(r+2)(r+3)(r+4)(r+5)(r+6)^2}$	0	$\frac{7r^7+112r^6+644r^5+1470r^4+357r^3-3262r^2-3648r-720}{r^2(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2(r+6)^3}$	$-\frac{1}{35280}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 + b_7x^7 + b_8x^8\dots \\
&= x\left(1+O\left(x^8\right)\right)\ln(x) + x\left(-x + \frac{x^2}{4} - \frac{x^3}{18} + \frac{x^4}{96} - \frac{x^5}{600} + \frac{x^6}{4320} - \frac{x^7}{35280} + O\left(x^8\right)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1x\left(1+O\left(x^8\right)\right) + c_2\left(x\left(1+O\left(x^8\right)\right)\ln(x)\right. \\
&\quad \left. + x\left(-x + \frac{x^2}{4} - \frac{x^3}{18} + \frac{x^4}{96} - \frac{x^5}{600} + \frac{x^6}{4320} - \frac{x^7}{35280} + O\left(x^8\right)\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1x\left(1+O\left(x^8\right)\right) \\
&\quad + c_2\left(x\left(1+O\left(x^8\right)\right)\ln(x) + x\left(-x + \frac{x^2}{4} - \frac{x^3}{18} + \frac{x^4}{96} - \frac{x^5}{600} + \frac{x^6}{4320} - \frac{x^7}{35280} + O\left(x^8\right)\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1 x\left(1+O\left(x^8\right)\right) + c_2\left(x\left(1+O\left(x^8\right)\right)\ln\left(x\right) + x\left(-x+\frac{x^2}{4}-\frac{x^3}{18}+\frac{x^4}{96}-\frac{x^5}{600}+\frac{x^6}{4320}-\frac{x^7}{35280}+O\left(x^8\right)\right)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x\left(1+O\left(x^8\right)\right) + c_2\left(x\left(1+O\left(x^8\right)\right)\ln\left(x\right) + x\left(-x+\frac{x^2}{4}-\frac{x^3}{18}+\frac{x^4}{96}-\frac{x^5}{600}+\frac{x^6}{4320}-\frac{x^7}{35280}+O\left(x^8\right)\right)\right)$$

Verified OK.

5.7.1 Maple step by step solution

Let's solve

$x^2y'' + (x^2 - x)\,y' + (1 - x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x-1)y}{x^2} - \frac{(x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x-1)y'}{x} - \frac{(x-1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{x-1}{x}, P_3(x) = -\frac{x-1}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 1$

  ◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$x^2 y'' + x(x-1)\, y' + (1-x)\, y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  ◦ Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0(-1+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k (k+r-1)^2 + a_{k-1}(k-2+r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(-1+r)^2 = 0$

- Values of r that satisfy the indicial equation

$r = 1$

- Each term in the series must be 0, giving the recursion relation

$a_k (k+r-1)^2 + a_{k-1}(k-2+r) = 0$

- Shift index using $k->k+1$

  $a_{k+1}(k+r)^2+a_k(k+r-1)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{a_k(k+r-1)}{(k+r)^2}$
- Recursion relation for $r=1$

  $a_{k+1}=-\frac{a_k k}{(k+1)^2}$
- Solution for $r=1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1}=-\frac{a_k k}{(k+1)^2}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 47

```
Order:=8;
dsolve(x^2*diff(y(x),x$2)+x*(x-1)*diff(y(x),x)+(1-x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\left((c_2\ln(x)+c_1)\left(1+\mathrm{O}\left(x^8\right)\right)+\left(-x+\frac{1}{4}x^2-\frac{1}{18}x^3+\frac{1}{96}x^4-\frac{1}{600}x^5+\frac{1}{4320}x^6-\frac{1}{35280}x^7+\mathrm{O}\left(x^8\right)\right)c_2\right)x$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 64

```
AsymptoticDSolveValue[x^2*y''[x]+x*(x-1)*y'[x]+(1-x)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(x\left(-\frac{x^7}{35280}+\frac{x^6}{4320}-\frac{x^5}{600}+\frac{x^4}{96}-\frac{x^3}{18}+\frac{x^2}{4}-x\right)+x\log(x)\right)+c_1x$$

5.8 problem 8

Internal problem ID [6955]
Internal file name [`OUTPUT/6198_Friday_August_12_2022_11_05_04_PM_84918379/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.6. Indicial Equation with Equal Roots. Exercises page 373
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x(x-2)\,y''+2(x-1)\,y'-2y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^2-2x\right)y''+(2x-2)\,y'-2y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{2x-2}{x\,(x-2)}$$
$$q(x)=-\frac{2}{x\,(x-2)}$$

Table 101: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2x-2}{x(x-2)}$	
singularity	type
$x = 0$	"regular"
$x = 2$	"regular"

$q(x) = -\frac{2}{x(x-2)}$	
singularity	type
$x = 0$	"regular"
$x = 2$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 2, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x(x-2)\, y'' + (2x-2)\, y' - 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\, (n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x(x-2) \left(\sum_{n=0}^{\infty} (n+r)\, (n+r-1)\, a_n x^{n+r-2}\right) + (2x-2) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2x^{n+r-1} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)\right) + \sum_{n=0}^{\infty} \left(-2(n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r) = \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-2x^{n+r-1} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-2(n+r) a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$-2x^{n+r-1} a_n(n+r)(n+r-1) - 2(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$-2x^{-1+r} a_0 r(-1+r) - 2r a_0 x^{-1+r} = 0$$

Or

$$\left(-2x^{-1+r} r(-1+r) - 2r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$-2x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$-2r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$-2x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} a_{n-1}(n+r-1)(n+r-2) - 2a_n(n+r)(n+r-1) \\ + 2a_{n-1}(n+r-1) - 2a_n(n+r) - 2a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(n^2+2nr+r^2-n-r-2)}{2n^2+4nr+2r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-1}(n^2-n-2)}{2n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{r^2+r-2}{2\left(r+1\right)^2}$$

Which for the root $r = 0$ becomes

$$a_1 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2+r-2}{2(r+1)^2}$	-1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(r+3)\, r(-1+r)}{4\left(r+2\right)\left(r+1\right)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2+r-2}{2(r+1)^2}$	-1
a_2	$\frac{(r+3)r(-1+r)}{4(r+2)(r+1)^2}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{(-1+r)\, r(r+4)}{8\,(r+3)\,(r+1)\,(r+2)}$$

Which for the root $r = 0$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2+r-2}{2(r+1)^2}$	-1
a_2	$\frac{(r+3)r(-1+r)}{4(r+2)(r+1)^2}$	0
a_3	$\frac{(-1+r)r(r+4)}{8(r+3)(r+1)(r+2)}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r(-1+r)\,(r+5)}{16\,(r+4)\,(r+1)\,(r+3)}$$

Which for the root $r = 0$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2+r-2}{2(r+1)^2}$	-1
a_2	$\frac{(r+3)r(-1+r)}{4(r+2)(r+1)^2}$	0
a_3	$\frac{(-1+r)r(r+4)}{8(r+3)(r+1)(r+2)}$	0
a_4	$\frac{r(-1+r)(r+5)}{16(r+4)(r+1)(r+3)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{(-1+r)\, r(r+6)}{32\,(r+5)\,(r+1)\,(r+4)}$$

Which for the root $r = 0$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2+r-2}{2(r+1)^2}$	-1
a_2	$\frac{(r+3)r(-1+r)}{4(r+2)(r+1)^2}$	0
a_3	$\frac{(-1+r)r(r+4)}{8(r+3)(r+1)(r+2)}$	0
a_4	$\frac{r(-1+r)(r+5)}{16(r+4)(r+1)(r+3)}$	0
a_5	$\frac{(-1+r)r(r+6)}{32(r+5)(r+1)(r+4)}$	0

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{r(-1+r)(r+7)}{64(r+6)(r+1)(r+5)}$$

Which for the root $r = 0$ becomes

$$a_6 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2+r-2}{2(r+1)^2}$	-1
a_2	$\frac{(r+3)r(-1+r)}{4(r+2)(r+1)^2}$	0
a_3	$\frac{(-1+r)r(r+4)}{8(r+3)(r+1)(r+2)}$	0
a_4	$\frac{r(-1+r)(r+5)}{16(r+4)(r+1)(r+3)}$	0
a_5	$\frac{(-1+r)r(r+6)}{32(r+5)(r+1)(r+4)}$	0
a_6	$\frac{r(-1+r)(r+7)}{64(r+6)(r+1)(r+5)}$	0

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{(-1+r)r(r+8)}{128(r+7)(r+1)(r+6)}$$

Which for the root $r = 0$ becomes

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2+r-2}{2(r+1)^2}$	-1
a_2	$\frac{(r+3)r(-1+r)}{4(r+2)(r+1)^2}$	0
a_3	$\frac{(-1+r)r(r+4)}{8(r+3)(r+1)(r+2)}$	0
a_4	$\frac{r(-1+r)(r+5)}{16(r+4)(r+1)(r+3)}$	0
a_5	$\frac{(-1+r)r(r+6)}{32(r+5)(r+1)(r+4)}$	0
a_6	$\frac{r(-1+r)(r+7)}{64(r+6)(r+1)(r+5)}$	0
a_7	$\frac{(-1+r)r(r+8)}{128(r+7)(r+1)(r+6)}$	0

Using the above table, then the first solution $y_1(x)$ becomes

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \ldots \\ &= 1 - x + O\left(x^8\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r=0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{r^2+r-2}{2(r+1)^2}$	-1	$\frac{r+5}{2(r+1)^3}$	$\frac{5}{2}$
b_2	$\frac{(r+3)r(-1+r)}{4(r+2)(r+1)^2}$	0	$\frac{r^3+7r^2+7r-3}{2(r+2)^2(r+1)^3}$	$-\frac{3}{8}$
b_3	$\frac{(-1+r)r(r+4)}{8(r+3)(r+1)(r+2)}$	0	$\frac{-3+\frac{3}{8}r^4+\frac{15}{4}r^3+\frac{9}{2}r+\frac{75}{8}r^2}{(r+3)^2(r+1)^2(r+2)^2}$	$-\frac{1}{12}$
b_4	$\frac{r(-1+r)(r+5)}{16(r+4)(r+1)(r+3)}$	0	$\frac{r^4+12r^3+38r^2+24r-15}{4(r+4)^2(r+1)^2(r+3)^2}$	$-\frac{5}{192}$
b_5	$\frac{(-1+r)r(r+6)}{32(r+5)(r+1)(r+4)}$	0	$\frac{\frac{5}{32}r^4+\frac{35}{16}r^3+\frac{25}{4}r+\frac{265}{32}r^2-\frac{15}{4}}{(r+5)^2(r+1)^2(r+4)^2}$	$-\frac{3}{320}$
b_6	$\frac{r(-1+r)(r+7)}{64(r+6)(r+1)(r+5)}$	0	$\frac{\frac{3}{32}r^4+\frac{3}{2}r^3+\frac{45}{8}r-\frac{105}{32}+\frac{105}{16}r^2}{(r+6)^2(r+1)^2(r+5)^2}$	$-\frac{7}{1920}$
b_7	$\frac{(-1+r)r(r+8)}{128(r+7)(r+1)(r+6)}$	0	$\frac{-\frac{21}{8}+\frac{7}{128}r^4+\frac{63}{64}r^3+\frac{147}{32}r+\frac{623}{128}r^2}{(r+7)^2(r+1)^2(r+6)^2}$	$-\frac{1}{672}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 + b_7x^7 + b_8x^8\ldots \\
&= \left(1 - x + O(x^8)\right)\ln(x) + \frac{5x}{2} - \frac{3x^2}{8} - \frac{x^3}{12} - \frac{5x^4}{192} - \frac{3x^5}{320} - \frac{7x^6}{1920} - \frac{x^7}{672} + O(x^8)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1\left(1 - x + O(x^8)\right) \\
&\quad + c_2\left(\left(1-x+O(x^8)\right)\ln(x) + \frac{5x}{2} - \frac{3x^2}{8} - \frac{x^3}{12} - \frac{5x^4}{192} - \frac{3x^5}{320} - \frac{7x^6}{1920} - \frac{x^7}{672} + O(x^8)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1 - x + O(x^8)\right) \\
&\quad + c_2\left(\left(1-x+O(x^8)\right)\ln(x) + \frac{5x}{2} - \frac{3x^2}{8} - \frac{x^3}{12} - \frac{5x^4}{192} - \frac{3x^5}{320} - \frac{7x^6}{1920} - \frac{x^7}{672} + O(x^8)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1\left(1-x+O\left(x^8\right)\right)+c_2\left(\left(1-x+O\left(x^8\right)\right)\ln\left(x\right)+\frac{5x}{2}-\frac{3x^2}{8}-\frac{x^3}{12}-\frac{5x^4}{192}-\frac{3x^5}{320}-\frac{7x^6}{1920}-\frac{x^7}{672}+O\left(x^8\right)\right) \tag{1}$$

Verification of solutions

$$y = c_1\left(1-x+O\left(x^8\right)\right)+c_2\left(\left(1-x+O\left(x^8\right)\right)\ln\left(x\right)+\frac{5x}{2}-\frac{3x^2}{8}-\frac{x^3}{12}-\frac{5x^4}{192}-\frac{3x^5}{320}-\frac{7x^6}{1920}-\frac{x^7}{672}+O\left(x^8\right)\right)$$

Verified OK.

5.8.1 Maple step by step solution

Let's solve

$x(x-2)\,y''+(2x-2)\,y'-2y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{2y}{x(x-2)}-\frac{2(x-1)y'}{x(x-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2(x-1)y'}{x(x-2)}-\frac{2y}{x(x-2)}=0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{2(x-1)}{x(x-2)},P_3(x)=-\frac{2}{x(x-2)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\Big|_{x=0}=1$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=0$

  - $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$x(x-2)\,y''+(2x-2)\,y'-2y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  ◦ Shift index using $k- >k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

Rewrite ODE with series expansions

$-2a_0r^2x^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)^2+a_k(k+r+2)\,(k+r-1)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-2r^2=0$

- Values of r that satisfy the indicial equation

$r=0$

- Each term in the series must be 0, giving the recursion relation

$-2a_{k+1}(k+1)^2+a_k(k+2)\,(k-1)=0$

- Recursion relation that defines series solution to ODE

$a_{k+1}=\frac{a_k(k+2)(k-1)}{2(k+1)^2}$

- Recursion relation for $r = 0$; series terminates at $k = 1$

$a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$

- Apply recursion relation for $k = 0$

  $a_1 = -a_0$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y = a_0 \cdot (1 - x)$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 47

```
Order:=8;
dsolve(x*(x-2)*diff(y(x),x$2)+2*(x-1)*diff(y(x),x)-2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln(x) + c_1)\left(1 - x + \mathrm{O}\left(x^8\right)\right) + \left(\frac{5}{2}x - \frac{3}{8}x^2 - \frac{1}{12}x^3 - \frac{5}{192}x^4 - \frac{3}{320}x^5 - \frac{7}{1920}x^6 - \frac{1}{672}x^7 + \mathrm{O}\left(x^8\right)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 71

```
AsymptoticDSolveValue[x*(x-2)*y''[x]+2*(x-1)*y'[x]-2*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2 \left(-\frac{x^7}{672} - \frac{7x^6}{1920} - \frac{3x^5}{320} - \frac{5x^4}{192} - \frac{x^3}{12} - \frac{3x^2}{8} + \frac{5x}{2} + (1-x)\log(x) \right) + c_1(1-x)$$

5.9 problem 9

Internal problem ID [6956]
Internal file name [`OUTPUT/6199_Friday_August_12_2022_11_05_06_PM_39063778/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.6. Indicial Equation with Equal Roots. Exercises page 373
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x(x-2)\,y''+2(x-1)\,y'-2y=0}$$

With the expansion point for the power series method at $x=2$.

The ode does not have its expansion point at $x=0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t=x-2$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left((t+2)^2-2t-4\right)\left(\frac{d^2}{dt^2}y(t)\right)+(2t+2)\left(\frac{d}{dt}y(t)\right)-2y(t)=0$$

With its expansion point and initial conditions now at $t=0$. The transformed ODE is now solved. The type of the expansion point is first determined. This is done on the

homogeneous part of the ODE.

$$\left(t^2+2t\right)\left(\frac{d^2}{dt^2}y(t)\right)+(2t+2)\left(\frac{d}{dt}y(t)\right)-2y(t)=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$\frac{d^2}{dt^2}y(t)+p(t)\frac{d}{dt}y(t)+q(t)y(t)=0$$

Where

$$p(t)=\frac{2t+2}{t\,(t+2)}$$
$$q(t)=-\frac{2}{t\,(t+2)}$$

Table 103: Table $p(t), q(t)$ singularites.

$p(t)=\frac{2t+2}{t(t+2)}$	
singularity	type
$t=-2$	"regular"
$t=0$	"regular"

$q(t)=-\frac{2}{t(t+2)}$	
singularity	type
$t=-2$	"regular"
$t=0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-2, 0, \infty]$

Irregular singular points : []

Since $t=0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$\left(\frac{d^2}{dt^2}y(t)\right)t(t+2)+(2t+2)\left(\frac{d}{dt}y(t)\right)-2y(t)=0$$

Let the solution be represented as Frobenius power series of the form

$$y(t)=\sum_{n=0}^{\infty}a_n t^{n+r}$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=0}^{\infty} (n+r)\, a_n t^{n+r-1}$$

$$\frac{d^2}{dt^2}y(t) = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n t^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n t^{n+r-2}\right) t(t+2) \\ + (2t+2)\left(\sum_{n=0}^{\infty} (n+r)\, a_n t^{n+r-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n t^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} t^{n+r} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2t^{n+r-1} a_n (n+r)\,(n+r-1)\right) \\ + \left(\sum_{n=0}^{\infty} 2t^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2(n+r)\, a_n t^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-2a_n t^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of t be $n+r-1$ in each summation term. Going over each summation term above with power of t in it which is not already t^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} t^{n+r} a_n (n+r)\,(n+r-1) = \sum_{n=1}^{\infty} a_{n-1} (n+r-1)\,(n+r-2)\, t^{n+r-1}$$

$$\sum_{n=0}^{\infty} 2t^{n+r} a_n (n+r) = \sum_{n=1}^{\infty} 2a_{n-1} (n+r-1)\, t^{n+r-1}$$

$$\sum_{n=0}^{\infty} \left(-2a_n t^{n+r}\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1} t^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of t are the same and equal to $n+r-1$.

$$
\begin{aligned}
&\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, t^{n+r-1}\right) \\
&+\left(\sum_{n=0}^{\infty} 2t^{n+r-1} a_n (n+r)(n+r-1)\right)+\left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, t^{n+r-1}\right) \\
&+\left(\sum_{n=0}^{\infty} 2(n+r)\, a_n t^{n+r-1}\right)+\sum_{n=1}^{\infty}\left(-2a_{n-1} t^{n+r-1}\right)=0
\end{aligned}
\tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
2t^{n+r-1} a_n (n+r)(n+r-1)+2(n+r)\, a_n t^{n+r-1}=0
$$

When $n=0$ the above becomes

$$
2t^{-1+r} a_0 r(-1+r)+2r a_0 t^{-1+r}=0
$$

Or

$$
\left(2t^{-1+r} r(-1+r)+2r\, t^{-1+r}\right) a_0=0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
2t^{-1+r} r^2=0
$$

Since the above is true for all t then the indicial equation becomes

$$
2r^2=0
$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= 0 \\
r_2 &= 0
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
2t^{-1+r} r^2=0
$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$
y_1(t)=\sum_{n=0}^{\infty} a_n t^{n+r} \tag{1A}
$$

Now the second solution y_2 is found using

$$y_2(t) = y_1(t)\ln(t) + \left(\sum_{n=1}^{\infty} b_n t^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(t)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &a_{n-1}(n+r-1)(n+r-2) + 2a_n(n+r)(n+r-1) \\ &+ 2a_{n-1}(n+r-1) + 2a_n(n+r) - 2a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n^2 + 2nr + r^2 - n - r - 2)}{2(n^2 + 2nr + r^2)} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{a_{n-1}(n^2 - n - 2)}{2n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-r^2 - r + 2}{2(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_1 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-r+2}{2(r+1)^2}$	1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(r+3)\,r(-1+r)}{4\,(r+2)\,(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-r+2}{2(r+1)^2}$	1
a_2	$\frac{(r+3)r(-1+r)}{4(r+2)(r+1)^2}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{(-1+r)\,r(r+4)}{8\,(r+3)\,(r+1)\,(r+2)}$$

Which for the root $r = 0$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-r+2}{2(r+1)^2}$	1
a_2	$\frac{(r+3)r(-1+r)}{4(r+2)(r+1)^2}$	0
a_3	$-\frac{(-1+r)r(r+4)}{8(r+3)(r+1)(r+2)}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r(-1+r)(r+5)}{16(r+4)(r+1)(r+3)}$$

Which for the root $r = 0$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-r+2}{2(r+1)^2}$	1
a_2	$\frac{(r+3)r(-1+r)}{4(r+2)(r+1)^2}$	0
a_3	$-\frac{(-1+r)r(r+4)}{8(r+3)(r+1)(r+2)}$	0
a_4	$\frac{r(-1+r)(r+5)}{16(r+4)(r+1)(r+3)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(-1+r)r(r+6)}{32(r+5)(r+1)(r+4)}$$

Which for the root $r = 0$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-r+2}{2(r+1)^2}$	1
a_2	$\frac{(r+3)r(-1+r)}{4(r+2)(r+1)^2}$	0
a_3	$-\frac{(-1+r)r(r+4)}{8(r+3)(r+1)(r+2)}$	0
a_4	$\frac{r(-1+r)(r+5)}{16(r+4)(r+1)(r+3)}$	0
a_5	$-\frac{(-1+r)r(r+6)}{32(r+5)(r+1)(r+4)}$	0

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{r(-1+r)(r+7)}{64(r+6)(r+1)(r+5)}$$

Which for the root $r = 0$ becomes

$$a_6 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-r+2}{2(r+1)^2}$	1
a_2	$\frac{(r+3)r(-1+r)}{4(r+2)(r+1)^2}$	0
a_3	$-\frac{(-1+r)r(r+4)}{8(r+3)(r+1)(r+2)}$	0
a_4	$\frac{r(-1+r)(r+5)}{16(r+4)(r+1)(r+3)}$	0
a_5	$-\frac{(-1+r)r(r+6)}{32(r+5)(r+1)(r+4)}$	0
a_6	$\frac{r(-1+r)(r+7)}{64(r+6)(r+1)(r+5)}$	0

For $n = 7$, using the above recursive equation gives

$$a_7 = -\frac{(-1+r)\, r(r+8)}{128\,(r+7)\,(r+1)\,(r+6)}$$

Which for the root $r = 0$ becomes

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-r+2}{2(r+1)^2}$	1
a_2	$\frac{(r+3)r(-1+r)}{4(r+2)(r+1)^2}$	0
a_3	$-\frac{(-1+r)r(r+4)}{8(r+3)(r+1)(r+2)}$	0
a_4	$\frac{r(-1+r)(r+5)}{16(r+4)(r+1)(r+3)}$	0
a_5	$-\frac{(-1+r)r(r+6)}{32(r+5)(r+1)(r+4)}$	0
a_6	$\frac{r(-1+r)(r+7)}{64(r+6)(r+1)(r+5)}$	0
a_7	$-\frac{(-1+r)r(r+8)}{128(r+7)(r+1)(r+6)}$	0

Using the above table, then the first solution $y_1(t)$ becomes

$$
\begin{aligned}
y_1(t) &= a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + a_7 t^7 + a_8 t^8 \dots \\
&= t + 1 + O(t^8)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(t) = y_1(t) \ln(t) + \left(\sum_{n=1}^{\infty} b_n t^{n+r} \right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = 0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{-r^2-r+2}{2(r+1)^2}$	1	$\frac{-r-5}{2(r+1)^3}$	$-\frac{5}{2}$
b_2	$\frac{(r+3)r(-1+r)}{4(r+2)(r+1)^2}$	0	$\frac{r^3+7r^2+7r-3}{2(r+2)^2(r+1)^3}$	$-\frac{3}{8}$
b_3	$-\frac{(-1+r)r(r+4)}{8(r+3)(r+1)(r+2)}$	0	$\frac{3-\frac{3}{8}r^4-\frac{15}{4}r^3-\frac{75}{8}r^2-\frac{9}{2}r}{(r+3)^2(r+1)^2(r+2)^2}$	$\frac{1}{12}$
b_4	$\frac{r(-1+r)(r+5)}{16(r+4)(r+1)(r+3)}$	0	$\frac{r^4+12r^3+38r^2+24r-15}{4(r+4)^2(r+1)^2(r+3)^2}$	$-\frac{5}{192}$
b_5	$-\frac{(-1+r)r(r+6)}{32(r+5)(r+1)(r+4)}$	0	$\frac{-\frac{5}{32}r^4-\frac{35}{16}r^3-\frac{265}{32}r^2-\frac{25}{4}r+\frac{15}{4}}{(r+5)^2(r+1)^2(r+4)^2}$	$\frac{3}{320}$
b_6	$\frac{r(-1+r)(r+7)}{64(r+6)(r+1)(r+5)}$	0	$\frac{\frac{3}{32}r^4+\frac{3}{2}r^3+\frac{105}{16}r^2+\frac{45}{8}r-\frac{105}{32}}{(r+6)^2(r+1)^2(r+5)^2}$	$-\frac{7}{1920}$
b_7	$-\frac{(-1+r)r(r+8)}{128(r+7)(r+1)(r+6)}$	0	$\frac{\frac{21}{8}-\frac{7}{128}r^4-\frac{63}{64}r^3-\frac{623}{128}r^2-\frac{147}{32}r}{(r+7)^2(r+1)^2(r+6)^2}$	$\frac{1}{672}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(t) &= y_1(t) \ln(t) + b_0 + b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5 + b_6 t^6 + b_7 t^7 + b_8 t^8 \dots \\
&= \left(t + 1 + O(t^8)\right) \ln(t) - \frac{5t}{2} - \frac{3t^2}{8} + \frac{t^3}{12} - \frac{5t^4}{192} + \frac{3t^5}{320} - \frac{7t^6}{1920} + \frac{t^7}{672} + O(t^8)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(t) &= c_1 y_1(t) + c_2 y_2(t) \\
&= c_1\left(t+1+O(t^8)\right) \\
&\quad + c_2\left(\left(t+1+O(t^8)\right)\ln(t) - \frac{5t}{2} - \frac{3t^2}{8} + \frac{t^3}{12} - \frac{5t^4}{192} + \frac{3t^5}{320} - \frac{7t^6}{1920} + \frac{t^7}{672} + O(t^8)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y(t) &= y_h \\
&= c_1\left(t+1+O(t^8)\right) \\
&\quad + c_2\left(\left(t+1+O(t^8)\right)\ln(t) - \frac{5t}{2} - \frac{3t^2}{8} + \frac{t^3}{12} - \frac{5t^4}{192} + \frac{3t^5}{320} - \frac{7t^6}{1920} + \frac{t^7}{672} + O(t^8)\right)
\end{aligned}
$$

Replacing t in the above with the original independent variable $x$susing $t = x - 2$ results in

$$
\begin{aligned}
y = c_1\left(x-1+O\left((x-2)^8\right)\right) + c_2\Bigg(&\left(x-1+O\left((x-2)^8\right)\right)\ln(x-2) - \frac{5x}{2} + 5 - \frac{3(x-2)^2}{8} \\
&+ \frac{(x-2)^3}{12} - \frac{5(x-2)^4}{192} + \frac{3(x-2)^5}{320} - \frac{7(x-2)^6}{1920} + \frac{(x-2)^7}{672} + O\left((x-2)^8\right)\Bigg)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
\begin{aligned}
y = c_1\left(x-1+O\left((x-2)^8\right)\right) + c_2\Bigg(&\left(x-1+O\left((x-2)^8\right)\right)\ln(x-2) - \frac{5x}{2} + 5 - \frac{3(x-2)^2}{8} \\
&+ \frac{(x-2)^3}{12} - \frac{5(x-2)^4}{192} + \frac{3(x-2)^5}{320} - \frac{7(x-2)^6}{1920} + \frac{(x-2)^7}{672} + O\left((x-2)^8\right)\Bigg)
\end{aligned} \tag{1}
$$

<u>Verification of solutions</u>

$$
\begin{aligned}
y = c_1\left(x-1+O\left((x-2)^8\right)\right) + c_2\Bigg(&\left(x-1+O\left((x-2)^8\right)\right)\ln(x-2) - \frac{5x}{2} + 5 - \frac{3(x-2)^2}{8} \\
&+ \frac{(x-2)^3}{12} - \frac{5(x-2)^4}{192} + \frac{3(x-2)^5}{320} - \frac{7(x-2)^6}{1920} + \frac{(x-2)^7}{672} + O\left((x-2)^8\right)\Bigg)
\end{aligned}
$$

Verified OK.

5.9.1 Maple step by step solution

Let's solve

$(x^2 - 2x)\, y'' + (2x - 2)\, y' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2y}{x(x-2)} - \frac{2(x-1)y'}{x(x-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2(x-1)y'}{x(x-2)} - \frac{2y}{x(x-2)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2(x-1)}{x(x-2)}, P_3(x) = -\frac{2}{x(x-2)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x(x - 2)\, y'' + (2x - 2)\, y' - 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y^{\prime}=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
- Convert $x^{m}\cdot y^{\prime\prime}$ to series expansion for $m=1..2$

  $x^{m}\cdot y^{\prime\prime}=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r-2+m}$
- Shift index using $k->k+2-m$

  $x^{m}\cdot y^{\prime\prime}=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-2a_{0}r^{2}x^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)^{2}+a_{k}(k+r+2)\,(k+r-1)\right) x^{k+r}\right)=0$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $-2r^{2}=0$
- Values of r that satisfy the indicial equation

  $r=0$
- Each term in the series must be 0, giving the recursion relation

  $-2a_{k+1}(k+1)^{2}+a_{k}(k+2)\,(k-1)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_{k}(k+2)(k-1)}{2(k+1)^{2}}$
- Recursion relation for $r=0$; series terminates at $k=1$

  $a_{k+1}=\frac{a_{k}(k+2)(k-1)}{2(k+1)^{2}}$
- Apply recursion relation for $k=0$

  $a_{1}=-a_{0}$
- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

  $y=a_{0}\cdot(1-x)$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 49

```
Order:=8;
dsolve(x*(x-2)*diff(y(x),x$2)+2*(x-1)*diff(y(x),x)-2*y(x)=0,y(x),type='series',x=2);
```

$$
\begin{aligned}
y(x) = {} & (\ln(-2+x)\, c_2 + c_1)\left(1 + (-2+x) + \mathrm{O}\left((-2+x)^8\right)\right) \\
& + \left(-\frac{5}{2}(-2+x) - \frac{3}{8}(-2+x)^2 + \frac{1}{12}(-2+x)^3 - \frac{5}{192}(-2+x)^4 + \frac{3}{320}(-2+x)^5 \right. \\
& \left. - \frac{7}{1920}(-2+x)^6 + \frac{1}{672}(-2+x)^7 + \mathrm{O}\left((-2+x)^8\right)\right) c_2
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 90

```
AsymptoticDSolveValue[x*(x-2)*y''[x]+2*(x-1)*y'[x]-2*y[x]==0,y[x],{x,2,7}]
```

$$
\begin{aligned}
y(x) \to {} & c_1(x-1) + c_2\left(\frac{1}{672}(x-2)^7 - \frac{7(x-2)^6}{1920} + \frac{3}{320}(x-2)^5 - \frac{5}{192}(x-2)^4 \right. \\
& \left. + \frac{1}{12}(x-2)^3 - \frac{3}{8}(x-2)^2 - 2(x-2) + \frac{2-x}{2} + (x-1)\log(x-2)\right)
\end{aligned}
$$

5.10 problem 10

Internal problem ID [6957]
Internal file name [`OUTPUT/6200_Friday_August_12_2022_11_05_09_PM_33212483/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.6. Indicial Equation with Equal Roots. Exercises page 373
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4(x-4)^2 y''+(x-4)(x-8) y'+yx=0}$$

With the expansion point for the power series method at $x=4$.

The ode does not have its expansion point at $x=0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t=x-4$$

The ode is converted to be in terms of the new independent variable t. This results in

$$4\left(\frac{d^2}{dt^2}y(t)\right)t^2+\left((t+4)^2-12t-16\right)\left(\frac{d}{dt}y(t)\right)+y(t)(t+4)=0$$

With its expansion point and initial conditions now at $t=0$. The transformed ODE is now solved. The type of the expansion point is first determined. This is done on the

homogeneous part of the ODE.

$$4\left(\frac{d^2}{dt^2}y(t)\right)t^2+\left(t^2-4t\right)\left(\frac{d}{dt}y(t)\right)+y(t)\,(t+4)=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$\frac{d^2}{dt^2}y(t)+p(t)\frac{d}{dt}y(t)+q(t)y(t)=0$$

Where

$$p(t)=\frac{t-4}{4t}$$
$$q(t)=\frac{t+4}{4t^2}$$

Table 105: Table $p(t), q(t)$ singularites.

$p(t)=\frac{t-4}{4t}$	
singularity	type
$t=0$	"regular"

$q(t)=\frac{t+4}{4t^2}$	
singularity	type
$t=0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $t=0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4\left(\frac{d^2}{dt^2}y(t)\right)t^2+\left(t^2-4t\right)\left(\frac{d}{dt}y(t)\right)+y(t)\,(t+4)=0$$

Let the solution be represented as Frobenius power series of the form

$$y(t)=\sum_{n=0}^{\infty}a_n t^{n+r}$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=0}^{\infty} (n+r)\, a_n t^{n+r-1}$$

$$\frac{d^2}{dt^2}y(t) = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n t^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &4\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n t^{n+r-2}\right) t^2 \\ &+ \left(t^2-4t\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n t^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n t^{n+r}\right)(t+4) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\begin{aligned} &\left(\sum_{n=0}^{\infty} 4t^{n+r} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} t^{1+n+r} a_n (n+r)\right) \\ &+ \sum_{n=0}^{\infty} \left(-4t^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} t^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} 4a_n t^{n+r}\right) = 0 \end{aligned} \tag{2A}$$

The next step is to make all powers of t be $n+r$ in each summation term. Going over each summation term above with power of t in it which is not already t^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} t^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1} (n+r-1)\, t^{n+r}$$

$$\sum_{n=0}^{\infty} t^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} t^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of t are the same and equal to $n+r$.

$$\begin{aligned} &\left(\sum_{n=0}^{\infty} 4t^{n+r} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} (n+r-1)\, t^{n+r}\right) \\ &+ \sum_{n=0}^{\infty} \left(-4t^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} t^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4a_n t^{n+r}\right) = 0 \end{aligned} \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$4t^{n+r}a_n(n+r)(n+r-1) - 4t^{n+r}a_n(n+r) + 4a_n t^{n+r} = 0$$

When $n = 0$ the above becomes

$$4t^r a_0 r(-1+r) - 4t^r a_0 r + 4a_0 t^r = 0$$

Or

$$(4t^r r(-1+r) - 4t^r r + 4t^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$4(-1+r)^2\, t^r = 0$$

Since the above is true for all t then the indicial equation becomes

$$4(-1+r)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$4(-1+r)^2\, t^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(t) = \sum_{n=0}^{\infty} a_n t^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(t) = y_1(t)\ln(t) + \left(\sum_{n=1}^{\infty} b_n t^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of

integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 1$, Eqs (1A,1B) become

$$y_1(t) = \sum_{n=0}^{\infty} a_n t^{1+n}$$

$$y_2(t) = y_1(t) \ln(t) + \left(\sum_{n=1}^{\infty} b_n t^{1+n}\right)$$

We start by finding the first solution $y_1(t)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \le n$ the recursive equation is

$$4a_n(n+r)(n+r-1) + a_{n-1}(n+r-1) - 4a_n(n+r) + a_{n-1} + 4a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n+r)}{4(n^2 + 2nr + r^2 - 2n - 2r + 1)} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-1}(1+n)}{4n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-r-1}{4r^2}$$

Which for the root $r = 1$ becomes

$$a_1 = -\frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r-1}{4r^2}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{2+r}{16(1+r)r^2}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{3}{32}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r-1}{4r^2}$	$-\frac{1}{2}$
a_2	$\frac{2+r}{16(1+r)r^2}$	$\frac{3}{32}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-3-r}{64(2+r)(1+r)r^2}$$

Which for the root $r = 1$ becomes

$$a_3 = -\frac{1}{96}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r-1}{4r^2}$	$-\frac{1}{2}$
a_2	$\frac{2+r}{16(1+r)r^2}$	$\frac{3}{32}$
a_3	$\frac{-3-r}{64(2+r)(1+r)r^2}$	$-\frac{1}{96}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4+r}{256(3+r)(2+r)(1+r)r^2}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{5}{6144}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r-1}{4r^2}$	$-\frac{1}{2}$
a_2	$\frac{2+r}{16(1+r)r^2}$	$\frac{3}{32}$
a_3	$\frac{-3-r}{64(2+r)(1+r)r^2}$	$-\frac{1}{96}$
a_4	$\frac{4+r}{256(3+r)(2+r)(1+r)r^2}$	$\frac{5}{6144}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-5-r}{1024\,(4+r)\,(3+r)\,(2+r)\,(1+r)\,r^2}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{1}{20480}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r-1}{4r^2}$	$-\frac{1}{2}$
a_2	$\frac{2+r}{16(1+r)r^2}$	$\frac{3}{32}$
a_3	$\frac{-3-r}{64(2+r)(1+r)r^2}$	$-\frac{1}{96}$
a_4	$\frac{4+r}{256(3+r)(2+r)(1+r)r^2}$	$\frac{5}{6144}$
a_5	$\frac{-5-r}{1024(4+r)(3+r)(2+r)(1+r)r^2}$	$-\frac{1}{20480}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{6+r}{4096\,(5+r)\,(4+r)\,(3+r)\,(2+r)\,(1+r)\,r^2}$$

Which for the root $r = 1$ becomes

$$a_6 = \frac{7}{2949120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r-1}{4r^2}$	$-\frac{1}{2}$
a_2	$\frac{2+r}{16(1+r)r^2}$	$\frac{3}{32}$
a_3	$\frac{-3-r}{64(2+r)(1+r)r^2}$	$-\frac{1}{96}$
a_4	$\frac{4+r}{256(3+r)(2+r)(1+r)r^2}$	$\frac{5}{6144}$
a_5	$\frac{-5-r}{1024(4+r)(3+r)(2+r)(1+r)r^2}$	$-\frac{1}{20480}$
a_6	$\frac{6+r}{4096(5+r)(4+r)(3+r)(2+r)(1+r)r^2}$	$\frac{7}{2949120}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{-7-r}{16384\,(6+r)\,(5+r)\,(4+r)\,(3+r)\,(2+r)\,(1+r)\,r^2}$$

Which for the root $r = 1$ becomes

$$a_7 = -\frac{1}{10321920}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r-1}{4r^2}$	$-\frac{1}{2}$
a_2	$\frac{2+r}{16(1+r)r^2}$	$\frac{3}{32}$
a_3	$\frac{-3-r}{64(2+r)(1+r)r^2}$	$-\frac{1}{96}$
a_4	$\frac{4+r}{256(3+r)(2+r)(1+r)r^2}$	$\frac{5}{6144}$
a_5	$\frac{-5-r}{1024(4+r)(3+r)(2+r)(1+r)r^2}$	$-\frac{1}{20480}$
a_6	$\frac{6+r}{4096(5+r)(4+r)(3+r)(2+r)(1+r)r^2}$	$\frac{7}{2949120}$
a_7	$\frac{-7-r}{16384(6+r)(5+r)(4+r)(3+r)(2+r)(1+r)r^2}$	$-\frac{1}{10321920}$

Using the above table, then the first solution $y_1(t)$ is

$$\begin{aligned} y_1(t) &= t\big(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + a_7 t^7 + a_8 t^8 \dots\big) \\ &= t\left(1 - \frac{t}{2} + \frac{3t^2}{32} - \frac{t^3}{96} + \frac{5t^4}{6144} - \frac{t^5}{20480} + \frac{7t^6}{2949120} - \frac{t^7}{10321920} + O(t^8)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(t) = y_1(t)\ln(t) + \left(\sum_{n=1}^{\infty} b_n t^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n($
b_0	1	1	N/A since b_n starts from 1	N/
b_1	$\frac{-r-1}{4r^2}$	$-\frac{1}{2}$	$\frac{2+r}{4r^3}$	$\frac{3}{4}$
b_2	$\frac{2+r}{16(1+r)r^2}$	$\frac{3}{32}$	$\frac{-2r^2-7r-4}{16(1+r)^2 r^3}$	$-\frac{1}{6}$
b_3	$\frac{-3-r}{64(2+r)(1+r)r^2}$	$-\frac{1}{96}$	$\frac{3r^3+18r^2+29r+12}{64(2+r)^2(1+r)^2 r^3}$	$\frac{31}{115}$
b_4	$\frac{4+r}{256(3+r)(2+r)(1+r)r^2}$	$\frac{5}{6144}$	$\frac{-2r^4-19r^3-59r^2-69r-24}{128(3+r)^2(2+r)^2(1+r)^2 r^3}$	-7
b_5	$\frac{-5-r}{1024(4+r)(3+r)(2+r)(1+r)r^2}$	$-\frac{1}{20480}$	$\frac{5r^5+70r^4+355r^3+800r^2+774r+240}{1024(4+r)^2(3+r)^2(2+r)^2(1+r)^2 r^3}$	$\frac{1}{122}$
b_6	$\frac{6+r}{4096(5+r)(4+r)(3+r)(2+r)(1+r)r^2}$	$\frac{7}{2949120}$	$\frac{-6r^6-117r^5-880r^4-3225r^3-5948r^2-5052r-1440}{4096(5+r)^2(4+r)^2(3+r)^2(2+r)^2(1+r)^2 r^3}$	-5
b_7	$\frac{-7-r}{16384(6+r)(5+r)(4+r)(3+r)(2+r)(1+r)r^2}$	$-\frac{1}{10321920}$	$\frac{7r^7+182r^6+1904r^5+10290r^4+30597r^3+49000r^2+37764r+10080}{16384(6+r)^2(5+r)^2(4+r)^2(3+r)^2(2+r)^2(1+r)^2 r^3}$	289

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(t) &= y_1(t)\ln(t) + b_0 + b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5 + b_6 t^6 + b_7 t^7 + b_8 t^8 \dots \\
&= t\left(1 - \frac{t}{2} + \frac{3t^2}{32} - \frac{t^3}{96} + \frac{5t^4}{6144} - \frac{t^5}{20480} + \frac{7t^6}{2949120} - \frac{t^7}{10321920} + O(t^8)\right)\ln(t) \\
&\quad + t\left(\frac{3t}{4} - \frac{13t^2}{64} + \frac{31t^3}{1152} - \frac{173t^4}{73728} + \frac{187t^5}{1228800} - \frac{463t^6}{58982400} + \frac{971t^7}{2890137600} + O(t^8)\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(t) = c_1 y_1(t) + c_2 y_2(t)$$

$$
\begin{aligned}
&= c_1 t\left(1-\frac{t}{2}+\frac{3t^2}{32}-\frac{t^3}{96}+\frac{5t^4}{6144}-\frac{t^5}{20480}+\frac{7t^6}{2949120}-\frac{t^7}{10321920}+O\left(t^8\right)\right)\\
&\quad + c_2\left(t\left(1-\frac{t}{2}+\frac{3t^2}{32}-\frac{t^3}{96}+\frac{5t^4}{6144}-\frac{t^5}{20480}+\frac{7t^6}{2949120}-\frac{t^7}{10321920}+O\left(t^8\right)\right)\ln\left(t\right)\right.\\
&\quad \left.+t\left(\frac{3t}{4}-\frac{13t^2}{64}+\frac{31t^3}{1152}-\frac{173t^4}{73728}+\frac{187t^5}{1228800}-\frac{463t^6}{58982400}+\frac{971t^7}{2890137600}+O\left(t^8\right)\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y(t) &= y_h\\
&= c_1 t\left(1-\frac{t}{2}+\frac{3t^2}{32}-\frac{t^3}{96}+\frac{5t^4}{6144}-\frac{t^5}{20480}+\frac{7t^6}{2949120}-\frac{t^7}{10321920}+O\left(t^8\right)\right)\\
&\quad + c_2\left(t\left(1-\frac{t}{2}+\frac{3t^2}{32}-\frac{t^3}{96}+\frac{5t^4}{6144}-\frac{t^5}{20480}+\frac{7t^6}{2949120}-\frac{t^7}{10321920}+O\left(t^8\right)\right)\ln\left(t\right)\right.\\
&\quad \left.+t\left(\frac{3t}{4}-\frac{13t^2}{64}+\frac{31t^3}{1152}-\frac{173t^4}{73728}+\frac{187t^5}{1228800}-\frac{463t^6}{58982400}+\frac{971t^7}{2890137600}+O\left(t^8\right)\right)\right)
\end{aligned}
$$

Replacing t in the above with the original independent variable $x$susing $t = x - 4$ results in

$$
\begin{aligned}
y &= c_1(x-4)\left(3-\frac{x}{2}+\frac{3(x-4)^2}{32}-\frac{(x-4)^3}{96}+\frac{5(x-4)^4}{6144}-\frac{(x-4)^5}{20480}+\frac{7(x-4)^6}{2949120}\right.\\
&\qquad \left.-\frac{(x-4)^7}{10321920}+O\left((x-4)^8\right)\right)\\
&\quad + c_2\left((x-4)\left(3-\frac{x}{2}+\frac{3(x-4)^2}{32}-\frac{(x-4)^3}{96}+\frac{5(x-4)^4}{6144}-\frac{(x-4)^5}{20480}+\frac{7(x-4)^6}{2949120}\right.\right.\\
&\quad \left.-\frac{(x-4)^7}{10321920}+O\left((x-4)^8\right)\right)\ln\left(x-4\right)+(x-4)\left(\frac{3x}{4}-3-\frac{13(x-4)^2}{64}+\frac{31(x-4)^3}{1152}\right.\\
&\quad \left.\left.-\frac{173(x-4)^4}{73728}+\frac{187(x-4)^5}{1228800}-\frac{463(x-4)^6}{58982400}+\frac{971(x-4)^7}{2890137600}+O\left((x-4)^8\right)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1(x-4)\left(3-\frac{x}{2}+\frac{3(x-4)^2}{32}-\frac{(x-4)^3}{96}+\frac{5(x-4)^4}{6144}-\frac{(x-4)^5}{20480}+\frac{7(x-4)^6}{2949120}\right.\\
& \left.-\frac{(x-4)^7}{10321920}+O\left((x-4)^8\right)\right)\\
& +c_2\left((x-4)\left(3-\frac{x}{2}+\frac{3(x-4)^2}{32}-\frac{(x-4)^3}{96}+\frac{5(x-4)^4}{6144}-\frac{(x-4)^5}{20480}+\frac{7(x-4)^6}{2949120}\right.\right.\\
& \left.-\frac{(x-4)^7}{10321920}+O\left((x-4)^8\right)\right)\ln(x-4)+(x-4)\left(\frac{3x}{4}-3-\frac{13(x-4)^2}{64}+\frac{31(x-4)^3}{1152}\right.\\
& \left.\left.-\frac{173(x-4)^4}{73728}+\frac{187(x-4)^5}{1228800}-\frac{463(x-4)^6}{58982400}+\frac{971(x-4)^7}{2890137600}+O\left((x-4)^8\right)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1(x-4)\left(3-\frac{x}{2}+\frac{3(x-4)^2}{32}-\frac{(x-4)^3}{96}+\frac{5(x-4)^4}{6144}-\frac{(x-4)^5}{20480}+\frac{7(x-4)^6}{2949120}\right.\\
& \left.-\frac{(x-4)^7}{10321920}+O\left((x-4)^8\right)\right)\\
& +c_2\left((x-4)\left(3-\frac{x}{2}+\frac{3(x-4)^2}{32}-\frac{(x-4)^3}{96}+\frac{5(x-4)^4}{6144}-\frac{(x-4)^5}{20480}+\frac{7(x-4)^6}{2949120}\right.\right.\\
& \left.-\frac{(x-4)^7}{10321920}+O\left((x-4)^8\right)\right)\ln(x-4)+(x-4)\left(\frac{3x}{4}-3-\frac{13(x-4)^2}{64}+\frac{31(x-4)^3}{1152}\right.\\
& \left.\left.-\frac{173(x-4)^4}{73728}+\frac{187(x-4)^5}{1228800}-\frac{463(x-4)^6}{58982400}+\frac{971(x-4)^7}{2890137600}+O\left((x-4)^8\right)\right)\right)
\end{aligned}
$$

Verified OK.

5.10.1 Maple step by step solution

Let's solve

$4(x-4)^2 y''+(x^2-12x+32)\,y'+yx=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{xy}{4(x-4)^2} - \frac{(x-8)y'}{4(x-4)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x-8)y'}{4(x-4)} + \frac{xy}{4(x-4)^2} = 0$

□ Check to see if $x_0 = 4$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{x-8}{4(x-4)}, P_3(x) = \frac{x}{4(x-4)^2}\right]$

  - $(x-4) \cdot P_2(x)$ is analytic at $x = 4$

    $\left((x-4) \cdot P_2(x)\right)\Big|_{x=4} = -1$

  - $(x-4)^2 \cdot P_3(x)$ is analytic at $x = 4$

    $\left((x-4)^2 \cdot P_3(x)\right)\Big|_{x=4} = 1$

  - $x = 4$is a regular singular point

    Check to see if $x_0 = 4$ is a regular singular point

    $x_0 = 4$

- Multiply by denominators

  $4(x-4)^2 y'' + (x-4)(x-8) y' + yx = 0$

- Change variables using $x = u + 4$ so that the regular singular point is at $u = 0$

  $4u^2\left(\frac{d^2}{du^2}y(u)\right) + (u^2 - 4u)\left(\frac{d}{du}y(u)\right) + (u+4)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  - Shift index using $k->k-m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^2 \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $$u^2 \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$4a_0(-1+r)^2\, u^r + \left(\sum_{k=1}^{\infty} \left(4a_k(k+r-1)^2 + a_{k-1}(k+r)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$4(-1+r)^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = 1$$

- Each term in the series must be 0, giving the recursion relation

  $$4a_k(k+r-1)^2 + a_{k-1}(k+r) = 0$$

- Shift index using $k- >k+1$

  $$4a_{k+1}(k+r)^2 + a_k(k+r+1) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{a_k(k+r+1)}{4(k+r)^2}$$

- Recursion relation for $r = 1$

  $$a_{k+1} = -\frac{a_k(k+2)}{4(k+1)^2}$$

- Solution for $r = 1$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+1}, a_{k+1} = -\frac{a_k(k+2)}{4(k+1)^2}\right]$$

- Revert the change of variables $u = x - 4$

  $$\left[y = \sum_{k=0}^{\infty} a_k(x-4)^{k+1}, a_{k+1} = -\frac{a_k(k+2)}{4(k+1)^2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 83

```
Order:=8;
dsolve(4*(x-4)^2*diff(y(x),x$2)+(x-4)*(x-8)*diff(y(x),x)+x*y(x)=0,y(x),type='series',x=4);
```

$$y(x) = \left((\ln(x-4)\, c_2 + c_1) \left(1 - \frac{1}{2}(x-4) + \frac{3}{32}(x-4)^2 - \frac{1}{96}(x-4)^3 + \frac{5}{6144}(x-4)^4 - \frac{1}{20480}(x-4)^5 + \frac{7}{2949120}(x-4)^6 - \frac{1}{10321920}(x-4)^7 + \mathrm{O}\left((x-4)^8\right)\right) + \left(\frac{3}{4}(x-4) - \frac{13}{64}(x-4)^2 + \frac{31}{1152}(x-4)^3 - \frac{173}{73728}(x-4)^4 + \frac{187}{1228800}(x-4)^5 - \frac{463}{58982400}(x-4)^6 + \frac{971}{2890137600}(x-4)^7 + \mathrm{O}\left((x-4)^8\right)\right) c_2 \right)(x-4)$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 222

```
AsymptoticDSolveValue[4*(x-4)^2*y''[x]+(x-4)*(x-8)*y'[x]+x*y[x]==0,y[x],{x,4,7}]
```

$$y(x) \to c_1\left(-\frac{(x-4)^7}{10321920}+\frac{7(x-4)^6}{2949120}-\frac{(x-4)^5}{20480}+\frac{5(x-4)^4}{6144}-\frac{1}{96}(x-4)^3+\frac{3}{32}(x-4)^2+\frac{4-x}{2}+1\right)(x-4)+c_2\left((x-4)\left(\frac{971(x-4)^7}{2890137600}-\frac{463(x-4)^6}{58982400}+\frac{187(x-4)^5}{1228800}-\frac{173(x-4)^4}{73728}+\frac{31(x-4)^3}{1152}-\frac{13}{64}(x-4)^2+\frac{4-x}{4}+x-4\right)+\left(-\frac{(x-4)^7}{10321920}+\frac{7(x-4)^6}{2949120}-\frac{(x-4)^5}{20480}+\frac{5(x-4)^4}{6144}-\frac{1}{96}(x-4)^3+\frac{3}{32}(x-4)^2+\frac{4-x}{2}+1\right)(x-4)\log(x-4)\right)$$

5.11 problem 11 (solved as direct Bessel)

Internal problem ID [6958]
Internal file name [`OUTPUT/6201_Friday_August_12_2022_11_05_12_PM_54095803/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.6. Indicial Equation with Equal Roots. Exercises page 373
Problem number: 11 (solved as direct Bessel).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + y' - yx = 0}$$

5.11.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + xy' - x^2y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + xy' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= 0\\ \beta &= i\\ n &= 0\\ \gamma &= 1\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = c_1 \operatorname{BesselI}(0, x) + c_2 \operatorname{BesselY}(0, ix)$$

Summary
The solution(s) found are the following

$$y = c_1 \operatorname{BesselI}(0, x) + c_2 \operatorname{BesselY}(0, ix) \tag{1}$$

Verification of solutions

$$y = c_1 \operatorname{BesselI}(0, x) + c_2 \operatorname{BesselY}(0, ix)$$

Verified OK.

5.11.2 Maple step by step solution

Let's solve

$xy'' + y' - yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} + y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} - y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = -1\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + y' - yx = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

  - Shift index using $k- >k - 1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

  - Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1}$

  - Shift index using $k- >k + 1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1} (k + r + 1)\, x^{k+r}$

  - Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r-1}$

  - Shift index using $k- >k + 1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k + r + 1)\,(k + r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r^2 x^{-1+r} + a_1 (1 + r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1} (k + r + 1)^2 - a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 0$
- Each term must be 0

  $a_1(1+r)^2 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 - a_{k-1} = 0$
- Shift index using $k- >k+1$

  $a_{k+2}(k+2)^2 - a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k}{(k+2)^2}$
- Recursion relation for $r = 0$

  $a_{k+2} = \frac{a_k}{(k+2)^2}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a_k}{(k+2)^2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x*diff(y(x),x$2)+diff(y(x),x)-x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \operatorname{BesselI}(0, x) + c_2 \operatorname{BesselK}(0, x)$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 26

```
DSolve[x*y''[x]+y'[x]-x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \operatorname{BesselJ}(0, ix) + c_2 \operatorname{BesselY}(0, -ix)$$

5.12 problem 11 (solved as series)

Internal problem ID [6959]
Internal file name [`OUTPUT/6202_Friday_August_12_2022_11_05_13_PM_51119002/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.6. Indicial Equation with Equal Roots. Exercises page 373
Problem number: 11 (solved as series).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + y' - yx = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + y' - yx = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -1$$

Table 108: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -1$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + y' - yx = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) x = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-x^{1+n+r}a_n\right)=\sum_{n=2}^{\infty}\left(-a_{n-2}x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty}x^{n+r-1}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}\right)+\sum_{n=2}^{\infty}\left(-a_{n-2}x^{n+r-1}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1}a_n(n+r)(n+r-1)+(n+r)a_nx^{n+r-1}=0$$

When $n=0$ the above becomes

$$x^{-1+r}a_0r(-1+r)+ra_0x^{-1+r}=0$$

Or

$$\left(x^{-1+r}r(-1+r)+r\,x^{-1+r}\right)a_0=0$$

Since $a_0\neq 0$ then the above simplifies to

$$x^{-1+r}r^2=0$$

Since the above is true for all x then the indicial equation becomes

$$r^2=0$$

Solving for r gives the roots of the indicial equation as

$$r_1=0$$
$$r_2=0$$

Since $a_0\neq 0$ then the indicial equation becomes

$$x^{-1+r}r^2=0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) - a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-2}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{(r+2)^2}$	$\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{(r+2)^2}$	$\frac{1}{4}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r+2)^2 (r+4)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{64}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{(r+2)^2}$	$\frac{1}{4}$
a_3	0	0
a_4	$\frac{1}{(r+2)^2(r+4)^2}$	$\frac{1}{64}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{(r+2)^2}$	$\frac{1}{4}$
a_3	0	0
a_4	$\frac{1}{(r+2)^2(r+4)^2}$	$\frac{1}{64}$
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{1}{(r+2)^2 (r+4)^2 (r+6)^2}$$

Which for the root $r = 0$ becomes

$$a_6 = \frac{1}{2304}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{(r+2)^2}$	$\frac{1}{4}$
a_3	0	0
a_4	$\frac{1}{(r+2)^2(r+4)^2}$	$\frac{1}{64}$
a_5	0	0
a_6	$\frac{1}{(r+2)^2(r+4)^2(r+6)^2}$	$\frac{1}{2304}$

For $n = 7$, using the above recursive equation gives

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{(r+2)^2}$	$\frac{1}{4}$
a_3	0	0
a_4	$\frac{1}{(r+2)^2(r+4)^2}$	$\frac{1}{64}$
a_5	0	0
a_6	$\frac{1}{(r+2)^2(r+4)^2(r+6)^2}$	$\frac{1}{2304}$
a_7	0	0

Using the above table, then the first solution $y_1(x)$ becomes

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots \\ &= \frac{x^2}{4} + 1 + \frac{x^4}{64} + \frac{x^6}{2304} + O(x^8) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r=0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$\frac{1}{(r+2)^2}$	$\frac{1}{4}$	$-\frac{2}{(r+2)^3}$	$-\frac{1}{4}$
b_3	0	0	0	0
b_4	$\frac{1}{(r+2)^2(r+4)^2}$	$\frac{1}{64}$	$\frac{-4r-12}{(r+2)^3(r+4)^3}$	$-\frac{3}{128}$
b_5	0	0	0	0
b_6	$\frac{1}{(r+2)^2(r+4)^2(r+6)^2}$	$\frac{1}{2304}$	$\frac{-6r^2-48r-88}{(r+2)^3(r+4)^3(r+6)^3}$	$-\frac{11}{13824}$
b_7	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 + b_7x^7 + b_8x^8 \ldots \\
&= \left(\frac{x^2}{4} + 1 + \frac{x^4}{64} + \frac{x^6}{2304} + O(x^8)\right)\ln(x) - \frac{x^2}{4} - \frac{3x^4}{128} - \frac{11x^6}{13824} + O(x^8)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1\left(\frac{x^2}{4} + 1 + \frac{x^4}{64} + \frac{x^6}{2304} + O(x^8)\right) \\
&\quad + c_2\left(\left(\frac{x^2}{4} + 1 + \frac{x^4}{64} + \frac{x^6}{2304} + O(x^8)\right)\ln(x) - \frac{x^2}{4} - \frac{3x^4}{128} - \frac{11x^6}{13824} + O(x^8)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1\left(\frac{x^2}{4} + 1 + \frac{x^4}{64} + \frac{x^6}{2304} + O(x^8)\right) \\
&\quad + c_2\left(\left(\frac{x^2}{4} + 1 + \frac{x^4}{64} + \frac{x^6}{2304} + O(x^8)\right)\ln(x) - \frac{x^2}{4} - \frac{3x^4}{128} - \frac{11x^6}{13824} + O(x^8)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\left(\frac{x^2}{4}+1+\frac{x^4}{64}+\frac{x^6}{2304}+O(x^8)\right) + c_2\left(\left(\frac{x^2}{4}+1+\frac{x^4}{64}+\frac{x^6}{2304}+O(x^8)\right)\ln(x)-\frac{x^2}{4}-\frac{3x^4}{128}-\frac{11x^6}{13824}+O(x^8)\right) \tag{1}$$

Verification of solutions

$$y = c_1\left(\frac{x^2}{4}+1+\frac{x^4}{64}+\frac{x^6}{2304}+O(x^8)\right) + c_2\left(\left(\frac{x^2}{4}+1+\frac{x^4}{64}+\frac{x^6}{2304}+O(x^8)\right)\ln(x)-\frac{x^2}{4}-\frac{3x^4}{128}-\frac{11x^6}{13824}+O(x^8)\right)$$

Verified OK.

5.12.1 Maple step by step solution

Let's solve

$xy''+y'-yx=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y'}{x}+y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y'}{x}-y=0$

□ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{1}{x}, P_3(x)=-1\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=1$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

  - $x=0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$xy'' + y' - yx = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

$$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

  ◦ Shift index using $k- >k - 1$

$$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

  ◦ Convert y' to series expansion

$$y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r-1}$$

  ◦ Shift index using $k- >k + 1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1}(k + r + 1)\, x^{k+r}$$

  ◦ Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\, x^{k+r-1}$$

  ◦ Shift index using $k- >k + 1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k + r + 1)\,(k + r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + a_1(1 + r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k + r + 1)^2 - a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r^2 = 0$

- Values of r that satisfy the indicial equation

$r = 0$

- Each term must be 0

  $a_1(1+r)^2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 - a_{k-1} = 0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2)^2 - a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k}{(k+2)^2}$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{a_k}{(k+2)^2}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a_k}{(k+2)^2}, a_1 = 0\right]$

<u>Maple trace</u>

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 47

```
Order:=8;
dsolve(x*diff(y(x),x$2)+diff(y(x),x)-x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln(x) + c_1)\left(1 + \frac{1}{4}x^2 + \frac{1}{64}x^4 + \frac{1}{2304}x^6 + \mathrm{O}\left(x^8\right)\right) + \left(-\frac{1}{4}x^2 - \frac{3}{128}x^4 - \frac{11}{13824}x^6 + \mathrm{O}\left(x^8\right)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 81

```
AsymptoticDSolveValue[x*y''[x]+y'[x]-x*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(\frac{x^6}{2304} + \frac{x^4}{64} + \frac{x^2}{4} + 1\right) + c_2\left(-\frac{11x^6}{13824} - \frac{3x^4}{128} - \frac{x^2}{4} + \left(\frac{x^6}{2304} + \frac{x^4}{64} + \frac{x^2}{4} + 1\right)\log(x)\right)$$

5.13 problem 12

Internal problem ID [6960]
Internal file name [`OUTPUT/6203_Friday_August_12_2022_11_05_15_PM_58281880/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.6. Indicial Equation with Equal Roots. Exercises page 373
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + \left(-x^2+1\right) y' - yx = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + \left(-x^2+1\right) y' - yx = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x^2-1}{x}$$
$$q(x) = -1$$

Table 110: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x^2-1}{x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = -1$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + (-x^2 + 1)\, y' - yx = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x\left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}\right) + (-x^2 + 1)\left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) x = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) = \sum_{n=2}^{\infty} \left(-a_{n-2}(n+r-2)\, x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=2}^{\infty} \left(-a_{n-2}(n+r-2)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-2}(n+r-2) + a_n(n+r) - a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}(n+r-1)}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-2}(n-1)}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1+r}{(r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1+r}{(r+2)^2}$	$\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1+r}{(r+2)^2}$	$\frac{1}{4}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(3+r)(1+r)}{(r+2)^2 (r+4)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{3}{64}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1+r}{(r+2)^2}$	$\frac{1}{4}$
a_3	0	0
a_4	$\frac{(3+r)(1+r)}{(r+2)^2(r+4)^2}$	$\frac{3}{64}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1+r}{(r+2)^2}$	$\frac{1}{4}$
a_3	0	0
a_4	$\frac{(3+r)(1+r)}{(r+2)^2(r+4)^2}$	$\frac{3}{64}$
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{(5+r)(3+r)(1+r)}{(r+2)^2 (r+4)^2 (r+6)^2}$$

Which for the root $r = 0$ becomes

$$a_6 = \frac{5}{768}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1+r}{(r+2)^2}$	$\frac{1}{4}$
a_3	0	0
a_4	$\frac{(3+r)(1+r)}{(r+2)^2(r+4)^2}$	$\frac{3}{64}$
a_5	0	0
a_6	$\frac{(5+r)(3+r)(1+r)}{(r+2)^2(r+4)^2(r+6)^2}$	$\frac{5}{768}$

For $n = 7$, using the above recursive equation gives

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1+r}{(r+2)^2}$	$\frac{1}{4}$
a_3	0	0
a_4	$\frac{(3+r)(1+r)}{(r+2)^2(r+4)^2}$	$\frac{3}{64}$
a_5	0	0
a_6	$\frac{(5+r)(3+r)(1+r)}{(r+2)^2(r+4)^2(r+6)^2}$	$\frac{5}{768}$
a_7	0	0

Using the above table, then the first solution $y_1(x)$ becomes

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots \\ &= \frac{x^2}{4} + 1 + \frac{3x^4}{64} + \frac{5x^6}{768} + O(x^8) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r} \right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = 0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$\frac{1+r}{(r+2)^2}$	$\frac{1}{4}$	$-\frac{r}{(r+2)^3}$	0
b_3	0	0	0	0
b_4	$\frac{(3+r)(1+r)}{(r+2)^2(r+4)^2}$	$\frac{3}{64}$	$\frac{-2r^3-12r^2-20r-4}{(r+2)^3(r+4)^3}$	$-\frac{1}{128}$
b_5	0	0	0	0
b_6	$\frac{(5+r)(3+r)(1+r)}{(r+2)^2(r+4)^2(r+6)^2}$	$\frac{5}{768}$	$\frac{-3r^5-48r^4-287r^3-774r^2-868r-216}{(r+2)^3(r+4)^3(r+6)^3}$	$-\frac{1}{512}$
b_7	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x) \ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \ldots \\
&= \left(\frac{x^2}{4} + 1 + \frac{3x^4}{64} + \frac{5x^6}{768} + O(x^8) \right) \ln(x) - \frac{x^4}{128} - \frac{x^6}{512} + O(x^8)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \left(\frac{x^2}{4} + 1 + \frac{3x^4}{64} + \frac{5x^6}{768} + O(x^8) \right) \\
&\quad + c_2 \left(\left(\frac{x^2}{4} + 1 + \frac{3x^4}{64} + \frac{5x^6}{768} + O(x^8) \right) \ln(x) - \frac{x^4}{128} - \frac{x^6}{512} + O(x^8) \right)
\end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(\frac{x^2}{4}+1+\frac{3x^4}{64}+\frac{5x^6}{768}+O(x^8)\right) \\
&\quad + c_2\left(\left(\frac{x^2}{4}+1+\frac{3x^4}{64}+\frac{5x^6}{768}+O(x^8)\right)\ln(x)-\frac{x^4}{128}-\frac{x^6}{512}+O(x^8)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1\left(\frac{x^2}{4}+1+\frac{3x^4}{64}+\frac{5x^6}{768}+O(x^8)\right) \\
& + c_2\left(\left(\frac{x^2}{4}+1+\frac{3x^4}{64}+\frac{5x^6}{768}+O(x^8)\right)\ln(x)-\frac{x^4}{128}-\frac{x^6}{512}+O(x^8)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1\left(\frac{x^2}{4}+1+\frac{3x^4}{64}+\frac{5x^6}{768}+O(x^8)\right) \\
& + c_2\left(\left(\frac{x^2}{4}+1+\frac{3x^4}{64}+\frac{5x^6}{768}+O(x^8)\right)\ln(x)-\frac{x^4}{128}-\frac{x^6}{512}+O(x^8)\right)
\end{aligned}
$$

Verified OK.

5.13.1 Maple step by step solution

Let's solve

$xy'' + (-x^2+1)\,y' - yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x^2-1)y'}{x} + y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x^2-1)y'}{x} - y = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

$\left[P_2(x) = -\frac{x^2-1}{x}, P_3(x) = -1\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + (-x^2 + 1)\, y' - yx = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

- Shift index using $k- >k - 1$

  $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$

- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\, x^{k+r-1}$

- Shift index using $k- >k + 1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + a_1(1+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)^2 - a_{k-1}(k+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 0$

- Each term must be 0

  $a_1(1+r)^2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 - a_{k-1}k = 0$

- Shift index using $k- >k+1$

  $a_{k+2}(k+2)^2 - a_k(k+1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k(k+1)}{(k+2)^2}$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{a_k(k+1)}{(k+2)^2}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a_k(k+1)}{(k+2)^2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
Order:=8;
dsolve(x*diff(y(x),x$2)+(1-x^2)*diff(y(x),x)-x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln(x) + c_1)\left(1 + \frac{1}{4}x^2 + \frac{3}{64}x^4 + \frac{5}{768}x^6 + \mathrm{O}\left(x^8\right)\right) + \left(-\frac{1}{128}x^4 - \frac{1}{512}x^6 + \mathrm{O}\left(x^8\right)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 74

```
AsymptoticDSolveValue[x*y''[x]+(1-x^2)*y'[x]-x*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(\frac{5x^6}{768} + \frac{3x^4}{64} + \frac{x^2}{4} + 1\right) + c_2\left(-\frac{x^6}{512} - \frac{x^4}{128} + \left(\frac{5x^6}{768} + \frac{3x^4}{64} + \frac{x^2}{4} + 1\right)\log(x)\right)$$

5.14 problem 14

Internal problem ID [6961]
Internal file name [`OUTPUT/6204_Friday_August_12_2022_11_05_17_PM_55982926/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.6. Indicial Equation with Equal Roots. Exercises page 373
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x(3+2x)\,y' + (3x+1)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(2x^2 + 3x\right)y' + (3x+1)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{3+2x}{x}$$
$$q(x) = \frac{3x+1}{x^2}$$

Table 112: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3+2x}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{3x+1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + \left(2x^2 + 3x\right) y' + (3x + 1) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) + \left(2x^2 + 3x\right) \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + (3x + 1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 3x^{1+n+r} a_n = \sum_{n=1}^{\infty} 3a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + 3x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + 3x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + 3x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r+1)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r+1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -1$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r+1)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = -1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-1}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n-1}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 2a_{n-1}(n+r-1) + 3a_n(n+r) + 3a_{n-1} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(2n + 2r + 1)}{n^2 + 2nr + r^2 + 2n + 2r + 1} \tag{4}$$

Which for the root $r = -1$ becomes

$$a_n = \frac{(1 - 2n)\, a_{n-1}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-2r - 3}{(r + 2)^2}$$

Which for the root $r = -1$ becomes

$$a_1 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r-3}{(r+2)^2}$	-1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4r^2 + 16r + 15}{(r + 2)^2 (r + 3)^2}$$

Which for the root $r = -1$ becomes

$$a_2 = \frac{3}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r-3}{(r+2)^2}$	-1
a_2	$\frac{4r^2+16r+15}{(r+2)^2(r+3)^2}$	$\frac{3}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-8r^3 - 60r^2 - 142r - 105}{(r+2)^2 (r+3)^2 (r+4)^2}$$

Which for the root $r = -1$ becomes

$$a_3 = -\frac{5}{12}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r-3}{(r+2)^2}$	-1
a_2	$\frac{4r^2+16r+15}{(r+2)^2(r+3)^2}$	$\frac{3}{4}$
a_3	$\frac{-8r^3-60r^2-142r-105}{(r+2)^2(r+3)^2(r+4)^2}$	$-\frac{5}{12}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16r^4 + 192r^3 + 824r^2 + 1488r + 945}{(r+2)^2 (r+3)^2 (r+4)^2 (r+5)^2}$$

Which for the root $r = -1$ becomes

$$a_4 = \frac{35}{192}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r-3}{(r+2)^2}$	-1
a_2	$\frac{4r^2+16r+15}{(r+2)^2(r+3)^2}$	$\frac{3}{4}$
a_3	$\frac{-8r^3-60r^2-142r-105}{(r+2)^2(r+3)^2(r+4)^2}$	$-\frac{5}{12}$
a_4	$\frac{16r^4+192r^3+824r^2+1488r+945}{(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$\frac{35}{192}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-32r^5 - 560r^4 - 3760r^3 - 12040r^2 - 18258r - 10395}{(r+2)^2(r+3)^2(r+4)^2(r+5)^2(r+6)^2}$$

Which for the root $r = -1$ becomes

$$a_5 = -\frac{21}{320}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r-3}{(r+2)^2}$	-1
a_2	$\frac{4r^2+16r+15}{(r+2)^2(r+3)^2}$	$\frac{3}{4}$
a_3	$\frac{-8r^3-60r^2-142r-105}{(r+2)^2(r+3)^2(r+4)^2}$	$-\frac{5}{12}$
a_4	$\frac{16r^4+192r^3+824r^2+1488r+945}{(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$\frac{35}{192}$
a_5	$\frac{-32r^5-560r^4-3760r^3-12040r^2-18258r-10395}{(r+2)^2(r+3)^2(r+4)^2(r+5)^2(r+6)^2}$	$-\frac{21}{320}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{64r^6 + 1536r^5 + 14800r^4 + 72960r^3 + 193036r^2 + 258144r + 135135}{(r+2)^2(r+3)^2(r+4)^2(r+5)^2(r+6)^2(r+7)^2}$$

Which for the root $r = -1$ becomes

$$a_6 = \frac{77}{3840}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r-3}{(r+2)^2}$	-1
a_2	$\frac{4r^2+16r+15}{(r+2)^2(r+3)^2}$	$\frac{3}{4}$
a_3	$\frac{-8r^3-60r^2-142r-105}{(r+2)^2(r+3)^2(r+4)^2}$	$-\frac{5}{12}$
a_4	$\frac{16r^4+192r^3+824r^2+1488r+945}{(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$\frac{35}{192}$
a_5	$\frac{-32r^5-560r^4-3760r^3-12040r^2-18258r-10395}{(r+2)^2(r+3)^2(r+4)^2(r+5)^2(r+6)^2}$	$-\frac{21}{320}$
a_6	$\frac{64r^6+1536r^5+14800r^4+72960r^3+193036r^2+258144r+135135}{(r+2)^2(r+3)^2(r+4)^2(r+5)^2(r+6)^2(r+7)^2}$	$\frac{77}{3840}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{-128r^7 - 4032r^6 - 52640r^5 - 367920r^4 - 1480472r^3 - 3411828r^2 - 4142430r - 2027025}{(r+2)^2 (r+3)^2 (r+4)^2 (r+5)^2 (r+6)^2 (r+7)^2 (8+r)^2}$$

Which for the root $r = -1$ becomes

$$a_7 = -\frac{143}{26880}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r-3}{(r+2)^2}$	-1
a_2	$\frac{4r^2+16r+15}{(r+2)^2(r+3)^2}$	$\frac{3}{4}$
a_3	$\frac{-8r^3-60r^2-142r-105}{(r+2)^2(r+3)^2(r+4)^2}$	$-\frac{5}{12}$
a_4	$\frac{16r^4+192r^3+824r^2+1488r+945}{(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$\frac{35}{192}$
a_5	$\frac{-32r^5-560r^4-3760r^3-12040r^2-18258r-10395}{(r+2)^2(r+3)^2(r+4)^2(r+5)^2(r+6)^2}$	$-\frac{21}{320}$
a_6	$\frac{64r^6+1536r^5+14800r^4+72960r^3+193036r^2+258144r+135135}{(r+2)^2(r+3)^2(r+4)^2(r+5)^2(r+6)^2(r+7)^2}$	$\frac{77}{3840}$
a_7	$\frac{-128r^7-4032r^6-52640r^5-367920r^4-1480472r^3-3411828r^2-4142430r-2027025}{(r+2)^2(r+3)^2(r+4)^2(r+5)^2(r+6)^2(r+7)^2(8+r)^2}$	$-\frac{143}{26880}$

Using the above table, then the first solution $y_1(x)$ is

$$y_1(x) = \frac{1}{x}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \dots\right)$$

$$= \frac{1 - x + \frac{3x^2}{4} - \frac{5x^3}{12} + \frac{35x^4}{192} - \frac{21x^5}{320} + \frac{77x^6}{3840} - \frac{143x^7}{26880} + O(x^8)}{x}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr}a_{n,r}$$

And the above is then evaluated at $r = -1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	$\frac{-2r-3}{(r+2)^2}$	-1	$\frac{2r+2}{(r+2)^3}$
b_2	$\frac{4r^2+16r+15}{(r+2)^2(r+3)^2}$	$\frac{3}{4}$	$\frac{-8r^3-48r^2-92r-54}{(r+2)^3(r+3)^3}$
b_3	$\frac{-8r^3-60r^2-142r-105}{(r+2)^2(r+3)^2(r+4)^2}$	$-\frac{5}{12}$	$\frac{24r^5+312r^4+1582r^3+3888r^2+4592r}{(r+2)^3(r+3)^3(r+4)^3}$
b_4	$\frac{16r^4+192r^3+824r^2+1488r+945}{(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$\frac{35}{192}$	$-\frac{4(16r^7+352r^6+3252r^5+16316r^4+}{(r+2)^3(r+3)^3}$
b_5	$\frac{-32r^5-560r^4-3760r^3-12040r^2-18258r-10395}{(r+2)^2(r+3)^2(r+4)^2(r+5)^2(r+6)^2}$	$-\frac{21}{320}$	$160r^9+5280r^8+76080r^7+627360r^6+$
b_6	$\frac{64r^6+1536r^5+14800r^4+72960r^3+193036r^2+258144r+135135}{(r+2)^2(r+3)^2(r+4)^2(r+5)^2(r+6)^2(r+7)^2}$	$\frac{77}{3840}$	$-384r^{11}-17664r^{10}-363520r^9-4413$
b_7	$\frac{-128r^7-4032r^6-52640r^5-367920r^4-1480472r^3-3411828r^2-4142430r-2027025}{(r+2)^2(r+3)^2(r+4)^2(r+5)^2(r+6)^2(r+7)^2(8+r)^2}$	$-\frac{143}{26880}$	$896r^{13}+54656r^{12}+1516704r^{11}+2533$

The above table gives all values of b_n needed. Hence the second solution is

$$y_2(x) = y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 + b_7x^7 + b_8x^8 \dots$$

$$= \frac{\left(1 - x + \frac{3x^2}{4} - \frac{5x^3}{12} + \frac{35x^4}{192} - \frac{21x^5}{320} + \frac{77x^6}{3840} - \frac{143x^7}{26880} + O(x^8)\right)\ln(x)}{x}$$

$$+ \frac{-\frac{x^2}{4} + \frac{x^3}{4} - \frac{19x^4}{128} + \frac{25x^5}{384} - \frac{317x^6}{13824} + \frac{469x^7}{69120} + O(x^8)}{x}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= \frac{c_1 \left(1 - x + \frac{3x^2}{4} - \frac{5x^3}{12} + \frac{35x^4}{192} - \frac{21x^5}{320} + \frac{77x^6}{3840} - \frac{143x^7}{26880} + O(x^8)\right)}{x} \\
&\quad + c_2 \left(\frac{\left(1 - x + \frac{3x^2}{4} - \frac{5x^3}{12} + \frac{35x^4}{192} - \frac{21x^5}{320} + \frac{77x^6}{3840} - \frac{143x^7}{26880} + O(x^8)\right) \ln(x)}{x} \right. \\
&\quad \left. + \frac{-\frac{x^2}{4} + \frac{x^3}{4} - \frac{19x^4}{128} + \frac{25x^5}{384} - \frac{317x^6}{13824} + \frac{469x^7}{69120} + O(x^8)}{x} \right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= \frac{c_1 \left(1 - x + \frac{3x^2}{4} - \frac{5x^3}{12} + \frac{35x^4}{192} - \frac{21x^5}{320} + \frac{77x^6}{3840} - \frac{143x^7}{26880} + O(x^8)\right)}{x} \\
&\quad + c_2 \left(\frac{\left(1 - x + \frac{3x^2}{4} - \frac{5x^3}{12} + \frac{35x^4}{192} - \frac{21x^5}{320} + \frac{77x^6}{3840} - \frac{143x^7}{26880} + O(x^8)\right) \ln(x)}{x} \right. \\
&\quad \left. + \frac{-\frac{x^2}{4} + \frac{x^3}{4} - \frac{19x^4}{128} + \frac{25x^5}{384} - \frac{317x^6}{13824} + \frac{469x^7}{69120} + O(x^8)}{x} \right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= \frac{c_1 \left(1 - x + \frac{3x^2}{4} - \frac{5x^3}{12} + \frac{35x^4}{192} - \frac{21x^5}{320} + \frac{77x^6}{3840} - \frac{143x^7}{26880} + O(x^8)\right)}{x} \\
&\quad + c_2 \left(\frac{\left(1 - x + \frac{3x^2}{4} - \frac{5x^3}{12} + \frac{35x^4}{192} - \frac{21x^5}{320} + \frac{77x^6}{3840} - \frac{143x^7}{26880} + O(x^8)\right) \ln(x)}{x} \right. \\
&\quad \left. + \frac{-\frac{x^2}{4} + \frac{x^3}{4} - \frac{19x^4}{128} + \frac{25x^5}{384} - \frac{317x^6}{13824} + \frac{469x^7}{69120} + O(x^8)}{x} \right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1\left(1-x+\frac{3x^2}{4}-\frac{5x^3}{12}+\frac{35x^4}{192}-\frac{21x^5}{320}+\frac{77x^6}{3840}-\frac{143x^7}{26880}+O(x^8)\right)}{x} + c_2\left(\frac{\left(1-x+\frac{3x^2}{4}-\frac{5x^3}{12}+\frac{35x^4}{192}-\frac{21x^5}{320}+\frac{77x^6}{3840}-\frac{143x^7}{26880}+O(x^8)\right)\ln(x)}{x} + \frac{-\frac{x^2}{4}+\frac{x^3}{4}-\frac{19x^4}{128}+\frac{25x^5}{384}-\frac{317x^6}{13824}+\frac{469x^7}{69120}+O(x^8)}{x}\right)$$

Verified OK.

5.14.1 Maple step by step solution

Let's solve

$x^2y''+(2x^2+3x)\,y'+(3x+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(3x+1)y}{x^2}-\frac{(3+2x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(3+2x)y'}{x}+\frac{(3x+1)y}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{3+2x}{x}, P_3(x)=\frac{3x+1}{x^2}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=3$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=1$

  - $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2y'' + x(3+2x)\,y' + (3x+1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- > k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k- > k + 1 - m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)^2\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)^2 + a_{k-1}(2k+1+2r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = -1$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)^2 + a_{k-1}(2k+1+2r) = 0$

- Shift index using $k- > k + 1$

  $a_{k+1}(k+2+r)^2 + a_k(2k+2r+3) = 0$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{a_k(2k+2r+3)}{(k+2+r)^2}$$

- Recursion relation for $r = -1$

  $$a_{k+1} = -\frac{a_k(2k+1)}{(k+1)^2}$$

- Solution for $r = -1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = -\frac{a_k(2k+1)}{(k+1)^2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 79

```
Order:=8;
dsolve(x^2*diff(y(x),x$2)+x*(3+2*x)*diff(y(x),x)+(1+3*x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{(c_2 \ln(x) + c_1)\left(1 - x + \frac{3}{4}x^2 - \frac{5}{12}x^3 + \frac{35}{192}x^4 - \frac{21}{320}x^5 + \frac{77}{3840}x^6 - \frac{143}{26880}x^7 + \mathrm{O}(x^8)\right) + \left(-\frac{1}{4}x^2 + \frac{1}{4}x^3 - \frac{19}{128}\right.}{x}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 161

```
AsymptoticDSolveValue[x^2*y''[x]+x*(3+2*x)*y'[x]+(1+3*x)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to \frac{c_1\left(-\frac{143x^7}{26880}+\frac{77x^6}{3840}-\frac{21x^5}{320}+\frac{35x^4}{192}-\frac{5x^3}{12}+\frac{3x^2}{4}-x+1\right)}{x} + c_2\left(\frac{\frac{469x^7}{69120}-\frac{317x^6}{13824}+\frac{25x^5}{384}-\frac{19x^4}{128}+\frac{x^3}{4}-\frac{x^2}{4}}{x} + \frac{\left(-\frac{143x^7}{26880}+\frac{77x^6}{3840}-\frac{21x^5}{320}+\frac{35x^4}{192}-\frac{5x^3}{12}+\frac{3x^2}{4}-x+1\right)\log(x)}{x}\right)$$

5.15 problem 15

Internal problem ID [6962]
Internal file name [`OUTPUT/6205_Friday_August_12_2022_11_05_19_PM_40367264/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.6. Indicial Equation with Equal Roots. Exercises page 373
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' + 8x(1+x)\,y' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4x^2y'' + \left(8x^2 + 8x\right)y' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{2x+2}{x}$$
$$q(x) = \frac{1}{4x^2}$$

Table 114: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2x+2}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2y'' + (8x^2 + 8x)\, y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (8x^2 + 8x) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 8x^{1+n+r} a_n (n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} 8x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} 8x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 8a_{n-1}(n+r-1)\, x^{n+r}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 8a_{n-1}(n+r-1)\, x^{n+r}\right) \\
&+ \left(\sum_{n=0}^{\infty} 8x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0
\end{aligned}
\tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
4x^{n+r} a_n (n+r)(n+r-1) + 8x^{n+r} a_n (n+r) + a_n x^{n+r} = 0
$$

When $n=0$ the above becomes

$$
4x^r a_0 r(-1+r) + 8x^r a_0 r + a_0 x^r = 0
$$

Or

$$
\left(4x^r r(-1+r) + 8x^r r + x^r\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
(2r+1)^2 x^r = 0
$$

Since the above is true for all x then the indicial equation becomes

$$
(2r+1)^2 = 0
$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -\frac{1}{2}$$
$$r_2 = -\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(2r+1)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = -\frac{1}{2}$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-\frac{1}{2}}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n-\frac{1}{2}}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$4a_n(n+r)(n+r-1) + 8a_{n-1}(n+r-1) + 8a_n(n+r) + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{8a_{n-1}(n+r-1)}{4n^2+8nr+4r^2+4n+4r+1} \tag{4}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_n = \frac{a_{n-1}(3-2n)}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{8r}{(2r+3)^2}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_1 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{8r}{(2r+3)^2}$	1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{64r(1+r)}{(2r+3)^2(5+2r)^2}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_2 = -\frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{8r}{(2r+3)^2}$	1
a_2	$\frac{64r(1+r)}{(2r+3)^2(5+2r)^2}$	$-\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{512r(1+r)(2+r)}{(2r+3)^2(5+2r)^2(7+2r)^2}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_3 = \frac{1}{12}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{8r}{(2r+3)^2}$	1
a_2	$\frac{64r(1+r)}{(2r+3)^2(5+2r)^2}$	$-\frac{1}{4}$
a_3	$-\frac{512r(1+r)(2+r)}{(2r+3)^2(5+2r)^2(7+2r)^2}$	$\frac{1}{12}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4096r(1+r)(2+r)(3+r)}{(2r+3)^2(5+2r)^2(7+2r)^2(9+2r)^2}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_4 = -\frac{5}{192}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{8r}{(2r+3)^2}$	1
a_2	$\frac{64r(1+r)}{(2r+3)^2(5+2r)^2}$	$-\frac{1}{4}$
a_3	$-\frac{512r(1+r)(2+r)}{(2r+3)^2(5+2r)^2(7+2r)^2}$	$\frac{1}{12}$
a_4	$\frac{4096r(1+r)(2+r)(3+r)}{(2r+3)^2(5+2r)^2(7+2r)^2(9+2r)^2}$	$-\frac{5}{192}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{32768r(1+r)(2+r)(3+r)(4+r)}{(2r+3)^2(5+2r)^2(7+2r)^2(9+2r)^2(11+2r)^2}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_5 = \frac{7}{960}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{8r}{(2r+3)^2}$	1
a_2	$\frac{64r(1+r)}{(2r+3)^2(5+2r)^2}$	$-\frac{1}{4}$
a_3	$-\frac{512r(1+r)(2+r)}{(2r+3)^2(5+2r)^2(7+2r)^2}$	$\frac{1}{12}$
a_4	$\frac{4096r(1+r)(2+r)(3+r)}{(2r+3)^2(5+2r)^2(7+2r)^2(9+2r)^2}$	$-\frac{5}{192}$
a_5	$-\frac{32768r(1+r)(2+r)(3+r)(4+r)}{(2r+3)^2(5+2r)^2(7+2r)^2(9+2r)^2(11+2r)^2}$	$\frac{7}{960}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{262144r(1+r)(2+r)(3+r)(4+r)(5+r)}{(2r+3)^2(5+2r)^2(7+2r)^2(9+2r)^2(11+2r)^2(13+2r)^2}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_6 = -\frac{7}{3840}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{8r}{(2r+3)^2}$	1
a_2	$\frac{64r(1+r)}{(2r+3)^2(5+2r)^2}$	$-\frac{1}{4}$
a_3	$-\frac{512r(1+r)(2+r)}{(2r+3)^2(5+2r)^2(7+2r)^2}$	$\frac{1}{12}$
a_4	$\frac{4096r(1+r)(2+r)(3+r)}{(2r+3)^2(5+2r)^2(7+2r)^2(9+2r)^2}$	$-\frac{5}{192}$
a_5	$-\frac{32768r(1+r)(2+r)(3+r)(4+r)}{(2r+3)^2(5+2r)^2(7+2r)^2(9+2r)^2(11+2r)^2}$	$\frac{7}{960}$
a_6	$\frac{262144r(1+r)(2+r)(3+r)(4+r)(5+r)}{(2r+3)^2(5+2r)^2(7+2r)^2(9+2r)^2(11+2r)^2(13+2r)^2}$	$-\frac{7}{3840}$

For $n = 7$, using the above recursive equation gives

$$a_7 = -\frac{2097152r(1+r)(2+r)(3+r)(4+r)(5+r)(6+r)}{(2r+3)^2(5+2r)^2(7+2r)^2(9+2r)^2(11+2r)^2(13+2r)^2(15+2r)^2}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_7 = \frac{11}{26880}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{8r}{(2r+3)^2}$	1
a_2	$\frac{64r(1+r)}{(2r+3)^2(5+2r)^2}$	$-\frac{1}{4}$
a_3	$-\frac{512r(1+r)(2+r)}{(2r+3)^2(5+2r)^2(7+2r)^2}$	$\frac{1}{12}$
a_4	$\frac{4096r(1+r)(2+r)(3+r)}{(2r+3)^2(5+2r)^2(7+2r)^2(9+2r)^2}$	$-\frac{5}{192}$
a_5	$-\frac{32768r(1+r)(2+r)(3+r)(4+r)}{(2r+3)^2(5+2r)^2(7+2r)^2(9+2r)^2(11+2r)^2}$	$\frac{7}{960}$
a_6	$\frac{262144r(1+r)(2+r)(3+r)(4+r)(5+r)}{(2r+3)^2(5+2r)^2(7+2r)^2(9+2r)^2(11+2r)^2(13+2r)^2}$	$-\frac{7}{3840}$
a_7	$-\frac{2097152r(1+r)(2+r)(3+r)(4+r)(5+r)(6+r)}{(2r+3)^2(5+2r)^2(7+2r)^2(9+2r)^2(11+2r)^2(13+2r)^2(15+2r)^2}$	$\frac{11}{26880}$

Using the above table, then the first solution $y_1(x)$ is

$$y_1(x) = \frac{1}{\sqrt{x}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right)$$
$$= \frac{1 + x - \frac{x^2}{4} + \frac{x^3}{12} - \frac{5x^4}{192} + \frac{7x^5}{960} - \frac{7x^6}{3840} + \frac{11x^7}{26880} + O(x^8)}{\sqrt{x}}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = -\frac{1}{2}$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	$-\frac{8r}{(2r+3)^2}$	1	$\frac{16r-24}{(2r+3)^3}$
b_2	$\frac{64r(1+r)}{(2r+3)^2(5+2r)^2}$	$-\frac{1}{4}$	$\frac{-512r^3-768r^2+896r+960}{(5+2r)^3(2r+3)^3}$
b_3	$-\frac{512r(1+r)(2+r)}{(2r+3)^2(5+2r)^2(7+2r)^2}$	$\frac{1}{12}$	$\frac{12288r^5+79872r^4+152576r^3+23040r^2-177152r-107520}{(5+2r)^3(2r+3)^3(7+2r)^3}$
b_4	$\frac{4096r(1+r)(2+r)(3+r)}{(2r+3)^2(5+2r)^2(7+2r)^2(9+2r)^2}$	$-\frac{5}{192}$	$-\frac{8192(32r^7+432r^6+2256r^5+5544r^4+5590r^3-1089r^2-}{(2r+3)^3(5+2r)^3(7+2r)^3(9+2r)^3}$
b_5	$-\frac{32768r(1+r)(2+r)(3+r)(4+r)}{(2r+3)^2(5+2r)^2(7+2r)^2(9+2r)^2(11+2r)^2}$	$\frac{7}{960}$	$\frac{5242880r^9+117964800r^8+1114112000r^7+5700321280r^6+}{(5-}$
b_6	$\frac{262144r(1+r)(2+r)(3+r)(4+r)(5+r)}{(2r+3)^2(5+2r)^2(7+2r)^2(9+2r)^2(11+2r)^2(13+2r)^2}$	$-\frac{7}{3840}$	$-\frac{262144(384r^{11}+12864r^{10}+188320r^9+1578960r^8+8334}{(2r+}$
b_7	$-\frac{2097152r(1+r)(2+r)(3+r)(4+r)(5+r)(6+r)}{(2r+3)^2(5+2r)^2(7+2r)^2(9+2r)^2(11+2r)^2(13+2r)^2(15+2r)^2}$	$\frac{11}{26880}$	$\frac{1879048192r^{13}+87375740928r^{12}+1819388411904r^{11}+22}{}$

The above table gives all values of b_n needed. Hence the second solution is

$$y_2(x) = y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots$$
$$= \frac{\left(1 + x - \frac{x^2}{4} + \frac{x^3}{12} - \frac{5x^4}{192} + \frac{7x^5}{960} - \frac{7x^6}{3840} + \frac{11x^7}{26880} + O(x^8)\right)\ln(x)}{\sqrt{x}}$$
$$+ \frac{-4x + \frac{3x^2}{4} - \frac{x^3}{4} + \frac{31x^4}{384} - \frac{3x^5}{128} + \frac{419x^6}{69120} - \frac{97x^7}{69120} + O(x^8)}{\sqrt{x}}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= \frac{c_1\left(1+x-\frac{x^2}{4}+\frac{x^3}{12}-\frac{5x^4}{192}+\frac{7x^5}{960}-\frac{7x^6}{3840}+\frac{11x^7}{26880}+O(x^8)\right)}{\sqrt{x}} \\
&\quad + c_2\left(\frac{\left(1+x-\frac{x^2}{4}+\frac{x^3}{12}-\frac{5x^4}{192}+\frac{7x^5}{960}-\frac{7x^6}{3840}+\frac{11x^7}{26880}+O(x^8)\right)\ln(x)}{\sqrt{x}}\right. \\
&\qquad \left.+\frac{-4x+\frac{3x^2}{4}-\frac{x^3}{4}+\frac{31x^4}{384}-\frac{3x^5}{128}+\frac{419x^6}{69120}-\frac{97x^7}{69120}+O(x^8)}{\sqrt{x}}\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= \frac{c_1\left(1+x-\frac{x^2}{4}+\frac{x^3}{12}-\frac{5x^4}{192}+\frac{7x^5}{960}-\frac{7x^6}{3840}+\frac{11x^7}{26880}+O(x^8)\right)}{\sqrt{x}} \\
&\quad + c_2\left(\frac{\left(1+x-\frac{x^2}{4}+\frac{x^3}{12}-\frac{5x^4}{192}+\frac{7x^5}{960}-\frac{7x^6}{3840}+\frac{11x^7}{26880}+O(x^8)\right)\ln(x)}{\sqrt{x}}\right. \\
&\qquad \left.+\frac{-4x+\frac{3x^2}{4}-\frac{x^3}{4}+\frac{31x^4}{384}-\frac{3x^5}{128}+\frac{419x^6}{69120}-\frac{97x^7}{69120}+O(x^8)}{\sqrt{x}}\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= \frac{c_1\left(1+x-\frac{x^2}{4}+\frac{x^3}{12}-\frac{5x^4}{192}+\frac{7x^5}{960}-\frac{7x^6}{3840}+\frac{11x^7}{26880}+O(x^8)\right)}{\sqrt{x}} \\
&\quad + c_2\left(\frac{\left(1+x-\frac{x^2}{4}+\frac{x^3}{12}-\frac{5x^4}{192}+\frac{7x^5}{960}-\frac{7x^6}{3840}+\frac{11x^7}{26880}+O(x^8)\right)\ln(x)}{\sqrt{x}}\right. \\
&\qquad \left.+\frac{-4x+\frac{3x^2}{4}-\frac{x^3}{4}+\frac{31x^4}{384}-\frac{3x^5}{128}+\frac{419x^6}{69120}-\frac{97x^7}{69120}+O(x^8)}{\sqrt{x}}\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\left(1+x-\frac{x^2}{4}+\frac{x^3}{12}-\frac{5x^4}{192}+\frac{7x^5}{960}-\frac{7x^6}{3840}+\frac{11x^7}{26880}+O(x^8)\right)}{\sqrt{x}} + c_2\left(\frac{\left(1+x-\frac{x^2}{4}+\frac{x^3}{12}-\frac{5x^4}{192}+\frac{7x^5}{960}-\frac{7x^6}{3840}+\frac{11x^7}{26880}+O(x^8)\right)\ln(x)}{\sqrt{x}} + \frac{-4x+\frac{3x^2}{4}-\frac{x^3}{4}+\frac{31x^4}{384}-\frac{3x^5}{128}+\frac{419x^6}{69120}-\frac{97x^7}{69120}+O(x^8)}{\sqrt{x}}\right)$$

Verified OK.

5.15.1 Maple step by step solution

Let's solve

$4x^2y''+(8x^2+8x)\,y'+y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{4x^2}-\frac{2(1+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2(1+x)y'}{x}+\frac{y}{4x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{2(1+x)}{x}, P_3(x)=\frac{1}{4x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\Big|_{x=0}=2$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\Big|_{x=0}=\frac{1}{4}$

  ◦ $x=0$is a regular singular point

    Check to see if $x_0=0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

  $4x^2 y'' + 8x(1+x)\, y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  - Shift index using $k- > k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 (1+2r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k (2k+2r+1)^2 + 8a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = -\frac{1}{2}$

- Each term in the series must be 0, giving the recursion relation

  $a_k (2k+2r+1)^2 + 8a_{k-1}(k+r-1) = 0$

- Shift index using $k- > k+1$

  $a_{k+1}(2k+3+2r)^2 + 8a_k (k+r) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{8a_k (k+r)}{(2k+3+2r)^2}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+1} = -\frac{8a_k \left(k-\frac{1}{2}\right)}{(2k+2)^2}$

- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = -\frac{8a_k \left(k-\frac{1}{2}\right)}{\left(2k+2\right)^2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 81

```
Order:=8;
dsolve(4*x^2*diff(y(x),x$2)+8*x*(x+1)*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{(c_2 \ln(x) + c_1)\left(1 + x - \frac{1}{4}x^2 + \frac{1}{12}x^3 - \frac{5}{192}x^4 + \frac{7}{960}x^5 - \frac{7}{3840}x^6 + \frac{11}{26880}x^7 + \mathrm{O}(x^8)\right) + \left((-4)x + \frac{3}{4}x^2 - \frac{1}{4}\right.}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 166

```
AsymptoticDSolveValue[4*x^2*y''[x]+8*x*(x+1)*y'[x]+y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to \frac{c_1\left(\frac{11x^7}{26880} - \frac{7x^6}{3840} + \frac{7x^5}{960} - \frac{5x^4}{192} + \frac{x^3}{12} - \frac{x^2}{4} + x + 1\right)}{\sqrt{x}} + c_2\left(\frac{-\frac{97x^7}{69120} + \frac{419x^6}{69120} - \frac{3x^5}{128} + \frac{31x^4}{384} - \frac{x^3}{4} + \frac{3x^2}{4} - 4x}{\sqrt{x}} + \frac{\left(\frac{11x^7}{26880} - \frac{7x^6}{3840} + \frac{7x^5}{960} - \frac{5x^4}{192} + \frac{x^3}{12} - \frac{x^2}{4} + x + 1\right)\log(x)}{\sqrt{x}}\right)$$

5.16 problem 16

Internal problem ID [6963]
Internal file name [`OUTPUT/6206_Friday_August_12_2022_11_05_21_PM_7363715/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.6. Indicial Equation with Equal Roots. Exercises page 373
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + 3x(1+x)\,y' + (-3x+1)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(3x^2 + 3x\right)y' + (-3x+1)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{3x+3}{x}$$
$$q(x) = -\frac{3x-1}{x^2}$$

Table 116: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3x+3}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{3x-1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + \left(3x^2 + 3x\right) y' + (-3x + 1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}\right) + \left(3x^2 + 3x\right) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (-3x + 1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-3x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 3x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-3x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-3a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \sum_{n=1}^{\infty} \left(-3a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + 3x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + 3x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + 3x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r+1)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r+1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -1$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r+1)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = -1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-1}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n-1}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 3a_{n-1}(n+r-1) + 3a_n(n+r) - 3a_{n-1} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{3a_{n-1}(n+r-2)}{n^2+2nr+r^2+2n+2r+1} \tag{4}$$

Which for the root $r = -1$ becomes

$$a_n = -\frac{3a_{n-1}(n-3)}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{3-3r}{(r+2)^2}$$

Which for the root $r = -1$ becomes

$$a_1 = 6$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3-3r}{(r+2)^2}$	6

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{9r(-1+r)}{(r+2)^2 (r+3)^2}$$

Which for the root $r = -1$ becomes

$$a_2 = \frac{9}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3-3r}{(r+2)^2}$	6
a_2	$\frac{9r(-1+r)}{(r+2)^2(r+3)^2}$	$\frac{9}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-27r^3 + 27r}{(r+2)^2 (r+3)^2 (r+4)^2}$$

Which for the root $r = -1$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3-3r}{(r+2)^2}$	6
a_2	$\frac{9r(-1+r)}{(r+2)^2(r+3)^2}$	$\frac{9}{2}$
a_3	$\frac{-27r^3+27r}{(r+2)^2(r+3)^2(r+4)^2}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{81r^3 - 81r}{(r+2)(r+3)^2 (r+4)^2 (r+5)^2}$$

Which for the root $r = -1$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3-3r}{(r+2)^2}$	6
a_2	$\frac{9r(-1+r)}{(r+2)^2(r+3)^2}$	$\frac{9}{2}$
a_3	$\frac{-27r^3+27r}{(r+2)^2(r+3)^2(r+4)^2}$	0
a_4	$\frac{81r^3-81r}{(r+2)(r+3)^2(r+4)^2(r+5)^2}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-243r^3 + 243r}{(r+2)(r+3)(r+4)^2(r+5)^2(r+6)^2}$$

Which for the root $r = -1$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3-3r}{(r+2)^2}$	6
a_2	$\frac{9r(-1+r)}{(r+2)^2(r+3)^2}$	$\frac{9}{2}$
a_3	$\frac{-27r^3+27r}{(r+2)^2(r+3)^2(r+4)^2}$	0
a_4	$\frac{81r^3-81r}{(r+2)(r+3)^2(r+4)^2(r+5)^2}$	0
a_5	$\frac{-243r^3+243r}{(r+2)(r+3)(r+4)^2(r+5)^2(r+6)^2}$	0

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{729r^3 - 729r}{(r+2)(r+3)(r+4)(r+5)^2(r+6)^2(r+7)^2}$$

Which for the root $r = -1$ becomes

$$a_6 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3-3r}{(r+2)^2}$	6
a_2	$\frac{9r(-1+r)}{(r+2)^2(r+3)^2}$	$\frac{9}{2}$
a_3	$\frac{-27r^3+27r}{(r+2)^2(r+3)^2(r+4)^2}$	0
a_4	$\frac{81r^3-81r}{(r+2)(r+3)^2(r+4)^2(r+5)^2}$	0
a_5	$\frac{-243r^3+243r}{(r+2)(r+3)(r+4)^2(r+5)^2(r+6)^2}$	0
a_6	$\frac{729r^3-729r}{(r+2)(r+3)(r+4)(r+5)^2(r+6)^2(r+7)^2}$	0

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{-2187r^3 + 2187r}{(r+2)(r+3)(r+4)(r+5)(r+6)^2(r+7)^2(8+r)^2}$$

Which for the root $r = -1$ becomes

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3-3r}{(r+2)^2}$	6
a_2	$\frac{9r(-1+r)}{(r+2)^2(r+3)^2}$	$\frac{9}{2}$
a_3	$\frac{-27r^3+27r}{(r+2)^2(r+3)^2(r+4)^2}$	0
a_4	$\frac{81r^3-81r}{(r+2)(r+3)^2(r+4)^2(r+5)^2}$	0
a_5	$\frac{-243r^3+243r}{(r+2)(r+3)(r+4)^2(r+5)^2(r+6)^2}$	0
a_6	$\frac{729r^3-729r}{(r+2)(r+3)(r+4)(r+5)^2(r+6)^2(r+7)^2}$	0
a_7	$\frac{-2187r^3+2187r}{(r+2)(r+3)(r+4)(r+5)(r+6)^2(r+7)^2(8+r)^2}$	0

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \frac{1}{x}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \dots\right) \\ &= \frac{1 + 6x + \frac{9x^2}{2} + O(x^8)}{x} \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr}a_{n,r}$$

And the above is then evaluated at $r = -1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	$\frac{3-3r}{(r+2)^2}$	6	$\frac{3r-12}{(r+2)^3}$
b_2	$\frac{9r(-1+r)}{(r+2)^2(r+3)^2}$	$\frac{9}{2}$	$\frac{-18r^3+27r^2+153r-54}{(r+2)^3(r+3)^3}$
b_3	$\frac{-27r^3+27r}{(r+2)^2(r+3)^2(r+4)^2}$	0	$\frac{81r^5+243r^4-837r^3-2673r^2-702r+648}{(r+2)^3(r+3)^3(r+4)^3}$
b_4	$\frac{81r^3-81r}{(r+2)(r+3)^2(r+4)^2(r+5)^2}$	0	$-\frac{162(2r^6+15r^5+9r^4-136r^3-263r^2-47r+60)}{(r+2)^2(r+3)^3(r+4)^3(r+5)^3}$
b_5	$\frac{-243r^3+243r}{(r+2)(r+3)(r+4)^2(r+5)^2(r+6)^2}$	0	$\frac{1215r^7+15795r^6+57105r^5-32805r^4-533628r^3-799470r^2-107892r+174960}{(r+2)^2(r+3)^2(r+4)^3(r+5)^3(r+6)^3}$
b_6	$\frac{729r^3-729r}{(r+2)(r+3)(r+4)(r+5)^2(r+6)^2(r+7)^2}$	0	$-\frac{729(6r^8+117r^7+796r^6+1800r^5-2852r^4-18789r^3-23870r^2-2568r+5040)}{(r+2)^2(r+3)^2(r+4)^2(r+5)^3(r+6)^3(r+7)^3}$
b_7	$\frac{-2187r^3+2187r}{(r+2)(r+3)(r+4)(r+5)(r+6)^2(r+7)^2(8+r)^2}$	0	$\frac{15309r^9+413343r^8+4286520r^7+20238498r^6+31061961r^5-85072113r^4-3828343}{(r+2)^2(r+3)^2(r+4)^2(r+5)^2(r+6)^3(r+7)^3(8}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \ldots \\
&= \frac{\left(1 + 6x + \frac{9x^2}{2} + O(x^8)\right)\ln(x)}{x} + \frac{-15x - \frac{81x^2}{4} - \frac{3x^3}{2} + \frac{9x^4}{32} - \frac{27x^5}{400} + \frac{27x^6}{1600} - \frac{81x^7}{19600} + O(x^8)}{x}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= \frac{c_1\left(1 + 6x + \frac{9x^2}{2} + O(x^8)\right)}{x} + c_2\left(\frac{\left(1 + 6x + \frac{9x^2}{2} + O(x^8)\right)\ln(x)}{x}\right. \\
&\quad \left. + \frac{-15x - \frac{81x^2}{4} - \frac{3x^3}{2} + \frac{9x^4}{32} - \frac{27x^5}{400} + \frac{27x^6}{1600} - \frac{81x^7}{19600} + O(x^8)}{x}\right)
\end{aligned}
$$

Hence the final solution is

$$
y = y_h
$$

$$
= \frac{c_1\left(1+6x+\frac{9x^2}{2}+O(x^8)\right)}{x} + c_2\left(\frac{\left(1+6x+\frac{9x^2}{2}+O(x^8)\right)\ln(x)}{x} + \frac{-15x-\frac{81x^2}{4}-\frac{3x^3}{2}+\frac{9x^4}{32}-\frac{27x^5}{400}+\frac{27x^6}{1600}-\frac{81x^7}{19600}+O(x^8)}{x}\right)
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\left(1+6x+\frac{9x^2}{2}+O(x^8)\right)}{x} + c_2\left(\frac{\left(1+6x+\frac{9x^2}{2}+O(x^8)\right)\ln(x)}{x} + \frac{-15x-\frac{81x^2}{4}-\frac{3x^3}{2}+\frac{9x^4}{32}-\frac{27x^5}{400}+\frac{27x^6}{1600}-\frac{81x^7}{19600}+O(x^8)}{x}\right) \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1\left(1+6x+\frac{9x^2}{2}+O(x^8)\right)}{x} + c_2\left(\frac{\left(1+6x+\frac{9x^2}{2}+O(x^8)\right)\ln(x)}{x} + \frac{-15x-\frac{81x^2}{4}-\frac{3x^3}{2}+\frac{9x^4}{32}-\frac{27x^5}{400}+\frac{27x^6}{1600}-\frac{81x^7}{19600}+O(x^8)}{x}\right)
$$

Verified OK.

5.16.1 Maple step by step solution

Let's solve

$x^2y'' + (3x^2+3x)\,y' + (-3x+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(3x-1)y}{x^2} - \frac{3(1+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3(1+x)y'}{x} - \frac{(3x-1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x)=\frac{3(1+x)}{x}, P_3(x)=-\frac{3x-1}{x^2}\right]$

- $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\Big|_{x=0}=3$

- $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=1$

- $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2y''+3x(1+x)\,y'+(-3x+1)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m\cdot y$ to series expansion for $m=0..1$

  $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m\cdot y'$ to series expansion for $m=1..2$

  $x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^2\cdot y''$ to series expansion

  $x^2\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

Rewrite ODE with series expansions

$$a_0(1+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)^2 + 3a_{k-1}(k-2+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = -1$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)^2 + 3a_{k-1}(k-2+r) = 0$

- Shift index using $k- >k+1$

  $a_{k+1}(k+2+r)^2 + 3a_k(k+r-1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{3a_k(k+r-1)}{(k+2+r)^2}$

- Recursion relation for $r = -1$; series terminates at $k = 2$

  $a_{k+1} = -\frac{3a_k(k-2)}{(k+1)^2}$

- Apply recursion relation for $k = 0$

  $a_1 = 6a_0$

- Apply recursion relation for $k = 1$

  $a_2 = \frac{3a_1}{4}$

- Express in terms of a_0

  $a_2 = \frac{9a_0}{2}$

- Terminating series solution of the ODE for $r = -1$. Use reduction of order to find the second

  $y = a_0 \cdot \left(1 + 6x + \frac{9}{2}x^2\right)$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 61

```
Order:=8;
dsolve(x^2*diff(y(x),x$2)+3*x*(1+x)*diff(y(x),x)+(1-3*x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{(c_2 \ln(x) + c_1)\left(1 + 6x + \frac{9}{2}x^2 + \mathrm{O}(x^8)\right) + \left((-15)x - \frac{81}{4}x^2 - \frac{3}{2}x^3 + \frac{9}{32}x^4 - \frac{27}{400}x^5 + \frac{27}{1600}x^6 - \frac{81}{19600}x^7 + \cdots\right.}{x}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 94

```
AsymptoticDSolveValue[x^2*y''[x]+3*x*(1+x)*y'[x]+(1-3*x)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to \frac{c_1\left(\frac{9x^2}{2} + 6x + 1\right)}{x} + c_2\left(\frac{\left(\frac{9x^2}{2} + 6x + 1\right)\log(x)}{x} + \frac{-\frac{81x^7}{19600} + \frac{27x^6}{1600} - \frac{27x^5}{400} + \frac{9x^4}{32} - \frac{3x^3}{2} - \frac{81x^2}{4} - 15x}{x}\right)$$

5.17 problem 17

Internal problem ID [6964]
Internal file name [`OUTPUT/6207_Friday_August_12_2022_11_05_23_PM_7488110/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.6. Indicial Equation with Equal Roots. Exercises page 373
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{xy'' + (1-x)\,y' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + (1-x)\,y' - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x-1}{x}$$
$$q(x) = -\frac{1}{x}$$

Table 118: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x-1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + (1 - x) y' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x\left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) + (1 - x) \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} (n+r-1) x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} (n+r-1) x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) + a_n(n+r) - a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{n+r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{1+r}$$

Which for the root $r = 0$ becomes

$$a_1 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(1+r)(2+r)}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	1
a_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{(3+r)(1+r)(2+r)}$$

Which for the root $r = 0$ becomes

$$a_3 = \frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	1
a_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{2}$
a_3	$\frac{1}{(3+r)(1+r)(2+r)}$	$\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(4+r)(3+r)(1+r)(2+r)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	1
a_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{2}$
a_3	$\frac{1}{(3+r)(1+r)(2+r)}$	$\frac{1}{6}$
a_4	$\frac{1}{(4+r)(3+r)(1+r)(2+r)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{(4+r)(3+r)(1+r)(2+r)(5+r)}$$

Which for the root $r = 0$ becomes

$$a_5 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	1
a_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{2}$
a_3	$\frac{1}{(3+r)(1+r)(2+r)}$	$\frac{1}{6}$
a_4	$\frac{1}{(4+r)(3+r)(1+r)(2+r)}$	$\frac{1}{24}$
a_5	$\frac{1}{(4+r)(3+r)(1+r)(2+r)(5+r)}$	$\frac{1}{120}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{1}{(4+r)(3+r)(1+r)(2+r)(5+r)(6+r)}$$

Which for the root $r = 0$ becomes

$$a_6 = \frac{1}{720}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	1
a_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{2}$
a_3	$\frac{1}{(3+r)(1+r)(2+r)}$	$\frac{1}{6}$
a_4	$\frac{1}{(4+r)(3+r)(1+r)(2+r)}$	$\frac{1}{24}$
a_5	$\frac{1}{(4+r)(3+r)(1+r)(2+r)(5+r)}$	$\frac{1}{120}$
a_6	$\frac{1}{(4+r)(3+r)(1+r)(2+r)(5+r)(6+r)}$	$\frac{1}{720}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{1}{(4+r)(3+r)(1+r)(2+r)(5+r)(6+r)(7+r)}$$

Which for the root $r = 0$ becomes

$$a_7 = \frac{1}{5040}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	1
a_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{2}$
a_3	$\frac{1}{(3+r)(1+r)(2+r)}$	$\frac{1}{6}$
a_4	$\frac{1}{(4+r)(3+r)(1+r)(2+r)}$	$\frac{1}{24}$
a_5	$\frac{1}{(4+r)(3+r)(1+r)(2+r)(5+r)}$	$\frac{1}{120}$
a_6	$\frac{1}{(4+r)(3+r)(1+r)(2+r)(5+r)(6+r)}$	$\frac{1}{720}$
a_7	$\frac{1}{(4+r)(3+r)(1+r)(2+r)(5+r)(6+r)(7+r)}$	$\frac{1}{5040}$

Using the above table, then the first solution $y_1(x)$ becomes

$$
\begin{aligned}
y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots \\
&= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r=0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{1}{1+r}$	1	$-\frac{1}{(1+r)^2}$	-1
b_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{2}$	$\frac{-3-2r}{(1+r)^2(2+r)^2}$	$-\frac{3}{4}$
b_3	$\frac{1}{(3+r)(1+r)(2+r)}$	$\frac{1}{6}$	$\frac{-3r^2-12r-11}{(3+r)^2(1+r)^2(2+r)^2}$	$-\frac{11}{36}$
b_4	$\frac{1}{(4+r)(3+r)(1+r)(2+r)}$	$\frac{1}{24}$	$\frac{-4r^3-30r^2-70r-50}{(4+r)^2(3+r)^2(1+r)^2(2+r)^2}$	$-\frac{25}{288}$
b_5	$\frac{1}{(4+r)(3+r)(1+r)(2+r)(5+r)}$	$\frac{1}{120}$	$\frac{-5r^4-60r^3-255r^2-450r-274}{(4+r)^2(3+r)^2(1+r)^2(2+r)^2(5+r)^2}$	$-\frac{137}{7200}$
b_6	$\frac{1}{(4+r)(3+r)(1+r)(2+r)(5+r)(6+r)}$	$\frac{1}{720}$	$\frac{-6r^5-105r^4-700r^3-2205r^2-3248r-1764}{(4+r)^2(3+r)^2(1+r)^2(2+r)^2(5+r)^2(6+r)^2}$	$-\frac{49}{14400}$
b_7	$\frac{1}{(4+r)(3+r)(1+r)(2+r)(5+r)(6+r)(7+r)}$	$\frac{1}{5040}$	$\frac{-7r^6-168r^5-1610r^4-7840r^3-20307r^2-26264r-13068}{(4+r)^2(3+r)^2(1+r)^2(2+r)^2(5+r)^2(6+r)^2(7+r)^2}$	$-\frac{121}{235200}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 + b_7x^7 + b_8x^8\dots \\
&= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8)\right)\ln(x) \\
&\quad - x - \frac{3x^2}{4} - \frac{11x^3}{36} - \frac{25x^4}{288} - \frac{137x^5}{7200} - \frac{49x^6}{14400} - \frac{121x^7}{235200} + O(x^8)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8)\right) \\
&\quad + c_2\left(\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8)\right)\ln(x) - x - \frac{3x^2}{4} \right. \\
&\quad \left. - \frac{11x^3}{36} - \frac{25x^4}{288} - \frac{137x^5}{7200} - \frac{49x^6}{14400} - \frac{121x^7}{235200} + O(x^8)\right)
\end{aligned}
$$

Hence the final solution is

$$y = y_h$$

$$
= c_1\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}+\frac{x^7}{5040}+O\left(x^8\right)\right) + c_2\left(\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}+\frac{x^7}{5040}+O\left(x^8\right)\right)\ln\left(x\right)-x-\frac{3x^2}{4} - \frac{11x^3}{36}-\frac{25x^4}{288}-\frac{137x^5}{7200}-\frac{49x^6}{14400}-\frac{121x^7}{235200}+O\left(x^8\right)\right)
$$

Summary

The solution(s) found are the following

$$
y = c_1\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}+\frac{x^7}{5040}+O\left(x^8\right)\right) + c_2\left(\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}+\frac{x^7}{5040}+O\left(x^8\right)\right)\ln\left(x\right)-x-\frac{3x^2}{4} - \frac{11x^3}{36}-\frac{25x^4}{288}-\frac{137x^5}{7200}-\frac{49x^6}{14400}-\frac{121x^7}{235200}+O\left(x^8\right)\right) \tag{1}
$$

Verification of solutions

$$
y = c_1\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}+\frac{x^7}{5040}+O\left(x^8\right)\right) + c_2\left(\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}+\frac{x^7}{5040}+O\left(x^8\right)\right)\ln\left(x\right)-x-\frac{3x^2}{4} - \frac{11x^3}{36}-\frac{25x^4}{288}-\frac{137x^5}{7200}-\frac{49x^6}{14400}-\frac{121x^7}{235200}+O\left(x^8\right)\right)
$$

Verified OK.

5.17.1 Maple step by step solution

Let's solve

$xy'' + (1-x)\,y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x-1)y'}{x} + \frac{y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x-1)y'}{x} - \frac{y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = -\frac{x-1}{x}, P_3(x) = -\frac{1}{x}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + (1-x)\,y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k- >k+1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)^2 - a_k(k+1+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+1)\left(a_{k+1}(k+1) - a_k\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k}{k+1}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 71

```
Order:=8;
dsolve(x*diff(y(x),x$2)+(1-x)*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln(x) + c_1)\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 + \mathrm{O}\left(x^8\right)\right) + \left(-x - \frac{3}{4}x^2 - \frac{11}{36}x^3 - \frac{25}{288}x^4 - \frac{137}{7200}x^5 - \frac{49}{14400}x^6 - \frac{121}{235200}x^7 + \mathrm{O}\left(x^8\right)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 149

```
AsymptoticDSolveValue[x*y''[x]+(1-x)*y'[x]-y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(\frac{x^7}{5040} + \frac{x^6}{720} + \frac{x^5}{120} + \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + x + 1\right) + c_2\left(-\frac{121x^7}{235200} - \frac{49x^6}{14400} - \frac{137x^5}{7200} - \frac{25x^4}{288} - \frac{11x^3}{36} - \frac{3x^2}{4} + \left(\frac{x^7}{5040} + \frac{x^6}{720} + \frac{x^5}{120} + \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + x + 1\right)\log(x) - x\right)$$

6 CHAPTER 18. Power series solutions. 18.8 Indicial Equation with Difference of Roots a Positive Integer: Nonlogarithmic Case. Exercises page 380

6.1 problem 1

Internal problem ID [6965]
Internal file name [`OUTPUT/6208_Friday_August_12_2022_11_05_25_PM_43924496/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.8 Indicial Equation with Difference of Roots a Positive Integer: Nonlogarithmic Case. Exercises page 380
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + 2x(x-2)\,y' + 2(2-3x)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(2x^2 - 4x\right)y' + (-6x+4)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{2x-4}{x}$$
$$q(x) = -\frac{2(3x-2)}{x^2}$$

Table 120: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2x-4}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{2(3x-2)}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + (2x^2 - 4x)\, y' + (-6x + 4)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}\right) + (2x^2 - 4x) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (-6x + 4) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)\right) \tag{2A}
$$

$$
+ \sum_{n=0}^{\infty} \left(-4x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-6x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r}
$$

$$
\sum_{n=0}^{\infty} \left(-6x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-6a_{n-1} x^{n+r}\right)
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r}\right) \tag{2B}
$$

$$
+ \sum_{n=0}^{\infty} \left(-4x^{n+r} a_n (n+r)\right) + \sum_{n=1}^{\infty} \left(-6a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r} a_n (n+r)(n+r-1) - 4x^{n+r} a_n (n+r) + 4a_n x^{n+r} = 0
$$

When $n=0$ the above becomes

$$
x^r a_0 r(-1+r) - 4x^r a_0 r + 4a_0 x^r = 0
$$

Or

$$
\left(x^r r(-1+r) - 4x^r r + 4x^r\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
\left(r^2 - 5r + 4\right) x^r = 0
$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 5r + 4 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 4$$
$$r_2 = 1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - 5r + 4\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^4 \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+4}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{1+n}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 2a_{n-1}(n+r-1) - 4a_n(n+r) - 6a_{n-1} + 4a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2a_{n-1}}{n+r-1} \tag{4}$$

Which for the root $r = 4$ becomes

$$a_n = -\frac{2a_{n-1}}{n+3} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 4$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{2}{r}$$

Which for the root $r = 4$ becomes

$$a_1 = -\frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{r}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4}{r(1+r)}$$

Which for the root $r = 4$ becomes

$$a_2 = \frac{1}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{r}$	$-\frac{1}{2}$
a_2	$\frac{4}{r(1+r)}$	$\frac{1}{5}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{8}{r(1+r)(2+r)}$$

Which for the root $r = 4$ becomes

$$a_3 = -\frac{1}{15}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{r}$	$-\frac{1}{2}$
a_2	$\frac{4}{r(1+r)}$	$\frac{1}{5}$
a_3	$-\frac{8}{r(1+r)(2+r)}$	$-\frac{1}{15}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{(1+r)(2+r)r(3+r)}$$

Which for the root $r = 4$ becomes

$$a_4 = \frac{2}{105}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{r}$	$-\frac{1}{2}$
a_2	$\frac{4}{r(1+r)}$	$\frac{1}{5}$
a_3	$-\frac{8}{r(1+r)(2+r)}$	$-\frac{1}{15}$
a_4	$\frac{16}{(1+r)(2+r)r(3+r)}$	$\frac{2}{105}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{32}{(2+r)r(3+r)(1+r)(4+r)}$$

Which for the root $r = 4$ becomes

$$a_5 = -\frac{1}{210}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{r}$	$-\frac{1}{2}$
a_2	$\frac{4}{r(1+r)}$	$\frac{1}{5}$
a_3	$-\frac{8}{r(1+r)(2+r)}$	$-\frac{1}{15}$
a_4	$\frac{16}{(1+r)(2+r)r(3+r)}$	$\frac{2}{105}$
a_5	$-\frac{32}{(2+r)r(3+r)(1+r)(4+r)}$	$-\frac{1}{210}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{64}{r(3+r)(1+r)(4+r)(5+r)(2+r)}$$

Which for the root $r = 4$ becomes

$$a_6 = \frac{1}{945}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{r}$	$-\frac{1}{2}$
a_2	$\frac{4}{r(1+r)}$	$\frac{1}{5}$
a_3	$-\frac{8}{r(1+r)(2+r)}$	$-\frac{1}{15}$
a_4	$\frac{16}{(1+r)(2+r)r(3+r)}$	$\frac{2}{105}$
a_5	$-\frac{32}{(2+r)r(3+r)(1+r)(4+r)}$	$-\frac{1}{210}$
a_6	$\frac{64}{r(3+r)(1+r)(4+r)(5+r)(2+r)}$	$\frac{1}{945}$

For $n = 7$, using the above recursive equation gives

$$a_7 = -\frac{128}{r(1+r)(4+r)(5+r)(2+r)(3+r)(6+r)}$$

Which for the root $r = 4$ becomes

$$a_7 = -\frac{1}{4725}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{r}$	$-\frac{1}{2}$
a_2	$\frac{4}{r(1+r)}$	$\frac{1}{5}$
a_3	$-\frac{8}{r(1+r)(2+r)}$	$-\frac{1}{15}$
a_4	$\frac{16}{(1+r)(2+r)r(3+r)}$	$\frac{2}{105}$
a_5	$-\frac{32}{(2+r)r(3+r)(1+r)(4+r)}$	$-\frac{1}{210}$
a_6	$\frac{64}{r(3+r)(1+r)(4+r)(5+r)(2+r)}$	$\frac{1}{945}$
a_7	$-\frac{128}{r(1+r)(4+r)(5+r)(2+r)(3+r)(6+r)}$	$-\frac{1}{4725}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^4\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\
&= x^4\left(1 - \frac{x}{2} + \frac{x^2}{5} - \frac{x^3}{15} + \frac{2x^4}{105} - \frac{x^5}{210} + \frac{x^6}{945} - \frac{x^7}{4725} + O(x^8)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}
a_N &= a_3 \\
&= -\frac{8}{r(1+r)(2+r)}
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{r\to r_2} -\frac{8}{r(1+r)(2+r)} &= \lim_{r\to 1} -\frac{8}{r(1+r)(2+r)} \\
&= -\frac{4}{3}
\end{aligned}$$

The limit is $-\frac{4}{3}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{1+n}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
b_n(n+r)(n+r-1) + 2b_{n-1}(n+r-1) - 4b_n(n+r) - 6b_{n-1} + 4b_n = 0 \tag{4}
$$

Which for for the root $r = 1$ becomes

$$
b_n(1+n)\,n + 2b_{n-1}n - 4b_n(1+n) - 6b_{n-1} + 4b_n = 0 \tag{4A}
$$

Solving for b_n from the recursive equation (4) gives

$$
b_n = -\frac{2b_{n-1}}{n+r-1} \tag{5}
$$

Which for the root $r = 1$ becomes

$$
b_n = -\frac{2b_{n-1}}{n} \tag{6}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$
b_1 = -\frac{2}{r}
$$

Which for the root $r = 1$ becomes

$$
b_1 = -2
$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{r}$	-2

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4}{r(1+r)}$$

Which for the root $r = 1$ becomes

$$b_2 = 2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{r}$	-2
b_2	$\frac{4}{r(1+r)}$	2

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{8}{r(1+r)(2+r)}$$

Which for the root $r = 1$ becomes

$$b_3 = -\frac{4}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{r}$	-2
b_2	$\frac{4}{r(1+r)}$	2
b_3	$-\frac{8}{r(1+r)(2+r)}$	$-\frac{4}{3}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16}{(1+r)(2+r)r(3+r)}$$

Which for the root $r = 1$ becomes

$$b_4 = \frac{2}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{r}$	-2
b_2	$\frac{4}{r(1+r)}$	2
b_3	$-\frac{8}{r(1+r)(2+r)}$	$-\frac{4}{3}$
b_4	$\frac{16}{(1+r)(2+r)r(3+r)}$	$\frac{2}{3}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{32}{(2+r)\, r\,(3+r)\,(1+r)\,(4+r)}$$

Which for the root $r = 1$ becomes

$$b_5 = -\frac{4}{15}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{r}$	-2
b_2	$\frac{4}{r(1+r)}$	2
b_3	$-\frac{8}{r(1+r)(2+r)}$	$-\frac{4}{3}$
b_4	$\frac{16}{(1+r)(2+r)r(3+r)}$	$\frac{2}{3}$
b_5	$-\frac{32}{(2+r)r(3+r)(1+r)(4+r)}$	$-\frac{4}{15}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{64}{r\,(3+r)\,(1+r)\,(4+r)\,(5+r)\,(2+r)}$$

Which for the root $r = 1$ becomes

$$b_6 = \frac{4}{45}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{r}$	-2
b_2	$\frac{4}{r(1+r)}$	2
b_3	$-\frac{8}{r(1+r)(2+r)}$	$-\frac{4}{3}$
b_4	$\frac{16}{(1+r)(2+r)r(3+r)}$	$\frac{2}{3}$
b_5	$-\frac{32}{(2+r)r(3+r)(1+r)(4+r)}$	$-\frac{4}{15}$
b_6	$\frac{64}{r(3+r)(1+r)(4+r)(5+r)(2+r)}$	$\frac{4}{45}$

For $n = 7$, using the above recursive equation gives

$$b_7 = -\frac{128}{r(1+r)(4+r)(5+r)(2+r)(3+r)(6+r)}$$

Which for the root $r = 1$ becomes

$$b_7 = -\frac{8}{315}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{r}$	-2
b_2	$\frac{4}{r(1+r)}$	2
b_3	$-\frac{8}{r(1+r)(2+r)}$	$-\frac{4}{3}$
b_4	$\frac{16}{(1+r)(2+r)r(3+r)}$	$\frac{2}{3}$
b_5	$-\frac{32}{(2+r)r(3+r)(1+r)(4+r)}$	$-\frac{4}{15}$
b_6	$\frac{64}{r(3+r)(1+r)(4+r)(5+r)(2+r)}$	$\frac{4}{45}$
b_7	$-\frac{128}{r(1+r)(4+r)(5+r)(2+r)(3+r)(6+r)}$	$-\frac{8}{315}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}y_2(x) &= x^4\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots\right)\\ &= x\left(1 - 2x + 2x^2 - \frac{4x^3}{3} + \frac{2x^4}{3} - \frac{4x^5}{15} + \frac{4x^6}{45} - \frac{8x^7}{315} + O(x^8)\right)\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1 y_1(x) + c_2 y_2(x)\\ &= c_1 x^4\left(1 - \frac{x}{2} + \frac{x^2}{5} - \frac{x^3}{15} + \frac{2x^4}{105} - \frac{x^5}{210} + \frac{x^6}{945} - \frac{x^7}{4725} + O(x^8)\right)\\ &\quad + c_2 x\left(1 - 2x + 2x^2 - \frac{4x^3}{3} + \frac{2x^4}{3} - \frac{4x^5}{15} + \frac{4x^6}{45} - \frac{8x^7}{315} + O(x^8)\right)\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y &= y_h\\ &= c_1 x^4\left(1 - \frac{x}{2} + \frac{x^2}{5} - \frac{x^3}{15} + \frac{2x^4}{105} - \frac{x^5}{210} + \frac{x^6}{945} - \frac{x^7}{4725} + O(x^8)\right)\\ &\quad + c_2 x\left(1 - 2x + 2x^2 - \frac{4x^3}{3} + \frac{2x^4}{3} - \frac{4x^5}{15} + \frac{4x^6}{45} - \frac{8x^7}{315} + O(x^8)\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}y &= c_1 x^4\left(1 - \frac{x}{2} + \frac{x^2}{5} - \frac{x^3}{15} + \frac{2x^4}{105} - \frac{x^5}{210} + \frac{x^6}{945} - \frac{x^7}{4725} + O(x^8)\right)\\ &\quad + c_2 x\left(1 - 2x + 2x^2 - \frac{4x^3}{3} + \frac{2x^4}{3} - \frac{4x^5}{15} + \frac{4x^6}{45} - \frac{8x^7}{315} + O(x^8)\right)\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}y &= c_1 x^4\left(1 - \frac{x}{2} + \frac{x^2}{5} - \frac{x^3}{15} + \frac{2x^4}{105} - \frac{x^5}{210} + \frac{x^6}{945} - \frac{x^7}{4725} + O(x^8)\right)\\ &\quad + c_2 x\left(1 - 2x + 2x^2 - \frac{4x^3}{3} + \frac{2x^4}{3} - \frac{4x^5}{15} + \frac{4x^6}{45} - \frac{8x^7}{315} + O(x^8)\right)\end{aligned}$$

Verified OK.

6.1.1 Maple step by step solution

Let's solve

$x^2 y'' + (2x^2 - 4x) y' + (-6x + 4) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2(3x-2)y}{x^2} - \frac{2(x-2)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2(x-2)y'}{x} - \frac{2(3x-2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{2(x-2)}{x}, P_3(x) = -\frac{2(3x-2)}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = -4$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 4$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' + 2x(x-2) y' + (-6x + 4) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- > k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-4+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)\,(k+r-4) + 2a_{k-1}(k+r-4)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-4+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{1, 4\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-4)\,(a_k(k+r-1) + 2a_{k-1}) = 0$

- Shift index using $k- > k + 1$

  $(k+r-3)\,(a_{k+1}(k+r) + 2a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{2a_k}{k+r}$

- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{2a_k}{k+1}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{2a_k}{k+1}\right]$

- Recursion relation for $r = 4$

  $a_{k+1} = -\frac{2a_k}{k+4}$

- Solution for $r=4$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+4}, a_{k+1}=-\frac{2a_k}{k+4}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+4}\right), a_{k+1}=-\frac{2a_k}{k+1}, b_{k+1}=-\frac{2b_k}{k+4}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 53

```
Order:=8;
dsolve(x^2*diff(y(x),x$2)+2*x*(x-2)*diff(y(x),x)+2*(2-3*x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=c_1x^4\left(1-\frac{1}{2}x+\frac{1}{5}x^2-\frac{1}{15}x^3+\frac{2}{105}x^4-\frac{1}{210}x^5+\frac{1}{945}x^6-\frac{1}{4725}x^7+\mathrm{O}\left(x^8\right)\right)$$
$$+c_2x\left(12-24x+24x^2-16x^3+8x^4-\frac{16}{5}x^5+\frac{16}{15}x^6-\frac{32}{105}x^7+\mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.101 (sec). Leaf size: 96

```
AsymptoticDSolveValue[x^2*y''[x]+2*x*(x-2)*y'[x]+2*(2-3*x)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(\frac{4x^7}{45} - \frac{4x^6}{15} + \frac{2x^5}{3} - \frac{4x^4}{3} + 2x^3 - 2x^2 + x\right) + c_2\left(\frac{x^{10}}{945} - \frac{x^9}{210} + \frac{2x^8}{105} - \frac{x^7}{15} + \frac{x^6}{5} - \frac{x^5}{2} + x^4\right)$$

6.2 problem 2

Internal problem ID [6966]
Internal file name [`OUTPUT/6209_Friday_August_12_2022_11_05_28_PM_13701037/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.8 Indicial Equation with Difference of Roots a Positive Integer: Nonlogarithmic Case. Exercises page 380
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(2x+1)\,y''+2x(1+6x)\,y'-2y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(2x^3+x^2\right)y''+\left(12x^2+2x\right)y'-2y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{2+12x}{x\,(2x+1)}$$
$$q(x)=-\frac{2}{x^2\,(2x+1)}$$

Table 122: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2+12x}{x(2x+1)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{1}{2}$	"regular"

$q(x) = -\frac{2}{x^2(2x+1)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{1}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{1}{2}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(2x+1)\, y'' + \left(12x^2 + 2x\right) y' - 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(2x+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(12x^2 + 2x\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 12x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 12x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 12a_{n-1}(n+r-1)\, x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 12a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + 2x^{n+r} a_n (n+r) - 2a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + 2x^r a_0 r - 2a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + 2x^r r - 2x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2+r-2\right)x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2+r-2 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 1 \\ r_2 &= -2 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2+r-2\right)x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x)\ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2} \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{1+n} \\ y_2(x) &= C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots

of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} 2a_{n-1}(n+r-1)(n+r-2)+a_n(n+r)(n+r-1) \\ +12a_{n-1}(n+r-1)+2a_n(n+r)-2a_n=0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2(n+r+4)\,a_{n-1}}{n+r+2} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{2(n+5)\,a_{n-1}}{n+3} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-10-2r}{3+r}$$

Which for the root $r = 1$ becomes

$$a_1 = -3$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-10-2r}{3+r}$	-3

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4r^2+44r+120}{(3+r)(r+4)}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{42}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-10-2r}{3+r}$	-3
a_2	$\frac{4r^2+44r+120}{(3+r)(r+4)}$	$\frac{42}{5}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-8r^2 - 104r - 336}{(3 + r)(r + 4)}$$

Which for the root $r = 1$ becomes

$$a_3 = -\frac{112}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-10-2r}{3+r}$	-3
a_2	$\frac{4r^2+44r+120}{(3+r)(r+4)}$	$\frac{42}{5}$
a_3	$\frac{-8r^2-104r-336}{(3+r)(r+4)}$	$-\frac{112}{5}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16r^2 + 240r + 896}{(3 + r)(r + 4)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{288}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-10-2r}{3+r}$	-3
a_2	$\frac{4r^2+44r+120}{(3+r)(r+4)}$	$\frac{42}{5}$
a_3	$\frac{-8r^2-104r-336}{(3+r)(r+4)}$	$-\frac{112}{5}$
a_4	$\frac{16r^2+240r+896}{(3+r)(r+4)}$	$\frac{288}{5}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-32r^2 - 544r - 2304}{(3+r)(r+4)}$$

Which for the root $r = 1$ becomes

$$a_5 = -144$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-10-2r}{3+r}$	-3
a_2	$\frac{4r^2+44r+120}{(3+r)(r+4)}$	$\frac{42}{5}$
a_3	$\frac{-8r^2-104r-336}{(3+r)(r+4)}$	$-\frac{112}{5}$
a_4	$\frac{16r^2+240r+896}{(3+r)(r+4)}$	$\frac{288}{5}$
a_5	$\frac{-32r^2-544r-2304}{(3+r)(r+4)}$	-144

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{64r^2 + 1216r + 5760}{(3+r)(r+4)}$$

Which for the root $r = 1$ becomes

$$a_6 = 352$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-10-2r}{3+r}$	-3
a_2	$\frac{4r^2+44r+120}{(3+r)(r+4)}$	$\frac{42}{5}$
a_3	$\frac{-8r^2-104r-336}{(3+r)(r+4)}$	$-\frac{112}{5}$
a_4	$\frac{16r^2+240r+896}{(3+r)(r+4)}$	$\frac{288}{5}$
a_5	$\frac{-32r^2-544r-2304}{(3+r)(r+4)}$	-144
a_6	$\frac{64r^2+1216r+5760}{(3+r)(r+4)}$	352

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{-128r^2 - 2688r - 14080}{(3+r)(r+4)}$$

Which for the root $r = 1$ becomes

$$a_7 = -\frac{4224}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-10-2r}{3+r}$	-3
a_2	$\frac{4r^2+44r+120}{(3+r)(r+4)}$	$\frac{42}{5}$
a_3	$\frac{-8r^2-104r-336}{(3+r)(r+4)}$	$-\frac{112}{5}$
a_4	$\frac{16r^2+240r+896}{(3+r)(r+4)}$	$\frac{288}{5}$
a_5	$\frac{-32r^2-544r-2304}{(3+r)(r+4)}$	-144
a_6	$\frac{64r^2+1216r+5760}{(3+r)(r+4)}$	352
a_7	$\frac{-128r^2-2688r-14080}{(3+r)(r+4)}$	$-\frac{4224}{5}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\ &= x\left(1 - 3x + \frac{42x^2}{5} - \frac{112x^3}{5} + \frac{288x^4}{5} - 144x^5 + 352x^6 - \frac{4224x^7}{5} + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_3 \\ &= \frac{-8r^2 - 104r - 336}{(3+r)(r+4)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{-8r^2 - 104r - 336}{(3+r)(r+4)} &= \lim_{r \to -2} \frac{-8r^2 - 104r - 336}{(3+r)(r+4)} \\ &= -80 \end{aligned}$$

The limit is -80. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-2} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} 2b_{n-1}(n+r-1)(n+r-2) + b_n(n+r)(n+r-1) \\ + 12b_{n-1}(n+r-1) + 2b_n(n+r) - 2b_n = 0 \end{aligned} \tag{4}$$

Which for for the root $r = -2$ becomes

$$2b_{n-1}(n-3)(n-4) + b_n(n-2)(n-3) + 12b_{n-1}(n-3) + 2b_n(n-2) - 2b_n = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{2(n+r+4)\, b_{n-1}}{n+r+2} \tag{5}$$

Which for the root $r = -2$ becomes

$$b_n = -\frac{2(n+2)\, b_{n-1}}{n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -2$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{2(5+r)}{3+r}$$

Which for the root $r = -2$ becomes

$$b_1 = -6$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-10-2r}{3+r}$	-6

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4r^2 + 44r + 120}{(3+r)(r+4)}$$

Which for the root $r = -2$ becomes

$$b_2 = 24$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-10-2r}{3+r}$	-6
b_2	$\frac{4r^2+44r+120}{(3+r)(r+4)}$	24

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{8(r^2 + 13r + 42)}{(3 + r)(r + 4)}$$

Which for the root $r = -2$ becomes

$$b_3 = -80$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-10-2r}{3+r}$	-6
b_2	$\frac{4r^2+44r+120}{(3+r)(r+4)}$	24
b_3	$\frac{-8r^2-104r-336}{(3+r)(r+4)}$	-80

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16r^2 + 240r + 896}{(3 + r)(r + 4)}$$

Which for the root $r = -2$ becomes

$$b_4 = 240$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-10-2r}{3+r}$	-6
b_2	$\frac{4r^2+44r+120}{(3+r)(r+4)}$	24
b_3	$\frac{-8r^2-104r-336}{(3+r)(r+4)}$	-80
b_4	$\frac{16r^2+240r+896}{(3+r)(r+4)}$	240

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{32(r^2 + 17r + 72)}{(3 + r)(r + 4)}$$

Which for the root $r = -2$ becomes

$$b_5 = -672$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-10-2r}{3+r}$	-6
b_2	$\frac{4r^2+44r+120}{(3+r)(r+4)}$	24
b_3	$\frac{-8r^2-104r-336}{(3+r)(r+4)}$	-80
b_4	$\frac{16r^2+240r+896}{(3+r)(r+4)}$	240
b_5	$\frac{-32r^2-544r-2304}{(3+r)(r+4)}$	-672

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{64r^2 + 1216r + 5760}{(3 + r)(r + 4)}$$

Which for the root $r = -2$ becomes

$$b_6 = 1792$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-10-2r}{3+r}$	-6
b_2	$\frac{4r^2+44r+120}{(3+r)(r+4)}$	24
b_3	$\frac{-8r^2-104r-336}{(3+r)(r+4)}$	-80
b_4	$\frac{16r^2+240r+896}{(3+r)(r+4)}$	240
b_5	$\frac{-32r^2-544r-2304}{(3+r)(r+4)}$	-672
b_6	$\frac{64r^2+1216r+5760}{(3+r)(r+4)}$	1792

For $n = 7$, using the above recursive equation gives

$$b_7 = -\frac{128(r^2 + 21r + 110)}{(3 + r)(r + 4)}$$

Which for the root $r = -2$ becomes

$$b_7 = -4608$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-10-2r}{3+r}$	-6
b_2	$\frac{4r^2+44r+120}{(3+r)(r+4)}$	24
b_3	$\frac{-8r^2-104r-336}{(3+r)(r+4)}$	-80
b_4	$\frac{16r^2+240r+896}{(3+r)(r+4)}$	240
b_5	$\frac{-32r^2-544r-2304}{(3+r)(r+4)}$	-672
b_6	$\frac{64r^2+1216r+5760}{(3+r)(r+4)}$	1792
b_7	$\frac{-128r^2-2688r-14080}{(3+r)(r+4)}$	-4608

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots\right) \\ &= \frac{1 - 6x + 24x^2 - 80x^3 + 240x^4 - 672x^5 + 1792x^6 - 4608x^7 + O(x^8)}{x^2} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x \left(1 - 3x + \frac{42x^2}{5} - \frac{112x^3}{5} + \frac{288x^4}{5} - 144x^5 + 352x^6 - \frac{4224x^7}{5} + O(x^8)\right) \\ &\quad + \frac{c_2(1 - 6x + 24x^2 - 80x^3 + 240x^4 - 672x^5 + 1792x^6 - 4608x^7 + O(x^8))}{x^2} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x \left(1 - 3x + \frac{42x^2}{5} - \frac{112x^3}{5} + \frac{288x^4}{5} - 144x^5 + 352x^6 - \frac{4224x^7}{5} + O(x^8)\right) \\ &\quad + \frac{c_2(1 - 6x + 24x^2 - 80x^3 + 240x^4 - 672x^5 + 1792x^6 - 4608x^7 + O(x^8))}{x^2} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 x \left(1 - 3x + \frac{42x^2}{5} - \frac{112x^3}{5} + \frac{288x^4}{5} - 144x^5 + 352x^6 - \frac{4224x^7}{5} + O(x^8)\right) + \frac{c_2(1 - 6x + 24x^2 - 80x^3 + 240x^4 - 672x^5 + 1792x^6 - 4608x^7 + O(x^8))}{x^2} \tag{1}$$

Verification of solutions

$$y = c_1 x \left(1 - 3x + \frac{42x^2}{5} - \frac{112x^3}{5} + \frac{288x^4}{5} - 144x^5 + 352x^6 - \frac{4224x^7}{5} + O(x^8)\right) + \frac{c_2(1 - 6x + 24x^2 - 80x^3 + 240x^4 - 672x^5 + 1792x^6 - 4608x^7 + O(x^8))}{x^2}$$

Verified OK.

6.2.1 Maple step by step solution

Let's solve

$x^2(2x+1)\, y'' + (12x^2 + 2x)\, y' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2y}{x^2(2x+1)} - \frac{2(1+6x)y'}{x(2x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2(1+6x)y'}{x(2x+1)} - \frac{2y}{x^2(2x+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2(1+6x)}{x(2x+1)}, P_3(x) = -\frac{2}{x^2(2x+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -2$

- $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(2x+1)\, y''+2x(1+6x)\, y'-2y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m\cdot y'$ to series expansion for $m=1..2$

  $x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m\cdot y''$ to series expansion for $m=2..3$

  $x^m\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)\,(-1+r)\, x^r+\left(\sum_{k=1}^{\infty}(a_k(k+r+2)\,(k+r-1)+2a_{k-1}(k+r-1)\,(k+4+r))\, x^{k+r}\right)=$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(-1+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{-2,1\}$

- Each term in the series must be 0, giving the recursion relation

  $((2k+2r+8)\, a_{k-1}+a_k(k+r+2))\,(k+r-1)=0$

- Shift index using $k->k+1$

  $((2k+2r+10)\, a_k+a_{k+1}(k+3+r))\,(k+r)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{2(k+r+5)a_k}{k+3+r}$

- Recursion relation for $r = -2$

  $a_{k+1} = -\frac{2(k+3)a_k}{k+1}$

- Solution for $r = -2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+1} = -\frac{2(k+3)a_k}{k+1}\right]$

- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{2(k+6)a_k}{k+4}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{2(k+6)a_k}{k+4}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+1} = -\frac{2(k+3)a_k}{k+1}, b_{k+1} = -\frac{2(k+6)b_k}{k+4}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 53

```
Order:=8;
dsolve(x^2*(1+2*x)*diff(y(x),x$2)+2*x*(1+6*x)*diff(y(x),x)-2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x \left(1 - 3x + \frac{42}{5}x^2 - \frac{112}{5}x^3 + \frac{288}{5}x^4 - 144x^5 + 352x^6 - \frac{4224}{5}x^7 + \mathrm{O}\left(x^8\right)\right) + \frac{c_2(12 - 72x + 288x^2 - 960x^3 + 2880x^4 - 8064x^5 + 21504x^6 - 55296x^7 + \mathrm{O}\left(x^8\right))}{x^2}$$

✓ Solution by Mathematica

Time used: 0.138 (sec). Leaf size: 76

```
AsymptoticDSolveValue[x^2*(1+2*x)*y''[x]+2*x*(1+6*x)*y'[x]-2*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(1792x^4 - 672x^3 + 240x^2 + \frac{1}{x^2} - 80x - \frac{6}{x} + 24\right) + c_2\left(352x^7 - 144x^6 + \frac{288x^5}{5} - \frac{112x^4}{5} + \frac{42x^3}{5} - 3x^2 + x\right)$$

6.3 problem 3

Internal problem ID [6967]
Internal file name [OUTPUT/6210_Friday_August_12_2022_11_05_30_PM_20629689/index.tex]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.8 Indicial Equation with Difference of Roots a Positive Integer: Nonlogarithmic Case. Exercises page 380
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x(3x+2)\,y' - 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(3x^2 + 2x\right)y' - 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{3x+2}{x}$$
$$q(x) = -\frac{2}{x^2}$$

Table 124: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3x+2}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{2}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + (3x^2 + 2x)\, y' - 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (3x^2 + 2x) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{1+n+r} a_n (n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} 3x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)\, x^{n+r}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)\, x^{n+r}\right) \\
&+ \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$
x^{n+r} a_n (n+r)(n+r-1) + 2x^{n+r} a_n (n+r) - 2a_n x^{n+r} = 0
$$

When $n = 0$ the above becomes

$$
x^r a_0 r(-1+r) + 2x^r a_0 r - 2a_0 x^r = 0
$$

Or

$$
\left(x^r r(-1+r) + 2x^r r - 2x^r\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
\left(r^2 + r - 2\right) x^r = 0
$$

Since the above is true for all x then the indicial equation becomes

$$
r^2 + r - 2 = 0
$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= 1 \\
r_2 &= -2
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
(r^2 + r - 2)\, x^r = 0
$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= x \left(\sum_{n=0}^{\infty} a_n x^n \right) \\
y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2}
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{1+n} \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2} \right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
a_n(n + r)(n + r - 1) + 3a_{n-1}(n + r - 1) + 2a_n(n + r) - 2a_n = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{3a_{n-1}}{n + r + 2} \tag{4}
$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{3a_{n-1}}{n+3} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{3}{3+r}$$

Which for the root $r = 1$ becomes

$$a_1 = -\frac{3}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{3+r}$	$-\frac{3}{4}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{9}{(3+r)(4+r)}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{9}{20}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{3+r}$	$-\frac{3}{4}$
a_2	$\frac{9}{(3+r)(4+r)}$	$\frac{9}{20}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{27}{(3+r)(4+r)(5+r)}$$

Which for the root $r = 1$ becomes

$$a_3 = -\frac{9}{40}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{3+r}$	$-\frac{3}{4}$
a_2	$\frac{9}{(3+r)(4+r)}$	$\frac{9}{20}$
a_3	$-\frac{27}{(3+r)(4+r)(5+r)}$	$-\frac{9}{40}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{81}{(4+r)(5+r)(3+r)(6+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{27}{280}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{3+r}$	$-\frac{3}{4}$
a_2	$\frac{9}{(3+r)(4+r)}$	$\frac{9}{20}$
a_3	$-\frac{27}{(3+r)(4+r)(5+r)}$	$-\frac{9}{40}$
a_4	$\frac{81}{(4+r)(5+r)(3+r)(6+r)}$	$\frac{27}{280}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{243}{(5+r)(3+r)(6+r)(4+r)(7+r)}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{81}{2240}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{3+r}$	$-\frac{3}{4}$
a_2	$\frac{9}{(3+r)(4+r)}$	$\frac{9}{20}$
a_3	$-\frac{27}{(3+r)(4+r)(5+r)}$	$-\frac{9}{40}$
a_4	$\frac{81}{(4+r)(5+r)(3+r)(6+r)}$	$\frac{27}{280}$
a_5	$-\frac{243}{(5+r)(3+r)(6+r)(4+r)(7+r)}$	$-\frac{81}{2240}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{729}{(3+r)(6+r)(4+r)(7+r)(5+r)(8+r)}$$

Which for the root $r = 1$ becomes

$$a_6 = \frac{27}{2240}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{3+r}$	$-\frac{3}{4}$
a_2	$\frac{9}{(3+r)(4+r)}$	$\frac{9}{20}$
a_3	$-\frac{27}{(3+r)(4+r)(5+r)}$	$-\frac{9}{40}$
a_4	$\frac{81}{(4+r)(5+r)(3+r)(6+r)}$	$\frac{27}{280}$
a_5	$-\frac{243}{(5+r)(3+r)(6+r)(4+r)(7+r)}$	$-\frac{81}{2240}$
a_6	$\frac{729}{(3+r)(6+r)(4+r)(7+r)(5+r)(8+r)}$	$\frac{27}{2240}$

For $n = 7$, using the above recursive equation gives

$$a_7 = -\frac{2187}{(3+r)(4+r)(7+r)(5+r)(8+r)(6+r)(9+r)}$$

Which for the root $r = 1$ becomes

$$a_7 = -\frac{81}{22400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{3+r}$	$-\frac{3}{4}$
a_2	$\frac{9}{(3+r)(4+r)}$	$\frac{9}{20}$
a_3	$-\frac{27}{(3+r)(4+r)(5+r)}$	$-\frac{9}{40}$
a_4	$\frac{81}{(4+r)(5+r)(3+r)(6+r)}$	$\frac{27}{280}$
a_5	$-\frac{243}{(5+r)(3+r)(6+r)(4+r)(7+r)}$	$-\frac{81}{2240}$
a_6	$\frac{729}{(3+r)(6+r)(4+r)(7+r)(5+r)(8+r)}$	$\frac{27}{2240}$
a_7	$-\frac{2187}{(3+r)(4+r)(7+r)(5+r)(8+r)(6+r)(9+r)}$	$-\frac{81}{22400}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\ &= x\left(1 - \frac{3x}{4} + \frac{9x^2}{20} - \frac{9x^3}{40} + \frac{27x^4}{280} - \frac{81x^5}{2240} + \frac{27x^6}{2240} - \frac{81x^7}{22400} + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_3 \\ &= -\frac{27}{(3+r)(4+r)(5+r)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} -\frac{27}{(3+r)(4+r)(5+r)} &= \lim_{r \to -2} -\frac{27}{(3+r)(4+r)(5+r)} \\ &= -\frac{9}{2} \end{aligned}$$

The limit is $-\frac{9}{2}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-2} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + 3b_{n-1}(n+r-1) + 2b_n(n+r) - 2b_n = 0 \tag{4}$$

Which for for the root $r = -2$ becomes

$$b_n(n-2)(n-3) + 3b_{n-1}(n-3) + 2b_n(n-2) - 2b_n = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{3b_{n-1}}{n+r+2} \tag{5}$$

Which for the root $r = -2$ becomes

$$b_n = -\frac{3b_{n-1}}{n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -2$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{3}{3+r}$$

Which for the root $r = -2$ becomes

$$b_1 = -3$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{3+r}$	-3

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{9}{(3+r)(4+r)}$$

Which for the root $r = -2$ becomes

$$b_2 = \frac{9}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{3+r}$	-3
b_2	$\frac{9}{(3+r)(4+r)}$	$\frac{9}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{27}{(3+r)(4+r)(5+r)}$$

Which for the root $r = -2$ becomes

$$b_3 = -\frac{9}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{3+r}$	-3
b_2	$\frac{9}{(3+r)(4+r)}$	$\frac{9}{2}$
b_3	$-\frac{27}{(3+r)(4+r)(5+r)}$	$-\frac{9}{2}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{81}{(4+r)(5+r)(3+r)(6+r)}$$

Which for the root $r = -2$ becomes

$$b_4 = \frac{27}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{3+r}$	-3
b_2	$\frac{9}{(3+r)(4+r)}$	$\frac{9}{2}$
b_3	$-\frac{27}{(3+r)(4+r)(5+r)}$	$-\frac{9}{2}$
b_4	$\frac{81}{(4+r)(5+r)(3+r)(6+r)}$	$\frac{27}{8}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{243}{(5+r)(3+r)(6+r)(4+r)(7+r)}$$

Which for the root $r = -2$ becomes

$$b_5 = -\frac{81}{40}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{3+r}$	-3
b_2	$\frac{9}{(3+r)(4+r)}$	$\frac{9}{2}$
b_3	$-\frac{27}{(3+r)(4+r)(5+r)}$	$-\frac{9}{2}$
b_4	$\frac{81}{(4+r)(5+r)(3+r)(6+r)}$	$\frac{27}{8}$
b_5	$-\frac{243}{(5+r)(3+r)(6+r)(4+r)(7+r)}$	$-\frac{81}{40}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{729}{(3+r)(6+r)(4+r)(7+r)(5+r)(8+r)}$$

Which for the root $r = -2$ becomes

$$b_6 = \frac{81}{80}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{3+r}$	-3
b_2	$\frac{9}{(3+r)(4+r)}$	$\frac{9}{2}$
b_3	$-\frac{27}{(3+r)(4+r)(5+r)}$	$-\frac{9}{2}$
b_4	$\frac{81}{(4+r)(5+r)(3+r)(6+r)}$	$\frac{27}{8}$
b_5	$-\frac{243}{(5+r)(3+r)(6+r)(4+r)(7+r)}$	$-\frac{81}{40}$
b_6	$\frac{729}{(3+r)(6+r)(4+r)(7+r)(5+r)(8+r)}$	$\frac{81}{80}$

For $n = 7$, using the above recursive equation gives

$$b_7 = -\frac{2187}{(3+r)(4+r)(7+r)(5+r)(8+r)(6+r)(9+r)}$$

Which for the root $r = -2$ becomes

$$b_7 = -\frac{243}{560}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{3+r}$	-3
b_2	$\frac{9}{(3+r)(4+r)}$	$\frac{9}{2}$
b_3	$-\frac{27}{(3+r)(4+r)(5+r)}$	$-\frac{9}{2}$
b_4	$\frac{81}{(4+r)(5+r)(3+r)(6+r)}$	$\frac{27}{8}$
b_5	$-\frac{243}{(5+r)(3+r)(6+r)(4+r)(7+r)}$	$-\frac{81}{40}$
b_6	$\frac{729}{(3+r)(6+r)(4+r)(7+r)(5+r)(8+r)}$	$\frac{81}{80}$
b_7	$-\frac{2187}{(3+r)(4+r)(7+r)(5+r)(8+r)(6+r)(9+r)}$	$-\frac{243}{560}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots\right) \\
&= \frac{1 - 3x + \frac{9x^2}{2} - \frac{9x^3}{2} + \frac{27x^4}{8} - \frac{81x^5}{40} + \frac{81x^6}{80} - \frac{243x^7}{560} + O(x^8)}{x^2}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1 - \frac{3x}{4} + \frac{9x^2}{20} - \frac{9x^3}{40} + \frac{27x^4}{280} - \frac{81x^5}{2240} + \frac{27x^6}{2240} - \frac{81x^7}{22400} + O(x^8)\right) \\
&\quad + \frac{c_2\left(1 - 3x + \frac{9x^2}{2} - \frac{9x^3}{2} + \frac{27x^4}{8} - \frac{81x^5}{40} + \frac{81x^6}{80} - \frac{243x^7}{560} + O(x^8)\right)}{x^2}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x\left(1 - \frac{3x}{4} + \frac{9x^2}{20} - \frac{9x^3}{40} + \frac{27x^4}{280} - \frac{81x^5}{2240} + \frac{27x^6}{2240} - \frac{81x^7}{22400} + O(x^8)\right) \\
&\quad + \frac{c_2\left(1 - 3x + \frac{9x^2}{2} - \frac{9x^3}{2} + \frac{27x^4}{8} - \frac{81x^5}{40} + \frac{81x^6}{80} - \frac{243x^7}{560} + O(x^8)\right)}{x^2}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1 x \left(1 - \frac{3x}{4} + \frac{9x^2}{20} - \frac{9x^3}{40} + \frac{27x^4}{280} - \frac{81x^5}{2240} + \frac{27x^6}{2240} - \frac{81x^7}{22400} + O(x^8)\right) + \frac{c_2 \left(1 - 3x + \frac{9x^2}{2} - \frac{9x^3}{2} + \frac{27x^4}{8} - \frac{81x^5}{40} + \frac{81x^6}{80} - \frac{243x^7}{560} + O(x^8)\right)}{x^2} \tag{1}$$

Verification of solutions

$$y = c_1 x \left(1 - \frac{3x}{4} + \frac{9x^2}{20} - \frac{9x^3}{40} + \frac{27x^4}{280} - \frac{81x^5}{2240} + \frac{27x^6}{2240} - \frac{81x^7}{22400} + O(x^8)\right) + \frac{c_2 \left(1 - 3x + \frac{9x^2}{2} - \frac{9x^3}{2} + \frac{27x^4}{8} - \frac{81x^5}{40} + \frac{81x^6}{80} - \frac{243x^7}{560} + O(x^8)\right)}{x^2}$$

Verified OK.

6.3.1 Maple step by step solution

Let's solve

$x^2 y'' + (3x^2 + 2x) y' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2y}{x^2} - \frac{(3x+2)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(3x+2)y'}{x} - \frac{2y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{3x+2}{x}, P_3(x) = -\frac{2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -2$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' + x(3x+2)\, y' - 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- > k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+2)\,(k+r-1) + 3a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 1\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\,(a_k(k+r+2) + 3a_{k-1}) = 0$

- Shift index using $k- > k+1$

  $(k+r)\,(a_{k+1}(k+3+r) + 3a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{3a_k}{k+3+r}$

- Recursion relation for $r = -2$

  $a_{k+1} = -\frac{3a_k}{k+1}$

- Solution for $r = -2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+1} = -\frac{3a_k}{k+1}\right]$

- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{3a_k}{k+4}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{3a_k}{k+4}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+1} = -\frac{3a_k}{k+1}, b_{k+1} = -\frac{3b_k}{k+4}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 53

```
Order:=8;
dsolve(x^2*diff(y(x),x$2)+x*(2+3*x)*diff(y(x),x)-2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x \left(1 - \frac{3}{4}x + \frac{9}{20}x^2 - \frac{9}{40}x^3 + \frac{27}{280}x^4 - \frac{81}{2240}x^5 + \frac{27}{2240}x^6 - \frac{81}{22400}x^7 + \mathrm{O}\left(x^8\right)\right) + \frac{c_2 \left(12 - 36x + 54x^2 - 54x^3 + \frac{81}{2}x^4 - \frac{243}{10}x^5 + \frac{243}{20}x^6 - \frac{729}{140}x^7 + \mathrm{O}\left(x^8\right)\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 92

```
AsymptoticDSolveValue[x^2*y''[x]+x*(2+3*x)*y'[x]-2*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1 \left(\frac{81x^4}{80} - \frac{81x^3}{40} + \frac{27x^2}{8} + \frac{1}{x^2} - \frac{9x}{2} - \frac{3}{x} + \frac{9}{2}\right) + c_2 \left(\frac{27x^7}{2240} - \frac{81x^6}{2240} + \frac{27x^5}{280} - \frac{9x^4}{40} + \frac{9x^3}{20} - \frac{3x^2}{4} + x\right)$$

6.4 problem 4

Internal problem ID [6968]
Internal file name [`OUTPUT/6211_Friday_August_12_2022_11_05_32_PM_10666093/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.8 Indicial Equation with Difference of Roots a Positive Integer: Nonlogarithmic Case. Exercises page 380
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

`[_Laguerre]`

$$\boxed{xy'' - (x+3)\,y' + 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + (-x-3)\,y' + 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x+3}{x}$$
$$q(x) = \frac{2}{x}$$

Table 126: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x+3}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{2}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + (-x - 3) y' + 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + (-x-3)\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-3(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \tag{2A}
$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1) x^{n+r-1}\right)
$$

$$
\sum_{n=0}^{\infty} 2a_n x^{n+r} = \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$
\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1) x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-3(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r-1} a_n (n+r)(n+r-1) - 3(n+r) a_n x^{n+r-1} = 0
$$

When $n=0$ the above becomes

$$
x^{-1+r} a_0 r(-1+r) - 3r a_0 x^{-1+r} = 0
$$

Or

$$
\left(x^{-1+r} r(-1+r) - 3r\, x^{-1+r}\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
r\, x^{-1+r}(-4+r) = 0
$$

Since the above is true for all x then the indicial equation becomes

$$r(-4+r)=0$$

Solving for r gives the roots of the indicial equation as

$$r_1=4$$
$$r_2=0$$

Since $a_0\neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(-4+r)=0$$

Solving for r gives the roots of the indicial equation as Since $r_1-r_2=4$ is an integer, then we can construct two linearly independent solutions

$$y_1(x)=x^{r_1}\left(\sum_{n=0}^{\infty}a_n x^n\right)$$
$$y_2(x)=Cy_1(x)\ln(x)+x^{r_2}\left(\sum_{n=0}^{\infty}b_n x^n\right)$$

Or

$$y_1(x)=x^4\left(\sum_{n=0}^{\infty}a_n x^n\right)$$
$$y_2(x)=Cy_1(x)\ln(x)+\left(\sum_{n=0}^{\infty}b_n x^n\right)$$

Or

$$y_1(x)=\sum_{n=0}^{\infty}a_n x^{n+4}$$
$$y_2(x)=Cy_1(x)\ln(x)+\left(\sum_{n=0}^{\infty}b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n=0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0=1$. For $1\leq n$ the recursive equation is

$$a_n(n+r)(n+r-1)-a_{n-1}(n+r-1)-3a_n(n+r)+2a_{n-1}=0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(n+r-3)}{n^2+2nr+r^2-4n-4r} \tag{4}$$

Which for the root $r = 4$ becomes

$$a_n = \frac{a_{n-1}(n+1)}{n\,(n+4)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 4$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-2+r}{r^2-2r-3}$$

Which for the root $r = 4$ becomes

$$a_1 = \frac{2}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{r^2-2r-3}$	$\frac{2}{5}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-1+r}{r^3-7r-6}$$

Which for the root $r = 4$ becomes

$$a_2 = \frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{r^2-2r-3}$	$\frac{2}{5}$
a_2	$\frac{-1+r}{r^3-7r-6}$	$\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{r}{(r+3)(r^3 - 7r - 6)}$$

Which for the root $r = 4$ becomes

$$a_3 = \frac{2}{105}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{r^2-2r-3}$	$\frac{2}{5}$
a_2	$\frac{-1+r}{r^3-7r-6}$	$\frac{1}{10}$
a_3	$\frac{r}{(r+3)(r^3-7r-6)}$	$\frac{2}{105}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r+4)(r+2)(r-3)(r+3)}$$

Which for the root $r = 4$ becomes

$$a_4 = \frac{1}{336}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{r^2-2r-3}$	$\frac{2}{5}$
a_2	$\frac{-1+r}{r^3-7r-6}$	$\frac{1}{10}$
a_3	$\frac{r}{(r+3)(r^3-7r-6)}$	$\frac{2}{105}$
a_4	$\frac{1}{(r+4)(r+2)(r-3)(r+3)}$	$\frac{1}{336}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{(r+4)(r^2 + 6r + 5)(r^2 - 9)}$$

Which for the root $r = 4$ becomes

$$a_5 = \frac{1}{2520}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{r^2-2r-3}$	$\frac{2}{5}$
a_2	$\frac{-1+r}{r^3-7r-6}$	$\frac{1}{10}$
a_3	$\frac{r}{(r+3)(r^3-7r-6)}$	$\frac{2}{105}$
a_4	$\frac{1}{(r+4)(r+2)(r-3)(r+3)}$	$\frac{1}{336}$
a_5	$\frac{1}{(r+4)(r^2+6r+5)(r^2-9)}$	$\frac{1}{2520}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{1}{(r+6)(r+2)(r-3)(r+5)(1+r)(r+4)}$$

Which for the root $r = 4$ becomes

$$a_6 = \frac{1}{21600}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{r^2-2r-3}$	$\frac{2}{5}$
a_2	$\frac{-1+r}{r^3-7r-6}$	$\frac{1}{10}$
a_3	$\frac{r}{(r+3)(r^3-7r-6)}$	$\frac{2}{105}$
a_4	$\frac{1}{(r+4)(r+2)(r-3)(r+3)}$	$\frac{1}{336}$
a_5	$\frac{1}{(r+4)(r^2+6r+5)(r^2-9)}$	$\frac{1}{2520}$
a_6	$\frac{1}{(r+6)(r+2)(r-3)(r+5)(1+r)(r+4)}$	$\frac{1}{21600}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{1}{(r+6)(r+2)(r-3)(r+5)(1+r)(r+7)(r+3)}$$

Which for the root $r = 4$ becomes

$$a_7 = \frac{1}{207900}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{r^2-2r-3}$	$\frac{2}{5}$
a_2	$\frac{-1+r}{r^3-7r-6}$	$\frac{1}{10}$
a_3	$\frac{r}{(r+3)(r^3-7r-6)}$	$\frac{2}{105}$
a_4	$\frac{1}{(r+4)(r+2)(r-3)(r+3)}$	$\frac{1}{336}$
a_5	$\frac{1}{(r+4)(r^2+6r+5)(r^2-9)}$	$\frac{1}{2520}$
a_6	$\frac{1}{(r+6)(r+2)(r-3)(r+5)(1+r)(r+4)}$	$\frac{1}{21600}$
a_7	$\frac{1}{(r+6)(r+2)(r-3)(r+5)(1+r)(r+7)(r+3)}$	$\frac{1}{207900}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^4\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\ &= x^4\left(1 + \frac{2x}{5} + \frac{x^2}{10} + \frac{2x^3}{105} + \frac{x^4}{336} + \frac{x^5}{2520} + \frac{x^6}{21600} + \frac{x^7}{207900} + O\left(x^8\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_4 \\ &= \frac{1}{(r+4)(r+2)(r-3)(r+3)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{1}{(r+4)(r+2)(r-3)(r+3)} &= \lim_{r\to 0} \frac{1}{(r+4)(r+2)(r-3)(r+3)} \\ &= -\frac{1}{72} \end{aligned}$$

The limit is $-\frac{1}{72}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^n \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) - b_{n-1}(n+r-1) - 3(n+r) b_n + 2b_{n-1} = 0 \tag{4}$$

Which for for the root $r = 0$ becomes

$$b_n n(n-1) - b_{n-1}(n-1) - 3nb_n + 2b_{n-1} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = \frac{b_{n-1}(n+r-3)}{n^2 + 2nr + r^2 - 4n - 4r} \tag{5}$$

Which for the root $r = 0$ becomes

$$b_n = \frac{b_{n-1}(n-3)}{n^2 - 4n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{-2+r}{r^2 - 2r - 3}$$

Which for the root $r = 0$ becomes

$$b_1 = \frac{2}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{r^2-2r-3}$	$\frac{2}{3}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{-1+r}{(r+2)(r^2-2r-3)}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{r^2-2r-3}$	$\frac{2}{3}$
b_2	$\frac{-1+r}{r^3-7r-6}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{r}{(r+3)(r+2)(r^2-2r-3)}$$

Which for the root $r = 0$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{r^2-2r-3}$	$\frac{2}{3}$
b_2	$\frac{-1+r}{r^3-7r-6}$	$\frac{1}{6}$
b_3	$\frac{r}{(r+3)(r+2)(1+r)(r-3)}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(r+4)(r+2)(r-3)(r+3)}$$

Which for the root $r = 0$ becomes

$$b_4 = -\frac{1}{72}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{r^2-2r-3}$	$\frac{2}{3}$
b_2	$\frac{-1+r}{r^3-7r-6}$	$\frac{1}{6}$
b_3	$\frac{r}{(r+3)(r+2)(1+r)(r-3)}$	0
b_4	$\frac{1}{(r+4)(r+2)(r^2-9)}$	$-\frac{1}{72}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{1}{(r+4)(r-3)(r+3)(r^2+6r+5)}$$

Which for the root $r = 0$ becomes

$$b_5 = -\frac{1}{180}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{r^2-2r-3}$	$\frac{2}{3}$
b_2	$\frac{-1+r}{r^3-7r-6}$	$\frac{1}{6}$
b_3	$\frac{r}{(r+3)(r+2)(1+r)(r-3)}$	0
b_4	$\frac{1}{(r+4)(r+2)(r^2-9)}$	$-\frac{1}{72}$
b_5	$\frac{1}{(r+4)(r^2+6r+5)(r^2-9)}$	$-\frac{1}{180}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{1}{(r^2+8r+12)(r-3)(r^2+6r+5)(r+4)}$$

Which for the root $r = 0$ becomes

$$b_6 = -\frac{1}{720}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{r^2-2r-3}$	$\frac{2}{3}$
b_2	$\frac{-1+r}{r^3-7r-6}$	$\frac{1}{6}$
b_3	$\frac{r}{(r+3)(r+2)(1+r)(r-3)}$	0
b_4	$\frac{1}{(r+4)(r+2)(r^2-9)}$	$-\frac{1}{72}$
b_5	$\frac{1}{(r+4)(r^2+6r+5)(r^2-9)}$	$-\frac{1}{180}$
b_6	$\frac{1}{(r+6)(r+2)(r-3)(r+5)(1+r)(r+4)}$	$-\frac{1}{720}$

For $n = 7$, using the above recursive equation gives

$$b_7 = \frac{1}{(r^2 + 8r + 12)(r - 3)(r^2 + 6r + 5)(r^2 + 10r + 21)}$$

Which for the root $r = 0$ becomes

$$b_7 = -\frac{1}{3780}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{r^2-2r-3}$	$\frac{2}{3}$
b_2	$\frac{-1+r}{r^3-7r-6}$	$\frac{1}{6}$
b_3	$\frac{r}{(r+3)(r+2)(1+r)(r-3)}$	0
b_4	$\frac{1}{(r+4)(r+2)(r^2-9)}$	$-\frac{1}{72}$
b_5	$\frac{1}{(r+4)(r^2+6r+5)(r^2-9)}$	$-\frac{1}{180}$
b_6	$\frac{1}{(r+6)(r+2)(r-3)(r+5)(1+r)(r+4)}$	$-\frac{1}{720}$
b_7	$\frac{1}{(r^2+8r+12)(r-3)(r^2+6r+5)(r^2+10r+21)}$	$-\frac{1}{3780}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\
&= 1 + \frac{2x}{3} + \frac{x^2}{6} - \frac{x^4}{72} - \frac{x^5}{180} - \frac{x^6}{720} - \frac{x^7}{3780} + O(x^8)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^4 \left(1 + \frac{2x}{5} + \frac{x^2}{10} + \frac{2x^3}{105} + \frac{x^4}{336} + \frac{x^5}{2520} + \frac{x^6}{21600} + \frac{x^7}{207900} + O(x^8)\right) \\
&+ c_2 \left(1 + \frac{2x}{3} + \frac{x^2}{6} - \frac{x^4}{72} - \frac{x^5}{180} - \frac{x^6}{720} - \frac{x^7}{3780} + O(x^8)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^4 \left(1 + \frac{2x}{5} + \frac{x^2}{10} + \frac{2x^3}{105} + \frac{x^4}{336} + \frac{x^5}{2520} + \frac{x^6}{21600} + \frac{x^7}{207900} + O(x^8)\right) \\
&+ c_2 \left(1 + \frac{2x}{3} + \frac{x^2}{6} - \frac{x^4}{72} - \frac{x^5}{180} - \frac{x^6}{720} - \frac{x^7}{3780} + O(x^8)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^4 \left(1 + \frac{2x}{5} + \frac{x^2}{10} + \frac{2x^3}{105} + \frac{x^4}{336} + \frac{x^5}{2520} + \frac{x^6}{21600} + \frac{x^7}{207900} + O(x^8)\right) \\
+ c_2 \left(1 + \frac{2x}{3} + \frac{x^2}{6} - \frac{x^4}{72} - \frac{x^5}{180} - \frac{x^6}{720} - \frac{x^7}{3780} + O(x^8)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1 x^4 \left(1 + \frac{2x}{5} + \frac{x^2}{10} + \frac{2x^3}{105} + \frac{x^4}{336} + \frac{x^5}{2520} + \frac{x^6}{21600} + \frac{x^7}{207900} + O(x^8)\right) \\
+ c_2 \left(1 + \frac{2x}{3} + \frac{x^2}{6} - \frac{x^4}{72} - \frac{x^5}{180} - \frac{x^6}{720} - \frac{x^7}{3780} + O(x^8)\right)
\end{aligned}
$$

Verified OK.

6.4.1 Maple step by step solution

Let's solve

$xy'' + (-x-3)\,y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y}{x} + \frac{(x+3)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x+3)y'}{x} + \frac{2y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x+3}{x}, P_3(x) = \frac{2}{x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -3$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $xy'' + (-x-3)\,y' + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-4+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k-3+r) - a_k(k+r-2)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-4+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 4\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k-3+r) - a_k(k+r-2) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r-2)}{(k+1+r)(k-3+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{a_k(k-2)}{(k+1)(k-3)}$

- Apply recursion relation for $k = 0$

  $a_1 = \frac{2a_0}{3}$

- Apply recursion relation for $k = 1$

  $a_2 = \frac{a_1}{4}$

- Express in terms of a_0

  $a_2 = \frac{a_0}{6}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y = a_0 \cdot \left(1 + \frac{2}{3}x + \frac{1}{6}x^2\right)$

- Recursion relation for $r = 4$

$$a_{k+1} = \frac{a_k(k+2)}{(k+5)(k+1)}$$

- Solution for $r = 4$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+4}, a_{k+1} = \frac{a_k(k+2)}{(k+5)(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y = a_0 \cdot \left(1 + \frac{2}{3}x + \frac{1}{6}x^2\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+4}\right), b_{k+1} = \frac{b_k(k+2)}{(k+5)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 50

```
Order:=8;
dsolve(x*diff(y(x),x$2)-(3+x)*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^4 \left(1 + \frac{2}{5}x + \frac{1}{10}x^2 + \frac{2}{105}x^3 + \frac{1}{336}x^4 + \frac{1}{2520}x^5 + \frac{1}{21600}x^6 + \frac{1}{207900}x^7 + O\left(x^8\right)\right) + c_2\left(-144 - 96x - 24x^2 + 2x^4 + \frac{4}{5}x^5 + \frac{1}{5}x^6 + \frac{4}{105}x^7 + O\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.101 (sec). Leaf size: 91

```
AsymptoticDSolveValue[x*y''[x]-(3+x)*y'[x]+2*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(-\frac{x^6}{720}-\frac{x^5}{180}-\frac{x^4}{72}+\frac{x^2}{6}+\frac{2x}{3}+1\right) + c_2\left(\frac{x^{10}}{21600}+\frac{x^9}{2520}+\frac{x^8}{336}+\frac{2x^7}{105}+\frac{x^6}{10}+\frac{2x^5}{5}+x^4\right)$$

6.5 problem 5

Internal problem ID [6969]
Internal file name [`OUTPUT/6212_Friday_August_12_2022_11_05_34_PM_18887529/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.8 Indicial Equation with Difference of Roots a Positive Integer: Nonlogarithmic Case. Exercises page 380
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x(1+x)\,y''+(x+5)\,y'-4y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^2+x\right)y''+(x+5)\,y'-4y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{x+5}{x\,(1+x)}$$
$$q(x)=-\frac{4}{x\,(1+x)}$$

Table 128: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x+5}{x(1+x)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

$q(x) = -\frac{4}{x(1+x)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x(1+x)\, y'' + (x+5)\, y' - 4y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} x(1+x) &\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) \\ &+ (x+5)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - 4\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 5(n+r)\, a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-4 a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1) &= \sum_{n=1}^{\infty} a_{n-1} (n+r-1)(n+r-2)\, x^{n+r-1} \\
\sum_{n=0}^{\infty} x^{n+r} a_n (n+r) &= \sum_{n=1}^{\infty} a_{n-1} (n+r-1)\, x^{n+r-1} \\
\sum_{n=0}^{\infty} \left(-4 a_n x^{n+r}\right) &= \sum_{n=1}^{\infty} \left(-4 a_{n-1} x^{n+r-1}\right)
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$
\begin{aligned}
&\left(\sum_{n=1}^{\infty} a_{n-1} (n+r-1)(n+r-2)\, x^{n+r-1}\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} (n+r-1)\, x^{n+r-1}\right) \\
&+ \left(\sum_{n=0}^{\infty} 5(n+r)\, a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-4 a_{n-1} x^{n+r-1}\right) = 0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r-1} a_n (n+r)(n+r-1) + 5(n+r)\, a_n x^{n+r-1} = 0
$$

When $n=0$ the above becomes

$$
x^{-1+r} a_0 r(-1+r) + 5 r a_0 x^{-1+r} = 0
$$

Or

$$\left(x^{-1+r} r(-1+r)+5 r\, x^{-1+r}\right) a_{0}=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(4+r)=0$$

Since the above is true for all x then the indicial equation becomes

$$r(4+r)=0$$

Solving for r gives the roots of the indicial equation as

$$r_{1}=0$$
$$r_{2}=-4$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(4+r)=0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 4$ is an integer, then we can construct two linearly independent solutions

$$y_{1}(x)=x^{r_{1}}\left(\sum_{n=0}^{\infty} a_{n} x^{n}\right)$$
$$y_{2}(x)=C y_{1}(x) \ln (x)+x^{r_{2}}\left(\sum_{n=0}^{\infty} b_{n} x^{n}\right)$$

Or

$$y_{1}(x)=\sum_{n=0}^{\infty} a_{n} x^{n}$$
$$y_{2}(x)=C y_{1}(x) \ln (x)+\frac{\sum_{n=0}^{\infty} b_{n} x^{n}}{x^{4}}$$

Or

$$y_{1}(x)=\sum_{n=0}^{\infty} a_{n} x^{n}$$
$$y_{2}(x)=C y_{1}(x) \ln (x)+\left(\sum_{n=0}^{\infty} b_{n} x^{n-4}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) \\ + a_{n-1}(n+r-1) + 5a_n(n+r) - 4a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n^2 + 2nr + r^2 - 2n - 2r - 3)}{n^2 + 2nr + r^2 + 4n + 4r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{a_{n-1}(n^2 - 2n - 3)}{n(n+4)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-r^2 + 4}{r^2 + 6r + 5}$$

Which for the root $r = 0$ becomes

$$a_1 = \frac{4}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+4}{r^2+6r+5}$	$\frac{4}{5}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^3 - 7r + 6}{r^3 + 12r^2 + 41r + 30}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+4}{r^2+6r+5}$	$\frac{4}{5}$
a_2	$\frac{r^3-7r+6}{r^3+12r^2+41r+30}$	$\frac{1}{5}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{(4+r)\, r(-1+r)\,(r-2)}{(r+7)\,(r+6)\,(r+5)\,(r+1)}$$

Which for the root $r = 0$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+4}{r^2+6r+5}$	$\frac{4}{5}$
a_2	$\frac{r^3-7r+6}{r^3+12r^2+41r+30}$	$\frac{1}{5}$
a_3	$-\frac{(4+r)r(-1+r)(r-2)}{(r+7)(r+6)(r+5)(r+1)}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(r-2)\, r(-1+r)}{(r+8)\,(r+6)\,(r+7)}$$

Which for the root $r = 0$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+4}{r^2+6r+5}$	$\frac{4}{5}$
a_2	$\frac{r^3-7r+6}{r^3+12r^2+41r+30}$	$\frac{1}{5}$
a_3	$-\frac{(4+r)r(-1+r)(r-2)}{(r+7)(r+6)(r+5)(r+1)}$	0
a_4	$\frac{(r-2)r(-1+r)}{(r+8)(r+6)(r+7)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{r(-1+r)\left(r^2-4\right)}{\left(r^2+14r+45\right)(r+7)(r+8)}$$

Which for the root $r = 0$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+4}{r^2+6r+5}$	$\frac{4}{5}$
a_2	$\frac{r^3-7r+6}{r^3+12r^2+41r+30}$	$\frac{1}{5}$
a_3	$-\frac{(4+r)r(-1+r)(r-2)}{(r+7)(r+6)(r+5)(r+1)}$	0
a_4	$\frac{(r-2)r(-1+r)}{(r+8)(r+6)(r+7)}$	0
a_5	$-\frac{r(-1+r)(r^2-4)}{(r^2+14r+45)(r+7)(r+8)}$	0

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{(r+3)\,r(-1+r)\left(r^2-4\right)}{(r+8)\left(r^2+14r+45\right)\left(r^2+16r+60\right)}$$

Which for the root $r = 0$ becomes

$$a_6 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+4}{r^2+6r+5}$	$\frac{4}{5}$
a_2	$\frac{r^3-7r+6}{r^3+12r^2+41r+30}$	$\frac{1}{5}$
a_3	$-\frac{(4+r)r(-1+r)(r-2)}{(r+7)(r+6)(r+5)(r+1)}$	0
a_4	$\frac{(r-2)r(-1+r)}{(r+8)(r+6)(r+7)}$	0
a_5	$-\frac{r(-1+r)(r^2-4)}{(r^2+14r+45)(r+7)(r+8)}$	0
a_6	$\frac{(r+3)r(-1+r)(r^2-4)}{(r+8)(r^2+14r+45)(r^2+16r+60)}$	0

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{-r^6 - 6r^5 - r^4 + 36r^3 + 20r^2 - 48r}{r^6 + 48r^5 + 946r^4 + 9792r^3 + 56113r^2 + 168720r + 207900}$$

Which for the root $r = 0$ becomes

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+4}{r^2+6r+5}$	$\frac{4}{5}$
a_2	$\frac{r^3-7r+6}{r^3+12r^2+41r+30}$	$\frac{1}{5}$
a_3	$-\frac{(4+r)r(-1+r)(r-2)}{(r+7)(r+6)(r+5)(r+1)}$	0
a_4	$\frac{(r-2)r(-1+r)}{(r+8)(r+6)(r+7)}$	0
a_5	$-\frac{r(-1+r)(r^2-4)}{(r^2+14r+45)(r+7)(r+8)}$	0
a_6	$\frac{(r+3)r(-1+r)(r^2-4)}{(r+8)(r^2+14r+45)(r^2+16r+60)}$	0
a_7	$\frac{-r^6-6r^5-r^4+36r^3+20r^2-48r}{r^6+48r^5+946r^4+9792r^3+56113r^2+168720r+207900}$	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots \\ &= 1 + \frac{4x}{5} + \frac{x^2}{5} + O(x^8) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_4 \\ &= \frac{(r-2)\, r(-1+r)}{(r+8)\,(r+6)\,(r+7)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{(r-2)\, r(-1+r)}{(r+8)\,(r+6)\,(r+7)} &= \lim_{r\to -4} \frac{(r-2)\, r(-1+r)}{(r+8)\,(r+6)\,(r+7)} \\ &= -5 \end{aligned}$$

The limit is -5. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-4} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} & b_{n-1}(n+r-1)\,(n+r-2) + b_n(n+r)\,(n+r-1) \\ & \quad + b_{n-1}(n+r-1) + 5(n+r)\, b_n - 4b_{n-1} = 0 \end{aligned} \tag{4}$$

Which for for the root $r = -4$ becomes

$$b_{n-1}(n-5)\,(n-6) + b_n(n-4)\,(n-5) + b_{n-1}(n-5) + 5(n-4)\, b_n - 4b_{n-1} = 0 \quad \text{(4A)}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{b_{n-1}(n^2 + 2nr + r^2 - 2n - 2r - 3)}{n^2 + 2nr + r^2 + 4n + 4r} \tag{5}$$

Which for the root $r = -4$ becomes

$$b_n = -\frac{b_{n-1}(n^2 - 10n + 21)}{n^2 - 4n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -4$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{r^2 - 4}{r^2 + 6r + 5}$$

Which for the root $r = -4$ becomes

$$b_1 = 4$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+4}{r^2+6r+5}$	4

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{r^3 - 7r + 6}{(r+6)(r^2 + 6r + 5)}$$

Which for the root $r = -4$ becomes

$$b_2 = 5$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+4}{r^2+6r+5}$	4
b_2	$\frac{r^3-7r+6}{r^3+12r^2+41r+30}$	5

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{(4+r)\, r(r^2 - 3r + 2)}{(r+7)\,(r+6)\,(r^2 + 6r + 5)}$$

Which for the root $r = -4$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+4}{r^2+6r+5}$	4
b_2	$\frac{r^3-7r+6}{r^3+12r^2+41r+30}$	5
b_3	$-\frac{(4+r)r(-1+r)(r-2)}{(r+7)(r+6)(r+5)(r+1)}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r(r^2 - 3r + 2)}{(r+8)\,(r+6)\,(r+7)}$$

Which for the root $r = -4$ becomes

$$b_4 = -5$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+4}{r^2+6r+5}$	4
b_2	$\frac{r^3-7r+6}{r^3+12r^2+41r+30}$	5
b_3	$-\frac{(4+r)r(-1+r)(r-2)}{(r+7)(r+6)(r+5)(r+1)}$	0
b_4	$\frac{r(r^2-3r+2)}{(r+8)(r+6)(r+7)}$	-5

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{(r+2)\, r(r^2 - 3r + 2)}{(r^2 + 14r + 45)\,(r+7)\,(r+8)}$$

Which for the root $r = -4$ becomes

$$b_5 = -4$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+4}{r^2+6r+5}$	4
b_2	$\frac{r^3-7r+6}{r^3+12r^2+41r+30}$	5
b_3	$-\frac{(4+r)r(-1+r)(r-2)}{(r+7)(r+6)(r+5)(r+1)}$	0
b_4	$\frac{r(r^2-3r+2)}{(r+8)(r+6)(r+7)}$	-5
b_5	$-\frac{r(-1+r)(r-2)(r+2)}{(9+r)(r+5)(r+7)(r+8)}$	-4

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{(r+3)(r+2)r(r^2-3r+2)}{(r+8)(r^2+14r+45)(r^2+16r+60)}$$

Which for the root $r = -4$ becomes

$$b_6 = -1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+4}{r^2+6r+5}$	4
b_2	$\frac{r^3-7r+6}{r^3+12r^2+41r+30}$	5
b_3	$-\frac{(4+r)r(-1+r)(r-2)}{(r+7)(r+6)(r+5)(r+1)}$	0
b_4	$\frac{r(r^2-3r+2)}{(r+8)(r+6)(r+7)}$	-5
b_5	$-\frac{r(-1+r)(r-2)(r+2)}{(9+r)(r+5)(r+7)(r+8)}$	-4
b_6	$\frac{(r+3)r(-1+r)(r-2)(r+2)}{(10+r)(r+6)(r+8)(r+5)(9+r)}$	-1

For $n = 7$, using the above recursive equation gives

$$b_7 = -\frac{(4+r)(r+3)(r+2)\,r(r^2-3r+2)}{(r^2+16r+60)(r^2+14r+45)(r^2+18r+77)}$$

Which for the root $r = -4$ becomes

$$b_7 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+4}{r^2+6r+5}$	4
b_2	$\frac{r^3-7r+6}{r^3+12r^2+41r+30}$	5
b_3	$-\frac{(4+r)r(-1+r)(r-2)}{(r+7)(r+6)(r+5)(r+1)}$	0
b_4	$\frac{r(r^2-3r+2)}{(r+8)(r+6)(r+7)}$	-5
b_5	$-\frac{r(-1+r)(r-2)(r+2)}{(9+r)(r+5)(r+7)(r+8)}$	-4
b_6	$\frac{(r+3)r(-1+r)(r-2)(r+2)}{(10+r)(r+6)(r+8)(r+5)(9+r)}$	-1
b_7	$-\frac{(4+r)(r+3)(r+2)r(r^2-3r+2)}{(r^2+16r+60)(r^2+14r+45)(r^2+18r+77)}$	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots\right) \\
&= \frac{1 + 4x + 5x^2 - 5x^4 - 4x^5 - x^6 + O(x^8)}{x^4}
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1 + \frac{4x}{5} + \frac{x^2}{5} + O(x^8)\right) + \frac{c_2(1 + 4x + 5x^2 - 5x^4 - 4x^5 - x^6 + O(x^8))}{x^4}
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1\left(1 + \frac{4x}{5} + \frac{x^2}{5} + O(x^8)\right) + \frac{c_2(1 + 4x + 5x^2 - 5x^4 - 4x^5 - x^6 + O(x^8))}{x^4}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\left(1+\frac{4x}{5}+\frac{x^2}{5}+O(x^8)\right)+\frac{c_2(1+4x+5x^2-5x^4-4x^5-x^6+O(x^8))}{x^4} \tag{1}$$

Verification of solutions

$$y = c_1\left(1+\frac{4x}{5}+\frac{x^2}{5}+O(x^8)\right)+\frac{c_2(1+4x+5x^2-5x^4-4x^5-x^6+O(x^8))}{x^4}$$

Verified OK.

6.5.1 Maple step by step solution

Let's solve

$x(1+x)\,y''+(x+5)\,y'-4y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{4y}{x(1+x)}-\frac{(x+5)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(x+5)y'}{x(1+x)}-\frac{4y}{x(1+x)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{x+5}{x(1+x)}, P_3(x)=-\frac{4}{x(1+x)}\right]$

  ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1}=-4$

  ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=-1$

- Multiply by denominators

  $x(1+x)\,y''+(x+5)\,y'-4y=0$
- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-u)\left(\frac{d^2}{du^2}y(u)\right)+(u+4)\left(\frac{d}{du}y(u)\right)-4y(u)=0$
- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$
  - Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-a_0r(-5+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-a_{k+1}(k+1+r)\,(k-4+r)+a_k(k+r+2)\,(k+r-2)\right)u^{k+r}\right)=$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-5+r)=0$
- Values of r that satisfy the indicial equation

  $r\in\{0,5\}$
- Each term in the series must be 0, giving the recursion relation

  $-a_{k+1}(k+1+r)\,(k-4+r)+a_k(k+r+2)\,(k+r-2)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k(k+r+2)(k+r-2)}{(k+1+r)(k-4+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{a_k(k+2)(k-2)}{(k+1)(k-4)}$

- Apply recursion relation for $k = 0$

  $a_1 = a_0$

- Apply recursion relation for $k = 1$

  $a_2 = \frac{a_1}{2}$

- Express in terms of a_0

  $a_2 = \frac{a_0}{2}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 + u + \frac{1}{2}u^2\right)$

- Revert the change of variables $u = 1 + x$

  $\left[y = a_0\left(\frac{5}{2} + 2x + \frac{1}{2}x^2\right)\right]$

- Recursion relation for $r = 5$

  $a_{k+1} = \frac{a_k(k+7)(k+3)}{(k+6)(k+1)}$

- Solution for $r = 5$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+5}, a_{k+1} = \frac{a_k(k+7)(k+3)}{(k+6)(k+1)}\right]$

- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k(1 + x)^{k+5}, a_{k+1} = \frac{a_k(k+7)(k+3)}{(k+6)(k+1)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0\left(\frac{5}{2} + 2x + \frac{1}{2}x^2\right) + \left(\sum_{k=0}^{\infty} b_k(1 + x)^{k+5}\right), b_{k+1} = \frac{b_k(k+7)(k+3)}{(k+6)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 38

```
Order:=8;
dsolve(x*(1+x)*diff(y(x),x$2)+(x+5)*diff(y(x),x)-4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\left(1+\frac{4}{5}x+\frac{1}{5}x^2+\mathrm{O}\left(x^8\right)\right) + \frac{c_2(-144-576x-720x^2+720x^4+576x^5+144x^6+\mathrm{O}\left(x^8\right))}{x^4}$$

✓ Solution by Mathematica

Time used: 0.13 (sec). Leaf size: 47

```
AsymptoticDSolveValue[x*(1+x)*y''[x]+(x+5)*y'[x]-4*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(\frac{x^2}{5}+\frac{4x}{5}+1\right)+c_1\left(\frac{1}{x^4}+\frac{4}{x^3}-x^2+\frac{5}{x^2}-4x-5\right)$$

6.6 problem 6

Internal problem ID [6970]
Internal file name [`OUTPUT/6213_Friday_August_12_2022_11_05_37_PM_5011505/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.8 Indicial Equation with Difference of Roots a Positive Integer: Nonlogarithmic Case. Exercises page 380
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x(1+x)\,y''+(x+5)\,y'-4y=0}$$

With the expansion point for the power series method at $x=-1$.

The ode does not have its expansion point at $x=0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t=1+x$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left((t-1)^2+t-1\right)\left(\frac{d^2}{dt^2}y(t)\right)+(t+4)\left(\frac{d}{dt}y(t)\right)-4y(t)=0$$

With its expansion point and initial conditions now at $t=0$. The transformed ODE is now solved. The type of the expansion point is first determined. This is done on the

homogeneous part of the ODE.

$$\left(t^2-t\right)\left(\frac{d^2}{dt^2}y(t)\right)+(t+4)\left(\frac{d}{dt}y(t)\right)-4y(t)=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$\frac{d^2}{dt^2}y(t)+p(t)\frac{d}{dt}y(t)+q(t)y(t)=0$$

Where

$$p(t)=\frac{t+4}{t\,(t-1)}$$
$$q(t)=-\frac{4}{t\,(t-1)}$$

Table 130: Table $p(t), q(t)$ singularites.

$p(t)=\frac{t+4}{t(t-1)}$	
singularity	type
$t=0$	"regular"
$t=1$	"regular"

$q(t)=-\frac{4}{t(t-1)}$	
singularity	type
$t=0$	"regular"
$t=1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 1, \infty]$

Irregular singular points : []

Since $t=0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$\left(\frac{d^2}{dt^2}y(t)\right)t(t-1)+(t+4)\left(\frac{d}{dt}y(t)\right)-4y(t)=0$$

Let the solution be represented as Frobenius power series of the form

$$y(t)=\sum_{n=0}^{\infty}a_n t^{n+r}$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=0}^{\infty} (n+r)\, a_n t^{n+r-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n t^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n t^{n+r-2}\right) t(t-1) + (t+4)\left(\sum_{n=0}^{\infty} (n+r)\, a_n t^{n+r-1}\right) - 4\left(\sum_{n=0}^{\infty} a_n t^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} t^{n+r} a_n (n+r)\,(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-t^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} t^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4(n+r)\, a_n t^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-4 a_n t^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of t be $n+r-1$ in each summation term. Going over each summation term above with power of t in it which is not already t^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} t^{n+r} a_n (n+r)\,(n+r-1) = \sum_{n=1}^{\infty} a_{n-1} (n+r-1)\,(n+r-2)\, t^{n+r-1}$$
$$\sum_{n=0}^{\infty} t^{n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1} (n+r-1)\, t^{n+r-1}$$
$$\sum_{n=0}^{\infty} \left(-4 a_n t^{n+r}\right) = \sum_{n=1}^{\infty} \left(-4 a_{n-1} t^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of t are the same and equal to $n+r-1$.

$$\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, t^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-t^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, t^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 4(n+r)\, a_n t^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-4a_{n-1} t^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$-t^{n+r-1} a_n (n+r)(n+r-1) + 4(n+r)\, a_n t^{n+r-1} = 0$$

When $n=0$ the above becomes

$$-t^{-1+r} a_0 r(-1+r) + 4r a_0 t^{-1+r} = 0$$

Or

$$\left(-t^{-1+r} r(-1+r) + 4r\, t^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, t^{-1+r}(5-r) = 0$$

Since the above is true for all t then the indicial equation becomes

$$-r(-5+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 5$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, t^{-1+r}(5-r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 5$ is an integer, then we can construct two linearly independent solutions

$$y_1(t) = t^{r_1} \left(\sum_{n=0}^{\infty} a_n t^n\right)$$
$$y_2(t) = C y_1(t) \ln(t) + t^{r_2} \left(\sum_{n=0}^{\infty} b_n t^n\right)$$

Or

$$y_1(t) = t^5 \left(\sum_{n=0}^{\infty} a_n t^n\right)$$
$$y_2(t) = C y_1(t) \ln(t) + \left(\sum_{n=0}^{\infty} b_n t^n\right)$$

Or

$$y_1(t) = \sum_{n=0}^{\infty} a_n t^{n+5}$$
$$y_2(t) = C y_1(t) \ln(t) + \left(\sum_{n=0}^{\infty} b_n t^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} & a_{n-1}(n+r-1)(n+r-2) - a_n(n+r)(n+r-1) \\ & + a_{n-1}(n+r-1) + 4a_n(n+r) - 4a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(n^2 + 2nr + r^2 - 2n - 2r - 3)}{n^2 + 2nr + r^2 - 5n - 5r} \tag{4}$$

Which for the root $r = 5$ becomes

$$a_n = \frac{(n+6)(n+2)\, a_{n-1}}{n(n+5)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 5$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{r^2 - 4}{r^2 - 3r - 4}$$

Which for the root $r = 5$ becomes

$$a_1 = \frac{7}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2-4}{r^2-3r-4}$	$\frac{7}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^3 - 7r + 6}{r^3 - 6r^2 + 5r + 12}$$

Which for the root $r = 5$ becomes

$$a_2 = 8$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2-4}{r^2-3r-4}$	$\frac{7}{2}$
a_2	$\frac{r^3-7r+6}{r^3-6r^2+5r+12}$	8

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{(r + 4)\, r(-1 + r)}{r^3 - 6r^2 + 5r + 12}$$

Which for the root $r = 5$ becomes

$$a_3 = 15$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2-4}{r^2-3r-4}$	$\frac{7}{2}$
a_2	$\frac{r^3-7r+6}{r^3-6r^2+5r+12}$	8
a_3	$\frac{(r+4)r(-1+r)}{r^3-6r^2+5r+12}$	15

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r(r+5)}{r^2 - 7r + 12}$$

Which for the root $r = 5$ becomes

$$a_4 = 25$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2-4}{r^2-3r-4}$	$\frac{7}{2}$
a_2	$\frac{r^3-7r+6}{r^3-6r^2+5r+12}$	8
a_3	$\frac{(r+4)r(-1+r)}{r^3-6r^2+5r+12}$	15
a_4	$\frac{r(r+5)}{r^2-7r+12}$	25

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{r^2 + 8r + 12}{r^2 - 7r + 12}$$

Which for the root $r = 5$ becomes

$$a_5 = \frac{77}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2-4}{r^2-3r-4}$	$\frac{7}{2}$
a_2	$\frac{r^3-7r+6}{r^3-6r^2+5r+12}$	8
a_3	$\frac{(r+4)r(-1+r)}{r^3-6r^2+5r+12}$	15
a_4	$\frac{r(r+5)}{r^2-7r+12}$	25
a_5	$\frac{r^2+8r+12}{r^2-7r+12}$	$\frac{77}{2}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{r^3 + 12r^2 + 41r + 42}{(r-3)(r+1)(r-4)}$$

Which for the root $r = 5$ becomes

$$a_6 = 56$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2-4}{r^2-3r-4}$	$\frac{7}{2}$
a_2	$\frac{r^3-7r+6}{r^3-6r^2+5r+12}$	8
a_3	$\frac{(r+4)r(-1+r)}{r^3-6r^2+5r+12}$	15
a_4	$\frac{r(r+5)}{r^2-7r+12}$	25
a_5	$\frac{r^2+8r+12}{r^2-7r+12}$	$\frac{77}{2}$
a_6	$\frac{r^3+12r^2+41r+42}{(r-3)(r+1)(r-4)}$	56

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{r^3 + 15r^2 + 68r + 96}{(r-4)(r+1)(r-3)}$$

Which for the root $r = 5$ becomes

$$a_7 = 78$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2-4}{r^2-3r-4}$	$\frac{7}{2}$
a_2	$\frac{r^3-7r+6}{r^3-6r^2+5r+12}$	8
a_3	$\frac{(r+4)r(-1+r)}{r^3-6r^2+5r+12}$	15
a_4	$\frac{r(r+5)}{r^2-7r+12}$	25
a_5	$\frac{r^2+8r+12}{r^2-7r+12}$	$\frac{77}{2}$
a_6	$\frac{r^3+12r^2+41r+42}{(r-3)(r+1)(r-4)}$	56
a_7	$\frac{r^3+15r^2+68r+96}{(r-4)(r+1)(r-3)}$	78

Using the above table, then the solution $y_1(t)$ is

$$\begin{aligned}y_1(t) &= t^5 \left(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + a_7 t^7 + a_8 t^8 \dots\right) \\ &= t^5 \left(1 + \frac{7t}{2} + 8t^2 + 15t^3 + 25t^4 + \frac{77t^5}{2} + 56t^6 + 78t^7 + O(t^8)\right)\end{aligned}$$

Now the second solution $y_2(t)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 5$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_5(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}a_N &= a_5 \\ &= \frac{r^2 + 8r + 12}{r^2 - 7r + 12}\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{r \to r_2} \frac{r^2 + 8r + 12}{r^2 - 7r + 12} &= \lim_{r \to 0} \frac{r^2 + 8r + 12}{r^2 - 7r + 12} \\ &= 1\end{aligned}$$

The limit is 1. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned}y_2(t) &= \sum_{n=0}^{\infty} b_n t^{n+r} \\ &= \sum_{n=0}^{\infty} b_n t^n\end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned}&b_{n-1}(n + r - 1)(n + r - 2) - b_n(n + r)(n + r - 1) \\ &\quad + b_{n-1}(n + r - 1) + 4(n + r) b_n - 4b_{n-1} = 0\end{aligned} \tag{4}$$

Which for for the root $r = 0$ becomes

$$b_{n-1}(n - 1)(n - 2) - b_n n(n - 1) + b_{n-1}(n - 1) + 4n b_n - 4b_{n-1} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = \frac{b_{n-1}(n^2 + 2nr + r^2 - 2n - 2r - 3)}{n^2 + 2nr + r^2 - 5n - 5r} \tag{5}$$

Which for the root $r = 0$ becomes

$$b_n = \frac{b_{n-1}(n^2 - 2n - 3)}{n^2 - 5n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{r^2 - 4}{r^2 - 3r - 4}$$

Which for the root $r = 0$ becomes

$$b_1 = 1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r^2-4}{r^2-3r-4}$	1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{r^3 - 7r + 6}{(r - 3)(r^2 - 3r - 4)}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r^2-4}{r^2-3r-4}$	1
b_2	$\frac{r^3-7r+6}{r^3-6r^2+5r+12}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{(r+4)\, r(-1+r)}{(r-3)\,(r^2-3r-4)}$$

Which for the root $r = 0$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r^2-4}{r^2-3r-4}$	1
b_2	$\frac{r^3-7r+6}{r^3-6r^2+5r+12}$	$\frac{1}{2}$
b_3	$\frac{(r+4)r(-1+r)}{r^3-6r^2+5r+12}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r(r+5)}{(r-4)\,(r-3)}$$

Which for the root $r = 0$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r^2-4}{r^2-3r-4}$	1
b_2	$\frac{r^3-7r+6}{r^3-6r^2+5r+12}$	$\frac{1}{2}$
b_3	$\frac{(r+4)r(-1+r)}{r^3-6r^2+5r+12}$	0
b_4	$\frac{r(r+5)}{(r-4)(r-3)}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{r^2+8r+12}{(r-4)\,(r-3)}$$

Which for the root $r = 0$ becomes

$$b_5 = 1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r^2-4}{r^2-3r-4}$	1
b_2	$\frac{r^3-7r+6}{r^3-6r^2+5r+12}$	$\frac{1}{2}$
b_3	$\frac{(r+4)r(-1+r)}{r^3-6r^2+5r+12}$	0
b_4	$\frac{r(r+5)}{(r-4)(r-3)}$	0
b_5	$\frac{r^2+8r+12}{(r-4)(r-3)}$	1

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{r^3 + 12r^2 + 41r + 42}{(r-3)(r+1)(r-4)}$$

Which for the root $r = 0$ becomes

$$b_6 = \frac{7}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r^2-4}{r^2-3r-4}$	1
b_2	$\frac{r^3-7r+6}{r^3-6r^2+5r+12}$	$\frac{1}{2}$
b_3	$\frac{(r+4)r(-1+r)}{r^3-6r^2+5r+12}$	0
b_4	$\frac{r(r+5)}{(r-4)(r-3)}$	0
b_5	$\frac{r^2+8r+12}{(r-4)(r-3)}$	1
b_6	$\frac{r^3+12r^2+41r+42}{(r-3)(r+1)(r-4)}$	$\frac{7}{2}$

For $n = 7$, using the above recursive equation gives

$$b_7 = \frac{r^3 + 15r^2 + 68r + 96}{(r-4)(r+1)(r-3)}$$

Which for the root $r = 0$ becomes

$$b_7 = 8$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r^2-4}{r^2-3r-4}$	1
b_2	$\frac{r^3-7r+6}{r^3-6r^2+5r+12}$	$\frac{1}{2}$
b_3	$\frac{(r+4)r(-1+r)}{r^3-6r^2+5r+12}$	0
b_4	$\frac{r(r+5)}{(r-4)(r-3)}$	0
b_5	$\frac{r^2+8r+12}{(r-4)(r-3)}$	1
b_6	$\frac{r^3+12r^2+41r+42}{(r-3)(r+1)(r-4)}$	$\frac{7}{2}$
b_7	$\frac{r^3+15r^2+68r+96}{(r-4)(r+1)(r-3)}$	8

Using the above table, then the solution $y_2(t)$ is

$$
\begin{aligned}
y_2(t) &= b_0 + b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5 + b_6 t^6 + b_7 t^7 + b_8 t^8 \dots \\
&= 1 + t + \frac{t^2}{2} + t^5 + \frac{7t^6}{2} + 8t^7 + O(t^8)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(t) &= c_1 y_1(t) + c_2 y_2(t) \\
&= c_1 t^5 \left(1 + \frac{7t}{2} + 8t^2 + 15t^3 + 25t^4 + \frac{77t^5}{2} + 56t^6 + 78t^7 + O(t^8)\right) \\
&\quad + c_2 \left(1 + t + \frac{t^2}{2} + t^5 + \frac{7t^6}{2} + 8t^7 + O(t^8)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y(t) &= y_h \\
&= c_1 t^5 \left(1 + \frac{7t}{2} + 8t^2 + 15t^3 + 25t^4 + \frac{77t^5}{2} + 56t^6 + 78t^7 + O(t^8)\right) \\
&\quad + c_2 \left(1 + t + \frac{t^2}{2} + t^5 + \frac{7t^6}{2} + 8t^7 + O(t^8)\right)
\end{aligned}
$$

Replacing t in the above with the original independent variable $x$using $t = 1 + x$ results in

$$y = c_1(1+x)^5 \left(\frac{9}{2} + \frac{7x}{2} + 8(1+x)^2 + 15(1+x)^3 + 25(1+x)^4 + \frac{77(1+x)^5}{2} + 56(1+x)^6 + 78(1+x)^7 + O\left((1+x)^8\right)\right) + c_2\left(2 + x + \frac{(1+x)^2}{2} + (1+x)^5 + \frac{7(1+x)^6}{2} + 8(1+x)^7 + O\left((1+x)^8\right)\right)$$

Summary

The solution(s) found are the following

$$y = c_1(1+x)^5 \left(\frac{9}{2} + \frac{7x}{2} + 8(1+x)^2 + 15(1+x)^3 + 25(1+x)^4 + \frac{77(1+x)^5}{2} + 56(1+x)^6 + 78(1+x)^7 + O\left((1+x)^8\right)\right) + c_2\left(2 + x + \frac{(1+x)^2}{2} + (1+x)^5 + \frac{7(1+x)^6}{2} + 8(1+x)^7 + O\left((1+x)^8\right)\right) \quad (1)$$

Verification of solutions

$$y = c_1(1+x)^5 \left(\frac{9}{2} + \frac{7x}{2} + 8(1+x)^2 + 15(1+x)^3 + 25(1+x)^4 + \frac{77(1+x)^5}{2} + 56(1+x)^6 + 78(1+x)^7 + O\left((1+x)^8\right)\right) + c_2\left(2 + x + \frac{(1+x)^2}{2} + (1+x)^5 + \frac{7(1+x)^6}{2} + 8(1+x)^7 + O\left((1+x)^8\right)\right)$$

Verified OK.

6.6.1 Maple step by step solution

Let's solve

$(x^2+x)\,y''+(x+5)\,y'-4y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{4y}{x(1+x)}-\frac{(x+5)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(x+5)y'}{x(1+x)}-\frac{4y}{x(1+x)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{x+5}{x(1+x)},P_3(x)=-\frac{4}{x(1+x)}\right]$

  - $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left.((1+x)\cdot P_2(x))\right|_{x=-1}=-4$

  - $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left.\left((1+x)^2\cdot P_3(x)\right)\right|_{x=-1}=0$

  - $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $x(1+x)\,y''+(x+5)\,y'-4y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-u)\left(\frac{d^2}{du^2}y(u)\right)+(u+4)\left(\frac{d}{du}y(u)\right)-4y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0 r(-5+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} (-a_{k+1}(k+1+r)(k-4+r) + a_k(k+r+2)(k+r-2))\, u^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-5+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 5\}$

- Each term in the series must be 0, giving the recursion relation

  $-a_{k+1}(k+1+r)(k-4+r) + a_k(k+r+2)(k+r-2) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r+2)(k+r-2)}{(k+1+r)(k-4+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{a_k(k+2)(k-2)}{(k+1)(k-4)}$

- Apply recursion relation for $k = 0$

  $a_1 = a_0$

- Apply recursion relation for $k = 1$

  $a_2 = \frac{a_1}{2}$

- Express in terms of a_0

  $a_2 = \frac{a_0}{2}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 + u + \frac{1}{2}u^2\right)$

- Revert the change of variables $u = 1 + x$

  $\left[y = a_0\left(\frac{5}{2} + 2x + \frac{1}{2}x^2\right)\right]$

- Recursion relation for $r = 5$

  $a_{k+1} = \frac{a_k(k+7)(k+3)}{(k+6)(k+1)}$

- Solution for $r = 5$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+5}, a_{k+1} = \frac{a_k(k+7)(k+3)}{(k+6)(k+1)}\right]$

- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k(1+x)^{k+5}, a_{k+1} = \frac{a_k(k+7)(k+3)}{(k+6)(k+1)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0\left(\frac{5}{2} + 2x + \frac{1}{2}x^2\right) + \left(\sum_{k=0}^{\infty} b_k(1+x)^{k+5}\right), b_{k+1} = \frac{b_k(k+7)(k+3)}{(k+6)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 50

```
Order:=8;
dsolve(x*(1+x)*diff(y(x),x$2)+(x+5)*diff(y(x),x)-4*y(x)=0,y(x),type='series',x=-1);
```

$$y(x) = c_1(x+1)^5\left(1+\frac{7}{2}(x+1)+8(x+1)^2+15(x+1)^3+25(x+1)^4+\frac{77}{2}(x+1)^5 + 56(x+1)^6+78(x+1)^7+\mathrm{O}\left((x+1)^8\right)\right)+c_2\left(2880+2880(x+1) + 1440(x+1)^2+2880(x+1)^5+10080(x+1)^6+23040(x+1)^7+\mathrm{O}\left((x+1)^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.12 (sec). Leaf size: 88

```
AsymptoticDSolveValue[x*(1+x)*y''[x]+(x+5)*y'[x]-4*y[x]==0,y[x],{x,-1,7}]
```

$$y(x) \to c_1\left(\frac{7}{2}(x+1)^6+(x+1)^5+\frac{1}{2}(x+1)^2+x+2\right)+c_2\left(56(x+1)^{11}+\frac{77}{2}(x+1)^{10} + 25(x+1)^9+15(x+1)^8+8(x+1)^7+\frac{7}{2}(x+1)^6+(x+1)^5\right)$$

6.7 problem 7

Internal problem ID [6971]
Internal file name [`OUTPUT/6214_Friday_August_12_2022_11_05_40_PM_79046404/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.8 Indicial Equation with Difference of Roots a Positive Integer: Nonlogarithmic Case. Exercises page 380
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x^2y' - 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + x^2y' - 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 1$$
$$q(x) = -\frac{2}{x^2}$$

Table 132: Table $p(x), q(x)$ singularites.

$p(x) = 1$	
singularity	type

$q(x) = -\frac{2}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + x^2y' - 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + x^2\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \quad \text{(2A)}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n(n+r) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\,(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n(n+r)\,(n+r-1) - 2a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - 2a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 2x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - r - 2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - r - 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - r - 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= x^2\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+2} \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
a_n(n+r)(n+r-1) + a_{n-1}(n+r-1) - 2a_n = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{a_{n-1}(n+r-1)}{n^2 + 2nr + r^2 - n - r - 2} \tag{4}
$$

Which for the root $r = 2$ becomes

$$
a_n = -\frac{a_{n-1}(1+n)}{n(n+3)} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{r}{r^2 + r - 2}$$

Which for the root $r = 2$ becomes

$$a_1 = -\frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{r^2+r-2}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1 + r}{r^3 + 4r^2 + r - 6}$$

Which for the root $r = 2$ becomes

$$a_2 = \frac{3}{20}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{r^2+r-2}$	$-\frac{1}{2}$
a_2	$\frac{1+r}{r^3+4r^2+r-6}$	$\frac{3}{20}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{r^3 + 6r^2 + 5r - 12}$$

Which for the root $r = 2$ becomes

$$a_3 = -\frac{1}{30}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{r^2+r-2}$	$-\frac{1}{2}$
a_2	$\frac{1+r}{r^3+4r^2+r-6}$	$\frac{3}{20}$
a_3	$-\frac{1}{r^3+6r^2+5r-12}$	$-\frac{1}{30}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{r^4 + 10r^3 + 27r^2 + 2r - 40}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{1}{168}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{r^2+r-2}$	$-\frac{1}{2}$
a_2	$\frac{1+r}{r^3+4r^2+r-6}$	$\frac{3}{20}$
a_3	$-\frac{1}{r^3+6r^2+5r-12}$	$-\frac{1}{30}$
a_4	$\frac{1}{r^4+10r^3+27r^2+2r-40}$	$\frac{1}{168}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{(r^2 + 9r + 18)(r^3 + 6r^2 + 3r - 10)}$$

Which for the root $r = 2$ becomes

$$a_5 = -\frac{1}{1120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{r^2+r-2}$	$-\frac{1}{2}$
a_2	$\frac{1+r}{r^3+4r^2+r-6}$	$\frac{3}{20}$
a_3	$-\frac{1}{r^3+6r^2+5r-12}$	$-\frac{1}{30}$
a_4	$\frac{1}{r^4+10r^3+27r^2+2r-40}$	$\frac{1}{168}$
a_5	$-\frac{1}{(r^2+9r+18)(r^3+6r^2+3r-10)}$	$-\frac{1}{1120}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{1}{(r^2 + 11r + 28)(r^2 + r - 2)(r^2 + 9r + 18)}$$

Which for the root $r = 2$ becomes

$$a_6 = \frac{1}{8640}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{r^2+r-2}$	$-\frac{1}{2}$
a_2	$\frac{1+r}{r^3+4r^2+r-6}$	$\frac{3}{20}$
a_3	$-\frac{1}{r^3+6r^2+5r-12}$	$-\frac{1}{30}$
a_4	$\frac{1}{r^4+10r^3+27r^2+2r-40}$	$\frac{1}{168}$
a_5	$-\frac{1}{(r^2+9r+18)(r^3+6r^2+3r-10)}$	$-\frac{1}{1120}$
a_6	$\frac{1}{(r^2+11r+28)(r^2+r-2)(r^2+9r+18)}$	$\frac{1}{8640}$

For $n = 7$, using the above recursive equation gives

$$a_7 = -\frac{1}{(8 + r)(5 + r)(r + 3)(2 + r)(-1 + r)(r + 7)(r + 4)}$$

Which for the root $r = 2$ becomes

$$a_7 = -\frac{1}{75600}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{r^2+r-2}$	$-\frac{1}{2}$
a_2	$\frac{1+r}{r^3+4r^2+r-6}$	$\frac{3}{20}$
a_3	$-\frac{1}{r^3+6r^2+5r-12}$	$-\frac{1}{30}$
a_4	$\frac{1}{r^4+10r^3+27r^2+2r-40}$	$\frac{1}{168}$
a_5	$-\frac{1}{(r^2+9r+18)(r^3+6r^2+3r-10)}$	$-\frac{1}{1120}$
a_6	$\frac{1}{(r^2+11r+28)(r^2+r-2)(r^2+9r+18)}$	$\frac{1}{8640}$
a_7	$-\frac{1}{(8+r)(5+r)(r+3)(2+r)(-1+r)(r+7)(r+4)}$	$-\frac{1}{75600}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^2\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\
&= x^2\left(1 - \frac{x}{2} + \frac{3x^2}{20} - \frac{x^3}{30} + \frac{x^4}{168} - \frac{x^5}{1120} + \frac{x^6}{8640} - \frac{x^7}{75600} + O(x^8)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_3 \\
&= -\frac{1}{r^3 + 6r^2 + 5r - 12}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r\to r_2} -\frac{1}{r^3 + 6r^2 + 5r - 12} &= \lim_{r\to -1} -\frac{1}{r^3 + 6r^2 + 5r - 12} \\
&= \frac{1}{12}
\end{aligned}
$$

The limit is $\frac{1}{12}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-1}
\end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + b_{n-1}(n+r-1) - 2b_n = 0 \tag{4}$$

Which for for the root $r = -1$ becomes

$$b_n(n-1)(n-2) + b_{n-1}(n-2) - 2b_n = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{b_{n-1}(n+r-1)}{n^2 + 2nr + r^2 - n - r - 2} \tag{5}$$

Which for the root $r = -1$ becomes

$$b_n = -\frac{b_{n-1}(n-2)}{n^2 - 3n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{r}{r^2 + r - 2}$$

Which for the root $r = -1$ becomes

$$b_1 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r}{r^2+r-2}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1+r}{(r^2+r-2)(r+3)}$$

Which for the root $r = -1$ becomes

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r}{r^2+r-2}$	$-\frac{1}{2}$
b_2	$\frac{1+r}{r^3+4r^2+r-6}$	0

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{1}{(r+4)(r+3)(-1+r)}$$

Which for the root $r = -1$ becomes

$$b_3 = \frac{1}{12}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r}{r^2+r-2}$	$-\frac{1}{2}$
b_2	$\frac{1+r}{r^3+4r^2+r-6}$	0
b_3	$-\frac{1}{(r+4)(r+3)(-1+r)}$	$\frac{1}{12}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(r+4)(-1+r)(r^2+7r+10)}$$

Which for the root $r = -1$ becomes

$$b_4 = -\frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r}{r^2+r-2}$	$-\frac{1}{2}$
b_2	$\frac{1+r}{r^3+4r^2+r-6}$	0
b_3	$-\frac{1}{(r+4)(r+3)(-1+r)}$	$\frac{1}{12}$
b_4	$\frac{1}{(5+r)(2+r)(r+4)(-1+r)}$	$-\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{1}{(-1+r)(r^2+7r+10)(r^2+9r+18)}$$

Which for the root $r = -1$ becomes

$$b_5 = \frac{1}{80}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r}{r^2+r-2}$	$-\frac{1}{2}$
b_2	$\frac{1+r}{r^3+4r^2+r-6}$	0
b_3	$-\frac{1}{(r+4)(r+3)(-1+r)}$	$\frac{1}{12}$
b_4	$\frac{1}{(5+r)(2+r)(r+4)(-1+r)}$	$-\frac{1}{24}$
b_5	$-\frac{1}{(6+r)(r+3)(5+r)(-1+r)(2+r)}$	$\frac{1}{80}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{1}{(r^2+9r+18)(2+r)(-1+r)(r^2+11r+28)}$$

Which for the root $r = -1$ becomes

$$b_6 = -\frac{1}{360}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r}{r^2+r-2}$	$-\frac{1}{2}$
b_2	$\frac{1+r}{r^3+4r^2+r-6}$	0
b_3	$-\frac{1}{(r+4)(r+3)(-1+r)}$	$\frac{1}{12}$
b_4	$\frac{1}{(5+r)(2+r)(r+4)(-1+r)}$	$-\frac{1}{24}$
b_5	$-\frac{1}{(6+r)(r+3)(5+r)(-1+r)(2+r)}$	$\frac{1}{80}$
b_6	$\frac{1}{(r+7)(r+4)(2+r)(-1+r)(6+r)(r+3)}$	$-\frac{1}{360}$

For $n = 7$, using the above recursive equation gives

$$b_7 = -\frac{1}{(r^2+11r+28)(-1+r)(2+r)(r+3)(r^2+13r+40)}$$

Which for the root $r = -1$ becomes

$$b_7 = \frac{1}{2016}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r}{r^2+r-2}$	$-\frac{1}{2}$
b_2	$\frac{1+r}{r^3+4r^2+r-6}$	0
b_3	$-\frac{1}{(r+4)(r+3)(-1+r)}$	$\frac{1}{12}$
b_4	$\frac{1}{(5+r)(2+r)(r+4)(-1+r)}$	$-\frac{1}{24}$
b_5	$-\frac{1}{(6+r)(r+3)(5+r)(-1+r)(2+r)}$	$\frac{1}{80}$
b_6	$\frac{1}{(r+7)(r+4)(2+r)(-1+r)(6+r)(r+3)}$	$-\frac{1}{360}$
b_7	$-\frac{1}{(8+r)(5+r)(r+3)(2+r)(-1+r)(r+7)(r+4)}$	$\frac{1}{2016}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^2\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots\right) \\
&= \frac{1 - \frac{x}{2} + \frac{x^3}{12} - \frac{x^4}{24} + \frac{x^5}{80} - \frac{x^6}{360} + \frac{x^7}{2016} + O(x^8)}{x}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^2 \left(1 - \frac{x}{2} + \frac{3x^2}{20} - \frac{x^3}{30} + \frac{x^4}{168} - \frac{x^5}{1120} + \frac{x^6}{8640} - \frac{x^7}{75600} + O(x^8)\right) \\
&\quad + \frac{c_2 \left(1 - \frac{x}{2} + \frac{x^3}{12} - \frac{x^4}{24} + \frac{x^5}{80} - \frac{x^6}{360} + \frac{x^7}{2016} + O(x^8)\right)}{x}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^2 \left(1 - \frac{x}{2} + \frac{3x^2}{20} - \frac{x^3}{30} + \frac{x^4}{168} - \frac{x^5}{1120} + \frac{x^6}{8640} - \frac{x^7}{75600} + O(x^8)\right) \\
&\quad + \frac{c_2 \left(1 - \frac{x}{2} + \frac{x^3}{12} - \frac{x^4}{24} + \frac{x^5}{80} - \frac{x^6}{360} + \frac{x^7}{2016} + O(x^8)\right)}{x}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1 x^2 \left(1 - \frac{x}{2} + \frac{3x^2}{20} - \frac{x^3}{30} + \frac{x^4}{168} - \frac{x^5}{1120} + \frac{x^6}{8640} - \frac{x^7}{75600} + O(x^8)\right) + \frac{c_2 \left(1 - \frac{x}{2} + \frac{x^3}{12} - \frac{x^4}{24} + \frac{x^5}{80} - \frac{x^6}{360} + \frac{x^7}{2016} + O(x^8)\right)}{x} \tag{1}$$

Verification of solutions

$$y = c_1 x^2 \left(1 - \frac{x}{2} + \frac{3x^2}{20} - \frac{x^3}{30} + \frac{x^4}{168} - \frac{x^5}{1120} + \frac{x^6}{8640} - \frac{x^7}{75600} + O(x^8)\right) + \frac{c_2 \left(1 - \frac{x}{2} + \frac{x^3}{12} - \frac{x^4}{24} + \frac{x^5}{80} - \frac{x^6}{360} + \frac{x^7}{2016} + O(x^8)\right)}{x}$$

Verified OK.

6.7.1 Maple step by step solution

Let's solve

$x^2 y'' + x^2 y' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -y' + \frac{2y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y' - \frac{2y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = 1, P_3(x) = -\frac{2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 0$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -2$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x^2y' - 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^2 \cdot y'$ to series expansion

  $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r+1}$

- Shift index using $k->k-1$

  $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) + a_{k-1}(k-1+r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)\,(k+r-2) + a_{k-1}(k-1+r) = 0$

- Shift index using $k->k+1$

  $a_{k+1}(k+2+r)\,(k-1+r) + a_k(k+r) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r)}{(k+2+r)(k-1+r)}$

- Recursion relation for $r=-1$; series terminates at $k=1$

  $a_{k+1} = -\frac{a_k(k-1)}{(k+1)(k-2)}$

- Apply recursion relation for $k=0$

  $a_1 = -\frac{a_0}{2}$

- Terminating series solution of the ODE for $r=-1$. Use reduction of order to find the second

  $y = a_0 \cdot \left(1-\frac{x}{2}\right)$

- Recursion relation for $r=2$

  $a_{k+1} = -\frac{a_k(k+2)}{(k+4)(k+1)}$

- Solution for $r=2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{a_k(k+2)}{(k+4)(k+1)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0 \cdot \left(1-\frac{x}{2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), b_{k+1} = -\frac{b_k(k+2)}{(k+4)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 53

```
Order:=8;
dsolve(x^2*diff(y(x),x$2)+x^2*diff(y(x),x)-2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^2 \left(1 - \frac{1}{2}x + \frac{3}{20}x^2 - \frac{1}{30}x^3 + \frac{1}{168}x^4 - \frac{1}{1120}x^5 + \frac{1}{8640}x^6 - \frac{1}{75600}x^7 + \mathrm{O}\left(x^8\right)\right) + \frac{c_2\left(12 - 6x + x^3 - \frac{1}{2}x^4 + \frac{3}{20}x^5 - \frac{1}{30}x^6 + \frac{1}{168}x^7 + \mathrm{O}\left(x^8\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.055 (sec). Leaf size: 91

```
AsymptoticDSolveValue[x^2*y''[x]+x^2*y'[x]-2*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(-\frac{x^5}{360} + \frac{x^4}{80} - \frac{x^3}{24} + \frac{x^2}{12} + \frac{1}{x} - \frac{1}{2}\right) + c_2\left(\frac{x^8}{8640} - \frac{x^7}{1120} + \frac{x^6}{168} - \frac{x^5}{30} + \frac{3x^4}{20} - \frac{x^3}{2} + x^2\right)$$

6.8 problem 8

Internal problem ID [6972]
Internal file name [`OUTPUT/6215_Friday_August_12_2022_11_05_42_PM_88058161/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.8 Indicial Equation with Difference of Roots a Positive Integer: Nonlogarithmic Case. Exercises page 380
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x(1-x)\,y''-3y'+2y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(-x^2+x)\,y''-3y'+2y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{3}{x\,(x-1)}$$
$$q(x)=-\frac{2}{x\,(x-1)}$$

Table 134: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

$q(x) = -\frac{2}{x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 1, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''x(x-1) - 3y' + 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x(x-1) - 3\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty}\left(-x^{n+r}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}x^{n+r-1}a_n(n+r)(n+r-1)\right) \\ +\sum_{n=0}^{\infty}\left(-3(n+r)a_n x^{n+r-1}\right)+\left(\sum_{n=0}^{\infty}2a_n x^{n+r}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-x^{n+r}a_n(n+r)(n+r-1)\right)=\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)(n+r-2)x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty}2a_n x^{n+r}=\sum_{n=1}^{\infty}2a_{n-1}x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)(n+r-2)x^{n+r-1}\right)+\left(\sum_{n=0}^{\infty}x^{n+r-1}a_n(n+r)(n+r-1)\right) \\ +\sum_{n=0}^{\infty}\left(-3(n+r)a_n x^{n+r-1}\right)+\left(\sum_{n=1}^{\infty}2a_{n-1}x^{n+r-1}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1}a_n(n+r)(n+r-1)-3(n+r)a_n x^{n+r-1}=0$$

When $n=0$ the above becomes

$$x^{-1+r}a_0 r(-1+r)-3ra_0x^{-1+r}=0$$

Or

$$\left(x^{-1+r}r(-1+r)-3r\,x^{-1+r}\right)a_0=0$$

Since $a_0\neq 0$ then the above simplifies to

$$r\,x^{-1+r}(-4+r)=0$$

Since the above is true for all x then the indicial equation becomes

$$r(-4+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 4$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(-4+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 4$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^4\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+4}$$
$$y_2(x) = C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$-a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) - 3a_n(n+r) + 2a_{n-1} = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{(n+r-3)\, a_{n-1}}{n-4+r} \tag{4}$$

Which for the root $r = 4$ becomes

$$a_n = \frac{(n+1)\, a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 4$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-2+r}{r-3}$$

Which for the root $r = 4$ becomes

$$a_1 = 2$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{r-3}$	2

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-1+r}{r-3}$$

Which for the root $r = 4$ becomes

$$a_2 = 3$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{r-3}$	2
a_2	$\frac{-1+r}{r-3}$	3

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{r}{r - 3}$$

Which for the root $r = 4$ becomes

$$a_3 = 4$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{r-3}$	2
a_2	$\frac{-1+r}{r-3}$	3
a_3	$\frac{r}{r-3}$	4

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1 + r}{r - 3}$$

Which for the root $r = 4$ becomes

$$a_4 = 5$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{r-3}$	2
a_2	$\frac{-1+r}{r-3}$	3
a_3	$\frac{r}{r-3}$	4
a_4	$\frac{1+r}{r-3}$	5

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{2 + r}{r - 3}$$

Which for the root $r = 4$ becomes

$$a_5 = 6$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{r-3}$	2
a_2	$\frac{-1+r}{r-3}$	3
a_3	$\frac{r}{r-3}$	4
a_4	$\frac{1+r}{r-3}$	5
a_5	$\frac{2+r}{r-3}$	6

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{3+r}{r-3}$$

Which for the root $r = 4$ becomes

$$a_6 = 7$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{r-3}$	2
a_2	$\frac{-1+r}{r-3}$	3
a_3	$\frac{r}{r-3}$	4
a_4	$\frac{1+r}{r-3}$	5
a_5	$\frac{2+r}{r-3}$	6
a_6	$\frac{3+r}{r-3}$	7

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{4+r}{r-3}$$

Which for the root $r = 4$ becomes

$$a_7 = 8$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{r-3}$	2
a_2	$\frac{-1+r}{r-3}$	3
a_3	$\frac{r}{r-3}$	4
a_4	$\frac{1+r}{r-3}$	5
a_5	$\frac{2+r}{r-3}$	6
a_6	$\frac{3+r}{r-3}$	7
a_7	$\frac{4+r}{r-3}$	8

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^4\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \ldots\right) \\ &= x^4\left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + 8x^7 + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_4 \\ &= \frac{1+r}{r-3} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{1+r}{r-3} &= \lim_{r\to 0} \frac{1+r}{r-3} \\ &= -\frac{1}{3} \end{aligned}$$

The limit is $-\frac{1}{3}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^n \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$-b_{n-1}(n + r - 1)(n + r - 2) + b_n(n + r)(n + r - 1) - 3(n + r) b_n + 2b_{n-1} = 0 \quad (4)$$

Which for for the root $r = 0$ becomes

$$-b_{n-1}(n - 1)(n - 2) + b_n n(n - 1) - 3nb_n + 2b_{n-1} = 0 \quad (4A)$$

Solving for b_n from the recursive equation (4) gives

$$b_n = \frac{(n + r - 3) b_{n-1}}{n - 4 + r} \quad (5)$$

Which for the root $r = 0$ becomes

$$b_n = \frac{(n - 3) b_{n-1}}{n - 4} \quad (6)$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{-2 + r}{r - 3}$$

Which for the root $r = 0$ becomes

$$b_1 = \frac{2}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{r-3}$	$\frac{2}{3}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{-1 + r}{r - 3}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{r-3}$	$\frac{2}{3}$
b_2	$\frac{-1+r}{r-3}$	$\frac{1}{3}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{r}{r-3}$$

Which for the root $r = 0$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{r-3}$	$\frac{2}{3}$
b_2	$\frac{-1+r}{r-3}$	$\frac{1}{3}$
b_3	$\frac{r}{r-3}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1+r}{r-3}$$

Which for the root $r = 0$ becomes

$$b_4 = -\frac{1}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{r-3}$	$\frac{2}{3}$
b_2	$\frac{-1+r}{r-3}$	$\frac{1}{3}$
b_3	$\frac{r}{r-3}$	0
b_4	$\frac{1+r}{r-3}$	$-\frac{1}{3}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{2+r}{r-3}$$

Which for the root $r = 0$ becomes

$$b_5 = -\frac{2}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{r-3}$	$\frac{2}{3}$
b_2	$\frac{-1+r}{r-3}$	$\frac{1}{3}$
b_3	$\frac{r}{r-3}$	0
b_4	$\frac{1+r}{r-3}$	$-\frac{1}{3}$
b_5	$\frac{2+r}{r-3}$	$-\frac{2}{3}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{3+r}{r-3}$$

Which for the root $r = 0$ becomes

$$b_6 = -1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{r-3}$	$\frac{2}{3}$
b_2	$\frac{-1+r}{r-3}$	$\frac{1}{3}$
b_3	$\frac{r}{r-3}$	0
b_4	$\frac{1+r}{r-3}$	$-\frac{1}{3}$
b_5	$\frac{2+r}{r-3}$	$-\frac{2}{3}$
b_6	$\frac{3+r}{r-3}$	-1

For $n = 7$, using the above recursive equation gives

$$b_7 = \frac{4+r}{r-3}$$

Which for the root $r = 0$ becomes

$$b_7 = -\frac{4}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{r-3}$	$\frac{2}{3}$
b_2	$\frac{-1+r}{r-3}$	$\frac{1}{3}$
b_3	$\frac{r}{r-3}$	0
b_4	$\frac{1+r}{r-3}$	$-\frac{1}{3}$
b_5	$\frac{2+r}{r-3}$	$-\frac{2}{3}$
b_6	$\frac{3+r}{r-3}$	-1
b_7	$\frac{4+r}{r-3}$	$-\frac{4}{3}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\ &= 1 + \frac{2x}{3} + \frac{x^2}{3} - \frac{x^4}{3} - \frac{2x^5}{3} - x^6 - \frac{4x^7}{3} + O(x^8) \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^4 \left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + 8x^7 + O(x^8)\right) \\
&\quad + c_2 \left(1 + \frac{2x}{3} + \frac{x^2}{3} - \frac{x^4}{3} - \frac{2x^5}{3} - x^6 - \frac{4x^7}{3} + O(x^8)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^4 \left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + 8x^7 + O(x^8)\right) \\
&\quad + c_2 \left(1 + \frac{2x}{3} + \frac{x^2}{3} - \frac{x^4}{3} - \frac{2x^5}{3} - x^6 - \frac{4x^7}{3} + O(x^8)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1 x^4 \left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + 8x^7 + O(x^8)\right) \\
& + c_2 \left(1 + \frac{2x}{3} + \frac{x^2}{3} - \frac{x^4}{3} - \frac{2x^5}{3} - x^6 - \frac{4x^7}{3} + O(x^8)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1 x^4 \left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + 8x^7 + O(x^8)\right) \\
& + c_2 \left(1 + \frac{2x}{3} + \frac{x^2}{3} - \frac{x^4}{3} - \frac{2x^5}{3} - x^6 - \frac{4x^7}{3} + O(x^8)\right)
\end{aligned}
$$

Verified OK.

6.8.1 Maple step by step solution

Let's solve

$-y''x(x-1) - 3y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2y}{x(x-1)} - \frac{3y'}{x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{3y'}{x(x-1)} - \frac{2y}{x(x-1)} = 0$

- Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{3}{x(x-1)}, P_3(x) = -\frac{2}{x(x-1)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -3$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$
  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $y''x(x-1) + 3y' - 2y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- Rewrite ODE with series expansions
  - Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$
  - Shift index using $k- >k+1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$
  - Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

    $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$
  - Shift index using $k- >k+2-m$

    $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

    Rewrite ODE with series expansions

$$-a_0 r(-4+r)\, x^{-1+r}+\left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(k-3+r)+a_k(k+1+r)\,(k+r-2)\right) x^{k+r}\right)=$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-4+r)=0$
- Values of r that satisfy the indicial equation

  $r\in\{0,4\}$
- Each term in the series must be 0, giving the recursion relation

  $(k+1+r)\left((-k-r+3)\, a_{k+1}+a_k(k+r-2)\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k(k+r-2)}{k-3+r}$
- Recursion relation for $r=0$; series terminates at $k=2$

  $a_{k+1}=\frac{a_k(k-2)}{k-3}$
- Apply recursion relation for $k=0$

  $a_1=\frac{2a_0}{3}$
- Apply recursion relation for $k=1$

  $a_2=\frac{a_1}{2}$
- Express in terms of a_0

  $a_2=\frac{a_0}{3}$
- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

  $y=a_0\cdot\left(1+\frac{2}{3}x+\frac{1}{3}x^2\right)$
- Recursion relation for $r=4$

  $a_{k+1}=\frac{a_k(k+2)}{k+1}$
- Solution for $r=4$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+4}, a_{k+1}=\frac{a_k(k+2)}{k+1}\right]$
- Combine solutions and rename parameters

  $\left[y=a_0\cdot\left(1+\frac{2}{3}x+\frac{1}{3}x^2\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+4}\right), b_{k+1}=\frac{b_k(k+2)}{k+1}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple
Time used: 0.031 (sec). Leaf size: 50

```
Order:=8;
dsolve(x*(1-x)*diff(y(x),x$2)-3*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^4 \left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + 8x^7 + \mathrm{O}\left(x^8\right)\right) + c_2 \left(-144 - 96x - 48x^2 + 48x^4 + 96x^5 + 144x^6 + 192x^7 + \mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica
Time used: 0.438 (sec). Leaf size: 77

```
AsymptoticDSolveValue[x*(1-x)*y''[x]-3*y'[x]+2*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1 \left(-x^6 - \frac{2x^5}{3} - \frac{x^4}{3} + \frac{x^2}{3} + \frac{2x}{3} + 1\right) + c_2 \left(7x^{10} + 6x^9 + 5x^8 + 4x^7 + 3x^6 + 2x^5 + x^4\right)$$

6.9 problem 9

Internal problem ID [6973]
Internal file name [OUTPUT/6216_Friday_August_12_2022_11_05_44_PM_68847940/index.tex]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.8 Indicial Equation with Difference of Roots a Positive Integer: Nonlogarithmic Case. Exercises page 380
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x(1-x)\,y''-3y'+2y=0}$$

With the expansion point for the power series method at $x=1$.

The ode does not have its expansion point at $x=0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t=x-1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left(-(t+1)^2+t+1\right)\left(\frac{d^2}{dt^2}y(t)\right)-3\frac{d}{dt}y(t)+2y(t)=0$$

With its expansion point and initial conditions now at $t=0$. The transformed ODE is now solved. The type of the expansion point is first determined. This is done on the

homogeneous part of the ODE.

$$(-t^2 - t)\left(\frac{d^2}{dt^2}y(t)\right) - 3\frac{d}{dt}y(t) + 2y(t) = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$\frac{d^2}{dt^2}y(t) + p(t)\frac{d}{dt}y(t) + q(t)y(t) = 0$$

Where

$$p(t) = \frac{3}{t(t+1)}$$
$$q(t) = -\frac{2}{t(t+1)}$$

Table 136: Table $p(t), q(t)$ singularites.

$p(t) = \frac{3}{t(t+1)}$	
singularity	type
$t = -1$	"regular"
$t = 0$	"regular"

$q(t) = -\frac{2}{t(t+1)}$	
singularity	type
$t = -1$	"regular"
$t = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, \infty]$

Irregular singular points : []

Since $t = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-\left(\frac{d^2}{dt^2}y(t)\right)t(t+1) - 3\frac{d}{dt}y(t) + 2y(t) = 0$$

Let the solution be represented as Frobenius power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^{n+r}$$

Then

$$
\begin{aligned}
\frac{d}{dt}y(t) &= \sum_{n=0}^{\infty} (n+r)\, a_n t^{n+r-1} \\
\frac{d^2}{dt^2}y(t) &= \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n t^{n+r-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\begin{aligned}
&-\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n t^{n+r-2}\right) t(t+1) \\
&- 3\left(\sum_{n=0}^{\infty} (n+r)\, a_n t^{n+r-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n t^{n+r}\right) = 0
\end{aligned} \tag{1}
$$

Which simplifies to

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \left(-t^{n+r} a_n (n+r)\,(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-t^{n+r-1} a_n (n+r)\,(n+r-1)\right) \\
&+ \sum_{n=0}^{\infty} \left(-3(n+r)\, a_n t^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 2a_n t^{n+r}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of t be $n+r-1$ in each summation term. Going over each summation term above with power of t in it which is not already t^{n+r-1} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} \left(-t^{n+r} a_n (n+r)\,(n+r-1)\right) &= \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\,(n+r-2)\, t^{n+r-1}\right) \\
\sum_{n=0}^{\infty} 2a_n t^{n+r} &= \sum_{n=1}^{\infty} 2a_{n-1} t^{n+r-1}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of t are the same and equal to $n+r-1$.

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\,(n+r-2)\, t^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-t^{n+r-1} a_n (n+r)\,(n+r-1)\right) \\
&+ \sum_{n=0}^{\infty} \left(-3(n+r)\, a_n t^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} t^{n+r-1}\right) = 0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$-t^{n+r-1}a_n(n+r)\,(n+r-1) - 3(n+r)\,a_n t^{n+r-1} = 0$$

When $n = 0$ the above becomes

$$-t^{-1+r}a_0 r(-1+r) - 3r a_0 t^{-1+r} = 0$$

Or

$$\left(-t^{-1+r}r(-1+r) - 3r\,t^{-1+r}\right)a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\,t^{-1+r}(-2-r) = 0$$

Since the above is true for all t then the indicial equation becomes

$$-r(2+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 0 \\ r_2 &= -2 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,t^{-1+r}(-2-r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(t) &= t^{r_1}\left(\sum_{n=0}^{\infty} a_n t^n\right) \\ y_2(t) &= C y_1(t)\ln(t) + t^{r_2}\left(\sum_{n=0}^{\infty} b_n t^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(t) &= \sum_{n=0}^{\infty} a_n t^n \\ y_2(t) &= C y_1(t)\ln(t) + \frac{\sum_{n=0}^{\infty} b_n t^n}{t^2} \end{aligned}$$

Or

$$
\begin{aligned}
y_1(t) &= \sum_{n=0}^{\infty} a_n t^n \\
y_2(t) &= C y_1(t) \ln(t) + \left(\sum_{n=0}^{\infty} b_n t^{n-2}\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
-a_{n-1}(n+r-1)(n+r-2) - a_n(n+r)(n+r-1) - 3a_n(n+r) + 2a_{n-1} = 0 \quad (3)
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{(n+r-3)\,a_{n-1}}{n+2+r} \tag{4}
$$

Which for the root $r = 0$ becomes

$$
a_n = -\frac{(n-3)\,a_{n-1}}{n+2} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$
a_1 = \frac{2-r}{3+r}
$$

Which for the root $r = 0$ becomes

$$
a_1 = \frac{2}{3}
$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2-r}{3+r}$	$\frac{2}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^2 - 3r + 2}{(3 + r)(4 + r)}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2-r}{3+r}$	$\frac{2}{3}$
a_2	$\frac{r^2-3r+2}{(3+r)(4+r)}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-r^3 + 3r^2 - 2r}{(3 + r)(4 + r)(5 + r)}$$

Which for the root $r = 0$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2-r}{3+r}$	$\frac{2}{3}$
a_2	$\frac{r^2-3r+2}{(3+r)(4+r)}$	$\frac{1}{6}$
a_3	$\frac{-r^3+3r^2-2r}{(3+r)(4+r)(5+r)}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(-1 + r)(-2 + r)\, r(1 + r)}{(3 + r)(4 + r)(5 + r)(6 + r)}$$

Which for the root $r = 0$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2-r}{3+r}$	$\frac{2}{3}$
a_2	$\frac{r^2-3r+2}{(3+r)(4+r)}$	$\frac{1}{6}$
a_3	$\frac{-r^3+3r^2-2r}{(3+r)(4+r)(5+r)}$	0
a_4	$\frac{(-1+r)(-2+r)r(1+r)}{(3+r)(4+r)(5+r)(6+r)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-r^5 + 5r^3 - 4r}{(3 + r)(4 + r)(5 + r)(6 + r)(7 + r)}$$

Which for the root $r = 0$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2-r}{3+r}$	$\frac{2}{3}$
a_2	$\frac{r^2-3r+2}{(3+r)(4+r)}$	$\frac{1}{6}$
a_3	$\frac{-r^3+3r^2-2r}{(3+r)(4+r)(5+r)}$	0
a_4	$\frac{(-1+r)(-2+r)r(1+r)}{(3+r)(4+r)(5+r)(6+r)}$	0
a_5	$\frac{-r^5+5r^3-4r}{(3+r)(4+r)(5+r)(6+r)(7+r)}$	0

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{r^5 - 5r^3 + 4r}{(4 + r)(5 + r)(6 + r)(7 + r)(8 + r)}$$

Which for the root $r = 0$ becomes

$$a_6 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2-r}{3+r}$	$\frac{2}{3}$
a_2	$\frac{r^2-3r+2}{(3+r)(4+r)}$	$\frac{1}{6}$
a_3	$\frac{-r^3+3r^2-2r}{(3+r)(4+r)(5+r)}$	0
a_4	$\frac{(-1+r)(-2+r)r(1+r)}{(3+r)(4+r)(5+r)(6+r)}$	0
a_5	$\frac{-r^5+5r^3-4r}{(3+r)(4+r)(5+r)(6+r)(7+r)}$	0
a_6	$\frac{r^5-5r^3+4r}{(4+r)(5+r)(6+r)(7+r)(8+r)}$	0

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{-r^5 + 5r^3 - 4r}{(5 + r)(6 + r)(7 + r)(8 + r)(9 + r)}$$

Which for the root $r = 0$ becomes

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2-r}{3+r}$	$\frac{2}{3}$
a_2	$\frac{r^2-3r+2}{(3+r)(4+r)}$	$\frac{1}{6}$
a_3	$\frac{-r^3+3r^2-2r}{(3+r)(4+r)(5+r)}$	0
a_4	$\frac{(-1+r)(-2+r)r(1+r)}{(3+r)(4+r)(5+r)(6+r)}$	0
a_5	$\frac{-r^5+5r^3-4r}{(3+r)(4+r)(5+r)(6+r)(7+r)}$	0
a_6	$\frac{r^5-5r^3+4r}{(4+r)(5+r)(6+r)(7+r)(8+r)}$	0
a_7	$\frac{-r^5+5r^3-4r}{(5+r)(6+r)(7+r)(8+r)(9+r)}$	0

Using the above table, then the solution $y_1(t)$ is

$$\begin{aligned} y_1(t) &= a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + a_7 t^7 + a_8 t^8 \dots \\ &= 1 + \frac{2t}{3} + \frac{t^2}{6} + O(t^8) \end{aligned}$$

Now the second solution $y_2(t)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_2 \\ &= \frac{r^2 - 3r + 2}{(3 + r)(4 + r)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{r^2 - 3r + 2}{(3 + r)(4 + r)} &= \lim_{r\to -2} \frac{r^2 - 3r + 2}{(3 + r)(4 + r)} \\ &= 6 \end{aligned}$$

The limit is 6. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(t) &= \sum_{n=0}^{\infty} b_n t^{n+r} \\ &= \sum_{n=0}^{\infty} b_n t^{n-2} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$-b_{n-1}(n + r - 1)(n + r - 2) - b_n(n + r)(n + r - 1) - 3(n + r) b_n + 2b_{n-1} = 0 \quad (4)$$

Which for for the root $r = -2$ becomes

$$-b_{n-1}(n - 3)(n - 4) - b_n(n - 2)(n - 3) - 3(n - 2) b_n + 2b_{n-1} = 0 \qquad (4\text{A})$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{(n + r - 3) b_{n-1}}{n + 2 + r} \qquad (5)$$

Which for the root $r = -2$ becomes

$$b_n = -\frac{(n-5)\, b_{n-1}}{n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -2$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{-2+r}{3+r}$$

Which for the root $r = -2$ becomes

$$b_1 = 4$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2-r}{3+r}$	4

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{r^2 - 3r + 2}{(3+r)(4+r)}$$

Which for the root $r = -2$ becomes

$$b_2 = 6$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2-r}{3+r}$	4
b_2	$\frac{r^2-3r+2}{(3+r)(4+r)}$	6

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{(r^2 - 3r + 2)\, r}{(4 + r)\,(3 + r)\,(5 + r)}$$

Which for the root $r = -2$ becomes

$$b_3 = 4$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2-r}{3+r}$	4
b_2	$\frac{r^2-3r+2}{(3+r)(4+r)}$	6
b_3	$\frac{-r^3+3r^2-2r}{(3+r)(4+r)(5+r)}$	4

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{(r^2 - 3r + 2)\, r(1 + r)}{(3 + r)\,(5 + r)\,(4 + r)\,(6 + r)}$$

Which for the root $r = -2$ becomes

$$b_4 = 1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2-r}{3+r}$	4
b_2	$\frac{r^2-3r+2}{(3+r)(4+r)}$	6
b_3	$\frac{-r^3+3r^2-2r}{(3+r)(4+r)(5+r)}$	4
b_4	$\frac{(-1+r)(-2+r)r(1+r)}{(3+r)(4+r)(5+r)(6+r)}$	1

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{(r^2 - 3r + 2)\, r(1 + r)\,(2 + r)}{(6 + r)\,(3 + r)\,(4 + r)\,(7 + r)\,(5 + r)}$$

Which for the root $r = -2$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2-r}{3+r}$	4
b_2	$\frac{r^2-3r+2}{(3+r)(4+r)}$	6
b_3	$\frac{-r^3+3r^2-2r}{(3+r)(4+r)(5+r)}$	4
b_4	$\frac{(-1+r)(-2+r)r(1+r)}{(3+r)(4+r)(5+r)(6+r)}$	1
b_5	$-\frac{(-2+r)(-1+r)r(1+r)(2+r)}{(6+r)(3+r)(4+r)(7+r)(5+r)}$	0

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{(r^2 - 3r + 2)\, r(1 + r)\, (2 + r)}{(7 + r)\, (5 + r)\, (4 + r)\, (8 + r)\, (6 + r)}$$

Which for the root $r = -2$ becomes

$$b_6 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2-r}{3+r}$	4
b_2	$\frac{r^2-3r+2}{(3+r)(4+r)}$	6
b_3	$\frac{-r^3+3r^2-2r}{(3+r)(4+r)(5+r)}$	4
b_4	$\frac{(-1+r)(-2+r)r(1+r)}{(3+r)(4+r)(5+r)(6+r)}$	1
b_5	$-\frac{(-2+r)(-1+r)r(1+r)(2+r)}{(6+r)(3+r)(4+r)(7+r)(5+r)}$	0
b_6	$\frac{r(-2+r)(2+r)(-1+r)(1+r)}{(4+r)(5+r)(6+r)(7+r)(8+r)}$	0

For $n = 7$, using the above recursive equation gives

$$b_7 = -\frac{(r^2 - 3r + 2)\, r(1 + r)\, (2 + r)}{(8 + r)\, (6 + r)\, (5 + r)\, (9 + r)\, (7 + r)}$$

Which for the root $r = -2$ becomes

$$b_7 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2-r}{3+r}$	4
b_2	$\frac{r^2-3r+2}{(3+r)(4+r)}$	6
b_3	$\frac{-r^3+3r^2-2r}{(3+r)(4+r)(5+r)}$	4
b_4	$\frac{(-1+r)(-2+r)r(1+r)}{(3+r)(4+r)(5+r)(6+r)}$	1
b_5	$-\frac{(-2+r)(-1+r)r(1+r)(2+r)}{(6+r)(3+r)(4+r)(7+r)(5+r)}$	0
b_6	$\frac{r(-2+r)(2+r)(-1+r)(1+r)}{(4+r)(5+r)(6+r)(7+r)(8+r)}$	0
b_7	$-\frac{r(-2+r)(2+r)(-1+r)(1+r)}{(5+r)(6+r)(7+r)(8+r)(9+r)}$	0

Using the above table, then the solution $y_2(t)$ is

$$\begin{aligned} y_2(t) &= 1\left(b_0 + b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5 + b_6 t^6 + b_7 t^7 + b_8 t^8 \dots\right) \\ &= \frac{1 + 4t + 6t^2 + 4t^3 + t^4 + O(t^8)}{t^2} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(t) &= c_1 y_1(t) + c_2 y_2(t) \\ &= c_1 \left(1 + \frac{2t}{3} + \frac{t^2}{6} + O(t^8)\right) + \frac{c_2(1 + 4t + 6t^2 + 4t^3 + t^4 + O(t^8))}{t^2} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y(t) &= y_h \\ &= c_1 \left(1 + \frac{2t}{3} + \frac{t^2}{6} + O(t^8)\right) + \frac{c_2(1 + 4t + 6t^2 + 4t^3 + t^4 + O(t^8))}{t^2} \end{aligned}$$

Replacing t in the above with the original independent variable $x$susing $t = x-1$ results in

$$y = c_1\left(\frac{1}{3}+\frac{2x}{3}+\frac{(x-1)^2}{6}+O\left((x-1)^8\right)\right) + \frac{c_2\left(-3+4x+6(x-1)^2+4(x-1)^3+(x-1)^4+O\left((x-1)^8\right)\right)}{(x-1)^2}$$

Summary
The solution(s) found are the following

$$y = c_1\left(\frac{1}{3}+\frac{2x}{3}+\frac{(x-1)^2}{6}+O\left((x-1)^8\right)\right) + \frac{c_2\left(-3+4x+6(x-1)^2+4(x-1)^3+(x-1)^4+O\left((x-1)^8\right)\right)}{(x-1)^2} \quad (1)$$

Verification of solutions

$$y = c_1\left(\frac{1}{3}+\frac{2x}{3}+\frac{(x-1)^2}{6}+O\left((x-1)^8\right)\right) + \frac{c_2\left(-3+4x+6(x-1)^2+4(x-1)^3+(x-1)^4+O\left((x-1)^8\right)\right)}{(x-1)^2}$$

Verified OK.

6.9.1 Maple step by step solution

Let's solve
$(-x^2+x)\,y''-3y'+2y=0$

- Highest derivative means the order of the ODE is 2
  y''
- Isolate 2nd derivative
  $y''=\frac{2y}{x(x-1)}-\frac{3y'}{x(x-1)}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $y''+\frac{3y'}{x(x-1)}-\frac{2y}{x(x-1)}=0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

$$\left[P_2(x) = \frac{3}{x(x-1)}, P_3(x) = -\frac{2}{x(x-1)}\right]$$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$(x \cdot P_2(x))\bigg|_{x=0} = -3$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$y''x(x-1) + 3y' - 2y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert y' to series expansion

  $$y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

  $$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0 r(-4+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(k-3+r) + a_k(k+1+r)\,(k+r-2)\right) x^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-r(-4+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 4\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+1+r)\left((-k-r+3)\, a_{k+1}+a_k(k+r-2)\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k(k+r-2)}{k-3+r}$

- Recursion relation for $r=0$; series terminates at $k=2$

  $a_{k+1}=\frac{a_k(k-2)}{k-3}$

- Apply recursion relation for $k=0$

  $a_1=\frac{2a_0}{3}$

- Apply recursion relation for $k=1$

  $a_2=\frac{a_1}{2}$

- Express in terms of a_0

  $a_2=\frac{a_0}{3}$

- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

  $y=a_0\cdot\left(1+\frac{2}{3}x+\frac{1}{3}x^2\right)$

- Recursion relation for $r=4$

  $a_{k+1}=\frac{a_k(k+2)}{k+1}$

- Solution for $r=4$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+4}, a_{k+1}=\frac{a_k(k+2)}{k+1}\right]$

- Combine solutions and rename parameters

  $\left[y=a_0\cdot\left(1+\frac{2}{3}x+\frac{1}{3}x^2\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+4}\right), b_{k+1}=\frac{b_k(k+2)}{k+1}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 38

```
Order:=8;
dsolve(x*(1-x)*diff(y(x),x$2)-3*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=1);
```

$$y(x) = c_1\left(1+\frac{2}{3}(x-1)+\frac{1}{6}(x-1)^2+\mathrm{O}\left((x-1)^8\right)\right) + \frac{c_2\left(-2-8(x-1)-12(x-1)^2-8(x-1)^3-2(x-1)^4+\mathrm{O}\left((x-1)^8\right)\right)}{(x-1)^2}$$

✓ Solution by Mathematica

Time used: 0.454 (sec). Leaf size: 52

```
AsymptoticDSolveValue[x*(1-x)*y''[x]-3*y'[x]+2*y[x]==0,y[x],{x,1,7}]
```

$$y(x) \to c_1\left((x-1)^2+4(x-1)+\frac{4}{x-1}+\frac{1}{(x-1)^2}+6\right)+c_2\left(\frac{1}{6}(x-1)^2+\frac{2(x-1)}{3}+1\right)$$

6.10 problem 10

Internal problem ID [6974]
Internal file name [`OUTPUT/6217_Friday_August_12_2022_11_05_47_PM_772028/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.8 Indicial Equation with Difference of Roots a Positive Integer: Nonlogarithmic Case. Exercises page 380
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{xy'' + (4 + 3x)\, y' + 3y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + (4 + 3x)\, y' + 3y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{4 + 3x}{x}$$
$$q(x) = \frac{3}{x}$$

Table 138: Table $p(x), q(x)$ singularites.

$p(x) = \frac{4+3x}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{3}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + (4 + 3x) y' + 3y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x\left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) + (4 + 3x) \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + 3\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} 4(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 3a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r) &= \sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)\, x^{n+r-1} \\
\sum_{n=0}^{\infty} 3a_n x^{n+r} &= \sum_{n=1}^{\infty} 3a_{n-1} x^{n+r-1}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)\, x^{n+r-1}\right) \\
&+ \left(\sum_{n=0}^{\infty} 4(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1} x^{n+r-1}\right) = 0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r-1} a_n (n+r)(n+r-1) + 4(n+r) a_n x^{n+r-1} = 0
$$

When $n=0$ the above becomes

$$
x^{-1+r} a_0 r(-1+r) + 4r a_0 x^{-1+r} = 0
$$

Or

$$
\left(x^{-1+r} r(-1+r) + 4r\, x^{-1+r}\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
r\, x^{-1+r}(3+r) = 0
$$

Since the above is true for all x then the indicial equation becomes

$$r(3+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 0 \\ r_2 &= -3 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(3+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\ y_2(x) &= C y_1(x)\ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^3} \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\ y_2(x) &= C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-3}\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 3a_{n-1}(n+r-1) + 4a_n(n+r) + 3a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{3a_{n-1}}{n+3+r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{3a_{n-1}}{n+3} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{3}{4+r}$$

Which for the root $r = 0$ becomes

$$a_1 = -\frac{3}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{4+r}$	$-\frac{3}{4}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{9}{(4+r)(5+r)}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{9}{20}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{4+r}$	$-\frac{3}{4}$
a_2	$\frac{9}{(4+r)(5+r)}$	$\frac{9}{20}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{27}{(4 + r)(5 + r)(6 + r)}$$

Which for the root $r = 0$ becomes

$$a_3 = -\frac{9}{40}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{4+r}$	$-\frac{3}{4}$
a_2	$\frac{9}{(4+r)(5+r)}$	$\frac{9}{20}$
a_3	$-\frac{27}{(4+r)(5+r)(6+r)}$	$-\frac{9}{40}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{81}{(5 + r)(6 + r)(4 + r)(7 + r)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{27}{280}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{4+r}$	$-\frac{3}{4}$
a_2	$\frac{9}{(4+r)(5+r)}$	$\frac{9}{20}$
a_3	$-\frac{27}{(4+r)(5+r)(6+r)}$	$-\frac{9}{40}$
a_4	$\frac{81}{(5+r)(6+r)(4+r)(7+r)}$	$\frac{27}{280}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{243}{(6 + r)(4 + r)(7 + r)(8 + r)(5 + r)}$$

Which for the root $r = 0$ becomes

$$a_5 = -\frac{81}{2240}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{4+r}$	$-\frac{3}{4}$
a_2	$\frac{9}{(4+r)(5+r)}$	$\frac{9}{20}$
a_3	$-\frac{27}{(4+r)(5+r)(6+r)}$	$-\frac{9}{40}$
a_4	$\frac{81}{(5+r)(6+r)(4+r)(7+r)}$	$\frac{27}{280}$
a_5	$-\frac{243}{(6+r)(4+r)(7+r)(8+r)(5+r)}$	$-\frac{81}{2240}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{729}{(4+r)(7+r)(8+r)(5+r)(6+r)(9+r)}$$

Which for the root $r = 0$ becomes

$$a_6 = \frac{27}{2240}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{4+r}$	$-\frac{3}{4}$
a_2	$\frac{9}{(4+r)(5+r)}$	$\frac{9}{20}$
a_3	$-\frac{27}{(4+r)(5+r)(6+r)}$	$-\frac{9}{40}$
a_4	$\frac{81}{(5+r)(6+r)(4+r)(7+r)}$	$\frac{27}{280}$
a_5	$-\frac{243}{(6+r)(4+r)(7+r)(8+r)(5+r)}$	$-\frac{81}{2240}$
a_6	$\frac{729}{(4+r)(7+r)(8+r)(5+r)(6+r)(9+r)}$	$\frac{27}{2240}$

For $n = 7$, using the above recursive equation gives

$$a_7 = -\frac{2187}{(4+r)(8+r)(5+r)(6+r)(9+r)(10+r)(7+r)}$$

Which for the root $r = 0$ becomes

$$a_7 = -\frac{81}{22400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{4+r}$	$-\frac{3}{4}$
a_2	$\frac{9}{(4+r)(5+r)}$	$\frac{9}{20}$
a_3	$-\frac{27}{(4+r)(5+r)(6+r)}$	$-\frac{9}{40}$
a_4	$\frac{81}{(5+r)(6+r)(4+r)(7+r)}$	$\frac{27}{280}$
a_5	$-\frac{243}{(6+r)(4+r)(7+r)(8+r)(5+r)}$	$-\frac{81}{2240}$
a_6	$\frac{729}{(4+r)(7+r)(8+r)(5+r)(6+r)(9+r)}$	$\frac{27}{2240}$
a_7	$-\frac{2187}{(4+r)(8+r)(5+r)(6+r)(9+r)(10+r)(7+r)}$	$-\frac{81}{22400}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots \\ &= 1 - \frac{3x}{4} + \frac{9x^2}{20} - \frac{9x^3}{40} + \frac{27x^4}{280} - \frac{81x^5}{2240} + \frac{27x^6}{2240} - \frac{81x^7}{22400} + O(x^8) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_3 \\ &= -\frac{27}{(4 + r)(5 + r)(6 + r)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} -\frac{27}{(4 + r)(5 + r)(6 + r)} &= \lim_{r \to -3} -\frac{27}{(4 + r)(5 + r)(6 + r)} \\ &= -\frac{9}{2} \end{aligned}$$

The limit is $-\frac{9}{2}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-3}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
b_n(n+r)(n+r-1) + 3b_{n-1}(n+r-1) + 4(n+r)b_n + 3b_{n-1} = 0 \tag{4}
$$

Which for for the root $r = -3$ becomes

$$
b_n(n-3)(n-4) + 3b_{n-1}(n-4) + 4(n-3)b_n + 3b_{n-1} = 0 \tag{4A}
$$

Solving for b_n from the recursive equation (4) gives

$$
b_n = -\frac{3b_{n-1}}{n+3+r} \tag{5}
$$

Which for the root $r = -3$ becomes

$$
b_n = -\frac{3b_{n-1}}{n} \tag{6}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -3$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$
b_1 = -\frac{3}{4+r}
$$

Which for the root $r = -3$ becomes

$$
b_1 = -3
$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{4+r}$	-3

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{9}{(4+r)(5+r)}$$

Which for the root $r = -3$ becomes

$$b_2 = \frac{9}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{4+r}$	-3
b_2	$\frac{9}{(4+r)(5+r)}$	$\frac{9}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{27}{(4+r)(5+r)(6+r)}$$

Which for the root $r = -3$ becomes

$$b_3 = -\frac{9}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{4+r}$	-3
b_2	$\frac{9}{(4+r)(5+r)}$	$\frac{9}{2}$
b_3	$-\frac{27}{(4+r)(5+r)(6+r)}$	$-\frac{9}{2}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{81}{(5+r)(6+r)(4+r)(7+r)}$$

Which for the root $r = -3$ becomes

$$b_4 = \frac{27}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{4+r}$	-3
b_2	$\frac{9}{(4+r)(5+r)}$	$\frac{9}{2}$
b_3	$-\frac{27}{(4+r)(5+r)(6+r)}$	$-\frac{9}{2}$
b_4	$\frac{81}{(5+r)(6+r)(4+r)(7+r)}$	$\frac{27}{8}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{243}{(6+r)(4+r)(7+r)(8+r)(5+r)}$$

Which for the root $r = -3$ becomes

$$b_5 = -\frac{81}{40}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{4+r}$	-3
b_2	$\frac{9}{(4+r)(5+r)}$	$\frac{9}{2}$
b_3	$-\frac{27}{(4+r)(5+r)(6+r)}$	$-\frac{9}{2}$
b_4	$\frac{81}{(5+r)(6+r)(4+r)(7+r)}$	$\frac{27}{8}$
b_5	$-\frac{243}{(6+r)(4+r)(7+r)(8+r)(5+r)}$	$-\frac{81}{40}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{729}{(4+r)(7+r)(8+r)(5+r)(6+r)(9+r)}$$

Which for the root $r = -3$ becomes

$$b_6 = \frac{81}{80}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{4+r}$	-3
b_2	$\frac{9}{(4+r)(5+r)}$	$\frac{9}{2}$
b_3	$-\frac{27}{(4+r)(5+r)(6+r)}$	$-\frac{9}{2}$
b_4	$\frac{81}{(5+r)(6+r)(4+r)(7+r)}$	$\frac{27}{8}$
b_5	$-\frac{243}{(6+r)(4+r)(7+r)(8+r)(5+r)}$	$-\frac{81}{40}$
b_6	$\frac{729}{(4+r)(7+r)(8+r)(5+r)(6+r)(9+r)}$	$\frac{81}{80}$

For $n = 7$, using the above recursive equation gives

$$b_7 = -\frac{2187}{(4+r)(8+r)(5+r)(6+r)(9+r)(10+r)(7+r)}$$

Which for the root $r = -3$ becomes

$$b_7 = -\frac{243}{560}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{4+r}$	-3
b_2	$\frac{9}{(4+r)(5+r)}$	$\frac{9}{2}$
b_3	$-\frac{27}{(4+r)(5+r)(6+r)}$	$-\frac{9}{2}$
b_4	$\frac{81}{(5+r)(6+r)(4+r)(7+r)}$	$\frac{27}{8}$
b_5	$-\frac{243}{(6+r)(4+r)(7+r)(8+r)(5+r)}$	$-\frac{81}{40}$
b_6	$\frac{729}{(4+r)(7+r)(8+r)(5+r)(6+r)(9+r)}$	$\frac{81}{80}$
b_7	$-\frac{2187}{(4+r)(8+r)(5+r)(6+r)(9+r)(10+r)(7+r)}$	$-\frac{243}{560}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots\right) \\ &= \frac{1 - 3x + \frac{9x^2}{2} - \frac{9x^3}{2} + \frac{27x^4}{8} - \frac{81x^5}{40} + \frac{81x^6}{80} - \frac{243x^7}{560} + O(x^8)}{x^3} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \left(1 - \frac{3x}{4} + \frac{9x^2}{20} - \frac{9x^3}{40} + \frac{27x^4}{280} - \frac{81x^5}{2240} + \frac{27x^6}{2240} - \frac{81x^7}{22400} + O(x^8)\right) \\ &+ \frac{c_2 \left(1 - 3x + \frac{9x^2}{2} - \frac{9x^3}{2} + \frac{27x^4}{8} - \frac{81x^5}{40} + \frac{81x^6}{80} - \frac{243x^7}{560} + O(x^8)\right)}{x^3} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 \left(1 - \frac{3x}{4} + \frac{9x^2}{20} - \frac{9x^3}{40} + \frac{27x^4}{280} - \frac{81x^5}{2240} + \frac{27x^6}{2240} - \frac{81x^7}{22400} + O(x^8)\right) \\ &+ \frac{c_2 \left(1 - 3x + \frac{9x^2}{2} - \frac{9x^3}{2} + \frac{27x^4}{8} - \frac{81x^5}{40} + \frac{81x^6}{80} - \frac{243x^7}{560} + O(x^8)\right)}{x^3} \end{aligned}$$

Summary
The solution(s) found are the following

$$
y = c_1 \left(1 - \frac{3x}{4} + \frac{9x^2}{20} - \frac{9x^3}{40} + \frac{27x^4}{280} - \frac{81x^5}{2240} + \frac{27x^6}{2240} - \frac{81x^7}{22400} + O(x^8)\right) + \frac{c_2 \left(1 - 3x + \frac{9x^2}{2} - \frac{9x^3}{2} + \frac{27x^4}{8} - \frac{81x^5}{40} + \frac{81x^6}{80} - \frac{243x^7}{560} + O(x^8)\right)}{x^3} \tag{1}
$$

Verification of solutions

$$
y = c_1 \left(1 - \frac{3x}{4} + \frac{9x^2}{20} - \frac{9x^3}{40} + \frac{27x^4}{280} - \frac{81x^5}{2240} + \frac{27x^6}{2240} - \frac{81x^7}{22400} + O(x^8)\right) + \frac{c_2 \left(1 - 3x + \frac{9x^2}{2} - \frac{9x^3}{2} + \frac{27x^4}{8} - \frac{81x^5}{40} + \frac{81x^6}{80} - \frac{243x^7}{560} + O(x^8)\right)}{x^3}
$$

Verified OK.

6.10.1 Maple step by step solution

Let's solve

$x y'' + (4+3x) y' + 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y}{x} - \frac{(4+3x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(4+3x)y'}{x} + \frac{3y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{4+3x}{x}, P_3(x) = \frac{3}{x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 4$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $xy''+(4+3x)\,y'+3y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m\cdot y'$ to series expansion for $m=0..1$

  $x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x\cdot y''$ to series expansion

  $x\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$

- Shift index using $k->k+1$

  $x\cdot y''=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(3+r)\,x^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(k+4+r)+3a_k(k+1+r)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(3+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{-3,0\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+1+r)\,(a_{k+1}(k+4+r)+3a_k)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{3a_k}{k+4+r}$

- Recursion relation for $r = -3$

  $a_{k+1} = -\frac{3a_k}{k+1}$

- Solution for $r = -3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+1} = -\frac{3a_k}{k+1}\right]$

- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{3a_k}{k+4}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{3a_k}{k+4}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+1} = -\frac{3a_k}{k+1}, b_{k+1} = -\frac{3b_k}{k+4}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 52

```
Order:=8;
dsolve(x*diff(y(x),x$2)+(4+3*x)*diff(y(x),x)+3*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\left(1 - \frac{3}{4}x + \frac{9}{20}x^2 - \frac{9}{40}x^3 + \frac{27}{280}x^4 - \frac{81}{2240}x^5 + \frac{27}{2240}x^6 - \frac{81}{22400}x^7 + \mathrm{O}\left(x^8\right)\right) + \frac{c_2\left(12 - 36x + 54x^2 - 54x^3 + \frac{81}{2}x^4 - \frac{243}{10}x^5 + \frac{243}{20}x^6 - \frac{729}{140}x^7 + \mathrm{O}\left(x^8\right)\right)}{x^3}$$

✓ Solution by Mathematica

Time used: 0.11 (sec). Leaf size: 90

```
AsymptoticDSolveValue[x*y''[x]+(4+3*x)*y'[x]+3*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(\frac{81x^3}{80} + \frac{1}{x^3} - \frac{81x^2}{40} - \frac{3}{x^2} + \frac{27x}{8} + \frac{9}{2x} - \frac{9}{2}\right) + c_2\left(\frac{27x^6}{2240} - \frac{81x^5}{2240} + \frac{27x^4}{280} - \frac{9x^3}{40} + \frac{9x^2}{20} - \frac{3x}{4} + 1\right)$$

6.11 problem 11

Internal problem ID [6975]
Internal file name [`OUTPUT/6218_Friday_August_12_2022_11_05_50_PM_79997059/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.8 Indicial Equation with Difference of Roots a Positive Integer: Nonlogarithmic Case. Exercises page 380
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' - 2(x+2)\,y' + 4y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + (-2x-4)\,y' + 4y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{2(x+2)}{x}$$
$$q(x) = \frac{4}{x}$$

Table 140: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{2(x+2)}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{4}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + (-2x - 4)\, y' + 4y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (-2x-4)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 4\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-4(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0 \tag{2A}
$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} \left(-2x^{n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1) x^{n+r-1}\right)
$$

$$
\sum_{n=0}^{\infty} 4a_n x^{n+r} = \sum_{n=1}^{\infty} 4a_{n-1} x^{n+r-1}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$
\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1) x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-4(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} 4a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}
$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$
x^{n+r-1} a_n (n+r)(n+r-1) - 4(n+r) a_n x^{n+r-1} = 0
$$

When $n = 0$ the above becomes

$$
x^{-1+r} a_0 r(-1+r) - 4r a_0 x^{-1+r} = 0
$$

Or

$$
\left(x^{-1+r} r(-1+r) - 4r\, x^{-1+r}\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
r\, x^{-1+r}(-5+r) = 0
$$

Since the above is true for all x then the indicial equation becomes

$$r(-5+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 5$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(-5+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 5$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^5\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+5}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - 2a_{n-1}(n+r-1) - 4a_n(n+r) + 4a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2a_{n-1}(n+r-3)}{n^2+2nr+r^2-5n-5r} \tag{4}$$

Which for the root $r=5$ becomes

$$a_n = \frac{2a_{n-1}(n+2)}{n\,(n+5)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r=5$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n=1$, using the above recursive equation gives

$$a_1 = \frac{-4+2r}{r^2-3r-4}$$

Which for the root $r=5$ becomes

$$a_1 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4+2r}{r^2-3r-4}$	1

For $n=2$, using the above recursive equation gives

$$a_2 = \frac{4(r-2)(-1+r)}{r^4-4r^3-7r^2+22r+24}$$

Which for the root $r=5$ becomes

$$a_2 = \frac{4}{7}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4+2r}{r^2-3r-4}$	1
a_2	$\frac{4(r-2)(-1+r)}{r^4-4r^3-7r^2+22r+24}$	$\frac{4}{7}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{8r(-1+r)}{r^5 - r^4 - 19r^3 + r^2 + 90r + 72}$$

Which for the root $r = 5$ becomes

$$a_3 = \frac{5}{21}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4+2r}{r^2-3r-4}$	1
a_2	$\frac{4(r-2)(-1+r)}{r^4-4r^3-7r^2+22r+24}$	$\frac{4}{7}$
a_3	$\frac{8r(-1+r)}{r^5-r^4-19r^3+r^2+90r+72}$	$\frac{5}{21}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16r}{r^5 + 2r^4 - 25r^3 - 50r^2 + 144r + 288}$$

Which for the root $r = 5$ becomes

$$a_4 = \frac{5}{63}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4+2r}{r^2-3r-4}$	1
a_2	$\frac{4(r-2)(-1+r)}{r^4-4r^3-7r^2+22r+24}$	$\frac{4}{7}$
a_3	$\frac{8r(-1+r)}{r^5-r^4-19r^3+r^2+90r+72}$	$\frac{5}{21}$
a_4	$\frac{16r}{r^5+2r^4-25r^3-50r^2+144r+288}$	$\frac{5}{63}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{32}{(r+5)(r^4 - 25r^2 + 144)}$$

Which for the root $r = 5$ becomes

$$a_5 = \frac{1}{45}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4+2r}{r^2-3r-4}$	1
a_2	$\frac{4(r-2)(-1+r)}{r^4-4r^3-7r^2+22r+24}$	$\frac{4}{7}$
a_3	$\frac{8r(-1+r)}{r^5-r^4-19r^3+r^2+90r+72}$	$\frac{5}{21}$
a_4	$\frac{16r}{r^5+2r^4-25r^3-50r^2+144r+288}$	$\frac{5}{63}$
a_5	$\frac{32}{(r+5)(r^4-25r^2+144)}$	$\frac{1}{45}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{64}{(r^2+7r+6)(r^3-3r^2-16r+48)(r+5)}$$

Which for the root $r = 5$ becomes

$$a_6 = \frac{8}{1485}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4+2r}{r^2-3r-4}$	1
a_2	$\frac{4(r-2)(-1+r)}{r^4-4r^3-7r^2+22r+24}$	$\frac{4}{7}$
a_3	$\frac{8r(-1+r)}{r^5-r^4-19r^3+r^2+90r+72}$	$\frac{5}{21}$
a_4	$\frac{16r}{r^5+2r^4-25r^3-50r^2+144r+288}$	$\frac{5}{63}$
a_5	$\frac{32}{(r+5)(r^4-25r^2+144)}$	$\frac{1}{45}$
a_6	$\frac{64}{(r^2+7r+6)(r^3-3r^2-16r+48)(r+5)}$	$\frac{8}{1485}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{128}{(r^2+9r+14)(r+5)(r^2-7r+12)(r^2+7r+6)}$$

Which for the root $r = 5$ becomes

$$a_7 = \frac{4}{3465}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4+2r}{r^2-3r-4}$	1
a_2	$\frac{4(r-2)(-1+r)}{r^4-4r^3-7r^2+22r+24}$	$\frac{4}{7}$
a_3	$\frac{8r(-1+r)}{r^5-r^4-19r^3+r^2+90r+72}$	$\frac{5}{21}$
a_4	$\frac{16r}{r^5+2r^4-25r^3-50r^2+144r+288}$	$\frac{5}{63}$
a_5	$\frac{32}{(r+5)(r^4-25r^2+144)}$	$\frac{1}{45}$
a_6	$\frac{64}{(r^2+7r+6)(r^3-3r^2-16r+48)(r+5)}$	$\frac{8}{1485}$
a_7	$\frac{128}{(r^2+9r+14)(r+5)(r^2-7r+12)(r^2+7r+6)}$	$\frac{4}{3465}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^5\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \ldots\right) \\
&= x^5\left(1 + x + \frac{4x^2}{7} + \frac{5x^3}{21} + \frac{5x^4}{63} + \frac{x^5}{45} + \frac{8x^6}{1485} + \frac{4x^7}{3465} + O(x^8)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 5$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_5(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_5 \\
&= \frac{32}{(r+5)(r^4 - 25r^2 + 144)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r \to r_2} \frac{32}{(r+5)(r^4 - 25r^2 + 144)} &= \lim_{r \to 0} \frac{32}{(r+5)(r^4 - 25r^2 + 144)} \\
&= \frac{2}{45}
\end{aligned}
$$

The limit is $\frac{2}{45}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^n \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) - 2b_{n-1}(n+r-1) - 4(n+r)b_n + 4b_{n-1} = 0 \tag{4}$$

Which for for the root $r = 0$ becomes

$$b_n n(n-1) - 2b_{n-1}(n-1) - 4nb_n + 4b_{n-1} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = \frac{2b_{n-1}(n+r-3)}{n^2 + 2nr + r^2 - 5n - 5r} \tag{5}$$

Which for the root $r = 0$ becomes

$$b_n = \frac{2b_{n-1}(n-3)}{n^2 - 5n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{-4 + 2r}{r^2 - 3r - 4}$$

Which for the root $r = 0$ becomes

$$b_1 = 1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4+2r}{r^2-3r-4}$	1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4(r-2)(-1+r)}{(r^2-3r-4)(r^2-r-6)}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4+2r}{r^2-3r-4}$	1
b_2	$\frac{4(r-2)(-1+r)}{r^4-4r^3-7r^2+22r+24}$	$\frac{1}{3}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{8r(-1+r)}{(r+3)(r^2-3r-4)(r^2-r-6)}$$

Which for the root $r = 0$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4+2r}{r^2-3r-4}$	1
b_2	$\frac{4(r-2)(-1+r)}{r^4-4r^3-7r^2+22r+24}$	$\frac{1}{3}$
b_3	$\frac{8r(-1+r)}{r^5-r^4-19r^3+r^2+90r+72}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16r}{(r+4)(r^2-r-6)(r-4)(r+3)}$$

Which for the root $r = 0$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4+2r}{r^2-3r-4}$	1
b_2	$\frac{4(r-2)(-1+r)}{r^4-4r^3-7r^2+22r+24}$	$\frac{1}{3}$
b_3	$\frac{8r(-1+r)}{r^5-r^4-19r^3+r^2+90r+72}$	0
b_4	$\frac{16r}{(r^2-r-6)(r+3)(r^2-16)}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{32}{(r+3)(r-4)(r-3)(r+4)(r+5)}$$

Which for the root $r = 0$ becomes

$$b_5 = \frac{2}{45}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4+2r}{r^2-3r-4}$	1
b_2	$\frac{4(r-2)(-1+r)}{r^4-4r^3-7r^2+22r+24}$	$\frac{1}{3}$
b_3	$\frac{8r(-1+r)}{r^5-r^4-19r^3+r^2+90r+72}$	0
b_4	$\frac{16r}{(r^2-r-6)(r+3)(r^2-16)}$	0
b_5	$\frac{32}{(r+5)(r^2-9)(r^2-16)}$	$\frac{2}{45}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{64}{(r-4)(r-3)(r+4)(r+5)(r^2+7r+6)}$$

Which for the root $r = 0$ becomes

$$b_6 = \frac{2}{45}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4+2r}{r^2-3r-4}$	1
b_2	$\frac{4(r-2)(-1+r)}{r^4-4r^3-7r^2+22r+24}$	$\frac{1}{3}$
b_3	$\frac{8r(-1+r)}{r^5-r^4-19r^3+r^2+90r+72}$	0
b_4	$\frac{16r}{(r^2-r-6)(r+3)(r^2-16)}$	0
b_5	$\frac{32}{(r+5)(r^2-9)(r^2-16)}$	$\frac{2}{45}$
b_6	$\frac{64}{(r-3)(r+5)(r^2+7r+6)(r^2-16)}$	$\frac{2}{45}$

For $n = 7$, using the above recursive equation gives

$$b_7 = \frac{128}{(r-4)(r-3)(r+5)(r^2+7r+6)(r^2+9r+14)}$$

Which for the root $r = 0$ becomes

$$b_7 = \frac{8}{315}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4+2r}{r^2-3r-4}$	1
b_2	$\frac{4(r-2)(-1+r)}{r^4-4r^3-7r^2+22r+24}$	$\frac{1}{3}$
b_3	$\frac{8r(-1+r)}{r^5-r^4-19r^3+r^2+90r+72}$	0
b_4	$\frac{16r}{(r^2-r-6)(r+3)(r^2-16)}$	0
b_5	$\frac{32}{(r+5)(r^2-9)(r^2-16)}$	$\frac{2}{45}$
b_6	$\frac{64}{(r-3)(r+5)(r^2+7r+6)(r^2-16)}$	$\frac{2}{45}$
b_7	$\frac{128}{(7+r)(2+r)(r+5)(r-3)(r-4)(6+r)(1+r)}$	$\frac{8}{315}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\ &= 1 + x + \frac{x^2}{3} + \frac{2x^5}{45} + \frac{2x^6}{45} + \frac{8x^7}{315} + O(x^8) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1 y_1(x) + c_2 y_2(x)\\ &= c_1 x^5 \left(1 + x + \frac{4x^2}{7} + \frac{5x^3}{21} + \frac{5x^4}{63} + \frac{x^5}{45} + \frac{8x^6}{1485} + \frac{4x^7}{3465} + O(x^8)\right)\\ &+ c_2 \left(1 + x + \frac{x^2}{3} + \frac{2x^5}{45} + \frac{2x^6}{45} + \frac{8x^7}{315} + O(x^8)\right)\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y &= y_h\\ &= c_1 x^5 \left(1 + x + \frac{4x^2}{7} + \frac{5x^3}{21} + \frac{5x^4}{63} + \frac{x^5}{45} + \frac{8x^6}{1485} + \frac{4x^7}{3465} + O(x^8)\right)\\ &+ c_2 \left(1 + x + \frac{x^2}{3} + \frac{2x^5}{45} + \frac{2x^6}{45} + \frac{8x^7}{315} + O(x^8)\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}y = c_1 x^5 \left(1 + x + \frac{4x^2}{7} + \frac{5x^3}{21} + \frac{5x^4}{63} + \frac{x^5}{45} + \frac{8x^6}{1485} + \frac{4x^7}{3465} + O(x^8)\right)\\ + c_2 \left(1 + x + \frac{x^2}{3} + \frac{2x^5}{45} + \frac{2x^6}{45} + \frac{8x^7}{315} + O(x^8)\right)\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}y = c_1 x^5 \left(1 + x + \frac{4x^2}{7} + \frac{5x^3}{21} + \frac{5x^4}{63} + \frac{x^5}{45} + \frac{8x^6}{1485} + \frac{4x^7}{3465} + O(x^8)\right)\\ + c_2 \left(1 + x + \frac{x^2}{3} + \frac{2x^5}{45} + \frac{2x^6}{45} + \frac{8x^7}{315} + O(x^8)\right)\end{aligned}$$

Verified OK.

6.11.1 Maple step by step solution

Let's solve

$xy'' + (-2x - 4)\, y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4y}{x} + \frac{2(x+2)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2(x+2)y'}{x} + \frac{4y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2(x+2)}{x}, P_3(x) = \frac{4}{x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -4$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + (-2x - 4)\, y' + 4y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x\cdot y''$ to series expansion

  $x\cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $x\cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-5+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k-4+r)-2a_k(k+r-2)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-5+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 5\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k-4+r)-2a_k(k+r-2) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k(k+r-2)}{(k+1+r)(k-4+r)}$

- Recursion relation for $r=0$; series terminates at $k=2$

  $a_{k+1} = \frac{2a_k(k-2)}{(k+1)(k-4)}$

- Apply recursion relation for $k=0$

  $a_1 = a_0$

- Apply recursion relation for $k=1$

  $a_2 = \frac{a_1}{3}$

- Express in terms of a_0

  $a_2 = \frac{a_0}{3}$

- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

  $y = a_0\cdot\left(1+x+\frac{1}{3}x^2\right)$

- Recursion relation for $r=5$

$a_{k+1} = \frac{2a_k(k+3)}{(k+6)(k+1)}$

- Solution for $r = 5$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+5}, a_{k+1} = \frac{2a_k(k+3)}{(k+6)(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y = a_0 \cdot \left(1 + x + \frac{1}{3}x^2\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+5}\right), b_{k+1} = \frac{2b_k(k+3)}{(k+6)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 48

```
Order:=8;
dsolve(x*diff(y(x),x$2)-2*(x+2)*diff(y(x),x)+4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^5 \left(1 + x + \frac{4}{7}x^2 + \frac{5}{21}x^3 + \frac{5}{63}x^4 + \frac{1}{45}x^5 + \frac{8}{1485}x^6 + \frac{4}{3465}x^7 + \mathrm{O}\left(x^8\right)\right) + c_2\left(2880 + 2880x + 960x^2 + 128x^5 + 128x^6 + \frac{512}{7}x^7 + \mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.11 (sec). Leaf size: 76

```
AsymptoticDSolveValue[x*y''[x]-2*(x+2)*y'[x]+4*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(\frac{2x^6}{45}+\frac{2x^5}{45}+\frac{x^2}{3}+x+1\right)+c_2\left(\frac{8x^{11}}{1485}+\frac{x^{10}}{45}+\frac{5x^9}{63}+\frac{5x^8}{21}+\frac{4x^7}{7}+x^6+x^5\right)$$

6.12 problem 12

Internal problem ID [6976]
Internal file name [`OUTPUT/6219_Friday_August_12_2022_11_05_52_PM_96052526/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.8 Indicial Equation with Difference of Roots a Positive Integer: Nonlogarithmic Case. Exercises page 380
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + (3+2x)\,y' + 4y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + (3+2x)\,y' + 4y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{3+2x}{x}$$
$$q(x) = \frac{4}{x}$$

Table 142: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3+2x}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{4}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + (3 + 2x)\, y' + 4y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x\left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}\right) + (3 + 2x) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + 4\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r) = \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} 4a_n x^{n+r} = \sum_{n=1}^{\infty} 4a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} 4a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + 3(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + 3r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + 3r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(2+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(2+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = -2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(2+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 2a_{n-1}(n+r-1) + 3a_n(n+r) + 4a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2a_{n-1}(n+r+1)}{n^2+2nr+r^2+2n+2r} \tag{4}$$

Which for the root $r=0$ becomes

$$a_n = -\frac{2a_{n-1}(n+1)}{n(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r=0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n=1$, using the above recursive equation gives

$$a_1 = \frac{-4-2r}{r^2+4r+3}$$

Which for the root $r=0$ becomes

$$a_1 = -\frac{4}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4-2r}{r^2+4r+3}$	$-\frac{4}{3}$

For $n=2$, using the above recursive equation gives

$$a_2 = \frac{4}{(r+4)(r+1)}$$

Which for the root $r=0$ becomes

$$a_2 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4-2r}{r^2+4r+3}$	$-\frac{4}{3}$
a_2	$\frac{4}{(r+4)(r+1)}$	1

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{8}{(r+1)(r+5)(3+r)}$$

Which for the root $r = 0$ becomes

$$a_3 = -\frac{8}{15}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4-2r}{r^2+4r+3}$	$-\frac{4}{3}$
a_2	$\frac{4}{(r+4)(r+1)}$	1
a_3	$-\frac{8}{(r+1)(r+5)(3+r)}$	$-\frac{8}{15}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{(r+1)(3+r)(r+6)(r+4)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{2}{9}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4-2r}{r^2+4r+3}$	$-\frac{4}{3}$
a_2	$\frac{4}{(r+4)(r+1)}$	1
a_3	$-\frac{8}{(r+1)(r+5)(3+r)}$	$-\frac{8}{15}$
a_4	$\frac{16}{(r+1)(3+r)(r+6)(r+4)}$	$\frac{2}{9}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{32}{(r+1)(3+r)(r+4)(r+7)(r+5)}$$

Which for the root $r = 0$ becomes

$$a_5 = -\frac{8}{105}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4-2r}{r^2+4r+3}$	$-\frac{4}{3}$
a_2	$\frac{4}{(r+4)(r+1)}$	1
a_3	$-\frac{8}{(r+1)(r+5)(3+r)}$	$-\frac{8}{15}$
a_4	$\frac{16}{(r+1)(3+r)(r+6)(r+4)}$	$\frac{2}{9}$
a_5	$-\frac{32}{(r+1)(3+r)(r+4)(r+7)(r+5)}$	$-\frac{8}{105}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{64}{(r+1)(3+r)(r+4)(r+5)(8+r)(r+6)}$$

Which for the root $r = 0$ becomes

$$a_6 = \frac{1}{45}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4-2r}{r^2+4r+3}$	$-\frac{4}{3}$
a_2	$\frac{4}{(r+4)(r+1)}$	1
a_3	$-\frac{8}{(r+1)(r+5)(3+r)}$	$-\frac{8}{15}$
a_4	$\frac{16}{(r+1)(3+r)(r+6)(r+4)}$	$\frac{2}{9}$
a_5	$-\frac{32}{(r+1)(3+r)(r+4)(r+7)(r+5)}$	$-\frac{8}{105}$
a_6	$\frac{64}{(r+1)(3+r)(r+4)(r+5)(8+r)(r+6)}$	$\frac{1}{45}$

For $n = 7$, using the above recursive equation gives

$$a_7 = -\frac{128}{(r+1)(3+r)(r+4)(r+5)(r+6)(9+r)(r+7)}$$

Which for the root $r = 0$ becomes

$$a_7 = -\frac{16}{2835}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4-2r}{r^2+4r+3}$	$-\frac{4}{3}$
a_2	$\frac{4}{(r+4)(r+1)}$	1
a_3	$-\frac{8}{(r+1)(r+5)(3+r)}$	$-\frac{8}{15}$
a_4	$\frac{16}{(r+1)(3+r)(r+6)(r+4)}$	$\frac{2}{9}$
a_5	$-\frac{32}{(r+1)(3+r)(r+4)(r+7)(r+5)}$	$-\frac{8}{105}$
a_6	$\frac{64}{(r+1)(3+r)(r+4)(r+5)(8+r)(r+6)}$	$\frac{1}{45}$
a_7	$-\frac{128}{(r+1)(3+r)(r+4)(r+5)(r+6)(9+r)(r+7)}$	$-\frac{16}{2835}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots \\ &= 1 - \frac{4x}{3} + x^2 - \frac{8x^3}{15} + \frac{2x^4}{9} - \frac{8x^5}{105} + \frac{x^6}{45} - \frac{16x^7}{2835} + O(x^8) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_2 \\ &= \frac{4}{(r+4)(r+1)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{4}{(r+4)(r+1)} &= \lim_{r \to -2} \frac{4}{(r+4)(r+1)} \\ &= -2 \end{aligned}$$

The limit is -2. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-2}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
b_n(n+r)(n+r-1) + 2b_{n-1}(n+r-1) + 3(n+r)b_n + 4b_{n-1} = 0 \tag{4}
$$

Which for for the root $r = -2$ becomes

$$
b_n(n-2)(n-3) + 2b_{n-1}(n-3) + 3(n-2)b_n + 4b_{n-1} = 0 \tag{4A}
$$

Solving for b_n from the recursive equation (4) gives

$$
b_n = -\frac{2b_{n-1}(n+r+1)}{n^2 + 2nr + r^2 + 2n + 2r} \tag{5}
$$

Which for the root $r = -2$ becomes

$$
b_n = -\frac{2b_{n-1}(n-1)}{n^2 - 2n} \tag{6}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -2$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$
b_1 = -\frac{2(2+r)}{r^2 + 4r + 3}
$$

Which for the root $r = -2$ becomes

$$
b_1 = 0
$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4-2r}{r^2+4r+3}$	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4}{(r+4)(r+1)}$$

Which for the root $r = -2$ becomes

$$b_2 = -2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4-2r}{r^2+4r+3}$	0
b_2	$\frac{4}{(r+4)(r+1)}$	-2

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{8}{(r+1)(r^2+8r+15)}$$

Which for the root $r = -2$ becomes

$$b_3 = \frac{8}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4-2r}{r^2+4r+3}$	0
b_2	$\frac{4}{(r+4)(r+1)}$	-2
b_3	$-\frac{8}{(r+1)(r+5)(3+r)}$	$\frac{8}{3}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16}{(r+1)(3+r)(r^2+10r+24)}$$

Which for the root $r = -2$ becomes

$$b_4 = -2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4-2r}{r^2+4r+3}$	0
b_2	$\frac{4}{(r+4)(r+1)}$	-2
b_3	$-\frac{8}{(r+1)(r+5)(3+r)}$	$\frac{8}{3}$
b_4	$\frac{16}{(r+1)(3+r)(r+6)(r+4)}$	-2

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{32}{(r+1)(3+r)(r+4)(r^2+12r+35)}$$

Which for the root $r = -2$ becomes

$$b_5 = \frac{16}{15}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4-2r}{r^2+4r+3}$	0
b_2	$\frac{4}{(r+4)(r+1)}$	-2
b_3	$-\frac{8}{(r+1)(r+5)(3+r)}$	$\frac{8}{3}$
b_4	$\frac{16}{(r+1)(3+r)(r+6)(r+4)}$	-2
b_5	$-\frac{32}{(r+1)(3+r)(r+4)(r+7)(r+5)}$	$\frac{16}{15}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{64}{(r+1)(3+r)(r+4)(r+5)(r^2+14r+48)}$$

Which for the root $r = -2$ becomes

$$b_6 = -\frac{4}{9}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4-2r}{r^2+4r+3}$	0
b_2	$\frac{4}{(r+4)(r+1)}$	-2
b_3	$-\frac{8}{(r+1)(r+5)(3+r)}$	$\frac{8}{3}$
b_4	$\frac{16}{(r+1)(3+r)(r+6)(r+4)}$	-2
b_5	$-\frac{32}{(r+1)(3+r)(r+4)(r+7)(r+5)}$	$\frac{16}{15}$
b_6	$\frac{64}{(r+1)(3+r)(r+4)(r+5)(8+r)(r+6)}$	$-\frac{4}{9}$

For $n = 7$, using the above recursive equation gives

$$b_7 = -\frac{128}{(r+1)(3+r)(r+4)(r+5)(r+6)(r^2+16r+63)}$$

Which for the root $r = -2$ becomes

$$b_7 = \frac{16}{105}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4-2r}{r^2+4r+3}$	0
b_2	$\frac{4}{(r+4)(r+1)}$	-2
b_3	$-\frac{8}{(r+1)(r+5)(3+r)}$	$\frac{8}{3}$
b_4	$\frac{16}{(r+1)(3+r)(r+6)(r+4)}$	-2
b_5	$-\frac{32}{(r+1)(3+r)(r+4)(r+7)(r+5)}$	$\frac{16}{15}$
b_6	$\frac{64}{(r+1)(3+r)(r+4)(r+5)(8+r)(r+6)}$	$-\frac{4}{9}$
b_7	$-\frac{128}{(r+1)(3+r)(r+4)(r+5)(r+6)(9+r)(r+7)}$	$\frac{16}{105}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots\right) \\
&= \frac{1 - 2x^2 + \frac{8x^3}{3} - 2x^4 + \frac{16x^5}{15} - \frac{4x^6}{9} + \frac{16x^7}{105} + O(x^8)}{x^2}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1 - \frac{4x}{3} + x^2 - \frac{8x^3}{15} + \frac{2x^4}{9} - \frac{8x^5}{105} + \frac{x^6}{45} - \frac{16x^7}{2835} + O(x^8)\right) \\
&+ \frac{c_2\left(1 - 2x^2 + \frac{8x^3}{3} - 2x^4 + \frac{16x^5}{15} - \frac{4x^6}{9} + \frac{16x^7}{105} + O(x^8)\right)}{x^2}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1 - \frac{4x}{3} + x^2 - \frac{8x^3}{15} + \frac{2x^4}{9} - \frac{8x^5}{105} + \frac{x^6}{45} - \frac{16x^7}{2835} + O(x^8)\right) \\
&+ \frac{c_2\left(1 - 2x^2 + \frac{8x^3}{3} - 2x^4 + \frac{16x^5}{15} - \frac{4x^6}{9} + \frac{16x^7}{105} + O(x^8)\right)}{x^2}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1\left(1-\frac{4x}{3}+x^2-\frac{8x^3}{15}+\frac{2x^4}{9}-\frac{8x^5}{105}+\frac{x^6}{45}-\frac{16x^7}{2835}+O(x^8)\right) + \frac{c_2\left(1-2x^2+\frac{8x^3}{3}-2x^4+\frac{16x^5}{15}-\frac{4x^6}{9}+\frac{16x^7}{105}+O(x^8)\right)}{x^2} \tag{1}$$

Verification of solutions

$$y = c_1\left(1-\frac{4x}{3}+x^2-\frac{8x^3}{15}+\frac{2x^4}{9}-\frac{8x^5}{105}+\frac{x^6}{45}-\frac{16x^7}{2835}+O(x^8)\right) + \frac{c_2\left(1-2x^2+\frac{8x^3}{3}-2x^4+\frac{16x^5}{15}-\frac{4x^6}{9}+\frac{16x^7}{105}+O(x^8)\right)}{x^2}$$

Verified OK.

6.12.1 Maple step by step solution

Let's solve

$xy'' + (3+2x)\,y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4y}{x} - \frac{(3+2x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(3+2x)y'}{x} + \frac{4y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{3+2x}{x}, P_3(x) = \frac{4}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 3$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

- ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $xy''+(3+2x)\,y'+4y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - ◦ Convert $x^m\cdot y'$ to series expansion for $m=0..1$

    $x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - ◦ Shift index using $k->k+1-m$

    $x^m\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - ◦ Convert $x\cdot y''$ to series expansion

    $x\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$

  - ◦ Shift index using $k->k+1$

    $x\cdot y''=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(2+r)\,x^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(k+3+r)+2a_k(k+r+2)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(2+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{-2,0\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k+3+r)+2a_k(k+r+2)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{2a_k(k+r+2)}{(k+1+r)(k+3+r)}$

- Recursion relation for $r = -2$

  $a_{k+1} = -\frac{2a_k k}{(k-1)(k+1)}$

- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 1$

  $a_{k+1} = -\frac{2a_k k}{(k-1)(k+1)}$

- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{2a_k(k+2)}{(k+1)(k+3)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{2a_k(k+2)}{(k+1)(k+3)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 50

```
Order:=8;
dsolve(x*diff(y(x),x$2)+(3+2*x)*diff(y(x),x)+4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\left(1 - \frac{4}{3}x + x^2 - \frac{8}{15}x^3 + \frac{2}{9}x^4 - \frac{8}{105}x^5 + \frac{1}{45}x^6 - \frac{16}{2835}x^7 + \mathrm{O}\left(x^8\right)\right) + \frac{c_2\left(-2 + 4x^2 - \frac{16}{3}x^3 + 4x^4 - \frac{32}{15}x^5 + \frac{8}{9}x^6 - \frac{32}{105}x^7 + \mathrm{O}\left(x^8\right)\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.106 (sec). Leaf size: 77

```
AsymptoticDSolveValue[x*y''[x]+(3+2*x)*y'[x]+4*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1 \left(-\frac{4x^4}{9} + \frac{16x^3}{15} - 2x^2 + \frac{1}{x^2} + \frac{8x}{3} - 2\right) + c_2 \left(\frac{x^6}{45} - \frac{8x^5}{105} + \frac{2x^4}{9} - \frac{8x^3}{15} + x^2 - \frac{4x}{3} + 1\right)$$

6.13 problem 13

Internal problem ID [6977]
Internal file name [`OUTPUT/6220_Friday_August_12_2022_11_05_54_PM_39873401/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.8 Indicial Equation with Difference of Roots a Positive Integer: Nonlogarithmic Case. Exercises page 380
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x(x+3)\,y''-9y'-6y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^2+3x)\,y''-6y-9y'=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=-\frac{9}{x\,(x+3)}$$
$$q(x)=-\frac{6}{x\,(x+3)}$$

Table 144: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{9}{x(x+3)}$	
singularity	type
$x = -3$	"regular"
$x = 0$	"regular"

$q(x) = -\frac{6}{x(x+3)}$	
singularity	type
$x = -3$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-3, 0, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x(x+3)\, y'' - 9y' - 6y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x(x+3)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) - 9\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - 6\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r-1} a_n (n+r)(n+r-1)\right) \tag{2A}$$
$$+ \sum_{n=0}^{\infty} \left(-9(n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-6a_n x^{n+r}\right) = 0$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} a_{n-1} (n+r-1)(n+r-2) x^{n+r-1}$$

$$\sum_{n=0}^{\infty} \left(-6a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-6a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=1}^{\infty} a_{n-1} (n+r-1)(n+r-2) x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r-1} a_n (n+r)(n+r-1)\right) \tag{2B}$$
$$+ \sum_{n=0}^{\infty} \left(-9(n+r) a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-6a_{n-1} x^{n+r-1}\right) = 0$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$3x^{n+r-1} a_n (n+r)(n+r-1) - 9(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$3x^{-1+r} a_0 r(-1+r) - 9r a_0 x^{-1+r} = 0$$

Or

$$\left(3x^{-1+r} r(-1+r) - 9r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$3r\, x^{-1+r} (-4+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$3r(-4+r)=0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 4$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$3r\,x^{-1+r}(-4+r)=0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 4$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^4\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+4}$$
$$y_2(x) = C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n=0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_{n-1}(n+r-1)(n+r-2) + 3a_n(n+r)(n+r-1) - 9a_n(n+r) - 6a_{n-1} = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(n+r+1)\,a_{n-1}}{3\,(n+r)} \tag{4}$$

Which for the root $r = 4$ becomes

$$a_n = -\frac{(n+5)\,a_{n-1}}{3n+12} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 4$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-2-r}{3r+3}$$

Which for the root $r = 4$ becomes

$$a_1 = -\frac{2}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-r}{3r+3}$	$-\frac{2}{5}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{3+r}{9r+9}$$

Which for the root $r = 4$ becomes

$$a_2 = \frac{7}{45}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-r}{3r+3}$	$-\frac{2}{5}$
a_2	$\frac{3+r}{9r+9}$	$\frac{7}{45}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-4 - r}{27r + 27}$$

Which for the root $r = 4$ becomes

$$a_3 = -\frac{8}{135}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-r}{3r+3}$	$-\frac{2}{5}$
a_2	$\frac{3+r}{9r+9}$	$\frac{7}{45}$
a_3	$\frac{-4-r}{27r+27}$	$-\frac{8}{135}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{5 + r}{81r + 81}$$

Which for the root $r = 4$ becomes

$$a_4 = \frac{1}{45}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-r}{3r+3}$	$-\frac{2}{5}$
a_2	$\frac{3+r}{9r+9}$	$\frac{7}{45}$
a_3	$\frac{-4-r}{27r+27}$	$-\frac{8}{135}$
a_4	$\frac{5+r}{81r+81}$	$\frac{1}{45}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-6 - r}{243r + 243}$$

Which for the root $r = 4$ becomes

$$a_5 = -\frac{2}{243}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-r}{3r+3}$	$-\frac{2}{5}$
a_2	$\frac{3+r}{9r+9}$	$\frac{7}{45}$
a_3	$\frac{-4-r}{27r+27}$	$-\frac{8}{135}$
a_4	$\frac{5+r}{81r+81}$	$\frac{1}{45}$
a_5	$\frac{-6-r}{243r+243}$	$-\frac{2}{243}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{7+r}{729r+729}$$

Which for the root $r = 4$ becomes

$$a_6 = \frac{11}{3645}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-r}{3r+3}$	$-\frac{2}{5}$
a_2	$\frac{3+r}{9r+9}$	$\frac{7}{45}$
a_3	$\frac{-4-r}{27r+27}$	$-\frac{8}{135}$
a_4	$\frac{5+r}{81r+81}$	$\frac{1}{45}$
a_5	$\frac{-6-r}{243r+243}$	$-\frac{2}{243}$
a_6	$\frac{7+r}{729r+729}$	$\frac{11}{3645}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{-8-r}{2187r+2187}$$

Which for the root $r = 4$ becomes

$$a_7 = -\frac{4}{3645}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-r}{3r+3}$	$-\frac{2}{5}$
a_2	$\frac{3+r}{9r+9}$	$\frac{7}{45}$
a_3	$\frac{-4-r}{27r+27}$	$-\frac{8}{135}$
a_4	$\frac{5+r}{81r+81}$	$\frac{1}{45}$
a_5	$\frac{-6-r}{243r+243}$	$-\frac{2}{243}$
a_6	$\frac{7+r}{729r+729}$	$\frac{11}{3645}$
a_7	$\frac{-8-r}{2187r+2187}$	$-\frac{4}{3645}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^4\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\ &= x^4\left(1 - \frac{2x}{5} + \frac{7x^2}{45} - \frac{8x^3}{135} + \frac{x^4}{45} - \frac{2x^5}{243} + \frac{11x^6}{3645} - \frac{4x^7}{3645} + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_4 \\ &= \frac{5+r}{81r+81} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{5+r}{81r+81} &= \lim_{r\to 0} \frac{5+r}{81r+81} \\ &= \frac{5}{81} \end{aligned}$$

The limit is $\frac{5}{81}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^n \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_{n-1}(n+r-1)(n+r-2) + 3b_n(n+r)(n+r-1) - 9(n+r)b_n - 6b_{n-1} = 0 \quad (4)$$

Which for for the root $r = 0$ becomes

$$b_{n-1}(n-1)(n-2) + 3b_n n(n-1) - 9nb_n - 6b_{n-1} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{(n+r+1)\,b_{n-1}}{3\,(n+r)} \tag{5}$$

Which for the root $r = 0$ becomes

$$b_n = -\frac{(n+1)\,b_{n-1}}{3n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{2+r}{3\,(r+1)}$$

Which for the root $r = 0$ becomes

$$b_1 = -\frac{2}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2-r}{3r+3}$	$-\frac{2}{3}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{3+r}{9r+9}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2-r}{3r+3}$	$-\frac{2}{3}$
b_2	$\frac{3+r}{9r+9}$	$\frac{1}{3}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{4+r}{27(r+1)}$$

Which for the root $r = 0$ becomes

$$b_3 = -\frac{4}{27}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2-r}{3r+3}$	$-\frac{2}{3}$
b_2	$\frac{3+r}{9r+9}$	$\frac{1}{3}$
b_3	$\frac{-4-r}{27r+27}$	$-\frac{4}{27}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{5+r}{81r+81}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{5}{81}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2-r}{3r+3}$	$-\frac{2}{3}$
b_2	$\frac{3+r}{9r+9}$	$\frac{1}{3}$
b_3	$\frac{-4-r}{27r+27}$	$-\frac{4}{27}$
b_4	$\frac{5+r}{81r+81}$	$\frac{5}{81}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{6+r}{243\,(r+1)}$$

Which for the root $r = 0$ becomes

$$b_5 = -\frac{2}{81}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2-r}{3r+3}$	$-\frac{2}{3}$
b_2	$\frac{3+r}{9r+9}$	$\frac{1}{3}$
b_3	$\frac{-4-r}{27r+27}$	$-\frac{4}{27}$
b_4	$\frac{5+r}{81r+81}$	$\frac{5}{81}$
b_5	$\frac{-6-r}{243r+243}$	$-\frac{2}{81}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{7+r}{729r+729}$$

Which for the root $r = 0$ becomes

$$b_6 = \frac{7}{729}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2-r}{3r+3}$	$-\frac{2}{3}$
b_2	$\frac{3+r}{9r+9}$	$\frac{1}{3}$
b_3	$\frac{-4-r}{27r+27}$	$-\frac{4}{27}$
b_4	$\frac{5+r}{81r+81}$	$\frac{5}{81}$
b_5	$\frac{-6-r}{243r+243}$	$-\frac{2}{81}$
b_6	$\frac{7+r}{729r+729}$	$\frac{7}{729}$

For $n = 7$, using the above recursive equation gives

$$b_7 = -\frac{8+r}{2187\,(r+1)}$$

Which for the root $r = 0$ becomes

$$b_7 = -\frac{8}{2187}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2-r}{3r+3}$	$-\frac{2}{3}$
b_2	$\frac{3+r}{9r+9}$	$\frac{1}{3}$
b_3	$\frac{-4-r}{27r+27}$	$-\frac{4}{27}$
b_4	$\frac{5+r}{81r+81}$	$\frac{5}{81}$
b_5	$\frac{-6-r}{243r+243}$	$-\frac{2}{81}$
b_6	$\frac{7+r}{729r+729}$	$\frac{7}{729}$
b_7	$\frac{-8-r}{2187r+2187}$	$-\frac{8}{2187}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\ &= 1 - \frac{2x}{3} + \frac{x^2}{3} - \frac{4x^3}{27} + \frac{5x^4}{81} - \frac{2x^5}{81} + \frac{7x^6}{729} - \frac{8x^7}{2187} + O\left(x^8\right) \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^4 \left(1 - \frac{2x}{5} + \frac{7x^2}{45} - \frac{8x^3}{135} + \frac{x^4}{45} - \frac{2x^5}{243} + \frac{11x^6}{3645} - \frac{4x^7}{3645} + O(x^8)\right) \\
&\quad + c_2 \left(1 - \frac{2x}{3} + \frac{x^2}{3} - \frac{4x^3}{27} + \frac{5x^4}{81} - \frac{2x^5}{81} + \frac{7x^6}{729} - \frac{8x^7}{2187} + O(x^8)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^4 \left(1 - \frac{2x}{5} + \frac{7x^2}{45} - \frac{8x^3}{135} + \frac{x^4}{45} - \frac{2x^5}{243} + \frac{11x^6}{3645} - \frac{4x^7}{3645} + O(x^8)\right) \\
&\quad + c_2 \left(1 - \frac{2x}{3} + \frac{x^2}{3} - \frac{4x^3}{27} + \frac{5x^4}{81} - \frac{2x^5}{81} + \frac{7x^6}{729} - \frac{8x^7}{2187} + O(x^8)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^4 &\left(1 - \frac{2x}{5} + \frac{7x^2}{45} - \frac{8x^3}{135} + \frac{x^4}{45} - \frac{2x^5}{243} + \frac{11x^6}{3645} - \frac{4x^7}{3645} + O(x^8)\right) \\
&+ c_2 \left(1 - \frac{2x}{3} + \frac{x^2}{3} - \frac{4x^3}{27} + \frac{5x^4}{81} - \frac{2x^5}{81} + \frac{7x^6}{729} - \frac{8x^7}{2187} + O(x^8)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1 x^4 &\left(1 - \frac{2x}{5} + \frac{7x^2}{45} - \frac{8x^3}{135} + \frac{x^4}{45} - \frac{2x^5}{243} + \frac{11x^6}{3645} - \frac{4x^7}{3645} + O(x^8)\right) \\
&+ c_2 \left(1 - \frac{2x}{3} + \frac{x^2}{3} - \frac{4x^3}{27} + \frac{5x^4}{81} - \frac{2x^5}{81} + \frac{7x^6}{729} - \frac{8x^7}{2187} + O(x^8)\right)
\end{aligned}
$$

Verified OK.

6.13.1 Maple step by step solution

Let's solve

$x(x+3)\, y''-9y'-6y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{6y}{x(x+3)}+\frac{9y'}{x(x+3)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{9y'}{x(x+3)}-\frac{6y}{x(x+3)}=0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=-\frac{9}{x(x+3)}, P_3(x)=-\frac{6}{x(x+3)}\right]$

- ○ $(x+3)\cdot P_2(x)$ is analytic at $x=-3$

  $\left((x+3)\cdot P_2(x)\right)\bigg|_{x=-3}=3$

- ○ $(x+3)^2\cdot P_3(x)$ is analytic at $x=-3$

  $\left((x+3)^2\cdot P_3(x)\right)\bigg|_{x=-3}=0$

- ○ $x=-3$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-3$

- Multiply by denominators

  $x(x+3)\, y''-9y'-6y=0$

- Change variables using $x=u-3$ so that the regular singular point is at $u=0$

  $(u^2-3u)\left(\frac{d^2}{du^2}y(u)\right)-9\frac{d}{du}y(u)-6y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $\frac{d}{du}y(u)$ to series expansion

  $\frac{d}{du}y(u) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1}$

- Shift index using $k->k+1$

  $\frac{d}{du}y(u) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-3a_0r(2+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} (-3a_{k+1}(k+1+r)\,(k+3+r) + a_k(k+r+2)\,(k+r-3))\, u^{k+r}\right) =$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-3r(2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 0\}$

- Each term in the series must be 0, giving the recursion relation

  $-3a_{k+1}(k+1+r)\,(k+3+r) + a_k(k+r+2)\,(k+r-3) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r+2)(k+r-3)}{3(k+1+r)(k+3+r)}$

- Recursion relation for $r = -2$; series terminates at $k = 5$

  $a_{k+1} = \frac{a_k k(k-5)}{3(k-1)(k+1)}$

- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 1$

  $a_{k+1} = \frac{a_k k(k-5)}{3(k-1)(k+1)}$

- Recursion relation for $r = 0$; series terminates at $k = 3$

  $a_{k+1} = \frac{a_k(k+2)(k-3)}{3(k+1)(k+3)}$

- Apply recursion relation for $k = 0$

$a_1 = -\frac{2a_0}{3}$

- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{4}$

- Express in terms of a_0

  $a_2 = \frac{a_0}{6}$

- Apply recursion relation for $k = 2$

  $a_3 = -\frac{4a_2}{45}$

- Express in terms of a_0

  $a_3 = -\frac{2a_0}{135}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - \frac{2}{3}u + \frac{1}{6}u^2 - \frac{2}{135}u^3\right)$

- Revert the change of variables $u = x + 3$

  $\left[y = a_0\left(\frac{1}{10} - \frac{1}{15}x + \frac{1}{30}x^2 - \frac{2}{135}x^3\right)\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 52

```
Order:=8;
dsolve(x*(x+3)*diff(y(x),x$2)-9*diff(y(x),x)-6*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1x^4\left(1 - \frac{2}{5}x + \frac{7}{45}x^2 - \frac{8}{135}x^3 + \frac{1}{45}x^4 - \frac{2}{243}x^5 + \frac{11}{3645}x^6 - \frac{4}{3645}x^7 + \mathrm{O}\left(x^8\right)\right) + c_2\left(-144 + 96x - 48x^2 + \frac{64}{3}x^3 - \frac{80}{9}x^4 + \frac{32}{9}x^5 - \frac{112}{81}x^6 + \frac{128}{243}x^7 + \mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.419 (sec). Leaf size: 98

```
AsymptoticDSolveValue[x*(x+3)*y''[x]-9*y'[x]-6*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(\frac{7x^6}{729} - \frac{2x^5}{81} + \frac{5x^4}{81} - \frac{4x^3}{27} + \frac{x^2}{3} - \frac{2x}{3} + 1\right) + c_2\left(\frac{11x^{10}}{3645} - \frac{2x^9}{243} + \frac{x^8}{45} - \frac{8x^7}{135} + \frac{7x^6}{45} - \frac{2x^5}{5} + x^4\right)$$

6.14 problem 14

Internal problem ID [6978]
Internal file name [`OUTPUT/6221_Friday_August_12_2022_11_05_57_PM_33879010/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.8 Indicial Equation with Difference of Roots a Positive Integer: Nonlogarithmic Case. Exercises page 380
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x(1-2x)\,y''-2(x+2)\,y'+8y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-2x^2+x\right)y''+\left(-2x-4\right)y'+8y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=\frac{2x+4}{x\left(2x-1\right)}\\ q(x)&=-\frac{8}{x\left(2x-1\right)}\end{aligned}$$

Table 146: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2x+4}{x(2x-1)}$	
singularity	type
$x = 0$	"regular"
$x = \frac{1}{2}$	"regular"

$q(x) = -\frac{8}{x(2x-1)}$	
singularity	type
$x = 0$	"regular"
$x = \frac{1}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, \frac{1}{2}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''x(2x-1) + (-2x-4)\, y' + 8y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x(2x-1) + (-2x-4)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 8\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty}\left(-2x^{n+r}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}x^{n+r-1}a_n(n+r)(n+r-1)\right)$$
$$+\sum_{n=0}^{\infty}\left(-2x^{n+r}a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-4(n+r)a_nx^{n+r-1}\right)+\left(\sum_{n=0}^{\infty}8a_nx^{n+r}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-2x^{n+r}a_n(n+r)(n+r-1)\right)=\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)(n+r-2)x^{n+r-1}\right)$$
$$\sum_{n=0}^{\infty}\left(-2x^{n+r}a_n(n+r)\right)=\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)x^{n+r-1}\right)$$
$$\sum_{n=0}^{\infty}8a_nx^{n+r}=\sum_{n=1}^{\infty}8a_{n-1}x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)(n+r-2)x^{n+r-1}\right)$$
$$+\left(\sum_{n=0}^{\infty}x^{n+r-1}a_n(n+r)(n+r-1)\right)+\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)x^{n+r-1}\right)$$
$$+\sum_{n=0}^{\infty}\left(-4(n+r)a_nx^{n+r-1}\right)+\left(\sum_{n=1}^{\infty}8a_{n-1}x^{n+r-1}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1}a_n(n+r)(n+r-1)-4(n+r)a_nx^{n+r-1}=0$$

When $n=0$ the above becomes

$$x^{-1+r}a_0r(-1+r)-4ra_0x^{-1+r}=0$$

Or

$$\left(x^{-1+r} r(-1+r) - 4r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-5+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-5+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 5$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(-5+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 5$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^5\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+5}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} -2a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) \\ -2a_{n-1}(n+r-1) - 4a_n(n+r) + 8a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2a_{n-1}(n^2 + 2nr + r^2 - 2n - 2r - 3)}{n^2 + 2nr + r^2 - 5n - 5r} \tag{4}$$

Which for the root $r = 5$ becomes

$$a_n = \frac{2(n+6)(n+2)a_{n-1}}{n(n+5)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 5$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{2r^2 - 8}{r^2 - 3r - 4}$$

Which for the root $r = 5$ becomes

$$a_1 = 7$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-8}{r^2-3r-4}$	7

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4r^3 - 28r + 24}{r^3 - 6r^2 + 5r + 12}$$

Which for the root $r = 5$ becomes

$$a_2 = 32$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-8}{r^2-3r-4}$	7
a_2	$\frac{4r^3-28r+24}{r^3-6r^2+5r+12}$	32

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{8(r+4)\,r(-1+r)}{r^3-6r^2+5r+12}$$

Which for the root $r = 5$ becomes

$$a_3 = 120$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-8}{r^2-3r-4}$	7
a_2	$\frac{4r^3-28r+24}{r^3-6r^2+5r+12}$	32
a_3	$\frac{8(r+4)r(-1+r)}{r^3-6r^2+5r+12}$	120

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16r(r+5)}{r^2-7r+12}$$

Which for the root $r = 5$ becomes

$$a_4 = 400$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-8}{r^2-3r-4}$	7
a_2	$\frac{4r^3-28r+24}{r^3-6r^2+5r+12}$	32
a_3	$\frac{8(r+4)r(-1+r)}{r^3-6r^2+5r+12}$	120
a_4	$\frac{16r(r+5)}{r^2-7r+12}$	400

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{32r^2 + 256r + 384}{r^2 - 7r + 12}$$

Which for the root $r = 5$ becomes

$$a_5 = 1232$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-8}{r^2-3r-4}$	7
a_2	$\frac{4r^3-28r+24}{r^3-6r^2+5r+12}$	32
a_3	$\frac{8(r+4)r(-1+r)}{r^3-6r^2+5r+12}$	120
a_4	$\frac{16r(r+5)}{r^2-7r+12}$	400
a_5	$\frac{32r^2+256r+384}{r^2-7r+12}$	1232

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{64r^3 + 768r^2 + 2624r + 2688}{(r-4)(r+1)(r-3)}$$

Which for the root $r = 5$ becomes

$$a_6 = 3584$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-8}{r^2-3r-4}$	7
a_2	$\frac{4r^3-28r+24}{r^3-6r^2+5r+12}$	32
a_3	$\frac{8(r+4)r(-1+r)}{r^3-6r^2+5r+12}$	120
a_4	$\frac{16r(r+5)}{r^2-7r+12}$	400
a_5	$\frac{32r^2+256r+384}{r^2-7r+12}$	1232
a_6	$\frac{64r^3+768r^2+2624r+2688}{(r-4)(r+1)(r-3)}$	3584

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{128r^3 + 1920r^2 + 8704r + 12288}{(r-4)(r+1)(r-3)}$$

Which for the root $r = 5$ becomes

$$a_7 = 9984$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-8}{r^2-3r-4}$	7
a_2	$\frac{4r^3-28r+24}{r^3-6r^2+5r+12}$	32
a_3	$\frac{8(r+4)r(-1+r)}{r^3-6r^2+5r+12}$	120
a_4	$\frac{16r(r+5)}{r^2-7r+12}$	400
a_5	$\frac{32r^2+256r+384}{r^2-7r+12}$	1232
a_6	$\frac{64r^3+768r^2+2624r+2688}{(r-4)(r+1)(r-3)}$	3584
a_7	$\frac{128r^3+1920r^2+8704r+12288}{(r-4)(r+1)(r-3)}$	9984

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^5\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \dots\right) \\ &= x^5\left(1 + 7x + 32x^2 + 120x^3 + 400x^4 + 1232x^5 + 3584x^6 + 9984x^7 + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 5$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_5(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_5 \\ &= \frac{32r^2 + 256r + 384}{r^2 - 7r + 12} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{32r^2 + 256r + 384}{r^2 - 7r + 12} &= \lim_{r \to 0} \frac{32r^2 + 256r + 384}{r^2 - 7r + 12} \\ &= 32 \end{aligned}$$

The limit is 32. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^n \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &-2b_{n-1}(n + r - 1)(n + r - 2) + b_n(n + r)(n + r - 1) \\ &- 2b_{n-1}(n + r - 1) - 4(n + r) b_n + 8b_{n-1} = 0 \end{aligned} \tag{4}$$

Which for for the root $r = 0$ becomes

$$-2b_{n-1}(n - 1)(n - 2) + b_n n(n - 1) - 2b_{n-1}(n - 1) - 4nb_n + 8b_{n-1} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = \frac{2b_{n-1}(n^2 + 2nr + r^2 - 2n - 2r - 3)}{n^2 + 2nr + r^2 - 5n - 5r} \tag{5}$$

Which for the root $r = 0$ becomes

$$b_n = \frac{2b_{n-1}(n^2 - 2n - 3)}{n^2 - 5n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{2r^2 - 8}{r^2 - 3r - 4}$$

Which for the root $r = 0$ becomes

$$b_1 = 2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2r^2-8}{r^2-3r-4}$	2

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4r^3 - 28r + 24}{(r - 3)(r^2 - 3r - 4)}$$

Which for the root $r = 0$ becomes

$$b_2 = 2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2r^2-8}{r^2-3r-4}$	2
b_2	$\frac{4r^3-28r+24}{r^3-6r^2+5r+12}$	2

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{8(r+4)\, r(-1+r)}{(r-3)\,(r^2-3r-4)}$$

Which for the root $r = 0$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2r^2-8}{r^2-3r-4}$	2
b_2	$\frac{4r^3-28r+24}{r^3-6r^2+5r+12}$	2
b_3	$\frac{8(r+4)r(-1+r)}{r^3-6r^2+5r+12}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16r(r+5)}{(r-4)\,(r-3)}$$

Which for the root $r = 0$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2r^2-8}{r^2-3r-4}$	2
b_2	$\frac{4r^3-28r+24}{r^3-6r^2+5r+12}$	2
b_3	$\frac{8(r+4)r(-1+r)}{r^3-6r^2+5r+12}$	0
b_4	$\frac{16r(r+5)}{(r-4)(r-3)}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{32r^2+256r+384}{(r-4)\,(r-3)}$$

Which for the root $r = 0$ becomes

$$b_5 = 32$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2r^2-8}{r^2-3r-4}$	2
b_2	$\frac{4r^3-28r+24}{r^3-6r^2+5r+12}$	2
b_3	$\frac{8(r+4)r(-1+r)}{r^3-6r^2+5r+12}$	0
b_4	$\frac{16r(r+5)}{(r-4)(r-3)}$	0
b_5	$\frac{32r^2+256r+384}{(r-4)(r-3)}$	32

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{64r^3 + 768r^2 + 2624r + 2688}{(r-4)(r+1)(r-3)}$$

Which for the root $r = 0$ becomes

$$b_6 = 224$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2r^2-8}{r^2-3r-4}$	2
b_2	$\frac{4r^3-28r+24}{r^3-6r^2+5r+12}$	2
b_3	$\frac{8(r+4)r(-1+r)}{r^3-6r^2+5r+12}$	0
b_4	$\frac{16r(r+5)}{(r-4)(r-3)}$	0
b_5	$\frac{32r^2+256r+384}{(r-4)(r-3)}$	32
b_6	$\frac{64r^3+768r^2+2624r+2688}{(r-4)(r+1)(r-3)}$	224

For $n = 7$, using the above recursive equation gives

$$b_7 = \frac{128r^3 + 1920r^2 + 8704r + 12288}{(r-4)(r+1)(r-3)}$$

Which for the root $r = 0$ becomes

$$b_7 = 1024$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2r^2-8}{r^2-3r-4}$	2
b_2	$\frac{4r^3-28r+24}{r^3-6r^2+5r+12}$	2
b_3	$\frac{8(r+4)r(-1+r)}{r^3-6r^2+5r+12}$	0
b_4	$\frac{16r(r+5)}{(r-4)(r-3)}$	0
b_5	$\frac{32r^2+256r+384}{(r-4)(r-3)}$	32
b_6	$\frac{64r^3+768r^2+2624r+2688}{(r-4)(r+1)(r-3)}$	224
b_7	$\frac{128r^3+1920r^2+8704r+12288}{(r-4)(r+1)(r-3)}$	1024

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\
&= 1 + 2x + 2x^2 + 32x^5 + 224x^6 + 1024x^7 + O(x^8)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^5 \left(1 + 7x + 32x^2 + 120x^3 + 400x^4 + 1232x^5 + 3584x^6 + 9984x^7 + O(x^8)\right) \\
&\quad + c_2 \left(1 + 2x + 2x^2 + 32x^5 + 224x^6 + 1024x^7 + O(x^8)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^5 \left(1 + 7x + 32x^2 + 120x^3 + 400x^4 + 1232x^5 + 3584x^6 + 9984x^7 + O(x^8)\right) \\
&\quad + c_2 \left(1 + 2x + 2x^2 + 32x^5 + 224x^6 + 1024x^7 + O(x^8)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^5 \left(1 + 7x + 32x^2 + 120x^3 + 400x^4 + 1232x^5 + 3584x^6 + 9984x^7 + O(x^8)\right) \\
+ c_2 \left(1 + 2x + 2x^2 + 32x^5 + 224x^6 + 1024x^7 + O(x^8)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = c_1 x^5 \left(1 + 7x + 32x^2 + 120x^3 + 400x^4 + 1232x^5 + 3584x^6 + 9984x^7 + O(x^8)\right) + c_2 \left(1 + 2x + 2x^2 + 32x^5 + 224x^6 + 1024x^7 + O(x^8)\right)$$

Verified OK.

6.14.1 Maple step by step solution

Let's solve

$-y''x(2x-1) + (-2x-4)y' + 8y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{8y}{x(2x-1)} - \frac{2(x+2)y'}{x(2x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2(x+2)y'}{x(2x-1)} - \frac{8y}{x(2x-1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2(x+2)}{x(2x-1)}, P_3(x) = -\frac{8}{x(2x-1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -4$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x(2x-1) + (2x+4)y' - 8y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

Rewrite ODE with series expansions

$-a_0 r(-5+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(k-4+r) + 2a_k(k+r+2)\,(k+r-2)\right) x^{k+r}\right) =$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-5+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 5\}$

- Each term in the series must be 0, giving the recursion relation

  $-a_{k+1}(k+1+r)\,(k-4+r) + 2a_k(k+r+2)\,(k+r-2) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k(k+r+2)(k+r-2)}{(k+1+r)(k-4+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{2a_k(k+2)(k-2)}{(k+1)(k-4)}$

- Apply recursion relation for $k = 0$

  $a_1 = 2a_0$

- Apply recursion relation for $k = 1$

  $a_2 = a_1$

- Express in terms of a_0

  $a_2 = 2a_0$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y = a_0 \cdot (2x^2 + 2x + 1)$

- Recursion relation for $r = 5$

  $a_{k+1} = \frac{2a_k(k+7)(k+3)}{(k+6)(k+1)}$

- Solution for $r = 5$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+5}, a_{k+1} = \frac{2a_k(k+7)(k+3)}{(k+6)(k+1)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0 \cdot (2x^2 + 2x + 1) + \left(\sum_{k=0}^{\infty} b_k x^{k+5}\right), b_{k+1} = \frac{2b_k(k+7)(k+3)}{(k+6)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 48

```
Order:=8;
dsolve(x*(1-2*x)*diff(y(x),x$2)-2*(2+x)*diff(y(x),x)+8*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^5 \left(1 + 7x + 32x^2 + 120x^3 + 400x^4 + 1232x^5 + 3584x^6 + 9984x^7 + O\left(x^8\right)\right) \\ + c_2 \left(2880 + 5760x + 5760x^2 + 92160x^5 + 645120x^6 + 2949120x^7 + O\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.123 (sec). Leaf size: 64

```
AsymptoticDSolveValue[x*(1-2*x)*y''[x]-2*(2+x)*y'[x]+8*y[x]==0,y[x],{x,0,7}]
```

$$
\begin{aligned}
y(x) \to\, & c_1\left(224x^6 + 32x^5 + 2x^2 + 2x + 1\right) \\
& + c_2\left(3584x^{11} + 1232x^{10} + 400x^9 + 120x^8 + 32x^7 + 7x^6 + x^5\right)
\end{aligned}
$$

6.15 problem 15

Internal problem ID [6979]
Internal file name [`OUTPUT/6222_Friday_August_12_2022_11_05_59_PM_73571264/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.8 Indicial Equation with Difference of Roots a Positive Integer: Nonlogarithmic Case. Exercises page 380
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + (x^3 - 1) y' + x^2 y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + (x^3 - 1) y' + x^2 y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x^3 - 1}{x}$$
$$q(x) = x$$

Table 148: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x^3-1}{x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = x$	
singularity	type
$x = \infty$	"regular"
$x = -\infty$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + (x^3 - 1) y' + x^2 y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x\left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) + (x^3 - 1) \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + x^2 \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r) = \sum_{n=3}^{\infty} a_{n-3}(n+r-3)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=3}^{\infty} a_{n-3} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=3}^{\infty} a_{n-3}(n+r-3)\, x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=3}^{\infty} a_{n-3} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) - (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) - r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) - r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-2+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-2+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(-2+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^2\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = 0$$

For $3 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_{n-3}(n+r-3) - a_n(n+r) + a_{n-3} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-3}}{n+r} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = -\frac{a_{n-3}}{n+2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{3+r}$$

Which for the root $r = 2$ becomes

$$a_3 = -\frac{1}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	$-\frac{1}{3+r}$	$-\frac{1}{5}$

For $n = 4$, using the above recursive equation gives

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	$-\frac{1}{3+r}$	$-\frac{1}{5}$
a_4	0	0

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	$-\frac{1}{3+r}$	$-\frac{1}{5}$
a_4	0	0
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{1}{(3+r)(6+r)}$$

Which for the root $r = 2$ becomes

$$a_6 = \frac{1}{40}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	$-\frac{1}{3+r}$	$-\frac{1}{5}$
a_4	0	0
a_5	0	0
a_6	$\frac{1}{(3+r)(6+r)}$	$\frac{1}{40}$

For $n = 7$, using the above recursive equation gives

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	$-\frac{1}{3+r}$	$-\frac{1}{5}$
a_4	0	0
a_5	0	0
a_6	$\frac{1}{(3+r)(6+r)}$	$\frac{1}{40}$
a_7	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^2\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \dots\right) \\
&= x^2\left(1 - \frac{x^3}{5} + \frac{x^6}{40} + O(x^8)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if

C is zero or not. This is done by finding $\lim_{r \to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_2 \\ &= 0 \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} 0 &= \lim_{r \to 0} 0 \\ &= 0 \end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^n \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

Substituting $n = 2$ in Eq(3) gives

$$b_2 = 0$$

For $3 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + b_{n-3}(n+r-3) - (n+r)b_n + b_{n-3} = 0 \tag{4}$$

Which for for the root $r = 0$ becomes

$$b_n n(n-1) + b_{n-3}(n-3) - nb_n + b_{n-3} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{b_{n-3}}{n+r} \tag{5}$$

Which for the root $r = 0$ becomes

$$b_n = -\frac{b_{n-3}}{n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{1}{3+r}$$

Which for the root $r = 0$ becomes

$$b_3 = -\frac{1}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	$-\frac{1}{3+r}$	$-\frac{1}{3}$

For $n = 4$, using the above recursive equation gives

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	$-\frac{1}{3+r}$	$-\frac{1}{3}$
b_4	0	0

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	$-\frac{1}{3+r}$	$-\frac{1}{3}$
b_4	0	0
b_5	0	0

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{1}{(3+r)(6+r)}$$

Which for the root $r = 0$ becomes

$$b_6 = \frac{1}{18}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	$-\frac{1}{3+r}$	$-\frac{1}{3}$
b_4	0	0
b_5	0	0
b_6	$\frac{1}{(3+r)(6+r)}$	$\frac{1}{18}$

For $n = 7$, using the above recursive equation gives

$$b_7 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	$-\frac{1}{3+r}$	$-\frac{1}{3}$
b_4	0	0
b_5	0	0
b_6	$\frac{1}{(3+r)(6+r)}$	$\frac{1}{18}$
b_7	0	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\
&= 1 - \frac{x^3}{3} + \frac{x^6}{18} + O(x^8)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^2 \left(1 - \frac{x^3}{5} + \frac{x^6}{40} + O(x^8)\right) + c_2 \left(1 - \frac{x^3}{3} + \frac{x^6}{18} + O(x^8)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^2 \left(1 - \frac{x^3}{5} + \frac{x^6}{40} + O(x^8)\right) + c_2 \left(1 - \frac{x^3}{3} + \frac{x^6}{18} + O(x^8)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^2 \left(1 - \frac{x^3}{5} + \frac{x^6}{40} + O(x^8)\right) + c_2 \left(1 - \frac{x^3}{3} + \frac{x^6}{18} + O(x^8)\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x^2 \left(1 - \frac{x^3}{5} + \frac{x^6}{40} + O(x^8)\right) + c_2 \left(1 - \frac{x^3}{3} + \frac{x^6}{18} + O(x^8)\right)
$$

Verified OK.

6.15.1 Maple step by step solution

Let's solve

$xy'' + (x^3 - 1)\, y' + x^2 y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^3-1)y'}{x} - yx$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x^3-1)y'}{x} + yx = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{x^3-1}{x}, P_3(x) = x\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + (x^3 - 1)\, y' + x^2 y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^2 \cdot y$ to series expansion

    $x^2 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+2}$

- Shift index using $k->k-2$

  $x^2 \cdot y = \sum_{k=2}^{\infty} a_{k-2} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$

- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-2+r)\, x^{-1+r} + a_1(1+r)\,(-1+r)\, x^r + a_2(2+r)\, r\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_{k+1}(k+1+r)\,(k+r-1)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- The coefficients of each power of x must be 0

  $[a_1(1+r)\,(-1+r) = 0, a_2(2+r)\, r = 0]$

- Solve for the dependent coefficient(s)

  $\{a_1 = 0, a_2 = 0\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\,(a_{k+1}(k+1+r) + a_{k-2}) = 0$

- Shift index using $k->k+2$

  $(k+1+r)\,(a_{k+3}(k+3+r) + a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+3} = -\frac{a_k}{k+3+r}$

- Recursion relation for $r = 0$

  $a_{k+3} = -\frac{a_k}{k+3}$
- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = -\frac{a_k}{k+3}, a_1 = 0, a_2 = 0\right]$$
- Recursion relation for $r = 2$

  $a_{k+3} = -\frac{a_k}{k+5}$
- Solution for $r = 2$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+3} = -\frac{a_k}{k+5}, a_1 = 0, a_2 = 0\right]$$
- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+3} = -\frac{a_k}{k+3}, a_1 = 0, a_2 = 0, b_{k+3} = -\frac{b_k}{k+5}, b_1 = 0, b_2 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 32

```
Order:=8;
dsolve(x*diff(y(x),x$2)+(x^3-1)*diff(y(x),x)+x^2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^2 \left(1 - \frac{1}{5}x^3 + \frac{1}{40}x^6 + \mathrm{O}\left(x^8\right)\right) + c_2\left(-2 + \frac{2}{3}x^3 - \frac{1}{9}x^6 + \mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 44

```
AsymptoticDSolveValue[x*y''[x]+(x^3-1)*y'[x]+x^2*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(\frac{x^6}{18} - \frac{x^3}{3} + 1\right) + c_2\left(\frac{x^8}{40} - \frac{x^5}{5} + x^2\right)$$

6.16 problem 16

Internal problem ID [6980]
Internal file name [`OUTPUT/6223_Friday_August_12_2022_11_06_01_PM_93983477/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.8 Indicial Equation with Difference of Roots a Positive Integer: Nonlogarithmic Case. Exercises page 380
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(4x-1)\,y''+x(1+5x)\,y'+3y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(4x^3-x^2\right)y''+\left(5x^2+x\right)y'+3y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{1+5x}{x\,(4x-1)}$$
$$q(x)=\frac{3}{x^2\,(4x-1)}$$

Table 150: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1+5x}{x(4x-1)}$	
singularity	type
$x = 0$	"regular"
$x = \frac{1}{4}$	"regular"

$q(x) = \frac{3}{x^2(4x-1)}$	
singularity	type
$x = 0$	"regular"
$x = \frac{1}{4}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, \frac{1}{4}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(4x-1)\, y'' + \left(5x^2 + x\right) y' + 3y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(4x-1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(5x^2+x\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 3\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 5x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 4x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} 4a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 5x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 5a_{n-1}(n+r-1)\, x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} 4a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 5a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$-x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) + 3a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$-x^r a_0 r(-1+r) + x^r a_0 r + 3a_0 x^r = 0$$

Or

$$\left(-x^r r(-1+r) + x^r r + 3x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(-r^2+2r+3\right)x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$-r^2+2r+3 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 3$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(-r^2+2r+3\right)x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 4$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^3\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x)\ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+3}$$
$$y_2(x) = C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots

of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
\begin{aligned}
&4a_{n-1}(n+r-1)(n+r-2) - a_n(n+r)(n+r-1) \\
&+ 5a_{n-1}(n+r-1) + a_n(n+r) + 3a_n = 0
\end{aligned}
\tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = \frac{a_{n-1}(4n^2 + 8nr + 4r^2 - 7n - 7r + 3)}{n^2 + 2nr + r^2 - 2n - 2r - 3} \tag{4}
$$

Which for the root $r = 3$ becomes

$$
a_n = \frac{a_{n-1}(4n^2 + 17n + 18)}{n(n+4)} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 3$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$
a_1 = \frac{4r^2 + r}{r^2 - 4}
$$

Which for the root $r = 3$ becomes

$$
a_1 = \frac{39}{5}
$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{4r^2+r}{r^2-4}$	$\frac{39}{5}$

For $n = 2$, using the above recursive equation gives

$$
a_2 = \frac{r(4r+1)(4r^2+9r+5)}{(r^2-4)(r^2+2r-3)}
$$

Which for the root $r = 3$ becomes

$$
a_2 = \frac{221}{5}
$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{4r^2+r}{r^2-4}$	$\frac{39}{5}$
a_2	$\frac{r(4r+1)(4r^2+9r+5)}{(r^2-4)(r^2+2r-3)}$	$\frac{221}{5}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{64r^4 + 304r^3 + 476r^2 + 281r + 45}{(r+4)(r+3)(-1+r)(r-2)}$$

Which for the root $r = 3$ becomes

$$a_3 = 221$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{4r^2+r}{r^2-4}$	$\frac{39}{5}$
a_2	$\frac{r(4r+1)(4r^2+9r+5)}{(r^2-4)(r^2+2r-3)}$	$\frac{221}{5}$
a_3	$\frac{64r^4+304r^3+476r^2+281r+45}{(r+4)(r+3)(-1+r)(r-2)}$	221

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{256r^4 + 1792r^3 + 4064r^2 + 3248r + 585}{(r+5)(r-2)(-1+r)(r+4)}$$

Which for the root $r = 3$ becomes

$$a_4 = \frac{16575}{16}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{4r^2+r}{r^2-4}$	$\frac{39}{5}$
a_2	$\frac{r(4r+1)(4r^2+9r+5)}{(r^2-4)(r^2+2r-3)}$	$\frac{221}{5}$
a_3	$\frac{64r^4+304r^3+476r^2+281r+45}{(r+4)(r+3)(-1+r)(r-2)}$	221
a_4	$\frac{256r^4+1792r^3+4064r^2+3248r+585}{(r+5)(r-2)(-1+r)(r+4)}$	$\frac{16575}{16}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1024r^5 + 11520r^4 + 46720r^3 + 82080r^2 + 57556r + 9945}{(r+6)(r+2)(-1+r)(r-2)(r+5)}$$

Which for the root $r = 3$ becomes

$$a_5 = \frac{224315}{48}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{4r^2+r}{r^2-4}$	$\frac{39}{5}$
a_2	$\frac{r(4r+1)(4r^2+9r+5)}{(r^2-4)(r^2+2r-3)}$	$\frac{221}{5}$
a_3	$\frac{64r^4+304r^3+476r^2+281r+45}{(r+4)(r+3)(-1+r)(r-2)}$	221
a_4	$\frac{256r^4+1792r^3+4064r^2+3248r+585}{(r+5)(r-2)(-1+r)(r+4)}$	$\frac{16575}{16}$
a_5	$\frac{1024r^5+11520r^4+46720r^3+82080r^2+57556r+9945}{(r+6)(r+2)(-1+r)(r-2)(r+5)}$	$\frac{224315}{48}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{4096r^6 + 67584r^5 + 428800r^4 + 1309440r^3 + 1953904r^2 + 1248456r + 208845}{(-1+r)(r+6)(r^2+10r+21)(r^2-4)}$$

Which for the root $r = 3$ becomes

$$a_6 = \frac{493493}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{4r^2+r}{r^2-4}$	$\frac{39}{5}$
a_2	$\frac{r(4r+1)(4r^2+9r+5)}{(r^2-4)(r^2+2r-3)}$	$\frac{221}{5}$
a_3	$\frac{64r^4+304r^3+476r^2+281r+45}{(r+4)(r+3)(-1+r)(r-2)}$	221
a_4	$\frac{256r^4+1792r^3+4064r^2+3248r+585}{(r+5)(r-2)(-1+r)(r+4)}$	$\frac{16575}{16}$
a_5	$\frac{1024r^5+11520r^4+46720r^3+82080r^2+57556r+9945}{(r+6)(r+2)(-1+r)(r-2)(r+5)}$	$\frac{224315}{48}$
a_6	$\frac{4096r^6+67584r^5+428800r^4+1309440r^3+1953904r^2+1248456r+208845}{(-1+r)(r+6)(r^2+10r+21)(r^2-4)}$	$\frac{493493}{24}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{16384r^7 + 372736r^6 + 3404800r^5 + 15957760r^4 + 40551616r^3 + 53841424r^2 + 32046780r + 5221125}{(r^2-4)(-1+r)(r^2+10r+21)(r^2+12r+32)}$$

Which for the root $r = 3$ becomes

$$a_7 = \frac{711399}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{4r^2+r}{r^2-4}$	$\frac{39}{5}$
a_2	$\frac{r(4r+1)(4r^2+9r+5)}{(r^2-4)(r^2+2r-3)}$	$\frac{221}{5}$
a_3	$\frac{64r^4+304r^3+476r^2+281r+45}{(r+4)(r+3)(-1+r)(r-2)}$	221
a_4	$\frac{256r^4+1792r^3+4064r^2+3248r+585}{(r+5)(r-2)(-1+r)(r+4)}$	$\frac{16575}{16}$
a_5	$\frac{1024r^5+11520r^4+46720r^3+82080r^2+57556r+9945}{(r+6)(r+2)(-1+r)(r-2)(r+5)}$	$\frac{224315}{48}$
a_6	$\frac{4096r^6+67584r^5+428800r^4+1309440r^3+1953904r^2+1248456r+208845}{(-1+r)(r+6)(r^2+10r+21)(r^2-4)}$	$\frac{493493}{24}$
a_7	$\frac{16384r^7+372736r^6+3404800r^5+15957760r^4+40551616r^3+53841424r^2+32046780r+5221125}{(r^2-4)(-1+r)(r^2+10r+21)(r^2+12r+32)}$	$\frac{711399}{8}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^3(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \dots) \\ &= x^3\left(1 + \frac{39x}{5} + \frac{221x^2}{5} + 221x^3 + \frac{16575x^4}{16} + \frac{224315x^5}{48} + \frac{493493x^6}{24} + \frac{711399x^7}{8} + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_4 \\ &= \frac{256r^4 + 1792r^3 + 4064r^2 + 3248r + 585}{(r+5)(r-2)(-1+r)(r+4)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{256r^4 + 1792r^3 + 4064r^2 + 3248r + 585}{(r+5)(r-2)(-1+r)(r+4)} &= \lim_{r\to -1} \frac{256r^4 + 1792r^3 + 4064r^2 + 3248r + 585}{(r+5)(r-2)(-1+r)(r+4)} \\ &= -\frac{15}{8} \end{aligned}$$

The limit is $-\frac{15}{8}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-1} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &4b_{n-1}(n+r-1)(n+r-2) - b_n(n+r)(n+r-1) \\ &+ 5b_{n-1}(n+r-1) + b_n(n+r) + 3b_n = 0 \end{aligned} \tag{4}$$

Which for for the root $r = -1$ becomes

$$4b_{n-1}(n-2)(n-3) - b_n(n-1)(n-2) + 5b_{n-1}(n-2) + b_n(n-1) + 3b_n = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = \frac{b_{n-1}(4n^2 + 8nr + 4r^2 - 7n - 7r + 3)}{n^2 + 2nr + r^2 - 2n - 2r - 3} \tag{5}$$

Which for the root $r = -1$ becomes

$$b_n = \frac{b_{n-1}(4n^2 - 15n + 14)}{n^2 - 4n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{r(4r+1)}{r^2 - 4}$$

Which for the root $r = -1$ becomes

$$b_1 = -1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{4r^2+r}{r^2-4}$	-1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{r(4r+1)\,(4r^2+9r+5)}{(r^2-4)\,(r^2+2r-3)}$$

Which for the root $r = -1$ becomes

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{4r^2+r}{r^2-4}$	-1
b_2	$\frac{r(4r+1)(4r^2+9r+5)}{(r^2-4)(r^2+2r-3)}$	0

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{(4r+9)(4r+1)(4r^2+9r+5)}{(r+4)(r^2+2r-3)(r-2)}$$

Which for the root $r = -1$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{4r^2+r}{r^2-4}$	-1
b_2	$\frac{r(4r+1)(4r^2+9r+5)}{(r^2-4)(r^2+2r-3)}$	0
b_3	$\frac{64r^4+304r^3+476r^2+281r+45}{(r+4)(r+3)(-1+r)(r-2)}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{(4r+5)(4r+1)(4r+9)(4r+13)}{(r+5)(r-2)(-1+r)(r+4)}$$

Which for the root $r = -1$ becomes

$$b_4 = -\frac{15}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{4r^2+r}{r^2-4}$	-1
b_2	$\frac{r(4r+1)(4r^2+9r+5)}{(r^2-4)(r^2+2r-3)}$	0
b_3	$\frac{64r^4+304r^3+476r^2+281r+45}{(r+4)(r+3)(-1+r)(r-2)}$	0
b_4	$\frac{(4r+5)(4r+1)(4r+9)(4r+13)}{(r+5)(r-2)(-1+r)(r+4)}$	$-\frac{15}{8}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{(4r+17)(4r+5)(4r+1)(4r+9)(4r+13)}{(-1+r)(r-2)(r+5)(r^2+8r+12)}$$

Which for the root $r = -1$ becomes

$$b_5 = -\frac{117}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{4r^2+r}{r^2-4}$	-1
b_2	$\frac{r(4r+1)(4r^2+9r+5)}{(r^2-4)(r^2+2r-3)}$	0
b_3	$\frac{64r^4+304r^3+476r^2+281r+45}{(r+4)(r+3)(-1+r)(r-2)}$	0
b_4	$\frac{(4r+5)(4r+1)(4r+9)(4r+13)}{(r+5)(r-2)(-1+r)(r+4)}$	$-\frac{15}{8}$
b_5	$\frac{(4r+17)(4r+5)(4r+1)(4r+9)(4r+13)}{(-1+r)(r-2)(r+5)(r^2+8r+12)}$	$-\frac{117}{8}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{(4r+21)(4r+17)(4r+5)(4r+1)(4r+9)(4r+13)}{(r-2)(-1+r)(r^2+8r+12)(r^2+10r+21)}$$

Which for the root $r = -1$ becomes

$$b_6 = -\frac{663}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{4r^2+r}{r^2-4}$	-1
b_2	$\frac{r(4r+1)(4r^2+9r+5)}{(r^2-4)(r^2+2r-3)}$	0
b_3	$\frac{64r^4+304r^3+476r^2+281r+45}{(r+4)(r+3)(-1+r)(r-2)}$	0
b_4	$\frac{(4r+5)(4r+1)(4r+9)(4r+13)}{(r+5)(r-2)(-1+r)(r+4)}$	$-\frac{15}{8}$
b_5	$\frac{(4r+17)(4r+5)(4r+1)(4r+9)(4r+13)}{(-1+r)(r-2)(r+5)(r^2+8r+12)}$	$-\frac{117}{8}$
b_6	$\frac{(4r+21)(4r+17)(4r+5)(4r+1)(4r+9)(4r+13)}{(r-2)(-1+r)(r^2+8r+12)(r^2+10r+21)}$	$-\frac{663}{8}$

For $n = 7$, using the above recursive equation gives

$$b_7 = \frac{(4r+25)(4r+21)(4r+17)(4r+5)(4r+1)(4r+9)(4r+13)}{(r+2)(-1+r)(r-2)(r^2+10r+21)(r^2+12r+32)}$$

Which for the root $r = -1$ becomes

$$b_7 = -\frac{3315}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{4r^2+r}{r^2-4}$	-1
b_2	$\frac{r(4r+1)(4r^2+9r+5)}{(r^2-4)(r^2+2r-3)}$	0
b_3	$\frac{64r^4+304r^3+476r^2+281r+45}{(r+4)(r+3)(-1+r)(r-2)}$	0
b_4	$\frac{(4r+5)(4r+1)(4r+9)(4r+13)}{(r+5)(r-2)(-1+r)(r+4)}$	$-\frac{15}{8}$
b_5	$\frac{(4r+17)(4r+5)(4r+1)(4r+9)(4r+13)}{(-1+r)(r-2)(r+5)(r^2+8r+12)}$	$-\frac{117}{8}$
b_6	$\frac{(4r+21)(4r+17)(4r+5)(4r+1)(4r+9)(4r+13)}{(r-2)(-1+r)(r^2+8r+12)(r^2+10r+21)}$	$-\frac{663}{8}$
b_7	$\frac{(4r+25)(4r+21)(4r+17)(4r+5)(4r+1)(4r+9)(4r+13)}{(-1+r)(r^2+10r+21)(r^2+12r+32)(r^2-4)}$	$-\frac{3315}{8}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^3(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 + b_7x^7 + b_8x^8 \dots) \\ &= \frac{1 - x - \frac{15x^4}{8} - \frac{117x^5}{8} - \frac{663x^6}{8} - \frac{3315x^7}{8} + O(x^8)}{x} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1x^3\left(1 + \frac{39x}{5} + \frac{221x^2}{5} + 221x^3 + \frac{16575x^4}{16} + \frac{224315x^5}{48} + \frac{493493x^6}{24} + \frac{711399x^7}{8}\right. \\ &\quad \left. + O(x^8)\right) + \frac{c_2\left(1 - x - \frac{15x^4}{8} - \frac{117x^5}{8} - \frac{663x^6}{8} - \frac{3315x^7}{8} + O(x^8)\right)}{x} \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^3 \left(1+\frac{39x}{5}+\frac{221x^2}{5}+221x^3+\frac{16575x^4}{16}+\frac{224315x^5}{48}+\frac{493493x^6}{24}+\frac{711399x^7}{8}\right. \\
&\quad \left.+O\left(x^8\right)\right)+\frac{c_2\left(1-x-\frac{15x^4}{8}-\frac{117x^5}{8}-\frac{663x^6}{8}-\frac{3315x^7}{8}+O\left(x^8\right)\right)}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^3 &\left(1+\frac{39x}{5}+\frac{221x^2}{5}+221x^3+\frac{16575x^4}{16}+\frac{224315x^5}{48}+\frac{493493x^6}{24}+\frac{711399x^7}{8}\right. \\
&\left.+O\left(x^8\right)\right)+\frac{c_2\left(1-x-\frac{15x^4}{8}-\frac{117x^5}{8}-\frac{663x^6}{8}-\frac{3315x^7}{8}+O\left(x^8\right)\right)}{x}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1 x^3 &\left(1+\frac{39x}{5}+\frac{221x^2}{5}+221x^3+\frac{16575x^4}{16}+\frac{224315x^5}{48}+\frac{493493x^6}{24}+\frac{711399x^7}{8}\right. \\
&\left.+O\left(x^8\right)\right)+\frac{c_2\left(1-x-\frac{15x^4}{8}-\frac{117x^5}{8}-\frac{663x^6}{8}-\frac{3315x^7}{8}+O\left(x^8\right)\right)}{x}
\end{aligned}
$$

Verified OK.

6.16.1 Maple step by step solution

Let's solve

$x^2(4x-1)\, y''+\left(5x^2+x\right) y'+3y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{3y}{x^2(4x-1)}-\frac{(1+5x)y'}{x(4x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(1+5x)y'}{x(4x-1)}+\frac{3y}{x^2(4x-1)}=0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{1+5x}{x(4x-1)}, P_3(x) = \frac{3}{x^2(4x-1)}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -3$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(4x-1)\, y'' + x(1+5x)\, y' + 3y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0(1+r)\,(-3+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r+1)\,(k+r-3) + a_{k-1}(k+r-1)\,(4k-3+4r)\right) x^{k+}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(1+r)(-3+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 3\}$
- Each term in the series must be 0, giving the recursion relation

  $4\left(k-\frac{3}{4}+r\right)(k+r-1) a_{k-1} - a_k(k+r+1)(k+r-3) = 0$
- Shift index using $k->k+1$

  $4\left(k+\frac{1}{4}+r\right)(k+r) a_k - a_{k+1}(k+2+r)(k-2+r) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{(4k+4r+1)(k+r)a_k}{(k+2+r)(k-2+r)}$
- Recursion relation for $r=-1$; series terminates at $k=1$

  $a_{k+1} = \frac{(4k-3)(k-1)a_k}{(k+1)(k-3)}$
- Apply recursion relation for $k=0$

  $a_1 = -a_0$
- Terminating series solution of the ODE for $r=-1$. Use reduction of order to find the second

  $y = a_0 \cdot (1-x)$
- Recursion relation for $r=3$

  $a_{k+1} = \frac{(4k+13)(k+3)a_k}{(k+5)(k+1)}$
- Solution for $r=3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = \frac{(4k+13)(k+3)a_k}{(k+5)(k+1)}\right]$
- Combine solutions and rename parameters

  $\left[y = a_0 \cdot (1-x) + \left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), b_{k+1} = \frac{(4k+13)(k+3)b_k}{(k+5)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 51

```
Order:=8;
dsolve(x^2*(4*x-1)*diff(y(x),x$2)+x*(5*x+1)*diff(y(x),x)+3*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^3 \left(1 + \frac{39}{5}x + \frac{221}{5}x^2 + 221x^3 + \frac{16575}{16}x^4 + \frac{224315}{48}x^5 + \frac{493493}{24}x^6 + \frac{711399}{8}x^7 + \mathrm{O}\left(x^8\right)\right) + \frac{c_2\left(-144 + 144x + 270x^4 + 2106x^5 + 11934x^6 + 59670x^7 + \mathrm{O}\left(x^8\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.171 (sec). Leaf size: 80

```
AsymptoticDSolveValue[x^2*(4*x-1)*y''[x]+x*(5*x+1)*y'[x]+3*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(-\frac{663x^5}{8} - \frac{117x^4}{8} - \frac{15x^3}{8} + \frac{1}{x} - 1\right) + c_2\left(\frac{493493x^9}{24} + \frac{224315x^8}{48} + \frac{16575x^7}{16} + 221x^6 + \frac{221x^5}{5} + \frac{39x^4}{5} + x^3\right)$$

7 CHAPTER 18. Power series solutions. 18.9 Indicial Equation with Difference of Roots a Positive Integer: Logarithmic Case. Exercises page 384

7.1 problem 1

Internal problem ID [6981]
Internal file name [`OUTPUT/6224_Friday_August_12_2022_11_06_04_PM_12870851/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.9 Indicial Equation with Difference of Roots a Positive Integer: Logarithmic Case. Exercises page 384
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{xy'' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 0$$
$$q(x) = \frac{1}{x}$$

Table 152: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n + r - 1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n + r - 1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) = 0$$

When $n = 0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) = 0$$

Or

$$x^{-1+r} a_0 r(-1+r) = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r(-1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= x\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+1} \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
a_n(n+r)(n+r-1) + a_{n-1} = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{a_{n-1}}{(n+r)(n+r-1)} \tag{4}
$$

Which for the root $r = 1$ becomes

$$
a_n = -\frac{a_{n-1}}{(n+1)\, n} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{(1+r)\,r}$$

Which for the root $r = 1$ becomes

$$a_1 = -\frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(1+r)r}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(1+r)^2\,r\,(2+r)}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{12}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(1+r)r}$	$-\frac{1}{2}$
a_2	$\frac{1}{(1+r)^2r(2+r)}$	$\frac{1}{12}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{(1+r)^2\,r\,(2+r)^2\,(3+r)}$$

Which for the root $r = 1$ becomes

$$a_3 = -\frac{1}{144}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(1+r)r}$	$-\frac{1}{2}$
a_2	$\frac{1}{(1+r)^2 r(2+r)}$	$\frac{1}{12}$
a_3	$-\frac{1}{(1+r)^2 r(2+r)^2 (3+r)}$	$-\frac{1}{144}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(1+r)^2 r(2+r)^2 (3+r)^2 (4+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{2880}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(1+r)r}$	$-\frac{1}{2}$
a_2	$\frac{1}{(1+r)^2 r(2+r)}$	$\frac{1}{12}$
a_3	$-\frac{1}{(1+r)^2 r(2+r)^2 (3+r)}$	$-\frac{1}{144}$
a_4	$\frac{1}{(1+r)^2 r(2+r)^2 (3+r)^2 (4+r)}$	$\frac{1}{2880}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{(1+r)^2 r(2+r)^2 (3+r)^2 (4+r)^2 (5+r)}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{1}{86400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(1+r)r}$	$-\frac{1}{2}$
a_2	$\frac{1}{(1+r)^2 r(2+r)}$	$\frac{1}{12}$
a_3	$-\frac{1}{(1+r)^2 r(2+r)^2(3+r)}$	$-\frac{1}{144}$
a_4	$\frac{1}{(1+r)^2 r(2+r)^2(3+r)^2(4+r)}$	$\frac{1}{2880}$
a_5	$-\frac{1}{(1+r)^2 r(2+r)^2(3+r)^2(4+r)^2(5+r)}$	$-\frac{1}{86400}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{1}{(1+r)^2 r (2+r)^2 (3+r)^2 (4+r)^2 (5+r)^2 (6+r)}$$

Which for the root $r = 1$ becomes

$$a_6 = \frac{1}{3628800}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(1+r)r}$	$-\frac{1}{2}$
a_2	$\frac{1}{(1+r)^2 r(2+r)}$	$\frac{1}{12}$
a_3	$-\frac{1}{(1+r)^2 r(2+r)^2(3+r)}$	$-\frac{1}{144}$
a_4	$\frac{1}{(1+r)^2 r(2+r)^2(3+r)^2(4+r)}$	$\frac{1}{2880}$
a_5	$-\frac{1}{(1+r)^2 r(2+r)^2(3+r)^2(4+r)^2(5+r)}$	$-\frac{1}{86400}$
a_6	$\frac{1}{(1+r)^2 r(2+r)^2(3+r)^2(4+r)^2(5+r)^2(6+r)}$	$\frac{1}{3628800}$

For $n = 7$, using the above recursive equation gives

$$a_7 = -\frac{1}{(1+r)^2 r (2+r)^2 (3+r)^2 (4+r)^2 (5+r)^2 (6+r)^2 (7+r)}$$

Which for the root $r = 1$ becomes

$$a_7 = -\frac{1}{203212800}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(1+r)r}$	$-\frac{1}{2}$
a_2	$\frac{1}{(1+r)^2 r(2+r)}$	$\frac{1}{12}$
a_3	$-\frac{1}{(1+r)^2 r(2+r)^2 (3+r)}$	$-\frac{1}{144}$
a_4	$\frac{1}{(1+r)^2 r(2+r)^2 (3+r)^2 (4+r)}$	$\frac{1}{2880}$
a_5	$-\frac{1}{(1+r)^2 r(2+r)^2 (3+r)^2 (4+r)^2 (5+r)}$	$-\frac{1}{86400}$
a_6	$\frac{1}{(1+r)^2 r(2+r)^2 (3+r)^2 (4+r)^2 (5+r)^2 (6+r)}$	$\frac{1}{3628800}$
a_7	$-\frac{1}{(1+r)^2 r(2+r)^2 (3+r)^2 (4+r)^2 (5+r)^2 (6+r)^2 (7+r)}$	$-\frac{1}{203212800}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\
&= x\left(1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^3}{144} + \frac{x^4}{2880} - \frac{x^5}{86400} + \frac{x^6}{3628800} - \frac{x^7}{203212800} + O(x^8)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_1 \\
&= -\frac{1}{(1+r)\, r}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r \to r_2} -\frac{1}{(1+r)\, r} &= \lim_{r \to 0} -\frac{1}{(1+r)\, r} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $xy'' + y = 0$ gives

$$\begin{aligned}
&x\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left.+\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) + Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}$$

Which can be written as

$$\begin{aligned}
&\left(\left(y_1''(x)\,x + y_1(x)\right)\ln(x) + x\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)\right)C \\
&+ x\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1''(x)\,x + y_1(x) = 0$$

Eq (7) simplifes to

$$x\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)C+x\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)$$
$$+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0 \tag{8}$$

Substituting $y_1=\sum_{n=0}^{\infty}a_n x^{n+r_1}$ into the above gives

$$\frac{\left(2\left(\sum_{n=0}^{\infty}x^{-1+n+r_1}a_n(n+r_1)\right)x-\left(\sum_{n=0}^{\infty}a_n x^{n+r_1}\right)\right)C}{x}$$
$$+\frac{\left(\sum_{n=0}^{\infty}x^{-2+n+r_2}b_n(n+r_2)(-1+n+r_2)\right)x^2+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)x}{x}=0 \tag{9}$$

Since $r_1=1$ and $r_2=0$ then the above becomes

$$\frac{\left(2\left(\sum_{n=0}^{\infty}x^{n}a_n(n+1)\right)x-\left(\sum_{n=0}^{\infty}a_n x^{n+1}\right)\right)C}{x}$$
$$+\frac{\left(\sum_{n=0}^{\infty}x^{-2+n}b_n n(n-1)\right)x^2+\left(\sum_{n=0}^{\infty}b_n x^{n}\right)x}{x}=0 \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty}2C\,x^n a_n(n+1)\right)+\sum_{n=0}^{\infty}(-C\,x^n a_n)+\left(\sum_{n=0}^{\infty}n\,x^{n-1}b_n(n-1)\right)+\left(\sum_{n=0}^{\infty}b_n x^n\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{n} a_n (n+1) = \sum_{n=1}^{\infty} 2C a_{n-1} n\, x^{n-1}$$

$$\sum_{n=0}^{\infty} \left(-C\, x^{n} a_n\right) = \sum_{n=1}^{\infty} \left(-C a_{n-1} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} b_n x^{n} = \sum_{n=1}^{\infty} b_{n-1} x^{n-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\left(\sum_{n=1}^{\infty} 2C a_{n-1} n\, x^{n-1}\right) + \sum_{n=1}^{\infty} \left(-C a_{n-1} x^{n-1}\right) + \left(\sum_{n=0}^{\infty} n\, x^{n-1} b_n (n-1)\right) + \left(\sum_{n=1}^{\infty} b_{n-1} x^{n-1}\right) = 0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = N$, where $N = 1$ which is the difference between the two roots, we are free to choose $b_1 = 0$. Hence for $n = 1$, Eq (2B) gives

$$C + 1 = 0$$

Which is solved for C. Solving for C gives

$$C = -1$$

For $n = 2$, Eq (2B) gives

$$3C a_1 + b_1 + 2b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$2b_2 + \frac{3}{2} = 0$$

Solving the above for b_2 gives

$$b_2 = -\frac{3}{4}$$

For $n = 3$, Eq (2B) gives

$$5C a_2 + b_2 + 6b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$6b_3 - \frac{7}{6} = 0$$

Solving the above for b_3 gives

$$b_3 = \frac{7}{36}$$

For $n = 4$, Eq (2B) gives

$$7Ca_3 + b_3 + 12b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$12b_4 + \frac{35}{144} = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{35}{1728}$$

For $n = 5$, Eq (2B) gives

$$9Ca_4 + b_4 + 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$20b_5 - \frac{101}{4320} = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{101}{86400}$$

For $n = 6$, Eq (2B) gives

$$11Ca_5 + b_5 + 30b_6 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$30b_6 + \frac{7}{5400} = 0$$

Solving the above for b_6 gives

$$b_6 = -\frac{7}{162000}$$

For $n = 7$, Eq (2B) gives

$$13Ca_6 + b_6 + 42b_7 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$42b_7 - \frac{283}{6048000} = 0$$

Solving the above for b_7 gives

$$b_7 = \frac{283}{254016000}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -1$ and all b_n, then the second solution becomes

$$y_2(x) = (-1)\left(x\left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^3}{144}+\frac{x^4}{2880}-\frac{x^5}{86400}+\frac{x^6}{3628800}-\frac{x^7}{203212800} + O(x^8)\right)\right)\ln(x) + 1 - \frac{3x^2}{4} + \frac{7x^3}{36} - \frac{35x^4}{1728} + \frac{101x^5}{86400} - \frac{7x^6}{162000} + \frac{283x^7}{254016000} + O(x^8)$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^3}{144}+\frac{x^4}{2880}-\frac{x^5}{86400}+\frac{x^6}{3628800}-\frac{x^7}{203212800}+O(x^8)\right) \\ &\quad + c_2\left((-1)\left(x\left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^3}{144}+\frac{x^4}{2880}-\frac{x^5}{86400}+\frac{x^6}{3628800}-\frac{x^7}{203212800}+O(x^8)\right)\right)\ln(x)\right. \\ &\quad \left. + 1 - \frac{3x^2}{4} + \frac{7x^3}{36} - \frac{35x^4}{1728} + \frac{101x^5}{86400} - \frac{7x^6}{162000} + \frac{283x^7}{254016000} + O(x^8)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x\left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^3}{144}+\frac{x^4}{2880}-\frac{x^5}{86400}+\frac{x^6}{3628800}-\frac{x^7}{203212800}+O(x^8)\right) \\ &\quad + c_2\left(-x\left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^3}{144}+\frac{x^4}{2880}-\frac{x^5}{86400}+\frac{x^6}{3628800}-\frac{x^7}{203212800}+O(x^8)\right)\ln(x)\right. \\ &\quad \left. + 1 - \frac{3x^2}{4} + \frac{7x^3}{36} - \frac{35x^4}{1728} + \frac{101x^5}{86400} - \frac{7x^6}{162000} + \frac{283x^7}{254016000} + O(x^8)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x \left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^3}{144}+\frac{x^4}{2880}-\frac{x^5}{86400}+\frac{x^6}{3628800}-\frac{x^7}{203212800}+O(x^8)\right) \\
+ c_2 \left(-x \left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^3}{144}+\frac{x^4}{2880}-\frac{x^5}{86400}+\frac{x^6}{3628800}-\frac{x^7}{203212800}\right.\right. \\
\left.\left. +O(x^8)\right)\ln(x)+1-\frac{3x^2}{4}+\frac{7x^3}{36}-\frac{35x^4}{1728}+\frac{101x^5}{86400}-\frac{7x^6}{162000}+\frac{283x^7}{254016000}+O(x^8)\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1 x \left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^3}{144}+\frac{x^4}{2880}-\frac{x^5}{86400}+\frac{x^6}{3628800}-\frac{x^7}{203212800}+O(x^8)\right) \\
+ c_2 \left(-x \left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^3}{144}+\frac{x^4}{2880}-\frac{x^5}{86400}+\frac{x^6}{3628800}-\frac{x^7}{203212800}+O(x^8)\right)\ln(x)\right. \\
\left. +1-\frac{3x^2}{4}+\frac{7x^3}{36}-\frac{35x^4}{1728}+\frac{101x^5}{86400}-\frac{7x^6}{162000}+\frac{283x^7}{254016000}+O(x^8)\right)
\end{aligned}
$$

Verified OK.

7.1.1 Maple step by step solution

Let's solve

$xy'' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = 0, P_3(x) = \frac{1}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 0$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Bigg|_{x=0} = 0$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- • Multiply by denominators

  $xy'' + y = 0$

- • Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - ○ Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-1}$

  - ○ Shift index using $k- >k+1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-1+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r) + a_k\right) x^{k+r}\right) = 0$

- • $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+r) = 0$

- • Values of r that satisfy the indicial equation

  $r \in \{0, 1\}$

- • Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+r) + a_k = 0$

- • Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{(k+1+r)(k+r)}$

- • Recursion relation for $r = 0$

  $a_{k+1} = -\frac{a_k}{(k+1)k}$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k}{(k+1) k}\right]$$

- Recursion relation for $r = 1$

  $$a_{k+1} = -\frac{a_k}{(k+2) (k+1)}$$

- Solution for $r = 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{a_k}{(k+2) (k+1)}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+1} = -\frac{a_k}{(k+1) k}, b_{k+1} = -\frac{b_k}{(k+2) (k+1)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 70

```
Order:=8;
dsolve(x*diff(y(x),x$2)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x\left(1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{144}x^3 + \frac{1}{2880}x^4 - \frac{1}{86400}x^5 + \frac{1}{3628800}x^6 - \frac{1}{203212800}x^7 + \mathrm{O}\left(x^8\right)\right) + c_2\left(\ln(x)\left(-x + \frac{1}{2}x^2 - \frac{1}{12}x^3 + \frac{1}{144}x^4 - \frac{1}{2880}x^5 + \frac{1}{86400}x^6 - \frac{1}{3628800}x^7 + \mathrm{O}\left(x^8\right)\right) + \left(1 - \frac{3}{4}x^2 + \frac{7}{36}x^3 - \frac{35}{1728}x^4 + \frac{101}{86400}x^5 - \frac{7}{162000}x^6 + \frac{283}{254016000}x^7 + \mathrm{O}\left(x^8\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 119

```
AsymptoticDSolveValue[x*y''[x]+y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(\frac{x(x^5 - 30x^4 + 600x^3 - 7200x^2 + 43200x - 86400)\log(x)}{86400} + \frac{-71x^6 + 1965x^5 - 35250x^4 + 360000x^3 - 1620000x^2 + 1296000x + 1296000}{1296000}\right) + c_2\left(\frac{x^7}{3628800} - \frac{x^6}{86400} + \frac{x^5}{2880} - \frac{x^4}{144} + \frac{x^3}{12} - \frac{x^2}{2} + x\right)$$

7.2 problem 2

Internal problem ID [6982]
Internal file name [`OUTPUT/6225_Friday_August_12_2022_11_06_06_PM_53530479/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.9 Indicial Equation with Difference of Roots a Positive Integer: Logarithmic Case. Exercises page 384
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 3xy' + (3 + 4x)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' - 3xy' + (3 + 4x)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{3}{x}$$
$$q(x) = \frac{3 + 4x}{x^2}$$

Table 154: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{3}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{3+4x}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' - 3xy' + (3 + 4x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} \right) - 3x \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \right) + (3+4x) \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3a_n x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 4x^{1+n+r} a_n = \sum_{n=1}^{\infty} 4a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3a_n x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 4a_{n-1} x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 3x^{n+r} a_n (n+r) + 3a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - 3x^r a_0 r + 3a_0 x^r = 0$$

Or

$$(x^r r(-1+r) - 3x^r r + 3x^r) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - 4r + 3\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 4r + 3 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 3$$
$$r_2 = 1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - 4r + 3\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^3 \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+3}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{1+n}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - 3a_n(n+r) + 3a_n + 4a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{4a_{n-1}}{n^2 + 2nr + r^2 - 4n - 4r + 3} \tag{4}$$

Which for the root $r = 3$ becomes

$$a_n = -\frac{4a_{n-1}}{n(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 3$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{4}{r(r-2)}$$

Which for the root $r = 3$ becomes

$$a_1 = -\frac{4}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{r(r-2)}$	$-\frac{4}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{16}{r(r-2)(r^2-1)}$$

Which for the root $r = 3$ becomes

$$a_2 = \frac{2}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{r(r-2)}$	$-\frac{4}{3}$
a_2	$\frac{16}{r(r-2)(r^2-1)}$	$\frac{2}{3}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{64}{r^6 - 5r^4 + 4r^2}$$

Which for the root $r = 3$ becomes

$$a_3 = -\frac{8}{45}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{r(r-2)}$	$-\frac{4}{3}$
a_2	$\frac{16}{r(r-2)(r^2-1)}$	$\frac{2}{3}$
a_3	$-\frac{64}{r^6-5r^4+4r^2}$	$-\frac{8}{45}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{256}{r^2 (r^4 - 5r^2 + 4) (r^2 + 4r + 3)}$$

Which for the root $r = 3$ becomes

$$a_4 = \frac{4}{135}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{r(r-2)}$	$-\frac{4}{3}$
a_2	$\frac{16}{r(r-2)(r^2-1)}$	$\frac{2}{3}$
a_3	$-\frac{64}{r^6-5r^4+4r^2}$	$-\frac{8}{45}$
a_4	$\frac{256}{r^2(r^4-5r^2+4)(r^2+4r+3)}$	$\frac{4}{135}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1024}{r^2 (r-2) (r+2)^2 (-1+r) (r+1)^2 (r+3) (r+4)}$$

Which for the root $r = 3$ becomes

$$a_5 = -\frac{16}{4725}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{r(r-2)}$	$-\frac{4}{3}$
a_2	$\frac{16}{r(r-2)(r^2-1)}$	$\frac{2}{3}$
a_3	$-\frac{64}{r^6-5r^4+4r^2}$	$-\frac{8}{45}$
a_4	$\frac{256}{r^2(r^4-5r^2+4)(r^2+4r+3)}$	$\frac{4}{135}$
a_5	$-\frac{1024}{r^2(r-2)(r+2)^2(-1+r)(r+1)^2(r+3)(r+4)}$	$-\frac{16}{4725}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{4096}{r^2 (r-2) (r+2)^2 (-1+r) (r+1)^2 (r+3)^2 (r+4) (r+5)}$$

Which for the root $r = 3$ becomes

$$a_6 = \frac{4}{14175}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{r(r-2)}$	$-\frac{4}{3}$
a_2	$\frac{16}{r(r-2)(r^2-1)}$	$\frac{2}{3}$
a_3	$-\frac{64}{r^6-5r^4+4r^2}$	$-\frac{8}{45}$
a_4	$\frac{256}{r^2(r^4-5r^2+4)(r^2+4r+3)}$	$\frac{4}{135}$
a_5	$-\frac{1024}{r^2(r-2)(r+2)^2(-1+r)(r+1)^2(r+3)(r+4)}$	$-\frac{16}{4725}$
a_6	$\frac{4096}{r^2(r-2)(r+2)^2(-1+r)(r+1)^2(r+3)^2(r+4)(r+5)}$	$\frac{4}{14175}$

For $n = 7$, using the above recursive equation gives

$$a_7 = -\frac{16384}{r^2 (r-2) (r+2)^2 (-1+r) (r+1)^2 (r+3)^2 (r+4)^2 (r+5) (r+6)}$$

Which for the root $r = 3$ becomes

$$a_7 = -\frac{16}{893025}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{r(r-2)}$	$-\frac{4}{3}$
a_2	$\frac{16}{r(r-2)(r^2-1)}$	$\frac{2}{3}$
a_3	$-\frac{64}{r^6-5r^4+4r^2}$	$-\frac{8}{45}$
a_4	$\frac{256}{r^2(r^4-5r^2+4)(r^2+4r+3)}$	$\frac{4}{135}$
a_5	$-\frac{1024}{r^2(r-2)(r+2)^2(-1+r)(r+1)^2(r+3)(r+4)}$	$-\frac{16}{4725}$
a_6	$\frac{4096}{r^2(r-2)(r+2)^2(-1+r)(r+1)^2(r+3)^2(r+4)(r+5)}$	$\frac{4}{14175}$
a_7	$-\frac{16384}{r^2(r-2)(r+2)^2(-1+r)(r+1)^2(r+3)^2(r+4)^2(r+5)(r+6)}$	$-\frac{16}{893025}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^3\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \dots\right) \\ &= x^3\left(1 - \frac{4x}{3} + \frac{2x^2}{3} - \frac{8x^3}{45} + \frac{4x^4}{135} - \frac{16x^5}{4725} + \frac{4x^6}{14175} - \frac{16x^7}{893025} + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\rightarrow r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_2 \\ &= \frac{16}{r(r-2)(r^2-1)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\rightarrow r_2} \frac{16}{r(r-2)(r^2-1)} &= \lim_{r\rightarrow 1} \frac{16}{r(r-2)(r^2-1)} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $x^2y'' - 3xy' + (3+4x)\,y = 0$ gives

$$\begin{aligned}
&x^2\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left.\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&\quad - 3x\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) \\
&\quad + (3+4x)\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0
\end{aligned}$$

Which can be written as

$$\begin{aligned}&\left(\left(x^{2} y_{1}''(x)-3 y_{1}'(x) x+(3+4 x) y_{1}(x)\right) \ln (x)+x^{2}\left(\frac{2 y_{1}'(x)}{x}-\frac{y_{1}(x)}{x^{2}}\right)\right.\\&\left.-3 y_{1}(x)\right) C+x^{2}\left(\sum_{n=0}^{\infty}\left(\frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)^{2}}{x^{2}}-\frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)}{x^{2}}\right)\right)\\&-3 x\left(\sum_{n=0}^{\infty} \frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)}{x}\right)+(3+4 x)\left(\sum_{n=0}^{\infty} b_{n} x^{n+r_{2}}\right)=0\end{aligned} \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$x^{2} y_{1}''(x)-3 y_{1}'(x) x+(3+4 x) y_{1}(x)=0$$

Eq (7) simplifes to

$$\begin{aligned}&\left(x^{2}\left(\frac{2 y_{1}'(x)}{x}-\frac{y_{1}(x)}{x^{2}}\right)-3 y_{1}(x)\right) C\\&+x^{2}\left(\sum_{n=0}^{\infty}\left(\frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)^{2}}{x^{2}}-\frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)}{x^{2}}\right)\right)\\&-3 x\left(\sum_{n=0}^{\infty} \frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)}{x}\right)+(3+4 x)\left(\sum_{n=0}^{\infty} b_{n} x^{n+r_{2}}\right)=0\end{aligned} \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\begin{aligned}&\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_{1}} a_{n}\left(n+r_{1}\right)\right) x-4\left(\sum_{n=0}^{\infty} a_{n} x^{n+r_{1}}\right)\right) C\\&+\left(\sum_{n=0}^{\infty} x^{-2+n+r_{2}} b_{n}\left(n+r_{2}\right)\left(-1+n+r_{2}\right)\right) x^{2}\\&-3\left(\sum_{n=0}^{\infty} x^{-1+n+r_{2}} b_{n}\left(n+r_{2}\right)\right) x+(3+4 x)\left(\sum_{n=0}^{\infty} b_{n} x^{n+r_{2}}\right)=0\end{aligned} \tag{9}$$

Since $r_1 = 3$ and $r_2 = 1$ then the above becomes

$$\begin{aligned}&\left(2\left(\sum_{n=0}^{\infty} x^{n+2} a_{n}(n+3)\right) x-4\left(\sum_{n=0}^{\infty} a_{n} x^{n+3}\right)\right) C+\left(\sum_{n=0}^{\infty} x^{n-1} b_{n}(1+n) n\right) x^{2}\\&-3\left(\sum_{n=0}^{\infty} x^{n} b_{n}(1+n)\right) x+(3+4 x)\left(\sum_{n=0}^{\infty} b_{n} x^{1+n}\right)=0\end{aligned} \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2C\, x^{n+3} a_n (n+3)\right) + \sum_{n=0}^{\infty} \left(-4C a_n x^{n+3}\right) + \left(\sum_{n=0}^{\infty} n\, x^{1+n} b_n (1+n)\right) \tag{2A}$$
$$+ \sum_{n=0}^{\infty} \left(-3x^{1+n} b_n (1+n)\right) + \left(\sum_{n=0}^{\infty} 3b_n x^{1+n}\right) + \left(\sum_{n=0}^{\infty} 4x^{n+2} b_n\right) = 0$$

The next step is to make all powers of x be $1+n$ in each summation term. Going over each summation term above with power of x in it which is not already x^{1+n} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{n+3} a_n (n+3) = \sum_{n=2}^{\infty} 2C a_{-2+n} (1+n)\, x^{1+n}$$
$$\sum_{n=0}^{\infty} \left(-4C a_n x^{n+3}\right) = \sum_{n=2}^{\infty} \left(-4C a_{-2+n} x^{1+n}\right)$$
$$\sum_{n=0}^{\infty} 4x^{n+2} b_n = \sum_{n=1}^{\infty} 4b_{n-1} x^{1+n}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $1+n$.

$$\left(\sum_{n=2}^{\infty} 2C a_{-2+n} (1+n)\, x^{1+n}\right) + \sum_{n=2}^{\infty} \left(-4C a_{-2+n} x^{1+n}\right) + \left(\sum_{n=0}^{\infty} n\, x^{1+n} b_n (1+n)\right) \tag{2B}$$
$$+ \sum_{n=0}^{\infty} \left(-3x^{1+n} b_n (1+n)\right) + \left(\sum_{n=0}^{\infty} 3b_n x^{1+n}\right) + \left(\sum_{n=1}^{\infty} 4b_{n-1} x^{1+n}\right) = 0$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n=1$, Eq (2B) gives

$$-b_1 + 4b_0 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-b_1 + 4 = 0$$

Solving the above for b_1 gives

$$b_1 = 4$$

For $n = N$, where $N = 2$ which is the difference between the two roots, we are free to choose $b_2 = 0$. Hence for $n = 2$, Eq (2B) gives

$$2C + 16 = 0$$

Which is solved for C. Solving for C gives

$$C = -8$$

For $n = 3$, Eq (2B) gives

$$4Ca_1 + 4b_2 + 3b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$3b_3 + \frac{128}{3} = 0$$

Solving the above for b_3 gives

$$b_3 = -\frac{128}{9}$$

For $n = 4$, Eq (2B) gives

$$6Ca_2 + 4b_3 + 8b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$8b_4 - \frac{800}{9} = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{100}{9}$$

For $n = 5$, Eq (2B) gives

$$8Ca_3 + 4b_4 + 15b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$15b_5 + \frac{2512}{45} = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{2512}{675}$$

For $n = 6$, Eq (2B) gives

$$10Ca_4 + 4b_5 + 24b_6 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$24b_6 - \frac{11648}{675} = 0$$

Solving the above for b_6 gives

$$b_6 = \frac{1456}{2025}$$

For $n = 7$, Eq (2B) gives

$$12Ca_5 + 4b_6 + 35b_7 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$35b_7 + \frac{45376}{14175} = 0$$

Solving the above for b_7 gives

$$b_7 = -\frac{45376}{496125}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -8$ and all b_n, then the second solution becomes

$$\begin{aligned}y_2(x) = (-8)\left(x^3\left(1-\frac{4x}{3}+\frac{2x^2}{3}-\frac{8x^3}{45}+\frac{4x^4}{135}-\frac{16x^5}{4725}+\frac{4x^6}{14175}-\frac{16x^7}{893025}+O(x^8)\right)\right)\ln(x)\\ + x\left(1+4x-\frac{128x^3}{9}+\frac{100x^4}{9}-\frac{2512x^5}{675}+\frac{1456x^6}{2025}-\frac{45376x^7}{496125}+O(x^8)\right)\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1y_1(x) + c_2y_2(x)\\ &= c_1x^3\left(1-\frac{4x}{3}+\frac{2x^2}{3}-\frac{8x^3}{45}+\frac{4x^4}{135}-\frac{16x^5}{4725}+\frac{4x^6}{14175}-\frac{16x^7}{893025}+O(x^8)\right)\\ &+c_2\left((-8)\left(x^3\left(1-\frac{4x}{3}+\frac{2x^2}{3}-\frac{8x^3}{45}+\frac{4x^4}{135}-\frac{16x^5}{4725}+\frac{4x^6}{14175}-\frac{16x^7}{893025}+O(x^8)\right)\right)\ln(x)\right.\\ &\left.+ x\left(1+4x-\frac{128x^3}{9}+\frac{100x^4}{9}-\frac{2512x^5}{675}+\frac{1456x^6}{2025}-\frac{45376x^7}{496125}+O(x^8)\right)\right)\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y &= y_h\\ &= c_1x^3\left(1-\frac{4x}{3}+\frac{2x^2}{3}-\frac{8x^3}{45}+\frac{4x^4}{135}-\frac{16x^5}{4725}+\frac{4x^6}{14175}-\frac{16x^7}{893025}+O(x^8)\right)\\ &+ c_2\left(-8x^3\left(1-\frac{4x}{3}+\frac{2x^2}{3}-\frac{8x^3}{45}+\frac{4x^4}{135}-\frac{16x^5}{4725}+\frac{4x^6}{14175}-\frac{16x^7}{893025}+O(x^8)\right)\ln(x)\right.\\ &\left.+ x\left(1+4x-\frac{128x^3}{9}+\frac{100x^4}{9}-\frac{2512x^5}{675}+\frac{1456x^6}{2025}-\frac{45376x^7}{496125}+O(x^8)\right)\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1 x^3 \left(1-\frac{4x}{3}+\frac{2x^2}{3}-\frac{8x^3}{45}+\frac{4x^4}{135}-\frac{16x^5}{4725}+\frac{4x^6}{14175}-\frac{16x^7}{893025}+O(x^8)\right) \\
& + c_2 \left(-8x^3 \left(1-\frac{4x}{3}+\frac{2x^2}{3}-\frac{8x^3}{45}+\frac{4x^4}{135}-\frac{16x^5}{4725}+\frac{4x^6}{14175}-\frac{16x^7}{893025}+O(x^8)\right)\ln(x) \right. \\
& \left. + x\left(1+4x-\frac{128x^3}{9}+\frac{100x^4}{9}-\frac{2512x^5}{675}+\frac{1456x^6}{2025}-\frac{45376x^7}{496125}+O(x^8)\right)\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1 x^3 \left(1-\frac{4x}{3}+\frac{2x^2}{3}-\frac{8x^3}{45}+\frac{4x^4}{135}-\frac{16x^5}{4725}+\frac{4x^6}{14175}-\frac{16x^7}{893025}+O(x^8)\right) \\
& + c_2 \left(-8x^3 \left(1-\frac{4x}{3}+\frac{2x^2}{3}-\frac{8x^3}{45}+\frac{4x^4}{135}-\frac{16x^5}{4725}+\frac{4x^6}{14175}-\frac{16x^7}{893025}+O(x^8)\right)\ln(x) \right. \\
& \left. + x\left(1+4x-\frac{128x^3}{9}+\frac{100x^4}{9}-\frac{2512x^5}{675}+\frac{1456x^6}{2025}-\frac{45376x^7}{496125}+O(x^8)\right)\right)
\end{aligned}
$$

Verified OK.

7.2.1 Maple step by step solution

Let's solve

$x^2 y'' - 3xy' + (3+4x)y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3y'}{x} - \frac{(3+4x)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{3y'}{x} + \frac{(3+4x)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{3}{x}, P_3(x) = \frac{3+4x}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -3$

◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 3$$

◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$$x^2y'' - 3xy' + (3 + 4x)\, y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

◦ Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$$

◦ Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1 + r)\,(-3 + r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k (k + r - 1)\,(k + r - 3) + 4a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1 + r)\,(-3 + r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{1, 3\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r-1)(k+r-3)+4a_{k-1}=0$$

- Shift index using $k->k+1$

$$a_{k+1}(k+r)(k-2+r)+4a_k=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=-\frac{4a_k}{(k+r)(k-2+r)}$$

- Recursion relation for $r=1$

$$a_{k+1}=-\frac{4a_k}{(k+1)(k-1)}$$

- Series not valid for $r=1$, division by 0 in the recursion relation at $k=1$

$$a_{k+1}=-\frac{4a_k}{(k+1)(k-1)}$$

- Recursion relation for $r=3$

$$a_{k+1}=-\frac{4a_k}{(k+3)(k+1)}$$

- Solution for $r=3$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1}=-\frac{4a_k}{(k+3)(k+1)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 73

```
Order:=8;
dsolve(x^2*diff(y(x),x$2)-3*x*diff(y(x),x)+(3+4*x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = x\left(c_1x^2\left(1-\frac{4}{3}x+\frac{2}{3}x^2-\frac{8}{45}x^3+\frac{4}{135}x^4-\frac{16}{4725}x^5+\frac{4}{14175}x^6-\frac{16}{893025}x^7+\mathrm{O}\left(x^8\right)\right)\right.$$
$$+c_2\left(\ln\left(x\right)\left(16x^2-\frac{64}{3}x^3+\frac{32}{3}x^4-\frac{128}{45}x^5+\frac{64}{135}x^6-\frac{256}{4725}x^7+\mathrm{O}\left(x^8\right)\right)\right.$$
$$\left.\left.+\left(-2-8x+\frac{256}{9}x^3-\frac{200}{9}x^4+\frac{5024}{675}x^5-\frac{2912}{2025}x^6+\frac{90752}{496125}x^7+\mathrm{O}\left(x^8\right)\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 121

```
AsymptoticDSolveValue[x^2*y''[x]-3*x*y'[x]+(3+4*x)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(\frac{x(1696x^6-8976x^5+27900x^4-39600x^3+8100x^2+8100x+2025)}{2025}\right.$$
$$\left.-\frac{8}{135}x^3\left(4x^4-24x^3+90x^2-180x+135\right)\log(x)\right)$$
$$+c_2\left(\frac{4x^9}{14175}-\frac{16x^8}{4725}+\frac{4x^7}{135}-\frac{8x^6}{45}+\frac{2x^5}{3}-\frac{4x^4}{3}+x^3\right)$$

7.3 problem 3

Internal problem ID [6983]
Internal file name [`OUTPUT/6226_Friday_August_12_2022_11_06_09_PM_76951881/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.9 Indicial Equation with Difference of Roots a Positive Integer: Logarithmic Case. Exercises page 384
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{2xy'' + 6y' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2xy'' + 6y' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{3}{x}$$
$$q(x) = \frac{1}{2x}$$

Table 156: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2xy'' + 6y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \right) + 6 \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1) \right) + \left(\sum_{n=0}^{\infty} 6(n+r) a_n x^{n+r-1} \right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 6(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$2x^{n+r-1} a_n (n+r)(n+r-1) + 6(n+r) a_n x^{n+r-1} = 0$$

When $n = 0$ the above becomes

$$2x^{-1+r} a_0 r(-1+r) + 6r a_0 x^{-1+r} = 0$$

Or

$$\left(2x^{-1+r} r(-1+r) + 6r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$2r\, x^{-1+r}(2+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r(2+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = -2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$2r\, x^{-1+r}(2+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\
y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2}
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
2a_n(n+r)(n+r-1) + 6a_n(n+r) + a_{n-1} = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{a_{n-1}}{2(n^2 + 2nr + r^2 + 2n + 2r)} \tag{4}
$$

Which for the root $r = 0$ becomes

$$
a_n = -\frac{a_{n-1}}{2n(n+2)} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{2r^2 + 8r + 6}$$

Which for the root $r = 0$ becomes

$$a_1 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+8r+6}$	$-\frac{1}{6}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{4r^4 + 40r^3 + 140r^2 + 200r + 96}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{96}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+8r+6}$	$-\frac{1}{6}$
a_2	$\frac{1}{4r^4+40r^3+140r^2+200r+96}$	$\frac{1}{96}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{8 (r + 3)^2 (r + 1) (r + 4) (2 + r) (r + 5)}$$

Which for the root $r = 0$ becomes

$$a_3 = -\frac{1}{2880}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+8r+6}$	$-\frac{1}{6}$
a_2	$\frac{1}{4r^4+40r^3+140r^2+200r+96}$	$\frac{1}{96}$
a_3	$-\frac{1}{8(r+3)^2(r+1)(r+4)(2+r)(r+5)}$	$-\frac{1}{2880}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{16\left(r+3\right)^2\left(r+1\right)\left(r+4\right)^2\left(2+r\right)\left(r+5\right)\left(r+6\right)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{138240}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+8r+6}$	$-\frac{1}{6}$
a_2	$\frac{1}{4r^4+40r^3+140r^2+200r+96}$	$\frac{1}{96}$
a_3	$-\frac{1}{8(r+3)^2(r+1)(r+4)(2+r)(r+5)}$	$-\frac{1}{2880}$
a_4	$\frac{1}{16(r+3)^2(r+1)(r+4)^2(2+r)(r+5)(r+6)}$	$\frac{1}{138240}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{32\left(r+3\right)^2\left(r+1\right)\left(r+4\right)^2\left(2+r\right)\left(r+5\right)^2\left(r+6\right)\left(r+7\right)}$$

Which for the root $r = 0$ becomes

$$a_5 = -\frac{1}{9676800}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+8r+6}$	$-\frac{1}{6}$
a_2	$\frac{1}{4r^4+40r^3+140r^2+200r+96}$	$\frac{1}{96}$
a_3	$-\frac{1}{8(r+3)^2(r+1)(r+4)(2+r)(r+5)}$	$-\frac{1}{2880}$
a_4	$\frac{1}{16(r+3)^2(r+1)(r+4)^2(2+r)(r+5)(r+6)}$	$\frac{1}{138240}$
a_5	$-\frac{1}{32(r+3)^2(r+1)(r+4)^2(2+r)(r+5)^2(r+6)(r+7)}$	$-\frac{1}{9676800}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{1}{64(r+3)^2(r+1)(r+4)^2(2+r)(r+5)^2(r+6)^2(r+7)(8+r)}$$

Which for the root $r = 0$ becomes

$$a_6 = \frac{1}{928972800}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+8r+6}$	$-\frac{1}{6}$
a_2	$\frac{1}{4r^4+40r^3+140r^2+200r+96}$	$\frac{1}{96}$
a_3	$-\frac{1}{8(r+3)^2(r+1)(r+4)(2+r)(r+5)}$	$-\frac{1}{2880}$
a_4	$\frac{1}{16(r+3)^2(r+1)(r+4)^2(2+r)(r+5)(r+6)}$	$\frac{1}{138240}$
a_5	$-\frac{1}{32(r+3)^2(r+1)(r+4)^2(2+r)(r+5)^2(r+6)(r+7)}$	$-\frac{1}{9676800}$
a_6	$\frac{1}{64(r+3)^2(r+1)(r+4)^2(2+r)(r+5)^2(r+6)^2(r+7)(8+r)}$	$\frac{1}{928972800}$

For $n = 7$, using the above recursive equation gives

$$a_7 = -\frac{1}{128(r+3)^2(r+1)(r+4)^2(2+r)(r+5)^2(r+6)^2(r+7)^2(8+r)(9+r)}$$

Which for the root $r = 0$ becomes

$$a_7 = -\frac{1}{117050572800}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+8r+6}$	$-\frac{1}{6}$
a_2	$\frac{1}{4r^4+40r^3+140r^2+200r+96}$	$\frac{1}{96}$
a_3	$-\frac{1}{8(r+3)^2(r+1)(r+4)(2+r)(r+5)}$	$-\frac{1}{2880}$
a_4	$\frac{1}{16(r+3)^2(r+1)(r+4)^2(2+r)(r+5)(r+6)}$	$\frac{1}{138240}$
a_5	$-\frac{1}{32(r+3)^2(r+1)(r+4)^2(2+r)(r+5)^2(r+6)(r+7)}$	$-\frac{1}{9676800}$
a_6	$\frac{1}{64(r+3)^2(r+1)(r+4)^2(2+r)(r+5)^2(r+6)^2(r+7)(8+r)}$	$\frac{1}{928972800}$
a_7	$-\frac{1}{128(r+3)^2(r+1)(r+4)^2(2+r)(r+5)^2(r+6)^2(r+7)^2(8+r)(9+r)}$	$-\frac{1}{117050572800}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \ldots \\
&= 1 - \frac{x}{6} + \frac{x^2}{96} - \frac{x^3}{2880} + \frac{x^4}{138240} - \frac{x^5}{9676800} + \frac{x^6}{928972800} - \frac{x^7}{117050572800} + O(x^8)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_2 \\
&= \frac{1}{4r^4 + 40r^3 + 140r^2 + 200r + 96}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r \to r_2} \frac{1}{4r^4 + 40r^3 + 140r^2 + 200r + 96} &= \lim_{r \to -2} \frac{1}{4r^4 + 40r^3 + 140r^2 + 200r + 96} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n + r_2} \right)
$$

Therefore

$$\frac{d}{dx}y_2(x) = Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)$$
$$= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right)$$
$$\frac{d^2}{dx^2}y_2(x) = Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)$$
$$= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)$$

Substituting these back into the given ode $2xy'' + 6y' + y = 0$ gives

$$2x\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) + 6Cy_1'(x)\ln(x) + \frac{6Cy_1(x)}{x} + 6\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0$$

Which can be written as

$$\left((2y_1''(x)\,x + 6y_1'(x) + y_1(x))\ln(x) + 2x\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + \frac{6y_1(x)}{x}\right)C + 2x\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) + 6\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$2y_1''(x)\,x + 6y_1'(x) + y_1(x) = 0$$

Eq (7) simplifes to

$$
\begin{aligned}
&\left(2x\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+\frac{6y_1(x)}{x}\right)C \\
&+2x\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&+6\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned}
\tag{8}
$$

Substituting $y_1=\sum_{n=0}^{\infty}a_n x^{n+r_1}$ into the above gives

$$
\begin{aligned}
&\frac{\left(4\left(\sum_{n=0}^{\infty}x^{-1+n+r_1}a_n(n+r_1)\right)x+4\left(\sum_{n=0}^{\infty}a_n x^{n+r_1}\right)\right)C}{x} \\
&+\frac{2\left(\sum_{n=0}^{\infty}x^{-2+n+r_2}b_n(n+r_2)(-1+n+r_2)\right)x^2+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)x+6\left(\sum_{n=0}^{\infty}x^{-1+n+r_2}b_n(n+r_2)\right)x}{x} \\
&=0
\end{aligned}
\tag{9}
$$

Since $r_1=0$ and $r_2=-2$ then the above becomes

$$
\begin{aligned}
&\frac{\left(4\left(\sum_{n=0}^{\infty}x^{n-1}a_n n\right)x+4\left(\sum_{n=0}^{\infty}a_n x^{n}\right)\right)C}{x} \\
&+\frac{2\left(\sum_{n=0}^{\infty}x^{-4+n}b_n(n-2)(-3+n)\right)x^2+\left(\sum_{n=0}^{\infty}b_n x^{n-2}\right)x+6\left(\sum_{n=0}^{\infty}x^{-3+n}b_n(n-2)\right)x}{x} \\
&=0
\end{aligned}
\tag{10}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty}4C\,x^{n-1}a_n n\right)+\left(\sum_{n=0}^{\infty}4C\,x^{n-1}a_n\right)+\left(\sum_{n=0}^{\infty}2x^{-3+n}b_n(n-2)(-3+n)\right) \\
&+\left(\sum_{n=0}^{\infty}b_n x^{n-2}\right)+\left(\sum_{n=0}^{\infty}6x^{-3+n}b_n(n-2)\right)=0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $-3+n$ in each summation term. Going over each summation term above with power of x in it which is not already x^{-3+n} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 4C\, x^{n-1} a_n n &= \sum_{n=2}^{\infty} 4C(n-2)\, a_{n-2} x^{-3+n} \\
\sum_{n=0}^{\infty} 4C\, x^{n-1} a_n &= \sum_{n=2}^{\infty} 4C a_{n-2} x^{-3+n} \\
\sum_{n=0}^{\infty} b_n x^{n-2} &= \sum_{n=1}^{\infty} b_{n-1} x^{-3+n}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $-3+n$.

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} 4C(n-2)\, a_{n-2} x^{-3+n}\right) + \left(\sum_{n=2}^{\infty} 4C a_{n-2} x^{-3+n}\right) \\
&+ \left(\sum_{n=0}^{\infty} 2x^{-3+n} b_n (n-2)\,(-3+n)\right) \\
&+ \left(\sum_{n=1}^{\infty} b_{n-1} x^{-3+n}\right) + \left(\sum_{n=0}^{\infty} 6x^{-3+n} b_n (n-2)\right) = 0
\end{aligned} \tag{2B}
$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n=1$, Eq (2B) gives

$$
-2b_1 + b_0 = 0
$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$
-2b_1 + 1 = 0
$$

Solving the above for b_1 gives

$$
b_1 = \frac{1}{2}
$$

For $n = N$, where $N = 2$ which is the difference between the two roots, we are free to choose $b_2 = 0$. Hence for $n = 2$, Eq (2B) gives

$$
4C + \frac{1}{2} = 0
$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{8}$$

For $n = 3$, Eq (2B) gives

$$8Ca_1 + b_2 + 6b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$6b_3 + \frac{1}{6} = 0$$

Solving the above for b_3 gives

$$b_3 = -\frac{1}{36}$$

For $n = 4$, Eq (2B) gives

$$12Ca_2 + b_3 + 16b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$16b_4 - \frac{25}{576} = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{25}{9216}$$

For $n = 5$, Eq (2B) gives

$$16Ca_3 + b_4 + 30b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$30b_5 + \frac{157}{46080} = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{157}{1382400}$$

For $n = 6$, Eq (2B) gives

$$20Ca_4 + b_5 + 48b_6 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$48b_6 - \frac{91}{691200} = 0$$

Solving the above for b_6 gives

$$b_6 = \frac{91}{33177600}$$

For $n = 7$, Eq (2B) gives

$$24Ca_5 + b_6 + 70b_7 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$70b_7 + \frac{709}{232243200} = 0$$

Solving the above for b_7 gives

$$b_7 = -\frac{709}{16257024000}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{8}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{1}{8}\left(1 - \frac{x}{6} + \frac{x^2}{96} - \frac{x^3}{2880} + \frac{x^4}{138240} - \frac{x^5}{9676800} + \frac{x^6}{928972800} - \frac{x^7}{117050572800} + O(x^8)\right)\ln(x) + \frac{1 + \frac{x}{2} - \frac{x^3}{36} + \frac{25x^4}{9216} - \frac{157x^5}{1382400} + \frac{91x^6}{33177600} - \frac{709x^7}{16257024000} + O(x^8)}{x^2}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1\left(1 - \frac{x}{6} + \frac{x^2}{96} - \frac{x^3}{2880} + \frac{x^4}{138240} - \frac{x^5}{9676800} + \frac{x^6}{928972800} - \frac{x^7}{117050572800} + O(x^8)\right) \\ &\quad + c_2\left(-\frac{1}{8}\left(1 - \frac{x}{6} + \frac{x^2}{96} - \frac{x^3}{2880} + \frac{x^4}{138240} - \frac{x^5}{9676800} + \frac{x^6}{928972800} - \frac{x^7}{117050572800} \right.\right. \\ &\quad \left.\left. + O(x^8)\right)\ln(x) + \frac{1 + \frac{x}{2} - \frac{x^3}{36} + \frac{25x^4}{9216} - \frac{157x^5}{1382400} + \frac{91x^6}{33177600} - \frac{709x^7}{16257024000} + O(x^8)}{x^2}\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1\left(1 - \frac{x}{6} + \frac{x^2}{96} - \frac{x^3}{2880} + \frac{x^4}{138240} - \frac{x^5}{9676800} + \frac{x^6}{928972800} - \frac{x^7}{117050572800} + O(x^8)\right) \\ &\quad + c_2\left(\left(-\frac{1}{8} + \frac{x}{48} - \frac{x^2}{768} + \frac{x^3}{23040} - \frac{x^4}{1105920} + \frac{x^5}{77414400} - \frac{x^6}{7431782400} + \frac{x^7}{936404582400} \right.\right. \\ &\quad \left.\left. - \frac{O(x^8)}{8}\right)\ln(x) + \frac{1 + \frac{x}{2} - \frac{x^3}{36} + \frac{25x^4}{9216} - \frac{157x^5}{1382400} + \frac{91x^6}{33177600} - \frac{709x^7}{16257024000} + O(x^8)}{x^2}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1\left(1-\frac{x}{6}+\frac{x^2}{96}-\frac{x^3}{2880}+\frac{x^4}{138240}-\frac{x^5}{9676800}+\frac{x^6}{928972800}-\frac{x^7}{117050572800}+O\left(x^8\right)\right)+c_2\left(\left(-\frac{1}{8}+\frac{x}{48}-\frac{x^2}{768}+\frac{x^3}{23040}-\frac{x^4}{1105920}+\frac{x^5}{77414400}-\frac{x^6}{7431782400}+\frac{x^7}{936404582400}-\frac{O(x^8)}{8}\right)\ln\left(x\right)+\frac{1+\frac{x}{2}-\frac{x^3}{36}+\frac{25x^4}{9216}-\frac{157x^5}{1382400}+\frac{91x^6}{33177600}-\frac{709x^7}{16257024000}+O(x^8)}{x^2}\right) \tag{1}$$

Verification of solutions

$$y = c_1\left(1-\frac{x}{6}+\frac{x^2}{96}-\frac{x^3}{2880}+\frac{x^4}{138240}-\frac{x^5}{9676800}+\frac{x^6}{928972800}-\frac{x^7}{117050572800}+O\left(x^8\right)\right)+c_2\left(\left(-\frac{1}{8}+\frac{x}{48}-\frac{x^2}{768}+\frac{x^3}{23040}-\frac{x^4}{1105920}+\frac{x^5}{77414400}-\frac{x^6}{7431782400}+\frac{x^7}{936404582400}-\frac{O(x^8)}{8}\right)\ln\left(x\right)+\frac{1+\frac{x}{2}-\frac{x^3}{36}+\frac{25x^4}{9216}-\frac{157x^5}{1382400}+\frac{91x^6}{33177600}-\frac{709x^7}{16257024000}+O(x^8)}{x^2}\right)$$

Verified OK.

7.3.1 Maple step by step solution

Let's solve

$2xy''+6y'+y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{3y'}{x}-\frac{y}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{3y'}{x}+\frac{y}{2x}=0$

☐ Check to see if $x_0=0$ is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{3}{x}, P_3(x) = \frac{1}{2x}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$(x \cdot P_2(x))\bigg|_{x=0} = 3$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$2xy'' + 6y' + y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert y' to series expansion

  $$y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$

- Shift index using $k- > k+1$

  $$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- > k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$2a_0 r(2+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(2a_{k+1}(k+1+r)\,(k+3+r) + a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $2r(2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-2, 0\}$
- Each term in the series must be 0, giving the recursion relation

  $2a_{k+1}(k+1+r)(k+3+r) + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{2(k+1+r)(k+3+r)}$
- Recursion relation for $r = -2$

  $a_{k+1} = -\frac{a_k}{2(k-1)(k+1)}$
- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 1$

  $a_{k+1} = -\frac{a_k}{2(k-1)(k+1)}$
- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{a_k}{2(k+1)(k+3)}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k}{2(k+1)(k+3)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 74

```
Order:=8;
dsolve(2*x*diff(y(x),x$2)+6*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1\left(1-\frac{1}{6}x+\frac{1}{96}x^2-\frac{1}{2880}x^3+\frac{1}{138240}x^4-\frac{1}{9676800}x^5+\frac{1}{928972800}x^6-\frac{1}{117050572800}x^7+\mathrm{O}\left(x^8\right)\right)x^2+c_2\left(\ln\left(x\right)\right.}{}$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 114

```
AsymptoticDSolveValue[2*x*y''[x]+6*y'[x]+y[x]==0,y[x],{x,0,7}]
```

$$\begin{aligned} y(x) &\to c_2\left(\frac{x^6}{928972800}-\frac{x^5}{9676800}+\frac{x^4}{138240}-\frac{x^3}{2880}+\frac{x^2}{96}-\frac{x}{6}+1\right) \\ &+ c_1\left(\frac{53x^6-2244x^5+55800x^4-633600x^3+1036800x^2+8294400x+16588800}{16588800x^2}\right. \\ &\left.-\frac{(x^4-48x^3+1440x^2-23040x+138240)\log(x)}{1105920}\right) \end{aligned}$$

7.4 problem 4

Internal problem ID [6984]
Internal file name [`OUTPUT/6227_Friday_August_12_2022_11_06_11_PM_53610816/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.9 Indicial Equation with Difference of Roots a Positive Integer: Logarithmic Case. Exercises page 384
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^{2}y''+2x(-x+2)\,y'-(3x+1)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4x^{2}y''+\left(-2x^{2}+4x\right)y'+(-3x-1)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=-\frac{x-2}{2x}\\ q(x)&=-\frac{3x+1}{4x^{2}}\end{aligned}$$

Table 158: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x-2}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{3x+1}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2y'' + \left(-2x^2 + 4x\right) y' + (-3x - 1) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2 \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) + \left(-2x^2 + 4x\right) \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + (-3x - 1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-3x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} \left(-3x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-3a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)\right) + \sum_{n=1}^{\infty} \left(-3a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r} a_n (n+r)(n+r-1) + 4x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$4x^r a_0 r(-1+r) + 4x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(4x^r r(-1+r) + 4x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2 - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = -\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(4r^2 - 1)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sqrt{x} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{\sqrt{x}}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$4a_n(n+r)(n+r-1) - 2a_{n-1}(n+r-1) + 4a_n(n+r) - 3a_{n-1} - a_n = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{2n + 2r - 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{a_{n-1}}{2n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{2r + 1}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2r+1}$	$\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{4r^2 + 8r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{1}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2r+1}$	$\frac{1}{2}$
a_2	$\frac{1}{4r^2+8r+3}$	$\frac{1}{8}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{8r^3 + 36r^2 + 46r + 15}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = \frac{1}{48}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2r+1}$	$\frac{1}{2}$
a_2	$\frac{1}{4r^2+8r+3}$	$\frac{1}{8}$
a_3	$\frac{1}{8r^3+36r^2+46r+15}$	$\frac{1}{48}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{16r^4 + 128r^3 + 344r^2 + 352r + 105}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{1}{384}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2r+1}$	$\frac{1}{2}$
a_2	$\frac{1}{4r^2+8r+3}$	$\frac{1}{8}$
a_3	$\frac{1}{8r^3+36r^2+46r+15}$	$\frac{1}{48}$
a_4	$\frac{1}{16r^4+128r^3+344r^2+352r+105}$	$\frac{1}{384}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{32r^5 + 400r^4 + 1840r^3 + 3800r^2 + 3378r + 945}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = \frac{1}{3840}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2r+1}$	$\frac{1}{2}$
a_2	$\frac{1}{4r^2+8r+3}$	$\frac{1}{8}$
a_3	$\frac{1}{8r^3+36r^2+46r+15}$	$\frac{1}{48}$
a_4	$\frac{1}{16r^4+128r^3+344r^2+352r+105}$	$\frac{1}{384}$
a_5	$\frac{1}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$	$\frac{1}{3840}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{1}{64r^6 + 1152r^5 + 8080r^4 + 27840r^3 + 48556r^2 + 39048r + 10395}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_6 = \frac{1}{46080}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2r+1}$	$\frac{1}{2}$
a_2	$\frac{1}{4r^2+8r+3}$	$\frac{1}{8}$
a_3	$\frac{1}{8r^3+36r^2+46r+15}$	$\frac{1}{48}$
a_4	$\frac{1}{16r^4+128r^3+344r^2+352r+105}$	$\frac{1}{384}$
a_5	$\frac{1}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$	$\frac{1}{3840}$
a_6	$\frac{1}{64r^6+1152r^5+8080r^4+27840r^3+48556r^2+39048r+10395}$	$\frac{1}{46080}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{1}{128r^7 + 3136r^6 + 31136r^5 + 160720r^4 + 459032r^3 + 709324r^2 + 528414r + 135135}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_7 = \frac{1}{645120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2r+1}$	$\frac{1}{2}$
a_2	$\frac{1}{4r^2+8r+3}$	$\frac{1}{8}$
a_3	$\frac{1}{8r^3+36r^2+46r+15}$	$\frac{1}{48}$
a_4	$\frac{1}{16r^4+128r^3+344r^2+352r+105}$	$\frac{1}{384}$
a_5	$\frac{1}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$	$\frac{1}{3840}$
a_6	$\frac{1}{64r^6+1152r^5+8080r^4+27840r^3+48556r^2+39048r+10395}$	$\frac{1}{46080}$
a_7	$\frac{1}{128r^7+3136r^6+31136r^5+160720r^4+459032r^3+709324r^2+528414r+135135}$	$\frac{1}{645120}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= \sqrt{x}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \dots\right) \\
&= \sqrt{x}\left(1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384} + \frac{x^5}{3840} + \frac{x^6}{46080} + \frac{x^7}{645120} + O(x^8)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}
a_N &= a_1 \\
&= \frac{1}{2r+1}
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{r\to r_2} \frac{1}{2r+1} &= \lim_{r\to -\frac{1}{2}} \frac{1}{2r+1} \\
&= \textit{undefined}
\end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}
\frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\
\frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $4x^2 y'' + (-2x^2 + 4x) y' + (-3x - 1) y = 0$ gives

$$\begin{aligned}
&4x^2 \left(C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2}\right. \\
&\left. + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&+ (-2x^2 + 4x) \left(C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) \\
&+ (-3x - 1) \left(C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0
\end{aligned}$$

Which can be written as

$$\begin{aligned}
&\left(\left(4x^2 y_1''(x) + \left(-2x^2+4x\right) y_1'(x) + \left(-3x-1\right) y_1(x)\right)\ln\left(x\right)\right.\\
&\left.+ 4x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + \frac{\left(-2x^2+4x\right) y_1(x)}{x}\right) C\\
&+ 4x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\
&+ \left(-2x^2+4x\right)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + \left(-3x-1\right)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$4x^2 y_1''(x) + \left(-2x^2+4x\right) y_1'(x) + \left(-3x-1\right) y_1(x) = 0$$

Eq (7) simplifes to

$$\begin{aligned}
&\left(4x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + \frac{\left(-2x^2+4x\right) y_1(x)}{x}\right) C\\
&+ 4x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\
&+ \left(-2x^2+4x\right)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + \left(-3x-1\right)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\begin{aligned}
&\left(8\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n(n+r_1)\right) x - 2\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right) x\right) C\\
&+ 4\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)\left(-1+n+r_2\right)\right) x^2\\
&+ \left(-2x^2+4x\right)\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) + \left(-3x-1\right)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{9}$$

Since $r_1 = \frac{1}{2}$ and $r_2 = -\frac{1}{2}$ then the above becomes

$$
\begin{aligned}
&\left(8\left(\sum_{n=0}^{\infty} x^{n-\frac{1}{2}} a_n \left(n+\frac{1}{2}\right)\right) x - 2\left(\sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}\right) x\right) C \\
&+ 4\left(\sum_{n=0}^{\infty} x^{-\frac{5}{2}+n} b_n \left(n-\frac{1}{2}\right)\left(-\frac{3}{2}+n\right)\right) x^2 \\
&+ \left(-2x^2+4x\right)\left(\sum_{n=0}^{\infty} x^{-\frac{3}{2}+n} b_n \left(n-\frac{1}{2}\right)\right) + (-3x-1)\left(\sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}\right) = 0
\end{aligned} \tag{10}
$$

Expanding $-\sqrt{x}$ as Taylor series around $x = 0$ and keeping only the first 8 terms gives

$$
\begin{aligned}
-\sqrt{x} &= -\sqrt{x} + \dots \\
&= -\sqrt{x}
\end{aligned}
$$

Expanding $\frac{2}{\sqrt{x}}$ as Taylor series around $x = 0$ and keeping only the first 8 terms gives

$$
\begin{aligned}
\frac{2}{\sqrt{x}} &= \frac{2}{\sqrt{x}} + \dots \\
&= \frac{2}{\sqrt{x}}
\end{aligned}
$$

Expanding $-3\sqrt{x}$ as Taylor series around $x = 0$ and keeping only the first 8 terms gives

$$
\begin{aligned}
-3\sqrt{x} &= -3\sqrt{x} + \dots \\
&= -3\sqrt{x}
\end{aligned}
$$

Expanding $-\frac{1}{\sqrt{x}}$ as Taylor series around $x = 0$ and keeping only the first 8 terms gives

$$
\begin{aligned}
-\frac{1}{\sqrt{x}} &= -\frac{1}{\sqrt{x}} + \dots \\
&= -\frac{1}{\sqrt{x}}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} (4+8n)\, C a_n x^{n+\frac{1}{2}}\right) + \sum_{n=0}^{\infty} \left(-2C\, x^{\frac{3}{2}+n} a_n\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n-\frac{1}{2}} b_n \left(4n^2-8n+3\right)\right) + \left(\sum_{n=0}^{\infty} x^{n+\frac{1}{2}} b_n (-2n+1)\right) \\
&+ \left(\sum_{n=0}^{\infty} (4n-2)\, b_n x^{n-\frac{1}{2}}\right) + \sum_{n=0}^{\infty} \left(-3x^{n+\frac{1}{2}} b_n\right) + \sum_{n=0}^{\infty} \left(-b_n x^{n-\frac{1}{2}}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n-\frac{1}{2}$ in each summation term. Going over each summation term above with power of x in it which is not already $x^{n-\frac{1}{2}}$ and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} (4+8n)\, C a_n x^{n+\frac{1}{2}} &= \sum_{n=1}^{\infty} C a_{n-1}(-4+8n)\, x^{n-\frac{1}{2}} \\
\sum_{n=0}^{\infty} \left(-2C\, x^{\frac{3}{2}+n} a_n\right) &= \sum_{n=2}^{\infty} \left(-2C a_{n-2} x^{n-\frac{1}{2}}\right) \\
\sum_{n=0}^{\infty} x^{n+\frac{1}{2}} b_n(-2n+1) &= \sum_{n=1}^{\infty} b_{n-1}(-2n+3)\, x^{n-\frac{1}{2}} \\
\sum_{n=0}^{\infty} \left(-3x^{n+\frac{1}{2}} b_n\right) &= \sum_{n=1}^{\infty} \left(-3b_{n-1} x^{n-\frac{1}{2}}\right)
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-\frac{1}{2}$.

$$
\begin{aligned}
&\left(\sum_{n=1}^{\infty} C a_{n-1}(-4+8n)\, x^{n-\frac{1}{2}}\right) + \sum_{n=2}^{\infty} \left(-2C a_{n-2} x^{n-\frac{1}{2}}\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n-\frac{1}{2}} b_n \left(4n^2-8n+3\right)\right) + \left(\sum_{n=1}^{\infty} b_{n-1}(-2n+3)\, x^{n-\frac{1}{2}}\right) \\
&+ \left(\sum_{n=0}^{\infty} (4n-2)\, b_n x^{n-\frac{1}{2}}\right) + \sum_{n=1}^{\infty} \left(-3b_{n-1} x^{n-\frac{1}{2}}\right) + \sum_{n=0}^{\infty} \left(-b_n x^{n-\frac{1}{2}}\right) = 0
\end{aligned} \tag{2B}
$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=N$, where $N=1$ which is the difference between the two roots, we are free to choose $b_1=0$. Hence for $n=1$, Eq (2B) gives

$$4C-2=0$$

Which is solved for C. Solving for C gives

$$C=\frac{1}{2}$$

For $n=2$, Eq (2B) gives

$$(-2a_0+12a_1)\, C-4b_1+8b_2=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$2 + 8b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = -\frac{1}{4}$$

For $n = 3$, Eq (2B) gives

$$(-2a_1 + 20a_2)\,C - 6b_2 + 24b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{9}{4} + 24b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = -\frac{3}{32}$$

For $n = 4$, Eq (2B) gives

$$(-2a_2 + 28a_3)\,C - 8b_3 + 48b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{11}{12} + 48b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{11}{576}$$

For $n = 5$, Eq (2B) gives

$$(-2a_3 + 36a_4)\,C - 10b_4 + 80b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{125}{576} + 80b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{25}{9216}$$

For $n = 6$, Eq (2B) gives

$$(-2a_4 + 44a_5)\,C - 12b_5 + 120b_6 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{137}{3840}+120b_6=0$$

Solving the above for b_6 gives

$$b_6=-\frac{137}{460800}$$

For $n=7$, Eq (2B) gives

$$(-2a_5+52a_6)\,C-14b_6+168b_7=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{343}{76800}+168b_7=0$$

Solving the above for b_7 gives

$$b_7=-\frac{49}{1843200}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x)=Cy_1(x)\ln(x)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)$$

Using the above value found for $C=\frac{1}{2}$ and all b_n, then the second solution becomes

$$y_2(x)=\frac{1}{2}\left(\sqrt{x}\left(1+\frac{x}{2}+\frac{x^2}{8}+\frac{x^3}{48}+\frac{x^4}{384}+\frac{x^5}{3840}+\frac{x^6}{46080}+\frac{x^7}{645120}+O(x^8)\right)\right)\ln(x)+\frac{1-\frac{x^2}{4}-\frac{3x^3}{32}-\frac{11x^4}{576}-\frac{25x^5}{9216}-\frac{137x^6}{460800}-\frac{49x^7}{1843200}+O(x^8)}{\sqrt{x}}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x)&=c_1y_1(x)+c_2y_2(x)\\&=c_1\sqrt{x}\left(1+\frac{x}{2}+\frac{x^2}{8}+\frac{x^3}{48}+\frac{x^4}{384}+\frac{x^5}{3840}+\frac{x^6}{46080}+\frac{x^7}{645120}+O(x^8)\right)\\&\quad+c_2\left(\frac{1}{2}\left(\sqrt{x}\left(1+\frac{x}{2}+\frac{x^2}{8}+\frac{x^3}{48}+\frac{x^4}{384}+\frac{x^5}{3840}+\frac{x^6}{46080}+\frac{x^7}{645120}+O(x^8)\right)\right)\ln(x)\right.\\&\qquad\left.+\frac{1-\frac{x^2}{4}-\frac{3x^3}{32}-\frac{11x^4}{576}-\frac{25x^5}{9216}-\frac{137x^6}{460800}-\frac{49x^7}{1843200}+O(x^8)}{\sqrt{x}}\right)\end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\sqrt{x}\left(1+\frac{x}{2}+\frac{x^2}{8}+\frac{x^3}{48}+\frac{x^4}{384}+\frac{x^5}{3840}+\frac{x^6}{46080}+\frac{x^7}{645120}+O(x^8)\right) \\
&\quad + c_2\left(\frac{\sqrt{x}\left(1+\frac{x}{2}+\frac{x^2}{8}+\frac{x^3}{48}+\frac{x^4}{384}+\frac{x^5}{3840}+\frac{x^6}{46080}+\frac{x^7}{645120}+O(x^8)\right)\ln(x)}{2}\right. \\
&\quad \left. + \frac{1-\frac{x^2}{4}-\frac{3x^3}{32}-\frac{11x^4}{576}-\frac{25x^5}{9216}-\frac{137x^6}{460800}-\frac{49x^7}{1843200}+O(x^8)}{\sqrt{x}}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\sqrt{x}\left(1+\frac{x}{2}+\frac{x^2}{8}+\frac{x^3}{48}+\frac{x^4}{384}+\frac{x^5}{3840}+\frac{x^6}{46080}+\frac{x^7}{645120}+O(x^8)\right) \\
&\quad + c_2\left(\frac{\sqrt{x}\left(1+\frac{x}{2}+\frac{x^2}{8}+\frac{x^3}{48}+\frac{x^4}{384}+\frac{x^5}{3840}+\frac{x^6}{46080}+\frac{x^7}{645120}+O(x^8)\right)\ln(x)}{2}\right. \\
&\quad \left. + \frac{1-\frac{x^2}{4}-\frac{3x^3}{32}-\frac{11x^4}{576}-\frac{25x^5}{9216}-\frac{137x^6}{460800}-\frac{49x^7}{1843200}+O(x^8)}{\sqrt{x}}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1\sqrt{x}\left(1+\frac{x}{2}+\frac{x^2}{8}+\frac{x^3}{48}+\frac{x^4}{384}+\frac{x^5}{3840}+\frac{x^6}{46080}+\frac{x^7}{645120}+O(x^8)\right) \\
&\quad + c_2\left(\frac{\sqrt{x}\left(1+\frac{x}{2}+\frac{x^2}{8}+\frac{x^3}{48}+\frac{x^4}{384}+\frac{x^5}{3840}+\frac{x^6}{46080}+\frac{x^7}{645120}+O(x^8)\right)\ln(x)}{2}\right. \\
&\quad \left. + \frac{1-\frac{x^2}{4}-\frac{3x^3}{32}-\frac{11x^4}{576}-\frac{25x^5}{9216}-\frac{137x^6}{460800}-\frac{49x^7}{1843200}+O(x^8)}{\sqrt{x}}\right)
\end{aligned}
$$

Verified OK.

7.4.1 Maple step by step solution

Let's solve

$4x^2y'' + (-2x^2 + 4x)\, y' + (-3x - 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(3x+1)y}{4x^2} + \frac{(x-2)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x-2)y'}{2x} - \frac{(3x+1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{x-2}{2x}, P_3(x) = -\frac{3x+1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{1}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' - 2x(x-2)\, y' + (-3x - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- > k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)\,(-1+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-1) - a_{k-1}(2k+2r+1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+2r)\,(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$(2k+2r+1)\,(2a_k k + 2a_k r - a_k - a_{k-1}) = 0$$

- Shift index using $k- > k+1$

$$(2k+2r+3)\,(2a_{k+1}(k+1) + 2a_{k+1} r - a_{k+1} - a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{2k+2r+1}$$

- Recursion relation for $r = -\frac{1}{2}$

$$a_{k+1} = \frac{a_k}{2k}$$

- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = \frac{a_k}{2k}\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = \frac{a_k}{2k+2}$$

- Solution for $r = \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k}{2k+2}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = \frac{a_k}{2k}, b_{k+1} = \frac{b_k}{2k+2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 79

```
Order:=8;
dsolve(4*x^2*diff(y(x),x$2)+2*x*(2-x)*diff(y(x),x)-(1+3*x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x\left(1 + \frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{48}x^3 + \frac{1}{384}x^4 + \frac{1}{3840}x^5 + \frac{1}{46080}x^6 + \frac{1}{645120}x^7 + \mathrm{O}\left(x^8\right)\right) + c_2\left(\ln\left(x\right)\left(\frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{16}x^3\right.\right.}{}$$

✓ Solution by Mathematica

Time used: 0.094 (sec). Leaf size: 141

```
AsymptoticDSolveValue[4*x^2*y''[x]+2*x*(2-x)*y'[x]-(1+3*x)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2 \left(\frac{x^{13/2}}{46080} + \frac{x^{11/2}}{3840} + \frac{x^{9/2}}{384} + \frac{x^{7/2}}{48} + \frac{x^{5/2}}{8} + \frac{x^{3/2}}{2} + \sqrt{x} \right) + c_1 \left(\frac{\sqrt{x}(x^5 + 10x^4 + 80x^3 + 480x^2 + 1920x + 3840) \log(x)}{7680} - \frac{137x^6 + 1250x^5 + 8800x^4 + 4320}{460800\sqrt{}} \right.$$

7.5 problem 5

Internal problem ID [6985]
Internal file name [`OUTPUT/6228_Friday_August_12_2022_11_06_14_PM_49739451/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.9 Indicial Equation with Difference of Roots a Positive Integer: Logarithmic Case. Exercises page 384
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - x(x+6)\,y' + 10y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(-x^2 - 6x\right) y' + 10y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x+6}{x}$$
$$q(x) = \frac{10}{x^2}$$

Table 160: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x+6}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{10}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + (-x^2 - 6x)\, y' + 10y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} \right) + (-x^2 - 6x) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \right) + 10 \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-6x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 10 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-6x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 10 a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 6x^{n+r} a_n (n+r) + 10 a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - 6x^r a_0 r + 10 a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 6x^r r + 10x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r-2)(r-5)\, x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r-2)(r-5) = 0$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= 5 \\
r_2 &= 2
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
(r-2)(r-5)x^r = 0
$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= x^5\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x)\ln(x) + x^2\left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+5} \\
y_2(x) &= C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+2}\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) - 6a_n(n+r) + 10a_n = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = \frac{a_{n-1}(n+r-1)}{n^2+2nr+r^2-7n-7r+10} \tag{4}
$$

Which for the root $r = 5$ becomes

$$a_n = \frac{a_{n-1}(n+4)}{n(n+3)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 5$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{r}{r^2 - 5r + 4}$$

Which for the root $r = 5$ becomes

$$a_1 = \frac{5}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{r^2-5r+4}$	$\frac{5}{4}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1+r}{r^3 - 8r^2 + 19r - 12}$$

Which for the root $r = 5$ becomes

$$a_2 = \frac{3}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{r^2-5r+4}$	$\frac{5}{4}$
a_2	$\frac{1+r}{r^3-8r^2+19r-12}$	$\frac{3}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{2 + r}{r^4 - 10r^3 + 35r^2 - 50r + 24}$$

Which for the root $r = 5$ becomes

$$a_3 = \frac{7}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{r^2-5r+4}$	$\frac{5}{4}$
a_2	$\frac{1+r}{r^3-8r^2+19r-12}$	$\frac{3}{4}$
a_3	$\frac{2+r}{r^4-10r^3+35r^2-50r+24}$	$\frac{7}{24}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{3 + r}{(-1 + r)^2 (r - 2) (r - 4) (r - 3)}$$

Which for the root $r = 5$ becomes

$$a_4 = \frac{1}{12}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{r^2-5r+4}$	$\frac{5}{4}$
a_2	$\frac{1+r}{r^3-8r^2+19r-12}$	$\frac{3}{4}$
a_3	$\frac{2+r}{r^4-10r^3+35r^2-50r+24}$	$\frac{7}{24}$
a_4	$\frac{3+r}{(-1+r)^2(r-2)(r-4)(r-3)}$	$\frac{1}{12}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{4 + r}{r (-1 + r)^2 (r - 2) (r - 4) (r - 3)}$$

Which for the root $r = 5$ becomes

$$a_5 = \frac{3}{160}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{r^2-5r+4}$	$\frac{5}{4}$
a_2	$\frac{1+r}{r^3-8r^2+19r-12}$	$\frac{3}{4}$
a_3	$\frac{2+r}{r^4-10r^3+35r^2-50r+24}$	$\frac{7}{24}$
a_4	$\frac{3+r}{(-1+r)^2(r-2)(r-4)(r-3)}$	$\frac{1}{12}$
a_5	$\frac{4+r}{r(-1+r)^2(r-2)(r-4)(r-3)}$	$\frac{3}{160}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{5+r}{(1+r)\,r\,(-1+r)^2\,(r-2)\,(r-4)\,(r-3)}$$

Which for the root $r = 5$ becomes

$$a_6 = \frac{1}{288}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{r^2-5r+4}$	$\frac{5}{4}$
a_2	$\frac{1+r}{r^3-8r^2+19r-12}$	$\frac{3}{4}$
a_3	$\frac{2+r}{r^4-10r^3+35r^2-50r+24}$	$\frac{7}{24}$
a_4	$\frac{3+r}{(-1+r)^2(r-2)(r-4)(r-3)}$	$\frac{1}{12}$
a_5	$\frac{4+r}{r(-1+r)^2(r-2)(r-4)(r-3)}$	$\frac{3}{160}$
a_6	$\frac{5+r}{(1+r)r(-1+r)^2(r-2)(r-4)(r-3)}$	$\frac{1}{288}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{6+r}{(1+r)\,r\,(-1+r)^2\,(r-4)\,(r-3)\,(r^2-4)}$$

Which for the root $r = 5$ becomes

$$a_7 = \frac{11}{20160}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{r^2-5r+4}$	$\frac{5}{4}$
a_2	$\frac{1+r}{r^3-8r^2+19r-12}$	$\frac{3}{4}$
a_3	$\frac{2+r}{r^4-10r^3+35r^2-50r+24}$	$\frac{7}{24}$
a_4	$\frac{3+r}{(-1+r)^2(r-2)(r-4)(r-3)}$	$\frac{1}{12}$
a_5	$\frac{4+r}{r(-1+r)^2(r-2)(r-4)(r-3)}$	$\frac{3}{160}$
a_6	$\frac{5+r}{(1+r)r(-1+r)^2(r-2)(r-4)(r-3)}$	$\frac{1}{288}$
a_7	$\frac{6+r}{(1+r)r(-1+r)^2(r-4)(r-3)(r^2-4)}$	$\frac{11}{20160}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}y_1(x) &= x^5 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\ &= x^5 \left(1 + \frac{5x}{4} + \frac{3x^2}{4} + \frac{7x^3}{24} + \frac{x^4}{12} + \frac{3x^5}{160} + \frac{x^6}{288} + \frac{11x^7}{20160} + O(x^8)\right)\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}a_N &= a_3 \\ &= \frac{2+r}{r^4 - 10r^3 + 35r^2 - 50r + 24}\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{r \to r_2} \frac{2+r}{r^4 - 10r^3 + 35r^2 - 50r + 24} &= \lim_{r \to 2} \frac{2+r}{r^4 - 10r^3 + 35r^2 - 50r + 24} \\ &= \textit{undefined}\end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\frac{d}{dx}y_2(x) = Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)$$

$$= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right)$$

$$\frac{d^2}{dx^2}y_2(x) = Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)$$

$$= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)$$

Substituting these back into the given ode $x^2y'' + (-x^2-6x)\,y' + 10y = 0$ gives

$$x^2\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) + (-x^2-6x)\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) + 10Cy_1(x)\ln(x) + 10\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0$$

Which can be written as

$$\left(\left(\left(x^{2} y_{1}''(x)+\left(-x^{2}-6 x\right) y_{1}'(x)+10 y_{1}(x)\right) \ln (x)+x^{2}\left(\frac{2 y_{1}'(x)}{x}-\frac{y_{1}(x)}{x^{2}}\right)\right.\right.$$
$$\left.+\frac{\left(-x^{2}-6 x\right) y_{1}(x)}{x}\right) C+x^{2}\left(\sum_{n=0}^{\infty}\left(\frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)^{2}}{x^{2}}-\frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)}{x^{2}}\right)\right) \tag{7}$$
$$+\left(-x^{2}-6 x\right)\left(\sum_{n=0}^{\infty} \frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)}{x}\right)+10\left(\sum_{n=0}^{\infty} b_{n} x^{n+r_{2}}\right)=0$$

But since $y_1(x)$ is a solution to the ode, then

$$x^{2} y_{1}''(x)+\left(-x^{2}-6 x\right) y_{1}'(x)+10 y_{1}(x)=0$$

Eq (7) simplifes to

$$\left(x^{2}\left(\frac{2 y_{1}'(x)}{x}-\frac{y_{1}(x)}{x^{2}}\right)+\frac{\left(-x^{2}-6 x\right) y_{1}(x)}{x}\right) C$$
$$+x^{2}\left(\sum_{n=0}^{\infty}\left(\frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)^{2}}{x^{2}}-\frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)}{x^{2}}\right)\right) \tag{8}$$
$$+\left(-x^{2}-6 x\right)\left(\sum_{n=0}^{\infty} \frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)}{x}\right)+10\left(\sum_{n=0}^{\infty} b_{n} x^{n+r_{2}}\right)=0$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_{1}} a_{n}\left(n+r_{1}\right)\right) x-(7+x)\left(\sum_{n=0}^{\infty} a_{n} x^{n+r_{1}}\right)\right) C$$
$$+\left(\sum_{n=0}^{\infty} x^{-2+n+r_{2}} b_{n}\left(n+r_{2}\right)\left(-1+n+r_{2}\right)\right) x^{2} \tag{9}$$
$$+\left(-x^{2}-6 x\right)\left(\sum_{n=0}^{\infty} x^{-1+n+r_{2}} b_{n}\left(n+r_{2}\right)\right)+10\left(\sum_{n=0}^{\infty} b_{n} x^{n+r_{2}}\right)=0$$

Since $r_1 = 5$ and $r_2 = 2$ then the above becomes

$$\begin{aligned}
&\left(2\left(\sum_{n=0}^{\infty} x^{n+4} a_n(n+5)\right) x-(7+x)\left(\sum_{n=0}^{\infty} a_n x^{n+5}\right)\right) C \\
&+\left(\sum_{n=0}^{\infty} x^n b_n(n+2)(1+n)\right) x^2 \\
&+\left(-x^2-6x\right)\left(\sum_{n=0}^{\infty} x^{1+n} b_n(n+2)\right)+10\left(\sum_{n=0}^{\infty} b_n x^{n+2}\right)=0
\end{aligned} \tag{10}$$

Which simplifies to

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{n+5} a_n(n+5)\right)+\sum_{n=0}^{\infty}\left(-7C\, x^{n+5} a_n\right)+\sum_{n=0}^{\infty}\left(-C\, x^{n+6} a_n\right) \\
&+\left(\sum_{n=0}^{\infty} x^{n+2} b_n\left(n^2+3n+2\right)\right)+\sum_{n=0}^{\infty}\left(-x^{n+3} b_n(n+2)\right) \\
&+\sum_{n=0}^{\infty}\left(-6x^{n+2} b_n(n+2)\right)+\left(\sum_{n=0}^{\infty} 10 b_n x^{n+2}\right)=0
\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n+2$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+2} and adjusting the power and the corresponding index gives

$$\begin{aligned}
\sum_{n=0}^{\infty} 2C\, x^{n+5} a_n(n+5) &= \sum_{n=3}^{\infty} 2C a_{n-3}(n+2)\, x^{n+2} \\
\sum_{n=0}^{\infty}\left(-7C\, x^{n+5} a_n\right) &= \sum_{n=3}^{\infty}\left(-7C a_{n-3} x^{n+2}\right) \\
\sum_{n=0}^{\infty}\left(-C\, x^{n+6} a_n\right) &= \sum_{n=4}^{\infty}\left(-C a_{n-4} x^{n+2}\right) \\
\sum_{n=0}^{\infty}\left(-x^{n+3} b_n(n+2)\right) &= \sum_{n=1}^{\infty}\left(-b_{n-1}(1+n)\, x^{n+2}\right)
\end{aligned}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+2$.

$$\left(\sum_{n=3}^{\infty} 2C a_{n-3}(n+2)\, x^{n+2}\right)+\sum_{n=3}^{\infty}\left(-7C a_{n-3} x^{n+2}\right)+\sum_{n=4}^{\infty}\left(-C a_{n-4} x^{n+2}\right) + \left(\sum_{n=0}^{\infty} x^{n+2} b_n\left(n^2+3n+2\right)\right)+\sum_{n=1}^{\infty}\left(-b_{n-1}(1+n)\, x^{n+2}\right) + \sum_{n=0}^{\infty}\left(-6x^{n+2} b_n(n+2)\right)+\left(\sum_{n=0}^{\infty} 10 b_n x^{n+2}\right)=0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$-2b_1-2b_0=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-2b_1-2=0$$

Solving the above for b_1 gives

$$b_1=-1$$

For $n=2$, Eq (2B) gives

$$-2b_2-3b_1=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-2b_2+3=0$$

Solving the above for b_2 gives

$$b_2=\frac{3}{2}$$

For $n=N$, where $N=3$ which is the difference between the two roots, we are free to choose $b_3=0$. Hence for $n=3$, Eq (2B) gives

$$3C-6=0$$

Which is solved for C. Solving for C gives

$$C=2$$

For $n=4$, Eq (2B) gives

$$(-a_0+5a_1)\, C-5b_3+4b_4=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{21}{2} + 4b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{21}{8}$$

For $n = 5$, Eq (2B) gives

$$(-a_1 + 7a_2)\, C - 6b_4 + 10b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{95}{4} + 10b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{19}{8}$$

For $n = 6$, Eq (2B) gives

$$(-a_2 + 9a_3)\, C - 7b_5 + 18b_6 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{163}{8} + 18b_6 = 0$$

Solving the above for b_6 gives

$$b_6 = -\frac{163}{144}$$

For $n = 7$, Eq (2B) gives

$$(-a_3 + 11a_4)\, C - 8b_6 + 28b_7 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{371}{36} + 28b_7 = 0$$

Solving the above for b_7 gives

$$b_7 = -\frac{53}{144}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = 2$ and all b_n, then the second solution becomes

$$y_2(x) = 2\left(x^5\left(1+\frac{5x}{4}+\frac{3x^2}{4}+\frac{7x^3}{24}+\frac{x^4}{12}+\frac{3x^5}{160}+\frac{x^6}{288}+\frac{11x^7}{20160}+O(x^8)\right)\right)\ln(x) + x^2\left(1-x+\frac{3x^2}{2}-\frac{21x^4}{8}-\frac{19x^5}{8}-\frac{163x^6}{144}-\frac{53x^7}{144}+O(x^8)\right)$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^5\left(1+\frac{5x}{4}+\frac{3x^2}{4}+\frac{7x^3}{24}+\frac{x^4}{12}+\frac{3x^5}{160}+\frac{x^6}{288}+\frac{11x^7}{20160}+O(x^8)\right) \\ &\quad + c_2\left(2\left(x^5\left(1+\frac{5x}{4}+\frac{3x^2}{4}+\frac{7x^3}{24}+\frac{x^4}{12}+\frac{3x^5}{160}+\frac{x^6}{288}+\frac{11x^7}{20160}+O(x^8)\right)\right)\ln(x) \right. \\ &\qquad \left. + x^2\left(1-x+\frac{3x^2}{2}-\frac{21x^4}{8}-\frac{19x^5}{8}-\frac{163x^6}{144}-\frac{53x^7}{144}+O(x^8)\right)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^5\left(1+\frac{5x}{4}+\frac{3x^2}{4}+\frac{7x^3}{24}+\frac{x^4}{12}+\frac{3x^5}{160}+\frac{x^6}{288}+\frac{11x^7}{20160}+O(x^8)\right) \\ &\quad + c_2\left(2x^5\left(1+\frac{5x}{4}+\frac{3x^2}{4}+\frac{7x^3}{24}+\frac{x^4}{12}+\frac{3x^5}{160}+\frac{x^6}{288}+\frac{11x^7}{20160}+O(x^8)\right)\ln(x) \right. \\ &\qquad \left. + x^2\left(1-x+\frac{3x^2}{2}-\frac{21x^4}{8}-\frac{19x^5}{8}-\frac{163x^6}{144}-\frac{53x^7}{144}+O(x^8)\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= c_1 x^5\left(1+\frac{5x}{4}+\frac{3x^2}{4}+\frac{7x^3}{24}+\frac{x^4}{12}+\frac{3x^5}{160}+\frac{x^6}{288}+\frac{11x^7}{20160}+O(x^8)\right) \\ &\quad + c_2\left(2x^5\left(1+\frac{5x}{4}+\frac{3x^2}{4}+\frac{7x^3}{24}+\frac{x^4}{12}+\frac{3x^5}{160}+\frac{x^6}{288}+\frac{11x^7}{20160}+O(x^8)\right)\ln(x) \right. \\ &\qquad \left. + x^2\left(1-x+\frac{3x^2}{2}-\frac{21x^4}{8}-\frac{19x^5}{8}-\frac{163x^6}{144}-\frac{53x^7}{144}+O(x^8)\right)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1 x^5 \left(1+\frac{5x}{4}+\frac{3x^2}{4}+\frac{7x^3}{24}+\frac{x^4}{12}+\frac{3x^5}{160}+\frac{x^6}{288}+\frac{11x^7}{20160}+O(x^8)\right) + c_2\left(2x^5\left(1+\frac{5x}{4}+\frac{3x^2}{4}+\frac{7x^3}{24}+\frac{x^4}{12}+\frac{3x^5}{160}+\frac{x^6}{288}+\frac{11x^7}{20160}+O(x^8)\right)\ln(x) + x^2\left(1-x+\frac{3x^2}{2}-\frac{21x^4}{8}-\frac{19x^5}{8}-\frac{163x^6}{144}-\frac{53x^7}{144}+O(x^8)\right)\right)$$

Verified OK.

7.5.1 Maple step by step solution

Let's solve

$x^2y''+(-x^2-6x)\,y'+10y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{10y}{x^2}+\frac{(x+6)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(x+6)y'}{x}+\frac{10y}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-\frac{x+6}{x}, P_3(x)=\frac{10}{x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=-6$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=10$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

$x^2 y'' - x(x+6)\, y' + 10y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  - Shift index using $k- > k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)\,(-5+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-2)\,(k+r-5) - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)\,(-5+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{2, 5\}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-2)\,(k+r-5) - a_{k-1}(k+r-1) = 0$

- Shift index using $k- > k+1$

  $a_{k+1}(k+r-1)\,(k-4+r) - a_k(k+r) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r)}{(k+r-1)(k-4+r)}$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k(k+2)}{(k+1)(k-2)}$

- Series not valid for $r = 2$, division by 0 in the recursion relation at $k = 2$

  $a_{k+1} = \frac{a_k(k+2)}{(k+1)(k-2)}$

- Recursion relation for $r = 5$

  $a_{k+1} = \frac{a_k(k+5)}{(k+4)(k+1)}$

- Solution for $r = 5$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+5}, a_{k+1} = \frac{a_k(k+5)}{(k+4)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 77

```
Order:=8;
dsolve(x^2*diff(y(x),x$2)-x*(6+x)*diff(y(x),x)+10*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = x^2\left(c_1x^3\left(1+\frac{5}{4}x+\frac{3}{4}x^2+\frac{7}{24}x^3+\frac{1}{12}x^4+\frac{3}{160}x^5+\frac{1}{288}x^6+\frac{11}{20160}x^7+O\left(x^8\right)\right)\right.$$
$$+c_2\left(\ln\left(x\right)\left(24x^3+30x^4+18x^5+7x^6+2x^7+O\left(x^8\right)\right)\right.$$
$$\left.\left.+\left(12-12x+18x^2+26x^3+x^4-9x^5-6x^6-\frac{9}{4}x^7+O\left(x^8\right)\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.053 (sec). Leaf size: 118

```
AsymptoticDSolveValue[x^2*y''[x]-x*(6+x)*y'[x]+10*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(\frac{1}{12}x^5(7x^3+18x^2+30x+24)\log(x) - \frac{1}{36}x^2(25x^6+45x^5+27x^4-54x^3-54x^2+36x-36)\right) + c_2\left(\frac{x^{11}}{288}+\frac{3x^{10}}{160}+\frac{x^9}{12}+\frac{7x^8}{24}+\frac{3x^7}{4}+\frac{5x^6}{4}+x^5\right)$$

7.6 problem 6

Internal problem ID [6986]
Internal file name [`OUTPUT/6229_Friday_August_12_2022_11_06_16_PM_1620668/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.9 Indicial Equation with Difference of Roots a Positive Integer: Logarithmic Case. Exercises page 384
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + (3 + 2x) y' + 8y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + (3 + 2x) y' + 8y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{3 + 2x}{x}$$
$$q(x) = \frac{8}{x}$$

Table 162: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3+2x}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{8}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + (3 + 2x)\, y' + 8y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x\left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}\right) + (3 + 2x)\left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + 8\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 8a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r) = \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} 8a_n x^{n+r} = \sum_{n=1}^{\infty} 8a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} 8a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + 3(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + 3r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + 3r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(2+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(2+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = -2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(2+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 2a_{n-1}(n+r-1) + 3a_n(n+r) + 8a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2a_{n-1}(n+r+3)}{n^2+2nr+r^2+2n+2r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{2a_{n-1}(n+3)}{n\,(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-8-2r}{r^2+4r+3}$$

Which for the root $r = 0$ becomes

$$a_1 = -\frac{8}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-8-2r}{r^2+4r+3}$	$-\frac{8}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{20+4r}{r^3+6r^2+11r+6}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{10}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-8-2r}{r^2+4r+3}$	$-\frac{8}{3}$
a_2	$\frac{20+4r}{r^3+6r^2+11r+6}$	$\frac{10}{3}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-48 - 8r}{(r+3)^2 (2+r)(r+1)}$$

Which for the root $r = 0$ becomes

$$a_3 = -\frac{8}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-8-2r}{r^2+4r+3}$	$-\frac{8}{3}$
a_2	$\frac{20+4r}{r^3+6r^2+11r+6}$	$\frac{10}{3}$
a_3	$\frac{-48-8r}{(r+3)^2(2+r)(r+1)}$	$-\frac{8}{3}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{112 + 16r}{(4+r)(r+3)^2 (2+r)(r+1)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{14}{9}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-8-2r}{r^2+4r+3}$	$-\frac{8}{3}$
a_2	$\frac{20+4r}{r^3+6r^2+11r+6}$	$\frac{10}{3}$
a_3	$\frac{-48-8r}{(r+3)^2(2+r)(r+1)}$	$-\frac{8}{3}$
a_4	$\frac{112+16r}{(4+r)(r+3)^2(2+r)(r+1)}$	$\frac{14}{9}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-256 - 32r}{(5+r)(4+r)(r+3)^2 (2+r)(r+1)}$$

Which for the root $r = 0$ becomes

$$a_5 = -\frac{32}{45}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-8-2r}{r^2+4r+3}$	$-\frac{8}{3}$
a_2	$\frac{20+4r}{r^3+6r^2+11r+6}$	$\frac{10}{3}$
a_3	$\frac{-48-8r}{(r+3)^2(2+r)(r+1)}$	$-\frac{8}{3}$
a_4	$\frac{112+16r}{(4+r)(r+3)^2(2+r)(r+1)}$	$\frac{14}{9}$
a_5	$\frac{-256-32r}{(5+r)(4+r)(r+3)^2(2+r)(r+1)}$	$-\frac{32}{45}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{576 + 64r}{(6+r)(5+r)(4+r)(r+3)^2(2+r)(r+1)}$$

Which for the root $r = 0$ becomes

$$a_6 = \frac{4}{15}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-8-2r}{r^2+4r+3}$	$-\frac{8}{3}$
a_2	$\frac{20+4r}{r^3+6r^2+11r+6}$	$\frac{10}{3}$
a_3	$\frac{-48-8r}{(r+3)^2(2+r)(r+1)}$	$-\frac{8}{3}$
a_4	$\frac{112+16r}{(4+r)(r+3)^2(2+r)(r+1)}$	$\frac{14}{9}$
a_5	$\frac{-256-32r}{(5+r)(4+r)(r+3)^2(2+r)(r+1)}$	$-\frac{32}{45}$
a_6	$\frac{576+64r}{(6+r)(5+r)(4+r)(r+3)^2(2+r)(r+1)}$	$\frac{4}{15}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{-1280 - 128r}{(7+r)(6+r)(5+r)(4+r)(r+3)^2(2+r)(r+1)}$$

Which for the root $r = 0$ becomes

$$a_7 = -\frac{16}{189}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-8-2r}{r^2+4r+3}$	$-\frac{8}{3}$
a_2	$\frac{20+4r}{r^3+6r^2+11r+6}$	$\frac{10}{3}$
a_3	$\frac{-48-8r}{(r+3)^2(2+r)(r+1)}$	$-\frac{8}{3}$
a_4	$\frac{112+16r}{(4+r)(r+3)^2(2+r)(r+1)}$	$\frac{14}{9}$
a_5	$\frac{-256-32r}{(5+r)(4+r)(r+3)^2(2+r)(r+1)}$	$-\frac{32}{45}$
a_6	$\frac{576+64r}{(6+r)(5+r)(4+r)(r+3)^2(2+r)(r+1)}$	$\frac{4}{15}$
a_7	$\frac{-1280-128r}{(7+r)(6+r)(5+r)(4+r)(r+3)^2(2+r)(r+1)}$	$-\frac{16}{189}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots \\ &= 1 - \frac{8x}{3} + \frac{10x^2}{3} - \frac{8x^3}{3} + \frac{14x^4}{9} - \frac{32x^5}{45} + \frac{4x^6}{15} - \frac{16x^7}{189} + O(x^8) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_2 \\ &= \frac{20 + 4r}{r^3 + 6r^2 + 11r + 6} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{20 + 4r}{r^3 + 6r^2 + 11r + 6} &= \lim_{r \to -2} \frac{20 + 4r}{r^3 + 6r^2 + 11r + 6} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}
\frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\
\frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2 C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= C y_1''(x) \ln(x) + \frac{2 C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $x y'' + (3+2x) y' + 8y = 0$ gives

$$\begin{aligned}
& x \left(C y_1''(x) \ln(x) + \frac{2 C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \right. \\
& \left. + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
& + (3+2x) \left(C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) \\
& + 8 C y_1(x) \ln(x) + 8 \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}$$

Which can be written as

$$
\begin{aligned}
&\Bigg(\left(y_1''(x)\,x+(3+2x)\,y_1'(x)+8y_1(x)\right)\ln(x)+x\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)\\
&+\frac{(3+2x)\,y_1(x)}{x}\Bigg)C+x\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\
&+(3+2x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+8\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned}
\tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$
y_1''(x)\,x+(3+2x)\,y_1'(x)+8y_1(x)=0
$$

Eq (7) simplifes to

$$
\begin{aligned}
&\left(x\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+\frac{(3+2x)\,y_1(x)}{x}\right)C\\
&+x\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\
&+(3+2x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+8\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned}
\tag{8}
$$

Substituting $y_1=\sum_{n=0}^{\infty}a_n x^{n+r_1}$ into the above gives

$$
\begin{aligned}
&\frac{\left(2\left(\sum_{n=0}^{\infty}x^{-1+n+r_1}a_n(n+r_1)\right)x+2(1+x)\left(\sum_{n=0}^{\infty}a_n x^{n+r_1}\right)\right)C}{x}\\
&+\frac{\left(\sum_{n=0}^{\infty}x^{-2+n+r_2}b_n(n+r_2)\,(-1+n+r_2)\right)x^2+(2x^2+3x)\left(\sum_{n=0}^{\infty}x^{-1+n+r_2}b_n(n+r_2)\right)+8\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)x}{x}\\
&=0
\end{aligned}
\tag{9}
$$

Since $r_1 = 0$ and $r_2 = -2$ then the above becomes

$$\frac{\left(2\left(\sum_{n=0}^{\infty} x^{n-1} a_n n\right) x + 2(1+x)\left(\sum_{n=0}^{\infty} a_n x^n\right)\right) C}{x} + \frac{\left(\sum_{n=0}^{\infty} x^{-4+n} b_n (n-2)(-3+n)\right) x^2 + (2x^2+3x)\left(\sum_{n=0}^{\infty} x^{-3+n} b_n (n-2)\right) + 8\left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) x}{x} = 0 \tag{10}$$

Which simplifies to

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{n-1} a_n n\right) + \left(\sum_{n=0}^{\infty} 2C\, x^{n-1} a_n\right) + \left(\sum_{n=0}^{\infty} 2C a_n x^n\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{-3+n} b_n (n^2 - 5n + 6)\right) + \left(\sum_{n=0}^{\infty} 2x^{n-2} b_n (n-2)\right) \\
&+ \left(\sum_{n=0}^{\infty} 3x^{-3+n} b_n (n-2)\right) + \left(\sum_{n=0}^{\infty} 8 b_n x^{n-2}\right) = 0
\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $-3+n$ in each summation term. Going over each summation term above with power of x in it which is not already x^{-3+n} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{n-1} a_n n = \sum_{n=2}^{\infty} 2C(n-2)\, a_{n-2} x^{-3+n}$$

$$\sum_{n=0}^{\infty} 2C\, x^{n-1} a_n = \sum_{n=2}^{\infty} 2C a_{n-2} x^{-3+n}$$

$$\sum_{n=0}^{\infty} 2C a_n x^n = \sum_{n=3}^{\infty} 2C a_{-3+n} x^{-3+n}$$

$$\sum_{n=0}^{\infty} 2x^{n-2} b_n (n-2) = \sum_{n=1}^{\infty} 2b_{n-1}(-3+n)\, x^{-3+n}$$

$$\sum_{n=0}^{\infty} 8 b_n x^{n-2} = \sum_{n=1}^{\infty} 8 b_{n-1} x^{-3+n}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $-3+n$.

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} 2C(n-2)\, a_{n-2} x^{-3+n}\right) + \left(\sum_{n=2}^{\infty} 2C a_{n-2} x^{-3+n}\right) \\
&+ \left(\sum_{n=3}^{\infty} 2C a_{-3+n} x^{-3+n}\right) + \left(\sum_{n=0}^{\infty} x^{-3+n} b_n \left(n^2-5n+6\right)\right) \\
&+ \left(\sum_{n=1}^{\infty} 2b_{n-1}(-3+n)\, x^{-3+n}\right) \\
&+ \left(\sum_{n=0}^{\infty} 3x^{-3+n} b_n (n-2)\right) + \left(\sum_{n=1}^{\infty} 8b_{n-1} x^{-3+n}\right) = 0
\end{aligned} \tag{2B}
$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$-b_1 + 4b_0 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-b_1 + 4 = 0$$

Solving the above for b_1 gives

$$b_1 = 4$$

For $n=N$, where $N=2$ which is the difference between the two roots, we are free to choose $b_2=0$. Hence for $n=2$, Eq (2B) gives

$$2C + 24 = 0$$

Which is solved for C. Solving for C gives

$$C = -12$$

For $n=3$, Eq (2B) gives

$$(2a_0 + 4a_1)\, C + 8b_2 + 3b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$104 + 3b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = -\frac{104}{3}$$

For $n = 4$, Eq (2B) gives

$$(2a_1 + 6a_2)\, C + 10b_3 + 8b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{1568}{3} + 8b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{196}{3}$$

For $n = 5$, Eq (2B) gives

$$(2a_2 + 8a_3)\, C + 12b_4 + 15b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$960 + 15b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = -64$$

For $n = 6$, Eq (2B) gives

$$2(a_3 + 5a_4)\, C + 14b_5 + 24b_6 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{3056}{3} + 24b_6 = 0$$

Solving the above for b_6 gives

$$b_6 = \frac{382}{9}$$

For $n = 7$, Eq (2B) gives

$$2(a_4 + 6a_5)\, C + 16b_6 + 35b_7 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{33488}{45} + 35b_7 = 0$$

Solving the above for b_7 gives

$$b_7 = -\frac{4784}{225}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -12$ and all b_n, then the second solution becomes

$$y_2(x) = (-12)\left(1 - \frac{8x}{3} + \frac{10x^2}{3} - \frac{8x^3}{3} + \frac{14x^4}{9} - \frac{32x^5}{45} + \frac{4x^6}{15} - \frac{16x^7}{189} + O(x^8)\right)\ln(x)$$
$$+ \frac{1 + 4x - \frac{104x^3}{3} + \frac{196x^4}{3} - 64x^5 + \frac{382x^6}{9} - \frac{4784x^7}{225} + O(x^8)}{x^2}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1\left(1 - \frac{8x}{3} + \frac{10x^2}{3} - \frac{8x^3}{3} + \frac{14x^4}{9} - \frac{32x^5}{45} + \frac{4x^6}{15} - \frac{16x^7}{189} + O(x^8)\right) \\ &\quad + c_2\left((-12)\left(1 - \frac{8x}{3} + \frac{10x^2}{3} - \frac{8x^3}{3} + \frac{14x^4}{9} - \frac{32x^5}{45} + \frac{4x^6}{15} - \frac{16x^7}{189} + O(x^8)\right)\ln(x)\right. \\ &\quad \left. + \frac{1 + 4x - \frac{104x^3}{3} + \frac{196x^4}{3} - 64x^5 + \frac{382x^6}{9} - \frac{4784x^7}{225} + O(x^8)}{x^2}\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1\left(1 - \frac{8x}{3} + \frac{10x^2}{3} - \frac{8x^3}{3} + \frac{14x^4}{9} - \frac{32x^5}{45} + \frac{4x^6}{15} - \frac{16x^7}{189} + O(x^8)\right) \\ &\quad + c_2\left(\left(-12 + 32x - 40x^2 + 32x^3 - \frac{56x^4}{3} + \frac{128x^5}{15} - \frac{16x^6}{5} + \frac{64x^7}{63} - 12O(x^8)\right)\ln(x)\right. \\ &\quad \left. + \frac{1 + 4x - \frac{104x^3}{3} + \frac{196x^4}{3} - 64x^5 + \frac{382x^6}{9} - \frac{4784x^7}{225} + O(x^8)}{x^2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\left(1-\frac{8x}{3}+\frac{10x^2}{3}-\frac{8x^3}{3}+\frac{14x^4}{9}-\frac{32x^5}{45}+\frac{4x^6}{15}-\frac{16x^7}{189}+O\left(x^8\right)\right) + c_2\left(\left(-12+32x-40x^2+32x^3-\frac{56x^4}{3}+\frac{128x^5}{15}-\frac{16x^6}{5}+\frac{64x^7}{63}-12O\left(x^8\right)\right)\ln\left(x\right) + \frac{1+4x-\frac{104x^3}{3}+\frac{196x^4}{3}-64x^5+\frac{382x^6}{9}-\frac{4784x^7}{225}+O(x^8)}{x^2}\right)$$

Verification of solutions

$$y = c_1\left(1-\frac{8x}{3}+\frac{10x^2}{3}-\frac{8x^3}{3}+\frac{14x^4}{9}-\frac{32x^5}{45}+\frac{4x^6}{15}-\frac{16x^7}{189}+O\left(x^8\right)\right) + c_2\left(\left(-12+32x-40x^2+32x^3-\frac{56x^4}{3}+\frac{128x^5}{15}-\frac{16x^6}{5}+\frac{64x^7}{63}-12O\left(x^8\right)\right)\ln\left(x\right) + \frac{1+4x-\frac{104x^3}{3}+\frac{196x^4}{3}-64x^5+\frac{382x^6}{9}-\frac{4784x^7}{225}+O(x^8)}{x^2}\right)$$

Verified OK.

7.6.1 Maple step by step solution

Let's solve

$xy''+(3+2x)\,y'+8y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{8y}{x}-\frac{(3+2x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(3+2x)y'}{x}+\frac{8y}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{3+2x}{x}, P_3(x)=\frac{8}{x}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$(x \cdot P_2(x))\Big|_{x=0} = 3$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + (3 + 2x)\, y' + 8y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k + 1 - m + r)\, x^{k+r}$

- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r-1}$

- Shift index using $k- >k + 1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k + 1 + r)\,(k + r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(2 + r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k + 1 + r)\,(k + 3 + r) + 2a_k(k + r + 4)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(2 + r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{-2, 0\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+3+r) + 2a_k(k+r+4) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{2a_k(k+r+4)}{(k+1+r)(k+3+r)}$

- Recursion relation for $r = -2$

  $a_{k+1} = -\frac{2a_k(k+2)}{(k-1)(k+1)}$

- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 1$

  $a_{k+1} = -\frac{2a_k(k+2)}{(k-1)(k+1)}$

- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{2a_k(k+4)}{(k+1)(k+3)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{2a_k(k+4)}{(k+1)(k+3)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 76

```
Order:=8;
dsolve(x*diff(y(x),x$2)+(3+2*x)*diff(y(x),x)+8*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1\left(1-\frac{8}{3}x+\frac{10}{3}x^2-\frac{8}{3}x^3+\frac{14}{9}x^4-\frac{32}{45}x^5+\frac{4}{15}x^6-\frac{16}{189}x^7+\mathrm{O}\left(x^8\right)\right)x^2+c_2\left(\ln\left(x\right)\left(24x^2-64x^3+80x^4-\right.\right.}{x}$$

✓ Solution by Mathematica

Time used: 0.107 (sec). Leaf size: 116

```
AsymptoticDSolveValue[x*y''[x]+(3+2*x)*y'[x]+8*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(\frac{4x^6}{15}-\frac{32x^5}{45}+\frac{14x^4}{9}-\frac{8x^3}{3}+\frac{10x^2}{3}-\frac{8x}{3}+1\right)+c_1\left(\frac{326x^6-480x^5+468x^4-216x^3-36x^2+36x+9}{9x^2}-\frac{4}{3}\left(14x^4-24x^3+30x^2-24x+9\right)\log(x)\right)$$

7.7 problem 7

Internal problem ID [6987]
Internal file name [`OUTPUT/6230_Friday_August_12_2022_11_06_19_PM_66811311/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.9 Indicial Equation with Difference of Roots a Positive Integer: Logarithmic Case. Exercises page 384
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[_Jacobi]
```

$$\boxed{x(1-x)\,y''+2(1-x)\,y'+2y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(-x^2+x)\,y''+(-2x+2)\,y'+2y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=\frac{2}{x}\\ q(x)&=-\frac{2}{x\,(x-1)}\end{aligned}$$

Table 164: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{2}{x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 1, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''x(x-1) + (-2x+2)\, y' + 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x(x-1) + (-2x+2)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r) (n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r) (n+r-1)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} (n+r-1) (n+r-2) x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-2x^{n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1} (n+r-1) x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} 2a_n x^{n+r} = \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\sum_{n=1}^{\infty} \left(-a_{n-1} (n+r-1) (n+r-2) x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r) (n+r-1)\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1} (n+r-1) x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r) (n+r-1) + 2(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + 2r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r)+2 r\, x^{-1+r}\right) a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(1+r)=0$$

Since the above is true for all x then the indicial equation becomes

$$r(1+r)=0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 0 \\ r_2 &= -1 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(1+r)=0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\ y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x} \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
\begin{aligned}
&-a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) \\
&- 2a_{n-1}(n+r-1) + 2a_n(n+r) + 2a_{n-1} = 0
\end{aligned}
\tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = \frac{(n+r-2)\, a_{n-1}}{n+r} \tag{4}
$$

Which for the root $r = 0$ becomes

$$
a_n = \frac{(n-2)\, a_{n-1}}{n} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$
a_1 = \frac{-1+r}{1+r}
$$

Which for the root $r = 0$ becomes

$$
a_1 = -1
$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{1+r}$	-1

For $n = 2$, using the above recursive equation gives

$$
a_2 = \frac{(-1+r)\, r}{(1+r)(2+r)}
$$

Which for the root $r = 0$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{1+r}$	-1
a_2	$\frac{(-1+r)r}{(1+r)(2+r)}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{r(-1+r)}{(2+r)(3+r)}$$

Which for the root $r = 0$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{1+r}$	-1
a_2	$\frac{(-1+r)r}{(1+r)(2+r)}$	0
a_3	$\frac{r(-1+r)}{(2+r)(3+r)}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r(-1+r)}{(3+r)(4+r)}$$

Which for the root $r = 0$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{1+r}$	-1
a_2	$\frac{(-1+r)r}{(1+r)(2+r)}$	0
a_3	$\frac{r(-1+r)}{(2+r)(3+r)}$	0
a_4	$\frac{r(-1+r)}{(3+r)(4+r)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{r(-1+r)}{(4+r)(5+r)}$$

Which for the root $r = 0$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{1+r}$	-1
a_2	$\frac{(-1+r)r}{(1+r)(2+r)}$	0
a_3	$\frac{r(-1+r)}{(2+r)(3+r)}$	0
a_4	$\frac{r(-1+r)}{(3+r)(4+r)}$	0
a_5	$\frac{r(-1+r)}{(4+r)(5+r)}$	0

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{r(-1+r)}{(5+r)(6+r)}$$

Which for the root $r = 0$ becomes

$$a_6 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{1+r}$	-1
a_2	$\frac{(-1+r)r}{(1+r)(2+r)}$	0
a_3	$\frac{r(-1+r)}{(2+r)(3+r)}$	0
a_4	$\frac{r(-1+r)}{(3+r)(4+r)}$	0
a_5	$\frac{r(-1+r)}{(4+r)(5+r)}$	0
a_6	$\frac{r(-1+r)}{(5+r)(6+r)}$	0

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{r(-1+r)}{(6+r)(7+r)}$$

Which for the root $r = 0$ becomes

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{1+r}$	-1
a_2	$\frac{(-1+r)r}{(1+r)(2+r)}$	0
a_3	$\frac{r(-1+r)}{(2+r)(3+r)}$	0
a_4	$\frac{r(-1+r)}{(3+r)(4+r)}$	0
a_5	$\frac{r(-1+r)}{(4+r)(5+r)}$	0
a_6	$\frac{r(-1+r)}{(5+r)(6+r)}$	0
a_7	$\frac{r(-1+r)}{(6+r)(7+r)}$	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots \\ &= 1 - x + O(x^8) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= \frac{-1+r}{1+r} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{-1+r}{1+r} &= \lim_{r \to -1} \frac{-1+r}{1+r} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned} \frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\ &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\ \frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\ &\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\ &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) \end{aligned}$$

Substituting these back into the given ode $-y''x(x-1)+(-2x+2)\,y'+2y=0$ gives

$$
\begin{aligned}
&-\left(Cy_1''(x)\ln(x)+\frac{2Cy_1'(x)}{x}-\frac{Cy_1(x)}{x^2}\right.\\
&\left.+\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x(x-1)\\
&+(-2x+2)\left(Cy_1'(x)\ln(x)+\frac{Cy_1(x)}{x}+\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)\\
&+2Cy_1(x)\ln(x)+2\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\Bigg(\left(-y_1''(x)\,x(x-1)+(-2x+2)\,y_1'(x)+2y_1(x)\right)\ln(x)\\
&-\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)x(x-1)+\frac{(-2x+2)\,y_1(x)}{x}\Bigg)C\\
&-\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x(x-1)\\
&+(-2x+2)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+2\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned}
\tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$
-y_1''(x)\,x(x-1)+(-2x+2)\,y_1'(x)+2y_1(x)=0
$$

Eq (7) simplifes to

$$
\begin{aligned}
&\left(-\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)x(x-1)+\frac{(-2x+2)\,y_1(x)}{x}\right)C\\
&-\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x(x-1)\\
&+(-2x+2)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+2\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned}
\tag{8}
$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\frac{\left(-2x(x-1)\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n(n+r_1)\right)-(x-1)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right)C}{x} \tag{9}$$
$$+\frac{(-x^3+x^2)\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)+(-2x^2+2x)\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right)+2\left(\sum_{n=0}^{\infty}\right.}{x}$$
$$=0$$

Since $r_1 = 0$ and $r_2 = -1$ then the above becomes

$$\frac{\left(-2x(x-1)\left(\sum_{n=0}^{\infty} x^{n-1} a_n n\right)-(x-1)\left(\sum_{n=0}^{\infty} a_n x^{n}\right)\right)C}{x} \tag{10}$$
$$+\frac{(-x^3+x^2)\left(\sum_{n=0}^{\infty} x^{-3+n} b_n(n-1)(n-2)\right)+(-2x^2+2x)\left(\sum_{n=0}^{\infty} x^{n-2} b_n(n-1)\right)+2\left(\sum_{n=0}^{\infty} b_n x^{n-1}\right)x}{x}$$
$$=0$$

Which simplifies to

$$\begin{aligned}\sum_{n=0}^{\infty}\left(-2C\,x^n a_n n\right)+\left(\sum_{n=0}^{\infty} 2Cn\,x^{n-1} a_n\right)+\sum_{n=0}^{\infty}\left(-C a_n x^n\right)+\left(\sum_{n=0}^{\infty} C\,x^{n-1} a_n\right)\\+\sum_{n=0}^{\infty}\left(-b_n x^{n-1}(n-1)(n-2)\right)+\left(\sum_{n=0}^{\infty} x^{n-2} b_n\left(n^2-3n+2\right)\right)\\+\sum_{n=0}^{\infty}\left(-2x^{n-1} b_n(n-1)\right)+\left(\sum_{n=0}^{\infty} 2x^{n-2} b_n(n-1)\right)+\left(\sum_{n=0}^{\infty} 2b_n x^{n-1}\right)=0\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n-2$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-2} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-2C\,x^n a_n n\right)=\sum_{n=2}^{\infty}\left(-2C(n-2)\,a_{n-2} x^{n-2}\right)$$

$$\sum_{n=0}^{\infty} 2Cn\,x^{n-1} a_n=\sum_{n=1}^{\infty} 2C(n-1)\,a_{n-1} x^{n-2}$$

$$\sum_{n=0}^{\infty} (-C a_n x^n) = \sum_{n=2}^{\infty} (-C a_{n-2} x^{n-2})$$

$$\sum_{n=0}^{\infty} C x^{n-1} a_n = \sum_{n=1}^{\infty} C a_{n-1} x^{n-2}$$

$$\sum_{n=0}^{\infty} \left(-b_n x^{n-1} (n-1)(n-2)\right) = \sum_{n=1}^{\infty} \left(-b_{n-1} (-3+n)(n-2) x^{n-2}\right)$$

$$\sum_{n=0}^{\infty} \left(-2 x^{n-1} b_n (n-1)\right) = \sum_{n=1}^{\infty} \left(-2 b_{n-1} (n-2) x^{n-2}\right)$$

$$\sum_{n=0}^{\infty} 2 b_n x^{n-1} = \sum_{n=1}^{\infty} 2 b_{n-1} x^{n-2}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-2$.

$$\begin{aligned} &\sum_{n=2}^{\infty} \left(-2C(n-2) a_{n-2} x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 2C(n-1) a_{n-1} x^{n-2}\right) \\ &\quad + \sum_{n=2}^{\infty} (-C a_{n-2} x^{n-2}) + \left(\sum_{n=1}^{\infty} C a_{n-1} x^{n-2}\right) \\ &\quad + \sum_{n=1}^{\infty} \left(-b_{n-1}(-3+n)(n-2) x^{n-2}\right) \\ &\quad + \left(\sum_{n=0}^{\infty} x^{n-2} b_n (n^2 - 3n + 2)\right) + \sum_{n=1}^{\infty} \left(-2 b_{n-1} (n-2) x^{n-2}\right) \\ &\quad + \left(\sum_{n=0}^{\infty} 2 x^{n-2} b_n (n-1)\right) + \left(\sum_{n=1}^{\infty} 2 b_{n-1} x^{n-2}\right) = 0 \end{aligned} \tag{2B}$$

For $n = 0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = N$, where $N = 1$ which is the difference between the two roots, we are free to choose $b_1 = 0$. Hence for $n = 1$, Eq (2B) gives

$$C + 2 = 0$$

Which is solved for C. Solving for C gives

$$C = -2$$

For $n = 2$, Eq (2B) gives

$$(-a_0 + 3a_1) C + 2b_1 + 2b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$8 + 2b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = -4$$

For $n = 3$, Eq (2B) gives

$$(-3a_1 + 5a_2) C + 6b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-6 + 6b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = 1$$

For $n = 4$, Eq (2B) gives

$$(-5a_2 + 7a_3) C - 4b_3 + 12b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-4 + 12b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{1}{3}$$

For $n = 5$, Eq (2B) gives

$$(-7a_3 + 9a_4) C - 10b_4 + 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{10}{3} + 20b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{1}{6}$$

For $n = 6$, Eq (2B) gives

$$(-9a_4 + 11a_5) C - 18b_5 + 30b_6 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3 + 30b_6 = 0$$

Solving the above for b_6 gives

$$b_6 = \frac{1}{10}$$

For $n = 7$, Eq (2B) gives

$$(-11a_5 + 13a_6) C - 28b_6 + 42b_7 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{14}{5} + 42b_7 = 0$$

Solving the above for b_7 gives

$$b_7 = \frac{1}{15}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -2$ and all b_n, then the second solution becomes

$$y_2(x) = (-2)\left(1 - x + O(x^8)\right) \ln(x) + \frac{1 - 4x^2 + x^3 + \frac{x^4}{3} + \frac{x^5}{6} + \frac{x^6}{10} + \frac{x^7}{15} + O(x^8)}{x}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1 - x + O(x^8)\right) \\
&\quad + c_2\left((-2)\left(1 - x + O(x^8)\right) \ln(x) + \frac{1 - 4x^2 + x^3 + \frac{x^4}{3} + \frac{x^5}{6} + \frac{x^6}{10} + \frac{x^7}{15} + O(x^8)}{x}\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1\left(1 - x + O(x^8)\right) \\
&\quad + c_2\left(\left(-2 + 2x - 2O(x^8)\right) \ln(x) + \frac{1 - 4x^2 + x^3 + \frac{x^4}{3} + \frac{x^5}{6} + \frac{x^6}{10} + \frac{x^7}{15} + O(x^8)}{x}\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1\left(1-x+O\left(x^8\right)\right) + c_2\left(\left(-2+2x-2O\left(x^8\right)\right)\ln\left(x\right)+\frac{1-4x^2+x^3+\frac{x^4}{3}+\frac{x^5}{6}+\frac{x^6}{10}+\frac{x^7}{15}+O\left(x^8\right)}{x}\right) \tag{1}$$

Verification of solutions

$$y = c_1\left(1-x+O\left(x^8\right)\right) + c_2\left(\left(-2+2x-2O\left(x^8\right)\right)\ln\left(x\right)+\frac{1-4x^2+x^3+\frac{x^4}{3}+\frac{x^5}{6}+\frac{x^6}{10}+\frac{x^7}{15}+O\left(x^8\right)}{x}\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 54

```
Order:=8;
dsolve(x*(1-x)*diff(y(x),x$2)+2*(1-x)*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{\ln\left(x\right)\left(\left(-2\right)x+2x^2+\mathrm{O}\left(x^8\right)\right)c_2+c_1\left(1-x+\mathrm{O}\left(x^8\right)\right)x+\left(1-4x^2+x^3+\frac{1}{3}x^4+\frac{1}{6}x^5+\frac{1}{10}x^6+\frac{1}{15}x^7+\right.}{x}$$

✓ Solution by Mathematica

Time used: 0.408 (sec). Leaf size: 60

```
AsymptoticDSolveValue[x*(1-x)*y''[x]+2*(1-x)*y'[x]+2*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1 \left(\frac{3x^6 + 5x^5 + 10x^4 + 30x^3 - 150x^2 + 30x + 30}{30x} + 2(x-1)\log(x) \right) + c_2(1-x)$$

7.8 problem 9

Internal problem ID [6988]
Internal file name [`OUTPUT/6231_Friday_August_12_2022_11_06_21_PM_82600808/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.9 Indicial Equation with Difference of Roots a Positive Integer: Logarithmic Case. Exercises page 384
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[_Jacobi]
```

$$\boxed{x(1-x)\,y''+2(1-x)\,y'+2y=0}$$

With the expansion point for the power series method at $x=1$.

The ode does not have its expansion point at $x=0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t=x-1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left(-(t+1)^2+t+1\right)\left(\frac{d^2}{dt^2}y(t)\right)-2t\left(\frac{d}{dt}y(t)\right)+2y(t)=0$$

With its expansion point and initial conditions now at $t=0$. The transformed ODE is now solved. The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-t^2-t\right)\left(\frac{d^2}{dt^2}y(t)\right)-2t\left(\frac{d}{dt}y(t)\right)+2y(t)=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$\frac{d^2}{dt^2}y(t) + p(t)\frac{d}{dt}y(t) + q(t)y(t) = 0$$

Where

$$p(t) = \frac{2}{t+1}$$
$$q(t) = -\frac{2}{t\,(t+1)}$$

Table 165: Table $p(t), q(t)$ singularites.

$p(t) = \frac{2}{t+1}$	
singularity	type
$t = -1$	"regular"

$q(t) = -\frac{2}{t(t+1)}$	
singularity	type
$t = -1$	"regular"
$t = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, \infty]$

Irregular singular points : []

Since $t = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-\left(\frac{d^2}{dt^2}y(t)\right)t(t+1) - 2t\left(\frac{d}{dt}y(t)\right) + 2y(t) = 0$$

Let the solution be represented as Frobenius power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^{n+r}$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=0}^{\infty} (n+r)\,a_n t^{n+r-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n t^{n+r-2}$$

Substituting the above back into the ode gives

$$
\begin{aligned}
&-\left(\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n t^{n+r-2}\right)t(t+1) \\
&-2t\left(\sum_{n=0}^{\infty}(n+r)\,a_n t^{n+r-1}\right)+2\left(\sum_{n=0}^{\infty}a_n t^{n+r}\right)=0
\end{aligned}
\tag{1}
$$

Which simplifies to

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\left(-t^{n+r}a_n(n+r)\,(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-t^{n+r-1}a_n(n+r)\,(n+r-1)\right) \\
&+\sum_{n=0}^{\infty}\left(-2t^{n+r}a_n(n+r)\right)+\left(\sum_{n=0}^{\infty}2a_n t^{n+r}\right)=0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of t be $n+r-1$ in each summation term. Going over each summation term above with power of t in it which is not already t^{n+r-1} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty}\left(-t^{n+r}a_n(n+r)\,(n+r-1)\right)&=\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)\,(n+r-2)\,t^{n+r-1}\right) \\
\sum_{n=0}^{\infty}\left(-2t^{n+r}a_n(n+r)\right)&=\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)\,t^{n+r-1}\right) \\
\sum_{n=0}^{\infty}2a_n t^{n+r}&=\sum_{n=1}^{\infty}2a_{n-1}t^{n+r-1}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of t are the same and equal to $n+r-1$.

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)\,(n+r-2)\,t^{n+r-1}\right)+\sum_{n=0}^{\infty}\left(-t^{n+r-1}a_n(n+r)\,(n+r-1)\right) \\
&+\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)\,t^{n+r-1}\right)+\left(\sum_{n=1}^{\infty}2a_{n-1}t^{n+r-1}\right)=0
\end{aligned}
\tag{2B}
$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$-t^{n+r-1}a_n(n+r)(n+r-1) = 0$$

When $n = 0$ the above becomes

$$-t^{-1+r}a_0r(-1+r) = 0$$

Or

$$-t^{-1+r}a_0r(-1+r) = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$-t^{-1+r}r(-1+r) = 0$$

Since the above is true for all t then the indicial equation becomes

$$-r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$-t^{-1+r}r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(t) = t^{r_1}\left(\sum_{n=0}^{\infty} a_n t^n\right)$$
$$y_2(t) = Cy_1(t)\ln(t) + t^{r_2}\left(\sum_{n=0}^{\infty} b_n t^n\right)$$

Or

$$y_1(t) = t\left(\sum_{n=0}^{\infty} a_n t^n\right)$$
$$y_2(t) = Cy_1(t)\ln(t) + \left(\sum_{n=0}^{\infty} b_n t^n\right)$$

Or

$$
\begin{aligned}
y_1(t) &= \sum_{n=0}^{\infty} a_n t^{n+1} \\
y_2(t) &= C y_1(t) \ln(t) + \left(\sum_{n=0}^{\infty} b_n t^n\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
-a_{n-1}(n+r-1)(n+r-2) - a_n(n+r)(n+r-1) - 2a_{n-1}(n+r-1) + 2a_{n-1} = 0 \quad (3)
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{a_{n-1}(n^2 + 2nr + r^2 - n - r - 2)}{(n+r)(n+r-1)} \tag{4}
$$

Which for the root $r = 1$ becomes

$$
a_n = -\frac{a_{n-1}(n^2 + n - 2)}{(n+1)n} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$
a_1 = \frac{-r^2 - r + 2}{(1+r)r}
$$

Which for the root $r = 1$ becomes

$$
a_1 = 0
$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-r+2}{(1+r)r}$	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^2 + 2r - 3}{(1 + r)^2}$$

Which for the root $r = 1$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-r+2}{(1+r)r}$	0
a_2	$\frac{r^2+2r-3}{(1+r)^2}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-r^2 - 3r + 4}{(2 + r)(1 + r)}$$

Which for the root $r = 1$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-r+2}{(1+r)r}$	0
a_2	$\frac{r^2+2r-3}{(1+r)^2}$	0
a_3	$\frac{-r^2-3r+4}{(2+r)(1+r)}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^2 + 4r - 5}{(r + 3)(1 + r)}$$

Which for the root $r = 1$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-r+2}{(1+r)r}$	0
a_2	$\frac{r^2+2r-3}{(1+r)^2}$	0
a_3	$\frac{-r^2-3r+4}{(2+r)(1+r)}$	0
a_4	$\frac{r^2+4r-5}{(r+3)(1+r)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-r^2 - 5r + 6}{(r+4)(1+r)}$$

Which for the root $r = 1$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-r+2}{(1+r)r}$	0
a_2	$\frac{r^2+2r-3}{(1+r)^2}$	0
a_3	$\frac{-r^2-3r+4}{(2+r)(1+r)}$	0
a_4	$\frac{r^2+4r-5}{(r+3)(1+r)}$	0
a_5	$\frac{-r^2-5r+6}{(r+4)(1+r)}$	0

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{r^2 + 6r - 7}{(r+5)(1+r)}$$

Which for the root $r = 1$ becomes

$$a_6 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-r+2}{(1+r)r}$	0
a_2	$\frac{r^2+2r-3}{(1+r)^2}$	0
a_3	$\frac{-r^2-3r+4}{(2+r)(1+r)}$	0
a_4	$\frac{r^2+4r-5}{(r+3)(1+r)}$	0
a_5	$\frac{-r^2-5r+6}{(r+4)(1+r)}$	0
a_6	$\frac{r^2+6r-7}{(r+5)(1+r)}$	0

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{-r^2 - 7r + 8}{(r+6)(1+r)}$$

Which for the root $r = 1$ becomes

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-r+2}{(1+r)r}$	0
a_2	$\frac{r^2+2r-3}{(1+r)^2}$	0
a_3	$\frac{-r^2-3r+4}{(2+r)(1+r)}$	0
a_4	$\frac{r^2+4r-5}{(r+3)(1+r)}$	0
a_5	$\frac{-r^2-5r+6}{(r+4)(1+r)}$	0
a_6	$\frac{r^2+6r-7}{(r+5)(1+r)}$	0
a_7	$\frac{-r^2-7r+8}{(r+6)(1+r)}$	0

Using the above table, then the solution $y_1(t)$ is

$$\begin{aligned} y_1(t) &= t\left(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + a_7 t^7 + a_8 t^8 \dots\right) \\ &= t\left(1 + O(t^8)\right) \end{aligned}$$

Now the second solution $y_2(t)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= \frac{-r^2 - r + 2}{(1+r)\, r} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{-r^2 - r + 2}{(1+r)\, r} &= \lim_{r\to 0} \frac{-r^2 - r + 2}{(1+r)\, r} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(t) = C y_1(t) \ln(t) + \left(\sum_{n=0}^{\infty} b_n t^{n+r_2}\right)$$

Therefore

$$\begin{aligned} \frac{d}{dt} y_2(t) &= C\left(\frac{d}{dt} y_1(t)\right) \ln(t) + \frac{C y_1(t)}{t} + \left(\sum_{n=0}^{\infty} \frac{b_n t^{n+r_2} (n + r_2)}{t}\right) \\ &= C\left(\frac{d}{dt} y_1(t)\right) \ln(t) + \frac{C y_1(t)}{t} + \left(\sum_{n=0}^{\infty} t^{-1+n+r_2} b_n (n + r_2)\right) \\ \frac{d^2}{dt^2} y_2(t) &= C\left(\frac{d^2}{dt^2} y_1(t)\right) \ln(t) + \frac{2C\left(\frac{d}{dt} y_1(t)\right)}{t} - \frac{C y_1(t)}{t^2} \\ &\quad + \sum_{n=0}^{\infty} \left(\frac{b_n t^{n+r_2} (n + r_2)^2}{t^2} - \frac{b_n t^{n+r_2} (n + r_2)}{t^2}\right) \\ &= C\left(\frac{d^2}{dt^2} y_1(t)\right) \ln(t) + \frac{2C\left(\frac{d}{dt} y_1(t)\right)}{t} - \frac{C y_1(t)}{t^2} + \left(\sum_{n=0}^{\infty} t^{-2+n+r_2} b_n (n + r_2) (-1 + n \right. \\ &\quad \left. + r_2)\right) \end{aligned}$$

Substituting these back into the given ode $-\left(\frac{d^2}{dt^2}y(t)\right)t(t+1)-2t\left(\frac{d}{dt}y(t)\right)+2y(t)=0$ gives

$$
\begin{aligned}
&-\left(C\left(\frac{d^2}{dt^2}y_1(t)\right)\ln(t)+\frac{2C\left(\frac{d}{dt}y_1(t)\right)}{t}-\frac{Cy_1(t)}{t^2}\right.\\
&\left.+\sum_{n=0}^{\infty}\left(\frac{b_n t^{n+r_2}(n+r_2)^2}{t^2}-\frac{b_n t^{n+r_2}(n+r_2)}{t^2}\right)\right)t(t+1)\\
&-2t\left(C\left(\frac{d}{dt}y_1(t)\right)\ln(t)+\frac{Cy_1(t)}{t}+\left(\sum_{n=0}^{\infty}\frac{b_n t^{n+r_2}(n+r_2)}{t}\right)\right)\\
&+2Cy_1(t)\ln(t)+2\left(\sum_{n=0}^{\infty}b_n t^{n+r_2}\right)=0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(-\left(\frac{d^2}{dt^2}y_1(t)\right)t(t+1)-2\left(\frac{d}{dt}y_1(t)\right)t+2y_1(t)\right)\ln(t)\right.\\
&\left.-\left(\frac{2\left(\frac{d}{dt}y_1(t)\right)}{t}-\frac{y_1(t)}{t^2}\right)t(t+1)-2y_1(t)\right)C\\
&-\left(\sum_{n=0}^{\infty}\left(\frac{b_n t^{n+r_2}(n+r_2)^2}{t^2}-\frac{b_n t^{n+r_2}(n+r_2)}{t^2}\right)\right)t(t+1)\\
&-2t\left(\sum_{n=0}^{\infty}\frac{b_n t^{n+r_2}(n+r_2)}{t}\right)+2\left(\sum_{n=0}^{\infty}b_n t^{n+r_2}\right)=0
\end{aligned}
\tag{7}
$$

But since $y_1(t)$ is a solution to the ode, then

$$
-\left(\frac{d^2}{dt^2}y_1(t)\right)t(t+1)-2\left(\frac{d}{dt}y_1(t)\right)t+2y_1(t)=0
$$

Eq (7) simplifes to

$$
\begin{aligned}
&\left(-\left(\frac{2\left(\frac{d}{dt}y_1(t)\right)}{t}-\frac{y_1(t)}{t^2}\right)t(t+1)-2y_1(t)\right)C\\
&-\left(\sum_{n=0}^{\infty}\left(\frac{b_n t^{n+r_2}(n+r_2)^2}{t^2}-\frac{b_n t^{n+r_2}(n+r_2)}{t^2}\right)\right)t(t+1)\\
&-2t\left(\sum_{n=0}^{\infty}\frac{b_n t^{n+r_2}(n+r_2)}{t}\right)+2\left(\sum_{n=0}^{\infty}b_n t^{n+r_2}\right)=0
\end{aligned}
\tag{8}
$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n t^{n+r_1}$ into the above gives

$$\frac{\left(-2t(t+1)\left(\sum_{n=0}^{\infty} t^{-1+n+r_1} a_n(n+r_1)\right)-(t-1)\left(\sum_{n=0}^{\infty} a_n t^{n+r_1}\right)\right) C}{t} + \frac{(-t^3-t^2)\left(\sum_{n=0}^{\infty} t^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)-2\left(\sum_{n=0}^{\infty} t^{-1+n+r_2} b_n(n+r_2)\right) t^2+2\left(\sum_{n=0}^{\infty} b_n t^{n+r_2}\right) t}{t} = 0 \tag{9}$$

Since $r_1 = 1$ and $r_2 = 0$ then the above becomes

$$\frac{\left(-2t(t+1)\left(\sum_{n=0}^{\infty} t^{n} a_n(n+1)\right)-(t-1)\left(\sum_{n=0}^{\infty} a_n t^{n+1}\right)\right) C}{t} + \frac{(-t^3-t^2)\left(\sum_{n=0}^{\infty} t^{-2+n} b_n n(n-1)\right)-2\left(\sum_{n=0}^{\infty} t^{n-1} b_n n\right) t^2+2\left(\sum_{n=0}^{\infty} b_n t^{n}\right) t}{t} = 0 \tag{10}$$

Which simplifies to

$$\begin{aligned}\sum_{n=0}^{\infty}\left(-2C\,t^{n+1} a_n(n+1)\right)+\sum_{n=0}^{\infty}\left(-2C\,t^{n} a_n(n+1)\right) \\ +\sum_{n=0}^{\infty}\left(-C\,t^{n+1} a_n\right)+\left(\sum_{n=0}^{\infty} C a_n t^n\right)+\sum_{n=0}^{\infty}\left(-t^n b_n n(n-1)\right) \\ +\sum_{n=0}^{\infty}\left(-n\,t^{n-1} b_n(n-1)\right)+\sum_{n=0}^{\infty}\left(-2t^n b_n n\right)+\left(\sum_{n=0}^{\infty} 2b_n t^n\right)=0\end{aligned} \tag{2A}$$

The next step is to make all powers of t be $n-1$ in each summation term. Going over each summation term above with power of t in it which is not already t^{n-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-2C\,t^{n+1} a_n(n+1)\right)=\sum_{n=2}^{\infty}\left(-2C a_{-2+n}(n-1)\,t^{n-1}\right)$$

$$\sum_{n=0}^{\infty}\left(-2C\,t^{n} a_n(n+1)\right)=\sum_{n=1}^{\infty}\left(-2C a_{n-1} n\,t^{n-1}\right)$$

$$\sum_{n=0}^{\infty}\left(-C\,t^{n+1} a_{n}\right) = \sum_{n=2}^{\infty}\left(-C a_{-2+n} t^{n-1}\right)$$

$$\sum_{n=0}^{\infty} C a_{n} t^{n} = \sum_{n=1}^{\infty} C a_{n-1} t^{n-1}$$

$$\sum_{n=0}^{\infty}\left(-t^{n} b_{n} n(n-1)\right) = \sum_{n=1}^{\infty}\left(-(n-1)\, b_{n-1}(-2+n)\, t^{n-1}\right)$$

$$\sum_{n=0}^{\infty}\left(-2 t^{n} b_{n} n\right) = \sum_{n=1}^{\infty}\left(-2(n-1)\, b_{n-1} t^{n-1}\right)$$

$$\sum_{n=0}^{\infty} 2 b_{n} t^{n} = \sum_{n=1}^{\infty} 2 b_{n-1} t^{n-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of t are the same and equal to $n-1$.

$$\begin{aligned}&\sum_{n=2}^{\infty}\left(-2 C a_{-2+n}(n-1)\, t^{n-1}\right) + \sum_{n=1}^{\infty}\left(-2 C a_{n-1} n\, t^{n-1}\right)\\&+\sum_{n=2}^{\infty}\left(-C a_{-2+n} t^{n-1}\right) + \left(\sum_{n=1}^{\infty} C a_{n-1} t^{n-1}\right)\\&+\sum_{n=1}^{\infty}\left(-(n-1)\, b_{n-1}(-2+n)\, t^{n-1}\right) + \sum_{n=0}^{\infty}\left(-n\, t^{n-1} b_{n}(n-1)\right)\\&+\sum_{n=1}^{\infty}\left(-2(n-1)\, b_{n-1} t^{n-1}\right) + \left(\sum_{n=1}^{\infty} 2 b_{n-1} t^{n-1}\right) = 0\end{aligned} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = N$, where $N = 1$ which is the difference between the two roots, we are free to choose $b_1 = 0$. Hence for $n = 1$, Eq (2B) gives

$$-C + 2 = 0$$

Which is solved for C. Solving for C gives

$$C = 2$$

For $n = 2$, Eq (2B) gives

$$(-3a_0 - 3a_1)\, C - 2b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-6 - 2b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = -3$$

For $n = 3$, Eq (2B) gives

$$(-5a_1 - 5a_2) C - 4b_2 - 6b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$12 - 6b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = 2$$

For $n = 4$, Eq (2B) gives

$$(-7a_2 - 7a_3) C - 10b_3 - 12b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-20 - 12b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{5}{3}$$

For $n = 5$, Eq (2B) gives

$$(-9a_3 - 9a_4) C - 18b_4 - 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$30 - 20b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{3}{2}$$

For $n = 6$, Eq (2B) gives

$$(-11a_4 - 11a_5) C - 28b_5 - 30b_6 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-42 - 30b_6 = 0$$

Solving the above for b_6 gives

$$b_6 = -\frac{7}{5}$$

For $n = 7$, Eq (2B) gives

$$(-13a_5 - 13a_6)\, C - 40b_6 - 42b_7 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$56 - 42b_7 = 0$$

Solving the above for b_7 gives

$$b_7 = \frac{4}{3}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(t) = C y_1(t) \ln(t) + \left(\sum_{n=0}^{\infty} b_n t^{n+r_2}\right)$$

Using the above value found for $C = 2$ and all b_n, then the second solution becomes

$$y_2(t) = 2\left(t\left(1 + O\left(t^8\right)\right)\right) \ln(t) + 1 - 3t^2 + 2t^3 - \frac{5t^4}{3} + \frac{3t^5}{2} - \frac{7t^6}{5} + \frac{4t^7}{3} + O\left(t^8\right)$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(t) &= c_1 y_1(t) + c_2 y_2(t) \\ &= c_1 t\left(1 + O\left(t^8\right)\right) + c_2 \left(2\left(t\left(1 + O\left(t^8\right)\right)\right) \ln(t) + 1 - 3t^2 + 2t^3 - \frac{5t^4}{3} + \frac{3t^5}{2} - \frac{7t^6}{5} + \frac{4t^7}{3} \right. \\ &\quad \left. + O\left(t^8\right)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y(t) &= y_h \\ &= c_1 t\left(1 + O\left(t^8\right)\right) + c_2 \left(2t\left(1 + O\left(t^8\right)\right) \ln(t) + 1 - 3t^2 + 2t^3 - \frac{5t^4}{3} + \frac{3t^5}{2} - \frac{7t^6}{5} + \frac{4t^7}{3} \right. \\ &\quad \left. + O\left(t^8\right)\right) \end{aligned}$$

Replacing t in the above with the original independent variable $x$susing $t = x-1$ results in

$$y = c_1(x-1)\left(1+O\left((x-1)^8\right)\right) + c_2\left(2(x-1)\left(1+O\left((x-1)^8\right)\right)\ln(x-1) + 1 - 3(x-1)^2 + 2(x-1)^3 - \frac{5(x-1)^4}{3} + \frac{3(x-1)^5}{2} - \frac{7(x-1)^6}{5} + \frac{4(x-1)^7}{3} + O\left((x-1)^8\right)\right)$$

Summary

The solution(s) found are the following

$$y = c_1(x-1)\left(1+O\left((x-1)^8\right)\right) + c_2\left(2(x-1)\left(1+O\left((x-1)^8\right)\right)\ln(x-1) + 1 - 3(x-1)^2 + 2(x-1)^3 - \frac{5(x-1)^4}{3} + \frac{3(x-1)^5}{2} - \frac{7(x-1)^6}{5} + \frac{4(x-1)^7}{3} + O\left((x-1)^8\right)\right) \tag{1}$$

Verification of solutions

$$y = c_1(x-1)\left(1+O\left((x-1)^8\right)\right) + c_2\left(2(x-1)\left(1+O\left((x-1)^8\right)\right)\ln(x-1) + 1 - 3(x-1)^2 + 2(x-1)^3 - \frac{5(x-1)^4}{3} + \frac{3(x-1)^5}{2} - \frac{7(x-1)^6}{5} + \frac{4(x-1)^7}{3} + O\left((x-1)^8\right)\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 50

```
Order:=8;
dsolve(x*(1-x)*diff(y(x),x$2)+2*(1-x)*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=1);
```

$$y(x) = \left(1 - 2(x-1) - 3(x-1)^2 + 2(x-1)^3 - \frac{5}{3}(x-1)^4 + \frac{3}{2}(x-1)^5 - \frac{7}{5}(x-1)^6 + \frac{4}{3}(x-1)^7 + \mathrm{O}\left((x-1)^8\right)\right) c_2 + c_1(x-1)\left(1 + \mathrm{O}\left((x-1)^8\right)\right) + \left(2(x-1) + \mathrm{O}\left((x-1)^8\right)\right) \ln(x-1)\, c_2$$

✓ Solution by Mathematica

Time used: 0.422 (sec). Leaf size: 69

```
AsymptoticDSolveValue[x*(1-x)*y''[x]+2*(1-x)*y'[x]+2*y[x]==0,y[x],{x,1,7}]
```

$$y(x) \to c_2(x-1) + c_1\left(\frac{1}{30}\left(-42(x-1)^6 + 45(x-1)^5 - 50(x-1)^4 + 60(x-1)^3 - 90(x-1)^2 - 90(x-1) + 30\right) + 2(x-1)\log(x-1)\right)$$

7.9 problem 10 (as direct Bessel)

Internal problem ID [6989]
Internal file name [`OUTPUT/6232_Friday_August_12_2022_11_06_25_PM_87387057/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.9 Indicial Equation with Difference of Roots a Positive Integer: Logarithmic Case. Exercises page 384
Problem number: 10 (as direct Bessel).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[_Bessel]
```

$$x^2y'' + xy' + \left(x^2 - 1\right)y = 0$$

7.9.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - 1\right)y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + xy' + \left(-n^2 + x^2\right)y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha(c_1 \operatorname{BesselJ}(n, \beta x^\gamma) + c_2 \operatorname{BesselY}(n, \beta x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = 0$$
$$\beta = 1$$
$$n = -1$$
$$\gamma = 1$$

Substituting all the above into (4) gives the solution as

$$y = -c_1 \operatorname{BesselJ}(1, x) - c_2 \operatorname{BesselY}(1, x)$$

Summary

The solution(s) found are the following

$$y = -c_1 \operatorname{BesselJ}(1, x) - c_2 \operatorname{BesselY}(1, x) \tag{1}$$

Verification of solutions

$$y = -c_1 \operatorname{BesselJ}(1, x) - c_2 \operatorname{BesselY}(1, x)$$

Verified OK.

7.9.2 Maple step by step solution

Let's solve

$x^2y'' + xy' + (x^2 - 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2-1)y}{x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(x^2-1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{x^2-1}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -1$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' + x y' + (x^2 - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $$a_0 (1 + r)\,(-1 + r)\, x^r + a_1 (2 + r)\, r\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k (k + r + 1)\,(k + r - 1) + a_{k-2}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + r)\,(-1 + r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 1\}$

- Each term must be 0

  $a_1(2+r)\,r = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)\,(k+r-1) + a_{k-2} = 0$

- Shift index using $k- >k+2$

  $a_{k+2}(k+3+r)\,(k+r+1) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+3+r)(k+r+1)}$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+2)k}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+2)k}, a_1 = 0\right]$

- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{a_k}{(k+4)(k+2)}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{(k+4)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+2} = -\frac{a_k}{(k+2)k}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+4)(k+2)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \operatorname{BesselJ}(1, x) + c_2 \operatorname{BesselY}(1, x)$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 18

```
DSolve[x^2*y''[x]+x*y'[x]+(x^2-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \operatorname{BesselJ}(1, x) + c_2 \operatorname{BesselY}(1, x)$$

7.10 problem 10 (as series)

Internal problem ID [6990]
Internal file name [OUTPUT/6233_Friday_August_12_2022_11_06_26_PM_22920425/index.tex]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.9 Indicial Equation with Difference of Roots a Positive Integer: Logarithmic Case. Exercises page 384
Problem number: 10 (as series).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[_Bessel]
```

$$\boxed{x^2y'' + xy' + (x^2 - 1)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + xy' + (x^2 - 1)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{x^2 - 1}{x^2}$$

Table 167: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x^2-1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + xy' + (x^2 - 1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} \right) + x \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \right) + (x^2 - 1) \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2-1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-2} - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2 - 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-2}}{n(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{r^2 + 4r + 3}$$

Which for the root $r = 1$ becomes

$$a_2 = -\frac{1}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+4r+3}$	$-\frac{1}{8}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+4r+3}$	$-\frac{1}{8}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r+3)^2 (1+r) (r+5)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{192}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+4r+3}$	$-\frac{1}{8}$
a_3	0	0
a_4	$\frac{1}{(r+3)^2(1+r)(r+5)}$	$\frac{1}{192}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+4r+3}$	$-\frac{1}{8}$
a_3	0	0
a_4	$\frac{1}{(r+3)^2(1+r)(r+5)}$	$\frac{1}{192}$
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = -\frac{1}{(r+3)^2 (1+r) (r+5)^2 (r+7)}$$

Which for the root $r = 1$ becomes

$$a_6 = -\frac{1}{9216}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+4r+3}$	$-\frac{1}{8}$
a_3	0	0
a_4	$\frac{1}{(r+3)^2(1+r)(r+5)}$	$\frac{1}{192}$
a_5	0	0
a_6	$-\frac{1}{(r+3)^2(1+r)(r+5)^2(r+7)}$	$-\frac{1}{9216}$

For $n = 7$, using the above recursive equation gives

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+4r+3}$	$-\frac{1}{8}$
a_3	0	0
a_4	$\frac{1}{(r+3)^2(1+r)(r+5)}$	$\frac{1}{192}$
a_5	0	0
a_6	$-\frac{1}{(r+3)^2(1+r)(r+5)^2(r+7)}$	$-\frac{1}{9216}$
a_7	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\ &= x\left(1 - \frac{x^2}{8} + \frac{x^4}{192} - \frac{x^6}{9216} + O\left(x^8\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\rightarrow r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_2 \\
&= -\frac{1}{r^2+4r+3}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r\rightarrow r_2} -\frac{1}{r^2+4r+3} &= \lim_{r\rightarrow -1} -\frac{1}{r^2+4r+3} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)
$$

Therefore

$$
\begin{aligned}
\frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\
\frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $x^2 y'' + xy' + (x^2 - 1)\, y = 0$ gives

$$
\begin{aligned}
& x^2 \left(C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \right. \\
& \left. + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2} \right) \right) \\
& + x \left(C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n+r_2)}{x} \right) \right) \\
& + (x^2 - 1) \left(C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2} \right) \right) = 0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
& \left(\left(x^2 y_1''(x) + y_1'(x)\, x + (x^2 - 1)\, y_1(x) \right) \ln(x) + x^2 \left(\frac{2 y_1'(x)}{x} - \frac{y_1(x)}{x^2} \right) + y_1(x) \right) C \\
& + x^2 \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2} \right) \right) \\
& + x \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n+r_2)}{x} \right) + (x^2 - 1) \left(\sum_{n=0}^{\infty} b_n x^{n+r_2} \right) = 0
\end{aligned} \tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$
x^2 y_1''(x) + y_1'(x)\, x + (x^2 - 1)\, y_1(x) = 0
$$

Eq (7) simplifes to

$$
\begin{aligned}
& \left(x^2 \left(\frac{2 y_1'(x)}{x} - \frac{y_1(x)}{x^2} \right) + y_1(x) \right) C \\
& + x^2 \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2} \right) \right) \\
& + x \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n+r_2)}{x} \right) + (x^2 - 1) \left(\sum_{n=0}^{\infty} b_n x^{n+r_2} \right) = 0
\end{aligned} \tag{8}
$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$
\begin{aligned}
&2x \left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) C + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x^2 \\
&+ \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) x^2 + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) x - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{9}
$$

Since $r_1 = 1$ and $r_2 = -1$ then the above becomes

$$
\begin{aligned}
&2x \left(\sum_{n=0}^{\infty} x^{n} a_n (n+1)\right) C + \left(\sum_{n=0}^{\infty} x^{-3+n} b_n (n-1)(n-2)\right) x^2 \\
&+ \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right) x^2 + \left(\sum_{n=0}^{\infty} x^{n-2} b_n (n-1)\right) x - \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right) = 0
\end{aligned} \tag{10}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{n+1} a_n (n+1)\right) + \left(\sum_{n=0}^{\infty} x^{n-1} b_n (n^2 - 3n + 2)\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n+1} b_n\right) + \left(\sum_{n=0}^{\infty} x^{n-1} b_n (n-1)\right) + \sum_{n=0}^{\infty} \left(-b_n x^{n-1}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 2C\, x^{n+1} a_n (n+1) &= \sum_{n=2}^{\infty} 2C a_{n-2} (n-1)\, x^{n-1} \\
\sum_{n=0}^{\infty} x^{n+1} b_n &= \sum_{n=2}^{\infty} b_{n-2} x^{n-1}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\left(\sum_{n=2}^{\infty} 2C a_{n-2}(n-1)\, x^{n-1}\right)+\left(\sum_{n=0}^{\infty} x^{n-1} b_n \left(n^2-3n+2\right)\right) + \left(\sum_{n=2}^{\infty} b_{n-2} x^{n-1}\right)+\left(\sum_{n=0}^{\infty} x^{n-1} b_n (n-1)\right)+\sum_{n=0}^{\infty} \left(-b_n x^{n-1}\right)=0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$-b_1=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-b_1=0$$

Solving the above for b_1 gives

$$b_1=0$$

For $n=N$, where $N=2$ which is the difference between the two roots, we are free to choose $b_2=0$. Hence for $n=2$, Eq (2B) gives

$$2C+1=0$$

Which is solved for C. Solving for C gives

$$C=-\frac{1}{2}$$

For $n=3$, Eq (2B) gives

$$4Ca_1+b_1+3b_3=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$3b_3=0$$

Solving the above for b_3 gives

$$b_3=0$$

For $n=4$, Eq (2B) gives

$$6Ca_2+b_2+8b_4=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$8b_4+\frac{3}{8}=0$$

Solving the above for b_4 gives

$$b_4 = -\frac{3}{64}$$

For $n = 5$, Eq (2B) gives

$$8Ca_3 + b_3 + 15b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$15b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

For $n = 6$, Eq (2B) gives

$$10Ca_4 + b_4 + 24b_6 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$24b_6 - \frac{7}{96} = 0$$

Solving the above for b_6 gives

$$b_6 = \frac{7}{2304}$$

For $n = 7$, Eq (2B) gives

$$12Ca_5 + b_5 + 35b_7 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$35b_7 = 0$$

Solving the above for b_7 gives

$$b_7 = 0$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{2}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{1}{2}\left(x\left(1 - \frac{x^2}{8} + \frac{x^4}{192} - \frac{x^6}{9216} + O(x^8)\right)\right)\ln(x) + \frac{1 - \frac{3x^4}{64} + \frac{7x^6}{2304} + O(x^8)}{x}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1-\frac{x^2}{8}+\frac{x^4}{192}-\frac{x^6}{9216}+O\left(x^8\right)\right) \\
&\quad + c_2\left(-\frac{1}{2}\left(x\left(1-\frac{x^2}{8}+\frac{x^4}{192}-\frac{x^6}{9216}+O\left(x^8\right)\right)\right)\ln\left(x\right)+\frac{1-\frac{3x^4}{64}+\frac{7x^6}{2304}+O\left(x^8\right)}{x}\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x\left(1-\frac{x^2}{8}+\frac{x^4}{192}-\frac{x^6}{9216}+O\left(x^8\right)\right) \\
&\quad + c_2\left(-\frac{x\left(1-\frac{x^2}{8}+\frac{x^4}{192}-\frac{x^6}{9216}+O\left(x^8\right)\right)\ln\left(x\right)}{2}+\frac{1-\frac{3x^4}{64}+\frac{7x^6}{2304}+O\left(x^8\right)}{x}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x\left(1-\frac{x^2}{8}+\frac{x^4}{192}-\frac{x^6}{9216}+O\left(x^8\right)\right) \\
&\quad + c_2\left(-\frac{x\left(1-\frac{x^2}{8}+\frac{x^4}{192}-\frac{x^6}{9216}+O\left(x^8\right)\right)\ln\left(x\right)}{2}+\frac{1-\frac{3x^4}{64}+\frac{7x^6}{2304}+O\left(x^8\right)}{x}\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x\left(1-\frac{x^2}{8}+\frac{x^4}{192}-\frac{x^6}{9216}+O\left(x^8\right)\right) \\
&\quad + c_2\left(-\frac{x\left(1-\frac{x^2}{8}+\frac{x^4}{192}-\frac{x^6}{9216}+O\left(x^8\right)\right)\ln\left(x\right)}{2}+\frac{1-\frac{3x^4}{64}+\frac{7x^6}{2304}+O\left(x^8\right)}{x}\right)
\end{aligned}
$$

Verified OK.

7.10.1 Maple step by step solution

Let's solve

$x^2y'' + xy' + (x^2 - 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2-1)y}{x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(x^2-1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{x^2-1}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -1$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + xy' + (x^2 - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

  $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+r)\,(-1+r)\, x^r + a_1(2+r)\, r\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+1)\,(k+r-1) + a_{k-2})\, x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+r)\,(-1+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-1, 1\}$$

- Each term must be 0

  $$a_1(2+r)\, r = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(k+r+1)\,(k+r-1) + a_{k-2} = 0$$

- Shift index using $k- >k+2$

  $$a_{k+2}(k+3+r)\,(k+r+1) + a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{a_k}{(k+3+r)(k+r+1)}$$

- Recursion relation for $r = -1$

  $$a_{k+2} = -\frac{a_k}{(k+2)k}$$

- Solution for $r = -1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+2)k}, a_1 = 0\right]$$

- Recursion relation for $r=1$

  $a_{k+2} = -\frac{a_k}{(k+4)(k+2)}$

- Solution for $r=1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{(k+4)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+2} = -\frac{a_k}{(k+2)k}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+4)(k+2)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 53

```
Order:=8;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x^2 \left(1 - \frac{1}{8}x^2 + \frac{1}{192}x^4 - \frac{1}{9216}x^6 + \mathrm{O}\left(x^8\right)\right) + c_2\left(\left(x^2 - \frac{1}{8}x^4 + \frac{1}{192}x^6 + \mathrm{O}\left(x^8\right)\right)\ln\left(x\right) + \left(-2 + \frac{3}{32}x^4 - \frac{7}{1152}x\right.\right.}{x}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 75

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]+(x^2-1)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(-\frac{x^7}{9216}+\frac{x^5}{192}-\frac{x^3}{8}+x\right) + c_1\left(\frac{5x^6-90x^4+288x^2+1152}{1152x}-\frac{1}{384}x\left(x^4-24x^2+192\right)\log(x)\right)$$

7.11 problem 11

Internal problem ID [6991]
Internal file name [`OUTPUT/6234_Friday_August_12_2022_11_06_28_PM_76142612/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.9 Indicial Equation with Difference of Roots a Positive Integer: Logarithmic Case. Exercises page 384
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 5xy' + (8+5x)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' - 5xy' + (8+5x)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{5}{x}$$
$$q(x) = \frac{8+5x}{x^2}$$

Table 169: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{5}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{8+5x}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' - 5xy' + (8 + 5x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2} \right) - 5x \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1} \right) + (8 + 5x) \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-5x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 8a_n x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 5x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 5x^{1+n+r} a_n = \sum_{n=1}^{\infty} 5a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-5x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 8a_n x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 5a_{n-1} x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 5x^{n+r} a_n (n+r) + 8a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - 5x^r a_0 r + 8a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 5x^r r + 8x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - 6r + 8\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 6r + 8 = 0$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= 4 \\
r_2 &= 2
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
\left(r^2 - 6r + 8\right) x^r = 0
$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= x^4 \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + x^2 \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+4} \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+2}\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
a_n(n+r)(n+r-1) - 5a_n(n+r) + 8a_n + 5a_{n-1} = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{5a_{n-1}}{n^2 + 2nr + r^2 - 6n - 6r + 8} \tag{4}
$$

Which for the root $r = 4$ becomes

$$a_n = -\frac{5a_{n-1}}{n(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 4$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{5}{r^2 - 4r + 3}$$

Which for the root $r = 4$ becomes

$$a_1 = -\frac{5}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{5}{r^2-4r+3}$	$-\frac{5}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{25}{(-1+r)(r-3)r(r-2)}$$

Which for the root $r = 4$ becomes

$$a_2 = \frac{25}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{5}{r^2-4r+3}$	$-\frac{5}{3}$
a_2	$\frac{25}{(-1+r)(r-3)r(r-2)}$	$\frac{25}{24}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{125}{(-1+r)^2 (r-3) r (r-2) (r+1)}$$

Which for the root $r = 4$ becomes

$$a_3 = -\frac{25}{72}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{5}{r^2-4r+3}$	$-\frac{5}{3}$
a_2	$\frac{25}{(-1+r)(r-3)r(r-2)}$	$\frac{25}{24}$
a_3	$-\frac{125}{(-1+r)^2(r-3)r(r-2)(r+1)}$	$-\frac{25}{72}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{625}{(-1+r)^2 (r-3) r^2 (r+1) (r^2-4)}$$

Which for the root $r = 4$ becomes

$$a_4 = \frac{125}{1728}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{5}{r^2-4r+3}$	$-\frac{5}{3}$
a_2	$\frac{25}{(-1+r)(r-3)r(r-2)}$	$\frac{25}{24}$
a_3	$-\frac{125}{(-1+r)^2(r-3)r(r-2)(r+1)}$	$-\frac{25}{72}$
a_4	$\frac{625}{(-1+r)^2(r-3)r^2(r+1)(r^2-4)}$	$\frac{125}{1728}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{3125}{(-1+r)^2 (r-3) r^2 (r+1)^2 (r-2) (r+2) (r+3)}$$

Which for the root $r = 4$ becomes

$$a_5 = -\frac{125}{12096}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{5}{r^2-4r+3}$	$-\frac{5}{3}$
a_2	$\frac{25}{(-1+r)(r-3)r(r-2)}$	$\frac{25}{24}$
a_3	$-\frac{125}{(-1+r)^2(r-3)r(r-2)(r+1)}$	$-\frac{25}{72}$
a_4	$\frac{625}{(-1+r)^2(r-3)r^2(r+1)(r^2-4)}$	$\frac{125}{1728}$
a_5	$-\frac{3125}{(-1+r)^2(r-3)r^2(r+1)^2(r-2)(r+2)(r+3)}$	$-\frac{125}{12096}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{15625}{(-1+r)^2 r^2 (r+1)^2 (r^2+6r+8)(r^2-9)(r^2-4)}$$

Which for the root $r = 4$ becomes

$$a_6 = \frac{625}{580608}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{5}{r^2-4r+3}$	$-\frac{5}{3}$
a_2	$\frac{25}{(-1+r)(r-3)r(r-2)}$	$\frac{25}{24}$
a_3	$-\frac{125}{(-1+r)^2(r-3)r(r-2)(r+1)}$	$-\frac{25}{72}$
a_4	$\frac{625}{(-1+r)^2(r-3)r^2(r+1)(r^2-4)}$	$\frac{125}{1728}$
a_5	$-\frac{3125}{(-1+r)^2(r-3)r^2(r+1)^2(r-2)(r+2)(r+3)}$	$-\frac{125}{12096}$
a_6	$\frac{15625}{(-1+r)^2r^2(r+1)^2(r^2+6r+8)(r^2-9)(r^2-4)}$	$\frac{625}{580608}$

For $n = 7$, using the above recursive equation gives

$$a_7 = -\frac{78125}{(-1+r)^2 r^2 (r+1)^2 (r+4)(r+2)^2 (r-3)(r+3)^2 (r-2)(r+5)}$$

Which for the root $r = 4$ becomes

$$a_7 = -\frac{3125}{36578304}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{5}{r^2-4r+3}$	$-\frac{5}{3}$
a_2	$\frac{25}{(-1+r)(r-3)r(r-2)}$	$\frac{25}{24}$
a_3	$-\frac{125}{(-1+r)^2(r-3)r(r-2)(r+1)}$	$-\frac{25}{72}$
a_4	$\frac{625}{(-1+r)^2(r-3)r^2(r+1)(r^2-4)}$	$\frac{125}{1728}$
a_5	$-\frac{3125}{(-1+r)^2(r-3)r^2(r+1)^2(r-2)(r+2)(r+3)}$	$-\frac{125}{12096}$
a_6	$\frac{15625}{(-1+r)^2r^2(r+1)^2(r^2+6r+8)(r^2-9)(r^2-4)}$	$\frac{625}{580608}$
a_7	$-\frac{78125}{(-1+r)^2r^2(r+1)^2(r+4)(r+2)^2(r-3)(r+3)^2(r-2)(r+5)}$	$-\frac{3125}{36578304}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^4\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \dots\right) \\
&= x^4\left(1 - \frac{5x}{3} + \frac{25x^2}{24} - \frac{25x^3}{72} + \frac{125x^4}{1728} - \frac{125x^5}{12096} + \frac{625x^6}{580608} - \frac{3125x^7}{36578304} + O(x^8)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}
a_N &= a_2 \\
&= \frac{25}{(-1+r)(r-3)r(r-2)}
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{r\to r_2} \frac{25}{(-1+r)(r-3)r(r-2)} &= \lim_{r\to 2} \frac{25}{(-1+r)(r-3)r(r-2)} \\
&= \textit{undefined}
\end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $x^2y'' - 5xy' + (8+5x)\,y = 0$ gives

$$\begin{aligned}
&x^2\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left.\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&\quad - 5x\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) \\
&\quad + (8+5x)\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0
\end{aligned}$$

Which can be written as

$$\left(\left(\left(x^{2} y_1''(x)-5 y_1'(x) x+(8+5 x) y_1(x)\right) \ln (x)+x^{2}\left(\frac{2 y_1'(x)}{x}-\frac{y_1(x)}{x^{2}}\right)\right.\right.$$
$$\left.\left.-5 y_1(x)\right) C+x^{2}\left(\sum_{n=0}^{\infty}\left(\frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)^{2}}{x^{2}}-\frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)}{x^{2}}\right)\right)\right)$$
$$-5 x\left(\sum_{n=0}^{\infty} \frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)}{x}\right)+(8+5 x)\left(\sum_{n=0}^{\infty} b_{n} x^{n+r_{2}}\right)=0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$x^{2} y_1''(x)-5 y_1'(x) x+(8+5 x) y_1(x)=0$$

Eq (7) simplifes to

$$\left(x^{2}\left(\frac{2 y_1'(x)}{x}-\frac{y_1(x)}{x^{2}}\right)-5 y_1(x)\right) C$$
$$+x^{2}\left(\sum_{n=0}^{\infty}\left(\frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)^{2}}{x^{2}}-\frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)}{x^{2}}\right)\right)$$
$$-5 x\left(\sum_{n=0}^{\infty} \frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)}{x}\right)+(8+5 x)\left(\sum_{n=0}^{\infty} b_{n} x^{n+r_{2}}\right)=0 \tag{8}$$

Substituting $y_1=\sum_{n=0}^{\infty} a_{n} x^{n+r_{1}}$ into the above gives

$$\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_{1}} a_{n}\left(n+r_{1}\right)\right) x-6\left(\sum_{n=0}^{\infty} a_{n} x^{n+r_{1}}\right)\right) C$$
$$+\left(\sum_{n=0}^{\infty} x^{-2+n+r_{2}} b_{n}\left(n+r_{2}\right)\left(-1+n+r_{2}\right)\right) x^{2}$$
$$-5\left(\sum_{n=0}^{\infty} x^{-1+n+r_{2}} b_{n}\left(n+r_{2}\right)\right) x+(8+5 x)\left(\sum_{n=0}^{\infty} b_{n} x^{n+r_{2}}\right)=0 \tag{9}$$

Since $r_1 = 4$ and $r_2 = 2$ then the above becomes

$$\left(2\left(\sum_{n=0}^{\infty} x^{3+n} a_n (n+4)\right) x - 6\left(\sum_{n=0}^{\infty} a_n x^{n+4}\right)\right) C + \left(\sum_{n=0}^{\infty} x^n b_n (n+2)(1+n)\right) x^2 - 5\left(\sum_{n=0}^{\infty} x^{1+n} b_n (n+2)\right) x + (8+5x)\left(\sum_{n=0}^{\infty} b_n x^{n+2}\right) = 0 \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2C\, x^{n+4} a_n (n+4)\right) + \sum_{n=0}^{\infty} \left(-6C a_n x^{n+4}\right) + \left(\sum_{n=0}^{\infty} x^{n+2} b_n \left(n^2+3n+2\right)\right) + \sum_{n=0}^{\infty} \left(-5x^{n+2} b_n (n+2)\right) + \left(\sum_{n=0}^{\infty} 8 b_n x^{n+2}\right) + \left(\sum_{n=0}^{\infty} 5x^{3+n} b_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+2$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+2} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{n+4} a_n (n+4) = \sum_{n=2}^{\infty} 2C a_{n-2} (n+2)\, x^{n+2}$$

$$\sum_{n=0}^{\infty} \left(-6C a_n x^{n+4}\right) = \sum_{n=2}^{\infty} \left(-6C a_{n-2} x^{n+2}\right)$$

$$\sum_{n=0}^{\infty} 5x^{3+n} b_n = \sum_{n=1}^{\infty} 5 b_{n-1} x^{n+2}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+2$.

$$\left(\sum_{n=2}^{\infty} 2Ca_{n-2}(n+2)\, x^{n+2}\right) + \sum_{n=2}^{\infty} \left(-6Ca_{n-2}x^{n+2}\right) + \left(\sum_{n=0}^{\infty} x^{n+2}b_n\left(n^2+3n+2\right)\right) + \sum_{n=0}^{\infty} \left(-5x^{n+2}b_n(n+2)\right) + \left(\sum_{n=0}^{\infty} 8b_n x^{n+2}\right) + \left(\sum_{n=1}^{\infty} 5b_{n-1}x^{n+2}\right) = 0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$-b_1+5b_0=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-b_1+5=0$$

Solving the above for b_1 gives

$$b_1=5$$

For $n=N$, where $N=2$ which is the difference between the two roots, we are free to choose $b_2=0$. Hence for $n=2$, Eq (2B) gives

$$2C+25=0$$

Which is solved for C. Solving for C gives

$$C=-\frac{25}{2}$$

For $n=3$, Eq (2B) gives

$$4Ca_1+5b_2+3b_3=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$3b_3+\frac{250}{3}=0$$

Solving the above for b_3 gives

$$b_3=-\frac{250}{9}$$

For $n=4$, Eq (2B) gives

$$6Ca_2+5b_3+8b_4=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$8b_4 - \frac{15625}{72} = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{15625}{576}$$

For $n = 5$, Eq (2B) gives

$$8Ca_3 + 5b_4 + 15b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$15b_5 + \frac{98125}{576} = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{19625}{1728}$$

For $n = 6$, Eq (2B) gives

$$10Ca_4 + 5b_5 + 24b_6 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$24b_6 - \frac{56875}{864} = 0$$

Solving the above for b_6 gives

$$b_6 = \frac{56875}{20736}$$

For $n = 7$, Eq (2B) gives

$$12Ca_5 + 5b_6 + 35b_7 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$35b_7 + \frac{2215625}{145152} = 0$$

Solving the above for b_7 gives

$$b_7 = -\frac{443125}{1016064}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{25}{2}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{25}{2}\left(x^4\left(1 - \frac{5x}{3} + \frac{25x^2}{24} - \frac{25x^3}{72} + \frac{125x^4}{1728} - \frac{125x^5}{12096} + \frac{625x^6}{580608} - \frac{3125x^7}{36578304} + O(x^8)\right)\right)\ln(x) + x^2\left(1 + 5x - \frac{250x^3}{9} + \frac{15625x^4}{576} - \frac{19625x^5}{1728} + \frac{56875x^6}{20736} - \frac{443125x^7}{1016064} + O(x^8)\right)$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^4\left(1 - \frac{5x}{3} + \frac{25x^2}{24} - \frac{25x^3}{72} + \frac{125x^4}{1728} - \frac{125x^5}{12096} + \frac{625x^6}{580608} - \frac{3125x^7}{36578304} + O(x^8)\right) \\ &\quad + c_2\left(-\frac{25}{2}\left(x^4\left(1 - \frac{5x}{3} + \frac{25x^2}{24} - \frac{25x^3}{72} + \frac{125x^4}{1728} - \frac{125x^5}{12096} + \frac{625x^6}{580608} - \frac{3125x^7}{36578304} + O(x^8)\right)\right)\ln(x) \right. \\ &\quad \left. + x^2\left(1 + 5x - \frac{250x^3}{9} + \frac{15625x^4}{576} - \frac{19625x^5}{1728} + \frac{56875x^6}{20736} - \frac{443125x^7}{1016064} + O(x^8)\right)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^4\left(1 - \frac{5x}{3} + \frac{25x^2}{24} - \frac{25x^3}{72} + \frac{125x^4}{1728} - \frac{125x^5}{12096} + \frac{625x^6}{580608} - \frac{3125x^7}{36578304} + O(x^8)\right) \\ &\quad + c_2\left(-\frac{25x^4\left(1 - \frac{5x}{3} + \frac{25x^2}{24} - \frac{25x^3}{72} + \frac{125x^4}{1728} - \frac{125x^5}{12096} + \frac{625x^6}{580608} - \frac{3125x^7}{36578304} + O(x^8)\right)\ln(x)}{2} \right. \\ &\quad \left. + x^2\left(1 + 5x - \frac{250x^3}{9} + \frac{15625x^4}{576} - \frac{19625x^5}{1728} + \frac{56875x^6}{20736} - \frac{443125x^7}{1016064} + O(x^8)\right)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^4 \left(1-\frac{5x}{3}+\frac{25x^2}{24}-\frac{25x^3}{72}+\frac{125x^4}{1728}-\frac{125x^5}{12096}+\frac{625x^6}{580608}-\frac{3125x^7}{36578304}+O\left(x^8\right)\right) \\
+ c_2 \left(-\frac{25x^4\left(1-\frac{5x}{3}+\frac{25x^2}{24}-\frac{25x^3}{72}+\frac{125x^4}{1728}-\frac{125x^5}{12096}+\frac{625x^6}{580608}-\frac{3125x^7}{36578304}+O(x^8)\right)\ln(x)}{2}\right. \\
\left. + x^2\left(1+5x-\frac{250x^3}{9}+\frac{15625x^4}{576}-\frac{19625x^5}{1728}+\frac{56875x^6}{20736}-\frac{443125x^7}{1016064}+O\left(x^8\right)\right)\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1 x^4 \left(1-\frac{5x}{3}+\frac{25x^2}{24}-\frac{25x^3}{72}+\frac{125x^4}{1728}-\frac{125x^5}{12096}+\frac{625x^6}{580608}-\frac{3125x^7}{36578304}+O\left(x^8\right)\right) \\
+ c_2 \left(-\frac{25x^4\left(1-\frac{5x}{3}+\frac{25x^2}{24}-\frac{25x^3}{72}+\frac{125x^4}{1728}-\frac{125x^5}{12096}+\frac{625x^6}{580608}-\frac{3125x^7}{36578304}+O(x^8)\right)\ln(x)}{2}\right. \\
\left. + x^2\left(1+5x-\frac{250x^3}{9}+\frac{15625x^4}{576}-\frac{19625x^5}{1728}+\frac{56875x^6}{20736}-\frac{443125x^7}{1016064}+O\left(x^8\right)\right)\right)
\end{aligned}
$$

Verified OK.

7.11.1 Maple step by step solution

Let's solve

$x^2y'' - 5xy' + (8+5x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{5y'}{x} - \frac{(8+5x)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{5y'}{x} + \frac{(8+5x)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = -\frac{5}{x}, P_3(x) = \frac{8+5x}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -5$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 8$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$x^2 y'' - 5xy' + (8 + 5x)\, y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k->k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

  $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-2+r)\,(-4+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-2)\,(k+r-4) + 5a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-2+r)\,(-4+r) = 0$$

- Values of r that satisfy the indicial equation

  $r \in \{2, 4\}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-2)(k+r-4) + 5a_{k-1} = 0$

- Shift index using $k->k+1$

  $a_{k+1}(k+r-1)(k-3+r) + 5a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{5a_k}{(k+r-1)(k-3+r)}$

- Recursion relation for $r = 2$

  $a_{k+1} = -\frac{5a_k}{(k+1)(k-1)}$

- Series not valid for $r = 2$, division by 0 in the recursion relation at $k = 1$

  $a_{k+1} = -\frac{5a_k}{(k+1)(k-1)}$

- Recursion relation for $r = 4$

  $a_{k+1} = -\frac{5a_k}{(k+3)(k+1)}$

- Solution for $r = 4$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+4}, a_{k+1} = -\frac{5a_k}{(k+3)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 77

```
Order:=8;
dsolve(x^2*diff(y(x),x$2)-5*x*diff(y(x),x)+(8+5*x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = x^2 \left(c_1 x^2 \left(1 - \frac{5}{3}x + \frac{25}{24}x^2 - \frac{25}{72}x^3 + \frac{125}{1728}x^4 - \frac{125}{12096}x^5 + \frac{625}{580608}x^6 - \frac{3125}{36578304}x^7 + \mathrm{O}\left(x^8\right) \right) + c_2 \left(\ln(x) \left(25x^2 - \frac{125}{3}x^3 + \frac{625}{24}x^4 - \frac{625}{72}x^5 + \frac{3125}{1728}x^6 - \frac{3125}{12096}x^7 + \mathrm{O}\left(x^8\right) \right) + \left(-2 - 10x + \frac{500}{9}x^3 - \frac{15625}{288}x^4 + \frac{19625}{864}x^5 - \frac{56875}{10368}x^6 + \frac{443125}{508032}x^7 + \mathrm{O}\left(x^8\right) \right) \right) \right)$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 123

```
AsymptoticDSolveValue[x^2*y''[x]-5*x*y'[x]+(8+5*x)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \rightarrow c_1 \left(\frac{x^2(33125x^6 - 140250x^5 + 348750x^4 - 396000x^3 + 64800x^2 + 51840x + 10368)}{10368} - \frac{25x^4(125x^4 - 600x^3 + 1800x^2 - 2880x + 1728)\log(x)}{3456} \right) + c_2 \left(\frac{625x^{10}}{580608} - \frac{125x^9}{12096} + \frac{125x^8}{1728} - \frac{25x^7}{72} + \frac{25x^6}{24} - \frac{5x^5}{3} + x^4 \right)$$

7.12 problem 12

Internal problem ID [6992]
Internal file name [`OUTPUT/6235_Friday_August_12_2022_11_06_30_PM_47983974/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.9 Indicial Equation with Difference of Roots a Positive Integer: Logarithmic Case. Exercises page 384
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

`[_Laguerre]`

$$\boxed{xy'' + (3 - x)\, y' - 5y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + (3 - x)\, y' - 5y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{-3 + x}{x}$$
$$q(x) = -\frac{5}{x}$$

Table 171: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{-3+x}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{5}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + (3 - x)\, y' - 5y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (3-x)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - 5\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-5 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-5 a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-5 a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-5 a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + 3(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + 3 r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + 3r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(2+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(2+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = -2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(2+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) + 3a_n(n+r) - 5a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(n+r+4)}{n^2+2nr+r^2+2n+2r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-1}(n+4)}{n\,(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{5+r}{r^2+4r+3}$$

Which for the root $r = 0$ becomes

$$a_1 = \frac{5}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{5+r}{r^2+4r+3}$	$\frac{5}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(5+r)(6+r)}{(r^2+4r+3)(r^2+6r+8)}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{5}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{5+r}{r^2+4r+3}$	$\frac{5}{3}$
a_2	$\frac{(5+r)(6+r)}{(r^2+4r+3)(r^2+6r+8)}$	$\frac{5}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{(6+r)(7+r)}{(r+3)^2 (r+1)(r+4)(2+r)}$$

Which for the root $r = 0$ becomes

$$a_3 = \frac{7}{12}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{5+r}{r^2+4r+3}$	$\frac{5}{3}$
a_2	$\frac{(5+r)(6+r)}{(r^2+4r+3)(r^2+6r+8)}$	$\frac{5}{4}$
a_3	$\frac{(6+r)(7+r)}{(r+3)^2(r+1)(r+4)(2+r)}$	$\frac{7}{12}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(7+r)(8+r)}{(r+4)^2 (r+3)^2 (r+1)(2+r)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{7}{36}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{5+r}{r^2+4r+3}$	$\frac{5}{3}$
a_2	$\frac{(5+r)(6+r)}{(r^2+4r+3)(r^2+6r+8)}$	$\frac{5}{4}$
a_3	$\frac{(6+r)(7+r)}{(r+3)^2(r+1)(r+4)(2+r)}$	$\frac{7}{12}$
a_4	$\frac{(7+r)(8+r)}{(r+4)^2(r+3)^2(r+1)(2+r)}$	$\frac{7}{36}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{(9+r)(8+r)}{(5+r)(r+4)^2 (r+3)^2 (r+1)(2+r)}$$

Which for the root $r = 0$ becomes

$$a_5 = \frac{1}{20}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{5+r}{r^2+4r+3}$	$\frac{5}{3}$
a_2	$\frac{(5+r)(6+r)}{(r^2+4r+3)(r^2+6r+8)}$	$\frac{5}{4}$
a_3	$\frac{(6+r)(7+r)}{(r+3)^2(r+1)(r+4)(2+r)}$	$\frac{7}{12}$
a_4	$\frac{(7+r)(8+r)}{(r+4)^2(r+3)^2(r+1)(2+r)}$	$\frac{7}{36}$
a_5	$\frac{(9+r)(8+r)}{(5+r)(r+4)^2(r+3)^2(r+1)(2+r)}$	$\frac{1}{20}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{(10+r)(9+r)}{(6+r)(5+r)(r+4)^2(r+3)^2(r+1)(2+r)}$$

Which for the root $r = 0$ becomes

$$a_6 = \frac{1}{96}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{5+r}{r^2+4r+3}$	$\frac{5}{3}$
a_2	$\frac{(5+r)(6+r)}{(r^2+4r+3)(r^2+6r+8)}$	$\frac{5}{4}$
a_3	$\frac{(6+r)(7+r)}{(r+3)^2(r+1)(r+4)(2+r)}$	$\frac{7}{12}$
a_4	$\frac{(7+r)(8+r)}{(r+4)^2(r+3)^2(r+1)(2+r)}$	$\frac{7}{36}$
a_5	$\frac{(9+r)(8+r)}{(5+r)(r+4)^2(r+3)^2(r+1)(2+r)}$	$\frac{1}{20}$
a_6	$\frac{(10+r)(9+r)}{(6+r)(5+r)(r+4)^2(r+3)^2(r+1)(2+r)}$	$\frac{1}{96}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{(11+r)(10+r)}{(7+r)(6+r)(5+r)(r+4)^2(r+3)^2(r+1)(2+r)}$$

Which for the root $r = 0$ becomes

$$a_7 = \frac{11}{6048}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{5+r}{r^2+4r+3}$	$\frac{5}{3}$
a_2	$\frac{(5+r)(6+r)}{(r^2+4r+3)(r^2+6r+8)}$	$\frac{5}{4}$
a_3	$\frac{(6+r)(7+r)}{(r+3)^2(r+1)(r+4)(2+r)}$	$\frac{7}{12}$
a_4	$\frac{(7+r)(8+r)}{(r+4)^2(r+3)^2(r+1)(2+r)}$	$\frac{7}{36}$
a_5	$\frac{(9+r)(8+r)}{(5+r)(r+4)^2(r+3)^2(r+1)(2+r)}$	$\frac{1}{20}$
a_6	$\frac{(10+r)(9+r)}{(6+r)(5+r)(r+4)^2(r+3)^2(r+1)(2+r)}$	$\frac{1}{96}$
a_7	$\frac{(11+r)(10+r)}{(7+r)(6+r)(5+r)(r+4)^2(r+3)^2(r+1)(2+r)}$	$\frac{11}{6048}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \ldots \\ &= 1 + \frac{5x}{3} + \frac{5x^2}{4} + \frac{7x^3}{12} + \frac{7x^4}{36} + \frac{x^5}{20} + \frac{x^6}{96} + \frac{11x^7}{6048} + O(x^8) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_2 \\ &= \frac{(5+r)(6+r)}{(r^2+4r+3)(r^2+6r+8)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{(5+r)(6+r)}{(r^2+4r+3)(r^2+6r+8)} &= \lim_{r \to -2} \frac{(5+r)(6+r)}{(r^2+4r+3)(r^2+6r+8)} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}
\frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\
\frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $xy'' + (3-x)y' - 5y = 0$ gives

$$\begin{aligned}
&x \Bigg(C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\
&+ \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\Bigg) \\
&+ (3-x)\left(C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) \\
&- 5C y_1(x) \ln(x) - 5\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}$$

Which can be written as

$$\left(\left(y_1''(x)\,x+(3-x)\,y_1'(x)-5y_1(x)\right)\ln(x)+x\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)\right.$$
$$\left.+\frac{(3-x)\,y_1(x)}{x}\right)C+x\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \tag{7}$$
$$+(3-x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)-5\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1''(x)\,x+(3-x)\,y_1'(x)-5y_1(x)=0$$

Eq (7) simplifes to

$$\left(x\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+\frac{(3-x)\,y_1(x)}{x}\right)C$$
$$+x\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \tag{8}$$
$$+(3-x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)-5\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0$$

Substituting $y_1=\sum_{n=0}^{\infty}a_n x^{n+r_1}$ into the above gives

$$\frac{\left(2\left(\sum_{n=0}^{\infty}x^{-1+n+r_1}a_n(n+r_1)\right)x-(x-2)\left(\sum_{n=0}^{\infty}a_n x^{n+r_1}\right)\right)C}{x} \tag{9}$$
$$+\frac{\left(\sum_{n=0}^{\infty}x^{-2+n+r_2}b_n(n+r_2)(-1+n+r_2)\right)x^2+(-x^2+3x)\left(\sum_{n=0}^{\infty}x^{-1+n+r_2}b_n(n+r_2)\right)-5\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)x}{x}$$
$$=0$$

Since $r_1 = 0$ and $r_2 = -2$ then the above becomes

$$\frac{\left(2\left(\sum_{n=0}^{\infty} x^{n-1} a_n n\right) x - (x-2)\left(\sum_{n=0}^{\infty} a_n x^n\right)\right) C}{x} \tag{10}$$
$$+ \frac{\left(\sum_{n=0}^{\infty} x^{-4+n} b_n (n-2)(-3+n)\right) x^2 + (-x^2+3x)\left(\sum_{n=0}^{\infty} x^{-3+n} b_n (n-2)\right) - 5\left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) x}{x}$$
$$= 0$$

Which simplifies to

$$\begin{aligned}\left(\sum_{n=0}^{\infty} 2C\, x^{n-1} a_n n\right) + \sum_{n=0}^{\infty} (-C a_n x^n) + \left(\sum_{n=0}^{\infty} 2C\, x^{n-1} a_n\right) \\ + \left(\sum_{n=0}^{\infty} x^{-3+n} b_n (n^2 - 5n + 6)\right) + \sum_{n=0}^{\infty} \left(-x^{n-2} b_n (n-2)\right) \\ + \left(\sum_{n=0}^{\infty} 3x^{-3+n} b_n (n-2)\right) + \sum_{n=0}^{\infty} \left(-5 b_n x^{n-2}\right) = 0\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $-3+n$ in each summation term. Going over each summation term above with power of x in it which is not already x^{-3+n} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{n-1} a_n n = \sum_{n=2}^{\infty} 2C(n-2)\, a_{n-2} x^{-3+n}$$

$$\sum_{n=0}^{\infty} (-C a_n x^n) = \sum_{n=3}^{\infty} \left(-C a_{-3+n} x^{-3+n}\right)$$

$$\sum_{n=0}^{\infty} 2C\, x^{n-1} a_n = \sum_{n=2}^{\infty} 2C a_{n-2} x^{-3+n}$$

$$\sum_{n=0}^{\infty} \left(-x^{n-2} b_n (n-2)\right) = \sum_{n=1}^{\infty} \left(-b_{n-1}(-3+n)\, x^{-3+n}\right)$$

$$\sum_{n=0}^{\infty} \left(-5 b_n x^{n-2}\right) = \sum_{n=1}^{\infty} \left(-5 b_{n-1} x^{-3+n}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $-3+n$.

$$\begin{aligned}
&\left(\sum_{n=2}^{\infty} 2C(n-2)\, a_{n-2} x^{-3+n}\right)+\sum_{n\,=3}^{\infty}\left(-C a_{-3+n} x^{-3+n}\right) \\
&+\left(\sum_{n=2}^{\infty} 2C a_{n-2} x^{-3+n}\right)+\left(\sum_{n=0}^{\infty} x^{-3+n} b_n\left(n^2-5n+6\right)\right) \\
&+\sum_{n\,=1}^{\infty}\left(-b_{n-1}(-3+n)\, x^{-3+n}\right) \\
&+\left(\sum_{n=0}^{\infty} 3x^{-3+n} b_n(n-2)\right)+\sum_{n\,=1}^{\infty}\left(-5 b_{n-1} x^{-3+n}\right)=0
\end{aligned} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$-b_1-3b_0=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-b_1-3=0$$

Solving the above for b_1 gives

$$b_1=-3$$

For $n=N$, where $N=2$ which is the difference between the two roots, we are free to choose $b_2=0$. Hence for $n=2$, Eq (2B) gives

$$2C+12=0$$

Which is solved for C. Solving for C gives

$$C=-6$$

For $n=3$, Eq (2B) gives

$$\left(-a_0+4a_1\right)C-5b_2+3b_3=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-34+3b_3=0$$

Solving the above for b_3 gives

$$b_3=\frac{34}{3}$$

For $n = 4$, Eq (2B) gives

$$(-a_1 + 6a_2) C - 6b_3 + 8b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-103 + 8b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{103}{8}$$

For $n = 5$, Eq (2B) gives

$$(-a_2 + 8a_3) C - 7b_4 + 15b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{885}{8} + 15b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{59}{8}$$

For $n = 6$, Eq (2B) gives

$$(-a_3 + 10a_4) C - 8b_5 + 24b_6 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{403}{6} + 24b_6 = 0$$

Solving the above for b_6 gives

$$b_6 = \frac{403}{144}$$

For $n = 7$, Eq (2B) gives

$$(-a_4 + 12a_5) C - 9b_6 + 35b_7 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{6629}{240} + 35b_7 = 0$$

Solving the above for b_7 gives

$$b_7 = \frac{947}{1200}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -6$ and all b_n, then the second solution becomes

$$y_2(x) = (-6)\left(1 + \frac{5x}{3} + \frac{5x^2}{4} + \frac{7x^3}{12} + \frac{7x^4}{36} + \frac{x^5}{20} + \frac{x^6}{96} + \frac{11x^7}{6048} + O(x^8)\right)\ln(x) + \frac{1 - 3x + \frac{34x^3}{3} + \frac{103x^4}{8} + \frac{59x^5}{8} + \frac{403x^6}{144} + \frac{947x^7}{1200} + O(x^8)}{x^2}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1 + \frac{5x}{3} + \frac{5x^2}{4} + \frac{7x^3}{12} + \frac{7x^4}{36} + \frac{x^5}{20} + \frac{x^6}{96} + \frac{11x^7}{6048} + O(x^8)\right) \\
&\quad + c_2\left((-6)\left(1 + \frac{5x}{3} + \frac{5x^2}{4} + \frac{7x^3}{12} + \frac{7x^4}{36} + \frac{x^5}{20} + \frac{x^6}{96} + \frac{11x^7}{6048} + O(x^8)\right)\ln(x)\right. \\
&\quad \left. + \frac{1 - 3x + \frac{34x^3}{3} + \frac{103x^4}{8} + \frac{59x^5}{8} + \frac{403x^6}{144} + \frac{947x^7}{1200} + O(x^8)}{x^2}\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1\left(1 + \frac{5x}{3} + \frac{5x^2}{4} + \frac{7x^3}{12} + \frac{7x^4}{36} + \frac{x^5}{20} + \frac{x^6}{96} + \frac{11x^7}{6048} + O(x^8)\right) \\
&\quad + c_2\left(\left(-6 - 10x - \frac{15x^2}{2} - \frac{7x^3}{2} - \frac{7x^4}{6} - \frac{3x^5}{10} - \frac{x^6}{16} - \frac{11x^7}{1008} - 6O(x^8)\right)\ln(x)\right. \\
&\quad \left. + \frac{1 - 3x + \frac{34x^3}{3} + \frac{103x^4}{8} + \frac{59x^5}{8} + \frac{403x^6}{144} + \frac{947x^7}{1200} + O(x^8)}{x^2}\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1\left(1+\frac{5x}{3}+\frac{5x^2}{4}+\frac{7x^3}{12}+\frac{7x^4}{36}+\frac{x^5}{20}+\frac{x^6}{96}+\frac{11x^7}{6048}+O(x^8)\right) \\
+ c_2\left(\left(-6-10x-\frac{15x^2}{2}-\frac{7x^3}{2}-\frac{7x^4}{6}-\frac{3x^5}{10}-\frac{x^6}{16}-\frac{11x^7}{1008}-6O(x^8)\right)\ln(x)(1)\right. \\
\left.+\frac{1-3x+\frac{34x^3}{3}+\frac{103x^4}{8}+\frac{59x^5}{8}+\frac{403x^6}{144}+\frac{947x^7}{1200}+O(x^8)}{x^2}\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1\left(1+\frac{5x}{3}+\frac{5x^2}{4}+\frac{7x^3}{12}+\frac{7x^4}{36}+\frac{x^5}{20}+\frac{x^6}{96}+\frac{11x^7}{6048}+O(x^8)\right) \\
+ c_2\left(\left(-6-10x-\frac{15x^2}{2}-\frac{7x^3}{2}-\frac{7x^4}{6}-\frac{3x^5}{10}-\frac{x^6}{16}-\frac{11x^7}{1008}-6O(x^8)\right)\ln(x)\right. \\
\left.+\frac{1-3x+\frac{34x^3}{3}+\frac{103x^4}{8}+\frac{59x^5}{8}+\frac{403x^6}{144}+\frac{947x^7}{1200}+O(x^8)}{x^2}\right)
\end{aligned}
$$

Verified OK.

7.12.1 Maple step by step solution

Let's solve

$xy'' + (3-x)\,y' - 5y = 0$

- Highest derivative means the order of the ODE is 2

  y''
- Isolate 2nd derivative

  $y'' = \frac{5y}{x} + \frac{(-3+x)y'}{x}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(-3+x)y'}{x} - \frac{5y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = -\frac{-3+x}{x}, P_3(x) = -\frac{5}{x}\right]$
- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$$(x \cdot P_2(x))\bigg|_{x=0} = 3$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$xy'' + (3 - x) y' - 5y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r) x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k + 1 - m + r) x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r) (k + r - 1) x^{k+r-1}$$

- Shift index using $k- >k + 1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k + 1 + r) (k + r) x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(2 + r) x^{-1+r} + \left(\sum_{k=0}^{\infty} (a_{k+1} (k + 1 + r) (k + 3 + r) - a_k (k + r + 5)) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(2 + r) = 0$$

- Values of r that satisfy the indicial equation

$r \in \{-2, 0\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+3+r) - a_k(k+r+5) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r+5)}{(k+1+r)(k+3+r)}$
- Recursion relation for $r = -2$

  $a_{k+1} = \frac{a_k(k+3)}{(k-1)(k+1)}$
- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 1$

  $a_{k+1} = \frac{a_k(k+3)}{(k-1)(k+1)}$
- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(k+5)}{(k+1)(k+3)}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k(k+5)}{(k+1)(k+3)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 76

```
Order:=8;
dsolve(x*diff(y(x),x$2)+(3-x)*diff(y(x),x)-5*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1\left(1+\frac{5}{3}x+\frac{5}{4}x^2+\frac{7}{12}x^3+\frac{7}{36}x^4+\frac{1}{20}x^5+\frac{1}{96}x^6+\frac{11}{6048}x^7+\mathrm{O}\left(x^8\right)\right)x^2+c_2\left(\ln\left(x\right)\left(12x^2+20x^3+15x^4+\right.\right.}{x^2}$$

✓ Solution by Mathematica

Time used: 0.103 (sec). Leaf size: 116

```
AsymptoticDSolveValue[x*y''[x]+(3-x)*y'[x]-5*y[x]==0,y[x],{x,0,7}]
```

$$y(x)\to c_2\left(\frac{x^6}{96}+\frac{x^5}{20}+\frac{7x^4}{36}+\frac{7x^3}{12}+\frac{5x^2}{4}+\frac{5x}{3}+1\right)+c_1\left(\frac{389x^6+1020x^5+1764x^4+1512x^3-72x^2-432x+144}{144x^2}-\frac{1}{6}\left(7x^4+21x^3+45x^2+60x+36\right)\log(x)\right)$$

7.13 problem 13

Internal problem ID [6993]
Internal file name [`OUTPUT/6236_Friday_August_12_2022_11_06_33_PM_16214539/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.9 Indicial Equation with Difference of Roots a Positive Integer: Logarithmic Case. Exercises page 384
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{9x^2y'' - 15xy' + 7(1+x)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$9x^2y'' - 15xy' + (7x+7)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{5}{3x}$$
$$q(x) = \frac{\frac{7x}{9} + \frac{7}{9}}{x^2}$$

Table 173: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{5}{3x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{\frac{7x}{9}+\frac{7}{9}}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$9x^2y'' - 15xy' + (7x + 7)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$9x^2 \left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}\right) - 15x \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (7x + 7) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-15x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 7x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} 7a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 7x^{1+n+r} a_n = \sum_{n=1}^{\infty} 7a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-15x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} 7a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 7a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$9x^{n+r} a_n (n+r)(n+r-1) - 15x^{n+r} a_n (n+r) + 7a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$9x^r a_0 r(-1+r) - 15x^r a_0 r + 7a_0 x^r = 0$$

Or

$$\left(9x^r r(-1+r) - 15x^r r + 7x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(9r^2 - 24r + 7\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$9r^2 - 24r + 7 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{7}{3}$$
$$r_2 = \frac{1}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(9r^2 - 24r + 7\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^{\frac{7}{3}} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{\frac{1}{3}} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{7}{3}}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+\frac{1}{3}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$9a_n(n+r)(n+r-1) - 15a_n(n+r) + 7a_{n-1} + 7a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{7a_{n-1}}{9n^2 + 18nr + 9r^2 - 24n - 24r + 7} \tag{4}$$

Which for the root $r = \frac{7}{3}$ becomes

$$a_n = -\frac{7a_{n-1}}{9n(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{7}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{7}{9r^2 - 6r - 8}$$

Which for the root $r = \frac{7}{3}$ becomes

$$a_1 = -\frac{7}{27}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{7}{9r^2-6r-8}$	$-\frac{7}{27}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{49}{(9r^2 - 6r - 8)(9r^2 + 12r - 5)}$$

Which for the root $r = \frac{7}{3}$ becomes

$$a_2 = \frac{49}{1944}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{7}{9r^2-6r-8}$	$-\frac{7}{27}$
a_2	$\frac{49}{(9r^2-6r-8)(9r^2+12r-5)}$	$\frac{49}{1944}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{343}{(9r^2 - 6r - 8)(9r^2 + 12r - 5)(9r^2 + 30r + 16)}$$

Which for the root $r = \frac{7}{3}$ becomes

$$a_3 = -\frac{343}{262440}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{7}{9r^2-6r-8}$	$-\frac{7}{27}$
a_2	$\frac{49}{(9r^2-6r-8)(9r^2+12r-5)}$	$\frac{49}{1944}$
a_3	$-\frac{343}{(9r^2-6r-8)(9r^2+12r-5)(9r^2+30r+16)}$	$-\frac{343}{262440}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{2401}{(9r^2 - 6r - 8)(9r^2 + 12r - 5)(9r^2 + 30r + 16)(9r^2 + 48r + 55)}$$

Which for the root $r = \frac{7}{3}$ becomes

$$a_4 = \frac{2401}{56687040}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{7}{9r^2-6r-8}$	$-\frac{7}{27}$
a_2	$\frac{49}{(9r^2-6r-8)(9r^2+12r-5)}$	$\frac{49}{1944}$
a_3	$-\frac{343}{(9r^2-6r-8)(9r^2+12r-5)(9r^2+30r+16)}$	$-\frac{343}{262440}$
a_4	$\frac{2401}{(9r^2-6r-8)(9r^2+12r-5)(9r^2+30r+16)(9r^2+48r+55)}$	$\frac{2401}{56687040}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{16807}{59049 \left(r + \frac{2}{3}\right)^2 \left(r + \frac{5}{3}\right)^2 \left(r + \frac{8}{3}\right)^2 \left(r + \frac{11}{3}\right) \left(r - \frac{1}{3}\right) \left(r - \frac{4}{3}\right) \left(r + \frac{14}{3}\right)}$$

Which for the root $r = \frac{7}{3}$ becomes

$$a_5 = -\frac{2401}{2550916800}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{7}{9r^2-6r-8}$	$-\frac{7}{27}$
a_2	$\frac{49}{(9r^2-6r-8)(9r^2+12r-5)}$	$\frac{49}{1944}$
a_3	$-\frac{343}{(9r^2-6r-8)(9r^2+12r-5)(9r^2+30r+16)}$	$-\frac{343}{262440}$
a_4	$\frac{2401}{(9r^2-6r-8)(9r^2+12r-5)(9r^2+30r+16)(9r^2+48r+55)}$	$\frac{2401}{56687040}$
a_5	$-\frac{16807}{59049\left(r+\frac{2}{3}\right)^2\left(r+\frac{5}{3}\right)^2\left(r+\frac{8}{3}\right)^2\left(r+\frac{11}{3}\right)\left(r-\frac{1}{3}\right)\left(r-\frac{4}{3}\right)\left(r+\frac{14}{3}\right)}$	$-\frac{2401}{2550916800}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{117649}{(3r+2)^2 (3r+5)^2 (3r+8)^2 (3r+11)(3r-1)(3r-4)(3r+14)(9r^2+84r+187)}$$

Which for the root $r = \frac{7}{3}$ becomes

$$a_6 = \frac{16807}{1101996057600}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{7}{9r^2-6r-8}$	$-\frac{7}{27}$
a_2	$\frac{49}{(9r^2-6r-8)(9r^2+12r-5)}$	$\frac{49}{1944}$
a_3	$-\frac{343}{(9r^2-6r-8)(9r^2+12r-5)(9r^2+30r+16)}$	$-\frac{343}{262440}$
a_4	$\frac{2401}{(9r^2-6r-8)(9r^2+12r-5)(9r^2+30r+16)(9r^2+48r+55)}$	$\frac{2401}{56687040}$
a_5	$-\frac{16807}{59049\left(r+\frac{2}{3}\right)^2\left(r+\frac{5}{3}\right)^2\left(r+\frac{8}{3}\right)^2\left(r+\frac{11}{3}\right)\left(r-\frac{1}{3}\right)\left(r-\frac{4}{3}\right)\left(r+\frac{14}{3}\right)}$	$-\frac{2401}{2550916800}$
a_6	$\frac{117649}{(3r+2)^2(3r+5)^2(3r+8)^2(3r+11)(3r-1)(3r-4)(3r+14)(9r^2+84r+187)}$	$\frac{16807}{1101996057600}$

For $n = 7$, using the above recursive equation gives

$$a_7 = -\frac{823543}{(3r+2)^2 (3r+5)^2 (3r+8)^2 (3r+11) (3r-1) (3r-4) (3r+14) (9r^2+84r+187) (9r^2+102r+}$$

Which for the root $r = \frac{7}{3}$ becomes

$$a_7 = -\frac{16807}{89261680665600}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{7}{9r^2-6r-8}$	$-\frac{7}{27}$
a_2	$\frac{49}{(9r^2-6r-8)(9r^2+12r-5)}$	$\frac{49}{1944}$
a_3	$-\frac{343}{(9r^2-6r-8)(9r^2+12r-5)(9r^2+30r+16)}$	$-\frac{343}{262440}$
a_4	$\frac{2401}{(9r^2-6r-8)(9r^2+12r-5)(9r^2+30r+16)(9r^2+48r+55)}$	$\frac{2401}{56687040}$
a_5	$-\frac{16807}{59049\left(r+\frac{2}{3}\right)^2\left(r+\frac{5}{3}\right)^2\left(r+\frac{8}{3}\right)^2\left(r+\frac{11}{3}\right)\left(r-\frac{1}{3}\right)\left(r-\frac{4}{3}\right)\left(r+\frac{14}{3}\right)}$	$-\frac{2401}{2550916800}$
a_6	$\frac{117649}{(3r+2)^2(3r+5)^2(3r+8)^2(3r+11)(3r-1)(3r-4)(3r+14)(9r^2+84r+187)}$	$\frac{16807}{1101996057600}$
a_7	$-\frac{823543}{(3r+2)^2(3r+5)^2(3r+8)^2(3r+11)(3r-1)(3r-4)(3r+14)(9r^2+84r+187)(9r^2+102r+280)}$	$-\frac{16807}{89261680665600}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{7}{3}}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \dots\right) \\ &= x^{\frac{7}{3}}\left(1 - \frac{7x}{27} + \frac{49x^2}{1944} - \frac{343x^3}{262440} + \frac{2401x^4}{56687040} - \frac{2401x^5}{2550916800} + \frac{16807x^6}{1101996057600} - \frac{16807x^7}{89261680665600}\right. \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_2 \\ &= \frac{49}{(9r^2-6r-8)(9r^2+12r-5)} \end{aligned}$$

Therefore

$$
\begin{aligned}
\lim_{r\to r_2} \frac{49}{(9r^2-6r-8)(9r^2+12r-5)} &= \lim_{r\to\frac{1}{3}} \frac{49}{(9r^2-6r-8)(9r^2+12r-5)} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)
$$

Therefore

$$
\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $9x^2y'' - 15xy' + (7x+7)y = 0$ gives

$$
\begin{aligned}
&9x^2\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left. + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&- 15x\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) \\
&+ (7x+7)\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(9x^2 y_1''(x) - 15y_1'(x)\,x + (7x+7)\,y_1(x)\right)\ln(x) + 9x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)\right. \\
&\left.- 15y_1(x)\right) C + 9x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&- 15x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + (7x+7)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$
9x^2 y_1''(x) - 15y_1'(x)\,x + (7x+7)\,y_1(x) = 0
$$

Eq (7) simplifes to

$$
\begin{aligned}
&\left(9x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) - 15y_1(x)\right) C \\
&+ 9x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&- 15x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + (7x+7)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{8}
$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$
\begin{aligned}
&\left(18\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) x - 24\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C \\
&+ 9\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x^2 \\
&- 15\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) x + (7x+7)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{9}
$$

Since $r_1 = \frac{7}{3}$ and $r_2 = \frac{1}{3}$ then the above becomes

$$\begin{aligned}
&\left(18\left(\sum_{n=0}^{\infty} x^{\frac{4}{3}+n} a_n \left(n+\frac{7}{3}\right)\right) x - 24\left(\sum_{n=0}^{\infty} a_n x^{n+\frac{7}{3}}\right)\right) C \\
&+ 9\left(\sum_{n=0}^{\infty} x^{-\frac{5}{3}+n} b_n \left(n+\frac{1}{3}\right)\left(-\frac{2}{3}+n\right)\right) x^2 \\
&- 15\left(\sum_{n=0}^{\infty} x^{-\frac{2}{3}+n} b_n \left(n+\frac{1}{3}\right)\right) x + (7x+7)\left(\sum_{n=0}^{\infty} b_n x^{n+\frac{1}{3}}\right) = 0
\end{aligned} \tag{10}$$

Expanding $7x^{\frac{4}{3}}$ as Taylor series around $x = 0$ and keeping only the first 8 terms gives

$$\begin{aligned}
7x^{\frac{4}{3}} &= 7x^{\frac{4}{3}} + \dots \\
&= 7x^{\frac{4}{3}}
\end{aligned}$$

Expanding $7x^{\frac{1}{3}}$ as Taylor series around $x = 0$ and keeping only the first 8 terms gives

$$\begin{aligned}
7x^{\frac{1}{3}} &= 7x^{\frac{1}{3}} + \dots \\
&= 7x^{\frac{1}{3}}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} (18n+42) C a_n x^{n+\frac{7}{3}}\right) + \sum_{n=0}^{\infty} \left(-24 C a_n x^{n+\frac{7}{3}}\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n+\frac{1}{3}} b_n \left(9n^2-3n-2\right)\right) + \left(\sum_{n=0}^{\infty} (-15n-5) b_n x^{n+\frac{1}{3}}\right) \\
&+ \left(\sum_{n=0}^{\infty} 7x^{\frac{4}{3}+n} b_n\right) + \left(\sum_{n=0}^{\infty} 7 b_n x^{n+\frac{1}{3}}\right) = 0
\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n + \frac{1}{3}$ in each summation term. Going over each summation term above with power of x in it which is not already $x^{n+\frac{1}{3}}$ and adjusting the power and the corresponding index gives

$$\begin{aligned}
\sum_{n=0}^{\infty} (18n+42) C a_n x^{n+\frac{7}{3}} &= \sum_{n=2}^{\infty} C a_{n-2} (18n+6)\, x^{n+\frac{1}{3}} \\
\sum_{n=0}^{\infty} \left(-24 C a_n x^{n+\frac{7}{3}}\right) &= \sum_{n=2}^{\infty} \left(-24 C a_{n-2} x^{n+\frac{1}{3}}\right)
\end{aligned}$$

$$\sum_{n=0}^{\infty} 7x^{\frac{4}{3}+n} b_n = \sum_{n=1}^{\infty} 7b_{n-1} x^{n+\frac{1}{3}}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+\frac{1}{3}$.

$$\left(\sum_{n=2}^{\infty} C a_{n-2}(18n+6)\, x^{n+\frac{1}{3}}\right) + \sum_{n=2}^{\infty} \left(-24 C a_{n-2} x^{n+\frac{1}{3}}\right) + \left(\sum_{n=0}^{\infty} x^{n+\frac{1}{3}} b_n \left(9n^2-3n-2\right)\right) + \left(\sum_{n=0}^{\infty} (-15n-5)\, b_n x^{n+\frac{1}{3}}\right) + \left(\sum_{n=1}^{\infty} 7b_{n-1} x^{n+\frac{1}{3}}\right) + \left(\sum_{n=0}^{\infty} 7b_n x^{n+\frac{1}{3}}\right) = 0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$-9b_1 + 7b_0 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-9b_1 + 7 = 0$$

Solving the above for b_1 gives

$$b_1 = \frac{7}{9}$$

For $n=N$, where $N=2$ which is the difference between the two roots, we are free to choose $b_2=0$. Hence for $n=2$, Eq (2B) gives

$$18C + \frac{49}{9} = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{49}{162}$$

For $n=3$, Eq (2B) gives

$$36Ca_1 + 7b_2 + 27b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$27b_3 + \frac{686}{243} = 0$$

Solving the above for b_3 gives

$$b_3 = -\frac{686}{6561}$$

For $n = 4$, Eq (2B) gives

$$54Ca_2 + 7b_3 + 72b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$72b_4 - \frac{60025}{52488} = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{60025}{3779136}$$

For $n = 5$, Eq (2B) gives

$$72Ca_3 + 7b_4 + 135b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$135b_5 + \frac{2638699}{18895680} = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{2638699}{2550916800}$$

For $n = 6$, Eq (2B) gives

$$90Ca_4 + 7b_5 + 216b_6 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$216b_6 - \frac{10706059}{1275458400} = 0$$

Solving the above for b_6 gives

$$b_6 = \frac{10706059}{275499014400}$$

For $n = 7$, Eq (2B) gives

$$108Ca_5 + 7b_6 + 315b_7 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$315b_7 + \frac{83413141}{275499014400} = 0$$

Solving the above for b_7 gives

$$b_7 = -\frac{11916163}{12397455648000}$$

Now that we found all b_n and C, we can calculate the second solution from

$$
y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2} \right)
$$

Using the above value found for $C = -\frac{49}{162}$ and all b_n, then the second solution becomes

$$
\begin{aligned}
y_2(x) = -\frac{49}{162} \left(x^{\frac{7}{3}} \left(1 - \frac{7x}{27} + \frac{49x^2}{1944} - \frac{343x^3}{262440} + \frac{2401x^4}{56687040} - \frac{2401x^5}{2550916800} + \frac{16807x^6}{1101996057600} \right.\right. \\
\left.\left. - \frac{16807x^7}{89261680665600} + O(x^8) \right) \right) \ln(x) + x^{\frac{1}{3}} \left(1 + \frac{7x}{9} - \frac{686x^3}{6561} + \frac{60025x^4}{3779136} \right. \\
\left. - \frac{2638699x^5}{2550916800} + \frac{10706059x^6}{275499014400} - \frac{11916163x^7}{12397455648000} + O(x^8) \right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{7}{3}} \left(1 - \frac{7x}{27} + \frac{49x^2}{1944} - \frac{343x^3}{262440} + \frac{2401x^4}{56687040} - \frac{2401x^5}{2550916800} + \frac{16807x^6}{1101996057600} \right. \\
&\quad \left. - \frac{16807x^7}{89261680665600} + O(x^8) \right) \\
&\quad + c_2 \left(-\frac{49}{162} \left(x^{\frac{7}{3}} \left(1 - \frac{7x}{27} + \frac{49x^2}{1944} - \frac{343x^3}{262440} + \frac{2401x^4}{56687040} - \frac{2401x^5}{2550916800} + \frac{16807x^6}{1101996057600} - \frac{16807}{892616806} \right.\right.\right. \\
&\quad + x^{\frac{1}{3}} \left(1 + \frac{7x}{9} - \frac{686x^3}{6561} + \frac{60025x^4}{3779136} - \frac{2638699x^5}{2550916800} + \frac{10706059x^6}{275499014400} \right. \\
&\quad \left.\left. - \frac{11916163x^7}{12397455648000} + O(x^8) \right) \right)
\end{aligned}
$$

Hence the final solution is

$$
y = y_h
$$

$$= c_1 x^{\frac{7}{3}} \left(1 - \frac{7x}{27} + \frac{49x^2}{1944} - \frac{343x^3}{262440} + \frac{2401x^4}{56687040} - \frac{2401x^5}{2550916800} + \frac{16807x^6}{1101996057600} - \frac{16807x^7}{89261680665600} + O(x^8)\right)$$

$$+ c_2 \left(-\frac{49x^{\frac{7}{3}} \left(1 - \frac{7x}{27} + \frac{49x^2}{1944} - \frac{343x^3}{262440} + \frac{2401x^4}{56687040} - \frac{2401x^5}{2550916800} + \frac{16807x^6}{1101996057600} - \frac{16807x^7}{89261680665600} + O(x^8)\right) \ln (}{162} + x^{\frac{1}{3}} \left(1 + \frac{7x}{9} - \frac{686x^3}{6561} + \frac{60025x^4}{3779136} - \frac{2638699x^5}{2550916800} + \frac{10706059x^6}{275499014400} - \frac{11916163x^7}{12397455648000} + O(x^8)\right)\right)$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{7}{3}} \left(1 - \frac{7x}{27} + \frac{49x^2}{1944} - \frac{343x^3}{262440} + \frac{2401x^4}{56687040} - \frac{2401x^5}{2550916800} + \frac{16807x^6}{1101996057600} - \frac{16807x^7}{89261680665600} + O(x^8)\right) \tag{1}$$

$$+ c_2 \left(-\frac{49x^{\frac{7}{3}} \left(1 - \frac{7x}{27} + \frac{49x^2}{1944} - \frac{343x^3}{262440} + \frac{2401x^4}{56687040} - \frac{2401x^5}{2550916800} + \frac{16807x^6}{1101996057600} - \frac{16807x^7}{89261680665600} + O(x^8)\right) \ln (x}{162} + x^{\frac{1}{3}} \left(1 + \frac{7x}{9} - \frac{686x^3}{6561} + \frac{60025x^4}{3779136} - \frac{2638699x^5}{2550916800} + \frac{10706059x^6}{275499014400} - \frac{11916163x^7}{12397455648000} + O(x^8)\right)\right)$$

Verification of solutions

$$y = c_1 x^{\frac{7}{3}} \left(1 - \frac{7x}{27} + \frac{49x^2}{1944} - \frac{343x^3}{262440} + \frac{2401x^4}{56687040} - \frac{2401x^5}{2550916800} + \frac{16807x^6}{1101996057600} - \frac{16807x^7}{89261680665600} + O(x^8)\right)$$
$$+ c_2 \left(-\frac{49x^{\frac{7}{3}} \left(1 - \frac{7x}{27} + \frac{49x^2}{1944} - \frac{343x^3}{262440} + \frac{2401x^4}{56687040} - \frac{2401x^5}{2550916800} + \frac{16807x^6}{1101996057600} - \frac{16807x^7}{89261680665600} + O(x^8)\right) \ln (x}{162} + x^{\frac{1}{3}} \left(1 + \frac{7x}{9} - \frac{686x^3}{6561} + \frac{60025x^4}{3779136} - \frac{2638699x^5}{2550916800} + \frac{10706059x^6}{275499014400} - \frac{11916163x^7}{12397455648000} + O(x^8)\right)\right)$$

Verified OK.

7.13.1 Maple step by step solution

Let's solve

$9x^2 y'' - 15x y' + (7x + 7) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{5y'}{3x} - \frac{7(1+x)y}{9x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{5y'}{3x} + \frac{7(1+x)y}{9x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{5}{3x}, P_3(x) = \frac{7(1+x)}{9x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -\frac{5}{3}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{7}{9}$

- $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $9x^2y''-15xy'+(7x+7)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m\cdot y$ to series expansion for $m=0..1$

  $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x\cdot y'$ to series expansion

  $x\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$

- Convert $x^2\cdot y''$ to series expansion

  $x^2\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+3r)\,(-7+3r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(3k+3r-1)\,(3k+3r-7)+7a_{k-1}\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+3r)\,(-7+3r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{\frac{1}{3},\frac{7}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $9\left(k+r-\frac{1}{3}\right)\left(k+r-\frac{7}{3}\right)a_k+7a_{k-1}=0$

- Shift index using $k->k+1$

  $9\left(k+\frac{2}{3}+r\right)\left(k-\frac{4}{3}+r\right)a_{k+1}+7a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{7a_k}{(3k+2+3r)(3k-4+3r)}$
- Recursion relation for $r = \frac{1}{3}$

  $a_{k+1} = -\frac{7a_k}{(3k+3)(3k-3)}$
- Series not valid for $r = \frac{1}{3}$, division by 0 in the recursion relation at $k = 1$

  $a_{k+1} = -\frac{7a_k}{(3k+3)(3k-3)}$
- Recursion relation for $r = \frac{7}{3}$

  $a_{k+1} = -\frac{7a_k}{(3k+9)(3k+3)}$
- Solution for $r = \frac{7}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{7}{3}}, a_{k+1} = -\frac{7a_k}{(3k+9)(3k+3)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.11 (sec). Leaf size: 77

```
Order:=8;
dsolve(9*x^2*diff(y(x),x$2)-15*x*diff(y(x),x)+7*(x+1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = x^{\frac{1}{3}}\left(x^2\left(1-\frac{7}{27}x+\frac{49}{1944}x^2-\frac{343}{262440}x^3+\frac{2401}{56687040}x^4-\frac{2401}{2550916800}x^5+\frac{16807}{1101996057600}x^6-\frac{16807}{89261680665600}x^7+\mathrm{O}\left(x^8\right)\right)c_1+c_2\left(\ln\left(x\right)\left(\frac{49}{81}x^2-\frac{343}{2187}x^3+\frac{2401}{157464}x^4-\frac{16807}{21257640}x^5+\frac{117649}{4591650240}x^6-\frac{117649}{206624260800}x^7+\mathrm{O}\left(x^8\right)\right)+\left(-2-\frac{14}{9}x+\frac{1372}{6561}x^3-\frac{60025}{1889568}x^4+\frac{2638699}{1275458400}x^5-\frac{10706059}{137749507200}x^6+\frac{11916163}{6198727824000}x^7+\mathrm{O}\left(x^8\right)\right)\right.\right.$$

✓ Solution by Mathematica

Time used: 0.064 (sec). Leaf size: 141

```
AsymptoticDSolveValue[9*x^2*y''[x]-15*x*y'[x]+7*(x+1)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(\frac{16807x^{25/3}}{1101996057600}-\frac{2401x^{22/3}}{2550916800}+\frac{2401x^{19/3}}{56687040}-\frac{343x^{16/3}}{262440}+\frac{49x^{13/3}}{1944}-\frac{7x^{10/3}}{27}+x^{7/3}\right)+c_1\left(\frac{\sqrt[3]{x}(6235397x^6-169717086x^5+2713009950x^4-19803722400x^3+20832487200x^2+10}{137749507200}\right.$$

7.14 problem 14

Internal problem ID [6994]
Internal file name [`OUTPUT/6237_Friday_August_12_2022_11_06_35_PM_61080954/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.9 Indicial Equation with Difference of Roots a Positive Integer: Logarithmic Case. Exercises page 384
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x(1-2x)\,y' - (1+x)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(-2x^2 + x\right)y' + (-x-1)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{2x-1}{x}$$
$$q(x) = -\frac{1+x}{x^2}$$

Table 175: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{2x-1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1+x}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + (-2x^2 + x)\, y' + (-x - 1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}\right) + (-2x^2 + x) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (-x - 1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2 x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-2 x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-2 a_{n-1} (n+r-1) x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-2 a_{n-1} (n+r-1) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n + r)(n + r - 1) - 2a_{n-1}(n + r - 1) + a_n(n + r) - a_{n-1} - a_n = 0 \qquad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(2n+2r-1)}{n^2+2nr+r^2-1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-1}(2n+1)}{n(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1+2r}{r(r+2)}$$

Which for the root $r = 1$ becomes

$$a_1 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+2r}{r(r+2)}$	1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4r^2+8r+3}{r(r+2)(r+3)(r+1)}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{5}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+2r}{r(r+2)}$	1
a_2	$\frac{4r^2+8r+3}{r(r+2)(r+3)(r+1)}$	$\frac{5}{8}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{8r^3 + 36r^2 + 46r + 15}{r(r+2)^2 (r+3)(r+1)(r+4)}$$

Which for the root $r = 1$ becomes

$$a_3 = \frac{7}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+2r}{r(r+2)}$	1
a_2	$\frac{4r^2+8r+3}{r(r+2)(r+3)(r+1)}$	$\frac{5}{8}$
a_3	$\frac{8r^3+36r^2+46r+15}{r(r+2)^2(r+3)(r+1)(r+4)}$	$\frac{7}{24}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16r^4 + 128r^3 + 344r^2 + 352r + 105}{r(r+2)^2 (r+3)^2 (r+1)(r+4)(r+5)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{7}{64}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+2r}{r(r+2)}$	1
a_2	$\frac{4r^2+8r+3}{r(r+2)(r+3)(r+1)}$	$\frac{5}{8}$
a_3	$\frac{8r^3+36r^2+46r+15}{r(r+2)^2(r+3)(r+1)(r+4)}$	$\frac{7}{24}$
a_4	$\frac{16r^4+128r^3+344r^2+352r+105}{r(r+2)^2(r+3)^2(r+1)(r+4)(r+5)}$	$\frac{7}{64}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{32r^5 + 400r^4 + 1840r^3 + 3800r^2 + 3378r + 945}{r(r+2)^2 (r+3)^2 (r+1)(r+4)^2 (r+5)(r+6)}$$

Which for the root $r = 1$ becomes

$$a_5 = \frac{11}{320}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+2r}{r(r+2)}$	1
a_2	$\frac{4r^2+8r+3}{r(r+2)(r+3)(r+1)}$	$\frac{5}{8}$
a_3	$\frac{8r^3+36r^2+46r+15}{r(r+2)^2(r+3)(r+1)(r+4)}$	$\frac{7}{24}$
a_4	$\frac{16r^4+128r^3+344r^2+352r+105}{r(r+2)^2(r+3)^2(r+1)(r+4)(r+5)}$	$\frac{7}{64}$
a_5	$\frac{32r^5+400r^4+1840r^3+3800r^2+3378r+945}{r(r+2)^2(r+3)^2(r+1)(r+4)^2(r+5)(r+6)}$	$\frac{11}{320}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{64r^6 + 1152r^5 + 8080r^4 + 27840r^3 + 48556r^2 + 39048r + 10395}{r(r+2)^2(r+3)^2(r+1)(r+4)^2(r+5)^2(r+6)(r+7)}$$

Which for the root $r = 1$ becomes

$$a_6 = \frac{143}{15360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+2r}{r(r+2)}$	1
a_2	$\frac{4r^2+8r+3}{r(r+2)(r+3)(r+1)}$	$\frac{5}{8}$
a_3	$\frac{8r^3+36r^2+46r+15}{r(r+2)^2(r+3)(r+1)(r+4)}$	$\frac{7}{24}$
a_4	$\frac{16r^4+128r^3+344r^2+352r+105}{r(r+2)^2(r+3)^2(r+1)(r+4)(r+5)}$	$\frac{7}{64}$
a_5	$\frac{32r^5+400r^4+1840r^3+3800r^2+3378r+945}{r(r+2)^2(r+3)^2(r+1)(r+4)^2(r+5)(r+6)}$	$\frac{11}{320}$
a_6	$\frac{64r^6+1152r^5+8080r^4+27840r^3+48556r^2+39048r+10395}{r(r+2)^2(r+3)^2(r+1)(r+4)^2(r+5)^2(r+6)(r+7)}$	$\frac{143}{15360}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{128r^7 + 3136r^6 + 31136r^5 + 160720r^4 + 459032r^3 + 709324r^2 + 528414r + 135135}{r(r+2)^2 (r+3)^2 (r+1)(r+4)^2 (r+5)^2 (r+6)^2 (r+7)(8+r)}$$

Which for the root $r = 1$ becomes

$$a_7 = \frac{143}{64512}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+2r}{r(r+2)}$	1
a_2	$\frac{4r^2+8r+3}{r(r+2)(r+3)(r+1)}$	$\frac{5}{8}$
a_3	$\frac{8r^3+36r^2+46r+15}{r(r+2)^2(r+3)(r+1)(r+4)}$	$\frac{7}{24}$
a_4	$\frac{16r^4+128r^3+344r^2+352r+105}{r(r+2)^2(r+3)^2(r+1)(r+4)(r+5)}$	$\frac{7}{64}$
a_5	$\frac{32r^5+400r^4+1840r^3+3800r^2+3378r+945}{r(r+2)^2(r+3)^2(r+1)(r+4)^2(r+5)(r+6)}$	$\frac{11}{320}$
a_6	$\frac{64r^6+1152r^5+8080r^4+27840r^3+48556r^2+39048r+10395}{r(r+2)^2(r+3)^2(r+1)(r+4)^2(r+5)^2(r+6)(r+7)}$	$\frac{143}{15360}$
a_7	$\frac{128r^7+3136r^6+31136r^5+160720r^4+459032r^3+709324r^2+528414r+135135}{r(r+2)^2(r+3)^2(r+1)(r+4)^2(r+5)^2(r+6)^2(r+7)(8+r)}$	$\frac{143}{64512}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\ &= x\left(1 + x + \frac{5x^2}{8} + \frac{7x^3}{24} + \frac{7x^4}{64} + \frac{11x^5}{320} + \frac{143x^6}{15360} + \frac{143x^7}{64512} + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_2 \\ &= \frac{4r^2 + 8r + 3}{r(r+2)(r+3)(r+1)} \end{aligned}$$

Therefore

$$\lim_{r\to r_2} \frac{4r^2+8r+3}{r\left(r+2\right)\left(r+3\right)\left(r+1\right)} = \lim_{r\to -1} \frac{4r^2+8r+3}{r\left(r+2\right)\left(r+3\right)\left(r+1\right)}$$
$$= \textit{undefined}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln\left(x\right) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln\left(x\right) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln\left(x\right) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln\left(x\right) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln\left(x\right) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)\left(-1+n+r_2\right)\right)
\end{aligned}$$

Substituting these back into the given ode $x^2y'' + \left(-2x^2+x\right)y' + \left(-x-1\right)y = 0$ gives

$$\begin{aligned}
&x^2\left(Cy_1''(x)\ln\left(x\right) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left.+ \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&+ \left(-2x^2+x\right)\left(Cy_1'(x)\ln\left(x\right) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) \\
&+ \left(-x-1\right)\left(Cy_1(x)\ln\left(x\right) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0
\end{aligned}$$

Which can be written as

$$\left(\left(x^2 y_1''(x)+\left(-2x^2+x\right) y_1'(x)+(-x-1)\, y_1(x)\right) \ln(x)+x^2\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)\right.$$
$$\left.+\frac{\left(-2x^2+x\right) y_1(x)}{x}\right) C+x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \tag{7}$$
$$+\left(-2x^2+x\right)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+(-x-1)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)=0$$

But since $y_1(x)$ is a solution to the ode, then

$$x^2 y_1''(x)+\left(-2x^2+x\right) y_1'(x)+(-x-1)\, y_1(x)=0$$

Eq (7) simplifes to

$$\left(x^2\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+\frac{\left(-2x^2+x\right) y_1(x)}{x}\right) C$$
$$+x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \tag{8}$$
$$+\left(-2x^2+x\right)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+(-x-1)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)=0$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\left(-2\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right) x+2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n(n+r_1)\right) x\right) C$$
$$+\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)\,(-1+n+r_2)\right) x^2 \tag{9}$$
$$+\left(-2x^2+x\right)\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right)+(-x-1)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)=0$$

Since $r_1 = 1$ and $r_2 = -1$ then the above becomes

$$\left(-2\left(\sum_{n=0}^{\infty} a_n x^{1+n}\right) x+2\left(\sum_{n=0}^{\infty} x^n a_n(1+n)\right) x\right) C + \left(\sum_{n=0}^{\infty} x^{-3+n} b_n(n-1)(-2+n)\right) x^2 + \left(-2x^2+x\right)\left(\sum_{n=0}^{\infty} x^{-2+n} b_n(n-1)\right)+(-x-1)\left(\sum_{n=0}^{\infty} b_n x^{n-1}\right)=0 \tag{10}$$

Which simplifies to

$$\sum_{n=0}^{\infty}\left(-2C\, x^{n+2} a_n\right)+\left(\sum_{n=0}^{\infty} 2C\, x^{1+n} a_n(1+n)\right) + \left(\sum_{n=0}^{\infty} x^{n-1} b_n\left(n^2-3n+2\right)\right)+\sum_{n=0}^{\infty}\left(-2x^n b_n(n-1)\right) + \left(\sum_{n=0}^{\infty} x^{n-1} b_n(n-1)\right)+\sum_{n=0}^{\infty}\left(-b_n x^n\right)+\sum_{n=0}^{\infty}\left(-b_n x^{n-1}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-2C\, x^{n+2} a_n\right)=\sum_{n=3}^{\infty}\left(-2C a_{-3+n} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} 2C\, x^{1+n} a_n(1+n)=\sum_{n=2}^{\infty} 2C a_{-2+n}(n-1)\, x^{n-1}$$

$$\sum_{n=0}^{\infty}\left(-2x^n b_n(n-1)\right)=\sum_{n=1}^{\infty}\left(-2b_{n-1}(-2+n)\, x^{n-1}\right)$$

$$\sum_{n=0}^{\infty}\left(-b_n x^n\right)=\sum_{n=1}^{\infty}\left(-b_{n-1} x^{n-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\begin{aligned}
&\sum_{n=3}^{\infty} \left(-2C a_{-3+n} x^{n-1}\right) + \left(\sum_{n=2}^{\infty} 2C a_{-2+n}(n-1)\, x^{n-1}\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n-1} b_n \left(n^2-3n+2\right)\right) + \sum_{n=1}^{\infty} \left(-2b_{n-1}(-2+n)\, x^{n-1}\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n-1} b_n (n-1)\right) + \sum_{n=1}^{\infty} \left(-b_{n-1} x^{n-1}\right) + \sum_{n=0}^{\infty} \left(-b_n x^{n-1}\right) = 0
\end{aligned} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n=1$, Eq (2B) gives

$$b_0 - b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$1 - b_1 = 0$$

Solving the above for b_1 gives

$$b_1 = 1$$

For $n = N$, where $N = 2$ which is the difference between the two roots, we are free to choose $b_2 = 0$. Hence for $n = 2$, Eq (2B) gives

$$2C - 1 = 0$$

Which is solved for C. Solving for C gives

$$C = \frac{1}{2}$$

For $n = 3$, Eq (2B) gives

$$(-2a_0 + 4a_1)\, C - 3b_2 + 3b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$1 + 3b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = -\frac{1}{3}$$

For $n = 4$, Eq (2B) gives

$$(-2a_1 + 6a_2) C - 5b_3 + 8b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{61}{24} + 8b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{61}{192}$$

For $n = 5$, Eq (2B) gives

$$(-2a_2 + 8a_3) C - 7b_4 + 15b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{177}{64} + 15b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{59}{320}$$

For $n = 6$, Eq (2B) gives

$$(-2a_3 + 10a_4) C - 9b_5 + 24b_6 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{919}{480} + 24b_6 = 0$$

Solving the above for b_6 gives

$$b_6 = -\frac{919}{11520}$$

For $n = 7$, Eq (2B) gives

$$(-2a_4 + 12a_5) C - 11b_6 + 35b_7 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{2245}{2304} + 35b_7 = 0$$

Solving the above for b_7 gives

$$b_7 = -\frac{449}{16128}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = \frac{1}{2}$ and all b_n, then the second solution becomes

$$y_2(x) = \frac{1}{2}\left(x\left(1 + x + \frac{5x^2}{8} + \frac{7x^3}{24} + \frac{7x^4}{64} + \frac{11x^5}{320} + \frac{143x^6}{15360} + \frac{143x^7}{64512} + O(x^8)\right)\right)\ln(x)$$
$$+ \frac{1 + x - \frac{x^3}{3} - \frac{61x^4}{192} - \frac{59x^5}{320} - \frac{919x^6}{11520} - \frac{449x^7}{16128} + O(x^8)}{x}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1 + x + \frac{5x^2}{8} + \frac{7x^3}{24} + \frac{7x^4}{64} + \frac{11x^5}{320} + \frac{143x^6}{15360} + \frac{143x^7}{64512} + O(x^8)\right) \\
&\quad + c_2\left(\frac{1}{2}\left(x\left(1+x+\frac{5x^2}{8} + \frac{7x^3}{24} + \frac{7x^4}{64} + \frac{11x^5}{320} + \frac{143x^6}{15360} + \frac{143x^7}{64512} + O(x^8)\right)\right)\ln(x)\right. \\
&\quad \left. + \frac{1 + x - \frac{x^3}{3} - \frac{61x^4}{192} - \frac{59x^5}{320} - \frac{919x^6}{11520} - \frac{449x^7}{16128} + O(x^8)}{x}\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x\left(1 + x + \frac{5x^2}{8} + \frac{7x^3}{24} + \frac{7x^4}{64} + \frac{11x^5}{320} + \frac{143x^6}{15360} + \frac{143x^7}{64512} + O(x^8)\right) \\
&\quad + c_2\left(\frac{x\left(1 + x + \frac{5x^2}{8} + \frac{7x^3}{24} + \frac{7x^4}{64} + \frac{11x^5}{320} + \frac{143x^6}{15360} + \frac{143x^7}{64512} + O(x^8)\right)\ln(x)}{2}\right. \\
&\quad \left. + \frac{1 + x - \frac{x^3}{3} - \frac{61x^4}{192} - \frac{59x^5}{320} - \frac{919x^6}{11520} - \frac{449x^7}{16128} + O(x^8)}{x}\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 x \left(1 + x + \frac{5x^2}{8} + \frac{7x^3}{24} + \frac{7x^4}{64} + \frac{11x^5}{320} + \frac{143x^6}{15360} + \frac{143x^7}{64512} + O(x^8)\right) + c_2 \left(\frac{x\left(1 + x + \frac{5x^2}{8} + \frac{7x^3}{24} + \frac{7x^4}{64} + \frac{11x^5}{320} + \frac{143x^6}{15360} + \frac{143x^7}{64512} + O(x^8)\right)\ln(x)}{2} + \frac{1 + x - \frac{x^3}{3} - \frac{61x^4}{192} - \frac{59x^5}{320} - \frac{919x^6}{11520} - \frac{449x^7}{16128} + O(x^8)}{x}\right) \tag{1}$$

Verification of solutions

$$y = c_1 x \left(1 + x + \frac{5x^2}{8} + \frac{7x^3}{24} + \frac{7x^4}{64} + \frac{11x^5}{320} + \frac{143x^6}{15360} + \frac{143x^7}{64512} + O(x^8)\right) + c_2 \left(\frac{x\left(1 + x + \frac{5x^2}{8} + \frac{7x^3}{24} + \frac{7x^4}{64} + \frac{11x^5}{320} + \frac{143x^6}{15360} + \frac{143x^7}{64512} + O(x^8)\right)\ln(x)}{2} + \frac{1 + x - \frac{x^3}{3} - \frac{61x^4}{192} - \frac{59x^5}{320} - \frac{919x^6}{11520} - \frac{449x^7}{16128} + O(x^8)}{x}\right)$$

Verified OK.

7.14.1 Maple step by step solution

Let's solve

$x^2 y'' + (-2x^2 + x) y' + (-x - 1) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(1+x)y}{x^2} + \frac{(2x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x-1)y'}{x} - \frac{(1+x)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{2x-1}{x}, P_3(x) = -\frac{1+x}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -1$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$x^2 y'' - x(2x-1)\, y' + (-x-1)\, y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k- >k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

  - Shift index using $k- >k + 1 - m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

  - Convert $x^2 \cdot y''$ to series expansion

    $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-1) - a_{k-1}(2k-1+2r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)(-1+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 1\}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)(k+r-1) - 2a_{k-1}\left(k-\frac{1}{2}+r\right) = 0$

- Shift index using $k->k+1$

  $a_{k+1}(k+2+r)(k+r) - 2a_k\left(k+\frac{1}{2}+r\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(2k+2r+1)}{(k+2+r)(k+r)}$

- Recursion relation for $r=-1$

  $a_{k+1} = \frac{a_k(2k-1)}{(k+1)(k-1)}$

- Series not valid for $r=-1$, division by 0 in the recursion relation at $k=1$

  $a_{k+1} = \frac{a_k(2k-1)}{(k+1)(k-1)}$

- Recursion relation for $r=1$

  $a_{k+1} = \frac{a_k(2k+3)}{(k+3)(k+1)}$

- Solution for $r=1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = \frac{a_k(2k+3)}{(k+3)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 75

```
Order:=8;
dsolve(x^2*diff(y(x),x$2)+x*(1-2*x)*diff(y(x),x)-(x+1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x^2 \left(1 + x + \frac{5}{8}x^2 + \frac{7}{24}x^3 + \frac{7}{64}x^4 + \frac{11}{320}x^5 + \frac{143}{15360}x^6 + \frac{143}{64512}x^7 + \mathrm{O}\left(x^8\right)\right) + c_2 \left(\ln\left(x\right)\left(-x^2 - x^3 - \frac{5}{8}x^4 - \cdots\right.\right.}{x}$$

✓ Solution by Mathematica

Time used: 0.109 (sec). Leaf size: 115

```
AsymptoticDSolveValue[x^2*y''[x]+x*(1-2*x)*y'[x]-(x+1)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1 \left(\frac{1}{384} x \left(21x^4 + 56x^3 + 120x^2 + 192x + 192\right) \log(x) - \frac{617x^6 + 1482x^5 + 2730x^4 + 3360x^3 + 1440x^2 - 5760x - 5760}{5760x}\right) + c_2 \left(\frac{143x^7}{15360} + \frac{11x^6}{320} + \frac{7x^5}{64} + \frac{7x^4}{24} + \frac{5x^3}{8} + x^2 + x\right)$$

8 CHAPTER 18. Power series solutions. 18.11 Many-Term Recurrence Relations. Exercises page 391

8.1 problem 1

Internal problem ID [6995]
Internal file name [`OUTPUT/6238_Thursday_August_18_2022_07_11_15_AM_9550685/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.11 Many-Term Recurrence Relations. Exercises page 391
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + 3xy' + \left(x^3 + x + 1\right) y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + 3xy' + \left(x^3 + x + 1\right) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= \frac{3}{x} \\ q(x) &= \frac{x^3 + x + 1}{x^2} \end{aligned}$$

Table 177: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x^3+x+1}{x^2}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + 3xy' + \left(x^3 + x + 1\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + 3x \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(x^3 + x + 1\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+3} a_n\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+3} a_n = \sum_{n=3}^{\infty} a_{n-3} x^{n+r}$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=3}^{\infty} a_{n-3} x^{n+r}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + 3x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + 3x^r a_0 r + a_0 x^r = 0$$

Or

$$(x^r r(-1+r) + 3x^r r + x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r+1)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r+1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -1$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r+1)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = -1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-1}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n-1}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = -\frac{1}{(r+2)^2}$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = \frac{1}{(r+2)^2 (r+3)^2}$$

For $3 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 3a_n(n+r) + a_{n-3} + a_{n-1} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-3} + a_{n-1}}{n^2 + 2nr + r^2 + 2n + 2r + 1} \tag{4}$$

Which for the root $r = -1$ becomes

$$a_n = \frac{-a_{n-3} - a_{n-1}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(r+2)^2}$	-1
a_2	$\frac{1}{(r+2)^2(r+3)^2}$	$\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-r^4 - 10r^3 - 37r^2 - 60r - 37}{(r+4)^2 (r+2)^2 (r+3)^2}$$

Which for the root $r = -1$ becomes

$$a_3 = -\frac{5}{36}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(r+2)^2}$	-1
a_2	$\frac{1}{(r+2)^2(r+3)^2}$	$\frac{1}{4}$
a_3	$\frac{-r^4-10r^3-37r^2-60r-37}{(r+4)^2(r+2)^2(r+3)^2}$	$-\frac{5}{36}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{2r^4 + 24r^3 + 110r^2 + 228r + 181}{(r+4)^2 (r+2)^2 (r+3)^2 (r+5)^2}$$

Which for the root $r = -1$ becomes

$$a_4 = \frac{41}{576}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(r+2)^2}$	-1
a_2	$\frac{1}{(r+2)^2(r+3)^2}$	$\frac{1}{4}$
a_3	$\frac{-r^4-10r^3-37r^2-60r-37}{(r+4)^2(r+2)^2(r+3)^2}$	$-\frac{5}{36}$
a_4	$\frac{2r^4+24r^3+110r^2+228r+181}{(r+4)^2(r+2)^2(r+3)^2(r+5)^2}$	$\frac{41}{576}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-3r^4 - 42r^3 - 231r^2 - 588r - 581}{(r+2)^2 (r+3)^2 (r+4)^2 (r+5)^2 (r+6)^2}$$

Which for the root $r = -1$ becomes

$$a_5 = -\frac{37}{2880}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(r+2)^2}$	-1
a_2	$\frac{1}{(r+2)^2(r+3)^2}$	$\frac{1}{4}$
a_3	$\frac{-r^4-10r^3-37r^2-60r-37}{(r+4)^2(r+2)^2(r+3)^2}$	$-\frac{5}{36}$
a_4	$\frac{2r^4+24r^3+110r^2+228r+181}{(r+4)^2(r+2)^2(r+3)^2(r+5)^2}$	$\frac{41}{576}$
a_5	$\frac{-3r^4-42r^3-231r^2-588r-581}{(r+2)^2(r+3)^2(r+4)^2(r+5)^2(r+6)^2}$	$-\frac{37}{2880}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{r^8 + 32r^7 + 438r^6 + 3344r^5 + 15557r^4 + 45136r^3 + 79828r^2 + 79008r + 33881}{(r+2)^2 (r+3)^2 (r+4)^2 (r+5)^2 (r+6)^2 (r+7)^2}$$

Which for the root $r = -1$ becomes

$$a_6 = \frac{437}{103680}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(r+2)^2}$	-1
a_2	$\frac{1}{(r+2)^2(r+3)^2}$	$\frac{1}{4}$
a_3	$\frac{-r^4-10r^3-37r^2-60r-37}{(r+4)^2(r+2)^2(r+3)^2}$	$-\frac{5}{36}$
a_4	$\frac{2r^4+24r^3+110r^2+228r+181}{(r+4)^2(r+2)^2(r+3)^2(r+5)^2}$	$\frac{41}{576}$
a_5	$\frac{-3r^4-42r^3-231r^2-588r-581}{(r+2)^2(r+3)^2(r+4)^2(r+5)^2(r+6)^2}$	$-\frac{37}{2880}$
a_6	$\frac{r^8+32r^7+438r^6+3344r^5+15557r^4+45136r^3+79828r^2+79008r+33881}{(r+2)^2(r+3)^2(r+4)^2(r+5)^2(r+6)^2(r+7)^2}$	$\frac{437}{103680}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{-3r^8 - 108r^7 - 1678r^6 - 14688r^5 - 79232r^4 - 269982r^3 - 568637r^2 - 678852r - 353165}{(r+7)^2 (r+2)^2 (r+3)^2 (r+4)^2 (r+5)^2 (r+6)^2 (8+r)^2}$$

Which for the root $r = -1$ becomes

$$a_7 = -\frac{7817}{5080320}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(r+2)^2}$	-1
a_2	$\frac{1}{(r+2)^2(r+3)^2}$	$\frac{1}{4}$
a_3	$\frac{-r^4-10r^3-37r^2-60r-37}{(r+4)^2(r+2)^2(r+3)^2}$	$-\frac{5}{36}$
a_4	$\frac{2r^4+24r^3+110r^2+228r+181}{(r+4)^2(r+2)^2(r+3)^2(r+5)^2}$	$\frac{41}{576}$
a_5	$\frac{-3r^4-42r^3-231r^2-588r-581}{(r+2)^2(r+3)^2(r+4)^2(r+5)^2(r+6)^2}$	$-\frac{37}{2880}$
a_6	$\frac{r^8+32r^7+438r^6+3344r^5+15557r^4+45136r^3+79828r^2+79008r+33881}{(r+2)^2(r+3)^2(r+4)^2(r+5)^2(r+6)^2(r+7)^2}$	$\frac{437}{103680}$
a_7	$\frac{-3r^8-108r^7-1678r^6-14688r^5-79232r^4-269982r^3-568637r^2-678852r-353165}{(r+7)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2(r+6)^2(8+r)^2}$	$-\frac{7817}{5080320}$

Using the above table, then the first solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \frac{1}{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\
&= \frac{1 - x + \frac{x^2}{4} - \frac{5x^3}{36} + \frac{41x^4}{576} - \frac{37x^5}{2880} + \frac{437x^6}{103680} - \frac{7817x^7}{5080320} + O(x^8)}{x}
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = -1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$
b_0	1	1	N/A since b_n starts from
b_1	$-\frac{1}{(r+2)^2}$	-1	$\frac{2}{(r+2)^3}$
b_2	$\frac{1}{(r+2)^2(r+3)^2}$	$\frac{1}{4}$	$\frac{-4r-10}{(r+2)^3(r+3)^3}$
b_3	$\frac{-r^4-10r^3-37r^2-60r-37}{(r+4)^2(r+2)^2(r+3)^2}$	$-\frac{5}{36}$	$2r^6+30r^5+186r^4+610r^3+1122r^2+$ / $(r+4)^3(r+2)^3(r+3)^3$
b_4	$\frac{2r^4+24r^3+110r^2+228r+181}{(r+4)^2(r+2)^2(r+3)^2(r+5)^2}$	$\frac{41}{576}$	$-8r^7-176r^6-1668r^5-8844r^4-283$ / $(r+2)^3(r+3)^3(r$
b_5	$\frac{-3r^4-42r^3-231r^2-588r-581}{(r+2)^2(r+3)^2(r+4)^2(r+5)^2(r+6)^2}$	$-\frac{37}{2880}$	$18r^8+534r^7+6978r^6+52542r^5+249$ / $(r+2)^3(r+$
b_6	$\frac{r^8+32r^7+438r^6+3344r^5+15557r^4+45136r^3+79828r^2+79008r+33881}{(r+2)^2(r+3)^2(r+4)^2(r+5)^2(r+6)^2(r+7)^2}$	$\frac{437}{103680}$	$-4r^{13}-214r^{12}-5220r^{11}-76822r^{10}$
b_7	$\frac{-3r^8-108r^7-1678r^6-14688r^5-79232r^4-269982r^3-568637r^2-678852r-353165}{(r+7)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2(r+6)^2(8+r)^2}$	$-\frac{7817}{5080320}$	$18r^{14}+1176r^{13}+35390r^{12}+650136$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 + b_7x^7 + b_8x^8\dots \\
&= \frac{\left(1 - x + \frac{x^2}{4} - \frac{5x^3}{36} + \frac{41x^4}{576} - \frac{37x^5}{2880} + \frac{437x^6}{103680} - \frac{7817x^7}{5080320} + O(x^8)\right)\ln(x)}{x} \\
&\quad + \frac{2x - \frac{3x^2}{4} + \frac{19x^3}{108} - \frac{593x^4}{3456} + \frac{3629x^5}{86400} - \frac{7733x^6}{1036800} + \frac{485257x^7}{118540800} + O(x^8)}{x}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= \frac{c_1\left(1 - x + \frac{x^2}{4} - \frac{5x^3}{36} + \frac{41x^4}{576} - \frac{37x^5}{2880} + \frac{437x^6}{103680} - \frac{7817x^7}{5080320} + O(x^8)\right)}{x} \\
&\quad + c_2\left(\frac{\left(1 - x + \frac{x^2}{4} - \frac{5x^3}{36} + \frac{41x^4}{576} - \frac{37x^5}{2880} + \frac{437x^6}{103680} - \frac{7817x^7}{5080320} + O(x^8)\right)\ln(x)}{x}\right. \\
&\quad \left. + \frac{2x - \frac{3x^2}{4} + \frac{19x^3}{108} - \frac{593x^4}{3456} + \frac{3629x^5}{86400} - \frac{7733x^6}{1036800} + \frac{485257x^7}{118540800} + O(x^8)}{x}\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= \frac{c_1\left(1 - x + \frac{x^2}{4} - \frac{5x^3}{36} + \frac{41x^4}{576} - \frac{37x^5}{2880} + \frac{437x^6}{103680} - \frac{7817x^7}{5080320} + O(x^8)\right)}{x} \\
&\quad + c_2\left(\frac{\left(1 - x + \frac{x^2}{4} - \frac{5x^3}{36} + \frac{41x^4}{576} - \frac{37x^5}{2880} + \frac{437x^6}{103680} - \frac{7817x^7}{5080320} + O(x^8)\right)\ln(x)}{x}\right. \\
&\quad \left. + \frac{2x - \frac{3x^2}{4} + \frac{19x^3}{108} - \frac{593x^4}{3456} + \frac{3629x^5}{86400} - \frac{7733x^6}{1036800} + \frac{485257x^7}{118540800} + O(x^8)}{x}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= \frac{c_1\left(1 - x + \frac{x^2}{4} - \frac{5x^3}{36} + \frac{41x^4}{576} - \frac{37x^5}{2880} + \frac{437x^6}{103680} - \frac{7817x^7}{5080320} + O(x^8)\right)}{x} \\
&\quad + c_2\left(\frac{\left(1 - x + \frac{x^2}{4} - \frac{5x^3}{36} + \frac{41x^4}{576} - \frac{37x^5}{2880} + \frac{437x^6}{103680} - \frac{7817x^7}{5080320} + O(x^8)\right)\ln(x)}{x}\right. \\
&\quad \left. + \frac{2x - \frac{3x^2}{4} + \frac{19x^3}{108} - \frac{593x^4}{3456} + \frac{3629x^5}{86400} - \frac{7733x^6}{1036800} + \frac{485257x^7}{118540800} + O(x^8)}{x}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= \frac{c_1\left(1 - x + \frac{x^2}{4} - \frac{5x^3}{36} + \frac{41x^4}{576} - \frac{37x^5}{2880} + \frac{437x^6}{103680} - \frac{7817x^7}{5080320} + O(x^8)\right)}{x} \\
&\quad + c_2\left(\frac{\left(1 - x + \frac{x^2}{4} - \frac{5x^3}{36} + \frac{41x^4}{576} - \frac{37x^5}{2880} + \frac{437x^6}{103680} - \frac{7817x^7}{5080320} + O(x^8)\right)\ln(x)}{x}\right. \\
&\quad \left. + \frac{2x - \frac{3x^2}{4} + \frac{19x^3}{108} - \frac{593x^4}{3456} + \frac{3629x^5}{86400} - \frac{7733x^6}{1036800} + \frac{485257x^7}{118540800} + O(x^8)}{x}\right)
\end{aligned}
$$

Verified OK.

8.1.1 Maple step by step solution

Let's solve

$x^2y'' + 3xy' + (x^3 + x + 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y'}{x} - \frac{(x^3+x+1)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y'}{x} + \frac{(x^3+x+1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{3}{x}, P_3(x) = \frac{x^3+x+1}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 3$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 1$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2y'' + 3xy' + (x^3 + x + 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..3$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x\cdot y'$ to series expansion

  $x\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r}$

- Convert $x^{2}\cdot y''$ to series expansion

  $x^{2}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(1+r)^{2} x^{r}+\left(a_{1}(2+r)^{2}+a_{0}\right) x^{1+r}+\left(a_{2}(3+r)^{2}+a_{1}\right) x^{2+r}+\left(\sum_{k=3}^{\infty} \left(a_{k}(k+r+1)^{2}+a_{k-1}+\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)^{2}=0$

- Values of r that satisfy the indicial equation

  $r=-1$

- The coefficients of each power of x must be 0

  $\left[a_{1}(2+r)^{2}+a_{0}=0, a_{2}(3+r)^{2}+a_{1}=0\right]$

- Solve for the dependent coefficient(s)

  $\left\{a_{1}=-\frac{a_{0}}{r^{2}+4r+4}, a_{2}=\frac{a_{0}}{r^{4}+10r^{3}+37r^{2}+60r+36}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k}(k+r+1)^{2}+a_{k-1}+a_{k-3}=0$

- Shift index using $k->k+3$

  $a_{k+3}(k+4+r)^{2}+a_{k+2}+a_{k}=0$

- Recursion relation that defines series solution to ODE

  $a_{k+3}=-\frac{a_{k+2}+a_{k}}{(k+4+r)^{2}}$

- Recursion relation for $r=-1$

  $a_{k+3}=-\frac{a_{k+2}+a_{k}}{(k+3)^{2}}$

- Solution for $r=-1$

  $\left[y=\sum_{k=0}^{\infty} a_{k} x^{k-1}, a_{k+3}=-\frac{a_{k+2}+a_{k}}{(k+3)^{2}}, a_{1}=-a_{0}, a_{2}=\frac{a_{0}}{4}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      trying to convert to an ODE of Bessel type
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 81

```
Order:=8;
dsolve(x^2*diff(y(x),x$2)+3*x*diff(y(x),x)+(1+x+x^3)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{(c_2 \ln(x) + c_1)\left(1 - x + \frac{1}{4}x^2 - \frac{5}{36}x^3 + \frac{41}{576}x^4 - \frac{37}{2880}x^5 + \frac{437}{103680}x^6 - \frac{7817}{5080320}x^7 + \mathrm{O}\left(x^8\right)\right) + \left(2x - \frac{3}{4}x^2 + \cdots\right.}{x}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 164

```
AsymptoticDSolveValue[x^2*y''[x]+3*x*y'[x]+(1+x+x^3)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to \frac{c_1\left(-\frac{7817x^7}{5080320} + \frac{437x^6}{103680} - \frac{37x^5}{2880} + \frac{41x^4}{576} - \frac{5x^3}{36} + \frac{x^2}{4} - x + 1\right)}{x} + c_2\left(\frac{\frac{485257x^7}{118540800} - \frac{7733x^6}{1036800} + \frac{3629x^5}{86400} - \frac{593x^4}{3456} + \frac{19x^3}{108} - \frac{3x^2}{4} + 2x}{x} + \frac{\left(-\frac{7817x^7}{5080320} + \frac{437x^6}{103680} - \frac{37x^5}{2880} + \frac{41x^4}{576} - \frac{5x^3}{36} + \frac{x^2}{4} - x + 1\right)\log(x)}{x}\right)$$

8.2 problem 2

Internal problem ID [6996]
Internal file name [OUTPUT/6239_Thursday_August_18_2022_07_11_17_AM_85928680/index.tex]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.11 Many-Term Recurrence Relations. Exercises page 391
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[_Jacobi]
```

$$\boxed{2x(1-x)\,y''+(1-2x)\,y'+(x+2)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-2x^2+2x\right)y''+(1-2x)\,y'+(x+2)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{2x-1}{2x\,(x-1)}$$
$$q(x)=-\frac{x+2}{2x\,(x-1)}$$

Table 179: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2x-1}{2x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

$q(x) = -\frac{x+2}{2x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 1]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-2y''x(x-1) + (1-2x)\,y' + (x+2)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) x(x-1) + (1-2x)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (x+2)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\begin{gathered}\sum_{n=0}^{\infty}\left(-2x^{n+r}a_n(n+r)\,(n+r-1)\right)+\left(\sum_{n=0}^{\infty}2x^{n+r-1}a_n(n+r)\,(n+r-1)\right)\\ +\sum_{n=0}^{\infty}\left(-2x^{n+r}a_n(n+r)\right)+\left(\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}\right)\\ +\left(\sum_{n=0}^{\infty}x^{1+n+r}a_n\right)+\left(\sum_{n=0}^{\infty}2a_n x^{n+r}\right)=0\end{gathered} \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-2x^{n+r}a_n(n+r)\,(n+r-1)\right)=\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)\,(n+r-2)\,x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty}\left(-2x^{n+r}a_n(n+r)\right)=\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)\,x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty}x^{1+n+r}a_n=\sum_{n=2}^{\infty}a_{n-2}x^{n+r-1}$$

$$\sum_{n=0}^{\infty}2a_n x^{n+r}=\sum_{n=1}^{\infty}2a_{n-1}x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\begin{gathered}\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)\,(n+r-2)\,x^{n+r-1}\right)\\ +\left(\sum_{n=0}^{\infty}2x^{n+r-1}a_n(n+r)\,(n+r-1)\right)\\ +\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)\,x^{n+r-1}\right)+\left(\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}\right)\\ +\left(\sum_{n=2}^{\infty}a_{n-2}x^{n+r-1}\right)+\left(\sum_{n=1}^{\infty}2a_{n-1}x^{n+r-1}\right)=0\end{gathered} \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$2x^{n+r-1}a_n(n+r)(n+r-1) + (n+r)\,a_n x^{n+r-1} = 0$$

When $n = 0$ the above becomes

$$2x^{-1+r}a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(2x^{-1+r}r(-1+r) + r\,x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\,x^{-1+r}(2r-1) = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(2r-1) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{-2 + 2r}{2r + 1}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} -2a_{n-1}(n + r - 1)(n + r - 2) + 2a_n(n + r)(n + r - 1) \\ - 2a_{n-1}(n + r - 1) + a_n(n + r) + a_{n-2} + 2a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2n^2 a_{n-1} + 4nra_{n-1} + 2r^2 a_{n-1} - 4na_{n-1} - 4ra_{n-1} - a_{n-2}}{2n^2 + 4nr + 2r^2 - n - r} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{(4n^2 - 4n - 3)\, a_{n-1} - 2a_{n-2}}{4n^2 + 2n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+2r}{2r+1}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4r^3 + 4r^2 - 10r - 1}{4r^3 + 16r^2 + 19r + 6}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = -\frac{9}{40}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+2r}{2r+1}$	$-\frac{1}{2}$
a_2	$\frac{4r^3+4r^2-10r-1}{4r^3+16r^2+19r+6}$	$-\frac{9}{40}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{8r^5 + 40r^4 + 32r^3 - 68r^2 - 66r + 6}{8r^5 + 76r^4 + 274r^3 + 461r^2 + 351r + 90}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = -\frac{149}{1680}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+2r}{2r+1}$	$-\frac{1}{2}$
a_2	$\frac{4r^3+4r^2-10r-1}{4r^3+16r^2+19r+6}$	$-\frac{9}{40}$
a_3	$\frac{8r^5+40r^4+32r^3-68r^2-66r+6}{8r^5+76r^4+274r^3+461r^2+351r+90}$	$-\frac{149}{1680}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16r^7 + 176r^6 + 664r^5 + 836r^4 - 520r^3 - 1816r^2 - 823r + 111}{16r^7 + 272r^6 + 1912r^5 + 7160r^4 + 15289r^3 + 18353r^2 + 11178r + 2520}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = -\frac{661}{13440}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+2r}{2r+1}$	$-\frac{1}{2}$
a_2	$\frac{4r^3+4r^2-10r-1}{4r^3+16r^2+19r+6}$	$-\frac{9}{40}$
a_3	$\frac{8r^5+40r^4+32r^3-68r^2-66r+6}{8r^5+76r^4+274r^3+461r^2+351r+90}$	$-\frac{149}{1680}$
a_4	$\frac{16r^7+176r^6+664r^5+836r^4-520r^3-1816r^2-823r+111}{16r^7+272r^6+1912r^5+7160r^4+15289r^3+18353r^2+11178r+2520}$	$-\frac{661}{13440}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{32r^9 + 608r^8 + 4608r^7 + 17376r^6 + 31368r^5 + 11664r^4 - 46046r^3 - 64544r^2 - 21156r + 3162}{(2r^2 + 19r + 45)(8r^5 + 76r^4 + 274r^3 + 461r^2 + 351r + 90)(2r^2 + 15r + 28)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = -\frac{16171}{492800}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+2r}{2r+1}$	$-\frac{1}{2}$
a_2	$\frac{4r^3+4r^2-10r-1}{4r^3+16r^2+19r+6}$	$-\frac{9}{40}$
a_3	$\frac{8r^5+40r^4+32r^3-68r^2-66r+6}{8r^5+76r^4+274r^3+461r^2+351r+90}$	$-\frac{149}{1680}$
a_4	$\frac{16r^7+176r^6+664r^5+836r^4-520r^3-1816r^2-823r+111}{16r^7+272r^6+1912r^5+7160r^4+15289r^3+18353r^2+11178r+2520}$	$-\frac{661}{13440}$
a_5	$\frac{32r^9+608r^8+4608r^7+17376r^6+31368r^5+11664r^4-46046r^3-64544r^2-21156r+3162}{(2r^2+19r+45)(8r^5+76r^4+274r^3+461r^2+351r+90)(2r^2+15r+28)}$	$-\frac{16171}{492800}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{64r^{11} + 1856r^{10} + 22880r^9 + 155440r^8 + 626048r^7 + 1462528r^6 + 1602128r^5 - 514244r^4 - 3483850}{(2r^2 + 23r + 66)(16r^7 + 272r^6 + 1912r^5 + 7160r^4 + 15289r^3 + 18353r^2 + 11178r + 2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_6 = -\frac{5530601}{230630400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+2r}{2r+1}$	$-\frac{1}{2}$
a_2	$\frac{4r^3+4r^2-10r-1}{4r^3+16r^2+19r+6}$	$-\frac{9}{40}$
a_3	$\frac{8r^5+40r^4+32r^3-68r^2-66r+6}{8r^5+76r^4+274r^3+461r^2+351r+90}$	$-\frac{149}{168}$
a_4	$\frac{16r^7+176r^6+664r^5+836r^4-520r^3-1816r^2-823r+111}{16r^7+272r^6+1912r^5+7160r^4+15289r^3+18353r^2+11178r+2520}$	$-\frac{66}{134}$
a_5	$\frac{32r^9+608r^8+4608r^7+17376r^6+31368r^5+11664r^4-46046r^3-64544r^2-21156r+3162}{(2r^2+19r+45)(8r^5+76r^4+274r^3+461r^2+351r+90)(2r^2+15r+28)}$	$-\frac{161}{492}$
a_6	$\frac{64r^{11}+1856r^{10}+22880r^9+155440r^8+626048r^7+1462528r^6+1602128r^5-514244r^4-3483850r^3-3417773r^2-917322r+146781}{(2r^2+23r+66)(16r^7+272r^6+1912r^5+7160r^4+15289r^3+18353r^2+11178r+2520)(2r^2+19r+45)}$	$-\frac{55}{230}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{128r^{13} + 5248r^{12} + 94720r^{11} + 987968r^{10} + 6558944r^9 + 28650144r^8 + 81361776r^7 + 137907936r^6 + \cdots}{(2r^2 + 27r + 91)(2r^2 + 23r + 66)(16r^7 + 272r^6 + 1912r^5 + 7160r^4 + \cdots}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_7 = -\frac{299137703}{16144128000}$$

And the table now becomes

n	$a_{n,r}$
a_0	1
a_1	$\frac{-2+2r}{2r+1}$
a_2	$\frac{4r^3+4r^2-10r-1}{4r^3+16r^2+19r+6}$
a_3	$\frac{8r^5+40r^4+32r^3-68r^2-66r+6}{8r^5+76r^4+274r^3+461r^2+351r+90}$
a_4	$\frac{16r^7+176r^6+664r^5+836r^4-520r^3-1816r^2-823r+111}{16r^7+272r^6+1912r^5+7160r^4+15289r^3+18353r^2+11178r+2520}$
a_5	$\frac{32r^9+608r^8+4608r^7+17376r^6+31368r^5+11664r^4-46046r^3-64544r^2-21156r+3162}{(2r^2+19r+45)(8r^5+76r^4+274r^3+461r^2+351r+90)(2r^2+15r+28)}$
a_6	$\frac{64r^{11}+1856r^{10}+22880r^9+155440r^8+626048r^7+1462528r^6+1602128r^5-514244r^4-3483850r^3-3417773r^2-917322r+146781}{(2r^2+23r+66)(16r^7+272r^6+1912r^5+7160r^4+15289r^3+18353r^2+11178r+2520)(2r^2+19r+45)}$
a_7	$\frac{128r^{13}+5248r^{12}+94720r^{11}+987968r^{10}+6558944r^9+28650144r^8+81361776r^7+137907936r^6+90592936r^5-126026704r^4-323164836r^3-\cdots}{(2r^2+27r+91)(2r^2+23r+66)(16r^7+272r^6+1912r^5+7160r^4+15289r^3+18353r^2+11178r+2520)(2r^2+19\cdots}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\ &= \sqrt{x}\left(1 - \frac{x}{2} - \frac{9x^2}{40} - \frac{149x^3}{1680} - \frac{661x^4}{13440} - \frac{16171x^5}{492800} - \frac{5530601x^6}{230630400} - \frac{299137703x^7}{16144128000} + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = \frac{-2 + 2r}{2r + 1}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &-2b_{n-1}(n + r - 1)(n + r - 2) + 2b_n(n + r)(n + r - 1) \\ &- 2b_{n-1}(n + r - 1) + (n + r) b_n + b_{n-2} + 2b_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{2n^2 b_{n-1} + 4nr b_{n-1} + 2r^2 b_{n-1} - 4n b_{n-1} - 4r b_{n-1} - b_{n-2}}{2n^2 + 4nr + 2r^2 - n - r} \tag{4}$$

Which for the root $r = 0$ becomes

$$b_n = \frac{2n^2 b_{n-1} - 4n b_{n-1} - b_{n-2}}{n(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+2r}{2r+1}$	-2

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4r^3 + 4r^2 - 10r - 1}{4r^3 + 16r^2 + 19r + 6}$$

Which for the root $r = 0$ becomes

$$b_2 = -\frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+2r}{2r+1}$	-2
b_2	$\frac{4r^3+4r^2-10r-1}{4r^3+16r^2+19r+6}$	$-\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{8r^5 + 40r^4 + 32r^3 - 68r^2 - 66r + 6}{8r^5 + 76r^4 + 274r^3 + 461r^2 + 351r + 90}$$

Which for the root $r = 0$ becomes

$$b_3 = \frac{1}{15}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+2r}{2r+1}$	-2
b_2	$\frac{4r^3+4r^2-10r-1}{4r^3+16r^2+19r+6}$	$-\frac{1}{6}$
b_3	$\frac{8r^5+40r^4+32r^3-68r^2-66r+6}{8r^5+76r^4+274r^3+461r^2+351r+90}$	$\frac{1}{15}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16r^7 + 176r^6 + 664r^5 + 836r^4 - 520r^3 - 1816r^2 - 823r + 111}{16r^7 + 272r^6 + 1912r^5 + 7160r^4 + 15289r^3 + 18353r^2 + 11178r + 2520}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{37}{840}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+2r}{2r+1}$	-2
b_2	$\frac{4r^3+4r^2-10r-1}{4r^3+16r^2+19r+6}$	$-\frac{1}{6}$
b_3	$\frac{8r^5+40r^4+32r^3-68r^2-66r+6}{8r^5+76r^4+274r^3+461r^2+351r+90}$	$\frac{1}{15}$
b_4	$\frac{16r^7+176r^6+664r^5+836r^4-520r^3-1816r^2-823r+111}{16r^7+272r^6+1912r^5+7160r^4+15289r^3+18353r^2+11178r+2520}$	$\frac{37}{840}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{32r^9 + 608r^8 + 4608r^7 + 17376r^6 + 31368r^5 + 11664r^4 - 46046r^3 - 64544r^2 - 21156r + 3162}{(2r^2 + 19r + 45)(8r^5 + 76r^4 + 274r^3 + 461r^2 + 351r + 90)(2r^2 + 15r + 28)}$$

Which for the root $r = 0$ becomes

$$b_5 = \frac{527}{18900}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+2r}{2r+1}$	-2
b_2	$\frac{4r^3+4r^2-10r-1}{4r^3+16r^2+19r+6}$	$-\frac{1}{6}$
b_3	$\frac{8r^5+40r^4+32r^3-68r^2-66r+6}{8r^5+76r^4+274r^3+461r^2+351r+90}$	$\frac{1}{15}$
b_4	$\frac{16r^7+176r^6+664r^5+836r^4-520r^3-1816r^2-823r+111}{16r^7+272r^6+1912r^5+7160r^4+15289r^3+18353r^2+11178r+2520}$	$\frac{37}{840}$
b_5	$\frac{32r^9+608r^8+4608r^7+17376r^6+31368r^5+11664r^4-46046r^3-64544r^2-21156r+3162}{(2r^2+19r+45)(8r^5+76r^4+274r^3+461r^2+351r+90)(2r^2+15r+28)}$	$\frac{527}{18900}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{64r^{11} + 1856r^{10} + 22880r^9 + 155440r^8 + 626048r^7 + 1462528r^6 + 1602128r^5 - 514244r^4 - 3483850}{(2r^2 + 23r + 66)\left(16r^7 + 272r^6 + 1912r^5 + 7160r^4 + 15289r^3 + 18353r^2 + 11178r + 2\right.}$$

Which for the root $r = 0$ becomes

$$b_6 = \frac{16309}{831600}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+2r}{2r+1}$	-2
b_2	$\frac{4r^3+4r^2-10r-1}{4r^3+16r^2+19r+6}$	$-\frac{1}{6}$
b_3	$\frac{8r^5+40r^4+32r^3-68r^2-66r+6}{8r^5+76r^4+274r^3+461r^2+351r+90}$	$\frac{1}{15}$
b_4	$\frac{16r^7+176r^6+664r^5+836r^4-520r^3-1816r^2-823r+111}{16r^7+272r^6+1912r^5+7160r^4+15289r^3+18353r^2+11178r+2520}$	$\frac{37}{840}$
b_5	$\frac{32r^9+608r^8+4608r^7+17376r^6+31368r^5+11664r^4-46046r^3-64544r^2-21156r+3162}{(2r^2+19r+45)(8r^5+76r^4+274r^3+461r^2+351r+90)(2r^2+15r+28)}$	$\frac{527}{18900}$
b_6	$\frac{64r^{11}+1856r^{10}+22880r^9+155440r^8+626048r^7+1462528r^6+1602128r^5-514244r^4-3483850r^3-3417773r^2-917322r+146781}{(2r^2+23r+66)(16r^7+272r^6+1912r^5+7160r^4+15289r^3+18353r^2+11178r+2520)(2r^2+19r+45)}$	$\frac{16309}{831600}$

For $n = 7$, using the above recursive equation gives

$$b_7 = \frac{128r^{13} + 5248r^{12} + 94720r^{11} + 987968r^{10} + 6558944r^9 + 28650144r^8 + 81361776r^7 + 137907936r^6}{(2r^2 + 27r + 91)(2r^2 + 23r + 66)\left(16r^7 + 272r^6 + 1912r^5 + 7160r^4 +\right.}$$

Which for the root $r = 0$ becomes

$$b_7 = \frac{14339}{970200}$$

And the table now becomes

n	$b_{n,r}$
b_0	1
b_1	$\frac{-2+2r}{2r+1}$
b_2	$\frac{4r^3+4r^2-10r-1}{4r^3+16r^2+19r+6}$
b_3	$\frac{8r^5+40r^4+32r^3-68r^2-66r+6}{8r^5+76r^4+274r^3+461r^2+351r+90}$
b_4	$\frac{16r^7+176r^6+664r^5+836r^4-520r^3-1816r^2-823r+111}{16r^7+272r^6+1912r^5+7160r^4+15289r^3+18353r^2+11178r+2520}$
b_5	$\frac{32r^9+608r^8+4608r^7+17376r^6+31368r^5+11664r^4-46046r^3-64544r^2-21156r+3162}{(2r^2+19r+45)(8r^5+76r^4+274r^3+461r^2+351r+90)(2r^2+15r+28)}$
b_6	$\frac{64r^{11}+1856r^{10}+22880r^9+155440r^8+626048r^7+1462528r^6+1602128r^5-514244r^4-3483850r^3-3417773r^2-917322r+146781}{(2r^2+23r+66)(16r^7+272r^6+1912r^5+7160r^4+15289r^3+18353r^2+11178r+2520)(2r^2+19r+45)}$
b_7	$\frac{128r^{13}+5248r^{12}+94720r^{11}+987968r^{10}+6558944r^9+28650144r^8+81361776r^7+137907936r^6+90592936r^5-126026704r^4-323164836r^3-}{(2r^2+27r+91)(2r^2+23r+66)(16r^7+272r^6+1912r^5+7160r^4+15289r^3+18353r^2+11178r+2520)(2r^2+19r}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\
&= 1 - 2x - \frac{x^2}{6} + \frac{x^3}{15} + \frac{37x^4}{840} + \frac{527x^5}{18900} + \frac{16309x^6}{831600} + \frac{14339x^7}{970200} + O(x^8)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \sqrt{x} \left(1 - \frac{x}{2} - \frac{9x^2}{40} - \frac{149x^3}{1680} - \frac{661x^4}{13440} - \frac{16171x^5}{492800} - \frac{5530601x^6}{230630400} - \frac{299137703x^7}{16144128000} \right. \\
&\quad \left. + O(x^8)\right) + c_2 \left(1 - 2x - \frac{x^2}{6} + \frac{x^3}{15} + \frac{37x^4}{840} + \frac{527x^5}{18900} + \frac{16309x^6}{831600} + \frac{14339x^7}{970200} + O(x^8)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \sqrt{x} \left(1 - \frac{x}{2} - \frac{9x^2}{40} - \frac{149x^3}{1680} - \frac{661x^4}{13440} - \frac{16171x^5}{492800} - \frac{5530601x^6}{230630400} - \frac{299137703x^7}{16144128000} + O(x^8)\right) \\
&\quad + c_2 \left(1 - 2x - \frac{x^2}{6} + \frac{x^3}{15} + \frac{37x^4}{840} + \frac{527x^5}{18900} + \frac{16309x^6}{831600} + \frac{14339x^7}{970200} + O(x^8)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x}\left(1-\frac{x}{2}-\frac{9x^2}{40}-\frac{149x^3}{1680}-\frac{661x^4}{13440}-\frac{16171x^5}{492800}-\frac{5530601x^6}{230630400}-\frac{299137703x^7}{16144128000}+O\left(x^8\right)\right)+c_2\left(1-2x-\frac{x^2}{6}+\frac{x^3}{15}+\frac{37x^4}{840}+\frac{527x^5}{18900}+\frac{16309x^6}{831600}+\frac{14339x^7}{970200}+O\left(x^8\right)\right) \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\left(1-\frac{x}{2}-\frac{9x^2}{40}-\frac{149x^3}{1680}-\frac{661x^4}{13440}-\frac{16171x^5}{492800}-\frac{5530601x^6}{230630400}-\frac{299137703x^7}{16144128000}+O\left(x^8\right)\right)+c_2\left(1-2x-\frac{x^2}{6}+\frac{x^3}{15}+\frac{37x^4}{840}+\frac{527x^5}{18900}+\frac{16309x^6}{831600}+\frac{14339x^7}{970200}+O\left(x^8\right)\right)$$

Verified OK.

8.2.1 Maple step by step solution

Let's solve

$-2y''x(x-1)+(1-2x)\,y'+(x+2)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(x+2)y}{2x(x-1)}-\frac{(2x-1)y'}{2x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(2x-1)y'}{2x(x-1)}-\frac{(x+2)y}{2x(x-1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{2x-1}{2x(x-1)}, P_3(x)=-\frac{x+2}{2x(x-1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=\frac{1}{2}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=0$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2y''x(x-1) + (2x-1)\,y' + (-x-2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  - Shift index using $k- >k + 1 - m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

    $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$

  - Shift index using $k- >k + 2 - m$

    $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 r(-1+2r)\, x^{-1+r} + (-a_1(1+r)\,(1+2r) + 2a_0(1+r)\,(-1+r))\, x^r + \left(\sum_{k=1}^{\infty} (-a_{k+1}(k+r+1\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-1+2r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{0, \frac{1}{2}\right\}$

- Each term must be 0

$$-a_1(1+r)(1+2r)+2a_0(1+r)(-1+r)=0$$

- Each term in the series must be 0, giving the recursion relation

$$-2(k+r+1)\left(k+\frac{1}{2}+r\right)a_{k+1}+2k^2a_k+4kra_k+2r^2a_k-2a_k-a_{k-1}=0$$

- Shift index using $k->k+1$

$$-2(k+2+r)\left(k+\frac{3}{2}+r\right)a_{k+2}+2(k+1)^2a_{k+1}+4(k+1)ra_{k+1}+2r^2a_{k+1}-2a_{k+1}-a_k=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=\frac{2k^2a_{k+1}+4kra_{k+1}+2r^2a_{k+1}+4ka_{k+1}+4ra_{k+1}-a_k}{(k+2+r)(2k+3+2r)}$$

- Recursion relation for $r=0$

$$a_{k+2}=\frac{2k^2a_{k+1}+4ka_{k+1}-a_k}{(k+2)(2k+3)}$$

- Solution for $r=0$

$$\left[y=\sum_{k=0}^{\infty}a_kx^k, a_{k+2}=\frac{2k^2a_{k+1}+4ka_{k+1}-a_k}{(k+2)(2k+3)}, -a_1-2a_0=0\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+2}=\frac{2k^2a_{k+1}+6ka_{k+1}-a_k+\frac{5}{2}a_{k+1}}{\left(k+\frac{5}{2}\right)(2k+4)}$$

- Solution for $r=\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_kx^{k+\frac{1}{2}}, a_{k+2}=\frac{2k^2a_{k+1}+6ka_{k+1}-a_k+\frac{5}{2}a_{k+1}}{\left(k+\frac{5}{2}\right)(2k+4)}, -3a_1-\frac{3a_0}{2}=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_kx^k\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+\frac{1}{2}}\right), a_{k+2}=\frac{2k^2a_{k+1}+4ka_{k+1}-a_k}{(k+2)(2k+3)}, -a_1-2a_0=0, b_{k+2}=\frac{2k^2b_{k+1}+6kb_{k+}}{\left(k+\frac{5}{2}\right)(2}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      Equivalence transformation and function parameters: {x = t}, {kappa = 12, mu = -8}
      <- Equivalence to the rational form of Mathieu ODE successful
   <- Mathieu successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 52

```
Order:=8;
dsolve(2*x*(1-x)*diff(y(x),x$2)+(1-2*x)*diff(y(x),x)+(2+x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\sqrt{x}\left(1-\frac{1}{2}x-\frac{9}{40}x^2-\frac{149}{1680}x^3-\frac{661}{13440}x^4-\frac{16171}{492800}x^5-\frac{5530601}{230630400}x^6-\frac{299137703}{16144128000}x^7+\mathrm{O}\left(x^8\right)\right)+c_2\left(1-2x-\frac{1}{6}x^2+\frac{1}{15}x^3+\frac{37}{840}x^4+\frac{527}{18900}x^5+\frac{16309}{831600}x^6+\frac{14339}{970200}x^7+\mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 111

```
AsymptoticDSolveValue[2*x*(1-x)*y''[x]+(1-2*x)*y'[x]+(2+x)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\sqrt{x}\left(-\frac{299137703x^7}{16144128000}-\frac{5530601x^6}{230630400}-\frac{16171x^5}{492800}-\frac{661x^4}{13440}-\frac{149x^3}{1680}-\frac{9x^2}{40}-\frac{x}{2}+1\right) + c_2\left(\frac{14339x^7}{970200}+\frac{16309x^6}{831600}+\frac{527x^5}{18900}+\frac{37x^4}{840}+\frac{x^3}{15}-\frac{x^2}{6}-2x+1\right)$$

8.3 problem 3

Internal problem ID [6997]
Internal file name [`OUTPUT/6240_Thursday_August_18_2022_07_11_20_AM_63408136/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.11 Many-Term Recurrence Relations. Exercises page 391
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + y' + x(1 + x) y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + y' + (x^2 + x) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = 1 + x$$

Table 181: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = 1 + x$	
singularity	type
$x = \infty$	"regular"
$x = -\infty$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + y' + (x^2 + x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (x^2 + x)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=3}^{\infty} a_{n-3} x^{n+r-1}$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=3}^{\infty} a_{n-3} x^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln (x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = -\frac{1}{(2+r)^2}$$

For $3 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-3} + a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-3} + a_{n-2}}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{-a_{n-3} - a_{n-2}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(2+r)^2}$	$-\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{(r+3)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = -\frac{1}{9}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(2+r)^2}$	$-\frac{1}{4}$
a_3	$-\frac{1}{(r+3)^2}$	$-\frac{1}{9}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(2+r)^2 (r+4)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{64}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(2+r)^2}$	$-\frac{1}{4}$
a_3	$-\frac{1}{(r+3)^2}$	$-\frac{1}{9}$
a_4	$\frac{1}{(2+r)^2(r+4)^2}$	$\frac{1}{64}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{2r^2 + 10r + 13}{(2 + r)^2 (r + 3)^2 (r + 5)^2}$$

Which for the root $r = 0$ becomes

$$a_5 = \frac{13}{900}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(2+r)^2}$	$-\frac{1}{4}$
a_3	$-\frac{1}{(r+3)^2}$	$-\frac{1}{9}$
a_4	$\frac{1}{(2+r)^2(r+4)^2}$	$\frac{1}{64}$
a_5	$\frac{2r^2+10r+13}{(2+r)^2(r+3)^2(r+5)^2}$	$\frac{13}{900}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{r^4 + 12r^3 + 51r^2 + 90r + 55}{(r + 3)^2 (2 + r)^2 (r + 4)^2 (r + 6)^2}$$

Which for the root $r = 0$ becomes

$$a_6 = \frac{55}{20736}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(2+r)^2}$	$-\frac{1}{4}$
a_3	$-\frac{1}{(r+3)^2}$	$-\frac{1}{9}$
a_4	$\frac{1}{(2+r)^2(r+4)^2}$	$\frac{1}{64}$
a_5	$\frac{2r^2+10r+13}{(2+r)^2(r+3)^2(r+5)^2}$	$\frac{13}{900}$
a_6	$\frac{r^4+12r^3+51r^2+90r+55}{(r+3)^2(2+r)^2(r+4)^2(r+6)^2}$	$\frac{55}{20736}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{-3r^4 - 42r^3 - 219r^2 - 504r - 433}{(2+r)^2 (r+4)^2 (r+3)^2 (r+5)^2 (r+7)^2}$$

Which for the root $r = 0$ becomes

$$a_7 = -\frac{433}{705600}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(2+r)^2}$	$-\frac{1}{4}$
a_3	$-\frac{1}{(r+3)^2}$	$-\frac{1}{9}$
a_4	$\frac{1}{(2+r)^2(r+4)^2}$	$\frac{1}{64}$
a_5	$\frac{2r^2+10r+13}{(2+r)^2(r+3)^2(r+5)^2}$	$\frac{13}{900}$
a_6	$\frac{r^4+12r^3+51r^2+90r+55}{(r+3)^2(2+r)^2(r+4)^2(r+6)^2}$	$\frac{55}{20736}$
a_7	$\frac{-3r^4-42r^3-219r^2-504r-433}{(2+r)^2(r+4)^2(r+3)^2(r+5)^2(r+7)^2}$	$-\frac{433}{705600}$

Using the above table, then the first solution $y_1(x)$ becomes

$$
\begin{aligned}
y_1(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \dots \\
&= 1 - \frac{x^2}{4} - \frac{x^3}{9} + \frac{x^4}{64} + \frac{13x^5}{900} + \frac{55x^6}{20736} - \frac{433x^7}{705600} + O(x^8)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	0	0	0
b_2	$-\frac{1}{(2+r)^2}$	$-\frac{1}{4}$	$\frac{2}{(2+r)^3}$
b_3	$-\frac{1}{(r+3)^2}$	$-\frac{1}{9}$	$\frac{2}{(r+3)^3}$
b_4	$\frac{1}{(2+r)^2(r+4)^2}$	$\frac{1}{64}$	$\frac{-4r-12}{(2+r)^3(r+4)^3}$
b_5	$\frac{2r^2+10r+13}{(2+r)^2(r+3)^2(r+5)^2}$	$\frac{13}{900}$	$\frac{-8r^4-90r^3-378r^2-710r-506}{(2+r)^3(r+3)^3(r+5)^3}$
b_6	$\frac{r^4+12r^3+51r^2+90r+55}{(r+3)^2(2+r)^2(r+4)^2(r+6)^2}$	$\frac{55}{20736}$	$\frac{-4r^7-90r^6-846r^5-4290r^4-12614r^3-21366r^2-19112r-6840}{(r+3)^3(2+r)^3(r+4)^3(r+6)^3}$
b_7	$\frac{-3r^4-42r^3-219r^2-504r-433}{(2+r)^2(r+4)^2(r+3)^2(r+5)^2(r+7)^2}$	$-\frac{433}{705600}$	$\frac{18r^8+546r^7+7176r^6+53424r^5+246616r^4+723366r^3+1317534r^2+1363404r+61410}{(2+r)^3(r+4)^3(r+3)^3(r+5)^3(r+7)^3}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 + b_7x^7 + b_8x^8 \dots \\
&= \left(1 - \frac{x^2}{4} - \frac{x^3}{9} + \frac{x^4}{64} + \frac{13x^5}{900} + \frac{55x^6}{20736} - \frac{433x^7}{705600} + O(x^8)\right)\ln(x) \\
&\quad + \frac{x^2}{4} + \frac{2x^3}{27} - \frac{3x^4}{128} - \frac{253x^5}{13500} - \frac{95x^6}{41472} + \frac{153527x^7}{148176000} + O(x^8)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \left(1 - \frac{x^2}{4} - \frac{x^3}{9} + \frac{x^4}{64} + \frac{13x^5}{900} + \frac{55x^6}{20736} - \frac{433x^7}{705600} + O(x^8)\right) \\
&\quad + c_2 \left(\left(1 - \frac{x^2}{4} - \frac{x^3}{9} + \frac{x^4}{64} + \frac{13x^5}{900} + \frac{55x^6}{20736} - \frac{433x^7}{705600} + O(x^8)\right) \ln(x) + \frac{x^2}{4} \right. \\
&\quad \left. + \frac{2x^3}{27} - \frac{3x^4}{128} - \frac{253x^5}{13500} - \frac{95x^6}{41472} + \frac{153527x^7}{148176000} + O(x^8)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \left(1 - \frac{x^2}{4} - \frac{x^3}{9} + \frac{x^4}{64} + \frac{13x^5}{900} + \frac{55x^6}{20736} - \frac{433x^7}{705600} + O(x^8)\right) \\
&\quad + c_2 \left(\left(1 - \frac{x^2}{4} - \frac{x^3}{9} + \frac{x^4}{64} + \frac{13x^5}{900} + \frac{55x^6}{20736} - \frac{433x^7}{705600} + O(x^8)\right) \ln(x) + \frac{x^2}{4} + \frac{2x^3}{27} \right. \\
&\quad \left. - \frac{3x^4}{128} - \frac{253x^5}{13500} - \frac{95x^6}{41472} + \frac{153527x^7}{148176000} + O(x^8)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 \left(1 - \frac{x^2}{4} - \frac{x^3}{9} + \frac{x^4}{64} + \frac{13x^5}{900} + \frac{55x^6}{20736} - \frac{433x^7}{705600} + O(x^8)\right) \\
&\quad + c_2 \left(\left(1 - \frac{x^2}{4} - \frac{x^3}{9} + \frac{x^4}{64} + \frac{13x^5}{900} + \frac{55x^6}{20736} - \frac{433x^7}{705600} + O(x^8)\right) \ln(x) + \frac{x^2}{4} \right. \\
&\quad \left. + \frac{2x^3}{27} - \frac{3x^4}{128} - \frac{253x^5}{13500} - \frac{95x^6}{41472} + \frac{153527x^7}{148176000} + O(x^8)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 \left(1 - \frac{x^2}{4} - \frac{x^3}{9} + \frac{x^4}{64} + \frac{13x^5}{900} + \frac{55x^6}{20736} - \frac{433x^7}{705600} + O(x^8)\right) \\
&\quad + c_2 \left(\left(1 - \frac{x^2}{4} - \frac{x^3}{9} + \frac{x^4}{64} + \frac{13x^5}{900} + \frac{55x^6}{20736} - \frac{433x^7}{705600} + O(x^8)\right) \ln(x) + \frac{x^2}{4} + \frac{2x^3}{27} \right. \\
&\quad \left. - \frac{3x^4}{128} - \frac{253x^5}{13500} - \frac{95x^6}{41472} + \frac{153527x^7}{148176000} + O(x^8)\right)
\end{aligned}
$$

Verified OK.

8.3.1 Maple step by step solution

Let's solve

$xy'' + y' + (x^2 + x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = (-x-1)\, y - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + (1+x)\, y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = 1 + x\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + y' + x(1+x)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 1..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert y' to series expansion

$$y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$

- Shift index using $k->k+1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + a_1 (1+r)^2 x^r + \left(a_2 (2+r)^2 + a_0\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_{k+1} (k+1+r)^2 + a_{k-1} + a_{k-2}\right) x^{k+}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 0$$

- The coefficients of each power of x must be 0

$$\left[a_1 (1+r)^2 = 0, a_2 (2+r)^2 + a_0 = 0\right]$$

- Solve for the dependent coefficient(s)

$$\left\{a_1 = 0, a_2 = -\frac{a_0}{r^2+4r+4}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1} (k+1)^2 + a_{k-1} + a_{k-2} = 0$$

- Shift index using $k->k+2$

$$a_{k+3} (k+3)^2 + a_{k+1} + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+3} = -\frac{a_{k+1} + a_k}{(k+3)^2}$$

- Recursion relation for $r = 0$

$a_{k+3} = -\frac{a_{k+1}+a_k}{(k+3)^2}$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = -\frac{a_{k+1}+a_k}{(k+3)^2}, a_1 = 0, a_2 = -\frac{a_0}{4}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      trying to convert to an ODE of Bessel type
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 65

```
Order:=8;
dsolve(x*diff(y(x),x$2)+diff(y(x),x)+x*(1+x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln(x) + c_1)\left(1 - \frac{1}{4}x^2 - \frac{1}{9}x^3 + \frac{1}{64}x^4 + \frac{13}{900}x^5 + \frac{55}{20736}x^6 - \frac{433}{705600}x^7 + \mathrm{O}\left(x^8\right)\right) + \left(\frac{1}{4}x^2 + \frac{2}{27}x^3 - \frac{3}{128}x^4 - \frac{253}{13500}x^5 - \frac{95}{41472}x^6 + \frac{153527}{148176000}x^7 + \mathrm{O}\left(x^8\right)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 144

```
AsymptoticDSolveValue[x*y''[x]+y'[x]+x*(1+x)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(-\frac{433x^7}{705600} + \frac{55x^6}{20736} + \frac{13x^5}{900} + \frac{x^4}{64} - \frac{x^3}{9} - \frac{x^2}{4} + 1\right) + c_2\left(\frac{153527x^7}{148176000} - \frac{95x^6}{41472} - \frac{253x^5}{13500} - \frac{3x^4}{128} + \frac{2x^3}{27} + \frac{x^2}{4} + \left(-\frac{433x^7}{705600} + \frac{55x^6}{20736} + \frac{13x^5}{900} + \frac{x^4}{64} - \frac{x^3}{9} - \frac{x^2}{4} + 1\right)\log(x)\right)$$

8.4 problem 4

Internal problem ID [6998]
Internal file name [`OUTPUT/6241_Thursday_August_18_2022_07_11_22_AM_33243587/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.11 Many-Term Recurrence Relations. Exercises page 391
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x(1+x)\,y' - (6x^2 - 3x + 1)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + (x^2 + x)\,y' + (-6x^2 + 3x - 1)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1+x}{x}$$
$$q(x) = -\frac{6x^2 - 3x + 1}{x^2}$$

Table 183: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1+x}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{6x^2-3x+1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + (x^2 + x) y' + (-6x^2 + 3x - 1) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + (x^2 + x) \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + (-6x^2 + 3x - 1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-6x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} 3x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1} (n+r-1) x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-6x^{n+r+2} a_n\right) = \sum_{n=2}^{\infty} \left(-6a_{n-2} x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} 3x^{1+n+r} a_n = \sum_{n=1}^{\infty} 3a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} (n+r-1) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=2}^{\infty} \left(-6a_{n-2} x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r) (n+r-1) + x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - a_0 x^r = 0$$

Or

$$(x^r r(-1+r) + x^r r - x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2-1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2-1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{-3 - r}{r\,(r + 2)}$$

For $2 \leq n$ the recursive equation is

$$a_n(n + r)\,(n + r - 1) + a_{n-1}(n + r - 1) + a_n(n + r) - 6a_{n-2} + 3a_{n-1} - a_n = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{na_{n-1} + ra_{n-1} - 6a_{n-2} + 2a_{n-1}}{n^2 + 2nr + r^2 - 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{-na_{n-1} + 6a_{n-2} - 3a_{n-1}}{n\,(n + 2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-r}{r(r+2)}$	$-\frac{4}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{7r + 12}{(r + 3)\,r\,(r + 2)}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{19}{12}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-r}{r(r+2)}$	$-\frac{4}{3}$
a_2	$\frac{7r+12}{(r+3)r(r+2)}$	$\frac{19}{12}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-57 - 13r}{(r+4)(r+2)(r+3)r}$$

Which for the root $r = 1$ becomes

$$a_3 = -\frac{7}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-r}{r(r+2)}$	$-\frac{4}{3}$
a_2	$\frac{7r+12}{(r+3)r(r+2)}$	$\frac{19}{12}$
a_3	$\frac{-57-13r}{(r+4)(r+2)(r+3)r}$	$-\frac{7}{6}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{55r + 210}{(r+5)(r+3)r(r+2)(r+4)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{53}{72}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-r}{r(r+2)}$	$-\frac{4}{3}$
a_2	$\frac{7r+12}{(r+3)r(r+2)}$	$\frac{19}{12}$
a_3	$\frac{-57-13r}{(r+4)(r+2)(r+3)r}$	$-\frac{7}{6}$
a_4	$\frac{55r+210}{(r+5)(r+3)r(r+2)(r+4)}$	$\frac{53}{72}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-795 - 133r}{(r+6)(r+4)(r+2)r(r+3)(r+5)}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{116}{315}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-r}{r(r+2)}$	$-\frac{4}{3}$
a_2	$\frac{7r+12}{(r+3)r(r+2)}$	$\frac{19}{12}$
a_3	$\frac{-57-13r}{(r+4)(r+2)(r+3)r}$	$-\frac{7}{6}$
a_4	$\frac{55r+210}{(r+5)(r+3)r(r+2)(r+4)}$	$\frac{53}{72}$
a_5	$\frac{-795-133r}{(r+6)(r+4)(r+2)r(r+3)(r+5)}$	$-\frac{116}{315}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{463r + 2784}{(r+7)(r+5)(r+3)\,r\,(r+2)(r+4)(r+6)}$$

Which for the root $r = 1$ becomes

$$a_6 = \frac{3247}{20160}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-r}{r(r+2)}$	$-\frac{4}{3}$
a_2	$\frac{7r+12}{(r+3)r(r+2)}$	$\frac{19}{12}$
a_3	$\frac{-57-13r}{(r+4)(r+2)(r+3)r}$	$-\frac{7}{6}$
a_4	$\frac{55r+210}{(r+5)(r+3)r(r+2)(r+4)}$	$\frac{53}{72}$
a_5	$\frac{-795-133r}{(r+6)(r+4)(r+2)r(r+3)(r+5)}$	$-\frac{116}{315}$
a_6	$\frac{463r+2784}{(r+7)(r+5)(r+3)r(r+2)(r+4)(r+6)}$	$\frac{3247}{20160}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{-9741 - 1261r}{(8+r)(r+6)(r+4)(r+2)\,r\,(r+3)(r+5)(r+7)}$$

Which for the root $r = 1$ becomes

$$a_7 = -\frac{5501}{90720}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-r}{r(r+2)}$	$-\frac{4}{3}$
a_2	$\frac{7r+12}{(r+3)r(r+2)}$	$\frac{19}{12}$
a_3	$\frac{-57-13r}{(r+4)(r+2)(r+3)r}$	$-\frac{7}{6}$
a_4	$\frac{55r+210}{(r+5)(r+3)r(r+2)(r+4)}$	$\frac{53}{72}$
a_5	$\frac{-795-133r}{(r+6)(r+4)(r+2)r(r+3)(r+5)}$	$-\frac{116}{315}$
a_6	$\frac{463r+2784}{(r+7)(r+5)(r+3)r(r+2)(r+4)(r+6)}$	$\frac{3247}{20160}$
a_7	$\frac{-9741-1261r}{(8+r)(r+6)(r+4)(r+2)r(r+3)(r+5)(r+7)}$	$-\frac{5501}{90720}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\ &= x\left(1 - \frac{4x}{3} + \frac{19x^2}{12} - \frac{7x^3}{6} + \frac{53x^4}{72} - \frac{116x^5}{315} + \frac{3247x^6}{20160} - \frac{5501x^7}{90720} + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_2 \\ &= \frac{7r+12}{(r+3)\,r\,(r+2)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{7r+12}{(r+3)\,r\,(r+2)} &= \lim_{r \to -1} \frac{7r+12}{(r+3)\,r\,(r+2)} \\ &= -\frac{5}{2} \end{aligned}$$

The limit is $-\frac{5}{2}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-1}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$
b_1 = -\frac{r+3}{r(r+2)}
$$

For $2 \leq n$ the recursive equation is

$$
b_n(n+r)(n+r-1) + b_{n-1}(n+r-1) + b_n(n+r) - 6b_{n-2} + 3b_{n-1} - b_n = 0 \quad (4)
$$

Which for for the root $r = -1$ becomes

$$
b_n(n-1)(n-2) + b_{n-1}(n-2) + b_n(n-1) - 6b_{n-2} + 3b_{n-1} - b_n = 0 \qquad (4A)
$$

Solving for b_n from the recursive equation (4) gives

$$
b_n = -\frac{n b_{n-1} + r b_{n-1} - 6b_{n-2} + 2b_{n-1}}{n^2 + 2nr + r^2 - 1} \tag{5}
$$

Which for the root $r = -1$ becomes

$$
b_n = -\frac{n b_{n-1} - 6b_{n-2} + b_{n-1}}{n^2 - 2n} \tag{6}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-3-r}{r(r+2)}$	2

For $n = 2$, using the above recursive equation gives

$$
b_2 = \frac{7r+12}{(r+3)r(r+2)}
$$

Which for the root $r = -1$ becomes

$$b_2 = -\frac{5}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-3-r}{r(r+2)}$	2
b_2	$\frac{7r+12}{(r+3)r(r+2)}$	$-\frac{5}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{13r + 57}{(r^2 + 6r + 8)(r + 3)r}$$

Which for the root $r = -1$ becomes

$$b_3 = \frac{22}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-3-r}{r(r+2)}$	2
b_2	$\frac{7r+12}{(r+3)r(r+2)}$	$-\frac{5}{2}$
b_3	$\frac{-57-13r}{(r+4)(r+2)(r+3)r}$	$\frac{22}{3}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{55r + 210}{(r^2 + 6r + 8)r(r^2 + 8r + 15)}$$

Which for the root $r = -1$ becomes

$$b_4 = -\frac{155}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-3-r}{r(r+2)}$	2
b_2	$\frac{7r+12}{(r+3)r(r+2)}$	$-\frac{5}{2}$
b_3	$\frac{-57-13r}{(r+4)(r+2)(r+3)r}$	$\frac{22}{3}$
b_4	$\frac{55r+210}{(r^2+6r+8)r(r^2+8r+15)}$	$-\frac{155}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{133r + 795}{(r^2 + 10r + 24)(r + 2) r (r + 3) (r + 5)}$$

Which for the root $r = -1$ becomes

$$b_5 = \frac{331}{60}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-3-r}{r(r+2)}$	2
b_2	$\frac{7r+12}{(r+3)r(r+2)}$	$-\frac{5}{2}$
b_3	$\frac{-57-13r}{(r+4)(r+2)(r+3)r}$	$\frac{22}{3}$
b_4	$\frac{55r+210}{(r^2+6r+8)r(r^2+8r+15)}$	$-\frac{155}{24}$
b_5	$\frac{-795-133r}{(r+6)(r+4)(r+2)r(r+3)(r+5)}$	$\frac{331}{60}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{463r + 2784}{(r + 3) r (r^2 + 6r + 8)(r + 6)(r^2 + 12r + 35)}$$

Which for the root $r = -1$ becomes

$$b_6 = -\frac{2321}{720}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-3-r}{r(r+2)}$	2
b_2	$\frac{7r+12}{(r+3)r(r+2)}$	$-\frac{5}{2}$
b_3	$\frac{-57-13r}{(r+4)(r+2)(r+3)r}$	$\frac{22}{3}$
b_4	$\frac{55r+210}{(r^2+6r+8)r(r^2+8r+15)}$	$-\frac{155}{24}$
b_5	$\frac{-795-133r}{(r+6)(r+4)(r+2)r(r+3)(r+5)}$	$\frac{331}{60}$
b_6	$\frac{463r+2784}{(r+7)(r+5)(r+3)r(r+2)(r+4)(r+6)}$	$-\frac{2321}{720}$

For $n = 7$, using the above recursive equation gives

$$b_7 = -\frac{1261r + 9741}{(r^2 + 14r + 48)(r + 4)(r + 2)r(r + 3)(r + 5)(r + 7)}$$

Which for the root $r = -1$ becomes

$$b_7 = \frac{106}{63}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-3-r}{r(r+2)}$	2
b_2	$\frac{7r+12}{(r+3)r(r+2)}$	$-\frac{5}{2}$
b_3	$\frac{-57-13r}{(r+4)(r+2)(r+3)r}$	$\frac{22}{3}$
b_4	$\frac{55r+210}{(r^2+6r+8)r(r^2+8r+15)}$	$-\frac{155}{24}$
b_5	$\frac{-795-133r}{(r+6)(r+4)(r+2)r(r+3)(r+5)}$	$\frac{331}{60}$
b_6	$\frac{463r+2784}{(r+7)(r+5)(r+3)r(r+2)(r+4)(r+6)}$	$-\frac{2321}{720}$
b_7	$\frac{-9741-1261r}{(8+r)(r+6)(r+4)(r+2)r(r+3)(r+5)(r+7)}$	$\frac{106}{63}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots\right) \\ &= \frac{1 + 2x - \frac{5x^2}{2} + \frac{22x^3}{3} - \frac{155x^4}{24} + \frac{331x^5}{60} - \frac{2321x^6}{720} + \frac{106x^7}{63} + O(x^8)}{x} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x \left(1 - \frac{4x}{3} + \frac{19x^2}{12} - \frac{7x^3}{6} + \frac{53x^4}{72} - \frac{116x^5}{315} + \frac{3247x^6}{20160} - \frac{5501x^7}{90720} + O(x^8)\right) \\
&\quad + \frac{c_2 \left(1 + 2x - \frac{5x^2}{2} + \frac{22x^3}{3} - \frac{155x^4}{24} + \frac{331x^5}{60} - \frac{2321x^6}{720} + \frac{106x^7}{63} + O(x^8)\right)}{x}
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x \left(1 - \frac{4x}{3} + \frac{19x^2}{12} - \frac{7x^3}{6} + \frac{53x^4}{72} - \frac{116x^5}{315} + \frac{3247x^6}{20160} - \frac{5501x^7}{90720} + O(x^8)\right) \\
&\quad + \frac{c_2 \left(1 + 2x - \frac{5x^2}{2} + \frac{22x^3}{3} - \frac{155x^4}{24} + \frac{331x^5}{60} - \frac{2321x^6}{720} + \frac{106x^7}{63} + O(x^8)\right)}{x}
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= c_1 x \left(1 - \frac{4x}{3} + \frac{19x^2}{12} - \frac{7x^3}{6} + \frac{53x^4}{72} - \frac{116x^5}{315} + \frac{3247x^6}{20160} - \frac{5501x^7}{90720} + O(x^8)\right) \\
&\quad + \frac{c_2 \left(1 + 2x - \frac{5x^2}{2} + \frac{22x^3}{3} - \frac{155x^4}{24} + \frac{331x^5}{60} - \frac{2321x^6}{720} + \frac{106x^7}{63} + O(x^8)\right)}{x}
\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}
y &= c_1 x \left(1 - \frac{4x}{3} + \frac{19x^2}{12} - \frac{7x^3}{6} + \frac{53x^4}{72} - \frac{116x^5}{315} + \frac{3247x^6}{20160} - \frac{5501x^7}{90720} + O(x^8)\right) \\
&\quad + \frac{c_2 \left(1 + 2x - \frac{5x^2}{2} + \frac{22x^3}{3} - \frac{155x^4}{24} + \frac{331x^5}{60} - \frac{2321x^6}{720} + \frac{106x^7}{63} + O(x^8)\right)}{x}
\end{aligned}$$

Verified OK.

8.4.1 Maple step by step solution

Let's solve

$x^2 y'' + (x^2 + x) y' + (-6x^2 + 3x - 1) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(6x^2-3x+1)y}{x^2} - \frac{(1+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(1+x)y'}{x} - \frac{(6x^2-3x+1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1+x}{x}, P_3(x) = -\frac{6x^2-3x+1}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -1$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' + x(1 + x) y' + (-6x^2 + 3x - 1) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+r)\,(-1+r)\, x^r + (a_1(2+r)\, r + a_0(3+r))\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+1)\,(k+r-1) + a_{k-1}(k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+r)\,(-1+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-1, 1\}$$

- Each term must be 0

  $$a_1(2+r)\, r + a_0(3+r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = -\frac{a_0(3+r)}{r(2+r)}$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(k+r+1)\,(k+r-1) + a_{k-1}(k+2+r) - 6a_{k-2} = 0$$

- Shift index using $k->k+2$

  $$a_{k+2}(k+r+3)\,(k+r+1) + a_{k+1}(k+4+r) - 6a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{k a_{k+1} + r a_{k+1} - 6a_k + 4a_{k+1}}{(k+r+3)(k+r+1)}$$

- Recursion relation for $r = -1$

  $$a_{k+2} = -\frac{k a_{k+1} - 6a_k + 3a_{k+1}}{(k+2)k}$$

- Series not valid for $r = -1$, division by 0 in the recursion relation at $k = 0$

  $a_{k+2} = -\frac{k a_{k+1} - 6 a_k + 3 a_{k+1}}{(k+2)k}$

- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{k a_{k+1} - 6 a_k + 5 a_{k+1}}{(k+4)(k+2)}$

- Solution for $r = 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{k a_{k+1} - 6 a_k + 5 a_{k+1}}{(k+4)(k+2)}, a_1 = -\frac{4a_0}{3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 53

```
Order:=8;
dsolve(x^2*diff(y(x),x$2)+x*(1+x)*diff(y(x),x)-(1-3*x+6*x^2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x \left(1 - \frac{4}{3}x + \frac{19}{12}x^2 - \frac{7}{6}x^3 + \frac{53}{72}x^4 - \frac{116}{315}x^5 + \frac{3247}{20160}x^6 - \frac{5501}{90720}x^7 + \mathrm{O}\left(x^8\right)\right) + \frac{c_2\left(-2 - 4x + 5x^2 - \frac{44}{3}x^3 + \frac{155}{12}x^4 - \frac{331}{30}x^5 + \frac{2321}{360}x^6 - \frac{212}{63}x^7 + \mathrm{O}\left(x^8\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.149 (sec). Leaf size: 92

```
AsymptoticDSolveValue[x^2*y''[x]+x*(1+x)*y'[x]-(1-3*x+6*x^2)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(-\frac{2321x^5}{720} + \frac{331x^4}{60} - \frac{155x^3}{24} + \frac{22x^2}{3} - \frac{5x}{2} + \frac{1}{x} + 2\right) + c_2\left(\frac{3247x^7}{20160} - \frac{116x^6}{315} + \frac{53x^5}{72} - \frac{7x^4}{6} + \frac{19x^3}{12} - \frac{4x^2}{3} + x\right)$$

8.5 problem 6

Internal problem ID [6999]
Internal file name [`OUTPUT/6242_Thursday_August_18_2022_07_11_25_AM_64086885/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.11 Many-Term Recurrence Relations. Exercises page 391
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + xy' + (x^4 + 1) y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + xy' + (x^4 + 1) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 1$$
$$q(x) = \frac{x^4 + 1}{x}$$

Table 185: Table $p(x), q(x)$ singularites.

$p(x) = 1$	
singularity	type

$q(x) = \frac{x^4+1}{x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + xy' + \left(x^4 + 1\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + x\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(x^4+1\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+4} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} x^{n+r+4} a_n = \sum_{n=5}^{\infty} a_{n-5} x^{n+r-1}$$

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \left(\sum_{n=5}^{\infty} a_{n-5} x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) = 0$$

Or

$$x^{-1+r} a_0 r(-1+r) = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r(-1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 1 \\ r_2 &= 0 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+1} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots

of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = -\frac{1}{r}$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = \frac{1}{r(1+r)}$$

Substituting $n = 3$ in Eq. (2B) gives

$$a_3 = -\frac{1}{r(1+r)(2+r)}$$

Substituting $n = 4$ in Eq. (2B) gives

$$a_4 = \frac{1}{r(1+r)(2+r)(3+r)}$$

For $5 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_{n-1}(n+r-1) + a_{n-5} + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{n a_{n-1} + r a_{n-1} + a_{n-5}}{(n+r)(n+r-1)} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{-n a_{n-1} - a_{n-5} - a_{n-1}}{(n+1)n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r}$	-1
a_2	$\frac{1}{r(1+r)}$	$\frac{1}{2}$
a_3	$-\frac{1}{r(1+r)(2+r)}$	$-\frac{1}{6}$
a_4	$\frac{1}{r(1+r)(2+r)(3+r)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-r^4 - 6r^3 - 11r^2 - 7r - 5}{r(1+r)(2+r)(3+r)(5+r)(4+r)}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r}$	-1
a_2	$\frac{1}{r(1+r)}$	$\frac{1}{2}$
a_3	$-\frac{1}{r(1+r)(2+r)}$	$-\frac{1}{6}$
a_4	$\frac{1}{r(1+r)(2+r)(3+r)}$	$\frac{1}{24}$
a_5	$\frac{-r^4-6r^3-11r^2-7r-5}{r(1+r)(2+r)(3+r)(5+r)(4+r)}$	$-\frac{1}{24}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{2r^5 + 27r^4 + 132r^3 + 298r^2 + 321r + 150}{r(1+r)(2+r)(3+r)(5+r)^2(4+r)(6+r)}$$

Which for the root $r = 1$ becomes

$$a_6 = \frac{31}{1008}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r}$	-1
a_2	$\frac{1}{r(1+r)}$	$\frac{1}{2}$
a_3	$-\frac{1}{r(1+r)(2+r)}$	$-\frac{1}{6}$
a_4	$\frac{1}{r(1+r)(2+r)(3+r)}$	$\frac{1}{24}$
a_5	$\frac{-r^4-6r^3-11r^2-7r-5}{r(1+r)(2+r)(3+r)(5+r)(4+r)}$	$-\frac{1}{24}$
a_6	$\frac{2r^5+27r^4+132r^3+298r^2+321r+150}{r(1+r)(2+r)(3+r)(5+r)^2(4+r)(6+r)}$	$\frac{31}{1008}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{-3r^6 - 66r^5 - 576r^4 - 2577r^3 - 6351r^2 - 8337r - 4650}{r(1+r)(2+r)(3+r)(5+r)^2(4+r)(6+r)^2(7+r)}$$

Which for the root $r = 1$ becomes

$$a_7 = -\frac{47}{3528}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r}$	-1
a_2	$\frac{1}{r(1+r)}$	$\frac{1}{2}$
a_3	$-\frac{1}{r(1+r)(2+r)}$	$-\frac{1}{6}$
a_4	$\frac{1}{r(1+r)(2+r)(3+r)}$	$\frac{1}{24}$
a_5	$\frac{-r^4-6r^3-11r^2-7r-5}{r(1+r)(2+r)(3+r)(5+r)(4+r)}$	$-\frac{1}{24}$
a_6	$\frac{2r^5+27r^4+132r^3+298r^2+321r+150}{r(1+r)(2+r)(3+r)(5+r)^2(4+r)(6+r)}$	$\frac{31}{1008}$
a_7	$\frac{-3r^6-66r^5-576r^4-2577r^3-6351r^2-8337r-4650}{r(1+r)(2+r)(3+r)(5+r)^2(4+r)(6+r)^2(7+r)}$	$-\frac{47}{3528}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\ &= x\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{24} + \frac{31x^6}{1008} - \frac{47x^7}{3528} + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= -\frac{1}{r} \end{aligned}$$

Therefore

$$\lim_{r\to r_2} -\frac{1}{r} = \lim_{r\to 0} -\frac{1}{r} \\ = \textit{undefined}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $xy'' + xy' + (x^4+1)\,y = 0$ gives

$$\begin{aligned}
&x\Bigg(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\Bigg) \\
&\quad + x\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) \\
&\quad + (x^4+1)\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0
\end{aligned}$$

Which can be written as

$$\left(\left(x y_1''(x)+y_1'(x)\, x+\left(x^4+1\right) y_1(x)\right) \ln (x)+x\left(\frac{2 y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+y_1(x)\right) C \\ +x\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\ +x\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+\left(x^4+1\right)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)=0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$x y_1''(x)+y_1'(x)\, x+\left(x^4+1\right) y_1(x)=0$$

Eq (7) simplifes to

$$\left(x\left(\frac{2 y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+y_1(x)\right) C \\ +x\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\ +x\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+\left(x^4+1\right)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)=0 \tag{8}$$

Substituting $y_1=\sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\frac{\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n(n+r_1)\right) x+(x-1)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C}{x} \\ +\frac{\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right) x^2+\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) x^2+\left(x^4+1\right)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) x}{x} \\ =0 \tag{9}$$

Since $r_1 = 1$ and $r_2 = 0$ then the above becomes

$$\frac{\left(2\left(\sum_{n=0}^{\infty} x^n a_n(n+1)\right) x + (x-1)\left(\sum_{n=0}^{\infty} a_n x^{n+1}\right)\right) C}{x} + \frac{\left(\sum_{n=0}^{\infty} x^{-2+n} b_n n(n-1)\right) x^2 + \left(\sum_{n=0}^{\infty} x^{n-1} b_n n\right) x^2 + (x^4+1)\left(\sum_{n=0}^{\infty} b_n x^n\right) x}{x} = 0 \tag{10}$$

Which simplifies to

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^n a_n(n+1)\right) + \left(\sum_{n=0}^{\infty} C\, x^{n+1} a_n\right) \\
&+ \sum_{n\,=0}^{\infty} (-C a_n x^n) + \left(\sum_{n=0}^{\infty} n\, x^{n-1} b_n(n-1)\right) \\
&+ \left(\sum_{n=0}^{\infty} x^n b_n n\right) + \left(\sum_{n=0}^{\infty} x^{n+4} b_n\right) + \left(\sum_{n=0}^{\infty} b_n x^n\right) = 0
\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\sum_{n\,=0}^{\infty} 2C\, x^n a_n(n+1) = \sum_{n=1}^{\infty} 2C a_{n-1} n\, x^{n-1}$$

$$\sum_{n\,=0}^{\infty} C\, x^{n+1} a_n = \sum_{n=2}^{\infty} C a_{-2+n} x^{n-1}$$

$$\sum_{n\,=0}^{\infty} (-C a_n x^n) = \sum_{n=1}^{\infty} \left(-C a_{n-1} x^{n-1}\right)$$

$$\sum_{n\,=0}^{\infty} x^n b_n n = \sum_{n=1}^{\infty} (n-1)\, b_{n-1} x^{n-1}$$

$$\sum_{n\,=0}^{\infty} x^{n+4} b_n = \sum_{n=5}^{\infty} b_{n-5} x^{n-1}$$

$$\sum_{n\,=0}^{\infty} b_n x^n = \sum_{n=1}^{\infty} b_{n-1} x^{n-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$
\begin{aligned}
&\left(\sum_{n=1}^{\infty} 2 C a_{n-1} n\, x^{n-1}\right)+\left(\sum_{n=2}^{\infty} C a_{-2+n} x^{n-1}\right)+\sum_{n\,=1}^{\infty}\left(-C a_{n-1} x^{n-1}\right) \\
&\quad+\left(\sum_{n=0}^{\infty} n\, x^{n-1} b_n(n-1)\right)+\left(\sum_{n=1}^{\infty}(n-1)\, b_{n-1} x^{n-1}\right) \qquad (2B) \\
&\quad+\left(\sum_{n=5}^{\infty} b_{n-5} x^{n-1}\right)+\left(\sum_{n=1}^{\infty} b_{n-1} x^{n-1}\right)=0
\end{aligned}
$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=N$, where $N=1$ which is the difference between the two roots, we are free to choose $b_1=0$. Hence for $n=1$, Eq (2B) gives

$$C+1=0$$

Which is solved for C. Solving for C gives

$$C=-1$$

For $n=2$, Eq (2B) gives

$$(a_0+3a_1)\,C+2b_1+2b_2=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$2+2b_2=0$$

Solving the above for b_2 gives

$$b_2=-1$$

For $n=3$, Eq (2B) gives

$$(a_1+5a_2)\,C+3b_2+6b_3=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{9}{2}+6b_3=0$$

Solving the above for b_3 gives

$$b_3=\frac{3}{4}$$

For $n = 4$, Eq (2B) gives

$$(a_2 + 7a_3)\,C + 4b_3 + 12b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{11}{3} + 12b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{11}{36}$$

For $n = 5$, Eq (2B) gives

$$(a_3 + 9a_4)\,C + b_0 + 5b_4 + 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{53}{72} + 20b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{53}{1440}$$

For $n = 6$, Eq (2B) gives

$$(a_4 + 11a_5)\,C + b_1 + 6b_5 + 30b_6 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{51}{80} + 30b_6 = 0$$

Solving the above for b_6 gives

$$b_6 = -\frac{17}{800}$$

For $n = 7$, Eq (2B) gives

$$(a_5 + 13a_6)\,C + b_2 + 7b_6 + 42b_7 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{75947}{50400} + 42b_7 = 0$$

Solving the above for b_7 gives

$$b_7 = \frac{75947}{2116800}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -1$ and all b_n, then the second solution becomes

$$y_2(x) = (-1)\left(x\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{24} + \frac{31x^6}{1008} - \frac{47x^7}{3528} + O(x^8)\right)\right)\ln(x) + 1 - x^2 + \frac{3x^3}{4} - \frac{11x^4}{36} + \frac{53x^5}{1440} - \frac{17x^6}{800} + \frac{75947x^7}{2116800} + O(x^8)$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{24} + \frac{31x^6}{1008} - \frac{47x^7}{3528} + O(x^8)\right) \\ &+ c_2\left((-1)\left(x\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{24} + \frac{31x^6}{1008} - \frac{47x^7}{3528} + O(x^8)\right)\right)\ln(x) \right. \\ &\left. + 1 - x^2 + \frac{3x^3}{4} - \frac{11x^4}{36} + \frac{53x^5}{1440} - \frac{17x^6}{800} + \frac{75947x^7}{2116800} + O(x^8)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{24} + \frac{31x^6}{1008} - \frac{47x^7}{3528} + O(x^8)\right) \\ &+ c_2\left(-x\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{24} + \frac{31x^6}{1008} - \frac{47x^7}{3528} + O(x^8)\right)\ln(x) + 1 - x^2 \right. \\ &\left. + \frac{3x^3}{4} - \frac{11x^4}{36} + \frac{53x^5}{1440} - \frac{17x^6}{800} + \frac{75947x^7}{2116800} + O(x^8)\right) \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$\begin{aligned} y &= c_1 x\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{24} + \frac{31x^6}{1008} - \frac{47x^7}{3528} + O(x^8)\right) \\ &+ c_2\left(-x\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{24} + \frac{31x^6}{1008} - \frac{47x^7}{3528} + O(x^8)\right)\ln(x) + 1 \right. \\ &\left. - x^2 + \frac{3x^3}{4} - \frac{11x^4}{36} + \frac{53x^5}{1440} - \frac{17x^6}{800} + \frac{75947x^7}{2116800} + O(x^8)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1 x \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{24} + \frac{31x^6}{1008} - \frac{47x^7}{3528} + O(x^8)\right) + c_2\left(-x\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{24} + \frac{31x^6}{1008} - \frac{47x^7}{3528} + O(x^8)\right)\ln(x) + 1 - x^2 + \frac{3x^3}{4} - \frac{11x^4}{36} + \frac{53x^5}{1440} - \frac{17x^6}{800} + \frac{75947x^7}{2116800} + O(x^8)\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      trying to convert to an ODE of Bessel type
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 70

```
Order:=8;
dsolve(x*diff(y(x),x$2)+x*diff(y(x),x)+(1+x^4)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x \left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{24}x^5 + \frac{31}{1008}x^6 - \frac{47}{3528}x^7 + \mathrm{O}\left(x^8\right)\right) + c_2 \left(\ln\left(x\right)\left(-x + x^2 - \frac{1}{2}x^3 + \frac{1}{6}x^4 - \frac{1}{24}x^5 + \frac{1}{24}x^6 - \frac{31}{1008}x^7 + \mathrm{O}\left(x^8\right)\right) + \left(1 - x + \frac{1}{4}x^3 - \frac{5}{36}x^4 - \frac{7}{1440}x^5 + \frac{49}{2400}x^6 + \frac{10847}{2116800}x^7 + \mathrm{O}\left(x^8\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.087 (sec). Leaf size: 114

```
AsymptoticDSolveValue[x*y''[x]+x*y'[x]+(1+x^4)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1 \left(\frac{1}{24}x\left(x^5 - x^4 + 4x^3 - 12x^2 + 24x - 24\right)\log(x) + \frac{-153x^6 + 265x^5 - 2200x^4 + 5400x^3 - 7200x^2 + 7200}{7200}\right) + c_2 \left(\frac{31x^7}{1008} - \frac{x^6}{24} + \frac{x^5}{24} - \frac{x^4}{6} + \frac{x^3}{2} - x^2 + x\right)$$

8.6 problem 8

Internal problem ID [7000]
Internal file name [`OUTPUT/6243_Thursday_August_18_2022_07_11_27_AM_31591721/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.11 Many-Term Recurrence Relations. Exercises page 391
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
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$$\boxed{x(x-2)^2 y'' - 2(x-2) y' + 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^3 - 4x^2 + 4x) y'' + (-2x + 4) y' + 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{2}{x(x-2)}$$
$$q(x) = \frac{2}{x(x-2)^2}$$

Table 186: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{2}{x(x-2)}$	
singularity	type
$x = 0$	"regular"
$x = 2$	"regular"

$q(x) = \frac{2}{x(x-2)^2}$	
singularity	type
$x = 0$	"regular"
$x = 2$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 2, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x\left(x^2 - 4x + 4\right) y'' + (-2x + 4)\, y' + 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x\left(x^2 - 4x + 4\right) \left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}\right) + (-2x + 4) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r) x^{n+r-1}$$

$$\sum_{n=0}^{\infty} \left(-4x^{n+r} a_n (n+r)(n+r-1)\right) = \sum_{n=1}^{\infty} \left(-4a_{n-1}(n+r-1)(n+r-2) x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-2x^{n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1) x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} 2a_n x^{n+r} = \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r) x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-4a_{n-1}(n+r-1)(n+r-2) x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1) x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 4(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$4x^{n+r-1}a_n(n+r)(n+r-1) + 4(n+r)\,a_n x^{n+r-1} = 0$$

When $n = 0$ the above becomes

$$4x^{-1+r}a_0 r(-1+r) + 4r a_0 x^{-1+r} = 0$$

Or

$$\left(4x^{-1+r}r(-1+r) + 4r\,x^{-1+r}\right)a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$4x^{-1+r}r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$4x^{-1+r}r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of

integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{2r^2 - r - 1}{2\left(1 + r\right)^2}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &a_{n-2}(n + r - 2)\left(n - 3 + r\right) - 4a_{n-1}(n + r - 1)\left(n + r - 2\right) \\ &+ 4a_n(n + r)\left(n + r - 1\right) - 2a_{n-1}(n + r - 1) + 4a_n(n + r) + 2a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{n^2a_{n-2} - 4n^2a_{n-1} + 2nra_{n-2} - 8nra_{n-1} + r^2a_{n-2} - 4r^2a_{n-1} - 5na_{n-2} + 10na_{n-1} - 5ra_{n-2} + 10r}{4\left(n^2 + 2nr + r^2\right)} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{\left(\left(a_{n-2} - 4a_{n-1}\right)n - 3a_{n-2} + 2a_{n-1}\right)\left(n - 2\right)}{4n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-r-1}{2(1+r)^2}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{3r^4 + 3r^3 - 4r^2 - 2r}{4\left(1 + r\right)^2\left(r + 2\right)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-r-1}{2(1+r)^2}$	$-\frac{1}{2}$
a_2	$\frac{3r^4+3r^3-4r^2-2r}{4(1+r)^2(r+2)^2}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{2r^5 + 7r^4 + 2r^3 - 8r^2 - 3r}{4(r+3)^2(r+2)^2(1+r)}$$

Which for the root $r = 0$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-r-1}{2(1+r)^2}$	$-\frac{1}{2}$
a_2	$\frac{3r^4+3r^3-4r^2-2r}{4(1+r)^2(r+2)^2}$	0
a_3	$\frac{2r^5+7r^4+2r^3-8r^2-3r}{4(r+3)^2(r+2)^2(1+r)}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{5r^6 + 35r^5 + 65r^4 - 5r^3 - 76r^2 - 24r}{16(r+4)^2(r+3)^2(r+2)(1+r)}$$

Which for the root $r = 0$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-r-1}{2(1+r)^2}$	$-\frac{1}{2}$
a_2	$\frac{3r^4+3r^3-4r^2-2r}{4(1+r)^2(r+2)^2}$	0
a_3	$\frac{2r^5+7r^4+2r^3-8r^2-3r}{4(r+3)^2(r+2)^2(1+r)}$	0
a_4	$\frac{5r^6+35r^5+65r^4-5r^3-76r^2-24r}{16(r+4)^2(r+3)^2(r+2)(1+r)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{6r^7 + 69r^6 + 265r^5 + 335r^4 - 127r^3 - 428r^2 - 120r}{32 (r+5)^2 (r+4)^2 (1+r) (r+2) (r+3)}$$

Which for the root $r = 0$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-r-1}{2(1+r)^2}$	$-\frac{1}{2}$
a_2	$\frac{3r^4+3r^3-4r^2-2r}{4(1+r)^2(r+2)^2}$	0
a_3	$\frac{2r^5+7r^4+2r^3-8r^2-3r}{4(r+3)^2(r+2)^2(1+r)}$	0
a_4	$\frac{5r^6+35r^5+65r^4-5r^3-76r^2-24r}{16(r+4)^2(r+3)^2(r+2)(1+r)}$	0
a_5	$\frac{6r^7+69r^6+265r^5+335r^4-127r^3-428r^2-120r}{32(r+5)^2(r+4)^2(1+r)(r+2)(r+3)}$	0

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{7r\left(r^6 + 18r^5 + 125r^4 + 420r^3 + 696r^2 + 504r + \frac{720}{7}\right)(-1+r)}{64 (r+6)^2 (r+5)^2 (1+r) (r+2) (r+3) (r+4)}$$

Which for the root $r = 0$ becomes

$$a_6 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-r-1}{2(1+r)^2}$	$-\frac{1}{2}$
a_2	$\frac{3r^4+3r^3-4r^2-2r}{4(1+r)^2(r+2)^2}$	0
a_3	$\frac{2r^5+7r^4+2r^3-8r^2-3r}{4(r+3)^2(r+2)^2(1+r)}$	0
a_4	$\frac{5r^6+35r^5+65r^4-5r^3-76r^2-24r}{16(r+4)^2(r+3)^2(r+2)(1+r)}$	0
a_5	$\frac{6r^7+69r^6+265r^5+335r^4-127r^3-428r^2-120r}{32(r+5)^2(r+4)^2(1+r)(r+2)(r+3)}$	0
a_6	$\frac{7r\left(r^6+18r^5+125r^4+420r^3+696r^2+504r+\frac{720}{7}\right)(-1+r)}{64(r+6)^2(r+5)^2(1+r)(r+2)(r+3)(r+4)}$	0

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{r(-1+r)\left(r^6+21r^5+168r^4+637r^3+1155r^2+882r+180\right)\left(\frac{7}{2}+r\right)}{16\left(r+7\right)^2\left(r+6\right)^2\left(r+3\right)\left(r+2\right)\left(1+r\right)\left(r+4\right)\left(r+5\right)}$$

Which for the root $r = 0$ becomes

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-r-1}{2(1+r)^2}$	$-\frac{1}{2}$
a_2	$\frac{3r^4+3r^3-4r^2-2r}{4(1+r)^2(r+2)^2}$	0
a_3	$\frac{2r^5+7r^4+2r^3-8r^2-3r}{4(r+3)^2(r+2)^2(1+r)}$	0
a_4	$\frac{5r^6+35r^5+65r^4-5r^3-76r^2-24r}{16(r+4)^2(r+3)^2(r+2)(1+r)}$	0
a_5	$\frac{6r^7+69r^6+265r^5+335r^4-127r^3-428r^2-120r}{32(r+5)^2(r+4)^2(1+r)(r+2)(r+3)}$	0
a_6	$\frac{7r\left(r^6+18r^5+125r^4+420r^3+696r^2+504r+\frac{720}{7}\right)(-1+r)}{64(r+6)^2(r+5)^2(1+r)(r+2)(r+3)(r+4)}$	0
a_7	$\frac{r(-1+r)\left(r^6+21r^5+168r^4+637r^3+1155r^2+882r+180\right)\left(\frac{7}{2}+r\right)}{16(r+7)^2(r+6)^2(r+3)(r+2)(1+r)(r+4)(r+5)}$	0

Using the above table, then the first solution $y_1(x)$ becomes

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots \\ &= 1 - \frac{x}{2} + O(x^8) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	$\frac{2r^2-r-1}{2(1+r)^2}$	$-\frac{1}{2}$	$\frac{5r+1}{2(1+r)^3}$
b_2	$\frac{3r^4+3r^3-4r^2-2r}{4(1+r)^2(r+2)^2}$	0	$\frac{15r^4+41r^3+24r^2-10r-4}{4(1+r)^3(r+2)^3}$
b_3	$\frac{2r^5+7r^4+2r^3-8r^2-3r}{4(r+3)^2(r+2)^2(1+r)}$	0	$\frac{15r^6+109r^5+278r^4+270r^3+27r^2-81r-18}{4(r+3)^3(r+2)^3(1+r)^2}$
b_4	$\frac{5r^6+35r^5+65r^4-5r^3-76r^2-24r}{16(r+4)^2(r+3)^2(r+2)(1+r)}$	0	$\frac{25r^8+345r^7+1900r^6+5262r^5+7429r^4+4251r^3-828r^2-1656r-}{8(r+4)^3(r+3)^3(r+2)^2(1+r)^2}$
b_5	$\frac{6r^7+69r^6+265r^5+335r^4-127r^3-428r^2-120r}{32(r+5)^2(r+4)^2(1+r)(r+2)(r+3)}$	0	$\frac{75r^{10}+1675r^9+15840r^8+82364r^7+255031r^6+471575r^5+4786}{32(r+5)^3(r+4)^3(1+r)^2(r+2)}$
b_6	$\frac{7r\left(r^6+18r^5+125r^4+420r^3+696r^2+504r+\frac{720}{7}\right)(-1+r)}{64(r+6)^2(r+5)^2(1+r)(r+2)(r+3)(r+4)}$	0	$\frac{105r^{12}+3451r^{11}+49700r^{10}+411754r^9+2163917r^8+7487227r^7}{64(r+6)^3(r}$
b_7	$\frac{r(-1+r)\left(r^6+21r^5+168r^4+637r^3+1155r^2+882r+180\right)\left(\frac{7}{2}+r\right)}{16(r+7)^2(r+6)^2(r+3)(r+2)(1+r)(r+4)(r+5)}$	0	$35r^{14}+1589r^{13}+32326r^{12}+388906r^{11}+3074148r^{10}+16764690$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 + b_7x^7 + b_8x^8\ldots \\
&= \left(1-\frac{x}{2}+O(x^8)\right)\ln(x) + \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{48} - \frac{x^4}{192} - \frac{x^5}{640} - \frac{x^6}{1920} - \frac{x^7}{5376} + O(x^8)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1\left(1-\frac{x}{2}+O(x^8)\right) \\
&\quad + c_2\left(\left(1-\frac{x}{2}+O(x^8)\right)\ln(x)+\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{48}-\frac{x^4}{192}-\frac{x^5}{640}-\frac{x^6}{1920}-\frac{x^7}{5376}+O(x^8)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1-\frac{x}{2}+O(x^8)\right) \\
&\quad + c_2\left(\left(1-\frac{x}{2}+O(x^8)\right)\ln(x)+\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{48}-\frac{x^4}{192}-\frac{x^5}{640}-\frac{x^6}{1920}-\frac{x^7}{5376}+O(x^8)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1\left(1-\frac{x}{2}+O\left(x^8\right)\right)+c_2\left(\left(1-\frac{x}{2}+O\left(x^8\right)\right)\ln\left(x\right)+\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{48}-\frac{x^4}{192}-\frac{x^5}{640}-\frac{x^6}{1920}-\frac{x^7}{5376}+O\left(x^8\right)\right) \tag{1}$$

Verification of solutions

$$y = c_1\left(1-\frac{x}{2}+O\left(x^8\right)\right)+c_2\left(\left(1-\frac{x}{2}+O\left(x^8\right)\right)\ln\left(x\right)+\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{48}-\frac{x^4}{192}-\frac{x^5}{640}-\frac{x^6}{1920}-\frac{x^7}{5376}+O\left(x^8\right)\right)$$

Verified OK.

8.6.1 Maple step by step solution

Let's solve

$x(x^2-4x+4)\,y''+(-2x+4)\,y'+2y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2y}{x(x^2-4x+4)}+\frac{2y'}{(x-2)x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{2y'}{(x-2)x}+\frac{2y}{x(x^2-4x+4)}=0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=-\frac{2}{x(x-2)},P_3(x)=\frac{2}{x(x^2-4x+4)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=1$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x(x-2)\, y''(x^2-4x+4) + (-2x^2+8x-8)\, y' + (2x-4)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- > k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- > k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..4$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- > k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-8a_0 r^2 x^{-1+r} + \left(-8a_1(1+r)^2 + 4a_0(3r^2-r-1)\right) x^r + \left(-8a_2(2+r)^2 + 4a_1(3r^2+5r+1) - 2a\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-8r^2 = 0$

- Values of r that satisfy the indicial equation

$r = 0$

- The coefficients of each power of x must be 0

  $[-8a_1(1+r)^2 + 4a_0(3r^2 - r - 1) = 0, -8a_2(2+r)^2 + 4a_1(3r^2 + 5r + 1) - 2a_0(1+3r)(-1+r)$

- Solve for the dependent coefficient(s)

  $\left\{a_1 = \frac{a_0(3r^2-r-1)}{2(r^2+2r+1)}, a_2 = \frac{a_0 r(6r^3+8r^2-3r-2)}{4(r^4+6r^3+13r^2+12r+4)}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 - 5k + 6) a_{k-2} + (-6k^2 + 16k - 8) a_{k-1} + (12k^2 - 4k - 4) a_k - 8a_{k+1}(k+1)^2 = 0$

- Shift index using $k- >k+2$

  $((k+2)^2 - 5k - 4) a_k + (-6(k+2)^2 + 16k + 24) a_{k+1} + (12(k+2)^2 - 4k - 12) a_{k+2} - 8a_{k+3}($

- Recursion relation that defines series solution to ODE

  $a_{k+3} = \frac{k^2a_k - 6k^2a_{k+1} + 12k^2a_{k+2} - ka_k - 8ka_{k+1} + 44ka_{k+2} + 36a_{k+2}}{8(k+3)^2}$

- Recursion relation for $r = 0$

  $a_{k+3} = \frac{k^2a_k - 6k^2a_{k+1} + 12k^2a_{k+2} - ka_k - 8ka_{k+1} + 44ka_{k+2} + 36a_{k+2}}{8(k+3)^2}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k^2a_k - 6k^2a_{k+1} + 12k^2a_{k+2} - ka_k - 8ka_{k+1} + 44ka_{k+2} + 36a_{k+2}}{8(k+3)^2}, a_1 = -\frac{a_0}{2}, a_2 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 47

```
Order:=8;
dsolve(x*(x-2)^2*diff(y(x),x$2)-2*(x-2)*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln (x) + c_1) \left(1 - \frac{1}{2}x + \mathrm{O}\left(x^8\right)\right) + \left(\frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{48}x^3 - \frac{1}{192}x^4 - \frac{1}{640}x^5 - \frac{1}{1920}x^6 - \frac{1}{5376}x^7 + \mathrm{O}\left(x^8\right)\right) c_2$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 75

```
AsymptoticDSolveValue[x*(x-2)^2*y''[x]-2*(x-2)*y'[x]+2*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2 \left(-\frac{x^7}{5376} - \frac{x^6}{1920} - \frac{x^5}{640} - \frac{x^4}{192} - \frac{x^3}{48} - \frac{x^2}{8} + \frac{x}{2} + \left(1 - \frac{x}{2}\right) \log(x)\right) + c_1 \left(1 - \frac{x}{2}\right)$$

8.7 problem 9

Internal problem ID [7001]
Internal file name [OUTPUT/6244_Thursday_August_18_2022_07_11_30_AM_32247685/index.tex]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.11 Many-Term Recurrence Relations. Exercises page 391
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x(x-2)^2 y'' - 2(x-2) y' + 2y = 0}$$

With the expansion point for the power series method at $x = 2$.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 2$$

The ode is converted to be in terms of the new independent variable t. This results in

$$(t+2)\, t^2 \left(\frac{d^2}{dt^2} y(t)\right) - 2t \left(\frac{d}{dt} y(t)\right) + 2y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. The transformed ODE is now solved. The type of the expansion point is first determined. This is done on the

homogeneous part of the ODE.

$$\left(t^3+2t^2\right)\left(\frac{d^2}{dt^2}y(t)\right)-2t\left(\frac{d}{dt}y(t)\right)+2y(t)=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$\frac{d^2}{dt^2}y(t)+p(t)\frac{d}{dt}y(t)+q(t)y(t)=0$$

Where

$$p(t)=-\frac{2}{t\,(t+2)}$$
$$q(t)=\frac{2}{(t+2)\,t^2}$$

Table 188: Table $p(t), q(t)$ singularites.

$p(t)=-\frac{2}{t(t+2)}$	
singularity	type
$t=-2$	"regular"
$t=0$	"regular"

$q(t)=\frac{2}{(t+2)t^2}$	
singularity	type
$t=-2$	"regular"
$t=0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-2, 0, \infty]$

Irregular singular points : $[]$

Since $t=0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$(t+2)\,t^2\left(\frac{d^2}{dt^2}y(t)\right)-2t\left(\frac{d}{dt}y(t)\right)+2y(t)=0$$

Let the solution be represented as Frobenius power series of the form

$$y(t)=\sum_{n=0}^{\infty}a_n t^{n+r}$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=0}^{\infty} (n+r)\, a_n t^{n+r-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n t^{n+r-2}$$

Substituting the above back into the ode gives

$$(t+2)\, t^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n t^{n+r-2}\right) - 2t \left(\sum_{n=0}^{\infty} (n+r)\, a_n t^{n+r-1}\right) + 2 \left(\sum_{n=0}^{\infty} a_n t^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} t^{1+n+r} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2t^{n+r} a_n (n+r)\,(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2t^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2a_n t^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of t be $n+r$ in each summation term. Going over each summation term above with power of t in it which is not already t^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} t^{1+n+r} a_n (n+r)\,(n+r-1) = \sum_{n=1}^{\infty} a_{n-1} (n+r-1)\,(n+r-2)\, t^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of t are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} a_{n-1} (n+r-1)\,(n+r-2)\, t^{n+r}\right) + \left(\sum_{n=0}^{\infty} 2t^{n+r} a_n (n+r)\,(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2t^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2a_n t^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$2t^{n+r}a_n(n+r)(n+r-1) - 2t^{n+r}a_n(n+r) + 2a_n t^{n+r} = 0$$

When $n = 0$ the above becomes

$$2t^r a_0 r(-1+r) - 2t^r a_0 r + 2a_0 t^r = 0$$

Or

$$(2t^r r(-1+r) - 2t^r r + 2t^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$2t^r(-1+r)^2 = 0$$

Since the above is true for all t then the indicial equation becomes

$$2(-1+r)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$2t^r(-1+r)^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(t) = \sum_{n=0}^{\infty} a_n t^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(t) = y_1(t)\ln(t) + \left(\sum_{n=1}^{\infty} b_n t^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of

integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 1$, Eqs (1A,1B) become

$$y_1(t) = \sum_{n=0}^{\infty} a_n t^{1+n}$$

$$y_2(t) = y_1(t) \ln(t) + \left(\sum_{n=1}^{\infty} b_n t^{1+n}\right)$$

We start by finding the first solution $y_1(t)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_{n-1}(n+r-1)(n+r-2) + 2a_n(n+r)(n+r-1) - 2a_n(n+r) + 2a_n = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(n+r-2)\, a_{n-1}}{2(n+r-1)} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{(n-1)\, a_{n-1}}{2n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1-r}{2r}$$

Which for the root $r = 1$ becomes

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-r}{2r}$	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-1 + r}{4 + 4r}$$

Which for the root $r = 1$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-r}{2r}$	0
a_2	$\frac{-1+r}{4+4r}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1 - r}{16 + 8r}$$

Which for the root $r = 1$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-r}{2r}$	0
a_2	$\frac{-1+r}{4+4r}$	0
a_3	$\frac{1-r}{16+8r}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{-1 + r}{48 + 16r}$$

Which for the root $r = 1$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-r}{2r}$	0
a_2	$\frac{-1+r}{4+4r}$	0
a_3	$\frac{1-r}{16+8r}$	0
a_4	$\frac{-1+r}{48+16r}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1-r}{128+32r}$$

Which for the root $r = 1$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-r}{2r}$	0
a_2	$\frac{-1+r}{4+4r}$	0
a_3	$\frac{1-r}{16+8r}$	0
a_4	$\frac{-1+r}{48+16r}$	0
a_5	$\frac{1-r}{128+32r}$	0

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{-1+r}{320+64r}$$

Which for the root $r = 1$ becomes

$$a_6 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-r}{2r}$	0
a_2	$\frac{-1+r}{4+4r}$	0
a_3	$\frac{1-r}{16+8r}$	0
a_4	$\frac{-1+r}{48+16r}$	0
a_5	$\frac{1-r}{128+32r}$	0
a_6	$\frac{-1+r}{320+64r}$	0

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{1-r}{768+128r}$$

Which for the root $r = 1$ becomes

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-r}{2r}$	0
a_2	$\frac{-1+r}{4+4r}$	0
a_3	$\frac{1-r}{16+8r}$	0
a_4	$\frac{-1+r}{48+16r}$	0
a_5	$\frac{1-r}{128+32r}$	0
a_6	$\frac{-1+r}{320+64r}$	0
a_7	$\frac{1-r}{768+128r}$	0

Using the above table, then the first solution $y_1(t)$ is

$$\begin{aligned} y_1(t) &= t\left(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + a_7 t^7 + a_8 t^8 \dots\right) \\ &= t\left(1 + O\left(t^8\right)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(t) = y_1(t) \ln(t) + \left(\sum_{n=1}^{\infty} b_n t^{n+r} \right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = 1)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{1-r}{2r}$	0	$-\frac{1}{2r^2}$	$-\frac{1}{2}$
b_2	$\frac{-1+r}{4+4r}$	0	$\frac{1}{2(1+r)^2}$	$\frac{1}{8}$
b_3	$\frac{1-r}{16+8r}$	0	$-\frac{3}{8(2+r)^2}$	$-\frac{1}{24}$
b_4	$\frac{-1+r}{48+16r}$	0	$\frac{1}{4(3+r)^2}$	$\frac{1}{64}$
b_5	$\frac{1-r}{128+32r}$	0	$-\frac{5}{32(4+r)^2}$	$-\frac{1}{160}$
b_6	$\frac{-1+r}{320+64r}$	0	$\frac{3}{32(5+r)^2}$	$\frac{1}{384}$
b_7	$\frac{1-r}{768+128r}$	0	$-\frac{7}{128(6+r)^2}$	$-\frac{1}{896}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned} y_2(t) &= y_1(t) \ln(t) + b_0 + b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5 + b_6 t^6 + b_7 t^7 + b_8 t^8 \dots \\ &= t\left(1 + O(t^8)\right) \ln(t) + t\left(-\frac{t}{2} + \frac{t^2}{8} - \frac{t^3}{24} + \frac{t^4}{64} - \frac{t^5}{160} + \frac{t^6}{384} - \frac{t^7}{896} + O(t^8)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(t) &= c_1 y_1(t) + c_2 y_2(t) \\ &= c_1 t\left(1 + O(t^8)\right) \\ &\quad + c_2 \left(t\left(1 + O(t^8)\right) \ln(t) + t\left(-\frac{t}{2} + \frac{t^2}{8} - \frac{t^3}{24} + \frac{t^4}{64} - \frac{t^5}{160} + \frac{t^6}{384} - \frac{t^7}{896} + O(t^8)\right)\right) \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y(t) &= y_h \\
&= c_1 t\left(1+O\left(t^8\right)\right) \\
&\quad + c_2\left(t\left(1+O\left(t^8\right)\right)\ln(t) + t\left(-\frac{t}{2}+\frac{t^2}{8}-\frac{t^3}{24}+\frac{t^4}{64}-\frac{t^5}{160}+\frac{t^6}{384}-\frac{t^7}{896}+O\left(t^8\right)\right)\right)
\end{aligned}
$$

Replacing t in the above with the original independent variable $x$susing $t = x-2$ results in

$$
\begin{aligned}
y = {} & c_1(x-2)\left(1+O\left((x-2)^8\right)\right)+c_2\left((x-2)\left(1+O\left((x-2)^8\right)\right)\ln(x-2)+(x-2)\left(-\frac{x}{2}+1\right.\right. \\
&\left.\left.+\frac{(x-2)^2}{8}-\frac{(x-2)^3}{24}+\frac{(x-2)^4}{64}-\frac{(x-2)^5}{160}+\frac{(x-2)^6}{384}-\frac{(x-2)^7}{896}+O\left((x-2)^8\right)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1(x-2)\left(1+O\left((x-2)^8\right)\right) \\
&+c_2\left((x-2)\left(1+O\left((x-2)^8\right)\right)\ln(x-2)+(x-2)\left(-\frac{x}{2}+1+\frac{(x-2)^2}{8}\right.\right. \\
&\left.\left.-\frac{(x-2)^3}{24}+\frac{(x-2)^4}{64}-\frac{(x-2)^5}{160}+\frac{(x-2)^6}{384}-\frac{(x-2)^7}{896}+O\left((x-2)^8\right)\right)\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1(x-2)\left(1+O\left((x-2)^8\right)\right)+c_2\left((x-2)\left(1+O\left((x-2)^8\right)\right)\ln(x-2)+(x-2)\left(-\frac{x}{2}+1\right.\right. \\
&\left.\left.+\frac{(x-2)^2}{8}-\frac{(x-2)^3}{24}+\frac{(x-2)^4}{64}-\frac{(x-2)^5}{160}+\frac{(x-2)^6}{384}-\frac{(x-2)^7}{896}+O\left((x-2)^8\right)\right)\right)
\end{aligned}
$$

Verified OK.

8.7.1 Maple step by step solution

Let's solve

$x(x-2)^2 y'' + (-2x+4) y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2y'}{(x-2)x} - \frac{2y}{x(x-2)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{(x-2)x} + \frac{2y}{x(x-2)^2} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2}{x(x-2)}, P_3(x) = \frac{2}{x(x-2)^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x=0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x=0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x(x-2)^2 y'' + (-2x+4) y' + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $$x^{m}\cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^{m}\cdot y''$ to series expansion for $m=1..3$

  $$x^{m}\cdot y'' = \sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^{m}\cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$4a_{0}r^{2}x^{-1+r} + \left(4a_{1}(1+r)^{2} - 2a_{0}(1+2r)\,(-1+r)\right) x^{r} + \left(\sum_{k=1}^{\infty} \left(4a_{k+1}(k+1+r)^{2} - 2a_{k}(2k+2r\right.\right.$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $$4r^{2} = 0$$

- Values of r that satisfy the indicial equation

  $$r = 0$$

- Each term must be 0

  $$4a_{1}(1+r)^{2} - 2a_{0}(1+2r)\,(-1+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(-4a_{k} + a_{k-1} + 4a_{k+1})\, k^{2} + (2a_{k} - 3a_{k-1} + 8a_{k+1})\, k + 2a_{k} + 2a_{k-1} + 4a_{k+1} = 0$$

- Shift index using $k->k+1$

  $$(-4a_{k+1} + a_{k} + 4a_{k+2})\,(k+1)^{2} + (2a_{k+1} - 3a_{k} + 8a_{k+2})\,(k+1) + 2a_{k+1} + 2a_{k} + 4a_{k+2} = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{k(ka_{k} - 4ka_{k+1} - a_{k} - 6a_{k+1})}{4(k^{2}+4k+4)}$$

- Recursion relation for $r=0$; series terminates at $k=0$

  $$a_{k+2} = -\frac{k(ka_{k} - 4ka_{k+1} - a_{k} - 6a_{k+1})}{4(k^{2}+4k+4)}$$

- Solution for $r=0$

  $$\left[y = \sum_{k=0}^{\infty} a_{k}x^{k}, a_{k+2} = -\frac{k(ka_{k} - 4ka_{k+1} - a_{k} - 6a_{k+1})}{4(k^{2}+4k+4)}, 4a_{1} + 2a_{0} = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 55

```
Order:=8;
dsolve(x*(x-2)^2*diff(y(x),x$2)-2*(x-2)*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=2);
```

$$y(x) = \left((\ln(-2+x) c_2 + c_1)\left(1 + \mathrm{O}\left((-2+x)^8\right)\right) + \left(-\frac{1}{2}(-2+x) + \frac{1}{8}(-2+x)^2 - \frac{1}{24}(-2+x)^3 + \frac{1}{64}(-2+x)^4 - \frac{1}{160}(-2+x)^5 + \frac{1}{384}(-2+x)^6 - \frac{1}{896}(-2+x)^7 + \mathrm{O}\left((-2+x)^8\right)\right) c_2\right)(-2+x)$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 90

```
AsymptoticDSolveValue[x*(x-2)^2*y''[x]-2*(x-2)*y'[x]+2*y[x]==0,y[x],{x,2,7}]
```

$$y(x) \to c_1(x-2) + c_2\left(\left(-\frac{1}{896}(x-2)^7 + \frac{1}{384}(x-2)^6 - \frac{1}{160}(x-2)^5 + \frac{1}{64}(x-2)^4 - \frac{1}{24}(x-2)^3 + \frac{1}{8}(x-2)^2 + \frac{2-x}{2}\right)(x-2) + (x-2)\log(x-2)\right)$$

8.8 problem 10

Internal problem ID [7002]
Internal file name [`OUTPUT/6245_Thursday_August_18_2022_07_11_33_AM_88339133/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. 18.11 Many-Term Recurrence Relations. Exercises page 391
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2xy'' + (1-x)\,y' - (1+x)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2xy'' + (1-x)\,y' + (-x-1)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= -\frac{x-1}{2x} \\ q(x) &= -\frac{1+x}{2x} \end{aligned}$$

Table 190: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x-1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1+x}{2x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2xy'' + (1 - x)\, y' + (-x - 1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x \left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}\right) + (1 - x) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (-x - 1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1}a_n(n+r)(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-x^{n+r}a_n(n+r)\right)+\left(\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}\right)+\sum_{n=0}^{\infty}\left(-x^{1+n+r}a_n\right)+\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-x^{n+r}a_n(n+r)\right)=\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)\,x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty}\left(-x^{1+n+r}a_n\right)=\sum_{n=2}^{\infty}\left(-a_{n-2}x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)=\sum_{n=1}^{\infty}\left(-a_{n-1}x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1}a_n(n+r)(n+r-1)\right)+\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)\,x^{n+r-1}\right)+\left(\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}\right)+\sum_{n=2}^{\infty}\left(-a_{n-2}x^{n+r-1}\right)+\sum_{n=1}^{\infty}\left(-a_{n-1}x^{n+r-1}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1}a_n(n+r)(n+r-1)+(n+r)\,a_n x^{n+r-1}=0$$

When $n=0$ the above becomes

$$2x^{-1+r}a_0 r(-1+r)+r a_0 x^{-1+r}=0$$

Or

$$\left(2x^{-1+r}r(-1+r)+r\,x^{-1+r}\right)a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\,x^{-1+r}(-1+2r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(-1+2r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{1}{2r+1}$$

For $2 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) + a_n(n+r) - a_{n-2} - a_{n-1} = 0 \qquad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{n a_{n-1} + r a_{n-1} + a_{n-2}}{2n^2 + 4nr + 2r^2 - n - r} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{2n a_{n-1} + 2a_{n-2} + a_{n-1}}{4n^2 + 2n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2r+1}$	$\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{3r + 3}{4r^3 + 16r^2 + 19r + 6}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{9}{40}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2r+1}$	$\frac{1}{2}$
a_2	$\frac{3r+3}{4r^3+16r^2+19r+6}$	$\frac{9}{40}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{5r^2 + 19r + 15}{8r^5 + 76r^4 + 274r^3 + 461r^2 + 351r + 90}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = \frac{103}{1680}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2r+1}$	$\frac{1}{2}$
a_2	$\frac{3r+3}{4r^3+16r^2+19r+6}$	$\frac{9}{40}$
a_3	$\frac{5r^2+19r+15}{8r^5+76r^4+274r^3+461r^2+351r+90}$	$\frac{103}{1680}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{11r^3 + 78r^2 + 169r + 105}{(2r^2 + 15r + 28)(4r^3 + 16r^2 + 19r + 6)(2r^2 + 11r + 15)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{187}{13440}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2r+1}$	$\frac{1}{2}$
a_2	$\frac{3r+3}{4r^3+16r^2+19r+6}$	$\frac{9}{40}$
a_3	$\frac{5r^2+19r+15}{8r^5+76r^4+274r^3+461r^2+351r+90}$	$\frac{103}{1680}$
a_4	$\frac{11r^3+78r^2+169r+105}{(2r^2+15r+28)(4r^3+16r^2+19r+6)(2r^2+11r+15)}$	$\frac{187}{13440}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{21r^4 + 246r^3 + 1014r^2 + 1707r + 945}{(2r^2 + 19r + 45)(8r^5 + 76r^4 + 274r^3 + 461r^2 + 351r + 90)(2r^2 + 15r + 28)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = \frac{247}{98560}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2r+1}$	$\frac{1}{2}$
a_2	$\frac{3r+3}{4r^3+16r^2+19r+6}$	$\frac{9}{40}$
a_3	$\frac{5r^2+19r+15}{8r^5+76r^4+274r^3+461r^2+351r+90}$	$\frac{103}{1680}$
a_4	$\frac{11r^3+78r^2+169r+105}{(2r^2+15r+28)(4r^3+16r^2+19r+6)(2r^2+11r+15)}$	$\frac{187}{13440}$
a_5	$\frac{21r^4+246r^3+1014r^2+1707r+945}{(2r^2+19r+45)(8r^5+76r^4+274r^3+461r^2+351r+90)(2r^2+15r+28)}$	$\frac{247}{98560}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{43r^5 + 737r^4 + 4805r^3 + 14722r^2 + 20787r + 10395}{(2r^2 + 23r + 66)(2r^2 + 19r + 45)(8r^5 + 76r^4 + 274r^3 + 461r^2 + 351r + 90)(2r^2 + 15r + 28)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_6 = \frac{17861}{46126080}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2r+1}$	$\frac{1}{2}$
a_2	$\frac{3r+3}{4r^3+16r^2+19r+6}$	$\frac{9}{40}$
a_3	$\frac{5r^2+19r+15}{8r^5+76r^4+274r^3+461r^2+351r+90}$	$\frac{103}{1680}$
a_4	$\frac{11r^3+78r^2+169r+105}{(2r^2+15r+28)(4r^3+16r^2+19r+6)(2r^2+11r+15)}$	$\frac{187}{13440}$
a_5	$\frac{21r^4+246r^3+1014r^2+1707r+945}{(2r^2+19r+45)(8r^5+76r^4+274r^3+461r^2+351r+90)(2r^2+15r+28)}$	$\frac{247}{98560}$
a_6	$\frac{43r^5+737r^4+4805r^3+14722r^2+20787r+10395}{(2r^2+23r+66)(2r^2+19r+45)(8r^5+76r^4+274r^3+461r^2+351r+90)(2r^2+15r+28)}$	$\frac{17861}{46126080}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{85r^6 + 2013r^5 + 19036r^4 + 91329r^3 + 231916r^2 + 290301r + 135135}{(2r^2 + 23r + 66)(2r^2 + 19r + 45)(8r^5 + 76r^4 + 274r^3 + 461r^2 + 351r + 90)(2r^2 + 15r + 28)(2r^2 +}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_7 = \frac{23767}{461260800}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2r+1}$	$\frac{1}{2}$
a_2	$\frac{3r+3}{4r^3+16r^2+19r+6}$	$\frac{9}{40}$
a_3	$\frac{5r^2+19r+15}{8r^5+76r^4+274r^3+461r^2+351r+90}$	$\frac{103}{1680}$
a_4	$\frac{11r^3+78r^2+169r+105}{(2r^2+15r+28)(4r^3+16r^2+19r+6)(2r^2+11r+15)}$	$\frac{187}{13440}$
a_5	$\frac{21r^4+246r^3+1014r^2+1707r+945}{(2r^2+19r+45)(8r^5+76r^4+274r^3+461r^2+351r+90)(2r^2+15r+28)}$	$\frac{247}{98560}$
a_6	$\frac{43r^5+737r^4+4805r^3+14722r^2+20787r+10395}{(2r^2+23r+66)(2r^2+19r+45)(8r^5+76r^4+274r^3+461r^2+351r+90)(2r^2+15r+28)}$	$\frac{17861}{46126080}$
a_7	$\frac{85r^6+2013r^5+19036r^4+91329r^3+231916r^2+290301r+135135}{(2r^2+23r+66)(2r^2+19r+45)(8r^5+76r^4+274r^3+461r^2+351r+90)(2r^2+15r+28)(2r^2+27r+91)}$	$\frac{23767}{461260800}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \ldots\right) \\
&= \sqrt{x}\left(1 + \frac{x}{2} + \frac{9x^2}{40} + \frac{103x^3}{1680} + \frac{187x^4}{13440} + \frac{247x^5}{98560} + \frac{17861x^6}{46126080} + \frac{23767x^7}{461260800} + O(x^8)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
b_1 = \frac{1}{2r+1}
$$

For $2 \leq n$ the recursive equation is

$$
2b_n(n+r)(n+r-1) - b_{n-1}(n+r-1) + (n+r)b_n - b_{n-2} - b_{n-1} = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = \frac{n b_{n-1} + r b_{n-1} + b_{n-2}}{2n^2 + 4nr + 2r^2 - n - r} \tag{4}
$$

Which for the root $r = 0$ becomes

$$
b_n = \frac{n b_{n-1} + b_{n-2}}{n(2n-1)} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2r+1}$	1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{3r+3}{4r^3+16r^2+19r+6}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2r+1}$	1
b_2	$\frac{3r+3}{4r^3+16r^2+19r+6}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{5r^2+19r+15}{8r^5+76r^4+274r^3+461r^2+351r+90}$$

Which for the root $r = 0$ becomes

$$b_3 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2r+1}$	1
b_2	$\frac{3r+3}{4r^3+16r^2+19r+6}$	$\frac{1}{2}$
b_3	$\frac{5r^2+19r+15}{8r^5+76r^4+274r^3+461r^2+351r+90}$	$\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{11r^3 + 78r^2 + 169r + 105}{(2r^2 + 15r + 28)(4r^3 + 16r^2 + 19r + 6)(2r^2 + 11r + 15)}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2r+1}$	1
b_2	$\frac{3r+3}{4r^3+16r^2+19r+6}$	$\frac{1}{2}$
b_3	$\frac{5r^2+19r+15}{8r^5+76r^4+274r^3+461r^2+351r+90}$	$\frac{1}{6}$
b_4	$\frac{11r^3+78r^2+169r+105}{(2r^2+15r+28)(4r^3+16r^2+19r+6)(2r^2+11r+15)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{21r^4 + 246r^3 + 1014r^2 + 1707r + 945}{(2r^2 + 19r + 45)(8r^5 + 76r^4 + 274r^3 + 461r^2 + 351r + 90)(2r^2 + 15r + 28)}$$

Which for the root $r = 0$ becomes

$$b_5 = \frac{1}{120}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2r+1}$	1
b_2	$\frac{3r+3}{4r^3+16r^2+19r+6}$	$\frac{1}{2}$
b_3	$\frac{5r^2+19r+15}{8r^5+76r^4+274r^3+461r^2+351r+90}$	$\frac{1}{6}$
b_4	$\frac{11r^3+78r^2+169r+105}{(2r^2+15r+28)(4r^3+16r^2+19r+6)(2r^2+11r+15)}$	$\frac{1}{24}$
b_5	$\frac{21r^4+246r^3+1014r^2+1707r+945}{(2r^2+19r+45)(8r^5+76r^4+274r^3+461r^2+351r+90)(2r^2+15r+28)}$	$\frac{1}{120}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{43r^5 + 737r^4 + 4805r^3 + 14722r^2 + 20787r + 10395}{(2r^2 + 23r + 66)(2r^2 + 19r + 45)(8r^5 + 76r^4 + 274r^3 + 461r^2 + 351r + 90)(2r^2 + 15r + 28)}$$

Which for the root $r = 0$ becomes

$$b_6 = \frac{1}{720}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2r+1}$	1
b_2	$\frac{3r+3}{4r^3+16r^2+19r+6}$	$\frac{1}{2}$
b_3	$\frac{5r^2+19r+15}{8r^5+76r^4+274r^3+461r^2+351r+90}$	$\frac{1}{6}$
b_4	$\frac{11r^3+78r^2+169r+105}{(2r^2+15r+28)(4r^3+16r^2+19r+6)(2r^2+11r+15)}$	$\frac{1}{24}$
b_5	$\frac{21r^4+246r^3+1014r^2+1707r+945}{(2r^2+19r+45)(8r^5+76r^4+274r^3+461r^2+351r+90)(2r^2+15r+28)}$	$\frac{1}{120}$
b_6	$\frac{43r^5+737r^4+4805r^3+14722r^2+20787r+10395}{(2r^2+23r+66)(2r^2+19r+45)(8r^5+76r^4+274r^3+461r^2+351r+90)(2r^2+15r+28)}$	$\frac{1}{720}$

For $n = 7$, using the above recursive equation gives

$$b_7 = \frac{85r^6 + 2013r^5 + 19036r^4 + 91329r^3 + 231916r^2 + 290301r + 135135}{(2r^2 + 23r + 66)(2r^2 + 19r + 45)(8r^5 + 76r^4 + 274r^3 + 461r^2 + 351r + 90)(2r^2 + 15r + 28)(2r^2 +}$$

Which for the root $r = 0$ becomes

$$b_7 = \frac{1}{5040}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2r+1}$	1
b_2	$\frac{3r+3}{4r^3+16r^2+19r+6}$	$\frac{1}{2}$
b_3	$\frac{5r^2+19r+15}{8r^5+76r^4+274r^3+461r^2+351r+90}$	$\frac{1}{6}$
b_4	$\frac{11r^3+78r^2+169r+105}{(2r^2+15r+28)(4r^3+16r^2+19r+6)(2r^2+11r+15)}$	$\frac{1}{24}$
b_5	$\frac{21r^4+246r^3+1014r^2+1707r+945}{(2r^2+19r+45)(8r^5+76r^4+274r^3+461r^2+351r+90)(2r^2+15r+28)}$	$\frac{1}{120}$
b_6	$\frac{43r^5+737r^4+4805r^3+14722r^2+20787r+10395}{(2r^2+23r+66)(2r^2+19r+45)(8r^5+76r^4+274r^3+461r^2+351r+90)(2r^2+15r+28)}$	$\frac{1}{720}$
b_7	$\frac{85r^6+2013r^5+19036r^4+91329r^3+231916r^2+290301r+135135}{(2r^2+23r+66)(2r^2+19r+45)(8r^5+76r^4+274r^3+461r^2+351r+90)(2r^2+15r+28)(2r^2+27r+91)}$	$\frac{1}{5040}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\
&= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \sqrt{x}\left(1+\frac{x}{2}+\frac{9x^2}{40}+\frac{103x^3}{1680}+\frac{187x^4}{13440}+\frac{247x^5}{98560}+\frac{17861x^6}{46126080}+\frac{23767x^7}{461260800}+O(x^8)\right) \\
&\quad + c_2\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 \sqrt{x}\left(1+\frac{x}{2}+\frac{9x^2}{40}+\frac{103x^3}{1680}+\frac{187x^4}{13440}+\frac{247x^5}{98560}+\frac{17861x^6}{46126080}+\frac{23767x^7}{461260800}+O(x^8)\right) \\
&\quad + c_2\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= c_1 \sqrt{x}\left(1+\frac{x}{2}+\frac{9x^2}{40}+\frac{103x^3}{1680}+\frac{187x^4}{13440}+\frac{247x^5}{98560}+\frac{17861x^6}{46126080}+\frac{23767x^7}{461260800}+O(x^8)\right) \\
&\quad + c_2\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8)\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}
y &= c_1 \sqrt{x}\left(1+\frac{x}{2}+\frac{9x^2}{40}+\frac{103x^3}{1680}+\frac{187x^4}{13440}+\frac{247x^5}{98560}+\frac{17861x^6}{46126080}+\frac{23767x^7}{461260800}+O(x^8)\right) \\
&\quad + c_2\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8)\right)
\end{aligned}$$

Verified OK.

8.8.1 Maple step by step solution

Let's solve

$2xy'' + (1-x)\,y' + (-x-1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(1+x)y}{2x} + \frac{(x-1)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x-1)y'}{2x} - \frac{(1+x)y}{2x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{x-1}{2x}, P_3(x) = -\frac{1+x}{2x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = \frac{1}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2xy'' + (1-x)\,y' + (-x-1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- > k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- > k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-1+2r)\, x^{-1+r} + (a_1(1+r)\,(1+2r) - a_0(1+r))\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(2k+1+2r) \cdot\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term must be 0

  $$a_1(1+r)\,(1+2r) - a_0(1+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+1+r)\,(2k+1+2r) - a_k(k+1+r) - a_{k-1} = 0$$

- Shift index using $k- > k+1$

  $$a_{k+2}(k+2+r)\,(2k+3+2r) - a_{k+1}(k+2+r) - a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = \frac{k a_{k+1} + r a_{k+1} + a_k + 2a_{k+1}}{(k+2+r)(2k+3+2r)}$$

- Recursion relation for $r = 0$

  $$a_{k+2} = \frac{k a_{k+1} + a_k + 2a_{k+1}}{(k+2)(2k+3)}$$

- Solution for $r = 0$

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=\frac{k a_{k+1}+a_k+2a_{k+1}}{(k+2)(2k+3)}, a_1-a_0=0\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+2}=\frac{k a_{k+1}+a_k+\frac{5}{2}a_{k+1}}{\left(k+\frac{5}{2}\right)(2k+4)}$$

- Solution for $r=\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2}=\frac{k a_{k+1}+a_k+\frac{5}{2}a_{k+1}}{\left(k+\frac{5}{2}\right)(2k+4)}, 3a_1-\frac{3a_0}{2}=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2}=\frac{k a_{k+1}+a_k+2a_{k+1}}{(k+2)(2k+3)}, a_1-a_0=0, b_{k+2}=\frac{k b_{k+1}+b_k+\frac{5}{2}b_{k+1}}{\left(k+\frac{5}{2}\right)(2k+4)}, 3b_1$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 52

```
Order:=8;
dsolve(2*x*diff(y(x),x$2)+(1-x)*diff(y(x),x)-(1+x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\sqrt{x}\left(1+\frac{1}{2}x+\frac{9}{40}x^2+\frac{103}{1680}x^3+\frac{187}{13440}x^4+\frac{247}{98560}x^5+\frac{17861}{46126080}x^6+\frac{23767}{461260800}x^7+\mathrm{O}\left(x^8\right)\right)+c_2\left(1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5+\frac{1}{720}x^6+\frac{1}{5040}x^7+\mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 75

```
AsymptoticDSolveValue[x*(x-2)^2*y''[x]-2*(x-2)*y'[x]+2*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(-\frac{x^7}{5376}-\frac{x^6}{1920}-\frac{x^5}{640}-\frac{x^4}{192}-\frac{x^3}{48}-\frac{x^2}{8}+\frac{x}{2}+\left(1-\frac{x}{2}\right)\log(x)\right)+c_1\left(1-\frac{x}{2}\right)$$

9 CHAPTER 18. Power series solutions. Miscellaneous Exercises. page 394

9.1 problem 1

Internal problem ID [7003]
Internal file name [`OUTPUT/6246_Thursday_August_18_2022_07_11_36_AM_97496273/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. Miscellaneous Exercises. page 394
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{xy'' - (x+2)\,y' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + (-x-2)\,y' - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x+2}{x}$$
$$q(x) = -\frac{1}{x}$$

Table 192: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x+2}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + (-x - 2)\, y' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &x \left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}\right) \\ &+ (-x - 2) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-2(n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-2(n+r) a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) - 2(n+r) a_n x^{n+r-1} = 0$$

When $n = 0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) - 2r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) - 2r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-3+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-3+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 3 \\ r_2 &= 0 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(-3+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x^3\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+3} \\ y_2(x) &= C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) - 2a_n(n+r) - a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{n - 3 + r} \tag{4}$$

Which for the root $r = 3$ becomes

$$a_n = \frac{a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 3$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{-2 + r}$$

Which for the root $r = 3$ becomes

$$a_1 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{-2+r}$	1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(-2 + r)(-1 + r)}$$

Which for the root $r = 3$ becomes

$$a_2 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{-2+r}$	1
a_2	$\frac{1}{(-2+r)(-1+r)}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{(-2+r)(-1+r)r}$$

Which for the root $r = 3$ becomes

$$a_3 = \frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{-2+r}$	1
a_2	$\frac{1}{(-2+r)(-1+r)}$	$\frac{1}{2}$
a_3	$\frac{1}{(-2+r)(-1+r)r}$	$\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(-2+r)r(r^2-1)}$$

Which for the root $r = 3$ becomes

$$a_4 = \frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{-2+r}$	1
a_2	$\frac{1}{(-2+r)(-1+r)}$	$\frac{1}{2}$
a_3	$\frac{1}{(-2+r)(-1+r)r}$	$\frac{1}{6}$
a_4	$\frac{1}{(-2+r)r(r^2-1)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{r^5 - 5r^3 + 4r}$$

Which for the root $r = 3$ becomes

$$a_5 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{-2+r}$	1
a_2	$\frac{1}{(-2+r)(-1+r)}$	$\frac{1}{2}$
a_3	$\frac{1}{(-2+r)(-1+r)r}$	$\frac{1}{6}$
a_4	$\frac{1}{(-2+r)r(r^2-1)}$	$\frac{1}{24}$
a_5	$\frac{1}{r^5-5r^3+4r}$	$\frac{1}{120}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{1}{r(r^4 - 5r^2 + 4)(3 + r)}$$

Which for the root $r = 3$ becomes

$$a_6 = \frac{1}{720}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{-2+r}$	1
a_2	$\frac{1}{(-2+r)(-1+r)}$	$\frac{1}{2}$
a_3	$\frac{1}{(-2+r)(-1+r)r}$	$\frac{1}{6}$
a_4	$\frac{1}{(-2+r)r(r^2-1)}$	$\frac{1}{24}$
a_5	$\frac{1}{r^5-5r^3+4r}$	$\frac{1}{120}$
a_6	$\frac{1}{r(r^4-5r^2+4)(3+r)}$	$\frac{1}{720}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{1}{r(r^4 - 5r^2 + 4)(3 + r)(4 + r)}$$

Which for the root $r = 3$ becomes

$$a_7 = \frac{1}{5040}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{-2+r}$	1
a_2	$\frac{1}{(-2+r)(-1+r)}$	$\frac{1}{2}$
a_3	$\frac{1}{(-2+r)(-1+r)r}$	$\frac{1}{6}$
a_4	$\frac{1}{(-2+r)r(r^2-1)}$	$\frac{1}{24}$
a_5	$\frac{1}{r^5-5r^3+4r}$	$\frac{1}{120}$
a_6	$\frac{1}{r(r^4-5r^2+4)(3+r)}$	$\frac{1}{720}$
a_7	$\frac{1}{r(r^4-5r^2+4)(3+r)(4+r)}$	$\frac{1}{5040}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^3\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \dots\right) \\ &= x^3\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_3 \\ &= \frac{1}{(-2+r)(-1+r)r} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{1}{(-2+r)(-1+r)r} &= \lim_{r\to 0} \frac{1}{(-2+r)(-1+r)r} \\ &= \mathit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$
\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $xy'' + (-x-2)y' - y = 0$ gives

$$
\begin{aligned}
& x\Bigg(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&+ \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\Bigg) \\
&+ (-x-2)\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) \\
&- Cy_1(x)\ln(x) - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\Bigg((xy_1''(x) + (-x-2)y_1'(x) - y_1(x))\ln(x) + x\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) \\
&+ \frac{(-x-2)y_1(x)}{x}\Bigg)C + x\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&+ (-x-2)\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$xy_1''(x) + (-x-2)\, y_1'(x) - y_1(x) = 0$$

Eq (7) simplifes to

$$\begin{aligned}
&\left(x\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + \frac{(-x-2)\, y_1(x)}{x}\right) C \\
&+ x\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&+ (-x-2)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\begin{aligned}
&\frac{\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n(n+r_1)\right) x - (x+3)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C}{x} \\
&+ \frac{\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right) x^2 + (-x^2-2x)\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) x}{x} \\
&= 0
\end{aligned} \tag{9}$$

Since $r_1 = 3$ and $r_2 = 0$ then the above becomes

$$\begin{aligned}
&\frac{\left(2\left(\sum_{n=0}^{\infty} x^{2+n} a_n(n+3)\right) x - (x+3)\left(\sum_{n=0}^{\infty} a_n x^{n+3}\right)\right) C}{x} \\
&+ \frac{\left(\sum_{n=0}^{\infty} x^{-2+n} b_n n(n-1)\right) x^2 + (-x^2-2x)\left(\sum_{n=0}^{\infty} x^{n-1} b_n n\right) - \left(\sum_{n=0}^{\infty} b_n x^n\right) x}{x} \\
&= 0
\end{aligned} \tag{10}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\,x^{2+n} a_n(n+3)\right)+\sum_{n=0}^{\infty}\left(-C\,x^{n+3} a_n\right) \\
&\quad+\sum_{n=0}^{\infty}\left(-3C\,x^{2+n} a_n\right)+\left(\sum_{n=0}^{\infty} n\,x^{n-1} b_n(n-1)\right) \\
&\quad+\sum_{n=0}^{\infty}\left(-x^{n} b_n n\right)+\sum_{n=0}^{\infty}\left(-2x^{n-1} b_n n\right)+\sum_{n=0}^{\infty}\left(-b_n x^{n}\right)=0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 2C\,x^{2+n} a_n(n+3) &= \sum_{n=3}^{\infty} 2C a_{n-3} n\,x^{n-1} \\
\sum_{n=0}^{\infty}\left(-C\,x^{n+3} a_n\right) &= \sum_{n=4}^{\infty}\left(-C a_{n-4} x^{n-1}\right) \\
\sum_{n=0}^{\infty}\left(-3C\,x^{2+n} a_n\right) &= \sum_{n=3}^{\infty}\left(-3C a_{n-3} x^{n-1}\right) \\
\sum_{n=0}^{\infty}\left(-x^{n} b_n n\right) &= \sum_{n=1}^{\infty}\left(-(n-1)\, b_{n-1} x^{n-1}\right) \\
\sum_{n=0}^{\infty}\left(-b_n x^{n}\right) &= \sum_{n=1}^{\infty}\left(-b_{n-1} x^{n-1}\right)
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$
\begin{aligned}
&\left(\sum_{n=3}^{\infty} 2C a_{n-3} n\,x^{n-1}\right)+\sum_{n=4}^{\infty}\left(-C a_{n-4} x^{n-1}\right)+\sum_{n=3}^{\infty}\left(-3C a_{n-3} x^{n-1}\right) \\
&\quad+\left(\sum_{n=0}^{\infty} n\,x^{n-1} b_n(n-1)\right)+\sum_{n=1}^{\infty}\left(-(n-1)\, b_{n-1} x^{n-1}\right) \\
&\quad+\sum_{n=0}^{\infty}\left(-2x^{n-1} b_n n\right)+\sum_{n=1}^{\infty}\left(-b_{n-1} x^{n-1}\right)=0
\end{aligned} \tag{2B}
$$

For $n = 0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = 1$, Eq (2B) gives

$$-2b_1 - b_0 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-2b_1 - 1 = 0$$

Solving the above for b_1 gives

$$b_1 = -\frac{1}{2}$$

For $n = 2$, Eq (2B) gives

$$-2b_2 - 2b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-2b_2 + 1 = 0$$

Solving the above for b_2 gives

$$b_2 = \frac{1}{2}$$

For $n = N$, where $N = 3$ which is the difference between the two roots, we are free to choose $b_3 = 0$. Hence for $n = 3$, Eq (2B) gives

$$3C - \frac{3}{2} = 0$$

Which is solved for C. Solving for C gives

$$C = \frac{1}{2}$$

For $n = 4$, Eq (2B) gives

$$(-a_0 + 5a_1)\, C - 4b_3 + 4b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$2 + 4b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{1}{2}$$

For $n = 5$, Eq (2B) gives

$$(-a_1 + 7a_2)\, C - 5b_4 + 10b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{15}{4}+10b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{3}{8}$$

For $n = 6$, Eq (2B) gives

$$(-a_2+9a_3)\, C - 6b_5 + 18b_6 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{11}{4}+18b_6 = 0$$

Solving the above for b_6 gives

$$b_6 = -\frac{11}{72}$$

For $n = 7$, Eq (2B) gives

$$(-a_3+11a_4)\, C - 7b_6 + 28b_7 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{175}{144}+28b_7 = 0$$

Solving the above for b_7 gives

$$b_7 = -\frac{25}{576}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = \frac{1}{2}$ and all b_n, then the second solution becomes

$$\begin{aligned} y_2(x) = {} & \frac{1}{2}\left(x^3\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}+\frac{x^7}{5040}+O(x^8)\right)\right)\ln(x) \\ & + 1 - \frac{x}{2} + \frac{x^2}{2} - \frac{x^4}{2} - \frac{3x^5}{8} - \frac{11x^6}{72} - \frac{25x^7}{576} + O(x^8) \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^3 \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8)\right) \\
&\quad + c_2 \left(\frac{1}{2}\left(x^3 \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8)\right)\right) \ln(x) + 1 \right. \\
&\quad \left. - \frac{x}{2} + \frac{x^2}{2} - \frac{x^4}{2} - \frac{3x^5}{8} - \frac{11x^6}{72} - \frac{25x^7}{576} + O(x^8)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^3 \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8)\right) \\
&\quad + c_2 \left(\frac{x^3 \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8)\right) \ln(x)}{2} + 1 - \frac{x}{2} + \frac{x^2}{2} \right. \\
&\quad \left. - \frac{x^4}{2} - \frac{3x^5}{8} - \frac{11x^6}{72} - \frac{25x^7}{576} + O(x^8)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^3 \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8)\right) \\
&\quad + c_2 \left(\frac{x^3 \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8)\right) \ln(x)}{2} + 1 - \frac{x}{2} \right. \\
&\quad \left. + \frac{x^2}{2} - \frac{x^4}{2} - \frac{3x^5}{8} - \frac{11x^6}{72} - \frac{25x^7}{576} + O(x^8)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = c_1 x^3 \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8)\right) + c_2 \left(\frac{x^3 \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8)\right) \ln(x)}{2} + 1 - \frac{x}{2} + \frac{x^2}{2} - \frac{x^4}{2} - \frac{3x^5}{8} - \frac{11x^6}{72} - \frac{25x^7}{576} + O(x^8)\right)$$

Verified OK.

9.1.1 Maple step by step solution

Let's solve

$xy'' + (-x - 2)\,y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{x} + \frac{(x+2)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x+2)y'}{x} - \frac{y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{x+2}{x}, P_3(x) = -\frac{1}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $xy'' + (-x-2)\,y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$

  ◦ Shift index using $k->k+1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-3+r)\,x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k-2+r) - a_k(k+1+r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 3\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+1+r)\,(a_{k+1}(k-2+r) - a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k-2+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k-2}$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 2$

$a_{k+1} = \frac{a_k}{k-2}$

- Recursion relation for $r = 3$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = \frac{a_k}{k+1}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 68

```
Order:=8;
dsolve(x*diff(y(x),x$2)-(2+x)*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^3 \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 + \mathrm{O}\left(x^8\right)\right) + c_2\left(\ln\left(x\right)\left(6x^3 + 6x^4 + 3x^5 + x^6 + \frac{1}{4}x^7 + \mathrm{O}\left(x^8\right)\right) + \left(12 - 6x + 6x^2 + 11x^3 + 5x^4 + x^5 - \frac{1}{16}x^7 + \mathrm{O}\left(x^8\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.095 (sec). Leaf size: 104

```
AsymptoticDSolveValue[x*y''[x]-(2+x)*y'[x]-y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1 \left(\frac{1}{12}\left(x^3+3x^2+6x+6\right)x^3\log(x)+\frac{1}{36}\left(-x^6+9x^4+27x^3+18x^2-18x+36\right)\right) + c_2\left(\frac{x^9}{720}+\frac{x^8}{120}+\frac{x^7}{24}+\frac{x^6}{6}+\frac{x^5}{2}+x^4+x^3\right)$$

9.2 problem 2

Internal problem ID [7004]
Internal file name [`OUTPUT/6247_Thursday_August_18_2022_07_11_39_AM_54095803/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. Miscellaneous Exercises. page 394
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

`[_Laguerre]`

$$\boxed{xy'' - (x + 2)\, y' - 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + (-x - 2)\, y' - 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x + 2}{x}$$
$$q(x) = -\frac{2}{x}$$

Table 194: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x+2}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{2}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + (-x - 2)\, y' - 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x \left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2} \right) + (-x - 2) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1} \right) - 2 \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-2(n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-2(n+r) a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) - 2(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) - 2r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) - 2r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-3+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-3+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 3$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(-3+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^3\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+3}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) - 2a_n(n+r) - 2a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(n+r+1)}{n^2+2nr+r^2-3n-3r} \tag{4}$$

Which for the root $r = 3$ becomes

$$a_n = \frac{a_{n-1}(n+4)}{n\,(n+3)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 3$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{2+r}{r^2-r-2}$$

Which for the root $r = 3$ becomes

$$a_1 = \frac{5}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2+r}{r^2-r-2}$	$\frac{5}{4}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{3+r}{r^3-2r^2-r+2}$$

Which for the root $r = 3$ becomes

$$a_2 = \frac{3}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2+r}{r^2-r-2}$	$\frac{5}{4}$
a_2	$\frac{3+r}{r^3-2r^2-r+2}$	$\frac{3}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{4+r}{r(r^3 - 2r^2 - r + 2)}$$

Which for the root $r = 3$ becomes

$$a_3 = \frac{7}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2+r}{r^2-r-2}$	$\frac{5}{4}$
a_2	$\frac{3+r}{r^3-2r^2-r+2}$	$\frac{3}{4}$
a_3	$\frac{4+r}{r(r^3-2r^2-r+2)}$	$\frac{7}{24}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{5+r}{(r+1)^2 r(-1+r)(r-2)}$$

Which for the root $r = 3$ becomes

$$a_4 = \frac{1}{12}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2+r}{r^2-r-2}$	$\frac{5}{4}$
a_2	$\frac{3+r}{r^3-2r^2-r+2}$	$\frac{3}{4}$
a_3	$\frac{4+r}{r(r^3-2r^2-r+2)}$	$\frac{7}{24}$
a_4	$\frac{5+r}{(r+1)^2r(-1+r)(r-2)}$	$\frac{1}{12}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{6+r}{(r+1)^2 r(-1+r)(r^2-4)}$$

Which for the root $r = 3$ becomes

$$a_5 = \frac{3}{160}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2+r}{r^2-r-2}$	$\frac{5}{4}$
a_2	$\frac{3+r}{r^3-2r^2-r+2}$	$\frac{3}{4}$
a_3	$\frac{4+r}{r(r^3-2r^2-r+2)}$	$\frac{7}{24}$
a_4	$\frac{5+r}{(r+1)^2 r(-1+r)(r-2)}$	$\frac{1}{12}$
a_5	$\frac{6+r}{(r+1)^2 r(-1+r)(r^2-4)}$	$\frac{3}{160}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{7+r}{(3+r)(r+1)^2 r(-1+r)(r^2-4)}$$

Which for the root $r = 3$ becomes

$$a_6 = \frac{1}{288}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2+r}{r^2-r-2}$	$\frac{5}{4}$
a_2	$\frac{3+r}{r^3-2r^2-r+2}$	$\frac{3}{4}$
a_3	$\frac{4+r}{r(r^3-2r^2-r+2)}$	$\frac{7}{24}$
a_4	$\frac{5+r}{(r+1)^2 r(-1+r)(r-2)}$	$\frac{1}{12}$
a_5	$\frac{6+r}{(r+1)^2 r(-1+r)(r^2-4)}$	$\frac{3}{160}$
a_6	$\frac{7+r}{(3+r)(r+1)^2 r(-1+r)(r^2-4)}$	$\frac{1}{288}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{8+r}{(4+r)(3+r)(r+1)^2 r(-1+r)(r^2-4)}$$

Which for the root $r = 3$ becomes

$$a_7 = \frac{11}{20160}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2+r}{r^2-r-2}$	$\frac{5}{4}$
a_2	$\frac{3+r}{r^3-2r^2-r+2}$	$\frac{3}{4}$
a_3	$\frac{4+r}{r(r^3-2r^2-r+2)}$	$\frac{7}{24}$
a_4	$\frac{5+r}{(r+1)^2 r(-1+r)(r-2)}$	$\frac{1}{12}$
a_5	$\frac{6+r}{(r+1)^2 r(-1+r)(r^2-4)}$	$\frac{3}{160}$
a_6	$\frac{7+r}{(3+r)(r+1)^2 r(-1+r)(r^2-4)}$	$\frac{1}{288}$
a_7	$\frac{8+r}{(4+r)(3+r)(r+1)^2 r(-1+r)(r^2-4)}$	$\frac{11}{20160}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^3\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \dots\right) \\
&= x^3\left(1 + \frac{5x}{4} + \frac{3x^2}{4} + \frac{7x^3}{24} + \frac{x^4}{12} + \frac{3x^5}{160} + \frac{x^6}{288} + \frac{11x^7}{20160} + O(x^8)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}
a_N &= a_3 \\
&= \frac{4+r}{r\left(r^3 - 2r^2 - r + 2\right)}
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{r\to r_2} \frac{4+r}{r\left(r^3 - 2r^2 - r + 2\right)} &= \lim_{r\to 0} \frac{4+r}{r\left(r^3 - 2r^2 - r + 2\right)} \\
&= \textit{undefined}
\end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $xy'' + (-x-2)y' - 2y = 0$ gives

$$\begin{aligned}
&x\Bigg(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&+ \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\Bigg) \\
&+ (-x-2)\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) \\
&- 2Cy_1(x)\ln(x) - 2\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}$$

Which can be written as

$$
\begin{aligned}
&\left(\left(x y_1''(x)+(-x-2)\, y_1'(x)-2 y_1(x)\right) \ln (x)+x\left(\frac{2 y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)\right. \\
&\left.+\frac{(-x-2)\, y_1(x)}{x}\right) C+x\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&+(-x-2)\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)-2\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)=0
\end{aligned} \tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$
x y_1''(x)+(-x-2)\, y_1'(x)-2 y_1(x)=0
$$

Eq (7) simplifes to

$$
\begin{aligned}
&\left(x\left(\frac{2 y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+\frac{(-x-2)\, y_1(x)}{x}\right) C \\
&+x\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&+(-x-2)\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)-2\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)=0
\end{aligned} \tag{8}
$$

Substituting $y_1=\sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$
\begin{aligned}
&\frac{\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n(n+r_1)\right) x-(x+3)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C}{x} \\
&+\frac{\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right) x^2+(-x^2-2x)\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right)-2\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)}{x} \\
&=0
\end{aligned} \tag{9}
$$

Since $r_1 = 3$ and $r_2 = 0$ then the above becomes

$$
\frac{\left(2\left(\sum_{n=0}^{\infty} x^{2+n} a_n (n+3)\right) x - (x+3)\left(\sum_{n=0}^{\infty} a_n x^{n+3}\right)\right) C}{x} + \frac{\left(\sum_{n=0}^{\infty} x^{-2+n} b_n n(n-1)\right) x^2 + (-x^2 - 2x)\left(\sum_{n=0}^{\infty} x^{n-1} b_n n\right) - 2\left(\sum_{n=0}^{\infty} b_n x^n\right) x}{x} = 0 \tag{10}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{2+n} a_n (n+3)\right) + \sum_{n=0}^{\infty} \left(-C\, x^{n+3} a_n\right) \\
&+ \sum_{n=0}^{\infty} \left(-3C\, x^{2+n} a_n\right) + \left(\sum_{n=0}^{\infty} n\, x^{n-1} b_n (n-1)\right) \\
&+ \sum_{n=0}^{\infty} \left(-x^n b_n n\right) + \sum_{n=0}^{\infty} \left(-2x^{n-1} b_n n\right) + \sum_{n=0}^{\infty} \left(-2 b_n x^n\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 2C\, x^{2+n} a_n (n+3) &= \sum_{n=3}^{\infty} 2C a_{n-3} n\, x^{n-1} \\
\sum_{n=0}^{\infty} \left(-C\, x^{n+3} a_n\right) &= \sum_{n=4}^{\infty} \left(-C a_{n-4} x^{n-1}\right) \\
\sum_{n=0}^{\infty} \left(-3C\, x^{2+n} a_n\right) &= \sum_{n=3}^{\infty} \left(-3C a_{n-3} x^{n-1}\right) \\
\sum_{n=0}^{\infty} \left(-x^n b_n n\right) &= \sum_{n=1}^{\infty} \left(-(n-1)\, b_{n-1} x^{n-1}\right) \\
\sum_{n=0}^{\infty} \left(-2 b_n x^n\right) &= \sum_{n=1}^{\infty} \left(-2 b_{n-1} x^{n-1}\right)
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\left(\sum_{n=3}^{\infty} 2Ca_{n-3}n\, x^{n-1}\right)+\sum_{n\,=4}^{\infty}\left(-Ca_{n-4}x^{n-1}\right)+\sum_{n\,=3}^{\infty}\left(-3Ca_{n-3}x^{n-1}\right)$$
$$+\left(\sum_{n=0}^{\infty} n\, x^{n-1}b_n(n-1)\right)+\sum_{n\,=1}^{\infty}\left(-(n-1)\, b_{n-1}x^{n-1}\right)$$
$$+\sum_{n\,=0}^{\infty}\left(-2x^{n-1}b_n n\right)+\sum_{n\,=1}^{\infty}\left(-2b_{n-1}x^{n-1}\right)=0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$-2b_1-2b_0=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-2b_1-2=0$$

Solving the above for b_1 gives

$$b_1=-1$$

For $n=2$, Eq (2B) gives

$$-2b_2-3b_1=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-2b_2+3=0$$

Solving the above for b_2 gives

$$b_2=\frac{3}{2}$$

For $n=N$, where $N=3$ which is the difference between the two roots, we are free to choose $b_3=0$. Hence for $n=3$, Eq (2B) gives

$$3C-6=0$$

Which is solved for C. Solving for C gives

$$C=2$$

For $n=4$, Eq (2B) gives

$$(-a_0+5a_1)\, C-5b_3+4b_4=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{21}{2} + 4b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{21}{8}$$

For $n = 5$, Eq (2B) gives

$$(-a_1 + 7a_2) C - 6b_4 + 10b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{95}{4} + 10b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{19}{8}$$

For $n = 6$, Eq (2B) gives

$$(-a_2 + 9a_3) C - 7b_5 + 18b_6 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{163}{8} + 18b_6 = 0$$

Solving the above for b_6 gives

$$b_6 = -\frac{163}{144}$$

For $n = 7$, Eq (2B) gives

$$(-a_3 + 11a_4) C - 8b_6 + 28b_7 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{371}{36} + 28b_7 = 0$$

Solving the above for b_7 gives

$$b_7 = -\frac{53}{144}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = 2$ and all b_n, then the second solution becomes

$$y_2(x) = 2\left(x^3\left(1+\frac{5x}{4}+\frac{3x^2}{4}+\frac{7x^3}{24}+\frac{x^4}{12}+\frac{3x^5}{160}+\frac{x^6}{288}+\frac{11x^7}{20160}+O(x^8)\right)\right)\ln(x) + 1 - x + \frac{3x^2}{2} - \frac{21x^4}{8} - \frac{19x^5}{8} - \frac{163x^6}{144} - \frac{53x^7}{144} + O(x^8)$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1x^3\left(1+\frac{5x}{4}+\frac{3x^2}{4}+\frac{7x^3}{24}+\frac{x^4}{12}+\frac{3x^5}{160}+\frac{x^6}{288}+\frac{11x^7}{20160}+O(x^8)\right) \\ &\quad + c_2\left(2\left(x^3\left(1+\frac{5x}{4}+\frac{3x^2}{4}+\frac{7x^3}{24}+\frac{x^4}{12}+\frac{3x^5}{160}+\frac{x^6}{288}+\frac{11x^7}{20160}+O(x^8)\right)\right)\ln(x) \right. \\ &\quad \left. + 1 - x + \frac{3x^2}{2} - \frac{21x^4}{8} - \frac{19x^5}{8} - \frac{163x^6}{144} - \frac{53x^7}{144} + O(x^8)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1x^3\left(1+\frac{5x}{4}+\frac{3x^2}{4}+\frac{7x^3}{24}+\frac{x^4}{12}+\frac{3x^5}{160}+\frac{x^6}{288}+\frac{11x^7}{20160}+O(x^8)\right) \\ &\quad + c_2\left(2x^3\left(1+\frac{5x}{4}+\frac{3x^2}{4}+\frac{7x^3}{24}+\frac{x^4}{12}+\frac{3x^5}{160}+\frac{x^6}{288}+\frac{11x^7}{20160}+O(x^8)\right)\ln(x) + 1 \right. \\ &\quad \left. - x + \frac{3x^2}{2} - \frac{21x^4}{8} - \frac{19x^5}{8} - \frac{163x^6}{144} - \frac{53x^7}{144} + O(x^8)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= c_1x^3\left(1+\frac{5x}{4}+\frac{3x^2}{4}+\frac{7x^3}{24}+\frac{x^4}{12}+\frac{3x^5}{160}+\frac{x^6}{288}+\frac{11x^7}{20160}+O(x^8)\right) \\ &\quad + c_2\left(2x^3\left(1+\frac{5x}{4}+\frac{3x^2}{4}+\frac{7x^3}{24}+\frac{x^4}{12}+\frac{3x^5}{160}+\frac{x^6}{288}+\frac{11x^7}{20160}+O(x^8)\right)\ln(x) + 1 \right. \\ &\quad \left. - x + \frac{3x^2}{2} - \frac{21x^4}{8} - \frac{19x^5}{8} - \frac{163x^6}{144} - \frac{53x^7}{144} + O(x^8)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1 x^3 \left(1+\frac{5x}{4}+\frac{3x^2}{4}+\frac{7x^3}{24}+\frac{x^4}{12}+\frac{3x^5}{160}+\frac{x^6}{288}+\frac{11x^7}{20160}+O(x^8)\right) + c_2\left(2x^3\left(1+\frac{5x}{4}+\frac{3x^2}{4}+\frac{7x^3}{24}+\frac{x^4}{12}+\frac{3x^5}{160}+\frac{x^6}{288}+\frac{11x^7}{20160}+O(x^8)\right)\ln(x)+1 - x+\frac{3x^2}{2}-\frac{21x^4}{8}-\frac{19x^5}{8}-\frac{163x^6}{144}-\frac{53x^7}{144}+O(x^8)\right)$$

Verified OK.

9.2.1 Maple step by step solution

Let's solve

$xy''+(-x-2)\,y'-2y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{2y}{x}+\frac{(x+2)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(x+2)y'}{x}-\frac{2y}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{x+2}{x}, P_3(x)=-\frac{2}{x}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=-2$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

$xy'' + (-x-2)\, y' - 2y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- ○ Convert $x \cdot y''$ to series expansion

$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

- ○ Shift index using $k->k+1$

$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0 r(-3+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} (a_{k+1}(k+1+r)\,(k-2+r) - a_k(k+r+2))\, x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-3+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{0, 3\}$

- Each term in the series must be 0, giving the recursion relation

$a_{k+1}(k+1+r)\,(k-2+r) - a_k(k+r+2) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = \frac{a_k(k+r+2)}{(k+1+r)(k-2+r)}$

- Recursion relation for $r = 0$

$a_{k+1} = \frac{a_k(k+2)}{(k+1)(k-2)}$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 2$

$$a_{k+1} = \frac{a_k(k+2)}{(k+1)(k-2)}$$

- Recursion relation for $r = 3$

$$a_{k+1} = \frac{a_k(k+5)}{(k+4)(k+1)}$$

- Solution for $r = 3$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = \frac{a_k(k+5)}{(k+4)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 70

```
Order:=8;
dsolve(x*diff(y(x),x$2)-(2+x)*diff(y(x),x)-2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^3 \left(1 + \frac{5}{4}x + \frac{3}{4}x^2 + \frac{7}{24}x^3 + \frac{1}{12}x^4 + \frac{3}{160}x^5 + \frac{1}{288}x^6 + \frac{11}{20160}x^7 + \mathrm{O}\left(x^8\right)\right)$$
$$+ c_2 \left(\ln(x)\left(24x^3 + 30x^4 + 18x^5 + 7x^6 + 2x^7 + \mathrm{O}\left(x^8\right)\right)\right.$$
$$\left. + \left(12 - 12x + 18x^2 + 26x^3 + x^4 - 9x^5 - 6x^6 - \frac{9}{4}x^7 + \mathrm{O}\left(x^8\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.103 (sec). Leaf size: 115

```
AsymptoticDSolveValue[x*y''[x]-(2+x)*y'[x]-2*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(\frac{1}{12}\left(7x^3+18x^2+30x+24\right)x^3\log(x) + \frac{1}{36}\left(-25x^6-45x^5-27x^4+54x^3+54x^2-36x+36\right)\right) + c_2\left(\frac{x^9}{288}+\frac{3x^8}{160}+\frac{x^7}{12}+\frac{7x^6}{24}+\frac{3x^5}{4}+\frac{5x^4}{4}+x^3\right)$$

9.3 problem 3

Internal problem ID [7005]
Internal file name [`OUTPUT/6248_Thursday_August_18_2022_07_11_41_AM_27159909/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. Miscellaneous Exercises. page 394
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + 2x^2y' - 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + 2x^2y' - 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 2$$
$$q(x) = -\frac{2}{x^2}$$

Table 196: Table $p(x), q(x)$ singularites.

$p(x) = 2$	
singularity	type

$q(x) = -\frac{2}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + 2x^2y' - 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + 2x^2 \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 2a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - 2a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 2x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - r - 2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - r - 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - r - 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = x^2 \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1} \right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 2a_{n-1}(n+r-1) - 2a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2a_{n-1}(n+r-1)}{n^2 + 2nr + r^2 - n - r - 2} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = -\frac{2a_{n-1}(1+n)}{n(n+3)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{2r}{r^2 + r - 2}$$

Which for the root $r = 2$ becomes

$$a_1 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2r}{r^2+r-2}$	-1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4 + 4r}{r^3 + 4r^2 + r - 6}$$

Which for the root $r = 2$ becomes

$$a_2 = \frac{3}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2r}{r^2+r-2}$	-1
a_2	$\frac{4+4r}{r^3+4r^2+r-6}$	$\frac{3}{5}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{8}{r^3 + 6r^2 + 5r - 12}$$

Which for the root $r = 2$ becomes

$$a_3 = -\frac{4}{15}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2r}{r^2+r-2}$	-1
a_2	$\frac{4+4r}{r^3+4r^2+r-6}$	$\frac{3}{5}$
a_3	$-\frac{8}{r^3+6r^2+5r-12}$	$-\frac{4}{15}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{r^4 + 10r^3 + 27r^2 + 2r - 40}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{2}{21}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2r}{r^2+r-2}$	-1
a_2	$\frac{4+4r}{r^3+4r^2+r-6}$	$\frac{3}{5}$
a_3	$-\frac{8}{r^3+6r^2+5r-12}$	$-\frac{4}{15}$
a_4	$\frac{16}{r^4+10r^3+27r^2+2r-40}$	$\frac{2}{21}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{32}{(r^2 + 9r + 18)(r^3 + 6r^2 + 3r - 10)}$$

Which for the root $r = 2$ becomes

$$a_5 = -\frac{1}{35}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2r}{r^2+r-2}$	-1
a_2	$\frac{4+4r}{r^3+4r^2+r-6}$	$\frac{3}{5}$
a_3	$-\frac{8}{r^3+6r^2+5r-12}$	$-\frac{4}{15}$
a_4	$\frac{16}{r^4+10r^3+27r^2+2r-40}$	$\frac{2}{21}$
a_5	$-\frac{32}{(r^2+9r+18)(r^3+6r^2+3r-10)}$	$-\frac{1}{35}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{64}{(r^2 + 11r + 28)(r^2 + r - 2)(r^2 + 9r + 18)}$$

Which for the root $r = 2$ becomes

$$a_6 = \frac{1}{135}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2r}{r^2+r-2}$	-1
a_2	$\frac{4+4r}{r^3+4r^2+r-6}$	$\frac{3}{5}$
a_3	$-\frac{8}{r^3+6r^2+5r-12}$	$-\frac{4}{15}$
a_4	$\frac{16}{r^4+10r^3+27r^2+2r-40}$	$\frac{2}{21}$
a_5	$-\frac{32}{(r^2+9r+18)(r^3+6r^2+3r-10)}$	$-\frac{1}{35}$
a_6	$\frac{64}{(r^2+11r+28)(r^2+r-2)(r^2+9r+18)}$	$\frac{1}{135}$

For $n = 7$, using the above recursive equation gives

$$a_7 = -\frac{128}{(8 + r)(5 + r)(r + 3)(2 + r)(-1 + r)(r + 7)(r + 4)}$$

Which for the root $r = 2$ becomes

$$a_7 = -\frac{8}{4725}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2r}{r^2+r-2}$	-1
a_2	$\frac{4+4r}{r^3+4r^2+r-6}$	$\frac{3}{5}$
a_3	$-\frac{8}{r^3+6r^2+5r-12}$	$-\frac{4}{15}$
a_4	$\frac{16}{r^4+10r^3+27r^2+2r-40}$	$\frac{2}{21}$
a_5	$-\frac{32}{(r^2+9r+18)(r^3+6r^2+3r-10)}$	$-\frac{1}{35}$
a_6	$\frac{64}{(r^2+11r+28)(r^2+r-2)(r^2+9r+18)}$	$\frac{1}{135}$
a_7	$-\frac{128}{(8+r)(5+r)(r+3)(2+r)(-1+r)(r+7)(r+4)}$	$-\frac{8}{4725}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}y_1(x) &= x^2\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \dots\right)\\ &= x^2\left(1 - x + \frac{3x^2}{5} - \frac{4x^3}{15} + \frac{2x^4}{21} - \frac{x^5}{35} + \frac{x^6}{135} - \frac{8x^7}{4725} + O(x^8)\right)\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}a_N &= a_3\\ &= -\frac{8}{r^3 + 6r^2 + 5r - 12}\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{r\to r_2} -\frac{8}{r^3 + 6r^2 + 5r - 12} &= \lim_{r\to -1} -\frac{8}{r^3 + 6r^2 + 5r - 12}\\ &= \frac{2}{3}\end{aligned}$$

The limit is $\frac{2}{3}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-1} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + 2b_{n-1}(n+r-1) - 2b_n = 0 \tag{4}$$

Which for for the root $r = -1$ becomes

$$b_n(n-1)(n-2) + 2b_{n-1}(n-2) - 2b_n = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{2b_{n-1}(n+r-1)}{n^2 + 2nr + r^2 - n - r - 2} \tag{5}$$

Which for the root $r = -1$ becomes

$$b_n = -\frac{2b_{n-1}(n-2)}{n^2 - 3n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{2r}{r^2 + r - 2}$$

Which for the root $r = -1$ becomes

$$b_1 = -1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2r}{r^2+r-2}$	-1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4 + 4r}{(r^2 + r - 2)(r + 3)}$$

Which for the root $r = -1$ becomes

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2r}{r^2+r-2}$	-1
b_2	$\frac{4+4r}{r^3+4r^2+r-6}$	0

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{8}{(r + 4)(r + 3)(-1 + r)}$$

Which for the root $r = -1$ becomes

$$b_3 = \frac{2}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2r}{r^2+r-2}$	-1
b_2	$\frac{4+4r}{r^3+4r^2+r-6}$	0
b_3	$-\frac{8}{(r+4)(r+3)(-1+r)}$	$\frac{2}{3}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16}{(r+4)(-1+r)(r^2+7r+10)}$$

Which for the root $r = -1$ becomes

$$b_4 = -\frac{2}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2r}{r^2+r-2}$	-1
b_2	$\frac{4+4r}{r^3+4r^2+r-6}$	0
b_3	$-\frac{8}{(r+4)(r+3)(-1+r)}$	$\frac{2}{3}$
b_4	$\frac{16}{(5+r)(2+r)(r+4)(-1+r)}$	$-\frac{2}{3}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{32}{(-1+r)(r^2+7r+10)(r^2+9r+18)}$$

Which for the root $r = -1$ becomes

$$b_5 = \frac{2}{5}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2r}{r^2+r-2}$	-1
b_2	$\frac{4+4r}{r^3+4r^2+r-6}$	0
b_3	$-\frac{8}{(r+4)(r+3)(-1+r)}$	$\frac{2}{3}$
b_4	$\frac{16}{(5+r)(2+r)(r+4)(-1+r)}$	$-\frac{2}{3}$
b_5	$-\frac{32}{(6+r)(r+3)(5+r)(-1+r)(2+r)}$	$\frac{2}{5}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{64}{(r^2 + 9r + 18)(2 + r)(-1 + r)(r^2 + 11r + 28)}$$

Which for the root $r = -1$ becomes

$$b_6 = -\frac{8}{45}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2r}{r^2+r-2}$	-1
b_2	$\frac{4+4r}{r^3+4r^2+r-6}$	0
b_3	$-\frac{8}{(r+4)(r+3)(-1+r)}$	$\frac{2}{3}$
b_4	$\frac{16}{(5+r)(2+r)(r+4)(-1+r)}$	$-\frac{2}{3}$
b_5	$-\frac{32}{(6+r)(r+3)(5+r)(-1+r)(2+r)}$	$\frac{2}{5}$
b_6	$\frac{64}{(r+7)(r+4)(2+r)(-1+r)(6+r)(r+3)}$	$-\frac{8}{45}$

For $n = 7$, using the above recursive equation gives

$$b_7 = -\frac{128}{(r^2 + 11r + 28)(-1 + r)(2 + r)(r + 3)(r^2 + 13r + 40)}$$

Which for the root $r = -1$ becomes

$$b_7 = \frac{4}{63}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2r}{r^2+r-2}$	-1
b_2	$\frac{4+4r}{r^3+4r^2+r-6}$	0
b_3	$-\frac{8}{(r+4)(r+3)(-1+r)}$	$\frac{2}{3}$
b_4	$\frac{16}{(5+r)(2+r)(r+4)(-1+r)}$	$-\frac{2}{3}$
b_5	$-\frac{32}{(6+r)(r+3)(5+r)(-1+r)(2+r)}$	$\frac{2}{5}$
b_6	$\frac{64}{(r+7)(r+4)(2+r)(-1+r)(6+r)(r+3)}$	$-\frac{8}{45}$
b_7	$-\frac{128}{(8+r)(5+r)(r+3)(2+r)(-1+r)(r+7)(r+4)}$	$\frac{4}{63}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^2\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots\right) \\
&= \frac{1 - x + \frac{2x^3}{3} - \frac{2x^4}{3} + \frac{2x^5}{5} - \frac{8x^6}{45} + \frac{4x^7}{63} + O(x^8)}{x}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^2 \left(1 - x + \frac{3x^2}{5} - \frac{4x^3}{15} + \frac{2x^4}{21} - \frac{x^5}{35} + \frac{x^6}{135} - \frac{8x^7}{4725} + O(x^8)\right) \\
&\quad + \frac{c_2\left(1 - x + \frac{2x^3}{3} - \frac{2x^4}{3} + \frac{2x^5}{5} - \frac{8x^6}{45} + \frac{4x^7}{63} + O(x^8)\right)}{x}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^2 \left(1 - x + \frac{3x^2}{5} - \frac{4x^3}{15} + \frac{2x^4}{21} - \frac{x^5}{35} + \frac{x^6}{135} - \frac{8x^7}{4725} + O(x^8)\right) \\
&\quad + \frac{c_2\left(1 - x + \frac{2x^3}{3} - \frac{2x^4}{3} + \frac{2x^5}{5} - \frac{8x^6}{45} + \frac{4x^7}{63} + O(x^8)\right)}{x}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 x^2 \left(1 - x + \frac{3x^2}{5} - \frac{4x^3}{15} + \frac{2x^4}{21} - \frac{x^5}{35} + \frac{x^6}{135} - \frac{8x^7}{4725} + O(x^8)\right) + \frac{c_2 \left(1 - x + \frac{2x^3}{3} - \frac{2x^4}{3} + \frac{2x^5}{5} - \frac{8x^6}{45} + \frac{4x^7}{63} + O(x^8)\right)}{x} \tag{1}
$$

Verification of solutions

$$
y = c_1 x^2 \left(1 - x + \frac{3x^2}{5} - \frac{4x^3}{15} + \frac{2x^4}{21} - \frac{x^5}{35} + \frac{x^6}{135} - \frac{8x^7}{4725} + O(x^8)\right) + \frac{c_2 \left(1 - x + \frac{2x^3}{3} - \frac{2x^4}{3} + \frac{2x^5}{5} - \frac{8x^6}{45} + \frac{4x^7}{63} + O(x^8)\right)}{x}
$$

Verified OK.

9.3.1 Maple step by step solution

Let's solve

$x^2 y'' + 2x^2 y' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -2y' + \frac{2y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 2y' - \frac{2y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = 2, P_3(x) = -\frac{2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -2$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + 2x^2y' - 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - ○ Convert $x^2 \cdot y'$ to series expansion

    $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r+1}$

  - ○ Shift index using $k- >k-1$

    $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$

  - ○ Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) + 2a_{k-1}(k-1+r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)\,(k+r-2) + 2a_{k-1}(k-1+r) = 0$

- Shift index using $k- >k+1$

  $a_{k+1}(k+2+r)\,(k-1+r) + 2a_k(k+r) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{2a_k(k+r)}{(k+2+r)(k-1+r)}$

- Recursion relation for $r = -1$; series terminates at $k = 1$

  $a_{k+1} = -\frac{2a_k(k-1)}{(k+1)(k-2)}$
- Apply recursion relation for $k = 0$

  $a_1 = -a_0$
- Terminating series solution of the ODE for $r = -1$. Use reduction of order to find the second

  $y = a_0 \cdot (1 - x)$
- Recursion relation for $r = 2$

  $a_{k+1} = -\frac{2a_k(k+2)}{(k+4)(k+1)}$
- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{2a_k(k+2)}{(k+4)(k+1)}\right]$
- Combine solutions and rename parameters

  $\left[y = a_0 \cdot (1 - x) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), b_{k+1} = -\frac{2b_k(k+2)}{(k+4)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 53

```
Order:=8;
dsolve(x^2*diff(y(x),x$2)+2*x^2*diff(y(x),x)-2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^2 \left(1 - x + \frac{3}{5}x^2 - \frac{4}{15}x^3 + \frac{2}{21}x^4 - \frac{1}{35}x^5 + \frac{1}{135}x^6 - \frac{8}{4725}x^7 + \mathrm{O}\left(x^8\right)\right) + \frac{c_2\left(12 - 12x + 8x^3 - 8x^4 + \frac{24}{5}x^5 - \frac{32}{15}x^6 + \frac{16}{21}x^7 + \mathrm{O}\left(x^8\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 87

```
AsymptoticDSolveValue[x^2*y''[x]+2*x^2*y'[x]-2*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(-\frac{8x^5}{45} + \frac{2x^4}{5} - \frac{2x^3}{3} + \frac{2x^2}{3} + \frac{1}{x} - 1\right) + c_2\left(\frac{x^8}{135} - \frac{x^7}{35} + \frac{2x^6}{21} - \frac{4x^5}{15} + \frac{3x^4}{5} - x^3 + x^2\right)$$

9.4 problem 4

Internal problem ID [7006]
Internal file name [`OUTPUT/6249_Thursday_August_18_2022_07_11_44_AM_35239012/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. Miscellaneous Exercises. page 394
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^{2}y''-x(2x+7)\,y'+2(x+5)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^{2}y''+\left(-2x^{2}-7x\right)y'+(2x+10)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=-\frac{2x+7}{2x}$$
$$q(x)=\frac{x+5}{x^{2}}$$

Table 198: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{2x+7}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x+5}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' + (-2x^2 - 7x)\, y' + (2x + 10)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} \right) + (-2x^2 - 7x) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \right) + (2x+10) \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-7x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} 10 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} 2x^{1+n+r} a_n = \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-7x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 10 a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$2x^{n+r} a_n (n+r)(n+r-1) - 7x^{n+r} a_n (n+r) + 10 a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$2x^r a_0 r(-1+r) - 7x^r a_0 r + 10 a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) - 7x^r r + 10x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 - 9r + 10\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 9r + 10 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{5}{2}$$
$$r_2 = 2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 - 9r + 10\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{5}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+2}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) - 2a_{n-1}(n+r-1) - 7a_n(n+r) + 2a_{n-1} + 10a_n = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2a_{n-1}}{2n+2r-5} \quad (4)$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_n = \frac{a_{n-1}}{n} \quad (5)$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{5}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{2}{-3 + 2r}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_1 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{-3+2r}$	1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4}{4r^2 - 8r + 3}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_2 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{-3+2r}$	1
a_2	$\frac{4}{4r^2-8r+3}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{8}{8r^3 - 12r^2 - 2r + 3}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_3 = \frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{-3+2r}$	1
a_2	$\frac{4}{4r^2-8r+3}$	$\frac{1}{2}$
a_3	$\frac{8}{8r^3-12r^2-2r+3}$	$\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{16r^4 - 40r^2 + 9}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_4 = \frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{-3+2r}$	1
a_2	$\frac{4}{4r^2-8r+3}$	$\frac{1}{2}$
a_3	$\frac{8}{8r^3-12r^2-2r+3}$	$\frac{1}{6}$
a_4	$\frac{16}{16r^4-40r^2+9}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{32}{(16r^4 - 40r^2 + 9)(5 + 2r)}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_5 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{-3+2r}$	1
a_2	$\frac{4}{4r^2-8r+3}$	$\frac{1}{2}$
a_3	$\frac{8}{8r^3-12r^2-2r+3}$	$\frac{1}{6}$
a_4	$\frac{16}{16r^4-40r^2+9}$	$\frac{1}{24}$
a_5	$\frac{32}{(16r^4-40r^2+9)(5+2r)}$	$\frac{1}{120}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{64}{(16r^4 - 40r^2 + 9)(5 + 2r)(7 + 2r)}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_6 = \frac{1}{720}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{-3+2r}$	1
a_2	$\frac{4}{4r^2-8r+3}$	$\frac{1}{2}$
a_3	$\frac{8}{8r^3-12r^2-2r+3}$	$\frac{1}{6}$
a_4	$\frac{16}{16r^4-40r^2+9}$	$\frac{1}{24}$
a_5	$\frac{32}{(16r^4-40r^2+9)(5+2r)}$	$\frac{1}{120}$
a_6	$\frac{64}{(16r^4-40r^2+9)(5+2r)(7+2r)}$	$\frac{1}{720}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{128}{(16r^4 - 40r^2 + 9)(5 + 2r)(7 + 2r)(9 + 2r)}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_7 = \frac{1}{5040}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{-3+2r}$	1
a_2	$\frac{4}{4r^2-8r+3}$	$\frac{1}{2}$
a_3	$\frac{8}{8r^3-12r^2-2r+3}$	$\frac{1}{6}$
a_4	$\frac{16}{16r^4-40r^2+9}$	$\frac{1}{24}$
a_5	$\frac{32}{(16r^4-40r^2+9)(5+2r)}$	$\frac{1}{120}$
a_6	$\frac{64}{(16r^4-40r^2+9)(5+2r)(7+2r)}$	$\frac{1}{720}$
a_7	$\frac{128}{(16r^4-40r^2+9)(5+2r)(7+2r)(9+2r)}$	$\frac{1}{5040}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^{\frac{5}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\
&= x^{\frac{5}{2}}\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
2b_n(n+r)(n+r-1) - 2b_{n-1}(n+r-1) - 7b_n(n+r) + 2b_{n-1} + 10b_n = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = \frac{2b_{n-1}}{2n+2r-5} \tag{4}
$$

Which for the root $r = 2$ becomes

$$
b_n = \frac{2b_{n-1}}{2n-1} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{2}{-3 + 2r}$$

Which for the root $r = 2$ becomes

$$b_1 = 2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{-3+2r}$	2

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4}{4r^2 - 8r + 3}$$

Which for the root $r = 2$ becomes

$$b_2 = \frac{4}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{-3+2r}$	2
b_2	$\frac{4}{4r^2-8r+3}$	$\frac{4}{3}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{8}{8r^3 - 12r^2 - 2r + 3}$$

Which for the root $r = 2$ becomes

$$b_3 = \frac{8}{15}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{-3+2r}$	2
b_2	$\frac{4}{4r^2-8r+3}$	$\frac{4}{3}$
b_3	$\frac{8}{8r^3-12r^2-2r+3}$	$\frac{8}{15}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16}{16r^4 - 40r^2 + 9}$$

Which for the root $r = 2$ becomes

$$b_4 = \frac{16}{105}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{-3+2r}$	2
b_2	$\frac{4}{4r^2-8r+3}$	$\frac{4}{3}$
b_3	$\frac{8}{8r^3-12r^2-2r+3}$	$\frac{8}{15}$
b_4	$\frac{16}{16r^4-40r^2+9}$	$\frac{16}{105}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{32}{(16r^4 - 40r^2 + 9)(5 + 2r)}$$

Which for the root $r = 2$ becomes

$$b_5 = \frac{32}{945}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{-3+2r}$	2
b_2	$\frac{4}{4r^2-8r+3}$	$\frac{4}{3}$
b_3	$\frac{8}{8r^3-12r^2-2r+3}$	$\frac{8}{15}$
b_4	$\frac{16}{16r^4-40r^2+9}$	$\frac{16}{105}$
b_5	$\frac{32}{(16r^4-40r^2+9)(5+2r)}$	$\frac{32}{945}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{64}{(16r^4 - 40r^2 + 9)(5 + 2r)(7 + 2r)}$$

Which for the root $r = 2$ becomes

$$b_6 = \frac{64}{10395}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{-3+2r}$	2
b_2	$\frac{4}{4r^2-8r+3}$	$\frac{4}{3}$
b_3	$\frac{8}{8r^3-12r^2-2r+3}$	$\frac{8}{15}$
b_4	$\frac{16}{16r^4-40r^2+9}$	$\frac{16}{105}$
b_5	$\frac{32}{(16r^4-40r^2+9)(5+2r)}$	$\frac{32}{945}$
b_6	$\frac{64}{(16r^4-40r^2+9)(5+2r)(7+2r)}$	$\frac{64}{10395}$

For $n = 7$, using the above recursive equation gives

$$b_7 = \frac{128}{(16r^4 - 40r^2 + 9)(5 + 2r)(7 + 2r)(9 + 2r)}$$

Which for the root $r = 2$ becomes

$$b_7 = \frac{128}{135135}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{-3+2r}$	2
b_2	$\frac{4}{4r^2-8r+3}$	$\frac{4}{3}$
b_3	$\frac{8}{8r^3-12r^2-2r+3}$	$\frac{8}{15}$
b_4	$\frac{16}{16r^4-40r^2+9}$	$\frac{16}{105}$
b_5	$\frac{32}{(16r^4-40r^2+9)(5+2r)}$	$\frac{32}{945}$
b_6	$\frac{64}{(16r^4-40r^2+9)(5+2r)(7+2r)}$	$\frac{64}{10395}$
b_7	$\frac{128}{(16r^4-40r^2+9)(5+2r)(7+2r)(9+2r)}$	$\frac{128}{135135}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= x^{\frac{5}{2}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots\right) \\
&= x^2\left(1 + 2x + \frac{4x^2}{3} + \frac{8x^3}{15} + \frac{16x^4}{105} + \frac{32x^5}{945} + \frac{64x^6}{10395} + \frac{128x^7}{135135} + O\left(x^8\right)\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{5}{2}}\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O\left(x^8\right)\right) \\
&\quad + c_2 x^2\left(1 + 2x + \frac{4x^2}{3} + \frac{8x^3}{15} + \frac{16x^4}{105} + \frac{32x^5}{945} + \frac{64x^6}{10395} + \frac{128x^7}{135135} + O\left(x^8\right)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{5}{2}}\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O\left(x^8\right)\right) \\
&\quad + c_2 x^2\left(1 + 2x + \frac{4x^2}{3} + \frac{8x^3}{15} + \frac{16x^4}{105} + \frac{32x^5}{945} + \frac{64x^6}{10395} + \frac{128x^7}{135135} + O\left(x^8\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= c_1 x^{\frac{5}{2}}\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O\left(x^8\right)\right) \\
&\quad + c_2 x^2\left(1 + 2x + \frac{4x^2}{3} + \frac{8x^3}{15} + \frac{16x^4}{105} + \frac{32x^5}{945} + \frac{64x^6}{10395} + \frac{128x^7}{135135} + O\left(x^8\right)\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}
y &= c_1 x^{\frac{5}{2}}\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O\left(x^8\right)\right) \\
&\quad + c_2 x^2\left(1 + 2x + \frac{4x^2}{3} + \frac{8x^3}{15} + \frac{16x^4}{105} + \frac{32x^5}{945} + \frac{64x^6}{10395} + \frac{128x^7}{135135} + O\left(x^8\right)\right)
\end{aligned}$$

Verified OK.

9.4.1 Maple step by step solution

Let's solve

$2x^2y'' + (-2x^2 - 7x)\, y' + (2x + 10)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x+5)y}{x^2} + \frac{(2x+7)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x+7)y'}{2x} + \frac{(x+5)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{2x+7}{2x}, P_3(x) = \frac{x+5}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -\frac{7}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 5$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2y'' - x(2x + 7)\, y' + (2x + 10)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- > k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-2+r)\,(-5+2r)\, x^r + \left(\sum_{k=1}^{\infty} (a_k(k+r-2)\,(2k+2r-5) - 2a_{k-1}(k+r-2))\, x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-2+r)\,(-5+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{2, \frac{5}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$2\left(\left(k+r-\frac{5}{2}\right) a_k - a_{k-1}\right)(k+r-2) = 0$$

- Shift index using $k- > k+1$

  $$2\left(\left(k-\frac{3}{2}+r\right) a_{k+1} - a_k\right)(k+r-1) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{2a_k}{2k-3+2r}$$

- Recursion relation for $r = 2$

  $$a_{k+1} = \frac{2a_k}{2k+1}$$

- Solution for $r = 2$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{2a_k}{2k+1}\right]$$

- Recursion relation for $r = \frac{5}{2}$

  $$a_{k+1} = \frac{2a_k}{2k+2}$$

- Solution for $r = \frac{5}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{2}}, a_{k+1} = \frac{2a_k}{2k+2}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{2}}\right), a_{k+1} = \frac{2a_k}{2k+1}, b_{k+1} = \frac{2b_k}{2k+2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 55

```
Order:=8;
dsolve(2*x^2*diff(y(x),x$2)-x*(2*x+7)*diff(y(x),x)+2*(x+5)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^2 \left(1 + 2x + \frac{4}{3}x^2 + \frac{8}{15}x^3 + \frac{16}{105}x^4 + \frac{32}{945}x^5 + \frac{64}{10395}x^6 + \frac{128}{135135}x^7 + \mathrm{O}\left(x^8\right)\right) + c_2 x^{\frac{5}{2}} \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 + \mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 110

```
AsymptoticDSolveValue[2*x^2*y''[x]-x*(2*x+7)*y'[x]+2*(x+5)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2 \left(\frac{128x^7}{135135} + \frac{64x^6}{10395} + \frac{32x^5}{945} + \frac{16x^4}{105} + \frac{8x^3}{15} + \frac{4x^2}{3} + 2x + 1\right) x^2 + c_1 \left(\frac{x^7}{5040} + \frac{x^6}{720} + \frac{x^5}{120} + \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + x + 1\right) x^{5/2}$$

9.5 problem 5

Internal problem ID [7007]
Internal file name [`OUTPUT/6250_Thursday_August_18_2022_07_11_46_AM_55982926/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. Miscellaneous Exercises. page 394
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x^2+1)\,y''+2x(x^2+3)\,y'+6y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^4+x^2)\,y''+(2x^3+6x)\,y'+6y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{2x^2+6}{x\,(x^2+1)}$$
$$q(x)=\frac{6}{x^2\,(x^2+1)}$$

Table 200: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2x^2+6}{x(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

$q(x) = \frac{6}{x^2(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -i, i, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x^2+1)\,y''+(2x^3+6x)\,y'+6y=0$$

Let the solution be represented as Frobenius power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y'=\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$
$$y''=\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(x^2+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + (2x^3+6x)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + 6\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) \\ + \left(\sum_{n=0}^{\infty} 2x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 6x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 6a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)(n+r-1) = \sum_{n=2}^{\infty} a_{n-2} (n+r-2)(n-3+r) x^{n+r}$$

$$\sum_{n=0}^{\infty} 2x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} 2a_{n-2} (n+r-2) x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} a_{n-2} (n+r-2)(n-3+r) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) \\ + \left(\sum_{n=2}^{\infty} 2a_{n-2} (n+r-2) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 6x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 6a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + 6x^{n+r} a_n (n+r) + 6a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + 6x^r a_0 r + 6a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + 6x^r r + 6x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 + 5r + 6\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 + 5r + 6 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= -2 \\ r_2 &= -3 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 + 5r + 6\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \frac{\sum_{n=0}^{\infty} a_n x^n}{x^2} \\ y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^3} \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n-2} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-3} \right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_{n-2}(n+r-2)(n-3+r)+a_n(n+r)(n+r-1) \\ +2a_{n-2}(n+r-2)+6a_n(n+r)+6a_n=0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n=-\frac{a_{n-2}(n^2+2nr+r^2-3n-3r+2)}{n^2+2nr+r^2+5n+5r+6} \tag{4}$$

Which for the root $r=-2$ becomes

$$a_n=-\frac{a_{n-2}(n-3)(n-4)}{n(n+1)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r=-2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n=2$, using the above recursive equation gives

$$a_2=-\frac{r(1+r)}{r^2+9r+20}$$

Which for the root $r=-2$ becomes

$$a_2=-\frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r(1+r)}{r^2+9r+20}$	$-\frac{1}{3}$

For $n=3$, using the above recursive equation gives

$$a_3=0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r(1+r)}{r^2+9r+20}$	$-\frac{1}{3}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r(1+r)(r+3)(r+2)}{(r+5)(r+4)(r+7)(r+6)}$$

Which for the root $r = -2$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r(1+r)}{r^2+9r+20}$	$-\frac{1}{3}$
a_3	0	0
a_4	$\frac{r(1+r)(r+3)(r+2)}{(r+5)(r+4)(r+7)(r+6)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r(1+r)}{r^2+9r+20}$	$-\frac{1}{3}$
a_3	0	0
a_4	$\frac{r(1+r)(r+3)(r+2)}{(r+5)(r+4)(r+7)(r+6)}$	0
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = -\frac{r(1+r)(r+3)(r+2)}{(r+9)(8+r)(r+6)(r+7)}$$

Which for the root $r = -2$ becomes

$$a_6 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r(1+r)}{r^2+9r+20}$	$-\frac{1}{3}$
a_3	0	0
a_4	$\frac{r(1+r)(r+3)(r+2)}{(r+5)(r+4)(r+7)(r+6)}$	0
a_5	0	0
a_6	$-\frac{r(1+r)(r+3)(r+2)}{(r+9)(8+r)(r+6)(r+7)}$	0

For $n = 7$, using the above recursive equation gives

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r(1+r)}{r^2+9r+20}$	$-\frac{1}{3}$
a_3	0	0
a_4	$\frac{r(1+r)(r+3)(r+2)}{(r+5)(r+4)(r+7)(r+6)}$	0
a_5	0	0
a_6	$-\frac{r(1+r)(r+3)(r+2)}{(r+9)(8+r)(r+6)(r+7)}$	0
a_7	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \frac{1}{x^2}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \dots\right) \\ &= \frac{1 - \frac{x^2}{3} + O(x^8)}{x^2} \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= 0 \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} 0 &= \lim_{r\to -3} 0 \\ &= 0 \end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-3} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &b_{n-2}(n + r - 2)(n - 3 + r) + b_n(n + r)(n + r - 1) \\ &+ 2b_{n-2}(n + r - 2) + 6b_n(n + r) + 6b_n = 0 \end{aligned} \tag{4}$$

Which for for the root $r = -3$ becomes

$$b_{n-2}(n-5)(n-6) + b_n(n-3)(n-4) + 2b_{n-2}(n-5) + 6b_n(n-3) + 6b_n = 0 \quad (4A)$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{b_{n-2}(n^2 + 2nr + r^2 - 3n - 3r + 2)}{n^2 + 2nr + r^2 + 5n + 5r + 6} \tag{5}$$

Which for the root $r = -3$ becomes

$$b_n = -\frac{b_{n-2}(n^2 - 9n + 20)}{n^2 - n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -3$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{r(1+r)}{r^2 + 9r + 20}$$

Which for the root $r = -3$ becomes

$$b_2 = -3$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r(1+r)}{r^2+9r+20}$	-3

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r(1+r)}{r^2+9r+20}$	-3
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r(1+r)\left(r^2+5r+6\right)}{\left(r^2+9r+20\right)\left(r^2+13r+42\right)}$$

Which for the root $r = -3$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r(1+r)}{r^2+9r+20}$	-3
b_3	0	0
b_4	$\frac{r(1+r)(r+3)(r+2)}{(r+5)(r+4)(r+7)(r+6)}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r(1+r)}{r^2+9r+20}$	-3
b_3	0	0
b_4	$\frac{r(1+r)(r+3)(r+2)}{(r+5)(r+4)(r+7)(r+6)}$	0
b_5	0	0

For $n = 6$, using the above recursive equation gives

$$b_6 = -\frac{r(1+r)(r^2+5r+6)}{(r^2+13r+42)(r^2+17r+72)}$$

Which for the root $r = -3$ becomes

$$b_6 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r(1+r)}{r^2+9r+20}$	-3
b_3	0	0
b_4	$\frac{r(1+r)(r+3)(r+2)}{(r+5)(r+4)(r+7)(r+6)}$	0
b_5	0	0
b_6	$-\frac{r(1+r)(r+3)(r+2)}{(r+9)(8+r)(r+6)(r+7)}$	0

For $n = 7$, using the above recursive equation gives

$$b_7 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r(1+r)}{r^2+9r+20}$	-3
b_3	0	0
b_4	$\frac{r(1+r)(r+3)(r+2)}{(r+5)(r+4)(r+7)(r+6)}$	0
b_5	0	0
b_6	$-\frac{r(1+r)(r+3)(r+2)}{(r+9)(8+r)(r+6)(r+7)}$	0
b_7	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= \frac{1}{x^2}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots\right) \\
&= \frac{1 - 3x^2 + O(x^8)}{x^3}
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= \frac{c_1\left(1 - \frac{x^2}{3} + O(x^8)\right)}{x^2} + \frac{c_2(1 - 3x^2 + O(x^8))}{x^3}
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= \frac{c_1\left(1 - \frac{x^2}{3} + O(x^8)\right)}{x^2} + \frac{c_2(1 - 3x^2 + O(x^8))}{x^3}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\left(1 - \frac{x^2}{3} + O(x^8)\right)}{x^2} + \frac{c_2(1 - 3x^2 + O(x^8))}{x^3} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\left(1 - \frac{x^2}{3} + O(x^8)\right)}{x^2} + \frac{c_2(1 - 3x^2 + O(x^8))}{x^3}$$

Verified OK.

9.5.1 Maple step by step solution

Let's solve

$x^2(x^2+1)\, y'' + (2x^3 + 6x)\, y' + 6y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$$y'' = -\frac{6y}{x^2(x^2+1)} - \frac{2(x^2+3)y'}{x(x^2+1)}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{2(x^2+3)y'}{x(x^2+1)} + \frac{6y}{x^2(x^2+1)} = 0$$

☐ Check to see if x_0 is a regular singular point

  - Define functions

$$\left[P_2(x) = \frac{2(x^2+3)}{x(x^2+1)}, P_3(x) = \frac{6}{x^2(x^2+1)}\right]$$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

$$(x \cdot P_2(x))\bigg|_{x=0} = 6$$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$(x^2 \cdot P_3(x))\bigg|_{x=0} = 6$$

  - $x = 0$is a regular singular point

Check to see if x_0 is a regular singular point

$$x_0 = 0$$

- Multiply by denominators

$$x^2(x^2+1)\,y'' + 2x(x^2+3)\,y' + 6y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

  - Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

  - Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

  - Shift index using $k- >k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(3+r)(2+r)\, x^r + a_1(4+r)(3+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+3)(k+r+2) + a_{k-2}(k-2+r)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(3+r)(2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-3, -2\}$
- Each term must be 0

  $a_1(4+r)(3+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+3)(k+r+2) + a_{k-2}(k-2+r)(k+r-1) = 0$
- Shift index using $k- >k+2$

  $a_{k+2}(k+5+r)(k+4+r) + a_k(k+r)(k+r+1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r)(k+r+1)}{(k+5+r)(k+4+r)}$
- Recursion relation for $r = -3$; series terminates at $k = 2$

  $a_{k+2} = -\frac{a_k(k-3)(k-2)}{(k+2)(k+1)}$
- Solution for $r = -3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = -\frac{a_k(k-3)(k-2)}{(k+2)(k+1)}, a_1 = 0\right]$
- Recursion relation for $r = -2$; series terminates at $k = 2$

  $a_{k+2} = -\frac{a_k(k-2)(k-1)}{(k+3)(k+2)}$
- Solution for $r = -2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{a_k(k-2)(k-1)}{(k+3)(k+2)}, a_1 = 0\right]$
- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-2}\right), a_{k+2} = -\frac{a_k (k-3)(k-2)}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{b_k (k-2)(k-1)}{(k+3)(k+2)}, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.031 (sec). Leaf size: 31

```
Order:=8;
dsolve(x^2*(1+x^2)*diff(y(x),x$2)+2*x*(3+x^2)*diff(y(x),x)+6*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 \left(1 - \frac{1}{3}x^2 + \mathrm{O}\left(x^8\right)\right) x + c_2 \left(1 - 3x^2 + \mathrm{O}\left(x^8\right)\right)}{x^3}$$

✓ Solution by Mathematica
Time used: 0.024 (sec). Leaf size: 26

```
AsymptoticDSolveValue[x^2*(1+x^2)*y''[x]+2*x*(3+x^2)*y'[x]+6*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \rightarrow c_1 \left(\frac{1}{x^3} - \frac{3}{x}\right) + c_2 \left(\frac{1}{x^2} - \frac{1}{3}\right)$$

9.6 problem 6

Internal problem ID [7008]
Internal file name [`OUTPUT/6251_Thursday_August_18_2022_07_11_48_AM_83478964/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. Miscellaneous Exercises. page 394
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[_Gegenbauer]
```

$$\boxed{\left(-x^2+1\right) y''-10 x y'-18 y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy'_0+\frac{x^2}{2}\,f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\,f'|_{x_0,y_0,y'_0}+\cdots \\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (249)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (250)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n \big|_{x_0, y_0, y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2(5xy'+9y)}{x^2-1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{92x^2y' + 216yx + 28y'}{(x^2-1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{-888x^3y' - 2304x^2y - 792xy' - 720y}{(x^2-1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(9240x^4 + 16128x^2 + 1512)\, y' + 25200xy\left(x^2 + \frac{23}{25}\right)}{(x^2-1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-104160x^5 - 297024x^3 - 82656x)\, y' - 292320\left(x^4 + \frac{262}{145}x^2 + \frac{5}{29}\right) y}{(x^2-1)^5} \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \frac{(1270080x^6 + 5334336x^4 + 2939328x^2 + 133056)\, y' + 3628800x\left(x^4 + \frac{74}{25}x^2 + \frac{21}{25}\right) y}{(x^2-1)^6} \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= \frac{(-16692480x^7 - 96526080x^5 - 87816960x^3 - 11854080x)\, y' - 48263040\left(x^6 + \frac{83}{19}x^4 + \frac{327}{133}x^2 + \frac{15}{133}\right) y}{(x^2-1)^7}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 18y(0) \\
F_1 &= 28y'(0) \\
F_2 &= 720y(0) \\
F_3 &= 1512y'(0) \\
F_4 &= 50400y(0) \\
F_5 &= 133056y'(0) \\
F_6 &= 5443200y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(135x^8 + 70x^6 + 30x^4 + 9x^2 + 1\right) y(0) + \left(x + \frac{14}{3}x^3 + \frac{63}{5}x^5 + \frac{132}{5}x^7\right) y'(0) + O\left(x^8\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
\left(-x^2 + 1\right) y'' - 10xy' - 18y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(-x^2 + 1\right) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - 10x \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 18\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}
$$

Which simplifies to

$$
\sum_{n=2}^{\infty} \left(-x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} \left(-10n a_n x^n\right) + \sum_{n=0}^{\infty} \left(-18 a_n x^n\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned}\sum_{n=2}^{\infty} (-x^n a_n n(n-1)) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) \\ + \sum_{n=1}^{\infty} (-10 n a_n x^n) + \sum_{n=0}^{\infty} (-18 a_n x^n) = 0\end{aligned} \tag{3}$$

$n = 0$ gives

$$2a_2 - 18a_0 = 0$$

$$a_2 = 9a_0$$

$n = 1$ gives

$$6a_3 - 28a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{14a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$-n a_n(n-1) + (n+2)\, a_{n+2}(n+1) - 10 n a_n - 18 a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_n(n^2 + 9n + 18)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$-40a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 30a_0$$

For $n = 3$ the recurrence equation gives

$$-54a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{63a_1}{5}$$

For $n = 4$ the recurrence equation gives

$$-70a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 70a_0$$

For $n = 5$ the recurrence equation gives

$$-88a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{132a_1}{5}$$

For $n = 6$ the recurrence equation gives

$$-108a_6 + 56a_8 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = 135a_0$$

For $n = 7$ the recurrence equation gives

$$-130a_7 + 72a_9 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{143a_1}{3}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x + 9a_0x^2 + \frac{14}{3}a_1x^3 + 30a_0x^4 + \frac{63}{5}a_1x^5 + 70a_0x^6 + \frac{132}{5}a_1x^7 + \dots$$

Collecting terms, the solution becomes

$$y = \left(70x^6 + 30x^4 + 9x^2 + 1\right)a_0 + \left(x + \frac{14}{3}x^3 + \frac{63}{5}x^5 + \frac{132}{5}x^7\right)a_1 + O\left(x^8\right) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(70x^6 + 30x^4 + 9x^2 + 1\right)c_1 + \left(x + \frac{14}{3}x^3 + \frac{63}{5}x^5 + \frac{132}{5}x^7\right)c_2 + O\left(x^8\right)$$

Summary

The solution(s) found are the following

$$y = \left(135x^8 + 70x^6 + 30x^4 + 9x^2 + 1\right)y(0) + \left(x + \frac{14}{3}x^3 + \frac{63}{5}x^5 + \frac{132}{5}x^7\right)y'(0) + O\left(x^8\right) \quad (1)$$

$$y = \left(70x^6 + 30x^4 + 9x^2 + 1\right)c_1 + \left(x + \frac{14}{3}x^3 + \frac{63}{5}x^5 + \frac{132}{5}x^7\right)c_2 + O\left(x^8\right) \quad (2)$$

Verification of solutions

$$y = (135x^8 + 70x^6 + 30x^4 + 9x^2 + 1)\, y(0) + \left(x + \frac{14}{3}x^3 + \frac{63}{5}x^5 + \frac{132}{5}x^7\right) y'(0) + O(x^8)$$

Verified OK.

$$y = (70x^6 + 30x^4 + 9x^2 + 1)\, c_1 + \left(x + \frac{14}{3}x^3 + \frac{63}{5}x^5 + \frac{132}{5}x^7\right) c_2 + O(x^8)$$

Verified OK.

9.6.1 Maple step by step solution

Let's solve

$(-x^2 + 1)\, y'' - 10xy' - 18y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{10xy'}{x^2-1} - \frac{18y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{10xy'}{x^2-1} + \frac{18y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{10x}{x^2-1}, P_3(x) = \frac{18}{x^2-1}\right]$

  ◦ $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

  $\left((1+x) \cdot P_2(x)\right)\Big|_{x=-1} = 5$

  ◦ $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $\left((1+x)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$

  ◦ $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

$y''(x^2-1)+10xy'+18y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(10u-10)\left(\frac{d}{du}y(u)\right)+18y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r(4+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)(k+5+r)+a_k(k+r+6)(k+r+3)\right)u^{k+r}\right)=$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-2r(4+r)=0$

- Values of r that satisfy the indicial equation

$r\in\{-4,0\}$

- Each term in the series must be 0, giving the recursion relation

$-2a_{k+1}(k+1+r)(k+5+r)+a_k(k+r+6)(k+r+3)=0$

- Recursion relation that defines series solution to ODE

$a_{k+1}=\frac{a_k(k+r+6)(k+r+3)}{2(k+1+r)(k+5+r)}$

- Recursion relation for $r=-4$; series terminates at $k=1$

$a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k-3)(k+1)}$

- Apply recursion relation for $k = 0$

  $a_1 = \frac{a_0}{3}$

- Terminating series solution of the ODE for $r = -4$. Use reduction of order to find the second

  $y(u) = a_0 \cdot \left(1 + \frac{u}{3}\right)$

- Revert the change of variables $u = 1 + x$

  $\left[y = a_0\left(\frac{4}{3} + \frac{x}{3}\right)\right]$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(k+6)(k+3)}{2(k+1)(k+5)}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(k+6)(k+3)}{2(k+1)(k+5)}\right]$

- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k(1 + x)^k, a_{k+1} = \frac{a_k(k+6)(k+3)}{2(k+1)(k+5)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0\left(\frac{4}{3} + \frac{x}{3}\right) + \left(\sum_{k=0}^{\infty} b_k(1 + x)^k\right), b_{k+1} = \frac{b_k(k+6)(k+3)}{2(k+1)(k+5)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
Order:=8;
dsolve((1-x^2)*diff(y(x),x$2)-10*x*diff(y(x),x)-18*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(70x^6 + 30x^4 + 9x^2 + 1\right) y(0) + \left(x + \frac{14}{3}x^3 + \frac{63}{5}x^5 + \frac{132}{5}x^7\right) D(y)(0) + O\left(x^8\right)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 50

```
AsymptoticDSolveValue[(1-x^2)*y''[x]-10*x*y'[x]-18*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(\frac{132x^7}{5} + \frac{63x^5}{5} + \frac{14x^3}{3} + x\right) + c_1\left(70x^6 + 30x^4 + 9x^2 + 1\right)$$

9.7 problem 7

Internal problem ID [7009]
Internal file name [`OUTPUT/6252_Thursday_August_18_2022_07_11_50_AM_88214269/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. Miscellaneous Exercises. page 394
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2xy'' + (2x+1)\,y' - 3y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2xy'' + (2x+1)\,y' - 3y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{2x+1}{2x}$$
$$q(x) = -\frac{3}{2x}$$

Table 203: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2x+1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{3}{2x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2xy'' + (2x + 1)\, y' - 3y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x\left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}\right) + (2x + 1)\left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) - 3\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-3a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r) = \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} \left(-3a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-3a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-3a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$2x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(2x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-1+2r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(-1+2r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) + 2a_{n-1}(n+r-1) + a_n(n+r) - 3a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(2n+2r-5)}{2n^2+4nr+2r^2-n-r} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{2a_{n-1}(n-2)}{2n^2+n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{3 - 2r}{2r^2 + 3r + 1}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = \frac{2}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3-2r}{2r^2+3r+1}$	$\frac{2}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4r^2 - 8r + 3}{4r^4 + 20r^3 + 35r^2 + 25r + 6}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3-2r}{2r^2+3r+1}$	$\frac{2}{3}$
a_2	$\frac{4r^2-8r+3}{4r^4+20r^3+35r^2+25r+6}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-4r^2 + 8r - 3}{4r^5 + 40r^4 + 155r^3 + 290r^2 + 261r + 90}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3-2r}{2r^2+3r+1}$	$\frac{2}{3}$
a_2	$\frac{4r^2-8r+3}{4r^4+20r^3+35r^2+25r+6}$	0
a_3	$\frac{-4r^2+8r-3}{4r^5+40r^4+155r^3+290r^2+261r+90}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4r^2 - 8r + 3}{4r^6 + 64r^5 + 415r^4 + 1390r^3 + 2521r^2 + 2326r + 840}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3-2r}{2r^2+3r+1}$	$\frac{2}{3}$
a_2	$\frac{4r^2-8r+3}{4r^4+20r^3+35r^2+25r+6}$	0
a_3	$\frac{-4r^2+8r-3}{4r^5+40r^4+155r^3+290r^2+261r+90}$	0
a_4	$\frac{4r^2-8r+3}{4r^6+64r^5+415r^4+1390r^3+2521r^2+2326r+840}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-4r^2 + 8r - 3}{4r^7 + 92r^6 + 883r^5 + 4565r^4 + 13651r^3 + 23423r^2 + 21102r + 7560}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3-2r}{2r^2+3r+1}$	$\frac{2}{3}$
a_2	$\frac{4r^2-8r+3}{4r^4+20r^3+35r^2+25r+6}$	0
a_3	$\frac{-4r^2+8r-3}{4r^5+40r^4+155r^3+290r^2+261r+90}$	0
a_4	$\frac{4r^2-8r+3}{4r^6+64r^5+415r^4+1390r^3+2521r^2+2326r+840}$	0
a_5	$\frac{-4r^2+8r-3}{4r^7+92r^6+883r^5+4565r^4+13651r^3+23423r^2+21102r+7560}$	0

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{4r^2 - 8r + 3}{4r^8 + 124r^7 + 1639r^6 + 12019r^5 + 53221r^4 + 144781r^3 + 234216r^2 + 203436r + 71280}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_6 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3-2r}{2r^2+3r+1}$	$\frac{2}{3}$
a_2	$\frac{4r^2-8r+3}{4r^4+20r^3+35r^2+25r+6}$	0
a_3	$\frac{-4r^2+8r-3}{4r^5+40r^4+155r^3+290r^2+261r+90}$	0
a_4	$\frac{4r^2-8r+3}{4r^6+64r^5+415r^4+1390r^3+2521r^2+2326r+840}$	0
a_5	$\frac{-4r^2+8r-3}{4r^7+92r^6+883r^5+4565r^4+13651r^3+23423r^2+21102r+7560}$	0
a_6	$\frac{4r^2-8r+3}{4r^8+124r^7+1639r^6+12019r^5+53221r^4+144781r^3+234216r^2+203436r+71280}$	0

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{-4r^2 + 8r - 3}{4r^9 + 160r^8 + 2775r^7 + 27300r^6 + 167202r^5 + 657720r^4 + 1650575r^3 + 2525300r^2 + 2110644r + 72}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3-2r}{2r^2+3r+1}$	$\frac{2}{3}$
a_2	$\frac{4r^2-8r+3}{4r^4+20r^3+35r^2+25r+6}$	0
a_3	$\frac{-4r^2+8r-3}{4r^5+40r^4+155r^3+290r^2+261r+90}$	0
a_4	$\frac{4r^2-8r+3}{4r^6+64r^5+415r^4+1390r^3+2521r^2+2326r+840}$	0
a_5	$\frac{-4r^2+8r-3}{4r^7+92r^6+883r^5+4565r^4+13651r^3+23423r^2+21102r+7560}$	0
a_6	$\frac{4r^2-8r+3}{4r^8+124r^7+1639r^6+12019r^5+53221r^4+144781r^3+234216r^2+203436r+71280}$	0
a_7	$\frac{-4r^2+8r-3}{4r^9+160r^8+2775r^7+27300r^6+167202r^5+657720r^4+1650575r^3+2525300r^2+2110644r+720720}$	0

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\
&= \sqrt{x}\left(1 + \frac{2x}{3} + O(x^8)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
2b_n(n+r)(n+r-1) + 2b_{n-1}(n+r-1) + (n+r)b_n - 3b_{n-1} = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = -\frac{b_{n-1}(2n+2r-5)}{2n^2+4nr+2r^2-n-r} \tag{4}
$$

Which for the root $r = 0$ becomes

$$
b_n = \frac{b_{n-1}(5-2n)}{2n^2-n} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{3 - 2r}{2r^2 + 3r + 1}$$

Which for the root $r = 0$ becomes

$$b_1 = 3$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{3-2r}{2r^2+3r+1}$	3

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4r^2 - 8r + 3}{4r^4 + 20r^3 + 35r^2 + 25r + 6}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{3-2r}{2r^2+3r+1}$	3
b_2	$\frac{4r^2-8r+3}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{-4r^2 + 8r - 3}{4r^5 + 40r^4 + 155r^3 + 290r^2 + 261r + 90}$$

Which for the root $r = 0$ becomes

$$b_3 = -\frac{1}{30}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{3-2r}{2r^2+3r+1}$	3
b_2	$\frac{4r^2-8r+3}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{2}$
b_3	$\frac{-4r^2+8r-3}{4r^5+40r^4+155r^3+290r^2+261r+90}$	$-\frac{1}{30}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{4r^2 - 8r + 3}{4r^6 + 64r^5 + 415r^4 + 1390r^3 + 2521r^2 + 2326r + 840}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{1}{280}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{3-2r}{2r^2+3r+1}$	3
b_2	$\frac{4r^2-8r+3}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{2}$
b_3	$\frac{-4r^2+8r-3}{4r^5+40r^4+155r^3+290r^2+261r+90}$	$-\frac{1}{30}$
b_4	$\frac{4r^2-8r+3}{4r^6+64r^5+415r^4+1390r^3+2521r^2+2326r+840}$	$\frac{1}{280}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{-4r^2 + 8r - 3}{4r^7 + 92r^6 + 883r^5 + 4565r^4 + 13651r^3 + 23423r^2 + 21102r + 7560}$$

Which for the root $r = 0$ becomes

$$b_5 = -\frac{1}{2520}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{3-2r}{2r^2+3r+1}$	3
b_2	$\frac{4r^2-8r+3}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{2}$
b_3	$\frac{-4r^2+8r-3}{4r^5+40r^4+155r^3+290r^2+261r+90}$	$-\frac{1}{30}$
b_4	$\frac{4r^2-8r+3}{4r^6+64r^5+415r^4+1390r^3+2521r^2+2326r+840}$	$\frac{1}{280}$
b_5	$\frac{-4r^2+8r-3}{4r^7+92r^6+883r^5+4565r^4+13651r^3+23423r^2+21102r+7560}$	$-\frac{1}{2520}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{4r^2 - 8r + 3}{4r^8 + 124r^7 + 1639r^6 + 12019r^5 + 53221r^4 + 144781r^3 + 234216r^2 + 203436r + 71280}$$

Which for the root $r = 0$ becomes

$$b_6 = \frac{1}{23760}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{3-2r}{2r^2+3r+1}$	3
b_2	$\frac{4r^2-8r+3}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{2}$
b_3	$\frac{-4r^2+8r-3}{4r^5+40r^4+155r^3+290r^2+261r+90}$	$-\frac{1}{30}$
b_4	$\frac{4r^2-8r+3}{4r^6+64r^5+415r^4+1390r^3+2521r^2+2326r+840}$	$\frac{1}{280}$
b_5	$\frac{-4r^2+8r-3}{4r^7+92r^6+883r^5+4565r^4+13651r^3+23423r^2+21102r+7560}$	$-\frac{1}{2520}$
b_6	$\frac{4r^2-8r+3}{4r^8+124r^7+1639r^6+12019r^5+53221r^4+144781r^3+234216r^2+203436r+71280}$	$\frac{1}{23760}$

For $n = 7$, using the above recursive equation gives

$$b_7 = \frac{-4r^2 + 8r - 3}{4r^9 + 160r^8 + 2775r^7 + 27300r^6 + 167202r^5 + 657720r^4 + 1650575r^3 + 2525300r^2 + 2110644r + 72}$$

Which for the root $r = 0$ becomes

$$b_7 = -\frac{1}{240240}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{3-2r}{2r^2+3r+1}$	3
b_2	$\frac{4r^2-8r+3}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{2}$
b_3	$\frac{-4r^2+8r-3}{4r^5+40r^4+155r^3+290r^2+261r+90}$	$-\frac{1}{30}$
b_4	$\frac{4r^2-8r+3}{4r^6+64r^5+415r^4+1390r^3+2521r^2+2326r+840}$	$\frac{1}{280}$
b_5	$\frac{-4r^2+8r-3}{4r^7+92r^6+883r^5+4565r^4+13651r^3+23423r^2+21102r+7560}$	$-\frac{1}{2520}$
b_6	$\frac{4r^2-8r+3}{4r^8+124r^7+1639r^6+12019r^5+53221r^4+144781r^3+234216r^2+203436r+71280}$	$\frac{1}{23760}$
b_7	$\frac{-4r^2+8r-3}{4r^9+160r^8+2775r^7+27300r^6+167202r^5+657720r^4+1650575r^3+2525300r^2+2110644r+720720}$	$-\frac{1}{240240}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\
&= 1 + 3x + \frac{x^2}{2} - \frac{x^3}{30} + \frac{x^4}{280} - \frac{x^5}{2520} + \frac{x^6}{23760} - \frac{x^7}{240240} + O(x^8)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \sqrt{x}\left(1 + \frac{2x}{3} + O(x^8)\right) \\
&\quad + c_2\left(1 + 3x + \frac{x^2}{2} - \frac{x^3}{30} + \frac{x^4}{280} - \frac{x^5}{2520} + \frac{x^6}{23760} - \frac{x^7}{240240} + O(x^8)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \sqrt{x}\left(1 + \frac{2x}{3} + O(x^8)\right) \\
&\quad + c_2\left(1 + 3x + \frac{x^2}{2} - \frac{x^3}{30} + \frac{x^4}{280} - \frac{x^5}{2520} + \frac{x^6}{23760} - \frac{x^7}{240240} + O(x^8)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y = c_1\sqrt{x}\left(1+\frac{2x}{3}+O(x^8)\right) \\
+ c_2\left(1+3x+\frac{x^2}{2}-\frac{x^3}{30}+\frac{x^4}{280}-\frac{x^5}{2520}+\frac{x^6}{23760}-\frac{x^7}{240240}+O(x^8)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1\sqrt{x}\left(1+\frac{2x}{3}+O(x^8)\right) \\
+ c_2\left(1+3x+\frac{x^2}{2}-\frac{x^3}{30}+\frac{x^4}{280}-\frac{x^5}{2520}+\frac{x^6}{23760}-\frac{x^7}{240240}+O(x^8)\right)
\end{aligned}
$$

Verified OK.

9.7.1 Maple step by step solution

Let's solve

$2xy'' + (2x+1)\,y' - 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(2x+1)y'}{2x} + \frac{3y}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2x+1)y'}{2x} - \frac{3y}{2x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2x+1}{2x}, P_3(x) = -\frac{3}{2x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{1}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  - $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$2xy'' + (2x+1)\,y' - 3y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  ◦ Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

  ◦ Shift index using $k->k+1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0 r(-1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k+1+2r) + a_k(2k+2r-3)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-1+2r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{0, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

$2(k+1+r)\left(k+\frac{1}{2}+r\right) a_{k+1} + 2\left(k+r-\frac{3}{2}\right) a_k = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = -\frac{(2k+2r-3)a_k}{(k+1+r)(2k+1+2r)}$

- Recursion relation for $r = 0$

$a_{k+1} = -\frac{(2k-3)a_k}{(k+1)(2k+1)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{(2k-3)a_k}{(k+1)(2k+1)}\right]$

- Recursion relation for $r = \frac{1}{2}$; series terminates at $k = 1$

  $a_{k+1} = -\frac{(2k-2)a_k}{\left(k+\frac{3}{2}\right)(2k+2)}$

- Apply recursion relation for $k = 0$

  $a_1 = \frac{2a_0}{3}$

- Terminating series solution of the ODE for $r = \frac{1}{2}$. Use reduction of order to find the second li

  $y = a_0 \cdot \left(1 + \frac{2x}{3}\right)$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + b_0 \cdot \left(1 + \frac{2x}{3}\right), a_{k+1} = -\frac{(2k-3)a_k}{(k+1)(2k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 40

```
Order:=8;
dsolve(2*x*diff(y(x),x$2)+(1+2*x)*diff(y(x),x)-3*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\sqrt{x}\left(1+\frac{2}{3}x+\mathrm{O}\left(x^8\right)\right) + c_2\left(1+3x+\frac{1}{2}x^2-\frac{1}{30}x^3+\frac{1}{280}x^4-\frac{1}{2520}x^5+\frac{1}{23760}x^6-\frac{1}{240240}x^7+\mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 69

```
AsymptoticDSolveValue[2*x*y''[x]+(1+2*x)*y'[x]-3*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(-\frac{x^7}{240240}+\frac{x^6}{23760}-\frac{x^5}{2520}+\frac{x^4}{280}-\frac{x^3}{30}+\frac{x^2}{2}+3x+1\right)+c_1\left(\frac{2x}{3}+1\right)\sqrt{x}$$

9.8 problem 8

Internal problem ID [7010]
Internal file name [`OUTPUT/6253_Thursday_August_18_2022_07_11_52_AM_64669449/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. Miscellaneous Exercises. page 394
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

`[_erf]`

$$\boxed{y'' + 2xy' - 8y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (253)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (254)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -2xy' + 8y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= 4x^2 y' - 16xy + 6y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= 4\left(-2x^3 - 5x\right) y' + 32y\left(x^2 + 1\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(16x^4 + 48x^2 + 12\right) y' + \left(-64x^3 - 96x\right) y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -32\left(\left(x^4 + 3x^2 + \frac{3}{4}\right) y' + \left(-4x^3 - 6x\right) y\right) x \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= 64\left(x^2 - \frac{1}{2}\right)\left(\left(x^4 + 3x^2 + \frac{3}{4}\right) y' + \left(-4x^3 - 6x\right) y\right) \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= -128\left(\left(x^4 + 3x^2 + \frac{3}{4}\right) y' + \left(-4x^3 - 6x\right) y\right) x\left(x^2 - \frac{3}{2}\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 8y(0) \\
F_1 &= 6y'(0) \\
F_2 &= 32y(0) \\
F_3 &= 12y'(0) \\
F_4 &= 0 \\
F_5 &= -24y'(0) \\
F_6 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + 4x^2 + \frac{4}{3}x^4\right) y(0) + \left(x + x^3 + \frac{1}{10}x^5 - \frac{1}{210}x^7\right) y'(0) + O(x^8)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -2x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 8\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 2n\, x^n a_n\right) + \sum_{n\,=0}^{\infty} (-8 a_n x^n) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} 2n\, x^n a_n\right) + \sum_{n=0}^{\infty} (-8a_n x^n) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - 8a_0 = 0$$

$$a_2 = 4a_0$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + 2na_n - 8a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{2a_n(n-4)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - 6a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = a_1$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 4a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{4a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{10}$$

For $n = 4$ the recurrence equation gives

$$30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 2a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{210}$$

For $n = 6$ the recurrence equation gives

$$56a_8 + 4a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = 0$$

For $n = 7$ the recurrence equation gives

$$72a_9 + 6a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{a_1}{2520}$$

And so on. Therefore the solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
y = a_0 + a_1 x + 4a_0 x^2 + a_1 x^3 + \frac{4}{3} a_0 x^4 + \frac{1}{10} a_1 x^5 - \frac{1}{210} a_1 x^7 + \dots
$$

Collecting terms, the solution becomes

$$
y = \left(1 + 4x^2 + \frac{4}{3} x^4\right) a_0 + \left(x + x^3 + \frac{1}{10} x^5 - \frac{1}{210} x^7\right) a_1 + O(x^8) \tag{3}
$$

At $x = 0$ the solution above becomes

$$
y = \left(1 + 4x^2 + \frac{4}{3} x^4\right) c_1 + \left(x + x^3 + \frac{1}{10} x^5 - \frac{1}{210} x^7\right) c_2 + O(x^8)
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \left(1 + 4x^2 + \frac{4}{3} x^4\right) y(0) + \left(x + x^3 + \frac{1}{10} x^5 - \frac{1}{210} x^7\right) y'(0) + O(x^8) \tag{1}
$$

$$
y = \left(1 + 4x^2 + \frac{4}{3} x^4\right) c_1 + \left(x + x^3 + \frac{1}{10} x^5 - \frac{1}{210} x^7\right) c_2 + O(x^8) \tag{2}
$$

<u>Verification of solutions</u>

$$
y = \left(1 + 4x^2 + \frac{4}{3} x^4\right) y(0) + \left(x + x^3 + \frac{1}{10} x^5 - \frac{1}{210} x^7\right) y'(0) + O(x^8)
$$

Verified OK.

$$
y = \left(1 + 4x^2 + \frac{4}{3} x^4\right) c_1 + \left(x + x^3 + \frac{1}{10} x^5 - \frac{1}{210} x^7\right) c_2 + O(x^8)
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
Order:=8;
dsolve(diff(y(x),x$2)+2*x*diff(y(x),x)-8*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(\frac{4}{3}x^4 + 4x^2 + 1\right) y(0) + \left(x + x^3 + \frac{1}{10}x^5 - \frac{1}{210}x^7\right) D(y)(0) + O(x^8)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 43

```
AsymptoticDSolveValue[y''[x]+2*x*y'[x]-8*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(\frac{4x^4}{3} + 4x^2 + 1\right) + c_2\left(-\frac{x^7}{210} + \frac{x^5}{10} + x^3 + x\right)$$

9.9 problem 9

Internal problem ID [7011]
Internal file name [`OUTPUT/6254_Thursday_August_18_2022_07_11_54_AM_79930221/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. Miscellaneous Exercises. page 394
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

`[[_2nd_order, _exact, _linear, _homogeneous]]`

$$\boxed{x(-x^2+1)\,y''-(x^2+7)\,y'+4yx=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(-x^3+x)\,y''+(-x^2-7)\,y'+4yx=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{x^2+7}{x\,(x^2-1)}$$
$$q(x)=-\frac{4}{x^2-1}$$

Table 205: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x^2+7}{x(x^2-1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"
$x = 1$	"regular"

$q(x) = -\frac{4}{x^2-1}$	
singularity	type
$x = -1$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, 1, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''x(x^2 - 1) + (-x^2 - 7) y' + 4yx = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} & -\left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) x(x^2 - 1) \\ & + (-x^2 - 7) \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + 4\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) x = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(-x^{1+n+r} a_n(n+r)(n+r-1)\right) \\
&+\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n(n+r)(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-x^{1+n+r} a_n(n+r)\right) \\
&+\sum_{n=0}^{\infty}\left(-7(n+r) a_n x^{n+r-1}\right)+\left(\sum_{n=0}^{\infty} 4 x^{1+n+r} a_n\right)=0
\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\begin{aligned}
\sum_{n=0}^{\infty}\left(-x^{1+n+r} a_n(n+r)(n+r-1)\right) &= \sum_{n=2}^{\infty}\left(-a_{n-2}(n+r-2)(n-3+r) x^{n+r-1}\right) \\
\sum_{n=0}^{\infty}\left(-x^{1+n+r} a_n(n+r)\right) &= \sum_{n=2}^{\infty}\left(-a_{n-2}(n+r-2) x^{n+r-1}\right) \\
\sum_{n=0}^{\infty} 4 x^{1+n+r} a_n &= \sum_{n=2}^{\infty} 4 a_{n-2} x^{n+r-1}
\end{aligned}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left(-a_{n-2}(n+r-2)(n-3+r) x^{n+r-1}\right) \\
&+\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n(n+r)(n+r-1)\right)+\sum_{n=2}^{\infty}\left(-a_{n-2}(n+r-2) x^{n+r-1}\right) \\
&+\sum_{n=0}^{\infty}\left(-7(n+r) a_n x^{n+r-1}\right)+\left(\sum_{n=2}^{\infty} 4 a_{n-2} x^{n+r-1}\right)=0
\end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n(n+r)(n+r-1)-7(n+r) a_n x^{n+r-1}=0$$

When $n = 0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) - 7r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) - 7r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-8+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-8+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 8$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(-8+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 8$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^8 \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+8}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} -a_{n-2}(n+r-2)(n-3+r) + a_n(n+r)(n+r-1) \\ - a_{n-2}(n+r-2) - 7a_n(n+r) + 4a_{n-2} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{(n+r-4)\, a_{n-2}}{n-8+r} \tag{4}$$

Which for the root $r = 8$ becomes

$$a_n = \frac{(n+4)\, a_{n-2}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 8$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-2+r}{-6+r}$$

Which for the root $r = 8$ becomes

$$a_2 = 3$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2+r}{-6+r}$	3

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2+r}{-6+r}$	3
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(-2+r)\,r}{(-6+r)\,(r-4)}$$

Which for the root $r = 8$ becomes

$$a_4 = 6$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2+r}{-6+r}$	3
a_3	0	0
a_4	$\frac{(-2+r)r}{(-6+r)(r-4)}$	6

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2+r}{-6+r}$	3
a_3	0	0
a_4	$\frac{(-2+r)r}{(-6+r)(r-4)}$	6
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{r(2+r)}{(-6+r)(r-4)}$$

Which for the root $r = 8$ becomes

$$a_6 = 10$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2+r}{-6+r}$	3
a_3	0	0
a_4	$\frac{(-2+r)r}{(-6+r)(r-4)}$	6
a_5	0	0
a_6	$\frac{r(2+r)}{(-6+r)(r-4)}$	10

For $n = 7$, using the above recursive equation gives

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2+r}{-6+r}$	3
a_3	0	0
a_4	$\frac{(-2+r)r}{(-6+r)(r-4)}$	6
a_5	0	0
a_6	$\frac{r(2+r)}{(-6+r)(r-4)}$	10
a_7	0	0

For $n = 8$, using the above recursive equation gives

$$a_8 = \frac{(2+r)(4+r)}{(-6+r)(r-4)}$$

Which for the root $r = 8$ becomes

$$a_8 = 15$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2+r}{-6+r}$	3
a_3	0	0
a_4	$\frac{(-2+r)r}{(-6+r)(r-4)}$	6
a_5	0	0
a_6	$\frac{r(2+r)}{(-6+r)(r-4)}$	10
a_7	0	0
a_8	$\frac{(2+r)(4+r)}{(-6+r)(r-4)}$	15

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^8\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 + a_9 x^9 \dots\right) \\ &= x^8\left(1 + 3x^2 + 6x^4 + 10x^6 + 15x^8 + O(x^9)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 8$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\rightarrow r_2} a_8(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_8 \\ &= \frac{(2+r)(4+r)}{(-6+r)(r-4)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\rightarrow r_2} \frac{(2+r)(4+r)}{(-6+r)(r-4)} &= \lim_{r\rightarrow 0} \frac{(2+r)(4+r)}{(-6+r)(r-4)} \\ &= \frac{1}{3} \end{aligned}$$

The limit is $\frac{1}{3}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^n \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} -b_{n-2}(n+r-2)(n-3+r) + b_n(n+r)(n+r-1) \\ - b_{n-2}(n+r-2) - 7(n+r) b_n + 4b_{n-2} = 0 \end{aligned} \tag{4}$$

Which for for the root $r = 0$ becomes

$$-b_{n-2}(n-2)(n-3) + b_n n(n-1) - b_{n-2}(n-2) - 7nb_n + 4b_{n-2} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = \frac{(n + r - 4)\, b_{n-2}}{n - 8 + r} \tag{5}$$

Which for the root $r = 0$ becomes

$$b_n = \frac{(n - 4)\, b_{n-2}}{n - 8} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{-2 + r}{-6 + r}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-2+r}{-6+r}$	$\frac{1}{3}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-2+r}{-6+r}$	$\frac{1}{3}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{(-2+r)\, r}{(-6+r)\,(r-4)}$$

Which for the root $r = 0$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-2+r}{-6+r}$	$\frac{1}{3}$
b_3	0	0
b_4	$\frac{(-2+r)r}{(-6+r)(r-4)}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-2+r}{-6+r}$	$\frac{1}{3}$
b_3	0	0
b_4	$\frac{(-2+r)r}{(-6+r)(r-4)}$	0
b_5	0	0

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{r(2+r)}{(-6+r)\,(r-4)}$$

Which for the root $r = 0$ becomes

$$b_6 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-2+r}{-6+r}$	$\frac{1}{3}$
b_3	0	0
b_4	$\frac{(-2+r)r}{(-6+r)(r-4)}$	0
b_5	0	0
b_6	$\frac{r(2+r)}{(-6+r)(r-4)}$	0

For $n = 7$, using the above recursive equation gives

$$b_7 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-2+r}{-6+r}$	$\frac{1}{3}$
b_3	0	0
b_4	$\frac{(-2+r)r}{(-6+r)(r-4)}$	0
b_5	0	0
b_6	$\frac{r(2+r)}{(-6+r)(r-4)}$	0
b_7	0	0

For $n = 8$, using the above recursive equation gives

$$b_8 = \frac{(2+r)(4+r)}{(-6+r)(r-4)}$$

Which for the root $r = 0$ becomes

$$b_8 = \frac{1}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-2+r}{-6+r}$	$\frac{1}{3}$
b_3	0	0
b_4	$\frac{(-2+r)r}{(-6+r)(r-4)}$	0
b_5	0	0
b_6	$\frac{r(2+r)}{(-6+r)(r-4)}$	0
b_7	0	0
b_8	$\frac{(2+r)(4+r)}{(-6+r)(r-4)}$	$\frac{1}{3}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 + b_9 x^9 \dots \\ &= 1 + \frac{x^2}{3} + \frac{x^8}{3} + O(x^9)\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^8 (1 + 3x^2 + 6x^4 + 10x^6 + 15x^8 + O(x^9)) + c_2 \left(1 + \frac{x^2}{3} + \frac{x^8}{3} + O(x^9)\right)\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y &= y_h \\ &= c_1 x^8 (1 + 3x^2 + 6x^4 + 10x^6 + 15x^8 + O(x^9)) + c_2 \left(1 + \frac{x^2}{3} + \frac{x^8}{3} + O(x^9)\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^8 (1 + 3x^2 + 6x^4 + 10x^6 + 15x^8 + O(x^9)) + c_2 \left(1 + \frac{x^2}{3} + \frac{x^8}{3} + O(x^9)\right) \quad (1)$$

Verification of solutions

$$y = c_1 x^8 (1 + 3x^2 + 6x^4 + 10x^6 + 15x^8 + O(x^9)) + c_2 \left(1 + \frac{x^2}{3} + \frac{x^8}{3} + O(x^9)\right)$$

Verified OK.

9.9.1 Maple step by step solution

Let's solve

$-y''x(x^2-1)+(-x^2-7)\,y'+4yx=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{4y}{x^2-1}-\frac{(x^2+7)y'}{x(x^2-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(x^2+7)y'}{x(x^2-1)}-\frac{4y}{x^2-1}=0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{x^2+7}{x(x^2-1)},P_3(x)=-\frac{4}{x^2-1}\right]$

  - $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((1+x)\cdot P_2(x)\right)\bigg|_{x=-1}=4$

  - $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((1+x)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$

  - $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $y''x(x^2-1)+(x^2+7)\,y'-4yx=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^3-3u^2+2u)\left(\frac{d^2}{du^2}y(u)\right)+(u^2-2u+8)\left(\frac{d}{du}y(u)\right)+(-4u+4)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k->k-m$

  $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$2a_0 r(3+r)\, u^{-1+r} + (2a_1(1+r)\,(4+r) - a_0(1+r)\,(-4+3r))\, u^r + \left(\sum_{k=1}^{\infty} (2a_{k+1}(k+r+1)\,(k+\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$2r(3+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-3, 0\}$$

- Each term must be 0

  $$2a_1(1+r)\,(4+r) - a_0(1+r)\,(-4+3r) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$-3\left(\left(a_k - \frac{a_{k-1}}{3} - \frac{2a_{k+1}}{3}\right) k + \left(a_k - \frac{a_{k-1}}{3} - \frac{2a_{k+1}}{3}\right) r - \frac{4a_k}{3} + a_{k-1} - \frac{8a_{k+1}}{3}\right)(k+r+1) = 0$$

- Shift index using $k->k+1$

  $$-3\left(\left(a_{k+1} - \frac{a_k}{3} - \frac{2a_{k+2}}{3}\right)(k+1) + \left(a_{k+1} - \frac{a_k}{3} - \frac{2a_{k+2}}{3}\right) r - \frac{4a_{k+1}}{3} + a_k - \frac{8a_{k+2}}{3}\right)(k+r+2) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k a_k - 3k a_{k+1} + r a_k - 3r a_{k+1} - 2a_k + a_{k+1}}{2(k+5+r)}$$

- Recursion relation for $r = -3$

$$a_{k+2} = -\frac{k a_k - 3k a_{k+1} - 5a_k + 10a_{k+1}}{2(k+2)}$$

- Solution for $r = -3$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-3}, a_{k+2} = -\frac{k a_k - 3k a_{k+1} - 5a_k + 10a_{k+1}}{2(k+2)}, -4a_1 - 26a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k-3}, a_{k+2} = -\frac{k a_k - 3k a_{k+1} - 5a_k + 10a_{k+1}}{2(k+2)}, -4a_1 - 26a_0 = 0\right]$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k a_k - 3k a_{k+1} - 2a_k + a_{k+1}}{2(k+5)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k}, a_{k+2} = -\frac{k a_k - 3k a_{k+1} - 2a_k + a_{k+1}}{2(k+5)}, 8a_1 + 4a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k}, a_{k+2} = -\frac{k a_k - 3k a_{k+1} - 2a_k + a_{k+1}}{2(k+5)}, 8a_1 + 4a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (1+x)^{k-3}\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k}\right), a_{k+2} = -\frac{k a_k - 3k a_{k+1} - 5a_k + 10a_{k+1}}{2(k+2)}, -4a_1 - 26a_0 = 0,\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 32

```
Order:=8;
dsolve(x*(1-x^2)*diff(y(x),x$2)-(7+x^2)*diff(y(x),x)+4*x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1x^8\left(1 + 3x^2 + 6x^4 + 10x^6 + \mathrm{O}\left(x^8\right)\right) + c_2\left(-203212800 - 67737600x^2 + \mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 38

```
AsymptoticDSolveValue[x*(1-x^2)*y''[x]-(7+x^2)*y'[x]+4*x*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(\frac{x^2}{3} + 1\right) + c_2\left(10x^{14} + 6x^{12} + 3x^{10} + x^8\right)$$

9.10 problem 10

Internal problem ID [7012]
Internal file name [`OUTPUT/6255_Thursday_August_18_2022_07_11_56_AM_92562973/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. Miscellaneous Exercises. page 394
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2y'' - x(2x+1)\,y' + (1+4x)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' + \left(-2x^2 - x\right)y' + (1+4x)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{2x+1}{2x}$$
$$q(x) = \frac{1+4x}{2x^2}$$

Table 207: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{2x+1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1+4x}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' + (-2x^2 - x)\, y' + (1 + 4x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2 \left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}\right) + (-2x^2 - x) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (1 + 4x) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} 4x^{1+n+r} a_n = \sum_{n=1}^{\infty} 4a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 4a_{n-1} x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 3r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = \frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) - 2a_{n-1}(n+r-1) - a_n(n+r) + a_n + 4a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2a_{n-1}(n+r-3)}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{2a_{n-1}(n-2)}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-4 + 2r}{r(2r + 1)}$$

Which for the root $r = 1$ becomes

$$a_1 = -\frac{2}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4+2r}{r(2r+1)}$	$-\frac{2}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4(-2 + r)(-1 + r)}{4r^4 + 12r^3 + 11r^2 + 3r}$$

Which for the root $r = 1$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4+2r}{r(2r+1)}$	$-\frac{2}{3}$
a_2	$\frac{4(-2+r)(-1+r)}{4r^4+12r^3+11r^2+3r}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{8(-2 + r)(-1 + r)}{8r^5 + 60r^4 + 170r^3 + 225r^2 + 137r + 30}$$

Which for the root $r = 1$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4+2r}{r(2r+1)}$	$-\frac{2}{3}$
a_2	$\frac{4(-2+r)(-1+r)}{4r^4+12r^3+11r^2+3r}$	0
a_3	$\frac{8(-2+r)(-1+r)}{8r^5+60r^4+170r^3+225r^2+137r+30}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16(-2+r)(-1+r)}{16r^6 + 208r^5 + 1080r^4 + 2840r^3 + 3929r^2 + 2637r + 630}$$

Which for the root $r = 1$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4+2r}{r(2r+1)}$	$-\frac{2}{3}$
a_2	$\frac{4(-2+r)(-1+r)}{4r^4+12r^3+11r^2+3r}$	0
a_3	$\frac{8(-2+r)(-1+r)}{8r^5+60r^4+170r^3+225r^2+137r+30}$	0
a_4	$\frac{16(-2+r)(-1+r)}{16r^6+208r^5+1080r^4+2840r^3+3929r^2+2637r+630}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{32(-2+r)(-1+r)}{32r^7 + 624r^6 + 5024r^5 + 21480r^4 + 52058r^3 + 70191r^2 + 47151r + 11340}$$

Which for the root $r = 1$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4+2r}{r(2r+1)}$	$-\frac{2}{3}$
a_2	$\frac{4(-2+r)(-1+r)}{4r^4+12r^3+11r^2+3r}$	0
a_3	$\frac{8(-2+r)(-1+r)}{8r^5+60r^4+170r^3+225r^2+137r+30}$	0
a_4	$\frac{16(-2+r)(-1+r)}{16r^6+208r^5+1080r^4+2840r^3+3929r^2+2637r+630}$	0
a_5	$\frac{32(-2+r)(-1+r)}{32r^7+624r^6+5024r^5+21480r^4+52058r^3+70191r^2+47151r+11340}$	0

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{64(-2+r)(-1+r)}{64r^8 + 1728r^7 + 19728r^6 + 123600r^5 + 460716r^4 + 1032852r^3 + 1332947r^2 + 874515r + 207900}$$

Which for the root $r = 1$ becomes

$$a_6 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4+2r}{r(2r+1)}$	$-\frac{2}{3}$
a_2	$\frac{4(-2+r)(-1+r)}{4r^4+12r^3+11r^2+3r}$	0
a_3	$\frac{8(-2+r)(-1+r)}{8r^5+60r^4+170r^3+225r^2+137r+30}$	0
a_4	$\frac{16(-2+r)(-1+r)}{16r^6+208r^5+1080r^4+2840r^3+3929r^2+2637r+630}$	0
a_5	$\frac{32(-2+r)(-1+r)}{32r^7+624r^6+5024r^5+21480r^4+52058r^3+70191r^2+47151r+11340}$	0
a_6	$\frac{64(-2+r)(-1+r)}{64r^8+1728r^7+19728r^6+123600r^5+460716r^4+1032852r^3+1332947r^2+874515r+207900}$	0

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{128(-2+r)(-1+r)}{128r^9 + 4544r^8 + 69472r^7 + 597296r^6 + 3161032r^5 + 10580276r^4 + 22101938r^3 + 27227409r^2 + 17}$$

Which for the root $r = 1$ becomes

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4+2r}{r(2r+1)}$	$-\frac{2}{3}$
a_2	$\frac{4(-2+r)(-1+r)}{4r^4+12r^3+11r^2+3r}$	0
a_3	$\frac{8(-2+r)(-1+r)}{8r^5+60r^4+170r^3+225r^2+137r+30}$	0
a_4	$\frac{16(-2+r)(-1+r)}{16r^6+208r^5+1080r^4+2840r^3+3929r^2+2637r+630}$	0
a_5	$\frac{32(-2+r)(-1+r)}{32r^7+624r^6+5024r^5+21480r^4+52058r^3+70191r^2+47151r+11340}$	0
a_6	$\frac{64(-2+r)(-1+r)}{64r^8+1728r^7+19728r^6+123600r^5+460716r^4+1032852r^3+1332947r^2+874515r+207900}$	0
a_7	$\frac{128(-2+r)(-1+r)}{128r^9+4544r^8+69472r^7+597296r^6+3161032r^5+10580276r^4+22101938r^3+27227409r^2+17338905r+4054050}$	0

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\
&= x\left(1 - \frac{2x}{3} + O\left(x^8\right)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
2b_n(n+r)(n+r-1) - 2b_{n-1}(n+r-1) - b_n(n+r) + b_n + 4b_{n-1} = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = \frac{2b_{n-1}(n+r-3)}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}
$$

Which for the root $r = \frac{1}{2}$ becomes

$$
b_n = \frac{b_{n-1}(2n-5)}{n(2n-1)} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{-4 + 2r}{r\,(2r + 1)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_1 = -3$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4+2r}{r(2r+1)}$	-3

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4(-2 + r)\,(-1 + r)}{4r^4 + 12r^3 + 11r^2 + 3r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4+2r}{r(2r+1)}$	-3
b_2	$\frac{4(-2+r)(-1+r)}{4r^4+12r^3+11r^2+3r}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{8(-2 + r)\,(-1 + r)}{8r^5 + 60r^4 + 170r^3 + 225r^2 + 137r + 30}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_3 = \frac{1}{30}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4+2r}{r(2r+1)}$	-3
b_2	$\frac{4(-2+r)(-1+r)}{4r^4+12r^3+11r^2+3r}$	$\frac{1}{2}$
b_3	$\frac{8(-2+r)(-1+r)}{8r^5+60r^4+170r^3+225r^2+137r+30}$	$\frac{1}{30}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16(-2+r)(-1+r)}{16r^6 + 208r^5 + 1080r^4 + 2840r^3 + 3929r^2 + 2637r + 630}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = \frac{1}{280}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4+2r}{r(2r+1)}$	-3
b_2	$\frac{4(-2+r)(-1+r)}{4r^4+12r^3+11r^2+3r}$	$\frac{1}{2}$
b_3	$\frac{8(-2+r)(-1+r)}{8r^5+60r^4+170r^3+225r^2+137r+30}$	$\frac{1}{30}$
b_4	$\frac{16(-2+r)(-1+r)}{16r^6+208r^5+1080r^4+2840r^3+3929r^2+2637r+630}$	$\frac{1}{280}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{32(-2+r)(-1+r)}{32r^7 + 624r^6 + 5024r^5 + 21480r^4 + 52058r^3 + 70191r^2 + 47151r + 11340}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_5 = \frac{1}{2520}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4+2r}{r(2r+1)}$	-3
b_2	$\frac{4(-2+r)(-1+r)}{4r^4+12r^3+11r^2+3r}$	$\frac{1}{2}$
b_3	$\frac{8(-2+r)(-1+r)}{8r^5+60r^4+170r^3+225r^2+137r+30}$	$\frac{1}{30}$
b_4	$\frac{16(-2+r)(-1+r)}{16r^6+208r^5+1080r^4+2840r^3+3929r^2+2637r+630}$	$\frac{1}{280}$
b_5	$\frac{32(-2+r)(-1+r)}{32r^7+624r^6+5024r^5+21480r^4+52058r^3+70191r^2+47151r+11340}$	$\frac{1}{2520}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{64(-2+r)(-1+r)}{64r^8 + 1728r^7 + 19728r^6 + 123600r^5 + 460716r^4 + 1032852r^3 + 1332947r^2 + 874515r + 207900}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_6 = \frac{1}{23760}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4+2r}{r(2r+1)}$	-3
b_2	$\frac{4(-2+r)(-1+r)}{4r^4+12r^3+11r^2+3r}$	$\frac{1}{2}$
b_3	$\frac{8(-2+r)(-1+r)}{8r^5+60r^4+170r^3+225r^2+137r+30}$	$\frac{1}{30}$
b_4	$\frac{16(-2+r)(-1+r)}{16r^6+208r^5+1080r^4+2840r^3+3929r^2+2637r+630}$	$\frac{1}{280}$
b_5	$\frac{32(-2+r)(-1+r)}{32r^7+624r^6+5024r^5+21480r^4+52058r^3+70191r^2+47151r+11340}$	$\frac{1}{2520}$
b_6	$\frac{64(-2+r)(-1+r)}{64r^8+1728r^7+19728r^6+123600r^5+460716r^4+1032852r^3+1332947r^2+874515r+207900}$	$\frac{1}{23760}$

For $n = 7$, using the above recursive equation gives

$$b_7 = \frac{128(-2+r)(-1+r)}{128r^9 + 4544r^8 + 69472r^7 + 597296r^6 + 3161032r^5 + 10580276r^4 + 22101938r^3 + 27227409r^2 + 17}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_7 = \frac{1}{240240}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4+2r}{r(2r+1)}$	-3
b_2	$\frac{4(-2+r)(-1+r)}{4r^4+12r^3+11r^2+3r}$	$\frac{1}{2}$
b_3	$\frac{8(-2+r)(-1+r)}{8r^5+60r^4+170r^3+225r^2+137r+30}$	$\frac{1}{30}$
b_4	$\frac{16(-2+r)(-1+r)}{16r^6+208r^5+1080r^4+2840r^3+3929r^2+2637r+630}$	$\frac{1}{280}$
b_5	$\frac{32(-2+r)(-1+r)}{32r^7+624r^6+5024r^5+21480r^4+52058r^3+70191r^2+47151r+11340}$	$\frac{1}{2520}$
b_6	$\frac{64(-2+r)(-1+r)}{64r^8+1728r^7+19728r^6+123600r^5+460716r^4+1032852r^3+1332947r^2+874515r+207900}$	$\frac{1}{23760}$
b_7	$\frac{128(-2+r)(-1+r)}{128r^9+4544r^8+69472r^7+597296r^6+3161032r^5+10580276r^4+22101938r^3+27227409r^2+17338905r+4054050}$	$\frac{1}{240240}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots\right) \\
&= \sqrt{x}\left(1 - 3x + \frac{x^2}{2} + \frac{x^3}{30} + \frac{x^4}{280} + \frac{x^5}{2520} + \frac{x^6}{23760} + \frac{x^7}{240240} + O(x^8)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1 - \frac{2x}{3} + O(x^8)\right) \\
&\quad + c_2\sqrt{x}\left(1 - 3x + \frac{x^2}{2} + \frac{x^3}{30} + \frac{x^4}{280} + \frac{x^5}{2520} + \frac{x^6}{23760} + \frac{x^7}{240240} + O(x^8)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x\left(1 - \frac{2x}{3} + O(x^8)\right) \\
&\quad + c_2\sqrt{x}\left(1 - 3x + \frac{x^2}{2} + \frac{x^3}{30} + \frac{x^4}{280} + \frac{x^5}{2520} + \frac{x^6}{23760} + \frac{x^7}{240240} + O(x^8)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1 x\left(1-\frac{2x}{3}+O\left(x^{8}\right)\right) \\
& + c_2\sqrt{x}\left(1-3x+\frac{x^{2}}{2}+\frac{x^{3}}{30}+\frac{x^{4}}{280}+\frac{x^{5}}{2520}+\frac{x^{6}}{23760}+\frac{x^{7}}{240240}+O\left(x^{8}\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1 x\left(1-\frac{2x}{3}+O\left(x^{8}\right)\right) \\
& + c_2\sqrt{x}\left(1-3x+\frac{x^{2}}{2}+\frac{x^{3}}{30}+\frac{x^{4}}{280}+\frac{x^{5}}{2520}+\frac{x^{6}}{23760}+\frac{x^{7}}{240240}+O\left(x^{8}\right)\right)
\end{aligned}
$$

Verified OK.

9.10.1 Maple step by step solution

Let's solve

$2x^{2}y''+\left(-2x^{2}-x\right)y'+\left(1+4x\right)y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(1+4x)y}{2x^{2}}+\frac{(2x+1)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(2x+1)y'}{2x}+\frac{(1+4x)y}{2x^{2}}=0$

☐ Check to see if $x_0=0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x)=-\frac{2x+1}{2x}, P_3(x)=\frac{1+4x}{2x^{2}}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=-\frac{1}{2}$

- ○ $x^{2}\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^{2}\cdot P_3(x)\right)\bigg|_{x=0}=\frac{1}{2}$

- ○ $x=0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$2x^2 y'' - x(2x+1)\, y' + (1+4x)\, y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k->k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+2r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)\,(k+r-1) - 2a_{k-1}(k-3+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(-1+2r)\,(-1+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{1, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

$2\left(k+r-\frac{1}{2}\right)(k+r-1)\, a_k - 2a_{k-1}(k-3+r) = 0$

- Shift index using $k- >k+1$

  $2\left(k+\frac{1}{2}+r\right)\left(k+r\right)a_{k+1}-2a_{k}\left(k+r-2\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{2a_{k}\left(k+r-2\right)}{\left(2k+1+2r\right)\left(k+r\right)}$

- Recursion relation for $r=1$; series terminates at $k=1$

  $a_{k+1}=\frac{2a_{k}\left(k-1\right)}{\left(2k+3\right)\left(k+1\right)}$

- Apply recursion relation for $k=0$

  $a_{1}=-\frac{2a_{0}}{3}$

- Terminating series solution of the ODE for $r=1$. Use reduction of order to find the second li

  $y=a_{0}\cdot\left(1-\frac{2x}{3}\right)$

- Recursion relation for $r=\frac{1}{2}$

  $a_{k+1}=\frac{2a_{k}\left(k-\frac{3}{2}\right)}{\left(2k+2\right)\left(k+\frac{1}{2}\right)}$

- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_{k}x^{k+\frac{1}{2}}, a_{k+1}=\frac{2a_{k}\left(k-\frac{3}{2}\right)}{\left(2k+2\right)\left(k+\frac{1}{2}\right)}\right]$

- Combine solutions and rename parameters

  $\left[y=a_{0}\cdot\left(1-\frac{2x}{3}\right)+\left(\sum_{k=0}^{\infty}b_{k}x^{k+\frac{1}{2}}\right), b_{k+1}=\frac{2b_{k}\left(k-\frac{3}{2}\right)}{\left(2k+2\right)\left(k+\frac{1}{2}\right)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 41

```
Order:=8;
dsolve(2*x^2*diff(y(x),x$2)-x*(1+2*x)*diff(y(x),x)+(1+4*x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\sqrt{x}\left(1-3x+\frac{1}{2}x^2+\frac{1}{30}x^3+\frac{1}{280}x^4+\frac{1}{2520}x^5+\frac{1}{23760}x^6+\frac{1}{240240}x^7+\mathrm{O}\left(x^8\right)\right) + c_2 x\left(1-\frac{2}{3}x+\mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 70

```
AsymptoticDSolveValue[2*x^2*y''[x]-x*(1+2*x)*y'[x]+(1+4*x)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\sqrt{x}\left(\frac{x^7}{240240} + \frac{x^6}{23760} + \frac{x^5}{2520} + \frac{x^4}{280} + \frac{x^3}{30} + \frac{x^2}{2} - 3x + 1\right) + c_1\left(1 - \frac{2x}{3}\right)x$$

9.11 problem 11

Internal problem ID [7013]
Internal file name [OUTPUT/6256_Thursday_August_18_2022_07_11_58_AM_54582995/index.tex]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. Miscellaneous Exercises. page 394
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{4x^2y'' - 2x(x+2)\,y' + (x+3)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4x^2y'' + \left(-2x^2 - 4x\right)y' + (x+3)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x+2}{2x}$$
$$q(x) = \frac{x+3}{4x^2}$$

Table 209: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x+2}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x+3}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2y'' + \left(-2x^2 - 4x\right) y' + (x + 3) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2 \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) + \left(-2x^2 - 4x\right) \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + (x + 3) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-4x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} 3a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1} (n+r-1)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1} (n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-4x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 3a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r} a_n (n+r)(n+r-1) - 4x^{n+r} a_n (n+r) + 3a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$4x^r a_0 r(-1+r) - 4x^r a_0 r + 3a_0 x^r = 0$$

Or

$$\left(4x^r r(-1+r) - 4x^r r + 3x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2 - 8r + 3\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - 8r + 3 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{3}{2}$$
$$r_2 = \frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(4r^2 - 8r + 3\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^{\frac{3}{2}}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \sqrt{x}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$4a_n(n+r)(n+r-1) - 2a_{n-1}(n+r-1) - 4a_n(n+r) + a_{n-1} + 3a_n = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{2n + 2r - 1} \tag{4}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_n = \frac{a_{n-1}}{2n + 2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{1 + 2r}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_1 = \frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+2r}$	$\frac{1}{4}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{4r^2 + 8r + 3}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_2 = \frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+2r}$	$\frac{1}{4}$
a_2	$\frac{1}{4r^2+8r+3}$	$\frac{1}{24}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{8r^3 + 36r^2 + 46r + 15}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_3 = \frac{1}{192}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+2r}$	$\frac{1}{4}$
a_2	$\frac{1}{4r^2+8r+3}$	$\frac{1}{24}$
a_3	$\frac{1}{8r^3+36r^2+46r+15}$	$\frac{1}{192}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{16r^4 + 128r^3 + 344r^2 + 352r + 105}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_4 = \frac{1}{1920}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+2r}$	$\frac{1}{4}$
a_2	$\frac{1}{4r^2+8r+3}$	$\frac{1}{24}$
a_3	$\frac{1}{8r^3+36r^2+46r+15}$	$\frac{1}{192}$
a_4	$\frac{1}{16r^4+128r^3+344r^2+352r+105}$	$\frac{1}{1920}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{32r^5 + 400r^4 + 1840r^3 + 3800r^2 + 3378r + 945}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_5 = \frac{1}{23040}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+2r}$	$\frac{1}{4}$
a_2	$\frac{1}{4r^2+8r+3}$	$\frac{1}{24}$
a_3	$\frac{1}{8r^3+36r^2+46r+15}$	$\frac{1}{192}$
a_4	$\frac{1}{16r^4+128r^3+344r^2+352r+105}$	$\frac{1}{1920}$
a_5	$\frac{1}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$	$\frac{1}{23040}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{1}{64r^6 + 1152r^5 + 8080r^4 + 27840r^3 + 48556r^2 + 39048r + 10395}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_6 = \frac{1}{322560}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+2r}$	$\frac{1}{4}$
a_2	$\frac{1}{4r^2+8r+3}$	$\frac{1}{24}$
a_3	$\frac{1}{8r^3+36r^2+46r+15}$	$\frac{1}{192}$
a_4	$\frac{1}{16r^4+128r^3+344r^2+352r+105}$	$\frac{1}{1920}$
a_5	$\frac{1}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$	$\frac{1}{23040}$
a_6	$\frac{1}{64r^6+1152r^5+8080r^4+27840r^3+48556r^2+39048r+10395}$	$\frac{1}{322560}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{1}{128r^7 + 3136r^6 + 31136r^5 + 160720r^4 + 459032r^3 + 709324r^2 + 528414r + 135135}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_7 = \frac{1}{5160960}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+2r}$	$\frac{1}{4}$
a_2	$\frac{1}{4r^2+8r+3}$	$\frac{1}{24}$
a_3	$\frac{1}{8r^3+36r^2+46r+15}$	$\frac{1}{192}$
a_4	$\frac{1}{16r^4+128r^3+344r^2+352r+105}$	$\frac{1}{1920}$
a_5	$\frac{1}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$	$\frac{1}{23040}$
a_6	$\frac{1}{64r^6+1152r^5+8080r^4+27840r^3+48556r^2+39048r+10395}$	$\frac{1}{322560}$
a_7	$\frac{1}{128r^7+3136r^6+31136r^5+160720r^4+459032r^3+709324r^2+528414r+135135}$	$\frac{1}{5160960}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{3}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\ &= x^{\frac{3}{2}}\left(1 + \frac{x}{4} + \frac{x^2}{24} + \frac{x^3}{192} + \frac{x^4}{1920} + \frac{x^5}{23040} + \frac{x^6}{322560} + \frac{x^7}{5160960} + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= \frac{1}{1 + 2r} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{1}{1 + 2r} &= \lim_{r\to \frac{1}{2}} \frac{1}{1 + 2r} \\ &= \frac{1}{2} \end{aligned}$$

The limit is $\frac{1}{2}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r}$$
$$= \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$4b_n(n+r)(n+r-1) - 2b_{n-1}(n+r-1) - 4b_n(n+r) + b_{n-1} + 3b_n = 0 \tag{4}$$

Which for for the root $r = \frac{1}{2}$ becomes

$$4b_n\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right) - 2b_{n-1}\left(n-\frac{1}{2}\right) - 4b_n\left(n+\frac{1}{2}\right) + b_{n-1} + 3b_n = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = \frac{b_{n-1}}{2n+2r-1} \tag{5}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = \frac{b_{n-1}}{2n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{1}{1+2r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_1 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{1+2r}$	$\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{(1+2r)(3+2r)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = \frac{1}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{1+2r}$	$\frac{1}{2}$
b_2	$\frac{1}{4r^2+8r+3}$	$\frac{1}{8}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{1}{(1+2r)(3+2r)(5+2r)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_3 = \frac{1}{48}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{1+2r}$	$\frac{1}{2}$
b_2	$\frac{1}{4r^2+8r+3}$	$\frac{1}{8}$
b_3	$\frac{1}{8r^3+36r^2+46r+15}$	$\frac{1}{48}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(1+2r)(3+2r)(5+2r)(7+2r)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = \frac{1}{384}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{1+2r}$	$\frac{1}{2}$
b_2	$\frac{1}{4r^2+8r+3}$	$\frac{1}{8}$
b_3	$\frac{1}{8r^3+36r^2+46r+15}$	$\frac{1}{48}$
b_4	$\frac{1}{16r^4+128r^3+344r^2+352r+105}$	$\frac{1}{384}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{1}{(1+2r)(3+2r)(5+2r)(9+2r)(7+2r)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_5 = \frac{1}{3840}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{1+2r}$	$\frac{1}{2}$
b_2	$\frac{1}{4r^2+8r+3}$	$\frac{1}{8}$
b_3	$\frac{1}{8r^3+36r^2+46r+15}$	$\frac{1}{48}$
b_4	$\frac{1}{16r^4+128r^3+344r^2+352r+105}$	$\frac{1}{384}$
b_5	$\frac{1}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$	$\frac{1}{3840}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{1}{(1+2r)(3+2r)(5+2r)(7+2r)(11+2r)(9+2r)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_6 = \frac{1}{46080}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{1+2r}$	$\frac{1}{2}$
b_2	$\frac{1}{4r^2+8r+3}$	$\frac{1}{8}$
b_3	$\frac{1}{8r^3+36r^2+46r+15}$	$\frac{1}{48}$
b_4	$\frac{1}{16r^4+128r^3+344r^2+352r+105}$	$\frac{1}{384}$
b_5	$\frac{1}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$	$\frac{1}{3840}$
b_6	$\frac{1}{(1+2r)(3+2r)(5+2r)(7+2r)(11+2r)(9+2r)}$	$\frac{1}{46080}$

For $n = 7$, using the above recursive equation gives

$$b_7 = \frac{1}{(1+2r)(3+2r)(5+2r)(9+2r)(7+2r)(11+2r)(13+2r)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_7 = \frac{1}{645120}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{1+2r}$	$\frac{1}{2}$
b_2	$\frac{1}{4r^2+8r+3}$	$\frac{1}{8}$
b_3	$\frac{1}{8r^3+36r^2+46r+15}$	$\frac{1}{48}$
b_4	$\frac{1}{16r^4+128r^3+344r^2+352r+105}$	$\frac{1}{384}$
b_5	$\frac{1}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$	$\frac{1}{3840}$
b_6	$\frac{1}{(1+2r)(3+2r)(5+2r)(7+2r)(11+2r)(9+2r)}$	$\frac{1}{46080}$
b_7	$\frac{1}{(1+2r)(3+2r)(5+2r)(9+2r)(7+2r)(11+2r)(13+2r)}$	$\frac{1}{645120}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\frac{3}{2}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots\right) \\ &= \sqrt{x}\left(1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384} + \frac{x^5}{3840} + \frac{x^6}{46080} + \frac{x^7}{645120} + O(x^8)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{3}{2}}\left(1 + \frac{x}{4} + \frac{x^2}{24} + \frac{x^3}{192} + \frac{x^4}{1920} + \frac{x^5}{23040} + \frac{x^6}{322560} + \frac{x^7}{5160960} + O(x^8)\right) \\ &+ c_2\sqrt{x}\left(1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384} + \frac{x^5}{3840} + \frac{x^6}{46080} + \frac{x^7}{645120} + O(x^8)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^{\frac{3}{2}}\left(1 + \frac{x}{4} + \frac{x^2}{24} + \frac{x^3}{192} + \frac{x^4}{1920} + \frac{x^5}{23040} + \frac{x^6}{322560} + \frac{x^7}{5160960} + O(x^8)\right) \\ &+ c_2\sqrt{x}\left(1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384} + \frac{x^5}{3840} + \frac{x^6}{46080} + \frac{x^7}{645120} + O(x^8)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 x^{\frac{3}{2}} \left(1+\frac{x}{4}+\frac{x^2}{24}+\frac{x^3}{192}+\frac{x^4}{1920}+\frac{x^5}{23040}+\frac{x^6}{322560}+\frac{x^7}{5160960}+O\left(x^8\right)\right) + c_2 \sqrt{x}\, \left(1+\frac{x}{2}+\frac{x^2}{8}+\frac{x^3}{48}+\frac{x^4}{384}+\frac{x^5}{3840}+\frac{x^6}{46080}+\frac{x^7}{645120}+O\left(x^8\right)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{3}{2}} \left(1+\frac{x}{4}+\frac{x^2}{24}+\frac{x^3}{192}+\frac{x^4}{1920}+\frac{x^5}{23040}+\frac{x^6}{322560}+\frac{x^7}{5160960}+O\left(x^8\right)\right) + c_2 \sqrt{x}\, \left(1+\frac{x}{2}+\frac{x^2}{8}+\frac{x^3}{48}+\frac{x^4}{384}+\frac{x^5}{3840}+\frac{x^6}{46080}+\frac{x^7}{645120}+O\left(x^8\right)\right)$$

Verified OK.

9.11.1 Maple step by step solution

Let's solve

$4x^2y'' + (-2x^2-4x)\,y' + (x+3)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x+3)y}{4x^2} + \frac{(x+2)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x+2)y'}{2x} + \frac{(x+3)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x+2}{2x}, P_3(x) = \frac{x+3}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{3}{4}$

  - $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$4x^2 y'' - 2x(x+2) y' + (x+3) y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ◦ Shift index using $k- > k - m$

$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1+m}$

- ◦ Shift index using $k- > k + 1 - m$

$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r) x^{k+r}$

- ◦ Convert $x^2 \cdot y''$ to series expansion

$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r}$

Rewrite ODE with series expansions

$a_0(-1+2r)(-3+2r) x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)(2k+2r-3) - a_{k-1}(2k+2r-3)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(-1+2r)(-3+2r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

$(2k+2r-3)(2k a_k + 2r a_k - a_k - a_{k-1}) = 0$

- Shift index using $k->k+1$

  $(2k+2r-1)\left(2(k+1)\,a_{k+1}+2r a_{k+1}-a_{k+1}-a_k\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k}{2k+1+2r}$

- Recursion relation for $r=\frac{1}{2}$

  $a_{k+1}=\frac{a_k}{2k+2}$

- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1}=\frac{a_k}{2k+2}\right]$

- Recursion relation for $r=\frac{3}{2}$

  $a_{k+1}=\frac{a_k}{2k+4}$

- Solution for $r=\frac{3}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1}=\frac{a_k}{2k+4}\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+1}=\frac{a_k}{2k+2}, b_{k+1}=\frac{b_k}{2k+4}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 55

```
Order:=8;
dsolve(4*x^2*diff(y(x),x$2)-2*x*(2+x)*diff(y(x),x)+(3+x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \sqrt{x}\left(x\left(1+\frac{1}{4}x+\frac{1}{24}x^2+\frac{1}{192}x^3+\frac{1}{1920}x^4+\frac{1}{23040}x^5+\frac{1}{322560}x^6+\frac{1}{5160960}x^7 + \mathrm{O}\left(x^8\right)\right)c_1 + \left(1+\frac{1}{2}x+\frac{1}{8}x^2+\frac{1}{48}x^3+\frac{1}{384}x^4+\frac{1}{3840}x^5+\frac{1}{46080}x^6+\frac{1}{645120}x^7+\mathrm{O}\left(x^8\right)\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.114 (sec). Leaf size: 130

```
AsymptoticDSolveValue[4*x^2*y''[x]-2*x*(2+x)*y'[x]+(3+x)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(\frac{x^{13/2}}{46080}+\frac{x^{11/2}}{3840}+\frac{x^{9/2}}{384}+\frac{x^{7/2}}{48}+\frac{x^{5/2}}{8}+\frac{x^{3/2}}{2} + \sqrt{x}\right) + c_2\left(\frac{x^{15/2}}{322560}+\frac{x^{13/2}}{23040}+\frac{x^{11/2}}{1920}+\frac{x^{9/2}}{192}+\frac{x^{7/2}}{24}+\frac{x^{5/2}}{4}+x^{3/2}\right)$$

9.12 problem 12

Internal problem ID [7014]
Internal file name [`OUTPUT/6257_Thursday_August_18_2022_07_12_01_AM_92905333/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. Miscellaneous Exercises. page 394
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' - x(x^2+1)\,y' + \left(-x^2+1\right)y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(-x^3 - x\right)y' + \left(-x^2+1\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x^2+1}{x}$$
$$q(x) = -\frac{x^2-1}{x^2}$$

Table 211: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x^2+1}{x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = -\frac{x^2-1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + (-x^3 - x)\, y' + (-x^2 + 1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}\right) + (-x^3 - x) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (-x^2 + 1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n (n+r)\right) = \sum_{n=2}^{\infty} \left(-a_{n-2}(n+r-2)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) = \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=2}^{\infty} \left(-a_{n-2}(n+r-2)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(-1+r)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(-1+r)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(-1+r)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+1}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-2}(n+r-2) - a_n(n+r) - a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}}{n+r-1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-2}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{1+r}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{1+r}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{1+r}$	$\frac{1}{2}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(3+r)(1+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{1+r}$	$\frac{1}{2}$
a_3	0	0
a_4	$\frac{1}{(3+r)(1+r)}$	$\frac{1}{8}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{1+r}$	$\frac{1}{2}$
a_3	0	0
a_4	$\frac{1}{(3+r)(1+r)}$	$\frac{1}{8}$
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{1}{(3+r)(1+r)(5+r)}$$

Which for the root $r = 1$ becomes

$$a_6 = \frac{1}{48}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{1+r}$	$\frac{1}{2}$
a_3	0	0
a_4	$\frac{1}{(3+r)(1+r)}$	$\frac{1}{8}$
a_5	0	0
a_6	$\frac{1}{(3+r)(1+r)(5+r)}$	$\frac{1}{48}$

For $n = 7$, using the above recursive equation gives

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{1+r}$	$\frac{1}{2}$
a_3	0	0
a_4	$\frac{1}{(3+r)(1+r)}$	$\frac{1}{8}$
a_5	0	0
a_6	$\frac{1}{(3+r)(1+r)(5+r)}$	$\frac{1}{48}$
a_7	0	0

Using the above table, then the first solution $y_1(x)$ is

$$y_1(x) = x\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \dots\right)$$
$$= x\left(1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + O(x^8)\right)$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr}a_{n,r}$$

And the above is then evaluated at $r = 1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r=1)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$\frac{1}{1+r}$	$\frac{1}{2}$	$-\frac{1}{(1+r)^2}$	$-\frac{1}{4}$
b_3	0	0	0	0
b_4	$\frac{1}{(3+r)(1+r)}$	$\frac{1}{8}$	$\frac{-4-2r}{(3+r)^2(1+r)^2}$	$-\frac{3}{32}$
b_5	0	0	0	0
b_6	$\frac{1}{(3+r)(1+r)(5+r)}$	$\frac{1}{48}$	$\frac{-3r^2-18r-23}{(3+r)^2(1+r)^2(5+r)^2}$	$-\frac{11}{576}$
b_7	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$y_2(x) = y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 + b_7x^7 + b_8x^8 \dots$$
$$= x\left(1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + O(x^8)\right)\ln(x) + x\left(-\frac{x^2}{4} - \frac{3x^4}{32} - \frac{11x^6}{576} + O(x^8)\right)$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1+\frac{x^2}{2}+\frac{x^4}{8}+\frac{x^6}{48}+O(x^8)\right) \\
&\quad + c_2\left(x\left(1+\frac{x^2}{2}+\frac{x^4}{8}+\frac{x^6}{48}+O(x^8)\right)\ln(x)+x\left(-\frac{x^2}{4}-\frac{3x^4}{32}-\frac{11x^6}{576}+O(x^8)\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x\left(1+\frac{x^2}{2}+\frac{x^4}{8}+\frac{x^6}{48}+O(x^8)\right) \\
&\quad + c_2\left(x\left(1+\frac{x^2}{2}+\frac{x^4}{8}+\frac{x^6}{48}+O(x^8)\right)\ln(x)+x\left(-\frac{x^2}{4}-\frac{3x^4}{32}-\frac{11x^6}{576}+O(x^8)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x\left(1+\frac{x^2}{2}+\frac{x^4}{8}+\frac{x^6}{48}+O(x^8)\right) \\
&\quad + c_2\left(x\left(1+\frac{x^2}{2}+\frac{x^4}{8}+\frac{x^6}{48}+O(x^8)\right)\ln(x)+x\left(-\frac{x^2}{4}-\frac{3x^4}{32}-\frac{11x^6}{576}+O(x^8)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x\left(1+\frac{x^2}{2}+\frac{x^4}{8}+\frac{x^6}{48}+O(x^8)\right) \\
&\quad + c_2\left(x\left(1+\frac{x^2}{2}+\frac{x^4}{8}+\frac{x^6}{48}+O(x^8)\right)\ln(x)+x\left(-\frac{x^2}{4}-\frac{3x^4}{32}-\frac{11x^6}{576}+O(x^8)\right)\right)
\end{aligned}
$$

Verified OK.

9.12.1 Maple step by step solution

Let's solve

$x^2y'' + (-x^3 - x)\,y' + (-x^2 + 1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x^2-1)y}{x^2} + \frac{(x^2+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x^2+1)y'}{x} - \frac{(x^2-1)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x^2+1}{x}, P_3(x) = -\frac{x^2-1}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 1$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2y'' - x(x^2 + 1)\,y' + (-x^2 + 1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x^{m}\cdot y'$ to series expansion for $m=1..3$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$
- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
- Convert $x^{2}\cdot y''$ to series expansion

  $x^{2}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(-1+r)^{2} x^{r}+a_{1} r^{2} x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_{k}(k+r-1)^{2}-a_{k-2}(k+r-1)\right) x^{k+r}\right)=0$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(-1+r)^{2}=0$
- Values of r that satisfy the indicial equation

  $r=1$
- Each term must be 0

  $a_{1} r^{2}=0$
- Solve for the dependent coefficient(s)

  $a_{1}=0$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_{k}(k+r-1)-a_{k-2}\right)=0$
- Shift index using $k->k+2$

  $(k+r+1)\left(a_{k+2}(k+r+1)-a_{k}\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{a_{k}}{k+r+1}$
- Recursion relation for $r=1$

  $a_{k+2}=\frac{a_{k}}{k+2}$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = \frac{a_k}{k+2}, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 51

```
Order:=8;
dsolve(x^2*diff(y(x),x$2)-x*(1+x^2)*diff(y(x),x)+(1-x^2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = x\left((c_2 \ln(x) + c_1)\left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6 + \mathrm{O}\left(x^8\right)\right) + \left(-\frac{1}{4}x^2 - \frac{3}{32}x^4 - \frac{11}{576}x^6 + \mathrm{O}\left(x^8\right)\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 86

```
AsymptoticDSolveValue[x^2*y''[x]-x*(1+x^2)*y'[x]+(1-x^2)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1 x\left(\frac{x^6}{48} + \frac{x^4}{8} + \frac{x^2}{2} + 1\right) + c_2\left(x\left(-\frac{11x^6}{576} - \frac{3x^4}{32} - \frac{x^2}{4}\right) + x\left(\frac{x^6}{48} + \frac{x^4}{8} + \frac{x^2}{2} + 1\right)\log(x)\right)$$

9.13 problem 13

Internal problem ID [7015]
Internal file name [`OUTPUT/6258_Thursday_August_18_2022_07_12_03_AM_35630610/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. Miscellaneous Exercises. page 394
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{2xy'' + y' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2xy'' + y' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{2x}$$
$$q(x) = \frac{1}{2x}$$

Table 213: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2xy'' + y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1} a_n(n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$2x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(2x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-1+2r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(-1+2r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{2n^2 + 4nr + 2r^2 - n - r} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{a_{n-1}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{2r^2 + 3r + 1}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = -\frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+3r+1}$	$-\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{4r^4 + 20r^3 + 35r^2 + 25r + 6}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{1}{30}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+3r+1}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{30}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{8r^6 + 84r^5 + 350r^4 + 735r^3 + 812r^2 + 441r + 90}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = -\frac{1}{630}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+3r+1}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{30}$
a_3	$-\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$-\frac{1}{630}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{16r^8 + 288r^7 + 2184r^6 + 9072r^5 + 22449r^4 + 33642r^3 + 29531r^2 + 13698r + 2520}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{1}{22680}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+3r+1}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{30}$
a_3	$-\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$-\frac{1}{630}$
a_4	$\frac{1}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{1}{22680}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{32r^{10} + 880r^9 + 10560r^8 + 72600r^7 + 315546r^6 + 902055r^5 + 1708465r^4 + 2102375r^3 + 1594197r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = -\frac{1}{1247400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+3r+1}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{30}$
a_3	$-\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$-\frac{1}{630}$
a_4	$\frac{1}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{1}{22680}$
a_5	$-\frac{1}{32r^{10}+880r^9+10560r^8+72600r^7+315546r^6+902055r^5+1708465r^4+2102375r^3+1594197r^2+664290r+113400}$	$-\frac{1}{1247400}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{1}{64 (r+4) \left(r+\frac{11}{2}\right) (r+3) (r+1) \left(\frac{3}{2}+r\right) \left(\frac{5}{2}+r\right) \left(r+\frac{9}{2}\right) \left(r+\frac{1}{2}\right) (r+5) \left(\frac{7}{2}+r\right) (r+6) (r+2)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_6 = \frac{1}{97297200}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+3r+1}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{30}$
a_3	$-\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$-\frac{1}{630}$
a_4	$\frac{1}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{1}{22680}$
a_5	$-\frac{1}{32r^{10}+880r^9+10560r^8+72600r^7+315546r^6+902055r^5+1708465r^4+2102375r^3+1594197r^2+664290r+113400}$	$-\frac{1}{1247400}$
a_6	$\frac{1}{64(r+4)\left(r+\frac{11}{2}\right)(r+3)(r+1)\left(\frac{3}{2}+r\right)\left(\frac{5}{2}+r\right)\left(r+\frac{9}{2}\right)\left(r+\frac{1}{2}\right)(r+5)\left(\frac{7}{2}+r\right)(r+6)(r+2)}$	$\frac{1}{97297200}$

For $n = 7$, using the above recursive equation gives

$$a_7 = -\frac{1}{(r+4) (2r+11) (r+3) (r+1) (3+2r) (5+2r) (2r+9) (2r+1) (r+5) (7+2r) (r+6) (r+2) (}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_7 = -\frac{1}{10216206000}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+3r+1}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{30}$
a_3	$-\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$-\frac{1}{630}$
a_4	$\frac{1}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{1}{22680}$
a_5	$-\frac{1}{32r^{10}+880r^9+10560r^8+72600r^7+315546r^6+902055r^5+1708465r^4+2102375r^3+1594197r^2+664290r+113400}$	$-\frac{1}{1247400}$
a_6	$\frac{1}{64(r+4)\left(r+\frac{11}{2}\right)(r+3)(r+1)\left(\frac{3}{2}+r\right)\left(\frac{5}{2}+r\right)\left(r+\frac{9}{2}\right)\left(r+\frac{1}{2}\right)(r+5)\left(\frac{7}{2}+r\right)(r+6)(r+2)}$	$\frac{1}{97297200}$
a_7	$-\frac{1}{(r+4)(2r+11)(r+3)(r+1)(3+2r)(5+2r)(2r+9)(2r+1)(r+5)(7+2r)(r+6)(r+2)(2r^2+27r+91)}$	$-\frac{1}{10216206000}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\
&= \sqrt{x}\left(1 - \frac{x}{3} + \frac{x^2}{30} - \frac{x^3}{630} + \frac{x^4}{22680} - \frac{x^5}{1247400} + \frac{x^6}{97297200} - \frac{x^7}{10216206000} + O\left(x^8\right)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
2b_n(n+r)(n+r-1) + (n+r)b_n + b_{n-1} = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = -\frac{b_{n-1}}{2n^2 + 4nr + 2r^2 - n - r} \tag{4}
$$

Which for the root $r = 0$ becomes

$$
b_n = -\frac{b_{n-1}}{n(2n-1)} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{1}{2r^2 + 3r + 1}$$

Which for the root $r = 0$ becomes

$$b_1 = -1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+3r+1}$	-1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{4r^4 + 20r^3 + 35r^2 + 25r + 6}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+3r+1}$	-1
b_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{1}{8r^6 + 84r^5 + 350r^4 + 735r^3 + 812r^2 + 441r + 90}$$

Which for the root $r = 0$ becomes

$$b_3 = -\frac{1}{90}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+3r+1}$	-1
b_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{6}$
b_3	$-\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$-\frac{1}{90}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{16r^8 + 288r^7 + 2184r^6 + 9072r^5 + 22449r^4 + 33642r^3 + 29531r^2 + 13698r + 2520}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{1}{2520}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+3r+1}$	-1
b_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{6}$
b_3	$-\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$-\frac{1}{90}$
b_4	$\frac{1}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{1}{2520}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{1}{32r^{10} + 880r^9 + 10560r^8 + 72600r^7 + 315546r^6 + 902055r^5 + 1708465r^4 + 2102375r^3 + 1594197r}$$

Which for the root $r = 0$ becomes

$$b_5 = -\frac{1}{113400}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+3r+1}$	-1
b_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{6}$
b_3	$-\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$-\frac{1}{90}$
b_4	$\frac{1}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{1}{2520}$
b_5	$-\frac{1}{32r^{10}+880r^9+10560r^8+72600r^7+315546r^6+902055r^5+1708465r^4+2102375r^3+1594197r^2+664290r+113400}$	$-\frac{1}{113400}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{1}{64 (r + 4) \left(r + \frac{11}{2}\right) (r + 3) (r + 1) \left(\frac{3}{2} + r\right) \left(\frac{5}{2} + r\right) \left(r + \frac{9}{2}\right) \left(r + \frac{1}{2}\right) (r + 5) \left(\frac{7}{2} + r\right) (r + 6) (r + 2)}$$

Which for the root $r = 0$ becomes

$$b_6 = \frac{1}{7484400}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+3r+1}$	-1
b_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{6}$
b_3	$-\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$-\frac{1}{90}$
b_4	$\frac{1}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{1}{2520}$
b_5	$-\frac{1}{32r^{10}+880r^9+10560r^8+72600r^7+315546r^6+902055r^5+1708465r^4+2102375r^3+1594197r^2+664290r+113400}$	$-\frac{1}{113400}$
b_6	$\frac{1}{64(r+4)\left(r+\frac{11}{2}\right)(r+3)(r+1)\left(\frac{3}{2}+r\right)\left(\frac{5}{2}+r\right)\left(r+\frac{9}{2}\right)\left(r+\frac{1}{2}\right)(r+5)\left(\frac{7}{2}+r\right)(r+6)(r+2)}$	$\frac{1}{7484400}$

For $n = 7$, using the above recursive equation gives

$$b_7 = -\frac{1}{(r + 4) (2r + 11) (r + 3) (r + 1) (3 + 2r) (5 + 2r) (2r + 9) (2r + 1) (r + 5) (7 + 2r) (r + 6) (r + 2) (}$$

Which for the root $r = 0$ becomes

$$b_7 = -\frac{1}{681080400}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+3r+1}$	-1
b_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{6}$
b_3	$-\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$-\frac{1}{90}$
b_4	$\frac{1}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{1}{2520}$
b_5	$-\frac{1}{32r^{10}+880r^9+10560r^8+72600r^7+315546r^6+902055r^5+1708465r^4+2102375r^3+1594197r^2+664290r+113400}$	$-\frac{1}{113400}$
b_6	$\frac{1}{64(r+4)\left(r+\frac{11}{2}\right)(r+3)(r+1)\left(\frac{3}{2}+r\right)\left(\frac{5}{2}+r\right)\left(r+\frac{9}{2}\right)\left(r+\frac{1}{2}\right)(r+5)\left(\frac{7}{2}+r\right)(r+6)(r+2)}$	$\frac{1}{7484400}$
b_7	$-\frac{1}{(r+4)(2r+11)(r+3)(r+1)(3+2r)(5+2r)(2r+9)(2r+1)(r+5)(7+2r)(r+6)(r+2)(2r^2+27r+91)}$	$-\frac{1}{681080400}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\
&= 1 - x + \frac{x^2}{6} - \frac{x^3}{90} + \frac{x^4}{2520} - \frac{x^5}{113400} + \frac{x^6}{7484400} - \frac{x^7}{681080400} + O(x^8)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \sqrt{x} \left(1 - \frac{x}{3} + \frac{x^2}{30} - \frac{x^3}{630} + \frac{x^4}{22680} - \frac{x^5}{1247400} + \frac{x^6}{97297200} - \frac{x^7}{10216206000} + O(x^8)\right) \\
&\quad + c_2 \left(1 - x + \frac{x^2}{6} - \frac{x^3}{90} + \frac{x^4}{2520} - \frac{x^5}{113400} + \frac{x^6}{7484400} - \frac{x^7}{681080400} + O(x^8)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \sqrt{x} \left(1 - \frac{x}{3} + \frac{x^2}{30} - \frac{x^3}{630} + \frac{x^4}{22680} - \frac{x^5}{1247400} + \frac{x^6}{97297200} - \frac{x^7}{10216206000} + O(x^8)\right) \\
&\quad + c_2 \left(1 - x + \frac{x^2}{6} - \frac{x^3}{90} + \frac{x^4}{2520} - \frac{x^5}{113400} + \frac{x^6}{7484400} - \frac{x^7}{681080400} + O(x^8)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1\sqrt{x}\left(1-\frac{x}{3}+\frac{x^2}{30}-\frac{x^3}{630}+\frac{x^4}{22680}-\frac{x^5}{1247400}+\frac{x^6}{97297200}-\frac{x^7}{10216206000}+O\left(x^8\right)\right) + c_2\left(1-x+\frac{x^2}{6}-\frac{x^3}{90}+\frac{x^4}{2520}-\frac{x^5}{113400}+\frac{x^6}{7484400}-\frac{x^7}{681080400}+O\left(x^8\right)\right) \tag{1}
$$

Verification of solutions

$$
y = c_1\sqrt{x}\left(1-\frac{x}{3}+\frac{x^2}{30}-\frac{x^3}{630}+\frac{x^4}{22680}-\frac{x^5}{1247400}+\frac{x^6}{97297200}-\frac{x^7}{10216206000}+O\left(x^8\right)\right) + c_2\left(1-x+\frac{x^2}{6}-\frac{x^3}{90}+\frac{x^4}{2520}-\frac{x^5}{113400}+\frac{x^6}{7484400}-\frac{x^7}{681080400}+O\left(x^8\right)\right)
$$

Verified OK.

9.13.1 Maple step by step solution

Let's solve

$2xy'' + y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{2x} - \frac{y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{2x} + \frac{y}{2x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{2x}, P_3(x) = \frac{1}{2x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{1}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$2xy'' + y' + y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert y' to series expansion

  $y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

  ◦ Shift index using $k- > k+1$

  $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$

  ◦ Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

  ◦ Shift index using $k- > k+1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0 r(-1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k+1+2r) + a_k\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-1+2r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{0, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

$2(k+1+r)\left(k+\frac{1}{2}+r\right) a_{k+1} + a_k = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = -\frac{a_k}{(k+1+r)(2k+1+2r)}$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{a_k}{(k+1)(2k+1)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k}{(k+1)(2k+1)}\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = -\frac{a_k}{\left(k+\frac{3}{2}\right)(2k+2)}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{a_k}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{a_k}{(k+1)(2k+1)}, b_{k+1} = -\frac{b_k}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 52

```
Order:=8;
dsolve(2*x*diff(y(x),x$2)+diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\sqrt{x}\left(1 - \frac{1}{3}x + \frac{1}{30}x^2 - \frac{1}{630}x^3 + \frac{1}{22680}x^4 - \frac{1}{1247400}x^5 + \frac{1}{97297200}x^6 - \frac{1}{10216206000}x^7 + \mathrm{O}\left(x^8\right)\right) + c_2\left(1 - x + \frac{1}{6}x^2 - \frac{1}{90}x^3 + \frac{1}{2520}x^4 - \frac{1}{113400}x^5 + \frac{1}{7484400}x^6 - \frac{1}{681080400}x^7 + \mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 111

```
AsymptoticDSolveValue[2*x*y''[x]+y'[x]+y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\sqrt{x}\left(-\frac{x^7}{10216206000} + \frac{x^6}{97297200} - \frac{x^5}{1247400} + \frac{x^4}{22680} - \frac{x^3}{630} + \frac{x^2}{30} - \frac{x}{3} + 1\right) + c_2\left(-\frac{x^7}{681080400} + \frac{x^6}{7484400} - \frac{x^5}{113400} + \frac{x^4}{2520} - \frac{x^3}{90} + \frac{x^2}{6} - x + 1\right)$$

9.14 problem 14

Internal problem ID [7016]
Internal file name [`OUTPUT/6259_Thursday_August_18_2022_07_12_06_AM_97693164/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. Miscellaneous Exercises. page 394
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x(x^2 - 3)\, y' + 4y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + (x^3 - 3x)\, y' + 4y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= \frac{x^2 - 3}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Table 215: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x^2-3}{x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = \frac{4}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + (x^3 - 3x)\, y' + 4y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (x^3 - 3x) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 4\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)\right) \\ + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} a_{n-2} (n+r-2)\, x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} (n+r-2)\, x^{n+r}\right) \\ + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4 a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 3x^{n+r} a_n (n+r) + 4 a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - 3x^r a_0 r + 4 a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 3x^r r + 4x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r-2)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r-2)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = 2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r-2)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 2$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+2}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_{n-2}(n+r-2) - 3a_n(n+r) + 4a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n+r-2} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = -\frac{a_{n-2}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{r}$$

Which for the root $r = 2$ becomes

$$a_2 = -\frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r}$	$-\frac{1}{2}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{r(2+r)}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{1}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{1}{r(2+r)}$	$\frac{1}{8}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{1}{r(2+r)}$	$\frac{1}{8}$
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = -\frac{1}{(4+r)\, r\, (2+r)}$$

Which for the root $r = 2$ becomes

$$a_6 = -\frac{1}{48}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{1}{r(2+r)}$	$\frac{1}{8}$
a_5	0	0
a_6	$-\frac{1}{(4+r)r(2+r)}$	$-\frac{1}{48}$

For $n = 7$, using the above recursive equation gives

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{1}{r(2+r)}$	$\frac{1}{8}$
a_5	0	0
a_6	$-\frac{1}{(4+r)r(2+r)}$	$-\frac{1}{48}$
a_7	0	0

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned}y_1(x) &= x^2\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\ &= x^2\left(1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + O\left(x^8\right)\right)\end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 2$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = 2)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$-\frac{1}{r}$	$-\frac{1}{2}$	$\frac{1}{r^2}$	$\frac{1}{4}$
b_3	0	0	0	0
b_4	$\frac{1}{r(2+r)}$	$\frac{1}{8}$	$\frac{-2-2r}{r^2(2+r)^2}$	$-\frac{3}{32}$
b_5	0	0	0	0
b_6	$-\frac{1}{(4+r)r(2+r)}$	$-\frac{1}{48}$	$\frac{3r^2+12r+8}{(4+r)^2 r^2 (2+r)^2}$	$\frac{11}{576}$
b_7	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned} y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\ &= x^2\left(1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + O(x^8)\right)\ln(x) + x^2\left(\frac{x^2}{4} - \frac{3x^4}{32} + \frac{11x^6}{576} + O(x^8)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^2\left(1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + O(x^8)\right) \\ &\quad + c_2\left(x^2\left(1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + O(x^8)\right)\ln(x) + x^2\left(\frac{x^2}{4} - \frac{3x^4}{32} + \frac{11x^6}{576} + O(x^8)\right)\right) \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^2 \left(1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + O(x^8)\right) \\
&\quad + c_2 \left(x^2 \left(1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + O(x^8)\right) \ln(x) + x^2 \left(\frac{x^2}{4} - \frac{3x^4}{32} + \frac{11x^6}{576} + O(x^8)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^2 &\left(1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + O(x^8)\right) \\
&+ c_2 \left(x^2 \left(1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + O(x^8)\right) \ln(x) + x^2 \left(\frac{x^2}{4} - \frac{3x^4}{32} + \frac{11x^6}{576} + O(x^8)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1 x^2 &\left(1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + O(x^8)\right) \\
&+ c_2 \left(x^2 \left(1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + O(x^8)\right) \ln(x) + x^2 \left(\frac{x^2}{4} - \frac{3x^4}{32} + \frac{11x^6}{576} + O(x^8)\right)\right)
\end{aligned}
$$

Verified OK.

9.14.1 Maple step by step solution

Let's solve

$x^2 y'' + (x^3 - 3x) y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4y}{x^2} - \frac{(x^2-3)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x^2-3)y'}{x} + \frac{4y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{x^2-3}{x}, P_3(x) = \frac{4}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -3$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 4$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x(x^2-3)\, y' + 4y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)^2\, x^r + a_1(-1+r)^2\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-2)^2 + a_{k-2}(k+r-2)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 2$

- Each term must be 0

  $a_1(-1+r)^2 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)\,(a_k(k+r-2)+a_{k-2}) = 0$

- Shift index using $k->k+2$

  $(k+r)\,(a_{k+2}(k+r)+a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{k+r}$

- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{a_k}{k+2}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k}{k+2}, a_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 57

```
Order:=8;
dsolve(x^2*diff(y(x),x$2)+x*(x^2-3)*diff(y(x),x)+4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = x^2\left((c_2 \ln(x) + c_1)\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \mathrm{O}\left(x^8\right)\right) + \left(\frac{1}{4}x^2 - \frac{3}{32}x^4 + \frac{11}{576}x^6 + \mathrm{O}\left(x^8\right)\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 92

```
AsymptoticDSolveValue[x^2*y''[x]+x*(x^2-3)*y'[x]+4*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(-\frac{x^6}{48} + \frac{x^4}{8} - \frac{x^2}{2} + 1\right)x^2 + c_2\left(\left(\frac{11x^6}{576} - \frac{3x^4}{32} + \frac{x^2}{4}\right)x^2 + \left(-\frac{x^6}{48} + \frac{x^4}{8} - \frac{x^2}{2} + 1\right)x^2\log(x)\right)$$

9.15 problem 15

Internal problem ID [7017]
Internal file name [OUTPUT/6260_Thursday_August_18_2022_07_12_07_AM_64728628/index.tex]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. Miscellaneous Exercises. page 394
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' - x^2y' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4x^2y'' - x^2y' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{4}$$
$$q(x) = \frac{1}{4x^2}$$

Table 217: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{4}$	
singularity	type

$q(x) = \frac{1}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2y'' - x^2y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) - x^2 \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)\,(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \quad \text{(2A)}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r} a_n (n+r)(n+r-1) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$4x^r a_0 r(-1+r) + a_0 x^r = 0$$

Or

$$\left(4x^r r(-1+r) + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(2r-1)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(2r-1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(2r-1)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = \frac{1}{2}$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+\frac{1}{2}}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$4a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(n+r-1)}{4n^2 + 8nr + 4r^2 - 4n - 4r + 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{a_{n-1}(2n-1)}{8n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{r}{(2r+1)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = \frac{1}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{(2r+1)^2}$	$\frac{1}{8}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r(1+r)}{(2r+1)^2 (3+2r)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{3}{256}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{(2r+1)^2}$	$\frac{1}{8}$
a_2	$\frac{r(1+r)}{(2r+1)^2(3+2r)^2}$	$\frac{3}{256}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{r(1+r)(2+r)}{(2r+1)^2 (3+2r)^2 (5+2r)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = \frac{5}{6144}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{(2r+1)^2}$	$\frac{1}{8}$
a_2	$\frac{r(1+r)}{(2r+1)^2(3+2r)^2}$	$\frac{3}{256}$
a_3	$\frac{r(1+r)(2+r)}{(2r+1)^2(3+2r)^2(5+2r)^2}$	$\frac{5}{6144}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r(1+r)(2+r)(3+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{35}{786432}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{(2r+1)^2}$	$\frac{1}{8}$
a_2	$\frac{r(1+r)}{(2r+1)^2(3+2r)^2}$	$\frac{3}{256}$
a_3	$\frac{r(1+r)(2+r)}{(2r+1)^2(3+2r)^2(5+2r)^2}$	$\frac{5}{6144}$
a_4	$\frac{r(1+r)(2+r)(3+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2}$	$\frac{35}{786432}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{r(1+r)(2+r)(3+r)(4+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2(2r+9)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = \frac{21}{10485760}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{(2r+1)^2}$	$\frac{1}{8}$
a_2	$\frac{r(1+r)}{(2r+1)^2(3+2r)^2}$	$\frac{3}{256}$
a_3	$\frac{r(1+r)(2+r)}{(2r+1)^2(3+2r)^2(5+2r)^2}$	$\frac{5}{6144}$
a_4	$\frac{r(1+r)(2+r)(3+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2}$	$\frac{35}{786432}$
a_5	$\frac{r(1+r)(2+r)(3+r)(4+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2(2r+9)^2}$	$\frac{21}{10485760}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{r(1+r)(2+r)(3+r)(4+r)(5+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2(2r+9)^2(2r+11)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_6 = \frac{77}{1006632960}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{(2r+1)^2}$	$\frac{1}{8}$
a_2	$\frac{r(1+r)}{(2r+1)^2(3+2r)^2}$	$\frac{3}{256}$
a_3	$\frac{r(1+r)(2+r)}{(2r+1)^2(3+2r)^2(5+2r)^2}$	$\frac{5}{6144}$
a_4	$\frac{r(1+r)(2+r)(3+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2}$	$\frac{35}{786432}$
a_5	$\frac{r(1+r)(2+r)(3+r)(4+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2(2r+9)^2}$	$\frac{21}{10485760}$
a_6	$\frac{r(1+r)(2+r)(3+r)(4+r)(5+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2(2r+9)^2(2r+11)^2}$	$\frac{77}{1006632960}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{r(1+r)(2+r)(3+r)(4+r)(5+r)(6+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2(2r+9)^2(2r+11)^2(2r+13)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_7 = \frac{143}{56371445760}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{(2r+1)^2}$	$\frac{1}{8}$
a_2	$\frac{r(1+r)}{(2r+1)^2(3+2r)^2}$	$\frac{3}{256}$
a_3	$\frac{r(1+r)(2+r)}{(2r+1)^2(3+2r)^2(5+2r)^2}$	$\frac{5}{6144}$
a_4	$\frac{r(1+r)(2+r)(3+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2}$	$\frac{35}{786432}$
a_5	$\frac{r(1+r)(2+r)(3+r)(4+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2(2r+9)^2}$	$\frac{21}{10485760}$
a_6	$\frac{r(1+r)(2+r)(3+r)(4+r)(5+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2(2r+9)^2(2r+11)^2}$	$\frac{77}{1006632960}$
a_7	$\frac{r(1+r)(2+r)(3+r)(4+r)(5+r)(6+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2(2r+9)^2(2r+11)^2(2r+13)^2}$	$\frac{143}{56371445760}$

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\ &= \sqrt{x}\left(1 + \frac{x}{8} + \frac{3x^2}{256} + \frac{5x^3}{6144} + \frac{35x^4}{786432} + \frac{21x^5}{10485760} + \frac{77x^6}{1006632960} + \frac{143x^7}{56371445760} + O(x^8)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = \frac{1}{2}$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	$\frac{r}{(2r+1)^2}$	$\frac{1}{8}$	$\frac{-2r+1}{(2r+1)^3}$
b_2	$\frac{r(1+r)}{(2r+1)^2(3+2r)^2}$	$\frac{3}{256}$	$\frac{-8r^3-12r^2-2r+3}{(2r+1)^3(3+2r)^3}$
b_3	$\frac{r(1+r)(2+r)}{(2r+1)^2(3+2r)^2(5+2r)^2}$	$\frac{5}{6144}$	$\frac{-24r^5-132r^4-250r^3-171r^2-2r+30}{(2r+1)^3(5+2r)^3(3+2r)^3}$
b_4	$\frac{r(1+r)(2+r)(3+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2}$	$\frac{35}{786432}$	$\frac{-64r^7-736r^6-3360r^5-7664r^4-8876r^3-4302r^2+198r-}{(2r+1)^3(3+2r)^3(5+2r)^3(7+2r)^3}$
b_5	$\frac{r(1+r)(2+r)(3+r)(4+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2(2r+9)^2}$	$\frac{21}{10485760}$	$\frac{-160r^9-3120r^8-25680r^7-115800r^6-309978r^5-49591}{(7+2r)^3(2r+1)^3(2r+9)^3(5-}$
b_6	$\frac{r(1+r)(2+r)(3+r)(4+r)(5+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2(2r+9)^2(2r+11)^2}$	$\frac{77}{1006632960}$	$\frac{-384r^{11}-11328r^{10}-146080r^9-1080720r^8-5057448r^7-}{(2r+1)^3(}$
b_7	$\frac{r(1+r)(2+r)(3+r)(4+r)(5+r)(6+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2(2r+9)^2(2r+11)^2(2r+13)^2}$	$\frac{143}{56371445760}$	$\frac{-896r^{13}-37184r^{12}-690144r^{11}-7558544r^{10}-54261480}{}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\
&= \sqrt{x}\left(1 + \frac{x}{8} + \frac{3x^2}{256} + \frac{5x^3}{6144} + \frac{35x^4}{786432} + \frac{21x^5}{10485760} + \frac{77x^6}{1006632960} + \frac{143x^7}{56371445760} \right. \\
&\quad \left. + O(x^8)\right)\ln(x) + \sqrt{x}\left(-\frac{x^2}{256} - \frac{x^3}{2048} - \frac{19x^4}{524288} - \frac{25x^5}{12582912} - \frac{317x^6}{3623878656} \right. \\
&\quad \left. - \frac{469x^7}{144955146240} + O(x^8)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\sqrt{x}\left(1 + \frac{x}{8} + \frac{3x^2}{256} + \frac{5x^3}{6144} + \frac{35x^4}{786432} + \frac{21x^5}{10485760} + \frac{77x^6}{1006632960} \right. \\
&\quad \left. + \frac{143x^7}{56371445760} + O(x^8)\right) + c_2\left(\sqrt{x}\left(1 + \frac{x}{8} + \frac{3x^2}{256} + \frac{5x^3}{6144} + \frac{35x^4}{786432} \right.\right. \\
&\quad \left. + \frac{21x^5}{10485760} + \frac{77x^6}{1006632960} + \frac{143x^7}{56371445760} + O(x^8)\right)\ln(x) + \sqrt{x}\left(-\frac{x^2}{256} \right. \\
&\quad \left.\left. - \frac{x^3}{2048} - \frac{19x^4}{524288} - \frac{25x^5}{12582912} - \frac{317x^6}{3623878656} - \frac{469x^7}{144955146240} + O(x^8)\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\sqrt{x}\left(1+\frac{x}{8}+\frac{3x^2}{256}+\frac{5x^3}{6144}+\frac{35x^4}{786432}+\frac{21x^5}{10485760}+\frac{77x^6}{1006632960}+\frac{143x^7}{56371445760}+O(x^8)\right) \\
&\quad + c_2\left(\sqrt{x}\left(1+\frac{x}{8}+\frac{3x^2}{256}+\frac{5x^3}{6144}+\frac{35x^4}{786432}+\frac{21x^5}{10485760}+\frac{77x^6}{1006632960}+\frac{143x^7}{56371445760}+O(x^8)\right)\ln(x) \right. \\
&\quad \left. +\sqrt{x}\left(-\frac{x^2}{256}-\frac{x^3}{2048}-\frac{19x^4}{524288}-\frac{25x^5}{12582912}-\frac{317x^6}{3623878656}-\frac{469x^7}{144955146240}+O(x^8)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\sqrt{x}\left(1+\frac{x}{8}+\frac{3x^2}{256}+\frac{5x^3}{6144}+\frac{35x^4}{786432}+\frac{21x^5}{10485760}+\frac{77x^6}{1006632960}+\frac{143x^7}{56371445760}+O(x^8)\right) \\
&\quad + c_2\left(\sqrt{x}\left(1+\frac{x}{8}+\frac{3x^2}{256}+\frac{5x^3}{6144}+\frac{35x^4}{786432}+\frac{21x^5}{10485760}+\frac{77x^6}{1006632960}+\frac{143x^7}{56371445760}+O(x^8)\right)\ln(x) \right. \\
&\quad \left. +\sqrt{x}\left(-\frac{x^2}{256}-\frac{x^3}{2048}-\frac{19x^4}{524288}-\frac{25x^5}{12582912}-\frac{317x^6}{3623878656}-\frac{469x^7}{144955146240}+O(x^8)\right)\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1\sqrt{x}\left(1+\frac{x}{8}+\frac{3x^2}{256}+\frac{5x^3}{6144}+\frac{35x^4}{786432}+\frac{21x^5}{10485760}+\frac{77x^6}{1006632960}+\frac{143x^7}{56371445760}+O(x^8)\right) \\
&\quad + c_2\left(\sqrt{x}\left(1+\frac{x}{8}+\frac{3x^2}{256}+\frac{5x^3}{6144}+\frac{35x^4}{786432}+\frac{21x^5}{10485760}+\frac{77x^6}{1006632960}+\frac{143x^7}{56371445760}+O(x^8)\right)\ln(x) \right. \\
&\quad \left. +\sqrt{x}\left(-\frac{x^2}{256}-\frac{x^3}{2048}-\frac{19x^4}{524288}-\frac{25x^5}{12582912}-\frac{317x^6}{3623878656}-\frac{469x^7}{144955146240}+O(x^8)\right)\right)
\end{aligned}
$$

Verified OK.

9.15.1 Maple step by step solution

Let's solve

$4x^2y'' - x^2y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{4} - \frac{y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{4} + \frac{y}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = -\frac{1}{4}, P_3(x) = \frac{1}{4x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 0$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{1}{4}$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' - x^2y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^2 \cdot y'$ to series expansion

  $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r+1}$

- ◦ Shift index using $k- >k - 1$

$$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+2r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)^2 - a_{k-1}(k-1+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = \frac{1}{2}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(2k+2r-1)^2 - a_{k-1}(k-1+r) = 0$
- Shift index using $k- > k+1$

  $a_{k+1}(2k+1+2r)^2 - a_k(k+r) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r)}{(2k+1+2r)^2}$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = \frac{a_k\left(k+\frac{1}{2}\right)}{(2k+2)^2}$
- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k\left(k+\frac{1}{2}\right)}{(2k+2)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 79

```
Order:=8;
dsolve(4*x^2*diff(y(x),x$2)-x^2*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \sqrt{x}\left((c_2\ln(x)+c_1)\left(1+\frac{1}{8}x+\frac{3}{256}x^2+\frac{5}{6144}x^3+\frac{35}{786432}x^4+\frac{21}{10485760}x^5+\frac{77}{1006632960}x^6+\frac{143}{56371445760}x^7+\mathrm{O}\left(x^8\right)\right)+\left(-\frac{1}{256}x^2-\frac{1}{2048}x^3-\frac{19}{524288}x^4-\frac{25}{12582912}x^5-\frac{317}{3623878656}x^6-\frac{469}{144955146240}x^7+\mathrm{O}\left(x^8\right)\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 171

```
AsymptoticDSolveValue[4*x^2*y''[x]-x^2*y'[x]+y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\sqrt{x}\left(\frac{143x^7}{56371445760}+\frac{77x^6}{1006632960}+\frac{21x^5}{10485760}+\frac{35x^4}{786432}+\frac{5x^3}{6144}+\frac{3x^2}{256}+\frac{x}{8}+1\right) + c_2\left(\sqrt{x}\left(-\frac{469x^7}{144955146240}-\frac{317x^6}{3623878656}-\frac{25x^5}{12582912}-\frac{19x^4}{524288}-\frac{x^3}{2048}-\frac{x^2}{256}\right) + \sqrt{x}\left(\frac{143x^7}{56371445760}+\frac{77x^6}{1006632960}+\frac{21x^5}{10485760}+\frac{35x^4}{786432}+\frac{5x^3}{6144}+\frac{3x^2}{256}+\frac{x}{8}+1\right)\log(x)\right)$$

9.16 problem 16

Internal problem ID [7018]
Internal file name [OUTPUT/6261_Thursday_August_18_2022_07_12_10_AM_34273/index.tex]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. Miscellaneous Exercises. page 394
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(x^2+1)\,y''-2y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy'_0+\frac{x^2}{2}\,f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\,f'|_{x_0,y_0,y'_0}+\cdots \\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (263) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (264) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{align}
F_0 &= f(x, y, y') \tag{4}\\
F_1 &= \frac{df}{dx} \nonumber\\
&= \frac{dF_0}{dx} \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \nonumber\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right) \nonumber\\
&= \frac{d}{dx}(F_1) \nonumber\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \nonumber\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0 \nonumber\\
&\vdots \nonumber\\
F_n &= \frac{d}{dx}(F_{n-1}) \nonumber\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \nonumber\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \tag{6}
\end{align}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n \big|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= \frac{2y}{x^2+1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{2x^2 y' - 4yx + 2y'}{\left(x^2+1\right)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{-8x(x^2+1)\,y' + 16x^2 y}{\left(x^2+1\right)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{40((x^2+1)\,y' - 2yx)\left(x^2 - \frac{1}{5}\right)}{\left(x^2+1\right)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -\frac{240((x^2+1)\,y' - 2yx)\left(x^2 - \frac{3}{5}\right)x}{\left(x^2+1\right)^5} \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \frac{1680((x^2+1)\,y' - 2yx)\left(x^4 - \frac{6}{5}x^2 + \frac{3}{35}\right)}{\left(x^2+1\right)^6} \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= -\frac{13440((x^2+1)\,y' - 2yx)\left(x^4 - 2x^2 + \frac{3}{7}\right)x}{\left(x^2+1\right)^7}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 2y(0) \\
F_1 &= 2y'(0) \\
F_2 &= 0 \\
F_3 &= -8y'(0) \\
F_4 &= 0 \\
F_5 &= 144y'(0) \\
F_6 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = (x^2 + 1)\, y(0) + \left(x + \frac{1}{3}x^3 - \frac{1}{15}x^5 + \frac{1}{35}x^7\right) y'(0) + O(x^8)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(x^2 + 1)\, y'' - 2y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
(x^2 + 1) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - 2\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=0}^{\infty} (-2 a_n x^n) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=0}^{\infty} (-2a_n x^n) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - 2a_0 = 0$$

$$a_2 = a_0$$

$n = 1$ gives

$$6a_3 - 2a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_1}{3}$$

For $2 \le n$, the recurrence equation is

$$na_n(n-1) + (n+2)\, a_{n+2}(n+1) - 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{(n-2)\, a_n}{n+2} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 0$$

For $n = 3$ the recurrence equation gives

$$4a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$10a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$18a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{35}$$

For $n = 6$ the recurrence equation gives

$$28a_6 + 56a_8 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = 0$$

For $n = 7$ the recurrence equation gives

$$40a_7 + 72a_9 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = -\frac{a_1}{63}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + a_0 x^2 + \frac{1}{3} a_1 x^3 - \frac{1}{15} a_1 x^5 + \frac{1}{35} a_1 x^7 + \dots$$

Collecting terms, the solution becomes

$$y = (x^2 + 1)\, a_0 + \left(x + \frac{1}{3}x^3 - \frac{1}{15}x^5 + \frac{1}{35}x^7\right) a_1 + O(x^8) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = (x^2 + 1)\, c_1 + \left(x + \frac{1}{3}x^3 - \frac{1}{15}x^5 + \frac{1}{35}x^7\right) c_2 + O(x^8)$$

<u>Summary</u>
The solution(s) found are the following

$$y = (x^2 + 1)\, y(0) + \left(x + \frac{1}{3}x^3 - \frac{1}{15}x^5 + \frac{1}{35}x^7\right) y'(0) + O(x^8) \tag{1}$$

$$y = (x^2 + 1)\, c_1 + \left(x + \frac{1}{3}x^3 - \frac{1}{15}x^5 + \frac{1}{35}x^7\right) c_2 + O(x^8) \tag{2}$$

<u>Verification of solutions</u>

$$y = (x^2 + 1)\, y(0) + \left(x + \frac{1}{3}x^3 - \frac{1}{15}x^5 + \frac{1}{35}x^7\right) y'(0) + O(x^8)$$

Verified OK.

$$y = (x^2 + 1)\, c_1 + \left(x + \frac{1}{3}x^3 - \frac{1}{15}x^5 + \frac{1}{35}x^7\right) c_2 + O(x^8)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 32

```
Order:=8;
dsolve((1+x^2)*diff(y(x),x$2)-2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (x^2+1)\,y(0) + \left(x + \frac{1}{3}x^3 - \frac{1}{15}x^5 + \frac{1}{35}x^7\right) D(y)\,(0) + O(x^8)$$

✓ Solution by Mathematica
Time used: 0.001 (sec). Leaf size: 38

```
AsymptoticDSolveValue[(1+x^2)*y''[x]-2*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1(x^2+1) + c_2\left(\frac{x^7}{35} - \frac{x^5}{15} + \frac{x^3}{3} + x\right)$$

9.17 problem 17

Internal problem ID [7019]
Internal file name [`OUTPUT/6262_Thursday_August_18_2022_07_12_11_AM_55985488/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. Miscellaneous Exercises. page 394
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2x^2y'' - x(2x+1)\,y' + (3x+1)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' + \left(-2x^2 - x\right) y' + (3x+1)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{2x+1}{2x}$$
$$q(x) = \frac{3x+1}{2x^2}$$

Table 219: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{2x+1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{3x+1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' + (-2x^2 - x)\, y' + (3x + 1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2 \left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}\right) + (-2x^2 - x) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (3x + 1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} 3x^{1+n+r} a_n = \sum_{n=1}^{\infty} 3a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 3r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = \frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(2r^2 - 3r + 1)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n + r)(n + r - 1) - 2a_{n-1}(n + r - 1) - a_n(n + r) + 3a_{n-1} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(2n + 2r - 5)}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-1}(2n - 3)}{n\,(2n + 1)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-3 + 2r}{r(2r + 1)}$$

Which for the root $r = 1$ becomes

$$a_1 = -\frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3+2r}{r(2r+1)}$	$-\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4r^2 - 8r + 3}{4r^4 + 12r^3 + 11r^2 + 3r}$$

Which for the root $r = 1$ becomes

$$a_2 = -\frac{1}{30}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3+2r}{r(2r+1)}$	$-\frac{1}{3}$
a_2	$\frac{4r^2-8r+3}{4r^4+12r^3+11r^2+3r}$	$-\frac{1}{30}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{4r^2 - 8r + 3}{4r^5 + 28r^4 + 71r^3 + 77r^2 + 30r}$$

Which for the root $r = 1$ becomes

$$a_3 = -\frac{1}{210}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3+2r}{r(2r+1)}$	$-\frac{1}{3}$
a_2	$\frac{4r^2-8r+3}{4r^4+12r^3+11r^2+3r}$	$-\frac{1}{30}$
a_3	$\frac{4r^2-8r+3}{4r^5+28r^4+71r^3+77r^2+30r}$	$-\frac{1}{210}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4r^2 - 8r + 3}{(2r^2 + 13r + 21)(2r^3 + 11r^2 + 19r + 10)\, r}$$

Which for the root $r = 1$ becomes

$$a_4 = -\frac{1}{1512}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3+2r}{r(2r+1)}$	$-\frac{1}{3}$
a_2	$\frac{4r^2-8r+3}{4r^4+12r^3+11r^2+3r}$	$-\frac{1}{30}$
a_3	$\frac{4r^2-8r+3}{4r^5+28r^4+71r^3+77r^2+30r}$	$-\frac{1}{210}$
a_4	$\frac{4r^2-8r+3}{(2r^2+13r+21)(2r^3+11r^2+19r+10)r}$	$-\frac{1}{1512}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{4r^2 - 8r + 3}{(2r^2 + 17r + 36)\, r\, (r^2 + 3r + 2)(2r^2 + 13r + 21)}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{1}{11880}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3+2r}{r(2r+1)}$	$-\frac{1}{3}$
a_2	$\frac{4r^2-8r+3}{4r^4+12r^3+11r^2+3r}$	$-\frac{1}{30}$
a_3	$\frac{4r^2-8r+3}{4r^5+28r^4+71r^3+77r^2+30r}$	$-\frac{1}{210}$
a_4	$\frac{4r^2-8r+3}{(2r^2+13r+21)(2r^3+11r^2+19r+10)r}$	$-\frac{1}{1512}$
a_5	$\frac{4r^2-8r+3}{(2r^2+17r+36)r(r^2+3r+2)(2r^2+13r+21)}$	$-\frac{1}{11880}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{4r^2 - 8r + 3}{(2r^2 + 21r + 55)(r + 3)(r^2 + 3r + 2) r (2r^2 + 17r + 36)}$$

Which for the root $r = 1$ becomes

$$a_6 = -\frac{1}{102960}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3+2r}{r(2r+1)}$	$-\frac{1}{3}$
a_2	$\frac{4r^2-8r+3}{4r^4+12r^3+11r^2+3r}$	$-\frac{1}{30}$
a_3	$\frac{4r^2-8r+3}{4r^5+28r^4+71r^3+77r^2+30r}$	$-\frac{1}{210}$
a_4	$\frac{4r^2-8r+3}{(2r^2+13r+21)(2r^3+11r^2+19r+10)r}$	$-\frac{1}{1512}$
a_5	$\frac{4r^2-8r+3}{(2r^2+17r+36)r(r^2+3r+2)(2r^2+13r+21)}$	$-\frac{1}{11880}$
a_6	$\frac{4r^2-8r+3}{(2r^2+21r+55)(r+3)(r^2+3r+2)r(2r^2+17r+36)}$	$-\frac{1}{102960}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{4r^2 - 8r + 3}{(2r^2 + 25r + 78)(r + 4) r (r^2 + 3r + 2)(r + 3)(2r^2 + 21r + 55)}$$

Which for the root $r = 1$ becomes

$$a_7 = -\frac{1}{982800}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3+2r}{r(2r+1)}$	$-\frac{1}{3}$
a_2	$\frac{4r^2-8r+3}{4r^4+12r^3+11r^2+3r}$	$-\frac{1}{30}$
a_3	$\frac{4r^2-8r+3}{4r^5+28r^4+71r^3+77r^2+30r}$	$-\frac{1}{210}$
a_4	$\frac{4r^2-8r+3}{(2r^2+13r+21)(2r^3+11r^2+19r+10)r}$	$-\frac{1}{1512}$
a_5	$\frac{4r^2-8r+3}{(2r^2+17r+36)r(r^2+3r+2)(2r^2+13r+21)}$	$-\frac{1}{11880}$
a_6	$\frac{4r^2-8r+3}{(2r^2+21r+55)(r+3)(r^2+3r+2)r(2r^2+17r+36)}$	$-\frac{1}{102960}$
a_7	$\frac{4r^2-8r+3}{(2r^2+25r+78)(r+4)r(r^2+3r+2)(r+3)(2r^2+21r+55)}$	$-\frac{1}{982800}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\ &= x\left(1 - \frac{x}{3} - \frac{x^2}{30} - \frac{x^3}{210} - \frac{x^4}{1512} - \frac{x^5}{11880} - \frac{x^6}{102960} - \frac{x^7}{982800} + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - 2b_{n-1}(n+r-1) - b_n(n+r) + 3b_{n-1} + b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-1}(2n+2r-5)}{2n^2+4nr+2r^2-3n-3r+1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = \frac{2b_{n-1}(n-2)}{n(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{-3 + 2r}{r (2r + 1)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_1 = -2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-3+2r}{r(2r+1)}$	-2

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4r^2 - 8r + 3}{4r^4 + 12r^3 + 11r^2 + 3r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-3+2r}{r(2r+1)}$	-2
b_2	$\frac{4r^2-8r+3}{4r^4+12r^3+11r^2+3r}$	0

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{4r^2 - 8r + 3}{4r^5 + 28r^4 + 71r^3 + 77r^2 + 30r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-3+2r}{r(2r+1)}$	-2
b_2	$\frac{4r^2-8r+3}{4r^4+12r^3+11r^2+3r}$	0
b_3	$\frac{4r^2-8r+3}{4r^5+28r^4+71r^3+77r^2+30r}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{4r^2 - 8r + 3}{(2r^2 + 13r + 21)(2r^3 + 11r^2 + 19r + 10) r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-3+2r}{r(2r+1)}$	-2
b_2	$\frac{4r^2-8r+3}{4r^4+12r^3+11r^2+3r}$	0
b_3	$\frac{4r^2-8r+3}{4r^5+28r^4+71r^3+77r^2+30r}$	0
b_4	$\frac{4r^2-8r+3}{(2r^2+13r+21)(2r^3+11r^2+19r+10)r}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{4r^2 - 8r + 3}{(2r^2 + 17r + 36) r (r^2 + 3r + 2)(2r^2 + 13r + 21)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-3+2r}{r(2r+1)}$	-2
b_2	$\frac{4r^2-8r+3}{4r^4+12r^3+11r^2+3r}$	0
b_3	$\frac{4r^2-8r+3}{4r^5+28r^4+71r^3+77r^2+30r}$	0
b_4	$\frac{4r^2-8r+3}{(2r^2+13r+21)(2r^3+11r^2+19r+10)r}$	0
b_5	$\frac{4r^2-8r+3}{(2r^2+17r+36)r(r^2+3r+2)(2r^2+13r+21)}$	0

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{4r^2 - 8r + 3}{(2r^2 + 21r + 55)(r + 3)(r^2 + 3r + 2) r (2r^2 + 17r + 36)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_6 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-3+2r}{r(2r+1)}$	-2
b_2	$\frac{4r^2-8r+3}{4r^4+12r^3+11r^2+3r}$	0
b_3	$\frac{4r^2-8r+3}{4r^5+28r^4+71r^3+77r^2+30r}$	0
b_4	$\frac{4r^2-8r+3}{(2r^2+13r+21)(2r^3+11r^2+19r+10)r}$	0
b_5	$\frac{4r^2-8r+3}{(2r^2+17r+36)r(r^2+3r+2)(2r^2+13r+21)}$	0
b_6	$\frac{4r^2-8r+3}{(2r^2+21r+55)(r+3)(r^2+3r+2)r(2r^2+17r+36)}$	0

For $n = 7$, using the above recursive equation gives

$$b_7 = \frac{4r^2 - 8r + 3}{(2r^2 + 25r + 78)(r + 4) r (r^2 + 3r + 2)(r + 3)(2r^2 + 21r + 55)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_7 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-3+2r}{r(2r+1)}$	-2
b_2	$\frac{4r^2-8r+3}{4r^4+12r^3+11r^2+3r}$	0
b_3	$\frac{4r^2-8r+3}{4r^5+28r^4+71r^3+77r^2+30r}$	0
b_4	$\frac{4r^2-8r+3}{(2r^2+13r+21)(2r^3+11r^2+19r+10)r}$	0
b_5	$\frac{4r^2-8r+3}{(2r^2+17r+36)r(r^2+3r+2)(2r^2+13r+21)}$	0
b_6	$\frac{4r^2-8r+3}{(2r^2+21r+55)(r+3)(r^2+3r+2)r(2r^2+17r+36)}$	0
b_7	$\frac{4r^2-8r+3}{(2r^2+25r+78)(r+4)r(r^2+3r+2)(r+3)(2r^2+21r+55)}$	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots\right) \\ &= \sqrt{x}\left(1 - 2x + O\left(x^8\right)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1 - \frac{x}{3} - \frac{x^2}{30} - \frac{x^3}{210} - \frac{x^4}{1512} - \frac{x^5}{11880} - \frac{x^6}{102960} - \frac{x^7}{982800} + O\left(x^8\right)\right) \\ &\quad + c_2\sqrt{x}\left(1 - 2x + O\left(x^8\right)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x\left(1 - \frac{x}{3} - \frac{x^2}{30} - \frac{x^3}{210} - \frac{x^4}{1512} - \frac{x^5}{11880} - \frac{x^6}{102960} - \frac{x^7}{982800} + O\left(x^8\right)\right) \\ &\quad + c_2\sqrt{x}\left(1 - 2x + O\left(x^8\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = c_1 x\left(1 - \frac{x}{3} - \frac{x^2}{30} - \frac{x^3}{210} - \frac{x^4}{1512} - \frac{x^5}{11880} - \frac{x^6}{102960} - \frac{x^7}{982800} + O\left(x^8\right)\right) \\ + c_2\sqrt{x}\left(1 - 2x + O\left(x^8\right)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1 x \left(1 - \frac{x}{3} - \frac{x^2}{30} - \frac{x^3}{210} - \frac{x^4}{1512} - \frac{x^5}{11880} - \frac{x^6}{102960} - \frac{x^7}{982800} + O(x^8)\right) + c_2 \sqrt{x}\,(1 - 2x + O(x^8))$$

Verified OK.

9.17.1 Maple step by step solution

Let's solve

$2x^2 y'' + (-2x^2 - x)\, y' + (3x + 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(3x+1)y}{2x^2} + \frac{(2x+1)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x+1)y'}{2x} + \frac{(3x+1)y}{2x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2x+1}{2x}, P_3(x) = \frac{3x+1}{2x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -\frac{1}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{1}{2}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $2x^2 y'' - x(2x + 1)\, y' + (3x + 1)\, y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+2r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)\,(k+r-1) - a_{k-1}(2k-5+2r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+2r)\,(-1+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{1, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$2\left(k+r-\frac{1}{2}\right)(k+r-1)\, a_k - 2a_{k-1}\left(k-\frac{5}{2}+r\right) = 0$$

- Shift index using $k- >k+1$

  $$2\left(k+\frac{1}{2}+r\right)(k+r)\, a_{k+1} - 2a_k\left(k-\frac{3}{2}+r\right) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k(2k+2r-3)}{(2k+1+2r)(k+r)}$$

- Recursion relation for $r = 1$

$a_{k+1} = \frac{a_k(2k-1)}{(2k+3)(k+1)}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = \frac{a_k(2k-1)}{(2k+3)(k+1)}\right]$

- Recursion relation for $r = \frac{1}{2}$; series terminates at $k = 1$

  $a_{k+1} = \frac{a_k(2k-2)}{(2k+2)(k+\frac{1}{2})}$

- Apply recursion relation for $k = 0$

  $a_1 = -2a_0$

- Terminating series solution of the ODE for $r = \frac{1}{2}$. Use reduction of order to find the second li

  $y = a_0 \cdot (1 - 2x)$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + b_0 \cdot (1 - 2x), a_{k+1} = \frac{a_k(2k-1)}{(2k+3)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 41

```
Order:=8;
dsolve(2*x^2*diff(y(x),x$2)-x*(1+2*x)*diff(y(x),x)+(1+3*x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\sqrt{x}\left(1-2x+\mathrm{O}\left(x^8\right)\right)+c_2x\left(1-\frac{1}{3}x-\frac{1}{30}x^2-\frac{1}{210}x^3-\frac{1}{1512}x^4-\frac{1}{11880}x^5-\frac{1}{102960}x^6-\frac{1}{982800}x^7+\mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 70

```
AsymptoticDSolveValue[2*x^2*y''[x]-x*(1+2*x)*y'[x]+(1+3*x)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1 x\left(-\frac{x^7}{982800}-\frac{x^6}{102960}-\frac{x^5}{11880}-\frac{x^4}{1512}-\frac{x^3}{210}-\frac{x^2}{30}-\frac{x}{3}+1\right)+c_2(1-2x)\sqrt{x}$$

9.18 problem 19

Internal problem ID [7020]
Internal file name [`OUTPUT/6263_Thursday_August_18_2022_07_12_13_AM_96052526/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. Miscellaneous Exercises. page 394
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{4x^2y'' + 3x^2y' + (3x + 1)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4x^2y'' + 3x^2y' + (3x + 1)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{3}{4}$$
$$q(x) = \frac{3x + 1}{4x^2}$$

Table 221: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{4}$	
singularity	type

$q(x) = \frac{3x+1}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2y'' + 3x^2y' + (3x + 1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2 \left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}\right) + 3x^2 \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (3x + 1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 3x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 3x^{1+n+r} a_n = \sum_{n=1}^{\infty} 3a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r} a_n (n+r)(n+r-1) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$4x^r a_0 r(-1+r) + a_0 x^r = 0$$

Or

$$\left(4x^r r(-1+r) + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(2r-1)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(2r-1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(2r-1)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = \frac{1}{2}$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+\frac{1}{2}}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \le n$ the recursive equation is

$$4a_n(n+r)(n+r-1) + 3a_{n-1}(n+r-1) + 3a_{n-1} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{3a_{n-1}(n+r)}{4n^2 + 8nr + 4r^2 - 4n - 4r + 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{3a_{n-1}(2n+1)}{8n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-3-3r}{(2r+1)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = -\frac{9}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-3r}{(2r+1)^2}$	$-\frac{9}{8}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{9(1+r)(2+r)}{(2r+1)^2(3+2r)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{135}{256}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-3r}{(2r+1)^2}$	$-\frac{9}{8}$
a_2	$\frac{9(1+r)(2+r)}{(2r+1)^2(3+2r)^2}$	$\frac{135}{256}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{27(3+r)(1+r)(2+r)}{(2r+1)^2(3+2r)^2(5+2r)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = -\frac{315}{2048}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-3r}{(2r+1)^2}$	$-\frac{9}{8}$
a_2	$\frac{9(1+r)(2+r)}{(2r+1)^2(3+2r)^2}$	$\frac{135}{256}$
a_3	$-\frac{27(3+r)(1+r)(2+r)}{(2r+1)^2(3+2r)^2(5+2r)^2}$	$-\frac{315}{2048}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{81(4+r)(3+r)(1+r)(2+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{8505}{262144}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-3r}{(2r+1)^2}$	$-\frac{9}{8}$
a_2	$\frac{9(1+r)(2+r)}{(2r+1)^2(3+2r)^2}$	$\frac{135}{256}$
a_3	$-\frac{27(3+r)(1+r)(2+r)}{(2r+1)^2(3+2r)^2(5+2r)^2}$	$-\frac{315}{2048}$
a_4	$\frac{81(4+r)(3+r)(1+r)(2+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2}$	$\frac{8505}{262144}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{243(5+r)(4+r)(3+r)(1+r)(2+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2(2r+9)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = -\frac{56133}{10485760}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-3r}{(2r+1)^2}$	$-\frac{9}{8}$
a_2	$\frac{9(1+r)(2+r)}{(2r+1)^2(3+2r)^2}$	$\frac{135}{256}$
a_3	$-\frac{27(3+r)(1+r)(2+r)}{(2r+1)^2(3+2r)^2(5+2r)^2}$	$-\frac{315}{2048}$
a_4	$\frac{81(4+r)(3+r)(1+r)(2+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2}$	$\frac{8505}{262144}$
a_5	$-\frac{243(5+r)(4+r)(3+r)(1+r)(2+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2(2r+9)^2}$	$-\frac{56133}{10485760}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{729(5+r)(4+r)(3+r)(1+r)(2+r)(6+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2(2r+9)^2(2r+11)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_6 = \frac{243243}{335544320}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-3r}{(2r+1)^2}$	$-\frac{9}{8}$
a_2	$\frac{9(1+r)(2+r)}{(2r+1)^2(3+2r)^2}$	$\frac{135}{256}$
a_3	$-\frac{27(3+r)(1+r)(2+r)}{(2r+1)^2(3+2r)^2(5+2r)^2}$	$-\frac{315}{2048}$
a_4	$\frac{81(4+r)(3+r)(1+r)(2+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2}$	$\frac{8505}{262144}$
a_5	$-\frac{243(5+r)(4+r)(3+r)(1+r)(2+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2(2r+9)^2}$	$-\frac{56133}{10485760}$
a_6	$\frac{729(5+r)(4+r)(3+r)(1+r)(2+r)(6+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2(2r+9)^2(2r+11)^2}$	$\frac{243243}{335544320}$

For $n = 7$, using the above recursive equation gives

$$a_7 = -\frac{2187(5+r)(4+r)(3+r)(1+r)(2+r)(6+r)(7+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2(2r+9)^2(2r+11)^2(2r+13)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_7 = -\frac{312741}{3758096384}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-3r}{(2r+1)^2}$	$-\frac{9}{8}$
a_2	$\frac{9(1+r)(2+r)}{(2r+1)^2(3+2r)^2}$	$\frac{135}{256}$
a_3	$-\frac{27(3+r)(1+r)(2+r)}{(2r+1)^2(3+2r)^2(5+2r)^2}$	$-\frac{315}{2048}$
a_4	$\frac{81(4+r)(3+r)(1+r)(2+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2}$	$\frac{8505}{262144}$
a_5	$-\frac{243(5+r)(4+r)(3+r)(1+r)(2+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2(2r+9)^2}$	$-\frac{56133}{10485760}$
a_6	$\frac{729(5+r)(4+r)(3+r)(1+r)(2+r)(6+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2(2r+9)^2(2r+11)^2}$	$\frac{243243}{335544320}$
a_7	$-\frac{2187(5+r)(4+r)(3+r)(1+r)(2+r)(6+r)(7+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2(2r+9)^2(2r+11)^2(2r+13)^2}$	$-\frac{312741}{3758096384}$

Using the above table, then the first solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\
&= \sqrt{x}\left(1 - \frac{9x}{8} + \frac{135x^2}{256} - \frac{315x^3}{2048} + \frac{8505x^4}{262144} - \frac{56133x^5}{10485760} + \frac{243243x^6}{335544320} \right. \\
&\qquad \left. - \frac{312741x^7}{3758096384} + O\left(x^8\right)\right)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = \frac{1}{2}$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	$\frac{-3-3r}{(2r+1)^2}$	$-\frac{9}{8}$	$\frac{6r+9}{(2r+1)^3}$
b_2	$\frac{9(1+r)(2+r)}{(2r+1)^2(3+2r)^2}$	$\frac{135}{256}$	$\frac{-72r^3-324r^2-450r-207}{(2r+1)^3(3+2r)^3}$
b_3	$-\frac{27(3+r)(1+r)(2+r)}{(2r+1)^2(3+2r)^2(5+2r)^2}$	$-\frac{315}{2048}$	$\frac{648r^5+6156r^4+22302r^3+38637r^2+32130r+10449}{(2r+1)^3(5+2r)^3(3+2r)^3}$
b_4	$\frac{81(4+r)(3+r)(1+r)(2+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2}$	$\frac{8505}{262144}$	$-\frac{162(32r^7+528r^6+3600r^5+13128r^4+27606r^3+33}{(2r+1)^3(3+2r)^3(5+2r)^3(7+2}$
b_5	$-\frac{243(5+r)(4+r)(3+r)(1+r)(2+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2(2r+9)^2}$	$-\frac{56133}{10485760}$	$\frac{38880r^9+991440r^8+10905840r^7+67797000r^6+2619}{(7+2r}$
b_6	$\frac{729(5+r)(4+r)(3+r)(1+r)(2+r)(6+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2(2r+9)^2(2r+11)^2}$	$\frac{243243}{335544320}$	$-\frac{729(384r^{11}+14016r^{10}+226720r^9+2142000r^8+13}{}$
b_7	$-\frac{2187(5+r)(4+r)(3+r)(1+r)(2+r)(6+r)(7+r)}{(2r+1)^2(3+2r)^2(5+2r)^2(7+2r)^2(2r+9)^2(2r+11)^2(2r+13)^2}$	$-\frac{312741}{3758096384}$	$\frac{1959552r^{13}+96997824r^{12}+2167754400r^{11}+2892372}{}$

The above table gives all values of b_n needed. Hence the second solution is

$$
y_2(x) = y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots
$$

$$
= \sqrt{x}\left(1-\frac{9x}{8}+\frac{135x^{2}}{256}-\frac{315x^{3}}{2048}+\frac{8505x^{4}}{262144}-\frac{56133x^{5}}{10485760}+\frac{243243x^{6}}{335544320}-\frac{312741x^{7}}{3758096384}+O\left(x^{8}\right)\right)\ln\left(x\right)+\sqrt{x}\left(\frac{3x}{2}-\frac{261x^{2}}{256}+\frac{729x^{3}}{2048}-\frac{44091x^{4}}{524288}+\frac{63099x^{5}}{4194304}-\frac{1454463x^{6}}{671088640}+\frac{1403811x^{7}}{5368709120}+O\left(x^{8}\right)\right)
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x)+c_2 y_2(x)\\
&= c_1\sqrt{x}\left(1-\frac{9x}{8}+\frac{135x^{2}}{256}-\frac{315x^{3}}{2048}+\frac{8505x^{4}}{262144}-\frac{56133x^{5}}{10485760}+\frac{243243x^{6}}{335544320}-\frac{312741x^{7}}{3758096384}+O\left(x^{8}\right)\right)+c_2\left(\sqrt{x}\left(1-\frac{9x}{8}+\frac{135x^{2}}{256}-\frac{315x^{3}}{2048}+\frac{8505x^{4}}{262144}-\frac{56133x^{5}}{10485760}+\frac{243243x^{6}}{335544320}-\frac{312741x^{7}}{3758096384}+O\left(x^{8}\right)\right)\ln\left(x\right)+\sqrt{x}\left(\frac{3x}{2}-\frac{261x^{2}}{256}+\frac{729x^{3}}{2048}-\frac{44091x^{4}}{524288}+\frac{63099x^{5}}{4194304}-\frac{1454463x^{6}}{671088640}+\frac{1403811x^{7}}{5368709120}+O\left(x^{8}\right)\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h\\
&= c_1\sqrt{x}\left(1-\frac{9x}{8}+\frac{135x^{2}}{256}-\frac{315x^{3}}{2048}+\frac{8505x^{4}}{262144}-\frac{56133x^{5}}{10485760}+\frac{243243x^{6}}{335544320}-\frac{312741x^{7}}{3758096384}+O\left(x^{8}\right)\right)+c_2\left(\sqrt{x}\left(1-\frac{9x}{8}+\frac{135x^{2}}{256}-\frac{315x^{3}}{2048}+\frac{8505x^{4}}{262144}-\frac{56133x^{5}}{10485760}+\frac{243243x^{6}}{335544320}-\frac{312741x^{7}}{3758096384}+O\left(x^{8}\right)\right)\ln\left(x\right)+\sqrt{x}\left(\frac{3x}{2}-\frac{261x^{2}}{256}+\frac{729x^{3}}{2048}-\frac{44091x^{4}}{524288}+\frac{63099x^{5}}{4194304}-\frac{1454463x^{6}}{671088640}+\frac{1403811x^{7}}{5368709120}+O\left(x^{8}\right)\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1\sqrt{x}\left(1-\frac{9x}{8}+\frac{135x^2}{256}-\frac{315x^3}{2048}+\frac{8505x^4}{262144}-\frac{56133x^5}{10485760}+\frac{243243x^6}{335544320}-\frac{312741x^7}{3758096384}+O(x^8)\right)+c_2\left(\sqrt{x}\left(1-\frac{9x}{8}+\frac{135x^2}{256}-\frac{315x^3}{2048}+\frac{8505x^4}{262144}-\frac{56133x^5}{10485760}+\frac{243243x^6}{335544320}-\frac{312741x^7}{3758096384}+O(x^8)\right)\ln(x)+\sqrt{x}\left(\frac{3x}{2}-\frac{261x^2}{256}+\frac{729x^3}{2048}-\frac{44091x^4}{524288}+\frac{63099x^5}{4194304}-\frac{1454463x^6}{671088640}+\frac{1403811x^7}{5368709120}+O(x^8)\right)\right) \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\left(1-\frac{9x}{8}+\frac{135x^2}{256}-\frac{315x^3}{2048}+\frac{8505x^4}{262144}-\frac{56133x^5}{10485760}+\frac{243243x^6}{335544320}-\frac{312741x^7}{3758096384}+O(x^8)\right)+c_2\left(\sqrt{x}\left(1-\frac{9x}{8}+\frac{135x^2}{256}-\frac{315x^3}{2048}+\frac{8505x^4}{262144}-\frac{56133x^5}{10485760}+\frac{243243x^6}{335544320}-\frac{312741x^7}{3758096384}+O(x^8)\right)\ln(x)+\sqrt{x}\left(\frac{3x}{2}-\frac{261x^2}{256}+\frac{729x^3}{2048}-\frac{44091x^4}{524288}+\frac{63099x^5}{4194304}-\frac{1454463x^6}{671088640}+\frac{1403811x^7}{5368709120}+O(x^8)\right)\right)$$

Verified OK.

9.18.1 Maple step by step solution

Let's solve

$4x^2y''+3x^2y'+(3x+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{3y'}{4}-\frac{(3x+1)y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{3y'}{4}+\frac{(3x+1)y}{4x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{3}{4}, P_3(x)=\frac{3x+1}{4x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 0$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{1}{4}$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$4x^2y'' + 3x^2y' + (3x+1)\,y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^2 \cdot y'$ to series expansion

  $$x^2 \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r+1}$$

- Shift index using $k- >k - 1$

  $$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k-1+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+2r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)^2 + 3a_{k-1}(k+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = \frac{1}{2}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(2k+2r-1)^2 + 3a_{k-1}(k+r) = 0$
- Shift index using $k- > k+1$

  $a_{k+1}(2k+1+2r)^2 + 3a_k(k+r+1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{3a_k(k+r+1)}{(2k+1+2r)^2}$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = -\frac{3a_k\left(k+\frac{3}{2}\right)}{(2k+2)^2}$
- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{3a_k\left(k+\frac{3}{2}\right)}{(2k+2)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 81

```
Order:=8;
dsolve(4*x^2*diff(y(x),x$2)+3*x^2*diff(y(x),x)+(1+3*x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \sqrt{x}\left((c_2 \ln(x) + c_1)\left(1 - \frac{9}{8}x + \frac{135}{256}x^2 - \frac{315}{2048}x^3 + \frac{8505}{262144}x^4 - \frac{56133}{10485760}x^5 + \frac{243243}{335544320}x^6 - \frac{312741}{3758096384}x^7 + \mathrm{O}\left(x^8\right)\right) + \left(\frac{3}{2}x - \frac{261}{256}x^2 + \frac{729}{2048}x^3 - \frac{44091}{524288}x^4 + \frac{63099}{4194304}x^5 - \frac{1454463}{671088640}x^6 + \frac{1403811}{5368709120}x^7 + \mathrm{O}\left(x^8\right)\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 176

```
AsymptoticDSolveValue[4*x^2*y''[x]+3*x^2*y'[x]+(1+3*x)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\sqrt{x}\left(-\frac{312741x^7}{3758096384} + \frac{243243x^6}{335544320} - \frac{56133x^5}{10485760} + \frac{8505x^4}{262144} - \frac{315x^3}{2048} + \frac{135x^2}{256} - \frac{9x}{8} + 1\right) + c_2\left(\sqrt{x}\left(\frac{1403811x^7}{5368709120} - \frac{1454463x^6}{671088640} + \frac{63099x^5}{4194304} - \frac{44091x^4}{524288} + \frac{729x^3}{2048} - \frac{261x^2}{256} + \frac{3x}{2}\right) + \sqrt{x}\left(-\frac{312741x^7}{3758096384} + \frac{243243x^6}{335544320} - \frac{56133x^5}{10485760} + \frac{8505x^4}{262144} - \frac{315x^3}{2048} + \frac{135x^2}{256} - \frac{9x}{8} + 1\right)\log(x)\right)$$

9.19 problem 20

Internal problem ID [7021]
Internal file name [`OUTPUT/6264_Thursday_August_18_2022_07_12_16_AM_39873401/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. Miscellaneous Exercises. page 394
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + (-x^2 + 1) y' + 2yx = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + (-x^2 + 1) y' + 2yx = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x^2 - 1}{x}$$
$$q(x) = 2$$

Table 223: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x^2-1}{x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = 2$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + (-x^2 + 1)\, y' + 2yx = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (-x^2+1)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) x = 0 \tag{1}$$

Which simplifies to

$$
\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n\right) = 0 \tag{2A}
$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) = \sum_{n=2}^{\infty} \left(-a_{n-2}(n+r-2) x^{n+r-1}\right)
$$

$$
\sum_{n=0}^{\infty} 2x^{1+n+r} a_n = \sum_{n=2}^{\infty} 2a_{n-2} x^{n+r-1}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$
\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=2}^{\infty} \left(-a_{n-2}(n+r-2) x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} 2a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0
$$

When $n=0$ the above becomes

$$
x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0
$$

Or

$$
\left(x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
x^{-1+r} r^2 = 0
$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-2}(n+r-2) + a_n(n+r) + 2a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}(n+r-4)}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-2}(n-4)}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-2+r}{(r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = -\frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2+r}{(r+2)^2}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2+r}{(r+2)^2}$	$-\frac{1}{2}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(-2+r)\, r}{(r+2)^2\,(r+4)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2+r}{(r+2)^2}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{(-2+r)r}{(r+2)^2(r+4)^2}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2+r}{(r+2)^2}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{(-2+r)r}{(r+2)^2(r+4)^2}$	0
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{(-2+r)\, r}{(r+2)\,(r+4)^2\,(r+6)^2}$$

Which for the root $r = 0$ becomes

$$a_6 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2+r}{(r+2)^2}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{(-2+r)r}{(r+2)^2(r+4)^2}$	0
a_5	0	0
a_6	$\frac{(-2+r)r}{(r+2)(r+4)^2(r+6)^2}$	0

For $n = 7$, using the above recursive equation gives

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2+r}{(r+2)^2}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{(-2+r)r}{(r+2)^2(r+4)^2}$	0
a_5	0	0
a_6	$\frac{(-2+r)r}{(r+2)(r+4)^2(r+6)^2}$	0
a_7	0	0

Using the above table, then the first solution $y_1(x)$ becomes

$$\begin{aligned}
y_1(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \ldots \\
&= 1 - \frac{x^2}{2} + O(x^8)
\end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r=0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$\frac{-2+r}{(r+2)^2}$	$-\frac{1}{2}$	$\frac{-r+6}{(r+2)^3}$	$\frac{3}{4}$
b_3	0	0	0	0
b_4	$\frac{(-2+r)r}{(r+2)^2(r+4)^2}$	0	$\frac{-2r^3+6r^2+28r-16}{(r+2)^3(r+4)^3}$	$-\frac{1}{32}$
b_5	0	0	0	0
b_6	$\frac{(-2+r)r}{(r+2)(r+4)^2(r+6)^2}$	0	$\frac{-3r^4-6r^3+76r^2+136r-96}{(r+2)^2(r+4)^3(r+6)^3}$	$-\frac{1}{576}$
b_7	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned} y_2(x) &= y_1(x) \ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \ldots \\ &= \left(1 - \frac{x^2}{2} + O(x^8)\right) \ln(x) + \frac{3x^2}{4} - \frac{x^4}{32} - \frac{x^6}{576} + O(x^8) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \left(1 - \frac{x^2}{2} + O(x^8)\right) + c_2 \left(\left(1 - \frac{x^2}{2} + O(x^8)\right) \ln(x) + \frac{3x^2}{4} - \frac{x^4}{32} - \frac{x^6}{576} + O(x^8)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 \left(1 - \frac{x^2}{2} + O(x^8)\right) + c_2 \left(\left(1 - \frac{x^2}{2} + O(x^8)\right) \ln(x) + \frac{3x^2}{4} - \frac{x^4}{32} - \frac{x^6}{576} + O(x^8)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1\left(1-\frac{x^2}{2}+O(x^8)\right)+c_2\left(\left(1-\frac{x^2}{2}+O(x^8)\right)\ln(x)+\frac{3x^2}{4}-\frac{x^4}{32}-\frac{x^6}{576}+O(x^8)\right) \tag{1}$$

Verification of solutions

$$y = c_1\left(1-\frac{x^2}{2}+O(x^8)\right)+c_2\left(\left(1-\frac{x^2}{2}+O(x^8)\right)\ln(x)+\frac{3x^2}{4}-\frac{x^4}{32}-\frac{x^6}{576}+O(x^8)\right)$$

Verified OK.

9.19.1 Maple step by step solution

Let's solve

$xy''+(-x^2+1)\,y'+2yx=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(x^2-1)y'}{x}-2y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(x^2-1)y'}{x}+2y=0$

□ Check to see if $x_0=0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=-\frac{x^2-1}{x}, P_3(x)=2\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\Big|_{x=0}=1$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\Big|_{x=0}=0$

- ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $xy'' + (-x^2+1)\, y' + 2yx = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

  - Shift index using $k->k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

  - Shift index using $k->k+1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r^2 x^{-1+r} + a_1 (1+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)^2 - a_{k-1}(k-3+r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 0$

- Each term must be 0

  $a_1 (1+r)^2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 - a_{k-1}(k-3) = 0$

- Shift index using $k- > k+1$

  $a_{k+2}(k+2)^2 - a_k(k-2) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k(k-2)}{(k+2)^2}$

- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+2} = \frac{a_k(k-2)}{(k+2)^2}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a_k(k-2)}{(k+2)^2}, a_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 39

```
Order:=8;
dsolve(x*diff(y(x),x$2)+(1-x^2)*diff(y(x),x)+2*x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln(x) + c_1)\left(1 - \frac{1}{2}x^2 + \mathrm{O}\left(x^8\right)\right) + \left(\frac{3}{4}x^2 - \frac{1}{32}x^4 - \frac{1}{576}x^6 + \mathrm{O}\left(x^8\right)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 53

```
AsymptoticDSolveValue[x*y''[x]+(1-x^2)*y'[x]+2*x*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(1-\frac{x^2}{2}\right)+c_2\left(-\frac{x^6}{576}-\frac{x^4}{32}+\frac{3x^2}{4}+\left(1-\frac{x^2}{2}\right)\log(x)\right)$$

9.20 problem 21

Internal problem ID [7022]
Internal file name [`OUTPUT/6265_Thursday_August_18_2022_07_12_18_AM_4049348/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. Miscellaneous Exercises. page 394
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{4x^2y'' + 2x^2y' - (x+3)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4x^2y'' + 2x^2y' + (-x-3)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{2}$$
$$q(x) = -\frac{x+3}{4x^2}$$

Table 225: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{2}$	
singularity	type

$q(x) = -\frac{x+3}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2y'' + 2x^2y' + (-x-3)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + 2x^2\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (-x-3)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n(n+r)\right) \\
+ \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-3a_n x^{n+r}\right) = 0 \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} 2x^{1+n+r} a_n(n+r) = \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r}
$$

$$
\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right)
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r}\right) \\
+ \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-3a_n x^{n+r}\right) = 0 \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
4x^{n+r} a_n(n+r)(n+r-1) - 3a_n x^{n+r} = 0
$$

When $n=0$ the above becomes

$$
4x^r a_0 r(-1+r) - 3a_0 x^r = 0
$$

Or

$$
\left(4x^r r(-1+r) - 3x^r\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
\left(4r^2 - 4r - 3\right) x^r = 0
$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - 4r - 3 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{3}{2}$$
$$r_2 = -\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(4r^2 - 4r - 3\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^{\frac{3}{2}} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{\sqrt{x}}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$4a_n(n+r)(n+r-1) + 2a_{n-1}(n+r-1) - a_{n-1} - 3a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{2n + 2r + 1} \tag{4}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_n = -\frac{a_{n-1}}{2n + 4} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{3 + 2r}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_1 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{3+2r}$	$-\frac{1}{6}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{4r^2 + 16r + 15}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_2 = \frac{1}{48}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{3+2r}$	$-\frac{1}{6}$
a_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{48}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{8r^3 + 60r^2 + 142r + 105}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_3 = -\frac{1}{480}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{3+2r}$	$-\frac{1}{6}$
a_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{48}$
a_3	$-\frac{1}{8r^3+60r^2+142r+105}$	$-\frac{1}{480}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{16r^4 + 192r^3 + 824r^2 + 1488r + 945}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_4 = \frac{1}{5760}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{3+2r}$	$-\frac{1}{6}$
a_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{48}$
a_3	$-\frac{1}{8r^3+60r^2+142r+105}$	$-\frac{1}{480}$
a_4	$\frac{1}{16r^4+192r^3+824r^2+1488r+945}$	$\frac{1}{5760}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{(8r^3 + 68r^2 + 174r + 135)(11 + 2r)(7 + 2r)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_5 = -\frac{1}{80640}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{3+2r}$	$-\frac{1}{6}$
a_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{48}$
a_3	$-\frac{1}{8r^3+60r^2+142r+105}$	$-\frac{1}{480}$
a_4	$\frac{1}{16r^4+192r^3+824r^2+1488r+945}$	$\frac{1}{5760}$
a_5	$-\frac{1}{(8r^3+68r^2+174r+135)(11+2r)(7+2r)}$	$-\frac{1}{80640}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{1}{(7+2r)(11+2r)(4r^2+16r+15)(9+2r)(13+2r)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_6 = \frac{1}{1290240}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{3+2r}$	$-\frac{1}{6}$
a_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{48}$
a_3	$-\frac{1}{8r^3+60r^2+142r+105}$	$-\frac{1}{480}$
a_4	$\frac{1}{16r^4+192r^3+824r^2+1488r+945}$	$\frac{1}{5760}$
a_5	$-\frac{1}{(8r^3+68r^2+174r+135)(11+2r)(7+2r)}$	$-\frac{1}{80640}$
a_6	$\frac{1}{(7+2r)(11+2r)(4r^2+16r+15)(9+2r)(13+2r)}$	$\frac{1}{1290240}$

For $n = 7$, using the above recursive equation gives

$$a_7 = -\frac{1}{(13+2r)(9+2r)(4r^2+16r+15)(7+2r)(11+2r)(15+2r)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_7 = -\frac{1}{23224320}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{3+2r}$	$-\frac{1}{6}$
a_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{48}$
a_3	$-\frac{1}{8r^3+60r^2+142r+105}$	$-\frac{1}{480}$
a_4	$\frac{1}{16r^4+192r^3+824r^2+1488r+945}$	$\frac{1}{5760}$
a_5	$-\frac{1}{(8r^3+68r^2+174r+135)(11+2r)(7+2r)}$	$-\frac{1}{80640}$
a_6	$\frac{1}{(7+2r)(11+2r)(4r^2+16r+15)(9+2r)(13+2r)}$	$\frac{1}{1290240}$
a_7	$-\frac{1}{(13+2r)(9+2r)(4r^2+16r+15)(7+2r)(11+2r)(15+2r)}$	$-\frac{1}{23224320}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{3}{2}}(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots) \\ &= x^{\frac{3}{2}}\left(1 - \frac{x}{6} + \frac{x^2}{48} - \frac{x^3}{480} + \frac{x^4}{5760} - \frac{x^5}{80640} + \frac{x^6}{1290240} - \frac{x^7}{23224320} + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_2 \\ &= \frac{1}{4r^2 + 16r + 15} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{1}{4r^2 + 16r + 15} &= \lim_{r \to -\frac{1}{2}} \frac{1}{4r^2 + 16r + 15} \\ &= \frac{1}{8} \end{aligned}$$

The limit is $\frac{1}{8}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
4b_n(n+r)(n+r-1) + 2b_{n-1}(n+r-1) - b_{n-1} - 3b_n = 0 \tag{4}
$$

Which for for the root $r = -\frac{1}{2}$ becomes

$$
4b_n\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right) + 2b_{n-1}\left(n-\frac{3}{2}\right) - b_{n-1} - 3b_n = 0 \tag{4A}
$$

Solving for b_n from the recursive equation (4) gives

$$
b_n = -\frac{b_{n-1}}{2n+2r+1} \tag{5}
$$

Which for the root $r = -\frac{1}{2}$ becomes

$$
b_n = -\frac{b_{n-1}}{2n} \tag{6}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$
b_1 = -\frac{1}{3+2r}
$$

Which for the root $r = -\frac{1}{2}$ becomes

$$
b_1 = -\frac{1}{2}
$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{3+2r}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{(3+2r)(5+2r)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_2 = \frac{1}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{3+2r}$	$-\frac{1}{2}$
b_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{8}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{1}{(5+2r)(7+2r)(3+2r)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_3 = -\frac{1}{48}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{3+2r}$	$-\frac{1}{2}$
b_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{8}$
b_3	$-\frac{1}{8r^3+60r^2+142r+105}$	$-\frac{1}{48}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(7+2r)(3+2r)(5+2r)(9+2r)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = \frac{1}{384}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{3+2r}$	$-\frac{1}{2}$
b_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{8}$
b_3	$-\frac{1}{8r^3+60r^2+142r+105}$	$-\frac{1}{48}$
b_4	$\frac{1}{16r^4+192r^3+824r^2+1488r+945}$	$\frac{1}{384}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{1}{(9+2r)(3+2r)(5+2r)(11+2r)(7+2r)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_5 = -\frac{1}{3840}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{3+2r}$	$-\frac{1}{2}$
b_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{8}$
b_3	$-\frac{1}{8r^3+60r^2+142r+105}$	$-\frac{1}{48}$
b_4	$\frac{1}{16r^4+192r^3+824r^2+1488r+945}$	$\frac{1}{384}$
b_5	$-\frac{1}{(9+2r)(3+2r)(5+2r)(11+2r)(7+2r)}$	$-\frac{1}{3840}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{1}{(7+2r)(11+2r)(3+2r)(5+2r)(9+2r)(13+2r)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_6 = \frac{1}{46080}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{3+2r}$	$-\frac{1}{2}$
b_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{8}$
b_3	$-\frac{1}{8r^3+60r^2+142r+105}$	$-\frac{1}{48}$
b_4	$\frac{1}{16r^4+192r^3+824r^2+1488r+945}$	$\frac{1}{384}$
b_5	$-\frac{1}{(9+2r)(3+2r)(5+2r)(11+2r)(7+2r)}$	$-\frac{1}{3840}$
b_6	$\frac{1}{(7+2r)(11+2r)(3+2r)(5+2r)(9+2r)(13+2r)}$	$\frac{1}{46080}$

For $n = 7$, using the above recursive equation gives

$$b_7 = -\frac{1}{(13+2r)(7+2r)(3+2r)(5+2r)(9+2r)(11+2r)(15+2r)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_7 = -\frac{1}{645120}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{3+2r}$	$-\frac{1}{2}$
b_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{8}$
b_3	$-\frac{1}{8r^3+60r^2+142r+105}$	$-\frac{1}{48}$
b_4	$\frac{1}{16r^4+192r^3+824r^2+1488r+945}$	$\frac{1}{384}$
b_5	$-\frac{1}{(9+2r)(3+2r)(5+2r)(11+2r)(7+2r)}$	$-\frac{1}{3840}$
b_6	$\frac{1}{(7+2r)(11+2r)(3+2r)(5+2r)(9+2r)(13+2r)}$	$\frac{1}{46080}$
b_7	$-\frac{1}{(13+2r)(7+2r)(3+2r)(5+2r)(9+2r)(11+2r)(15+2r)}$	$-\frac{1}{645120}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^{\frac{3}{2}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots\right) \\
&= \frac{1 - \frac{x}{2} + \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{384} - \frac{x^5}{3840} + \frac{x^6}{46080} - \frac{x^7}{645120} + O(x^8)}{\sqrt{x}}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{3}{2}}\left(1 - \frac{x}{6} + \frac{x^2}{48} - \frac{x^3}{480} + \frac{x^4}{5760} - \frac{x^5}{80640} + \frac{x^6}{1290240} - \frac{x^7}{23224320} + O(x^8)\right) \\
&\quad + \frac{c_2\left(1 - \frac{x}{2} + \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{384} - \frac{x^5}{3840} + \frac{x^6}{46080} - \frac{x^7}{645120} + O(x^8)\right)}{\sqrt{x}}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{3}{2}}\left(1 - \frac{x}{6} + \frac{x^2}{48} - \frac{x^3}{480} + \frac{x^4}{5760} - \frac{x^5}{80640} + \frac{x^6}{1290240} - \frac{x^7}{23224320} + O(x^8)\right) \\
&\quad + \frac{c_2\left(1 - \frac{x}{2} + \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{384} - \frac{x^5}{3840} + \frac{x^6}{46080} - \frac{x^7}{645120} + O(x^8)\right)}{\sqrt{x}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{3}{2}} \left(1-\frac{x}{6}+\frac{x^2}{48}-\frac{x^3}{480}+\frac{x^4}{5760}-\frac{x^5}{80640}+\frac{x^6}{1290240}-\frac{x^7}{23224320}+O\left(x^8\right)\right) + \frac{c_2 \left(1-\frac{x}{2}+\frac{x^2}{8}-\frac{x^3}{48}+\frac{x^4}{384}-\frac{x^5}{3840}+\frac{x^6}{46080}-\frac{x^7}{645120}+O\left(x^8\right)\right)}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{3}{2}} \left(1-\frac{x}{6}+\frac{x^2}{48}-\frac{x^3}{480}+\frac{x^4}{5760}-\frac{x^5}{80640}+\frac{x^6}{1290240}-\frac{x^7}{23224320}+O\left(x^8\right)\right) + \frac{c_2 \left(1-\frac{x}{2}+\frac{x^2}{8}-\frac{x^3}{48}+\frac{x^4}{384}-\frac{x^5}{3840}+\frac{x^6}{46080}-\frac{x^7}{645120}+O\left(x^8\right)\right)}{\sqrt{x}}$$

Verified OK.

9.20.1 Maple step by step solution

Let's solve

$4x^2y''+2x^2y'+(-x-3)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y'}{2}+\frac{(x+3)y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y'}{2}-\frac{(x+3)y}{4x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{1}{2}, P_3(x)=-\frac{x+3}{4x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=0$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=-\frac{3}{4}$

- $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4x^2y''+2x^2y'+(-x-3)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m=0..1$

    $x^m \cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^2 \cdot y'$ to series expansion

    $x^2 \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r+1}$

  - Shift index using $k->k-1$

    $x^2 \cdot y'=\sum_{k=1}^{\infty} a_{k-1}(k-1+r)\,x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-3+2r)\,x^r+\left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-3)+a_{k-1}(2k+2r-3)\right) x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-3+2r)=0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

$$(2k+2r-3)(2ka_k+2ra_k+a_k+a_{k-1})=0$$

- Shift index using $k->k+1$

$$(2k+2r-1)(2(k+1)a_{k+1}+2ra_{k+1}+a_{k+1}+a_k)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=-\frac{a_k}{2k+3+2r}$$

- Recursion relation for $r=-\frac{1}{2}$

$$a_{k+1}=-\frac{a_k}{2k+2}$$

- Solution for $r=-\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1}=-\frac{a_k}{2k+2}\right]$$

- Recursion relation for $r=\frac{3}{2}$

$$a_{k+1}=-\frac{a_k}{2k+6}$$

- Solution for $r=\frac{3}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1}=-\frac{a_k}{2k+6}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+1}=-\frac{a_k}{2k+2}, b_{k+1}=-\frac{b_k}{2k+6}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 55

```
Order:=8;
dsolve(4*x^2*diff(y(x),x$2)+2*x^2*diff(y(x),x)-(x+3)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x^2 \left(1 - \frac{1}{6}x + \frac{1}{48}x^2 - \frac{1}{480}x^3 + \frac{1}{5760}x^4 - \frac{1}{80640}x^5 + \frac{1}{1290240}x^6 - \frac{1}{23224320}x^7 + \mathrm{O}\left(x^8\right)\right) + c_2\left(-2 + x - \frac{1}{4}x^2 + \right.}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.106 (sec). Leaf size: 130

```
AsymptoticDSolveValue[4*x^2*y''[x]+2*x^2*y'[x]-(x+3)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(\frac{x^{11/2}}{46080} - \frac{x^{9/2}}{3840} + \frac{x^{7/2}}{384} - \frac{x^{5/2}}{48} + \frac{x^{3/2}}{8} - \frac{\sqrt{x}}{2} + \frac{1}{\sqrt{x}}\right) + c_2\left(\frac{x^{15/2}}{1290240} - \frac{x^{13/2}}{80640} + \frac{x^{11/2}}{5760} - \frac{x^{9/2}}{480} + \frac{x^{7/2}}{48} - \frac{x^{5/2}}{6} + x^{3/2}\right)$$

9.21 problem 22

Internal problem ID [7023]
Internal file name [OUTPUT/6266_Thursday_August_18_2022_07_12_20_AM_33879010/index.tex]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. Miscellaneous Exercises. page 394
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x(-x^2+1)\, y''+5(-x^2+1)\, y'-4yx=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(-x^3+x)\, y''+(-5x^2+5)\, y'-4yx=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{5}{x}$$
$$q(x)=\frac{4}{x^2-1}$$

Table 227: Table $p(x), q(x)$ singularites.

$p(x) = \frac{5}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{4}{x^2-1}$	
singularity	type
$x = -1$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -1, 1, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''x(x^2-1) + (-5x^2+5)\, y' - 4yx = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x(x^2-1) + (-5x^2+5)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - 4\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) x = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-5x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 5(n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-4x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)(n+r-1)\right) = \sum_{n=2}^{\infty} \left(-a_{n-2}(n+r-2)(n-3+r)\, x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-5x^{1+n+r} a_n (n+r)\right) = \sum_{n=2}^{\infty} \left(-5a_{n-2}(n+r-2)\, x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-4x^{1+n+r} a_n\right) = \sum_{n=2}^{\infty} \left(-4a_{n-2} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\sum_{n=2}^{\infty} \left(-a_{n-2}(n+r-2)(n-3+r)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=2}^{\infty} \left(-5a_{n-2}(n+r-2)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 5(n+r) a_n x^{n+r-1}\right) + \sum_{n=2}^{\infty} \left(-4a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + 5(n+r) a_n x^{n+r-1} = 0$$

When $n = 0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + 5r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + 5r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(4+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(4+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 0 \\ r_2 &= -4 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(4+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 4$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\ y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^4} \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-4}\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} -a_{n-2}(n+r-2)(n-3+r) + a_n(n+r)(n+r-1) \\ -5a_{n-2}(n+r-2) + 5a_n(n+r) - 4a_{n-2} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{(n+r)\,a_{n-2}}{n+4+r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{n a_{n-2}}{n+4} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{2+r}{6+r}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2+r}{6+r}$	$\frac{1}{3}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2+r}{6+r}$	$\frac{1}{3}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^2 + 6r + 8}{(6 + r)(8 + r)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2+r}{6+r}$	$\frac{1}{3}$
a_3	0	0
a_4	$\frac{r^2+6r+8}{(6+r)(8+r)}$	$\frac{1}{6}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2+r}{6+r}$	$\frac{1}{3}$
a_3	0	0
a_4	$\frac{r^2+6r+8}{(6+r)(8+r)}$	$\frac{1}{6}$
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{r^2 + 6r + 8}{(8 + r)(10 + r)}$$

Which for the root $r = 0$ becomes

$$a_6 = \frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2+r}{6+r}$	$\frac{1}{3}$
a_3	0	0
a_4	$\frac{r^2+6r+8}{(6+r)(8+r)}$	$\frac{1}{6}$
a_5	0	0
a_6	$\frac{r^2+6r+8}{(8+r)(10+r)}$	$\frac{1}{10}$

For $n = 7$, using the above recursive equation gives

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2+r}{6+r}$	$\frac{1}{3}$
a_3	0	0
a_4	$\frac{r^2+6r+8}{(6+r)(8+r)}$	$\frac{1}{6}$
a_5	0	0
a_6	$\frac{r^2+6r+8}{(8+r)(10+r)}$	$\frac{1}{10}$
a_7	0	0

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots \\
&= 1 + \frac{x^2}{3} + \frac{x^4}{6} + \frac{x^6}{10} + O(x^8)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_4 \\
&= \frac{r^2 + 6r + 8}{(6+r)(8+r)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r \to r_2} \frac{r^2 + 6r + 8}{(6+r)(8+r)} &= \lim_{r \to -4} \frac{r^2 + 6r + 8}{(6+r)(8+r)} \\
&= 0
\end{aligned}
$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-4}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &-b_{n-2}(n+r-2)(n-3+r)+b_n(n+r)(n+r-1) \\ &-5b_{n-2}(n+r-2)+5(n+r)b_n-4b_{n-2}=0 \end{aligned} \tag{4}$$

Which for for the root $r = -4$ becomes

$$-b_{n-2}(n-6)(n-7)+b_n(n-4)(n-5)-5b_{n-2}(n-6)+5(n-4)b_n-4b_{n-2}=0 \quad \text{(4A)}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = \frac{(n+r)\,b_{n-2}}{n+4+r} \tag{5}$$

Which for the root $r = -4$ becomes

$$b_n = \frac{(n-4)\,b_{n-2}}{n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -4$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{2+r}{6+r}$$

Which for the root $r = -4$ becomes

$$b_2 = -1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{2+r}{6+r}$	-1

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{2+r}{6+r}$	-1
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r^2 + 6r + 8}{(6 + r)(8 + r)}$$

Which for the root $r = -4$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{2+r}{6+r}$	-1
b_3	0	0
b_4	$\frac{r^2+6r+8}{(6+r)(8+r)}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{2+r}{6+r}$	-1
b_3	0	0
b_4	$\frac{r^2+6r+8}{(6+r)(8+r)}$	0
b_5	0	0

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{r^2 + 6r + 8}{(8 + r)(10 + r)}$$

Which for the root $r = -4$ becomes

$$b_6 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{2+r}{6+r}$	-1
b_3	0	0
b_4	$\frac{r^2+6r+8}{(6+r)(8+r)}$	0
b_5	0	0
b_6	$\frac{r^2+6r+8}{(8+r)(10+r)}$	0

For $n = 7$, using the above recursive equation gives

$$b_7 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{2+r}{6+r}$	-1
b_3	0	0
b_4	$\frac{r^2+6r+8}{(6+r)(8+r)}$	0
b_5	0	0
b_6	$\frac{r^2+6r+8}{(8+r)(10+r)}$	0
b_7	0	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots\right) \\
&= \frac{1 - x^2 + O(x^8)}{x^4}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1 + \frac{x^2}{3} + \frac{x^4}{6} + \frac{x^6}{10} + O(x^8)\right) + \frac{c_2(1 - x^2 + O(x^8))}{x^4}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1 + \frac{x^2}{3} + \frac{x^4}{6} + \frac{x^6}{10} + O(x^8)\right) + \frac{c_2(1 - x^2 + O(x^8))}{x^4}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\left(1 + \frac{x^2}{3} + \frac{x^4}{6} + \frac{x^6}{10} + O(x^8)\right) + \frac{c_2(1 - x^2 + O(x^8))}{x^4} \tag{1}
$$

Verification of solutions

$$
y = c_1\left(1 + \frac{x^2}{3} + \frac{x^4}{6} + \frac{x^6}{10} + O(x^8)\right) + \frac{c_2(1 - x^2 + O(x^8))}{x^4}
$$

Verified OK.

9.21.1 Maple step by step solution

Let's solve

$-y''x(x^2-1)+(-5x^2+5)\,y'-4yx=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{4y}{x^2-1}-\frac{5y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{5y'}{x}+\frac{4y}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{5}{x}, P_3(x)=\frac{4}{x^2-1}\right]$

  - $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1}=0$

  - $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  - $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $y''x(x^2-1)+(5x^2-5)\,y'+4yx=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^3-3u^2+2u)\left(\frac{d^2}{du^2}y(u)\right)+(5u^2-10u)\left(\frac{d}{du}y(u)\right)+(4u-4)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

$$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- >k-m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$2a_0 r(-1+r)\, u^{-1+r} + (2a_1(1+r)\, r - a_0(4+3r)\,(1+r))\, u^r + \left(\sum_{k=1}^{\infty} (2a_{k+1}(k+r+1)\,(k+r) - a$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$2r(-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 1\}$$

- Each term must be 0

$$2a_1(1+r)\, r - a_0(4+3r)\,(1+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$-3\left(\left(a_k - \frac{a_{k-1}}{3} - \frac{2a_{k+1}}{3}\right) k + \left(a_k - \frac{a_{k-1}}{3} - \frac{2a_{k+1}}{3}\right) r + \frac{4a_k}{3} - \frac{a_{k-1}}{3}\right)(k+r+1) = 0$$

- Shift index using $k- >k+1$

$$-3\left(\left(a_{k+1} - \frac{a_k}{3} - \frac{2a_{k+2}}{3}\right)(k+1) + \left(a_{k+1} - \frac{a_k}{3} - \frac{2a_{k+2}}{3}\right) r + \frac{4a_{k+1}}{3} - \frac{a_k}{3}\right)(k+r+2) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k a_k - 3k a_{k+1} + r a_k - 3r a_{k+1} + 2a_k - 7a_{k+1}}{2(k+r+1)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k a_k - 3k a_{k+1} + 2a_k - 7a_{k+1}}{2(k+1)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k a_k - 3k a_{k+1} + 2a_k - 7a_{k+1}}{2(k+1)}, -4a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^k, a_{k+2} = -\frac{k a_k - 3k a_{k+1} + 2a_k - 7a_{k+1}}{2(k+1)}, -4a_0 = 0\right]$$

- Recursion relation for $r = 1$

$$a_{k+2} = -\frac{k a_k - 3k a_{k+1} + 3a_k - 10a_{k+1}}{2(k+2)}$$

- Solution for $r = 1$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+1}, a_{k+2} = -\frac{k a_k - 3k a_{k+1} + 3a_k - 10a_{k+1}}{2(k+2)}, 4a_1 - 14a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+1}, a_{k+2} = -\frac{k a_k - 3k a_{k+1} + 3a_k - 10a_{k+1}}{2(k+2)}, 4a_1 - 14a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (1+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k+1}\right), a_{k+2} = -\frac{k a_k - 3k a_{k+1} + 2a_k - 7a_{k+1}}{2(k+1)}, -4a_0 = 0, b_{k+2} = -\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 32

```
Order:=8;
dsolve(x*(1-x^2)*diff(y(x),x$2)+5*(1-x^2)*diff(y(x),x)-4*x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 \left(1 + \frac{1}{3}x^2 + \frac{1}{6}x^4 + \frac{1}{10}x^6 + \mathrm{O}\left(x^8\right)\right) + \frac{c_2(-144 + 144x^2 + \mathrm{O}\left(x^8\right))}{x^4}$$

✓ Solution by Mathematica
Time used: 0.024 (sec). Leaf size: 42

```
AsymptoticDSolveValue[x*(1-x^2)*y''[x]+5*(1-x^2)*y'[x]-4*x*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1 \left(\frac{1}{x^4} - \frac{1}{x^2}\right) + c_2 \left(\frac{x^6}{10} + \frac{x^4}{6} + \frac{x^2}{3} + 1\right)$$

9.22 problem 23

Internal problem ID [7024]
Internal file name [`OUTPUT/6267_Thursday_August_18_2022_07_12_22_AM_12870851/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. Miscellaneous Exercises. page 394
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x^2y'' + x(x+3)\,y' + (2x+1)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(x^2 + 3x\right)y' + (2x+1)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x+3}{x}$$
$$q(x) = \frac{2x+1}{x^2}$$

Table 229: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x+3}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{2x+1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + (x^2 + 3x)\, y' + (2x + 1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}\right) + (x^2 + 3x) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (2x + 1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1} (n+r-1) x^{n+r}
$$

$$
\sum_{n=0}^{\infty} 2x^{1+n+r} a_n = \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} (n+r-1) x^{n+r}\right) \\
&+ \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r} a_n (n+r)(n+r-1) + 3x^{n+r} a_n (n+r) + a_n x^{n+r} = 0
$$

When $n=0$ the above becomes

$$
x^r a_0 r(-1+r) + 3x^r a_0 r + a_0 x^r = 0
$$

Or

$$
\left(x^r r(-1+r) + 3x^r r + x^r\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
(r+1)^2 x^r = 0
$$

Since the above is true for all x then the indicial equation becomes

$$(r+1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -1$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r+1)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = -1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-1}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n-1}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_{n-1}(n+r-1) + 3a_n(n+r) + 2a_{n-1} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{1+n+r} \tag{4}$$

Which for the root $r = -1$ becomes

$$a_n = -\frac{a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{2+r}$$

Which for the root $r = -1$ becomes

$$a_1 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2+r}$	-1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(2+r)(3+r)}$$

Which for the root $r = -1$ becomes

$$a_2 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2+r}$	-1
a_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{(4+r)(2+r)(3+r)}$$

Which for the root $r = -1$ becomes

$$a_3 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2+r}$	-1
a_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{2}$
a_3	$-\frac{1}{(4+r)(2+r)(3+r)}$	$-\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(4+r)(2+r)(3+r)(5+r)}$$

Which for the root $r = -1$ becomes

$$a_4 = \frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2+r}$	-1
a_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{2}$
a_3	$-\frac{1}{(4+r)(2+r)(3+r)}$	$-\frac{1}{6}$
a_4	$\frac{1}{(4+r)(2+r)(3+r)(5+r)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{(4+r)(2+r)(3+r)(5+r)(6+r)}$$

Which for the root $r = -1$ becomes

$$a_5 = -\frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2+r}$	-1
a_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{2}$
a_3	$-\frac{1}{(4+r)(2+r)(3+r)}$	$-\frac{1}{6}$
a_4	$\frac{1}{(4+r)(2+r)(3+r)(5+r)}$	$\frac{1}{24}$
a_5	$-\frac{1}{(4+r)(2+r)(3+r)(5+r)(6+r)}$	$-\frac{1}{120}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{1}{(4+r)(2+r)(3+r)(5+r)(6+r)(7+r)}$$

Which for the root $r = -1$ becomes

$$a_6 = \frac{1}{720}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2+r}$	-1
a_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{2}$
a_3	$-\frac{1}{(4+r)(2+r)(3+r)}$	$-\frac{1}{6}$
a_4	$\frac{1}{(4+r)(2+r)(3+r)(5+r)}$	$\frac{1}{24}$
a_5	$-\frac{1}{(4+r)(2+r)(3+r)(5+r)(6+r)}$	$-\frac{1}{120}$
a_6	$\frac{1}{(4+r)(2+r)(3+r)(5+r)(6+r)(7+r)}$	$\frac{1}{720}$

For $n = 7$, using the above recursive equation gives

$$a_7 = -\frac{1}{(8+r)(4+r)(2+r)(3+r)(5+r)(6+r)(7+r)}$$

Which for the root $r = -1$ becomes

$$a_7 = -\frac{1}{5040}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2+r}$	-1
a_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{2}$
a_3	$-\frac{1}{(4+r)(2+r)(3+r)}$	$-\frac{1}{6}$
a_4	$\frac{1}{(4+r)(2+r)(3+r)(5+r)}$	$\frac{1}{24}$
a_5	$-\frac{1}{(4+r)(2+r)(3+r)(5+r)(6+r)}$	$-\frac{1}{120}$
a_6	$\frac{1}{(4+r)(2+r)(3+r)(5+r)(6+r)(7+r)}$	$\frac{1}{720}$
a_7	$-\frac{1}{(8+r)(4+r)(2+r)(3+r)(5+r)(6+r)(7+r)}$	$-\frac{1}{5040}$

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \frac{1}{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\ &= \frac{1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \frac{x^6}{720} - \frac{x^7}{5040} + O(x^8)}{x} \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = -1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r=-1)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$-\frac{1}{2+r}$	-1	$\frac{1}{(2+r)^2}$	1
b_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{2}$	$\frac{-5-2r}{(2+r)^2(3+r)^2}$	$-\frac{3}{4}$
b_3	$-\frac{1}{(4+r)(2+r)(3+r)}$	$-\frac{1}{6}$	$\frac{3r^2+18r+26}{(4+r)^2(2+r)^2(3+r)^2}$	$\frac{11}{36}$
b_4	$\frac{1}{(4+r)(2+r)(3+r)(5+r)}$	$\frac{1}{24}$	$\frac{-4r^3-42r^2-142r-154}{(4+r)^2(2+r)^2(3+r)^2(5+r)^2}$	$-\frac{25}{288}$
b_5	$-\frac{1}{(4+r)(2+r)(3+r)(5+r)(6+r)}$	$-\frac{1}{120}$	$\frac{5r^4+80r^3+465r^2+1160r+1044}{(4+r)^2(2+r)^2(3+r)^2(5+r)^2(6+r)^2}$	$\frac{137}{7200}$
b_6	$\frac{1}{(4+r)(2+r)(3+r)(5+r)(6+r)(7+r)}$	$\frac{1}{720}$	$\frac{-6r^5-135r^4-1180r^3-4995r^2-10208r-8028}{(4+r)^2(2+r)^2(3+r)^2(5+r)^2(6+r)^2(7+r)^2}$	$-\frac{49}{14400}$
b_7	$-\frac{1}{(8+r)(4+r)(2+r)(3+r)(5+r)(6+r)(7+r)}$	$-\frac{1}{5040}$	$\frac{7r^6+210r^5+2555r^4+16100r^3+55272r^2+97720r+69264}{(8+r)^2(4+r)^2(2+r)^2(3+r)^2(5+r)^2(6+r)^2(7+r)^2}$	$\frac{121}{235200}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\
&= \frac{\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \frac{x^6}{720} - \frac{x^7}{5040} + O(x^8)\right)\ln(x)}{x} \\
&\quad + \frac{x - \frac{3x^2}{4} + \frac{11x^3}{36} - \frac{25x^4}{288} + \frac{137x^5}{7200} - \frac{49x^6}{14400} + \frac{121x^7}{235200} + O(x^8)}{x}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= \frac{c_1\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \frac{x^6}{720} - \frac{x^7}{5040} + O(x^8)\right)}{x} \\
&\quad + c_2\left(\frac{\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \frac{x^6}{720} - \frac{x^7}{5040} + O(x^8)\right)\ln(x)}{x}\right. \\
&\quad \left. + \frac{x - \frac{3x^2}{4} + \frac{11x^3}{36} - \frac{25x^4}{288} + \frac{137x^5}{7200} - \frac{49x^6}{14400} + \frac{121x^7}{235200} + O(x^8)}{x}\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= \frac{c_1\left(1-x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}-\frac{x^5}{120}+\frac{x^6}{720}-\frac{x^7}{5040}+O(x^8)\right)}{x} \\
&\quad + c_2\left(\frac{\left(1-x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}-\frac{x^5}{120}+\frac{x^6}{720}-\frac{x^7}{5040}+O(x^8)\right)\ln(x)}{x}\right. \\
&\quad \left. + \frac{x-\frac{3x^2}{4}+\frac{11x^3}{36}-\frac{25x^4}{288}+\frac{137x^5}{7200}-\frac{49x^6}{14400}+\frac{121x^7}{235200}+O(x^8)}{x}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= \frac{c_1\left(1-x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}-\frac{x^5}{120}+\frac{x^6}{720}-\frac{x^7}{5040}+O(x^8)\right)}{x} \\
&\quad + c_2\left(\frac{\left(1-x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}-\frac{x^5}{120}+\frac{x^6}{720}-\frac{x^7}{5040}+O(x^8)\right)\ln(x)}{x}\right. \\
&\quad \left. + \frac{x-\frac{3x^2}{4}+\frac{11x^3}{36}-\frac{25x^4}{288}+\frac{137x^5}{7200}-\frac{49x^6}{14400}+\frac{121x^7}{235200}+O(x^8)}{x}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= \frac{c_1\left(1-x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}-\frac{x^5}{120}+\frac{x^6}{720}-\frac{x^7}{5040}+O(x^8)\right)}{x} \\
&\quad + c_2\left(\frac{\left(1-x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}-\frac{x^5}{120}+\frac{x^6}{720}-\frac{x^7}{5040}+O(x^8)\right)\ln(x)}{x}\right. \\
&\quad \left. + \frac{x-\frac{3x^2}{4}+\frac{11x^3}{36}-\frac{25x^4}{288}+\frac{137x^5}{7200}-\frac{49x^6}{14400}+\frac{121x^7}{235200}+O(x^8)}{x}\right)
\end{aligned}
$$

Verified OK.

9.22.1 Maple step by step solution

Let's solve

$x^2y'' + (x^2 + 3x)\, y' + (2x + 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x+3)y'}{x} - \frac{(2x+1)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x+3)y'}{x} + \frac{(2x+1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{x+3}{x}, P_3(x) = \frac{2x+1}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 3$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 1$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x(x + 3)\, y' + (2x + 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) \, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r) \, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r) \, (k+r-1) \, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 (1+r)^2 \, x^r + \left(\sum_{k=1}^{\infty} \left(a_k (k+r+1)^2 + a_{k-1} (k+r+1) \right) x^{k+r} \right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = -1$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r+1) \left(a_k (k+r+1) + a_{k-1} \right) = 0$$

- Shift index using $k->k+1$

$$(k+r+2) \left(a_{k+1} (k+r+2) + a_k \right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k}{k+r+2}$$

- Recursion relation for $r = -1$

$$a_{k+1} = -\frac{a_k}{k+1}$$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = -\frac{a_k}{k+1} \right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 81

```
Order:=8;
dsolve(x^2*diff(y(x),x$2)+x*(3+x)*diff(y(x),x)+(1+2*x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{(c_2 \ln(x) + c_1)\left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5 + \frac{1}{720}x^6 - \frac{1}{5040}x^7 + \mathrm{O}(x^8)\right) + \left(x - \frac{3}{4}x^2 + \frac{11}{36}x^3 - \frac{25}{288}\right.}{x}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 162

```
AsymptoticDSolveValue[x^2*y''[x]+x*(3+x)*y'[x]+(1+2*x)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to \frac{c_1\left(-\frac{x^7}{5040} + \frac{x^6}{720} - \frac{x^5}{120} + \frac{x^4}{24} - \frac{x^3}{6} + \frac{x^2}{2} - x + 1\right)}{x} + c_2\left(\frac{\frac{121x^7}{235200} - \frac{49x^6}{14400} + \frac{137x^5}{7200} - \frac{25x^4}{288} + \frac{11x^3}{36} - \frac{3x^2}{4} + x}{x} + \frac{\left(-\frac{x^7}{5040} + \frac{x^6}{720} - \frac{x^5}{120} + \frac{x^4}{24} - \frac{x^3}{6} + \frac{x^2}{2} - x + 1\right)\log(x)}{x}\right)$$

9.23 problem 24

Internal problem ID [7025]
Internal file name [`OUTPUT/6268_Thursday_August_18_2022_07_12_24_AM_26965091/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. Miscellaneous Exercises. page 394
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_Bessel, _modified]]
```

$$\boxed{x^2y'' + xy' - \left(x^2+4\right)y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + xy' + \left(-x^2-4\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -\frac{x^2+4}{x^2}$$

Table 231: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{x^2+4}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + x y' + \left(-x^2 - 4\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + x \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(-x^2 - 4\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\right) \\ +\sum_{n=0}^{\infty}\left(-x^{n+r+2} a_n\right)+\sum_{n=0}^{\infty}\left(-4 a_n x^{n+r}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-x^{n+r+2} a_n\right)=\sum_{n=2}^{\infty}\left(-a_{n-2} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\right) \\ +\sum_{n=2}^{\infty}\left(-a_{n-2} x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-4 a_n x^{n+r}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n(n+r)(n+r-1)+x^{n+r} a_n(n+r)-4 a_n x^{n+r}=0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r)+x^r a_0 r-4 a_0 x^r=0$$

Or

$$\left(x^r r(-1+r)+x^r r-4 x^r\right) a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2-4\right) x^r=0$$

Since the above is true for all x then the indicial equation becomes

$$r^2-4=0$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= 2 \\
r_2 &= -2
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
(r^2 - 4)\, x^r = 0
$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 4$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2}
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+2} \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
a_n(n+r)(n+r-1) + a_n(n+r) - a_{n-2} - 4a_n = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}}{n^2 + 2nr + r^2 - 4} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = \frac{a_{n-2}}{n(n+4)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{r(r+4)}$$

Which for the root $r = 2$ becomes

$$a_2 = \frac{1}{12}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{r(r+4)}$	$\frac{1}{12}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{r(r+4)}$	$\frac{1}{12}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{r(r+4)(r+6)(r+2)}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{1}{384}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{r(r+4)}$	$\frac{1}{12}$
a_3	0	0
a_4	$\frac{1}{r(r+4)(r+6)(r+2)}$	$\frac{1}{384}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{r(r+4)}$	$\frac{1}{12}$
a_3	0	0
a_4	$\frac{1}{r(r+4)(r+6)(r+2)}$	$\frac{1}{384}$
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{1}{r(r+4)^2(r+6)(r+2)(8+r)}$$

Which for the root $r = 2$ becomes

$$a_6 = \frac{1}{23040}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{r(r+4)}$	$\frac{1}{12}$
a_3	0	0
a_4	$\frac{1}{r(r+4)(r+6)(r+2)}$	$\frac{1}{384}$
a_5	0	0
a_6	$\frac{1}{r(r+4)^2(r+6)(r+2)(8+r)}$	$\frac{1}{23040}$

For $n = 7$, using the above recursive equation gives

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{r(r+4)}$	$\frac{1}{12}$
a_3	0	0
a_4	$\frac{1}{r(r+4)(r+6)(r+2)}$	$\frac{1}{384}$
a_5	0	0
a_6	$\frac{1}{r(r+4)^2(r+6)(r+2)(8+r)}$	$\frac{1}{23040}$
a_7	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^2\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \ldots\right) \\ &= x^2\left(1 + \frac{x^2}{12} + \frac{x^4}{384} + \frac{x^6}{23040} + O\left(x^8\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\rightarrow r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_4 \\
&= \frac{1}{r(r+4)(r+6)(r+2)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r\rightarrow r_2} \frac{1}{r(r+4)(r+6)(r+2)} &= \lim_{r\rightarrow -2} \frac{1}{r(r+4)(r+6)(r+2)} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)
$$

Therefore

$$
\begin{aligned}
\frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\
\frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $x^2y'' + xy' + (-x^2 - 4)\, y = 0$ gives

$$x^2\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) + x\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) + (-x^2-4)\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0$$

Which can be written as

$$\left(\left(x^2y_1''(x) + y_1'(x)\, x + (-x^2-4)\, y_1(x)\right)\ln(x) + x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + y_1(x)\right)C + x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) + x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + (-x^2-4)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$x^2y_1''(x) + y_1'(x)\, x + (-x^2-4)\, y_1(x) = 0$$

Eq (7) simplifes to

$$\left(x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + y_1(x)\right)C + x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) + x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + (-x^2-4)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$
\begin{aligned}
&2x\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n(n+r_1)\right) C - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) x^2 \\
&+ \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)\,(-1+n+r_2)\right) x^2 \\
&+ \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) x - 4\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{9}
$$

Since $r_1 = 2$ and $r_2 = -2$ then the above becomes

$$
\begin{aligned}
&2x\left(\sum_{n=0}^{\infty} x^{1+n} a_n(n+2)\right) C - \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) x^2 \\
&+ \left(\sum_{n=0}^{\infty} x^{-4+n} b_n(n-2)\,(-3+n)\right) x^2 \\
&+ \left(\sum_{n=0}^{\infty} x^{-3+n} b_n(n-2)\right) x - 4\left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) = 0
\end{aligned} \tag{10}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{n+2} a_n(n+2)\right) + \sum_{n=0}^{\infty} (-b_n x^n) + \left(\sum_{n=0}^{\infty} x^{n-2} b_n\left(n^2-5n+6\right)\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n-2} b_n(n-2)\right) + \sum_{n=0}^{\infty} \left(-4 b_n x^{n-2}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n-2$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-2} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} 2C\, x^{n+2} a_n(n+2) = \sum_{n=4}^{\infty} 2C a_{-4+n}(n-2)\, x^{n-2}
$$

$$
\sum_{n=0}^{\infty} (-b_n x^n) = \sum_{n=2}^{\infty} \left(-b_{n-2} x^{n-2}\right)
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-2$.

$$\left(\sum_{n=4}^{\infty} 2Ca_{-4+n}(n-2)\, x^{n-2}\right)+\sum_{n=2}^{\infty}\left(-b_{n-2}x^{n-2}\right)+\left(\sum_{n=0}^{\infty} x^{n-2}b_n\left(n^2-5n+6\right)\right) \\ +\left(\sum_{n=0}^{\infty} x^{n-2}b_n(n-2)\right)+\sum_{n=0}^{\infty}\left(-4b_n x^{n-2}\right)=0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$-3b_1=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_1=0$$

Solving the above for b_1 gives

$$b_1=0$$

For $n=2$, Eq (2B) gives

$$-b_0-4b_2=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-1-4b_2=0$$

Solving the above for b_2 gives

$$b_2=-\frac{1}{4}$$

For $n=3$, Eq (2B) gives

$$-b_1-3b_3=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_3=0$$

Solving the above for b_3 gives

$$b_3=0$$

For $n=N$, where $N=4$ which is the difference between the two roots, we are free to choose $b_4=0$. Hence for $n=4$, Eq (2B) gives

$$4C+\frac{1}{4}=0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{16}$$

For $n = 5$, Eq (2B) gives

$$6Ca_1 - b_3 + 5b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$5b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

For $n = 6$, Eq (2B) gives

$$8Ca_2 - b_4 + 12b_6 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$12b_6 - \frac{1}{24} = 0$$

Solving the above for b_6 gives

$$b_6 = \frac{1}{288}$$

For $n = 7$, Eq (2B) gives

$$10Ca_3 - b_5 + 21b_7 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$21b_7 = 0$$

Solving the above for b_7 gives

$$b_7 = 0$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{16}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{1}{16}\left(x^2\left(1 + \frac{x^2}{12} + \frac{x^4}{384} + \frac{x^6}{23040} + O(x^8)\right)\right)\ln(x) + \frac{1 - \frac{x^2}{4} + \frac{x^6}{288} + O(x^8)}{x^2}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^2 \left(1 + \frac{x^2}{12} + \frac{x^4}{384} + \frac{x^6}{23040} + O\left(x^8\right)\right) \\
&\quad + c_2 \left(-\frac{1}{16}\left(x^2\left(1 + \frac{x^2}{12} + \frac{x^4}{384} + \frac{x^6}{23040} + O\left(x^8\right)\right)\right)\ln(x) + \frac{1 - \frac{x^2}{4} + \frac{x^6}{288} + O(x^8)}{x^2}\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x^2 \left(1 + \frac{x^2}{12} + \frac{x^4}{384} + \frac{x^6}{23040} + O\left(x^8\right)\right) \\
&\quad + c_2 \left(-\frac{x^2\left(1 + \frac{x^2}{12} + \frac{x^4}{384} + \frac{x^6}{23040} + O(x^8)\right)\ln(x)}{16} + \frac{1 - \frac{x^2}{4} + \frac{x^6}{288} + O(x^8)}{x^2}\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= c_1 x^2 \left(1 + \frac{x^2}{12} + \frac{x^4}{384} + \frac{x^6}{23040} + O\left(x^8\right)\right) \\
&\quad + c_2 \left(-\frac{x^2\left(1 + \frac{x^2}{12} + \frac{x^4}{384} + \frac{x^6}{23040} + O(x^8)\right)\ln(x)}{16} + \frac{1 - \frac{x^2}{4} + \frac{x^6}{288} + O(x^8)}{x^2}\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}
y &= c_1 x^2 \left(1 + \frac{x^2}{12} + \frac{x^4}{384} + \frac{x^6}{23040} + O\left(x^8\right)\right) \\
&\quad + c_2 \left(-\frac{x^2\left(1 + \frac{x^2}{12} + \frac{x^4}{384} + \frac{x^6}{23040} + O(x^8)\right)\ln(x)}{16} + \frac{1 - \frac{x^2}{4} + \frac{x^6}{288} + O(x^8)}{x^2}\right)
\end{aligned}$$

Verified OK.

9.23.1 Maple step by step solution

Let's solve

$x^2y'' + xy' + (-x^2 - 4)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x^2+4)y}{x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} - \frac{(x^2+4)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = -\frac{x^2+4}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -4$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + xy' + (-x^2 - 4)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)\,(-2+r)\, x^r+a_1(3+r)\,(-1+r)\, x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k(k+r+2)\,(k+r-2)-a_{k-2}\right) x^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{-2,2\}$

- Each term must be 0

  $a_1(3+r)\,(-1+r)=0$

- Solve for the dependent coefficient(s)

  $a_1=0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)\,(k+r-2)-a_{k-2}=0$

- Shift index using $k->k+2$

  $a_{k+2}(k+4+r)\,(k+r)-a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{a_k}{(k+4+r)(k+r)}$

- Recursion relation for $r=-2$

  $a_{k+2}=\frac{a_k}{(k+2)(k-2)}$

- Series not valid for $r=-2$, division by 0 in the recursion relation at $k=2$

  $a_{k+2}=\frac{a_k}{(k+2)(k-2)}$

- Recursion relation for $r=2$

$a_{k+2} = \frac{a_k}{(k+6)(k+2)}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{a_k}{(k+6)(k+2)}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 53

```
Order:=8;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)-(x^2+4)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x^4 \left(1 + \frac{1}{12}x^2 + \frac{1}{384}x^4 + \frac{1}{23040}x^6 + O\left(x^8\right)\right) + c_2 \left(\ln\left(x\right)\left(9x^4 + \frac{3}{4}x^6 + O\left(x^8\right)\right) + \left(-144 + 36x^2 - \frac{1}{2}x^6 + O\right.\right.}{x^2}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 74

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]-(x^2+4)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1 \left(\frac{11x^6 + 36x^4 - 576x^2 + 2304}{2304x^2} - \frac{1}{192} x^2 (x^2 + 12) \log(x) \right) + c_2 \left(\frac{x^8}{23040} + \frac{x^6}{384} + \frac{x^4}{12} + x^2 \right)$$

9.24 problem 25

Internal problem ID [7026]
Internal file name [`OUTPUT/6269_Thursday_August_18_2022_07_12_27_AM_68378690/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. Miscellaneous Exercises. page 394
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x(1-2x)\,y''-2(x+2)\,y'+18y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-2x^2+x\right)y''+(-2x-4)\,y'+18y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{2x+4}{x\,(2x-1)}$$
$$q(x)=-\frac{18}{x\,(2x-1)}$$

Table 233: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2x+4}{x(2x-1)}$	
singularity	type
$x = 0$	"regular"
$x = \frac{1}{2}$	"regular"

$q(x) = -\frac{18}{x(2x-1)}$	
singularity	type
$x = 0$	"regular"
$x = \frac{1}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, \frac{1}{2}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''x(2x-1) + (-2x-4)\,y' + 18y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) x(2x-1) + (-2x-4)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + 18\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty}\left(-2x^{n+r}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}x^{n+r-1}a_n(n+r)(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-2x^{n+r}a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-4(n+r)a_nx^{n+r-1}\right)+\left(\sum_{n=0}^{\infty}18a_nx^{n+r}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-2x^{n+r}a_n(n+r)(n+r-1)\right)=\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)(n+r-2)x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty}\left(-2x^{n+r}a_n(n+r)\right)=\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty}18a_nx^{n+r}=\sum_{n=1}^{\infty}18a_{n-1}x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)(n+r-2)x^{n+r-1}\right)+\left(\sum_{n=0}^{\infty}x^{n+r-1}a_n(n+r)(n+r-1)\right)+\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)x^{n+r-1}\right)+\sum_{n=0}^{\infty}\left(-4(n+r)a_nx^{n+r-1}\right)+\left(\sum_{n=1}^{\infty}18a_{n-1}x^{n+r-1}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1}a_n(n+r)(n+r-1)-4(n+r)a_nx^{n+r-1}=0$$

When $n = 0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) - 4r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) - 4r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-5+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-5+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 5$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(-5+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 5$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^5\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+5}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &-2a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) \\ &- 2a_{n-1}(n+r-1) - 4a_n(n+r) + 18a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2a_{n-1}(n^2 + 2nr + r^2 - 2n - 2r - 8)}{n^2 + 2nr + r^2 - 5n - 5r} \tag{4}$$

Which for the root $r = 5$ becomes

$$a_n = \frac{2a_{n-1}(n^2 + 8n + 7)}{n(n+5)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 5$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{2r^2 - 18}{r^2 - 3r - 4}$$

Which for the root $r = 5$ becomes

$$a_1 = \frac{16}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-18}{r^2-3r-4}$	$\frac{16}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4r^3 + 20r^2 - 8r - 96}{r^3 - r^2 - 10r - 8}$$

Which for the root $r = 5$ becomes

$$a_2 = \frac{144}{7}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-18}{r^2-3r-4}$	$\frac{16}{3}$
a_2	$\frac{4r^3+20r^2-8r-96}{r^3-r^2-10r-8}$	$\frac{144}{7}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{8r^3 + 64r^2 + 88r - 160}{r^3 - r^2 - 10r - 8}$$

Which for the root $r = 5$ becomes

$$a_3 = \frac{480}{7}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-18}{r^2-3r-4}$	$\frac{16}{3}$
a_2	$\frac{4r^3+20r^2-8r-96}{r^3-r^2-10r-8}$	$\frac{144}{7}$
a_3	$\frac{8r^3+64r^2+88r-160}{r^3-r^2-10r-8}$	$\frac{480}{7}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16(r + 6)\, r(r + 5)}{r^3 - r^2 - 10r - 8}$$

Which for the root $r = 5$ becomes

$$a_4 = \frac{4400}{21}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-18}{r^2-3r-4}$	$\frac{16}{3}$
a_2	$\frac{4r^3+20r^2-8r-96}{r^3-r^2-10r-8}$	$\frac{144}{7}$
a_3	$\frac{8r^3+64r^2+88r-160}{r^3-r^2-10r-8}$	$\frac{480}{7}$
a_4	$\frac{16(r+6)r(r+5)}{r^3-r^2-10r-8}$	$\frac{4400}{21}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{32(r+6)(r+7)}{r^2-2r-8}$$

Which for the root $r = 5$ becomes

$$a_5 = \frac{4224}{7}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-18}{r^2-3r-4}$	$\frac{16}{3}$
a_2	$\frac{4r^3+20r^2-8r-96}{r^3-r^2-10r-8}$	$\frac{144}{7}$
a_3	$\frac{8r^3+64r^2+88r-160}{r^3-r^2-10r-8}$	$\frac{480}{7}$
a_4	$\frac{16(r+6)r(r+5)}{r^3-r^2-10r-8}$	$\frac{4400}{21}$
a_5	$\frac{32(r+6)(r+7)}{r^2-2r-8}$	$\frac{4224}{7}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{64(r+7)(r+8)}{(r+1)(r-4)}$$

Which for the root $r = 5$ becomes

$$a_6 = 1664$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-18}{r^2-3r-4}$	$\frac{16}{3}$
a_2	$\frac{4r^3+20r^2-8r-96}{r^3-r^2-10r-8}$	$\frac{144}{7}$
a_3	$\frac{8r^3+64r^2+88r-160}{r^3-r^2-10r-8}$	$\frac{480}{7}$
a_4	$\frac{16(r+6)r(r+5)}{r^3-r^2-10r-8}$	$\frac{4400}{21}$
a_5	$\frac{32(r+6)(r+7)}{r^2-2r-8}$	$\frac{4224}{7}$
a_6	$\frac{64(r+7)(r+8)}{(r+1)(r-4)}$	1664

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{128(r+9)(r+3)(r+8)}{(r+2)(r+1)(r-4)}$$

Which for the root $r = 5$ becomes

$$a_7 = \frac{13312}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-18}{r^2-3r-4}$	$\frac{16}{3}$
a_2	$\frac{4r^3+20r^2-8r-96}{r^3-r^2-10r-8}$	$\frac{144}{7}$
a_3	$\frac{8r^3+64r^2+88r-160}{r^3-r^2-10r-8}$	$\frac{480}{7}$
a_4	$\frac{16(r+6)r(r+5)}{r^3-r^2-10r-8}$	$\frac{4400}{21}$
a_5	$\frac{32(r+6)(r+7)}{r^2-2r-8}$	$\frac{4224}{7}$
a_6	$\frac{64(r+7)(r+8)}{(r+1)(r-4)}$	1664
a_7	$\frac{128(r+9)(r+3)(r+8)}{(r+2)(r+1)(r-4)}$	$\frac{13312}{3}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^5\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \dots\right) \\ &= x^5\left(1 + \frac{16x}{3} + \frac{144x^2}{7} + \frac{480x^3}{7} + \frac{4400x^4}{21} + \frac{4224x^5}{7} + 1664x^6 + \frac{13312x^7}{3} + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 5$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_5(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_5 \\ &= \frac{32(r+6)(r+7)}{r^2 - 2r - 8} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{32(r+6)(r+7)}{r^2 - 2r - 8} &= \lim_{r \to 0} \frac{32(r+6)(r+7)}{r^2 - 2r - 8} \\ &= -168 \end{aligned}$$

The limit is -168. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^n \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &-2b_{n-1}(n+r-1)(n+r-2) + b_n(n+r)(n+r-1) \\ &- 2b_{n-1}(n+r-1) - 4(n+r)b_n + 18b_{n-1} = 0 \end{aligned} \tag{4}$$

Which for for the root $r = 0$ becomes

$$-2b_{n-1}(n-1)(n-2) + b_n n(n-1) - 2b_{n-1}(n-1) - 4nb_n + 18b_{n-1} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = \frac{2b_{n-1}(n^2 + 2nr + r^2 - 2n - 2r - 8)}{n^2 + 2nr + r^2 - 5n - 5r} \tag{5}$$

Which for the root $r = 0$ becomes

$$b_n = \frac{2b_{n-1}(n^2 - 2n - 8)}{n^2 - 5n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{2r^2 - 18}{r^2 - 3r - 4}$$

Which for the root $r = 0$ becomes

$$b_1 = \frac{9}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2r^2-18}{r^2-3r-4}$	$\frac{9}{2}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4r^3 + 20r^2 - 8r - 96}{(r + 2)(r^2 - 3r - 4)}$$

Which for the root $r = 0$ becomes

$$b_2 = 12$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2r^2-18}{r^2-3r-4}$	$\frac{9}{2}$
b_2	$\frac{4r^3+20r^2-8r-96}{r^3-r^2-10r-8}$	12

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{8r^3 + 64r^2 + 88r - 160}{(r+2)(r^2 - 3r - 4)}$$

Which for the root $r = 0$ becomes

$$b_3 = 20$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2r^2-18}{r^2-3r-4}$	$\frac{9}{2}$
b_2	$\frac{4r^3+20r^2-8r-96}{r^3-r^2-10r-8}$	12
b_3	$\frac{8r^3+64r^2+88r-160}{r^3-r^2-10r-8}$	20

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16(r+6)\,r(r+5)}{(r+2)(r^2 - 3r - 4)}$$

Which for the root $r = 0$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2r^2-18}{r^2-3r-4}$	$\frac{9}{2}$
b_2	$\frac{4r^3+20r^2-8r-96}{r^3-r^2-10r-8}$	12
b_3	$\frac{8r^3+64r^2+88r-160}{r^3-r^2-10r-8}$	20
b_4	$\frac{16(r+6)r(r+5)}{r^3-r^2-10r-8}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{32(r+6)(r+7)}{(r-4)(r+2)}$$

Which for the root $r = 0$ becomes

$$b_5 = -168$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2r^2-18}{r^2-3r-4}$	$\frac{9}{2}$
b_2	$\frac{4r^3+20r^2-8r-96}{r^3-r^2-10r-8}$	12
b_3	$\frac{8r^3+64r^2+88r-160}{r^3-r^2-10r-8}$	20
b_4	$\frac{16(r+6)r(r+5)}{r^3-r^2-10r-8}$	0
b_5	$\frac{32(r+6)(r+7)}{(r-4)(r+2)}$	-168

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{64(r+7)(r+8)}{(r+1)(r-4)}$$

Which for the root $r = 0$ becomes

$$b_6 = -896$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2r^2-18}{r^2-3r-4}$	$\frac{9}{2}$
b_2	$\frac{4r^3+20r^2-8r-96}{r^3-r^2-10r-8}$	12
b_3	$\frac{8r^3+64r^2+88r-160}{r^3-r^2-10r-8}$	20
b_4	$\frac{16(r+6)r(r+5)}{r^3-r^2-10r-8}$	0
b_5	$\frac{32(r+6)(r+7)}{(r-4)(r+2)}$	-168
b_6	$\frac{64(r+7)(r+8)}{(r+1)(r-4)}$	-896

For $n = 7$, using the above recursive equation gives

$$b_7 = \frac{128(r^2+12r+27)(r+8)}{(r+2)(r+1)(r-4)}$$

Which for the root $r = 0$ becomes

$$b_7 = -3456$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2r^2-18}{r^2-3r-4}$	$\frac{9}{2}$
b_2	$\frac{4r^3+20r^2-8r-96}{r^3-r^2-10r-8}$	12
b_3	$\frac{8r^3+64r^2+88r-160}{r^3-r^2-10r-8}$	20
b_4	$\frac{16(r+6)r(r+5)}{r^3-r^2-10r-8}$	0
b_5	$\frac{32(r+6)(r+7)}{(r-4)(r+2)}$	-168
b_6	$\frac{64(r+7)(r+8)}{(r+1)(r-4)}$	-896
b_7	$\frac{128(r+9)(r+3)(r+8)}{(r+2)(r+1)(r-4)}$	-3456

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\ &= 1 + \frac{9x}{2} + 12x^2 + 20x^3 - 168x^5 - 896x^6 - 3456x^7 + O(x^8) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^5 \left(1 + \frac{16x}{3} + \frac{144x^2}{7} + \frac{480x^3}{7} + \frac{4400x^4}{21} + \frac{4224x^5}{7} + 1664x^6 + \frac{13312x^7}{3} + O(x^8)\right) \\ &\quad + c_2 \left(1 + \frac{9x}{2} + 12x^2 + 20x^3 - 168x^5 - 896x^6 - 3456x^7 + O(x^8)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^5 \left(1 + \frac{16x}{3} + \frac{144x^2}{7} + \frac{480x^3}{7} + \frac{4400x^4}{21} + \frac{4224x^5}{7} + 1664x^6 + \frac{13312x^7}{3} + O(x^8)\right) \\ &\quad + c_2 \left(1 + \frac{9x}{2} + 12x^2 + 20x^3 - 168x^5 - 896x^6 - 3456x^7 + O(x^8)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^5 \left(1+\frac{16x}{3}+\frac{144x^2}{7}+\frac{480x^3}{7}+\frac{4400x^4}{21}+\frac{4224x^5}{7}+1664x^6+\frac{13312x^7}{3}+O(x^8)\right) + c_2\left(1+\frac{9x}{2}+12x^2+20x^3-168x^5-896x^6-3456x^7+O(x^8)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^5 \left(1+\frac{16x}{3}+\frac{144x^2}{7}+\frac{480x^3}{7}+\frac{4400x^4}{21}+\frac{4224x^5}{7}+1664x^6+\frac{13312x^7}{3}+O(x^8)\right) + c_2\left(1+\frac{9x}{2}+12x^2+20x^3-168x^5-896x^6-3456x^7+O(x^8)\right)$$

Verified OK.

9.24.1 Maple step by step solution

Let's solve

$-y''x(2x-1)+(-2x-4)\,y'+18y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{18y}{x(2x-1)}-\frac{2(x+2)y'}{x(2x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2(x+2)y'}{x(2x-1)}-\frac{18y}{x(2x-1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{2(x+2)}{x(2x-1)}, P_3(x)=-\frac{18}{x(2x-1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\bigg|_{x=0}=-4$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\bigg|_{x=0}=0$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y'' x(2x-1) + (2x+4)\, y' - 18y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 r(-5+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(k-4+r) + 2a_k(k+r+3)\,(k+r-3)\right) x^{k+r}\right) =$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-5+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 5\}$

- Each term in the series must be 0, giving the recursion relation

  $-a_{k+1}(k+1+r)\,(k-4+r) + 2a_k(k+r+3)\,(k+r-3) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k(k+r+3)(k+r-3)}{(k+1+r)(k-4+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 3$

  $a_{k+1} = \frac{2a_k(k+3)(k-3)}{(k+1)(k-4)}$
- Apply recursion relation for $k = 0$

  $a_1 = \frac{9a_0}{2}$
- Apply recursion relation for $k = 1$

  $a_2 = \frac{8a_1}{3}$
- Express in terms of a_0

  $a_2 = 12a_0$
- Apply recursion relation for $k = 2$

  $a_3 = \frac{5a_2}{3}$
- Express in terms of a_0

  $a_3 = 20a_0$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y = a_0 \cdot \left(1 + \frac{9}{2}x + 12x^2 + 20x^3\right)$
- Recursion relation for $r = 5$

  $a_{k+1} = \frac{2a_k(k+8)(k+2)}{(k+6)(k+1)}$
- Solution for $r = 5$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+5}, a_{k+1} = \frac{2a_k(k+8)(k+2)}{(k+6)(k+1)}\right]$
- Combine solutions and rename parameters

  $\left[y = a_0 \cdot \left(1 + \frac{9}{2}x + 12x^2 + 20x^3\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+5}\right), b_{k+1} = \frac{2b_k(k+8)(k+2)}{(k+6)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 50

```
Order:=8;
dsolve(x*(1-2*x)*diff(y(x),x$2)-2*(2+x)*diff(y(x),x)+18*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^5 \left(1 + \frac{16}{3}x + \frac{144}{7}x^2 + \frac{480}{7}x^3 + \frac{4400}{21}x^4 + \frac{4224}{7}x^5 + 1664x^6 + \frac{13312}{3}x^7 + \mathrm{O}\left(x^8\right)\right) + c_2\left(2880 + 12960x + 34560x^2 + 57600x^3 - 483840x^5 - 2580480x^6 - 9953280x^7 + \mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.128 (sec). Leaf size: 81

```
AsymptoticDSolveValue[x*(1-2*x)*y''[x]-2*(2+x)*y'[x]+18*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(-896x^6 - 168x^5 + 20x^3 + 12x^2 + \frac{9x}{2} + 1\right) + c_2\left(1664x^{11} + \frac{4224x^{10}}{7} + \frac{4400x^9}{21} + \frac{480x^8}{7} + \frac{144x^7}{7} + \frac{16x^6}{3} + x^5\right)$$

9.25 problem 26

Internal problem ID [7027]
Internal file name [`OUTPUT/6270_Thursday_August_18_2022_07_12_29_AM_86533897/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. Miscellaneous Exercises. page 394
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{xy'' + (-x+2)\,y' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + (-x+2)\,y' - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x-2}{x}$$
$$q(x) = -\frac{1}{x}$$

Table 235: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x-2}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + (-x+2)\,y' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + (-x+2)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2(n+r)\, a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 2(n+r)\, a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + 2(n+r)\, a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + 2r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + 2r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) + 2a_n(n+r) - a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{n+1+r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-1}}{n+1} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{2+r}$$

Which for the root $r = 0$ becomes

$$a_1 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+r}$	$\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(2+r)(3+r)}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+r}$	$\frac{1}{2}$
a_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{(2+r)(3+r)(4+r)}$$

Which for the root $r = 0$ becomes

$$a_3 = \frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+r}$	$\frac{1}{2}$
a_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{6}$
a_3	$\frac{1}{(2+r)(3+r)(4+r)}$	$\frac{1}{24}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(2+r)(3+r)(4+r)(5+r)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+r}$	$\frac{1}{2}$
a_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{6}$
a_3	$\frac{1}{(2+r)(3+r)(4+r)}$	$\frac{1}{24}$
a_4	$\frac{1}{(2+r)(3+r)(4+r)(5+r)}$	$\frac{1}{120}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{(2+r)(3+r)(4+r)(6+r)(5+r)}$$

Which for the root $r = 0$ becomes

$$a_5 = \frac{1}{720}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+r}$	$\frac{1}{2}$
a_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{6}$
a_3	$\frac{1}{(2+r)(3+r)(4+r)}$	$\frac{1}{24}$
a_4	$\frac{1}{(2+r)(3+r)(4+r)(5+r)}$	$\frac{1}{120}$
a_5	$\frac{1}{(2+r)(3+r)(4+r)(6+r)(5+r)}$	$\frac{1}{720}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{1}{(2+r)(3+r)(4+r)(5+r)(7+r)(6+r)}$$

Which for the root $r = 0$ becomes

$$a_6 = \frac{1}{5040}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+r}$	$\frac{1}{2}$
a_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{6}$
a_3	$\frac{1}{(2+r)(3+r)(4+r)}$	$\frac{1}{24}$
a_4	$\frac{1}{(2+r)(3+r)(4+r)(5+r)}$	$\frac{1}{120}$
a_5	$\frac{1}{(2+r)(3+r)(4+r)(6+r)(5+r)}$	$\frac{1}{720}$
a_6	$\frac{1}{(2+r)(3+r)(4+r)(5+r)(7+r)(6+r)}$	$\frac{1}{5040}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{1}{(2+r)(3+r)(4+r)(6+r)(5+r)(8+r)(7+r)}$$

Which for the root $r = 0$ becomes

$$a_7 = \frac{1}{40320}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+r}$	$\frac{1}{2}$
a_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{6}$
a_3	$\frac{1}{(2+r)(3+r)(4+r)}$	$\frac{1}{24}$
a_4	$\frac{1}{(2+r)(3+r)(4+r)(5+r)}$	$\frac{1}{120}$
a_5	$\frac{1}{(2+r)(3+r)(4+r)(6+r)(5+r)}$	$\frac{1}{720}$
a_6	$\frac{1}{(2+r)(3+r)(4+r)(5+r)(7+r)(6+r)}$	$\frac{1}{5040}$
a_7	$\frac{1}{(2+r)(3+r)(4+r)(6+r)(5+r)(8+r)(7+r)}$	$\frac{1}{40320}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots \\ &= 1 + \frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} + \frac{x^5}{720} + \frac{x^6}{5040} + \frac{x^7}{40320} + O(x^8) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= \frac{1}{2+r} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{1}{2+r} &= \lim_{r \to -1} \frac{1}{2+r} \\ &= 1 \end{aligned}$$

The limit is 1. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-1}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
b_n(n+r)(n+r-1) - b_{n-1}(n+r-1) + 2(n+r) b_n - b_{n-1} = 0 \tag{4}
$$

Which for for the root $r = -1$ becomes

$$
b_n(n-1)(n-2) - b_{n-1}(n-2) + 2(n-1) b_n - b_{n-1} = 0 \tag{4A}
$$

Solving for b_n from the recursive equation (4) gives

$$
b_n = \frac{b_{n-1}}{n+1+r} \tag{5}
$$

Which for the root $r = -1$ becomes

$$
b_n = \frac{b_{n-1}}{n} \tag{6}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$
b_1 = \frac{1}{2+r}
$$

Which for the root $r = -1$ becomes

$$
b_1 = 1
$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+r}$	1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{(2+r)(3+r)}$$

Which for the root $r = -1$ becomes

$$b_2 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+r}$	1
b_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{1}{(2+r)(3+r)(4+r)}$$

Which for the root $r = -1$ becomes

$$b_3 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+r}$	1
b_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{2}$
b_3	$\frac{1}{(2+r)(3+r)(4+r)}$	$\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(2+r)(3+r)(4+r)(5+r)}$$

Which for the root $r = -1$ becomes

$$b_4 = \frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+r}$	1
b_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{2}$
b_3	$\frac{1}{(2+r)(3+r)(4+r)}$	$\frac{1}{6}$
b_4	$\frac{1}{(2+r)(3+r)(4+r)(5+r)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{1}{(2+r)(3+r)(4+r)(6+r)(5+r)}$$

Which for the root $r = -1$ becomes

$$b_5 = \frac{1}{120}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+r}$	1
b_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{2}$
b_3	$\frac{1}{(2+r)(3+r)(4+r)}$	$\frac{1}{6}$
b_4	$\frac{1}{(2+r)(3+r)(4+r)(5+r)}$	$\frac{1}{24}$
b_5	$\frac{1}{(2+r)(3+r)(4+r)(6+r)(5+r)}$	$\frac{1}{120}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{1}{(2+r)(3+r)(4+r)(5+r)(7+r)(6+r)}$$

Which for the root $r = -1$ becomes

$$b_6 = \frac{1}{720}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+r}$	1
b_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{2}$
b_3	$\frac{1}{(2+r)(3+r)(4+r)}$	$\frac{1}{6}$
b_4	$\frac{1}{(2+r)(3+r)(4+r)(5+r)}$	$\frac{1}{24}$
b_5	$\frac{1}{(2+r)(3+r)(4+r)(6+r)(5+r)}$	$\frac{1}{120}$
b_6	$\frac{1}{(2+r)(3+r)(4+r)(5+r)(7+r)(6+r)}$	$\frac{1}{720}$

For $n = 7$, using the above recursive equation gives

$$b_7 = \frac{1}{(2+r)(3+r)(4+r)(6+r)(5+r)(8+r)(7+r)}$$

Which for the root $r = -1$ becomes

$$b_7 = \frac{1}{5040}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+r}$	1
b_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{2}$
b_3	$\frac{1}{(2+r)(3+r)(4+r)}$	$\frac{1}{6}$
b_4	$\frac{1}{(2+r)(3+r)(4+r)(5+r)}$	$\frac{1}{24}$
b_5	$\frac{1}{(2+r)(3+r)(4+r)(6+r)(5+r)}$	$\frac{1}{120}$
b_6	$\frac{1}{(2+r)(3+r)(4+r)(5+r)(7+r)(6+r)}$	$\frac{1}{720}$
b_7	$\frac{1}{(2+r)(3+r)(4+r)(6+r)(5+r)(8+r)(7+r)}$	$\frac{1}{5040}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots\right) \\
&= \frac{1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8)}{x}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \left(1 + \frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} + \frac{x^5}{720} + \frac{x^6}{5040} + \frac{x^7}{40320} + O(x^8)\right) \\
&\quad + \frac{c_2 \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8)\right)}{x}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \left(1 + \frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} + \frac{x^5}{720} + \frac{x^6}{5040} + \frac{x^7}{40320} + O(x^8)\right) \\
&\quad + \frac{c_2 \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8)\right)}{x}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1\left(1+\frac{x}{2}+\frac{x^2}{6}+\frac{x^3}{24}+\frac{x^4}{120}+\frac{x^5}{720}+\frac{x^6}{5040}+\frac{x^7}{40320}+O(x^8)\right) + \frac{c_2\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}+\frac{x^7}{5040}+O(x^8)\right)}{x} \tag{1}$$

Verification of solutions

$$y = c_1\left(1+\frac{x}{2}+\frac{x^2}{6}+\frac{x^3}{24}+\frac{x^4}{120}+\frac{x^5}{720}+\frac{x^6}{5040}+\frac{x^7}{40320}+O(x^8)\right) + \frac{c_2\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}+\frac{x^7}{5040}+O(x^8)\right)}{x}$$

Verified OK.

9.25.1 Maple step by step solution

Let's solve

$xy'' + (-x+2)\,y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{x} + \frac{(x-2)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x-2)y'}{x} - \frac{y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ∘ Define functions

    $\left[P_2(x) = -\frac{x-2}{x}, P_3(x) = -\frac{1}{x}\right]$

  ∘ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

  ∘ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x y'' + (-x+2) y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r) x^{k+r}$

- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r) (k+r-1) x^{k+r-1}$

- Shift index using $k->k+1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r) (k+r) x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(1+r) x^{-1+r} + \left(\sum_{k=0}^{\infty} (a_{k+1}(k+1+r)(k+2+r) - a_k(k+1+r)) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+1+r)(a_{k+1}(k+2+r) - a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+2+r}$

- Recursion relation for $r = -1$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = \frac{a_k}{k+1}\right]$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+2}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k}{k+2}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 52

```
Order:=8;
dsolve(x*diff(y(x),x$2)+(2-x)*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\left(1 + \frac{1}{2}x + \frac{1}{6}x^2 + \frac{1}{24}x^3 + \frac{1}{120}x^4 + \frac{1}{720}x^5 + \frac{1}{5040}x^6 + \frac{1}{40320}x^7 + \mathrm{O}\left(x^8\right)\right) + \frac{c_2\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 + \mathrm{O}\left(x^8\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.078 (sec). Leaf size: 90

```
AsymptoticDSolveValue[x*y''[x]+(2-x)*y'[x]-y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(\frac{x^5}{720}+\frac{x^4}{120}+\frac{x^3}{24}+\frac{x^2}{6}+\frac{x}{2}+\frac{1}{x}+1\right)+c_2\left(\frac{x^6}{5040}+\frac{x^5}{720}+\frac{x^4}{120}+\frac{x^3}{24}+\frac{x^2}{6}+\frac{x}{2}+1\right)$$

9.26 problem 27

Internal problem ID [7028]
Internal file name [`OUTPUT/6271_Thursday_August_18_2022_07_12_32_AM_76458216/index.tex`]

Book: Elementary differential equations. Rainville, Bedient, Bedient. Prentice Hall. NJ. 8th edition. 1997.
Section: CHAPTER 18. Power series solutions. Miscellaneous Exercises. page 394
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 3xy' + 4(1+x)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' - 3xy' + (4x+4)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{3}{x}$$
$$q(x) = \frac{4x+4}{x^2}$$

Table 237: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{3}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{4x+4}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' - 3xy' + (4x + 4)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}\right) - 3x \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (4x + 4) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 4x^{1+n+r} a_n = \sum_{n=1}^{\infty} 4a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} 4a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 3x^{n+r} a_n (n+r) + 4a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - 3x^r a_0 r + 4a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 3x^r r + 4x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r-2)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r-2)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = 2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r-2)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 2$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+2}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - 3a_n(n+r) + 4a_{n-1} + 4a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{4a_{n-1}}{n^2 + 2nr + r^2 - 4n - 4r + 4} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = -\frac{4a_{n-1}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{4}{(-1+r)^2}$$

Which for the root $r = 2$ becomes

$$a_1 = -4$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{(-1+r)^2}$	-4

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{16}{(-1+r)^2 r^2}$$

Which for the root $r = 2$ becomes

$$a_2 = 4$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{(-1+r)^2}$	-4
a_2	$\frac{16}{(-1+r)^2 r^2}$	4

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{64}{(-1+r)^2 r^2 (r+1)^2}$$

Which for the root $r = 2$ becomes

$$a_3 = -\frac{16}{9}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{(-1+r)^2}$	-4
a_2	$\frac{16}{(-1+r)^2 r^2}$	4
a_3	$-\frac{64}{(-1+r)^2 r^2 (r+1)^2}$	$-\frac{16}{9}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{256}{(-1+r)^2 r^2 (r+1)^2 (r+2)^2}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{4}{9}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{(-1+r)^2}$	-4
a_2	$\frac{16}{(-1+r)^2 r^2}$	4
a_3	$-\frac{64}{(-1+r)^2 r^2 (r+1)^2}$	$-\frac{16}{9}$
a_4	$\frac{256}{(-1+r)^2 r^2 (r+1)^2 (r+2)^2}$	$\frac{4}{9}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1024}{(-1+r)^2 r^2 (r+1)^2 (r+2)^2 (r+3)^2}$$

Which for the root $r = 2$ becomes

$$a_5 = -\frac{16}{225}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{(-1+r)^2}$	-4
a_2	$\frac{16}{(-1+r)^2 r^2}$	4
a_3	$-\frac{64}{(-1+r)^2 r^2 (r+1)^2}$	$-\frac{16}{9}$
a_4	$\frac{256}{(-1+r)^2 r^2 (r+1)^2 (r+2)^2}$	$\frac{4}{9}$
a_5	$-\frac{1024}{(-1+r)^2 r^2 (r+1)^2 (r+2)^2 (r+3)^2}$	$-\frac{16}{225}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{4096}{(-1+r)^2 r^2 (r+1)^2 (r+2)^2 (r+3)^2 (r+4)^2}$$

Which for the root $r = 2$ becomes

$$a_6 = \frac{16}{2025}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{(-1+r)^2}$	-4
a_2	$\frac{16}{(-1+r)^2 r^2}$	4
a_3	$-\frac{64}{(-1+r)^2 r^2 (r+1)^2}$	$-\frac{16}{9}$
a_4	$\frac{256}{(-1+r)^2 r^2 (r+1)^2 (r+2)^2}$	$\frac{4}{9}$
a_5	$-\frac{1024}{(-1+r)^2 r^2 (r+1)^2 (r+2)^2 (r+3)^2}$	$-\frac{16}{225}$
a_6	$\frac{4096}{(-1+r)^2 r^2 (r+1)^2 (r+2)^2 (r+3)^2 (r+4)^2}$	$\frac{16}{2025}$

For $n = 7$, using the above recursive equation gives

$$a_7 = -\frac{16384}{\left(-1+r\right)^2 r^2 \left(r+1\right)^2 \left(r+2\right)^2 \left(r+3\right)^2 \left(r+4\right)^2 \left(r+5\right)^2}$$

Which for the root $r = 2$ becomes

$$a_7 = -\frac{64}{99225}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{(-1+r)^2}$	-4
a_2	$\frac{16}{(-1+r)^2 r^2}$	4
a_3	$-\frac{64}{(-1+r)^2 r^2 (r+1)^2}$	$-\frac{16}{9}$
a_4	$\frac{256}{(-1+r)^2 r^2 (r+1)^2 (r+2)^2}$	$\frac{4}{9}$
a_5	$-\frac{1024}{(-1+r)^2 r^2 (r+1)^2 (r+2)^2 (r+3)^2}$	$-\frac{16}{225}$
a_6	$\frac{4096}{(-1+r)^2 r^2 (r+1)^2 (r+2)^2 (r+3)^2 (r+4)^2}$	$\frac{16}{2025}$
a_7	$-\frac{16384}{(-1+r)^2 r^2 (r+1)^2 (r+2)^2 (r+3)^2 (r+4)^2 (r+5)^2}$	$-\frac{64}{99225}$

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^2 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\ &= x^2 \left(4x^2 - 4x + 1 - \frac{16x^3}{9} + \frac{4x^4}{9} - \frac{16x^5}{225} + \frac{16x^6}{2025} - \frac{64x^7}{99225} + O(x^8)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 2$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	$-\frac{4}{(-1+r)^2}$	-4	$\frac{8}{(-1+r)^3}$
b_2	$\frac{16}{(-1+r)^2 r^2}$	4	$\frac{-64r+32}{(-1+r)^3 r^3}$
b_3	$-\frac{64}{(-1+r)^2 r^2 (r+1)^2}$	$-\frac{16}{9}$	$\frac{384r^2-128}{(-1+r)^3 r^3 (r+1)^3}$
b_4	$\frac{256}{(-1+r)^2 r^2 (r+1)^2 (r+2)^2}$	$\frac{4}{9}$	$\frac{-2048r^3-3072r^2+1024r+1024}{(-1+r)^3 r^3 (r+1)^3 (r+2)^3}$
b_5	$-\frac{1024}{(-1+r)^2 r^2 (r+1)^2 (r+2)^2 (r+3)^2}$	$-\frac{16}{225}$	$\frac{10240r^4+40960r^3+30720r^2-20480r-12288}{(-1+r)^3 r^3 (r+1)^3 (r+2)^3 (r+3)^3}$
b_6	$\frac{4096}{(-1+r)^2 r^2 (r+1)^2 (r+2)^2 (r+3)^2 (r+4)^2}$	$\frac{16}{2025}$	$-\frac{49152\left(r^4+6r^3+\frac{23}{3}r^2-4r-\frac{8}{3}\right)\left(\frac{3}{2}+r\right)}{(-1+r)^3 r^3 (r+1)^3 (r+2)^3 (r+3)^3 (r+4)^3}$
b_7	$-\frac{16384}{(-1+r)^2 r^2 (r+1)^2 (r+2)^2 (r+3)^2 (r+4)^2 (r+5)^2}$	$-\frac{64}{99225}$	$\frac{229376r^6+2752512r^5+11468800r^4+18350080r^3+4816896r^2-10092544}{(-1+r)^3 r^3 (r+1)^3 (r+2)^3 (r+3)^3 (r+4)^3 (r+5)^3}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \ldots \\
&= x^2\left(4x^2 - 4x + 1 - \frac{16x^3}{9} + \frac{4x^4}{9} - \frac{16x^5}{225} + \frac{16x^6}{2025} - \frac{64x^7}{99225} + O(x^8)\right)\ln(x) \\
&\quad + x^2\left(-12x^2 + 8x + \frac{176x^3}{27} - \frac{50x^4}{27} + \frac{1096x^5}{3375} - \frac{392x^6}{10125} + \frac{3872x^7}{1157625} + O(x^8)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^2\left(4x^2 - 4x + 1 - \frac{16x^3}{9} + \frac{4x^4}{9} - \frac{16x^5}{225} + \frac{16x^6}{2025} - \frac{64x^7}{99225} + O(x^8)\right) \\
&\quad + c_2\left(x^2\left(4x^2 - 4x + 1 - \frac{16x^3}{9} + \frac{4x^4}{9} - \frac{16x^5}{225} + \frac{16x^6}{2025} - \frac{64x^7}{99225} + O(x^8)\right)\ln(x)\right. \\
&\quad \left. + x^2\left(-12x^2 + 8x + \frac{176x^3}{27} - \frac{50x^4}{27} + \frac{1096x^5}{3375} - \frac{392x^6}{10125} + \frac{3872x^7}{1157625} + O(x^8)\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
y = y_h
$$

$$= c_1 x^2 \left(4x^2 - 4x + 1 - \frac{16x^3}{9} + \frac{4x^4}{9} - \frac{16x^5}{225} + \frac{16x^6}{2025} - \frac{64x^7}{99225} + O(x^8)\right)$$
$$+ c_2 \left(x^2 \left(4x^2 - 4x + 1 - \frac{16x^3}{9} + \frac{4x^4}{9} - \frac{16x^5}{225} + \frac{16x^6}{2025} - \frac{64x^7}{99225} + O(x^8)\right) \ln(x)$$
$$+ x^2 \left(-12x^2 + 8x + \frac{176x^3}{27} - \frac{50x^4}{27} + \frac{1096x^5}{3375} - \frac{392x^6}{10125} + \frac{3872x^7}{1157625} + O(x^8)\right)\right)$$

Summary

The solution(s) found are the following

$$y = c_1 x^2 \left(4x^2 - 4x + 1 - \frac{16x^3}{9} + \frac{4x^4}{9} - \frac{16x^5}{225} + \frac{16x^6}{2025} - \frac{64x^7}{99225} + O(x^8)\right)$$
$$+ c_2 \left(x^2 \left(4x^2 - 4x + 1 - \frac{16x^3}{9} + \frac{4x^4}{9} - \frac{16x^5}{225} + \frac{16x^6}{2025} - \frac{64x^7}{99225} + O(x^8)\right) \ln(x) \right. \tag{1}$$
$$\left. + x^2 \left(-12x^2 + 8x + \frac{176x^3}{27} - \frac{50x^4}{27} + \frac{1096x^5}{3375} - \frac{392x^6}{10125} + \frac{3872x^7}{1157625} + O(x^8)\right)\right)$$

Verification of solutions

$$y = c_1 x^2 \left(4x^2 - 4x + 1 - \frac{16x^3}{9} + \frac{4x^4}{9} - \frac{16x^5}{225} + \frac{16x^6}{2025} - \frac{64x^7}{99225} + O(x^8)\right)$$
$$+ c_2 \left(x^2 \left(4x^2 - 4x + 1 - \frac{16x^3}{9} + \frac{4x^4}{9} - \frac{16x^5}{225} + \frac{16x^6}{2025} - \frac{64x^7}{99225} + O(x^8)\right) \ln(x)$$
$$+ x^2 \left(-12x^2 + 8x + \frac{176x^3}{27} - \frac{50x^4}{27} + \frac{1096x^5}{3375} - \frac{392x^6}{10125} + \frac{3872x^7}{1157625} + O(x^8)\right)\right)$$

Verified OK.

9.26.1 Maple step by step solution

Let's solve

$x^2 y'' - 3xy' + (4x + 4) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3y'}{x} - \frac{4(1+x)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{3y'}{x} + \frac{4(1+x)y}{x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $$\left[P_2(x) = -\frac{3}{x}, P_3(x) = \frac{4(1+x)}{x^2}\right]$$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -3$$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 4$$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $$x_0 = 0$$
- Multiply by denominators

  $$x^2 y'' - 3xy' + (4x + 4)\, y = 0$$
- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$
  - Shift index using $k- >k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$
  - Convert $x \cdot y'$ to series expansion

    $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$$
  - Convert $x^2 \cdot y''$ to series expansion

    $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\, (k + r - 1)\, x^{k+r}$$

    Rewrite ODE with series expansions

$$a_0(-2+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-2)^2 + 4a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 2$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-2)^2 + 4a_{k-1} = 0$

- Shift index using $k- >k+1$

  $a_{k+1}(k+r-1)^2 + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{4a_k}{(k+r-1)^2}$

- Recursion relation for $r = 2$

  $a_{k+1} = -\frac{4a_k}{(k+1)^2}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{4a_k}{(k+1)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 81

```
Order:=8;
dsolve(x^2*diff(y(x),x$2)-3*x*diff(y(x),x)+4*(1+x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left((c_2 \ln(x) + c_1)\left(1 - 4x + 4x^2 - \frac{16}{9}x^3 + \frac{4}{9}x^4 - \frac{16}{225}x^5 + \frac{16}{2025}x^6 - \frac{64}{99225}x^7 + \mathrm{O}\left(x^8\right)\right) + \left(8x - 12x^2 + \frac{176}{27}x^3 - \frac{50}{27}x^4 + \frac{1096}{3375}x^5 - \frac{392}{10125}x^6 + \frac{3872}{1157625}x^7 + \mathrm{O}\left(x^8\right)\right)c_2\right)x^2$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 158

```
AsymptoticDSolveValue[x^2*y''[x]-3*x*y'[x]+4*(1+x)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(-\frac{64x^7}{99225} + \frac{16x^6}{2025} - \frac{16x^5}{225} + \frac{4x^4}{9} - \frac{16x^3}{9} + 4x^2 - 4x + 1\right)x^2 + c_2\left(\left(\frac{3872x^7}{1157625} - \frac{392x^6}{10125} + \frac{1096x^5}{3375} - \frac{50x^4}{27} + \frac{176x^3}{27} - 12x^2 + 8x\right)x^2 + \left(-\frac{64x^7}{99225} + \frac{16x^6}{2025} - \frac{16x^5}{225} + \frac{4x^4}{9} - \frac{16x^3}{9} + 4x^2 - 4x + 1\right)x^2 \log(x)\right)$$