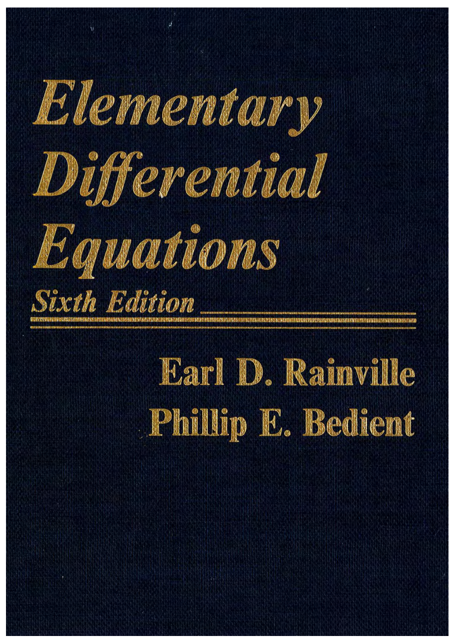
A Solution Manual For

Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.



Nasser M. Abbasi

May 15, 2024

Contents

1 CHAPTER 16. Nonlinear equations. Section 94. Factoring the left member. EXERCISES Page 309

1.1 problem 1

Internal problem ID [6767]
Internal file name [`OUTPUT/6014_Monday_July_25_2022_01_59_32_AM_9550685/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 94. Factoring the left member. EXERCISES Page 309
Problem number: 1.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_nonlinear_p_but_separable"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x^2 {y'}^2 - y^2 = 0}$$

1.1.1 Solving as first order nonlinear p but separable ode

The ode has the form

$$(y')^{\frac{n}{m}} = f(x)g(y) \tag{1}$$

Where $n = 2, m = 1, f = \frac{1}{x^2}, g = y^2$. Hence the ode is

$$(y')^2 = \frac{y^2}{x^2}$$

Solving for y' from (1) gives

$$y' = \sqrt{fg}$$
$$y' = -\sqrt{fg}$$

To be able to solve as separable ode, we have to now assume that $f > 0, g > 0$.

$$\frac{1}{x^2} > 0$$
$$y^2 > 0$$

Under the above assumption the differential equations become separable and can be written as

$$y' = \sqrt{f}\,\sqrt{g}$$
$$y' = -\sqrt{f}\,\sqrt{g}$$

Therefore

$$\frac{1}{\sqrt{g}}\,dy = \left(\sqrt{f}\right)\,dx$$
$$-\frac{1}{\sqrt{g}}\,dy = \left(\sqrt{f}\right)\,dx$$

Replacing $f(x), g(y)$ by their values gives

$$\frac{1}{\sqrt{y^2}}\,dy = \left(\sqrt{\frac{1}{x^2}}\right)\,dx$$
$$-\frac{1}{\sqrt{y^2}}\,dy = \left(\sqrt{\frac{1}{x^2}}\right)\,dx$$

Integrating now gives the solutions.

$$\int \frac{1}{\sqrt{y^2}}dy = \int \sqrt{\frac{1}{x^2}}dx + c_1$$
$$\int -\frac{1}{\sqrt{y^2}}dy = \int \sqrt{\frac{1}{x^2}}dx + c_1$$

Integrating gives

$$\frac{y\ln(y)}{\sqrt{y^2}} = \sqrt{\frac{1}{x^2}}\,x\ln(x) + c_1$$
$$-\frac{y\ln(y)}{\sqrt{y^2}} = \sqrt{\frac{1}{x^2}}\,x\ln(x) + c_1$$

Therefore

$$\frac{y\ln(y)}{\sqrt{y^2}} = \sqrt{\frac{1}{x^2}}\,x\ln(x) + c_1$$

$$-\frac{y\ln(y)}{\sqrt{y^2}} = \sqrt{\frac{1}{x^2}}\,x\ln(x) + c_1$$

Summary
The solution(s) found are the following

$$\frac{y \ln (y)}{\sqrt{y^{2}}} = \sqrt{\frac{1}{x^{2}}}\, x \ln (x) + c_{1} \tag{1}$$

$$-\frac{y \ln (y)}{\sqrt{y^{2}}} = \sqrt{\frac{1}{x^{2}}}\, x \ln (x) + c_{1} \tag{2}$$

Verification of solutions

$$\frac{y \ln (y)}{\sqrt{y^{2}}} = \sqrt{\frac{1}{x^{2}}}\, x \ln (x) + c_{1}$$

Verified OK. `{0 < 1/x^2, 0 < y^2}`

$$-\frac{y \ln (y)}{\sqrt{y^{2}}} = \sqrt{\frac{1}{x^{2}}}\, x \ln (x) + c_{1}$$

Verified OK. `{0 < 1/x^2, 0 < y^2}`

1.1.2 Maple step by step solution

Let's solve

$x^{2} {y'}^{2} - y^{2} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \frac{1}{x} dx + c_{1}$

- Evaluate integral

  $\ln (y) = \ln (x) + c_{1}$

- Solve for y

  $y = x\, \mathrm{e}^{c_{1}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x)^2-y(x)^2=0,y(x), singsol=all)
```

$$y(x) = c_1 x$$
$$y(x) = \frac{c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 24

```
DSolve[x^2*(y'[x])^2-y[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1}{x}$$
$$y(x) \to c_1 x$$
$$y(x) \to 0$$

1.2 problem 2

Internal problem ID [6768]
Internal file name [`OUTPUT/6015_Monday_July_25_2022_01_59_33_AM_45017633/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 94. Factoring the left member. EXERCISES Page 309
Problem number: 2.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"exact", "linear", "quadrature", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{xy'^2 - (3y + 2x)\, y' + 6y = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = 2 \tag{1}$$
$$y' = \frac{3y}{x} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int 2 \, \mathrm{d}x \\ &= c_1 + 2x \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 + 2x \tag{1}$$

Verification of solutions

$$y = c_1 + 2x$$

Verified OK.

Solving equation (2)

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{3y}{x} \end{aligned}$$

Where $f(x) = \frac{3}{x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y}\,dy &= \frac{3}{x}\,dx \\ \int \frac{1}{y}\,dy &= \int \frac{3}{x}\,dx \\ \ln(y) &= 3\ln(x) + c_2 \\ y &= e^{3\ln(x)+c_2} \\ &= c_2 x^3 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2 x^3 \tag{1}$$

Verification of solutions

$$y = c_2 x^3$$

Verified OK.

1.2.1 Maple step by step solution

Let's solve

$x{y'}^2 - (3y + 2x)\,y' + 6y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

$$\int \left(x{y'}^2 - (3y+2x)\,y' + 6y\right) dx = \int 0 dx + c_1$$

- Cannot compute integral

$$\int \left(x{y'}^2 - (3y+2x)\,y' + 6y\right) dx = c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(x*diff(y(x),x)^2-(2*x+3*y(x))*diff(y(x),x)+6*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1x^3$$
$$y(x) = c_1 + 2x$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 26

```
DSolve[x*(y'[x])^2-(2*x+3*y[x])*y'[x]+6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1x^3$$
$$y(x) \to 2x + c_1$$
$$y(x) \to 0$$

1.3 problem 3

Internal problem ID [6769]
Internal file name [OUTPUT/6016_Monday_July_25_2022_01_59_35_AM_69854974/index.tex]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 94. Factoring the left member. EXERCISES Page 309
Problem number: 3.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{x^2 {y'}^2 - 5xyy' + 6y^2 = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{2y}{x} \tag{1}$$
$$y' = \frac{3y}{x} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{2y}{x} \end{aligned}$$

Where $f(x) = \frac{2}{x}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\,dy &= \frac{2}{x}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{2}{x}\,dx \\
\ln\left(y\right) &= 2\ln\left(x\right) + c_1 \\
y &= \mathrm{e}^{2\ln(x)+c_1} \\
&= c_1 x^2
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 x^2 \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1 x^2$$

Verified OK.
<u>Solving equation (2)</u>

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x, y) \\
&= f(x)g(y) \\
&= \frac{3y}{x}
\end{aligned}
$$

Where $f(x) = \frac{3}{x}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\,dy &= \frac{3}{x}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{3}{x}\,dx \\
\ln\left(y\right) &= 3\ln\left(x\right) + c_2 \\
y &= \mathrm{e}^{3\ln(x)+c_2} \\
&= c_2 x^3
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_2 x^3 \tag{1}$$

Verification of solutions

$$y = c_2 x^3$$

Verified OK.

1.3.1 Maple step by step solution

Let's solve

$x^2 {y'}^2 - 5xyy' + 6y^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{3}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \frac{3}{x} dx + c_1$

- Evaluate integral

  $\ln(y) = 3\ln(x) + c_1$

- Solve for y

  $y = e^{c_1} x^3$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(x^2*diff(y(x),x)^2-5*x*y(x)*diff(y(x),x)+6*y(x)^2=0,y(x), singsol=all)
```

$$y(x) = c_1x^3$$
$$y(x) = c_1x^2$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 26

```
DSolve[x^2*(y'[x])^2-5*x*y[x]*y'[x]+6*y[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1x^2$$
$$y(x) \to c_1x^3$$
$$y(x) \to 0$$

1.4 problem 4

Internal problem ID [6770]
Internal file name [OUTPUT/6017_Monday_July_25_2022_01_59_37_AM_73862004/index.tex]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 94. Factoring the left member. EXERCISES Page 309
Problem number: 4.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "differentialType", "homogeneousTypeD2", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x^2 {y'}^2 + x y' - y^2 - y = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{y}{x} \tag{1}$$

$$y' = -\frac{1+y}{x} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\, dy &= \frac{1}{x}\, dx \\
\int \frac{1}{y}\, dy &= \int \frac{1}{x}\, dx \\
\ln(y) &= \ln(x) + c_1 \\
y &= e^{\ln(x)+c_1} \\
&= c_1 x
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x \tag{1}$$

Verification of solutions

$$y = c_1 x$$

Verified OK.

Solving equation (2)

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x, y) \\
&= f(x)g(y) \\
&= \frac{-y-1}{x}
\end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = -y - 1$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{-y-1}\, dy &= \frac{1}{x}\, dx \\
\int \frac{1}{-y-1}\, dy &= \int \frac{1}{x}\, dx \\
-\ln(1+y) &= \ln(x) + c_2
\end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{1+y} = e^{\ln(x)+c_2}$$

Which simplifies to

$$\frac{1}{1+y} = c_3 x$$

Which simplifies to

$$y = -\frac{(c_3 x\, e^{c_2} - 1)\, e^{-c_2}}{c_3 x}$$

Summary

The solution(s) found are the following

$$y = -\frac{(c_3 x\, e^{c_2} - 1)\, e^{-c_2}}{c_3 x} \tag{1}$$

Verification of solutions

$$y = -\frac{(c_3 x\, e^{c_2} - 1)\, e^{-c_2}}{c_3 x}$$

Verified OK.

1.4.1 Maple step by step solution

Let's solve

$x^2 {y'}^2 + x y' - y^2 - y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \frac{1}{x} dx + c_1$

- Evaluate integral

  $\ln(y) = \ln(x) + c_1$

- Solve for y

  $y = x\, e^{c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x)^2+x*diff(y(x),x)-y(x)^2-y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x$$
$$y(x) = \frac{-x + c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 31

```
DSolve[x^2*(y'[x])^2+x*y'[x]-y[x]^2-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x$$
$$y(x) \to -1 + \frac{c_1}{x}$$
$$y(x) \to -1$$
$$y(x) \to 0$$

1.5 problem 5

Internal problem ID [6771]
Internal file name [`OUTPUT/6018_Monday_July_25_2022_01_59_38_AM_27299389/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 94. Factoring the left member. EXERCISES Page 309
Problem number: 5.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"exact", "linear", "quadrature", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{xy'^2 + (1 - yx^2)\, y' - xy = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = xy \tag{1}$$
$$y' = -\frac{1}{x} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= xy \end{aligned}$$

Where $f(x) = x$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y} \, dy &= x \, dx \\
\int \frac{1}{y} \, dy &= \int x \, dx \\
\ln(y) &= \frac{x^2}{2} + c_1 \\
y &= e^{\frac{x^2}{2} + c_1} \\
&= c_1 e^{\frac{x^2}{2}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 e^{\frac{x^2}{2}} \tag{1}$$

Verification of solutions

$$y = c_1 e^{\frac{x^2}{2}}$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$
\begin{aligned}
y &= \int -\frac{1}{x} \, dx \\
&= -\ln(x) + c_2
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = -\ln(x) + c_2 \tag{1}$$

Verification of solutions

$$y = -\ln(x) + c_2$$

Verified OK.

1.5.1 Maple step by step solution

Let's solve

$xy'^2 + (1 - yx^2)\, y' - xy = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \left(xy'^2 + (1 - yx^2)\, y' - xy\right) dx = \int 0 dx + c_1$

- Cannot compute integral

  $\int \left(xy'^2 + (1 - yx^2)\, y' - xy\right) dx = c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(x*diff(y(x),x)^2+(1-x^2*y(x))*diff(y(x),x)-x*y(x)=0,y(x), singsol=all)
```

$$y(x) = -\ln(x) + c_1$$
$$y(x) = e^{\frac{x^2}{2}} c_1$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 28

```
DSolve[x*(y'[x])^2+(1-x^2*y[x])*y'[x]-x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{\frac{x^2}{2}}$$
$$y(x) \to -\log(x) + c_1$$

1.6 problem 6

Internal problem ID [6772]
Internal file name [`OUTPUT/6019_Monday_July_25_2022_01_59_42_AM_10893776/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 94. Factoring the left member. EXERCISES Page 309
Problem number: 6.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"exact", "linear", "quadrature", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y'^2 - (yx^2 + 3)\, y' + 3yx^2 = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = 3 \tag{1}$$
$$y' = yx^2 \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int 3\, \mathrm{d}x \\ &= 3x + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = 3x + c_1 \tag{1}$$

Verification of solutions

$$y = 3x + c_1$$

Verified OK.

Solving equation (2)

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= y\,x^2 \end{aligned}$$

Where $f(x) = x^2$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y}\,dy &= x^2\,dx \\ \int \frac{1}{y}\,dy &= \int x^2\,dx \\ \ln(y) &= \frac{x^3}{3} + c_2 \\ y &= \mathrm{e}^{\frac{x^3}{3}+c_2} \\ &= c_2\mathrm{e}^{\frac{x^3}{3}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2\mathrm{e}^{\frac{x^3}{3}} \tag{1}$$

Verification of solutions

$$y = c_2\mathrm{e}^{\frac{x^3}{3}}$$

Verified OK.

1.6.1 Maple step by step solution

Let's solve

$y'^2 - (yx^2 + 3)\,y' + 3yx^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \left(y'^2 - (yx^2+3)\, y' + 3yx^2\right) dx = \int 0 dx + c_1$
- Cannot compute integral

  $\int \left(y'^2 - (yx^2+3)\, y' + 3yx^2\right) dx = c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x)^2-(x^2*y(x)+3)*diff(y(x),x)+3*x^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{\frac{x^3}{3}}$$
$$y(x) = 3x + c_1$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 27

```
DSolve[(y'[x])^2-(x^2*y[x]+3)*y'[x]+3*x^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{\frac{x^3}{3}}$$
$$y(x) \to 3x + c_1$$

1.7 problem 7

Internal problem ID [6773]
Internal file name [`OUTPUT/6020_Monday_July_25_2022_01_59_43_AM_41766277/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 94. Factoring the left member. EXERCISES Page 309
Problem number: 7.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{xy'^2 - (xy+1)\, y' + y = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = y \tag{1}$$

$$y' = \frac{1}{x} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} \int \frac{1}{y} dy &= x + c_1 \\ \ln(y) &= x + c_1 \\ y &= e^{x+c_1} \\ y &= c_1 e^x \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{x} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{x}$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{1}{x} \, \mathrm{d}x \\ &= \ln(x) + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \ln(x) + c_2 \tag{1}$$

Verification of solutions

$$y = \ln(x) + c_2$$

Verified OK.

1.7.1 Maple step by step solution

Let's solve

$xy'^2 - (xy+1)\,y' + y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \left(xy'^2 - (xy+1)\,y' + y\right) dx = \int 0 dx + c_1$

- Cannot compute integral

  $\int \left(xy'^2 - (xy+1)\,y' + y\right) dx = c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x*diff(y(x),x)^2-(1+x*y(x))*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = \ln(x) + c_1$$
$$y(x) = e^x c_1$$

✓ Solution by Mathematica
Time used: 0.01 (sec). Leaf size: 20

```
DSolve[x*(y'[x])^2-(1+x*y[x])*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x$$
$$y(x) \to \log(x) + c_1$$

1.8 problem 8

Internal problem ID [6774]
Internal file name [`OUTPUT/6021_Monday_July_25_2022_01_59_46_AM_52332013/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 94. Factoring the left member. EXERCISES Page 309
Problem number: 8.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_nonlinear_p_but_separable"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y'^2 - y^2x^2 = 0}$$

1.8.1 Solving as first order nonlinear p but separable ode

The ode has the form

$$(y')^{\frac{n}{m}} = f(x)g(y) \tag{1}$$

Where $n = 2, m = 1, f = x^2, g = y^2$. Hence the ode is

$$(y')^2 = y^2x^2$$

Solving for y' from (1) gives

$$y' = \sqrt{fg}$$
$$y' = -\sqrt{fg}$$

To be able to solve as separable ode, we have to now assume that $f > 0, g > 0$.

$$x^2 > 0$$
$$y^2 > 0$$

Under the above assumption the differential equations become separable and can be written as

$$y' = \sqrt{f}\,\sqrt{g}$$
$$y' = -\sqrt{f}\,\sqrt{g}$$

Therefore

$$\frac{1}{\sqrt{g}}\,dy = \left(\sqrt{f}\right)\,dx$$
$$-\frac{1}{\sqrt{g}}\,dy = \left(\sqrt{f}\right)\,dx$$

Replacing $f(x), g(y)$ by their values gives

$$\frac{1}{\sqrt{y^2}}\,dy = \left(\sqrt{x^2}\right)\,dx$$
$$-\frac{1}{\sqrt{y^2}}\,dy = \left(\sqrt{x^2}\right)\,dx$$

Integrating now gives the solutions.

$$\int \frac{1}{\sqrt{y^2}}dy = \int \sqrt{x^2}dx + c_1$$
$$\int -\frac{1}{\sqrt{y^2}}dy = \int \sqrt{x^2}dx + c_1$$

Integrating gives

$$\frac{y\ln(y)}{\sqrt{y^2}} = \frac{x\sqrt{x^2}}{2} + c_1$$
$$-\frac{y\ln(y)}{\sqrt{y^2}} = \frac{x\sqrt{x^2}}{2} + c_1$$

Therefore

$$\frac{y\ln(y)}{\sqrt{y^2}} = \frac{x\sqrt{x^2}}{2} + c_1$$

$$-\frac{y\ln(y)}{\sqrt{y^2}} = \frac{x\sqrt{x^2}}{2} + c_1$$

Summary

The solution(s) found are the following

$$\frac{y\ln(y)}{\sqrt{y^2}} = \frac{x\sqrt{x^2}}{2} + c_1 \tag{1}$$
$$-\frac{y\ln(y)}{\sqrt{y^2}} = \frac{x\sqrt{x^2}}{2} + c_1 \tag{2}$$

Verification of solutions

$$\frac{y\ln(y)}{\sqrt{y^2}} = \frac{x\sqrt{x^2}}{2} + c_1$$

Verified OK. `{0 < x^2, 0 < y^2}`

$$-\frac{y\ln(y)}{\sqrt{y^2}} = \frac{x\sqrt{x^2}}{2} + c_1$$

Verified OK. `{0 < x^2, 0 < y^2}`

1.8.2 Maple step by step solution

Let's solve

$y'^2 - y^2x^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int x dx + c_1$

- Evaluate integral

  $\ln(y) = \frac{x^2}{2} + c_1$

- Solve for y

  $y = e^{\frac{x^2}{2}+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x)^2-x^2*y(x)^2=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-\frac{x^2}{2}} c_1$$
$$y(x) = \mathrm{e}^{\frac{x^2}{2}} c_1$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 38

```
DSolve[(y'[x])^2-x^2*y[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-\frac{x^2}{2}}$$
$$y(x) \to c_1 e^{\frac{x^2}{2}}$$
$$y(x) \to 0$$

1.9 problem 9

Internal problem ID [6775]
Internal file name [`OUTPUT/6022_Monday_July_25_2022_01_59_47_AM_53691888/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 94. Factoring the left member. EXERCISES Page 309
Problem number: 9.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{(x+y)^2 y'^2 - y^2 = 0}$$

1.9.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$(x+y)^2 p^2 - y^2 = 0$$

Solving for y from the above results in

$$y = -\frac{px}{1+p} \tag{1A}$$

$$y = -\frac{px}{-1+p} \tag{2A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. Each of the above ode's is dAlembert ode which is now solved. Solving ode 1A Taking derivative of (*) w.r.t. x gives

$$
\begin{aligned}
p &= f + (xf' + g')\frac{dp}{dx} \\
p - f &= (xf' + g')\frac{dp}{dx}
\end{aligned} \tag{2}
$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= -\frac{p}{1+p} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p + \frac{p}{1+p} = x\left(-\frac{1}{1+p} + \frac{p}{(1+p)^2}\right)p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p + \frac{p}{1+p} = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= -2 \\
p &= 0
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= -2x \\
y &= 0
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) + \frac{p(x)}{1+p(x)}}{x\left(-\frac{1}{1+p(x)} + \frac{p(x)}{(1+p(x))^2}\right)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp}x(p) = \frac{x(p)\left(-\frac{1}{1+p} + \frac{p}{(1+p)^2}\right)}{p + \frac{p}{1+p}} \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = \frac{1}{(2+p)\,p\,(1+p)}$$
$$q(p) = 0$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{x(p)}{(2+p)\,p\,(1+p)} = 0$$

The integrating factor μ is

$$\mu = e^{\int \frac{1}{(2+p)p(1+p)}dp}$$
$$= e^{-\ln(1+p)+\frac{\ln(p)}{2}+\frac{\ln(2+p)}{2}}$$

Which simplifies to

$$\mu = \frac{\sqrt{p}\,\sqrt{2+p}}{1+p}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}p}\mu x = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{\sqrt{p}\,\sqrt{2+p}\,x}{1+p}\right) = 0$$

Integrating gives

$$\frac{\sqrt{p}\,\sqrt{2+p}\,x}{1+p} = c_3$$

Dividing both sides by the integrating factor $\mu = \frac{\sqrt{p}\,\sqrt{2+p}}{1+p}$ results in

$$x(p) = \frac{c_3(1+p)}{\sqrt{p}\,\sqrt{2+p}}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = -\frac{y}{x+y}$$

Substituting the above in the solution for x found above gives

$$x = \frac{c_3 x}{(x+y)\sqrt{-\frac{y}{x+y}}\sqrt{\frac{2x+y}{x+y}}}$$

<u>Solving ode 2A</u> Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= -\frac{p}{-1+p} \\ g &= 0 \end{aligned}$$

Hence (2) becomes

$$p + \frac{p}{-1+p} = x\left(-\frac{1}{-1+p} + \frac{p}{(-1+p)^2}\right)p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p + \frac{p}{-1+p} = 0$$

Solving for p from the above gives

$$\begin{aligned} p &= 0 \\ p &= 0 \end{aligned}$$

Removing solutions for p which leads to undefined results and substituting these in (1A) gives

$$y = 0$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) + \frac{p(x)}{-1+p(x)}}{x\left(-\frac{1}{-1+p(x)} + \frac{p(x)}{(-1+p(x))^2}\right)} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{x(p)\left(-\frac{1}{-1+p} + \frac{p}{(-1+p)^2}\right)}{p + \frac{p}{-1+p}} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$\begin{aligned} p(p) &= -\frac{1}{p^2(-1+p)} \\ q(p) &= 0 \end{aligned}$$

Hence the ode is

$$\frac{d}{dp}x(p) - \frac{x(p)}{p^2(-1+p)} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{1}{p^2(-1+p)}dp} \\ &= \mathrm{e}^{-\ln(-1+p)+\ln(p)-\frac{1}{p}} \end{aligned}$$

Which simplifies to

$$\mu = \frac{p\,\mathrm{e}^{-\frac{1}{p}}}{-1+p}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}p}\mu x &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{p\,\mathrm{e}^{-\frac{1}{p}}x}{-1+p}\right) &= 0 \end{aligned}$$

Integrating gives

$$\frac{p\,\mathrm{e}^{-\frac{1}{p}}x}{-1+p} = c_5$$

Dividing both sides by the integrating factor $\mu = \frac{p\,\mathrm{e}^{-\frac{1}{p}}}{-1+p}$ results in

$$x(p) = \frac{c_5(-1+p)\,\mathrm{e}^{\frac{1}{p}}}{p}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = \frac{y}{x+y}$$

Substituting the above in the solution for x found above gives

$$x = -\frac{c_5 x\,\mathrm{e}^{\frac{x+y}{y}}}{y}$$

Summary

The solution(s) found are the following

$$y = -2x \tag{1}$$
$$y = 0 \tag{2}$$
$$x = \frac{c_3 x}{(x+y)\sqrt{-\frac{y}{x+y}}\sqrt{\frac{2x+y}{x+y}}} \tag{3}$$
$$y = 0 \tag{4}$$
$$x = -\frac{c_5 x\,\mathrm{e}^{\frac{x+y}{y}}}{y} \tag{5}$$

Verification of solutions

$$y = -2x$$

Verified OK.

$$y = 0$$

Verified OK.

$$x = \frac{c_3 x}{(x+y)\sqrt{-\frac{y}{x+y}}\,\sqrt{\frac{2x+y}{x+y}}}$$

Verified OK.

$$y = 0$$

Verified OK.

$$x = -\frac{c_5 x\,\mathrm{e}^{\frac{x+y}{y}}}{y}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 47

```
dsolve((x+y(x))^2*diff(y(x),x)^2=y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{x}{\text{LambertW}\left(x\,\mathrm{e}^{c_1}\right)}$$
$$y(x) = -x - \sqrt{x^2 + 2c_1}$$
$$y(x) = -x + \sqrt{x^2 + 2c_1}$$

✓ Solution by Mathematica

Time used: 4.023 (sec). Leaf size: 101

```
DSolve[(x+y[x])^2*(y'[x])^2==y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x - \sqrt{x^2 + e^{2c_1}}$$
$$y(x) \to -x + \sqrt{x^2 + e^{2c_1}}$$
$$y(x) \to \frac{x}{W\left(e^{-c_1}x\right)}$$
$$y(x) \to 0$$
$$y(x) \to -\sqrt{x^2} - x$$
$$y(x) \to \sqrt{x^2} - x$$

1.10 problem 10

Internal problem ID [6776]
Internal file name [`OUTPUT/6023_Monday_July_25_2022_01_59_50_AM_35329907/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 94. Factoring the left member. EXERCISES Page 309
Problem number: 10.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"exact", "quadrature", "separable", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{yy'^2 + (x - y^2)\, y' - xy = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = y \tag{1}$$
$$y' = -\frac{x}{y} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned}\int \frac{1}{y} dy &= x + c_1 \\ \ln(y) &= x + c_1 \\ y &= e^{x+c_1} \\ y &= c_1 e^x\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x \tag{1}$$

Verification of solutions

$$y = c_1 e^x$$

Verified OK.

Solving equation (2)

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= -\frac{x}{y} \end{aligned}$$

Where $f(x) = -x$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{1}{y}} \, dy &= -x \, dx \\ \int \frac{1}{\frac{1}{y}} \, dy &= \int -x \, dx \\ \frac{y^2}{2} &= -\frac{x^2}{2} + c_2 \end{aligned}$$

Which results in

$$\begin{aligned} y &= \sqrt{-x^2 + 2c_2} \\ y &= -\sqrt{-x^2 + 2c_2} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sqrt{-x^2 + 2c_2} \tag{1}$$
$$y = -\sqrt{-x^2 + 2c_2} \tag{2}$$

Verification of solutions

$$y = \sqrt{-x^2 + 2c_2}$$

Verified OK.

$$y = -\sqrt{-x^2 + 2c_2}$$

Verified OK.

1.10.1 Maple step by step solution

Let's solve

$yy'^2 + (x - y^2)\, y' - xy = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int 1 dx + c_1$

- Evaluate integral

  $\ln(y) = x + c_1$

- Solve for y

  $y = e^{x+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve(y(x)*diff(y(x),x)^2+(x-y(x)^2)*diff(y(x),x)-x*y(x)=0,y(x), singsol=all)
```

$$
\begin{aligned}
y(x) &= \sqrt{-x^2 + c_1} \\
y(x) &= -\sqrt{-x^2 + c_1} \\
y(x) &= \mathrm{e}^{x} c_1
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.134 (sec). Leaf size: 54

```
DSolve[y[x]*(y'[x])^2+(x-y[x]^2)*y'[x]-x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to c_1 e^x \\
y(x) &\to -\sqrt{-x^2 + 2c_1} \\
y(x) &\to \sqrt{-x^2 + 2c_1} \\
y(x) &\to 0
\end{aligned}
$$

1.11 problem 11

Internal problem ID [6777]
Internal file name [OUTPUT/6024_Monday_July_25_2022_01_59_51_AM_43122919/index.tex]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 94. Factoring the left member. EXERCISES Page 309
Problem number: 11.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"exact", "linear", "riccati", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y'^2 - xy(x+y)\,y' + x^3y^3 = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = xy^2 \tag{1}$$
$$y' = yx^2 \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= y^2x \end{aligned}$$

Where $f(x) = x$ and $g(y) = y^2$. Integrating both sides gives

$$\frac{1}{y^2}\,dy = x\,dx$$

$$\int \frac{1}{y^2}\,dy = \int x\,dx$$

$$-\frac{1}{y} = \frac{x^2}{2} + c_1$$

Which results in

$$y = -\frac{2}{x^2 + 2c_1}$$

Summary

The solution(s) found are the following

$$y = -\frac{2}{x^2 + 2c_1} \tag{1}$$

Verification of solutions

$$y = -\frac{2}{x^2 + 2c_1}$$

Verified OK.

Solving equation (2)

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= y\,x^2 \end{aligned}$$

Where $f(x) = x^2$ and $g(y) = y$. Integrating both sides gives

$$\frac{1}{y}\,dy = x^2\,dx$$

$$\int \frac{1}{y}\,dy = \int x^2\,dx$$

$$\ln(y) = \frac{x^3}{3} + c_2$$

$$\begin{aligned} y &= \mathrm{e}^{\frac{x^3}{3} + c_2} \\ &= c_2 \mathrm{e}^{\frac{x^3}{3}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2 \mathrm{e}^{\frac{x^3}{3}} \tag{1}$$

Verification of solutions

$$y = c_2 \mathrm{e}^{\frac{x^3}{3}}$$

Verified OK.

1.11.1 Maple step by step solution

Let's solve

$y'^2 - xy(x+y)\, y' + x^3y^3 = 0$

- Highest derivative means the order of the ODE is 1

  y'
- Separate variables

  $\frac{y'}{y} = x^2$
- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int x^2dx + c_1$
- Evaluate integral

  $\ln(y) = \frac{x^3}{3} + c_1$
- Solve for y

  $y = \mathrm{e}^{\frac{x^3}{3}+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x)^2-x*y(x)*(x+y(x))*diff(y(x),x)+x^3*y(x)^3=0,y(x), singsol=all)
```

$$y(x) = -\frac{2}{x^2 - 2c_1}$$
$$y(x) = c_1 \mathrm{e}^{\frac{x^3}{3}}$$

✓ Solution by Mathematica

Time used: 0.122 (sec). Leaf size: 38

```
DSolve[(y'[x])^2-x*y[x]*(x+y[x])*y'[x]+x^3*y[x]^3==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to c_1 e^{\frac{x^3}{3}}$$
$$y(x) \to -\frac{2}{x^2 + 2c_1}$$
$$y(x) \to 0$$

1.12 problem 12

Internal problem ID [6778]
Internal file name [`OUTPUT/6025_Monday_July_25_2022_01_59_52_AM_16739784/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 94. Factoring the left member. EXERCISES Page 309
Problem number: 12.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{(4x-y)\,{y'}^2+6(x-y)\,y'-5y=-2x}$$

1.12.1 Solving as dAlembert ode

Let $p=y'$ the ode becomes

$$(-y+4x)\,p^2+6(x-y)\,p-5y=-2x$$

Solving for y from the above results in

$$y=\frac{2(2p+1)\,x}{p+5} \tag{1A}$$

This has the form

$$y=xf(p)+g(p) \tag{*}$$

Where f,g are functions of $p=y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p=f+(xf'+g')\frac{dp}{dx}$$
$$p-f=(xf'+g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= \frac{4p+2}{p+5} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p - \frac{4p+2}{p+5} = x\left(\frac{4}{p+5} - \frac{4p+2}{(p+5)^2}\right)p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p - \frac{4p+2}{p+5} = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= 1 \\
p &= -2
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= -2x \\
y &= x
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) - \frac{4p(x)+2}{p(x)+5}}{x\left(\frac{4}{p(x)+5} - \frac{4p(x)+2}{(p(x)+5)^2}\right)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp}x(p) = \frac{x(p)\left(\frac{4}{p+5} - \frac{4p+2}{(p+5)^2}\right)}{p - \frac{4p+2}{p+5}} \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
\frac{d}{dp}x(p) + p(p)x(p) = q(p)
$$

Where here

$$
\begin{aligned}
p(p) &= -\frac{18}{p^3 + 6p^2 + 3p - 10} \\
q(p) &= 0
\end{aligned}
$$

Hence the ode is

$$
\frac{d}{dp}x(p) - \frac{18x(p)}{p^3 + 6p^2 + 3p - 10} = 0
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{18}{p^3+6p^2+3p-10}dp} \\
&= e^{-\ln(p-1)+2\ln(p+2)-\ln(p+5)}
\end{aligned}
$$

Which simplifies to

$$
\mu = \frac{(p+2)^2}{(p-1)(p+5)}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}p}\mu x &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{(p+2)^2 x}{(p-1)(p+5)}\right) &= 0
\end{aligned}
$$

Integrating gives

$$
\frac{(p+2)^2 x}{(p-1)(p+5)} = c_3
$$

Dividing both sides by the integrating factor $\mu = \frac{(p+2)^2}{(p-1)(p+5)}$ results in

$$
x(p) = \frac{c_3(p-1)(p+5)}{(p+2)^2}
$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$
p = -\frac{2x - 5y}{4x - y}
$$

Substituting the above in the solution for x found above gives

$$x = -\frac{12x(x-y)\, c_3}{(2x+y)^2}$$

Summary

The solution(s) found are the following

$$y = -2x \tag{1}$$
$$y = x \tag{2}$$
$$x = -\frac{12x(x-y)\, c_3}{(2x+y)^2} \tag{3}$$

Verification of solutions

$$y = -2x$$

Verified OK.

$$y = x$$

Verified OK.

$$x = -\frac{12x(x-y)\, c_3}{(2x+y)^2}$$

Verified OK.

1.12.2 Maple step by step solution

Let's solve

$(4x-y)\, y'^2 + 6(x-y)\, y' - 5y = -2x$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \left((4x-y)\, y'^2 + 6(x-y)\, y' - 5y\right) dx = \int -2x dx + c_1$

- Cannot compute integral

  $\int \left((4x-y)\, y'^2 + 6(x-y)\, y' - 5y\right) dx = -x^2 + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 55

```
dsolve((4*x-y(x))*diff(y(x),x)^2+6*(x-y(x))*diff(y(x),x)+2*x-5*y(x)=0,y(x), singsol=all)
```

$$
\begin{aligned}
y(x) &= -x + c_1 \\
y(x) &= \frac{-4c_1x + \sqrt{-12c_1x + 1} + 1}{2c_1} \\
y(x) &= \frac{-4c_1x - \sqrt{-12c_1x + 1} + 1}{2c_1}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 1.077 (sec). Leaf size: 90

```
DSolve[(4*x-y[x])*(y'[x])^2+6*(x-y[x])*y'[x]+2*x-5*y[x]==0,y[x],x,IncludeSingularSolutions -
```

$$
\begin{aligned}
y(x) &\to \frac{1}{2}\left(-4x - e^{\frac{c_1}{2}}\sqrt{12x + e^{c_1}} - e^{c_1}\right) \\
y(x) &\to \frac{1}{2}\left(-4x + e^{\frac{c_1}{2}}\sqrt{12x + e^{c_1}} - e^{c_1}\right) \\
y(x) &\to -x + c_1
\end{aligned}
$$

1.13 problem 13

Internal problem ID [6779]
Internal file name [OUTPUT/6026_Monday_July_25_2022_01_59_56_AM_71060998/index.tex]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 94. Factoring the left member. EXERCISES Page 309
Problem number: 13.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{(x-y)^2 {y'}^2 - y^2 = 0}$$

1.13.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$(x-y)^2 p^2 - y^2 = 0$$

Solving for y from the above results in

$$y = \frac{px}{-1+p} \tag{1A}$$

$$y = \frac{px}{1+p} \tag{2A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. Each of the above ode's is dAlembert ode which is now solved. Solving ode 1A Taking derivative of (*) w.r.t. x gives

$$
\begin{aligned}
p &= f + (xf' + g')\frac{dp}{dx} \\
p - f &= (xf' + g')\frac{dp}{dx} \qquad (2)
\end{aligned}
$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= \frac{p}{-1 + p} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p - \frac{p}{-1 + p} = x\left(\frac{1}{-1 + p} - \frac{p}{(-1 + p)^2}\right) p'(x) \qquad (2A)
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p - \frac{p}{-1 + p} = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= 0 \\
p &= 2
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= 0 \\
y &= 2x
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) - \frac{p(x)}{-1+p(x)}}{x\left(\frac{1}{-1+p(x)} - \frac{p(x)}{(-1+p(x))^2}\right)} \qquad (3)
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp}x(p) = \frac{x(p)\left(\frac{1}{-1+p} - \frac{p}{(-1+p)^2}\right)}{p - \frac{p}{-1+p}} \qquad (4)
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = \frac{1}{(-2+p)\,p\,(-1+p)}$$
$$q(p) = 0$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{x(p)}{(-2+p)\,p\,(-1+p)} = 0$$

The integrating factor μ is

$$\mu = e^{\int \frac{1}{(-2+p)p(-1+p)}dp}$$
$$= e^{-\ln(-1+p)+\frac{\ln(p)}{2}+\frac{\ln(-2+p)}{2}}$$

Which simplifies to

$$\mu = \frac{\sqrt{p}\,\sqrt{-2+p}}{-1+p}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}p}\mu x = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{\sqrt{p}\,\sqrt{-2+p}\,x}{-1+p}\right) = 0$$

Integrating gives

$$\frac{\sqrt{p}\,\sqrt{-2+p}\,x}{-1+p} = c_3$$

Dividing both sides by the integrating factor $\mu = \frac{\sqrt{p}\,\sqrt{-2+p}}{-1+p}$ results in

$$x(p) = \frac{c_3(-1+p)}{\sqrt{p}\,\sqrt{-2+p}}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = -\frac{y}{x-y}$$

Substituting the above in the solution for x found above gives

$$x = -\frac{c_3 x}{(x-y)\sqrt{-\frac{y}{x-y}}\sqrt{\frac{-2x+y}{x-y}}}$$

Solving ode 2A Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= \frac{p}{1+p} \\ g &= 0 \end{aligned}$$

Hence (2) becomes

$$p - \frac{p}{1+p} = x\left(\frac{1}{1+p} - \frac{p}{(1+p)^2}\right)p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{p}{1+p} = 0$$

Solving for p from the above gives

$$\begin{aligned} p &= 0 \\ p &= 0 \end{aligned}$$

Removing solutions for p which leads to undefined results and substituting these in (1A) gives

$$y = 0$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - \frac{p(x)}{1+p(x)}}{x\left(\frac{1}{1+p(x)} - \frac{p(x)}{(1+p(x))^2}\right)} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{x(p)\left(\frac{1}{1+p} - \frac{p}{(1+p)^2}\right)}{p - \frac{p}{1+p}} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = -\frac{1}{p^2(1+p)}$$
$$q(p) = 0$$

Hence the ode is

$$\frac{d}{dp}x(p) - \frac{x(p)}{p^2(1+p)} = 0$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int -\frac{1}{p^2(1+p)}dp}$$
$$= \mathrm{e}^{-\ln(1+p)+\frac{1}{p}+\ln(p)}$$

Which simplifies to

$$\mu = \frac{p\,\mathrm{e}^{\frac{1}{p}}}{1+p}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}p}\mu x = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{p\,\mathrm{e}^{\frac{1}{p}}x}{1+p}\right) = 0$$

Integrating gives

$$\frac{p\,\mathrm{e}^{\frac{1}{p}}x}{1+p} = c_5$$

Dividing both sides by the integrating factor $\mu = \frac{p\,\mathrm{e}^{\frac{1}{p}}}{1+p}$ results in

$$x(p) = \frac{c_5(1+p)\,\mathrm{e}^{-\frac{1}{p}}}{p}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = \frac{y}{x-y}$$

Substituting the above in the solution for x found above gives

$$x = \frac{c_5 x\,\mathrm{e}^{\frac{-x+y}{y}}}{y}$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$

$$y = 2x \tag{2}$$

$$x = -\frac{c_3 x}{(x-y)\sqrt{-\frac{y}{x-y}}\sqrt{\frac{-2x+y}{x-y}}} \tag{3}$$

$$y = 0 \tag{4}$$

$$x = \frac{c_5 x\,\mathrm{e}^{\frac{-x+y}{y}}}{y} \tag{5}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$y = 2x$$

Verified OK.

$$x = -\frac{c_3 x}{(x-y)\sqrt{-\frac{y}{x-y}}\sqrt{\frac{-2x+y}{x-y}}}$$

Verified OK.

$$y = 0$$

Verified OK.

$$x = \frac{c_5 x \,\mathrm{e}^{\frac{-x+y}{y}}}{y}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 47

```
dsolve((x-y(x))^2*diff(y(x),x)^2=y(x)^2,y(x), singsol=all)
```

$$
\begin{aligned}
y(x) &= x-\sqrt{x^{2}-2c_{1}} \\
y(x) &= x+\sqrt{x^{2}-2c_{1}} \\
y(x) &= -\frac{x}{\operatorname{LambertW}\left(-x\,\mathrm{e}^{-c_{1}}\right)}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 4.446 (sec). Leaf size: 99

```
DSolve[(x-y[x])^2*(y'[x])^2==y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to x-\sqrt{x^{2}-e^{2c_{1}}} \\
y(x) &\to x+\sqrt{x^{2}-e^{2c_{1}}} \\
y(x) &\to -\frac{x}{W\left(-e^{-c_{1}}x\right)} \\
y(x) &\to 0 \\
y(x) &\to x-\sqrt{x^{2}} \\
y(x) &\to \sqrt{x^{2}}+x
\end{aligned}
$$

1.14 problem 14

Internal problem ID [6780]
Internal file name [`OUTPUT/6027_Monday_July_25_2022_01_59_58_AM_88349000/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 94. Factoring the left member. EXERCISES Page 309
Problem number: 14.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"exact", "quadrature", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{xy{y'}^2+\left(x y^2-1\right) y'-y=0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = -y \tag{1}$$
$$y' = \frac{1}{xy} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int -\frac{1}{y}dy = \int dx$$
$$-\ln\left(y\right) = x + c_1$$

Raising both side to exponential gives

$$\frac{1}{y} = \mathrm{e}^{x+c_1}$$

Which simplifies to

$$\frac{1}{y} = c_2 e^{x}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-x}}{c_2} \tag{1}$$

Verification of solutions

$$y = \frac{e^{-x}}{c_2}$$

Verified OK.

Solving equation (2)

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{1}{yx} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\, dy = \frac{1}{x}\, dx$$

$$\int \frac{1}{\frac{1}{y}}\, dy = \int \frac{1}{x}\, dx$$

$$\frac{y^2}{2} = \ln(x) + c_3$$

Which results in

$$\begin{aligned} y &= \sqrt{2\ln(x) + 2c_3} \\ y &= -\sqrt{2\ln(x) + 2c_3} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sqrt{2\ln(x) + 2c_3} \tag{1}$$
$$y = -\sqrt{2\ln(x) + 2c_3} \tag{2}$$

Verification of solutions

$$y = \sqrt{2\ln(x) + 2c_3}$$

Verified OK.

$$y = -\sqrt{2\ln(x) + 2c_3}$$

Verified OK.

1.14.1 Maple step by step solution

Let's solve

$xyy'^2 + (xy^2 - 1)y' - y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int (-1)\,dx + c_1$

- Evaluate integral

  $\ln(y) = -x + c_1$

- Solve for y

  $y = e^{-x+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 34

```
dsolve(x*y(x)*diff(y(x),x)^2+(x*y(x)^2-1)*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$
\begin{aligned}
y(x) &= \sqrt{2\ln(x) + c_1} \\
y(x) &= -\sqrt{2\ln(x) + c_1} \\
y(x) &= c_1 \mathrm{e}^{-x}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.108 (sec). Leaf size: 57

```
DSolve[x*y[x]*(y'[x])^2+(x*y[x]^2-1)*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to c_1 e^{-x} \\
y(x) &\to -\sqrt{2}\sqrt{\log(x) + c_1} \\
y(x) &\to \sqrt{2}\sqrt{\log(x) + c_1} \\
y(x) &\to 0
\end{aligned}
$$

1.15 problem 15

Internal problem ID [6781]
Internal file name [`OUTPUT/6028_Tuesday_July_26_2022_05_04_37_AM_9550685/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 94. Factoring the left member. EXERCISES Page 309
Problem number: 15.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{\left(x^2+y^2\right)^2 y'^2-4y^2x^2=0}$$

1.15.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$\left(x^2+y^2\right)^2 p^2-4y^2x^2=0$$

Solving for y from the above results in

$$y = \frac{\left(1+\sqrt{-p^2+1}\right)x}{p} \tag{1A}$$

$$y = -\frac{\left(-1+\sqrt{-p^2+1}\right)x}{p} \tag{2A}$$

$$y = \frac{\left(-1+\sqrt{-p^2+1}\right)x}{p} \tag{3A}$$

$$y = -\frac{\left(1+\sqrt{-p^2+1}\right)x}{p} \tag{4A}$$

This has the form

$$y = xf(p)+g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. Each of the above ode's is dAlembert ode which is now solved. Solving ode 1A Taking derivative of (*) w.r.t. x gives

$$
\begin{aligned}
p &= f + (xf' + g')\frac{dp}{dx} \\
p - f &= (xf' + g')\frac{dp}{dx}
\end{aligned} \tag{2}
$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= \frac{1 + \sqrt{-p^2 + 1}}{p} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p - \frac{1 + \sqrt{-p^2 + 1}}{p} = x\left(-\frac{1}{\sqrt{-p^2 + 1}} - \frac{1 + \sqrt{-p^2 + 1}}{p^2}\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p - \frac{1 + \sqrt{-p^2 + 1}}{p} = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= 1 \\
p &= -1
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= -x \\
y &= x
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) - \frac{1 + \sqrt{-p(x)^2 + 1}}{p(x)}}{x\left(-\frac{1}{\sqrt{-p(x)^2 + 1}} - \frac{1 + \sqrt{-p(x)^2 + 1}}{p(x)^2}\right)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{x(p)\left(-\frac{1}{\sqrt{-p^2+1}} - \frac{1+\sqrt{-p^2+1}}{p^2}\right)}{p - \frac{1+\sqrt{-p^2+1}}{p}} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = -\frac{1+\sqrt{-p^2+1}}{\sqrt{-p^2+1}\,p\left(-p^2+\sqrt{-p^2+1}+1\right)}$$

$$q(p) = 0$$

Hence the ode is

$$\frac{d}{dp}x(p) - \frac{x(p)\left(1+\sqrt{-p^2+1}\right)}{p\sqrt{-p^2+1}\left(-p^2+\sqrt{-p^2+1}+1\right)} = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{1+\sqrt{-p^2+1}}{\sqrt{-p^2+1}\,p\left(-p^2+\sqrt{-p^2+1}+1\right)}dp} \\ &= \mathrm{e}^{\frac{\ln(p+1)}{2}+\frac{\ln(p-1)}{2}-\ln(p)}\end{aligned}$$

Which simplifies to

$$\mu = \frac{\sqrt{p+1}\,\sqrt{p-1}}{p}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}\mu x &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{\sqrt{p+1}\,\sqrt{p-1}\,x}{p}\right) &= 0\end{aligned}$$

Integrating gives

$$\frac{\sqrt{p+1}\,\sqrt{p-1}\,x}{p} = c_3$$

Dividing both sides by the integrating factor $\mu = \frac{\sqrt{p+1}\sqrt{p-1}}{p}$ results in

$$x(p) = \frac{c_3 p}{\sqrt{p+1}\sqrt{p-1}}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = \frac{2xy}{x^2+y^2}$$

Substituting the above in the solution for x found above gives

$$x = \frac{2c_3xy}{\sqrt{\frac{(x+y)^2}{x^2+y^2}}\sqrt{-\frac{(x-y)^2}{x^2+y^2}}\,(x^2+y^2)}$$

<u>Solving ode 2A</u> Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= \frac{1-\sqrt{-p^2+1}}{p} \\ g &= 0 \end{aligned}$$

Hence (2) becomes

$$p - \frac{1-\sqrt{-p^2+1}}{p} = x\left(\frac{1}{\sqrt{-p^2+1}} - \frac{1-\sqrt{-p^2+1}}{p^2}\right)p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{1-\sqrt{-p^2+1}}{p} = 0$$

Solving for p from the above gives

$$\begin{aligned} p &= 1 \\ p &= -1 \end{aligned}$$

Substituting these in (1A) gives

$$y = -x$$
$$y = x$$

The general solution is found when $\frac{\mathrm{dp}}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - \frac{1-\sqrt{-p(x)^2+1}}{p(x)}}{x\left(\frac{1}{\sqrt{-p(x)^2+1}} - \frac{1-\sqrt{-p(x)^2+1}}{p(x)^2}\right)} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{x(p)\left(\frac{1}{\sqrt{-p^2+1}} - \frac{1-\sqrt{-p^2+1}}{p^2}\right)}{p - \frac{1-\sqrt{-p^2+1}}{p}} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = \frac{-1+\sqrt{-p^2+1}}{\sqrt{-p^2+1}\,p\left(p^2+\sqrt{-p^2+1}-1\right)}$$
$$q(p) = 0$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{x(p)\left(-1+\sqrt{-p^2+1}\right)}{\sqrt{-p^2+1}\,p\left(p^2+\sqrt{-p^2+1}-1\right)} = 0$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int \frac{-1+\sqrt{-p^2+1}}{\sqrt{-p^2+1}\,p\left(p^2+\sqrt{-p^2+1}-1\right)}dp}$$
$$= \mathrm{e}^{\frac{\ln(p+1)}{2}+\frac{\ln(p-1)}{2}-\ln(p)}$$

Which simplifies to

$$\mu = \frac{\sqrt{p+1}\,\sqrt{p-1}}{p}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}p}\mu x = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{\sqrt{p+1}\,\sqrt{p-1}\,x}{p}\right) = 0$$

Integrating gives

$$\frac{\sqrt{p+1}\,\sqrt{p-1}\,x}{p} = c_6$$

Dividing both sides by the integrating factor $\mu = \frac{\sqrt{p+1}\,\sqrt{p-1}}{p}$ results in

$$x(p) = \frac{c_6 p}{\sqrt{p+1}\,\sqrt{p-1}}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = \frac{2xy}{x^2+y^2}$$

Substituting the above in the solution for x found above gives

$$x = \frac{2c_6 xy}{\sqrt{\frac{(x+y)^2}{x^2+y^2}}\,\sqrt{-\frac{(x-y)^2}{x^2+y^2}}\,(x^2+y^2)}$$

Solving ode 3A Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = \frac{-1+\sqrt{-p^2+1}}{p}$$
$$g = 0$$

Hence (2) becomes

$$p-\frac{-1+\sqrt{-p^{2}+1}}{p}=x\left(-\frac{1}{\sqrt{-p^{2}+1}}-\frac{-1+\sqrt{-p^{2}+1}}{p^{2}}\right)p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx}=0$ in the above which gives

$$p-\frac{-1+\sqrt{-p^{2}+1}}{p}=0$$

No singular solution are found

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x}\neq 0$. From eq. (2A). This results in

$$p'(x)=\frac{p(x)-\frac{-1+\sqrt{-p(x)^{2}+1}}{p(x)}}{x\left(-\frac{1}{\sqrt{-p(x)^{2}+1}}-\frac{-1+\sqrt{-p(x)^{2}+1}}{p(x)^{2}}\right)} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p)=\frac{x(p)\left(-\frac{1}{\sqrt{-p^{2}+1}}-\frac{-1+\sqrt{-p^{2}+1}}{p^{2}}\right)}{p-\frac{-1+\sqrt{-p^{2}+1}}{p}} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p)+p(p)x(p)=q(p)$$

Where here

$$p(p)=-\frac{-1+\sqrt{-p^{2}+1}}{\left(p^{2}-\sqrt{-p^{2}+1}+1\right)\sqrt{-p^{2}+1}\,p}$$

$$q(p)=0$$

Hence the ode is

$$\frac{d}{dp}x(p)-\frac{\left(-1+\sqrt{-p^{2}+1}\right)x(p)}{\left(p^{2}-\sqrt{-p^{2}+1}+1\right)\sqrt{-p^{2}+1}\,p}=0$$

The integrating factor μ is

$$\mu=\mathrm{e}^{\int -\frac{-1+\sqrt{-p^{2}+1}}{\left(p^{2}-\sqrt{-p^{2}+1}+1\right)\sqrt{-p^{2}+1}\,p}dp}$$

The ode becomes

$$\frac{d}{dp}\mu x = 0$$

$$\frac{d}{dp}\left(e^{\int -\frac{-1+\sqrt{-p^2+1}}{\left(p^2-\sqrt{-p^2+1}+1\right)\sqrt{-p^2+1}\,p}dp} x\right) = 0$$

Integrating gives

$$e^{\int -\frac{-1+\sqrt{-p^2+1}}{\left(p^2-\sqrt{-p^2+1}+1\right)\sqrt{-p^2+1}\,p}dp} x = c_8$$

Dividing both sides by the integrating factor $\mu = e^{\int -\frac{-1+\sqrt{-p^2+1}}{\left(p^2-\sqrt{-p^2+1}+1\right)\sqrt{-p^2+1}\,p}dp}$ results in

$$x(p) = c_8 e^{-\left(\int \frac{-1+\sqrt{-p^2+1}}{\sqrt{-p^2+1}\,p\left(-p^2+\sqrt{-p^2+1}-1\right)}dp\right)}$$

Since the solution $x(p)$ has unresolved integral, unable to continue.

Solving ode 4A Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= \frac{-1-\sqrt{-p^2+1}}{p} \\ g &= 0 \end{aligned}$$

Hence (2) becomes

$$p - \frac{-1-\sqrt{-p^2+1}}{p} = x\left(\frac{1}{\sqrt{-p^2+1}} - \frac{-1-\sqrt{-p^2+1}}{p^2}\right)p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{-1-\sqrt{-p^2+1}}{p} = 0$$

Solving for p from the above gives

$$p = i\sqrt{3}$$
$$p = -i\sqrt{3}$$

Substituting these in (1A) gives

$$y = -i\sqrt{3}\, x$$
$$y = i\sqrt{3}\, x$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - \frac{-1-\sqrt{-p(x)^2+1}}{p(x)}}{x\left(\frac{1}{\sqrt{-p(x)^2+1}} - \frac{-1-\sqrt{-p(x)^2+1}}{p(x)^2}\right)} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{x(p)\left(\frac{1}{\sqrt{-p^2+1}} - \frac{-1-\sqrt{-p^2+1}}{p^2}\right)}{p - \frac{-1-\sqrt{-p^2+1}}{p}} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = -\frac{1+\sqrt{-p^2+1}}{\sqrt{-p^2+1}\, p\left(p^2+\sqrt{-p^2+1}+1\right)}$$
$$q(p) = 0$$

Hence the ode is

$$\frac{d}{dp}x(p) - \frac{x(p)\left(1+\sqrt{-p^2+1}\right)}{p\sqrt{-p^2+1}\,\left(p^2+\sqrt{-p^2+1}+1\right)} = 0$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int -\frac{1+\sqrt{-p^2+1}}{\sqrt{-p^2+1}\, p\left(p^2+\sqrt{-p^2+1}+1\right)}dp}$$

The ode becomes

$$\frac{d}{dp}\mu x = 0$$

$$\frac{d}{dp}\left(e^{\int -\frac{1+\sqrt{-p^2+1}}{\sqrt{-p^2+1}\, p\left(p^2+\sqrt{-p^2+1}+1\right)}dp} x\right) = 0$$

Integrating gives

$$e^{\int -\frac{1+\sqrt{-p^2+1}}{\sqrt{-p^2+1}\, p\left(p^2+\sqrt{-p^2+1}+1\right)}dp} x = \_C10$$

Dividing both sides by the integrating factor $\mu = e^{\int -\frac{1+\sqrt{-p^2+1}}{\sqrt{-p^2+1}\, p\left(p^2+\sqrt{-p^2+1}+1\right)}dp}$ results in

$$x(p) = \_C10\, e^{\int \frac{1+\sqrt{-p^2+1}}{\sqrt{-p^2+1}\, p\left(p^2+\sqrt{-p^2+1}+1\right)}dp}$$

Since the solution $x(p)$ has unresolved integral, unable to continue.

Summary

The solution(s) found are the following

$$y = -x \tag{1}$$

$$y = x \tag{2}$$

$$x = \frac{2c_3 x y}{\sqrt{\frac{(x+y)^2}{x^2+y^2}}\, \sqrt{-\frac{(x-y)^2}{x^2+y^2}}\, (x^2+y^2)} \tag{3}$$

$$y = -x \tag{4}$$

$$y = x \tag{5}$$

$$x = \frac{2c_6 x y}{\sqrt{\frac{(x+y)^2}{x^2+y^2}}\, \sqrt{-\frac{(x-y)^2}{x^2+y^2}}\, (x^2+y^2)} \tag{6}$$

$$y = -i\sqrt{3}\, x \tag{7}$$

$$y = i\sqrt{3}\, x \tag{8}$$

Verification of solutions

$$y = -x$$

Verified OK.

$$y = x$$

Verified OK.

$$x = \frac{2c_3 x y}{\sqrt{\frac{(x+y)^2}{x^2+y^2}}\,\sqrt{-\frac{(x-y)^2}{x^2+y^2}}\,\left(x^2+y^2\right)}$$

Verified OK.

$$y = -x$$

Verified OK.

$$y = x$$

Verified OK.

$$x = \frac{2c_6 x y}{\sqrt{\frac{(x+y)^2}{x^2+y^2}}\,\sqrt{-\frac{(x-y)^2}{x^2+y^2}}\,\left(x^2+y^2\right)}$$

Verified OK.

$$y = -i\sqrt{3}\,x$$

Verified OK.

$$y = i\sqrt{3}\,x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.141 (sec). Leaf size: 255

```
dsolve((x^2+y(x)^2)^2*diff(y(x),x)^2=4*x^2*y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{1-\sqrt{4x^2c_1^2+1}}{2c_1}$$

$$y(x) = \frac{1+\sqrt{4x^2c_1^2+1}}{2c_1}$$

$$y(x) = -\frac{2\left(c_1x^2-\frac{\left(4+4\sqrt{4c_1^3x^6+1}\right)^{\frac{2}{3}}}{4}\right)}{\sqrt{c_1}\left(4+4\sqrt{4c_1^3x^6+1}\right)^{\frac{1}{3}}}$$

$$y(x) = -\frac{\left(1+i\sqrt{3}\right)\left(4+4\sqrt{4c_1^3x^6+1}\right)^{\frac{1}{3}}}{4\sqrt{c_1}} - \frac{\left(i\sqrt{3}-1\right)x^2\sqrt{c_1}}{\left(4+4\sqrt{4c_1^3x^6+1}\right)^{\frac{1}{3}}}$$

$$y(x) = \frac{4i\sqrt{3}\,c_1x^2+i\sqrt{3}\left(4+4\sqrt{4c_1^3x^6+1}\right)^{\frac{2}{3}}+4c_1x^2-\left(4+4\sqrt{4c_1^3x^6+1}\right)^{\frac{2}{3}}}{4\left(4+4\sqrt{4c_1^3x^6+1}\right)^{\frac{1}{3}}\sqrt{c_1}}$$

✓ Solution by Mathematica

Time used: 15.845 (sec). Leaf size: 345

```
DSolve[(x^2+y[x]^2)^2*(y'[x])^2==4*x^2*y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(-\sqrt{4x^2+e^{2c_1}}-e^{c_1}\right)$$
$$y(x) \to \frac{1}{2}\left(\sqrt{4x^2+e^{2c_1}}-e^{c_1}\right)$$
$$y(x) \to \frac{\sqrt[3]{\sqrt{4x^6+e^{6c_1}}+e^{3c_1}}}{\sqrt[3]{2}}-\frac{\sqrt[3]{2}x^2}{\sqrt[3]{\sqrt{4x^6+e^{6c_1}}+e^{3c_1}}}$$
$$y(x) \to \frac{i2^{2/3}\left(\sqrt{3}+i\right)\left(\sqrt{4x^6+e^{6c_1}}+e^{3c_1}\right)^{2/3}+\sqrt[3]{2}\left(2+2i\sqrt{3}\right)x^2}{4\sqrt[3]{\sqrt{4x^6+e^{6c_1}}+e^{3c_1}}}$$
$$y(x) \to \frac{\left(1-i\sqrt{3}\right)x^2}{2^{2/3}\sqrt[3]{\sqrt{4x^6+e^{6c_1}}+e^{3c_1}}}-\frac{\left(1+i\sqrt{3}\right)\sqrt[3]{\sqrt{4x^6+e^{6c_1}}+e^{3c_1}}}{2\sqrt[3]{2}}$$
$$y(x) \to 0$$

1.16 problem 16

Internal problem ID [6782]
Internal file name [`OUTPUT/6029_Tuesday_July_26_2022_05_04_40_AM_96206336/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 94. Factoring the left member. EXERCISES Page 309
Problem number: 16.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{(x+y)^2 {y'}^2+\left(2y^2+xy-x^2\right) y'+y(-x+y)=0}$$

1.16.1 Solving as dAlembert ode

Let $p=y'$ the ode becomes

$$(x+y)^2 p^2+\left(-x^2+xy+2y^2\right) p+y(-x+y)=0$$

Solving for y from the above results in

$$y=-\frac{(p-1)\,x}{1+p} \tag{1A}$$
$$y=-\frac{xp}{1+p} \tag{2A}$$

This has the form

$$y=xf(p)+g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. Each of the above ode's is dAlembert ode which is now solved. Solving ode 1A Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= \frac{-p+1}{1+p} \\ g &= 0 \end{aligned}$$

Hence (2) becomes

$$p - \frac{-p+1}{1+p} = x\left(-\frac{1}{1+p} - \frac{-p+1}{(1+p)^2}\right)p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{-p+1}{1+p} = 0$$

Solving for p from the above gives

$$\begin{aligned} p &= \sqrt{2} - 1 \\ p &= -1 - \sqrt{2} \end{aligned}$$

Substituting these in (1A) gives

$$\begin{aligned} y &= -x - x\sqrt{2} \\ y &= -x + x\sqrt{2} \end{aligned}$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - \frac{-p(x)+1}{1+p(x)}}{x\left(-\frac{1}{1+p(x)} - \frac{-p(x)+1}{(1+p(x))^2}\right)} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{x(p)\left(-\frac{1}{1+p} - \frac{-p+1}{(1+p)^2}\right)}{p - \frac{-p+1}{1+p}} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = \frac{2}{(p^2 + 2p - 1)(1 + p)}$$
$$q(p) = 0$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{2x(p)}{(p^2 + 2p - 1)(1 + p)} = 0$$

The integrating factor μ is

$$\mu = e^{\int \frac{2}{(p^2+2p-1)(1+p)}dp}$$
$$= e^{-\ln(1+p)+\frac{\ln(p^2+2p-1)}{2}}$$

Which simplifies to

$$\mu = \frac{\sqrt{p^2 + 2p - 1}}{1 + p}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}p}\mu x = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{\sqrt{p^2 + 2p - 1}\, x}{1 + p}\right) = 0$$

Integrating gives

$$\frac{\sqrt{p^2 + 2p - 1}\, x}{1 + p} = c_3$$

Dividing both sides by the integrating factor $\mu = \frac{\sqrt{p^2+2p-1}}{1+p}$ results in

$$x(p) = \frac{c_3(1 + p)}{\sqrt{p^2 + 2p - 1}}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = \frac{x - y}{x + y}$$

Substituting the above in the solution for x found above gives

$$x = \frac{c_3 x \sqrt{2}}{(x + y)\sqrt{\frac{x^2 - 2xy - y^2}{(x+y)^2}}}$$

<u>Solving ode 2A</u> Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= -\frac{p}{1 + p} \\ g &= 0 \end{aligned}$$

Hence (2) becomes

$$p + \frac{p}{1 + p} = x\left(-\frac{1}{1 + p} + \frac{p}{(1 + p)^2}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p + \frac{p}{1 + p} = 0$$

Solving for p from the above gives

$$\begin{aligned} p &= -2 \\ p &= 0 \end{aligned}$$

Substituting these in (1A) gives

$$\begin{aligned} y &= -2x \\ y &= 0 \end{aligned}$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) + \frac{p(x)}{1+p(x)}}{x\left(-\frac{1}{1+p(x)} + \frac{p(x)}{(1+p(x))^2}\right)} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{x(p)\left(-\frac{1}{1+p} + \frac{p}{(1+p)^2}\right)}{p + \frac{p}{1+p}} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = \frac{1}{(2+p)\,p\,(1+p)}$$
$$q(p) = 0$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{x(p)}{(2+p)\,p\,(1+p)} = 0$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int \frac{1}{(2+p)p(1+p)}dp}$$
$$= \mathrm{e}^{-\ln(1+p)+\frac{\ln(p)}{2}+\frac{\ln(2+p)}{2}}$$

Which simplifies to

$$\mu = \frac{\sqrt{p}\,\sqrt{2+p}}{1+p}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}p}\mu x = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{\sqrt{p}\,\sqrt{2+p}\,x}{1+p}\right) = 0$$

Integrating gives

$$\frac{\sqrt{p}\,\sqrt{2+p}\,x}{1+p} = c_6$$

Dividing both sides by the integrating factor $\mu = \frac{\sqrt{p}\,\sqrt{2+p}}{1+p}$ results in

$$x(p) = \frac{c_6(1+p)}{\sqrt{p}\,\sqrt{2+p}}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = -\frac{y}{x+y}$$

Substituting the above in the solution for x found above gives

$$x = \frac{c_6 x}{(x+y)\sqrt{-\frac{y}{x+y}}\,\sqrt{\frac{2x+y}{x+y}}}$$

Summary

The solution(s) found are the following

$$y = -x - x\sqrt{2} \tag{1}$$

$$y = -x + x\sqrt{2} \tag{2}$$

$$x = \frac{c_3 x\sqrt{2}}{(x+y)\sqrt{\frac{x^2-2xy-y^2}{(x+y)^2}}} \tag{3}$$

$$y = -2x \tag{4}$$

$$y = 0 \tag{5}$$

$$x = \frac{c_6 x}{(x+y)\sqrt{-\frac{y}{x+y}}\,\sqrt{\frac{2x+y}{x+y}}} \tag{6}$$

Verification of solutions

$$y = -x - x\sqrt{2}$$

Verified OK.

$$y = -x + x\sqrt{2}$$

Verified OK.

$$x = \frac{c_3 x\sqrt{2}}{(x+y)\sqrt{\frac{x^2-2xy-y^2}{(x+y)^2}}}$$

Verified OK.

$$y = -2x$$

Verified OK.

$$y = 0$$

Verified OK.

$$x = \frac{c_6 x}{(x+y)\sqrt{-\frac{y}{x+y}}\sqrt{\frac{2x+y}{x+y}}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 85

```
dsolve((y(x)+x)^2*diff(y(x),x)^2+(2*y(x)^2+x*y(x)-x^2)*diff(y(x),x)+y(x)*(y(x)-x)=0,y(x), si
```

$$y(x) = -x - \sqrt{x^2 + 2c_1}$$
$$y(x) = -x + \sqrt{x^2 + 2c_1}$$
$$y(x) = \frac{-c_1 x - \sqrt{2x^2 c_1^2 + 1}}{c_1}$$
$$y(x) = \frac{-c_1 x + \sqrt{2x^2 c_1^2 + 1}}{c_1}$$

✓ Solution by Mathematica

Time used: 0.492 (sec). Leaf size: 172

```
DSolve[(y[x]+x)^2*(y'[x])^2+(2*y[x]^2+x*y[x]-x^2)*y'[x]+y[x]*(y[x]-x)==0,y[x],x,IncludeSingu
```

$$y(x) \to -x - \sqrt{x^2 + e^{2c_1}}$$
$$y(x) \to -x + \sqrt{x^2 + e^{2c_1}}$$
$$y(x) \to -x - \sqrt{2x^2 + e^{2c_1}}$$
$$y(x) \to -x + \sqrt{2x^2 + e^{2c_1}}$$
$$y(x) \to -\sqrt{x^2} - x$$
$$y(x) \to \sqrt{x^2} - x$$
$$y(x) \to -\sqrt{2}\sqrt{x^2} - x$$
$$y(x) \to \sqrt{2}\sqrt{x^2} - x$$

1.17 problem 17

Internal problem ID [6783]
Internal file name [OUTPUT/6030_Tuesday_July_26_2022_05_04_42_AM_73131458/index.tex]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 94. Factoring the left member. EXERCISES Page 309
Problem number: 17.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{xy(x^2+y^2)\left(y'^2-1\right)-y'(x^4+y^2x^2+y^4)=0}$$

1.17.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$xy(x^2+y^2)\,(p^2-1)-p(x^4+y^2x^2+y^4)=0$$

Solving for y from the above results in

$$y = \frac{\left(-1+\sqrt{-4p^2+1}\right)x}{2p} \tag{1A}$$

$$y = -\frac{\left(1+\sqrt{-4p^2+1}\right)x}{2p} \tag{2A}$$

$$y = \left(\frac{p}{2}+\frac{\sqrt{p^2-4}}{2}\right)x \tag{3A}$$

$$y = \left(\frac{p}{2}-\frac{\sqrt{p^2-4}}{2}\right)x \tag{4A}$$

This has the form

$$y = xf(p)+g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. Each of the above ode's is dAlembert ode which is now solved. Solving ode 1A Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= \frac{-1 + \sqrt{-4p^2 + 1}}{2p} \\ g &= 0 \end{aligned}$$

Hence (2) becomes

$$p - \frac{-1 + \sqrt{-4p^2 + 1}}{2p} = x\left(-\frac{-1 + \sqrt{-4p^2 + 1}}{2p^2} - \frac{2}{\sqrt{-4p^2 + 1}}\right)p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{-1 + \sqrt{-4p^2 + 1}}{2p} = 0$$

No singular solution are found

The general solution is found when $\frac{\mathrm{dp}}{\mathrm{dx}} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - \frac{-1+\sqrt{-4p(x)^2+1}}{2p(x)}}{x\left(-\frac{-1+\sqrt{-4p(x)^2+1}}{2p(x)^2} - \frac{2}{\sqrt{-4p(x)^2+1}}\right)} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{x(p)\left(-\frac{-1+\sqrt{-4p^2+1}}{2p^2} - \frac{2}{\sqrt{-4p^2+1}}\right)}{p - \frac{-1+\sqrt{-4p^2+1}}{2p}} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$
\begin{aligned}
p(p) &= -\frac{1-\sqrt{-4p^2+1}}{p\sqrt{-4p^2+1}\,\left(-2p^2+\sqrt{-4p^2+1}-1\right)} \\
q(p) &= 0
\end{aligned}
$$

Hence the ode is

$$
\frac{d}{dp}x(p) - \frac{x(p)\left(1-\sqrt{-4p^2+1}\right)}{p\sqrt{-4p^2+1}\,\left(-2p^2+\sqrt{-4p^2+1}-1\right)} = 0
$$

The integrating factor μ is

$$
\mu = \mathrm{e}^{\int -\frac{1-\sqrt{-4p^2+1}}{p\sqrt{-4p^2+1}\left(-2p^2+\sqrt{-4p^2+1}-1\right)}dp}
$$

The ode becomes

$$
\frac{\mathrm{d}}{\mathrm{d}p}\mu x = 0
$$

$$
\frac{\mathrm{d}}{\mathrm{d}p}\left(\mathrm{e}^{\int -\frac{1-\sqrt{-4p^2+1}}{p\sqrt{-4p^2+1}\left(-2p^2+\sqrt{-4p^2+1}-1\right)}dp} x\right) = 0
$$

Integrating gives

$$
\mathrm{e}^{\int -\frac{1-\sqrt{-4p^2+1}}{p\sqrt{-4p^2+1}\left(-2p^2+\sqrt{-4p^2+1}-1\right)}dp} x = c_2
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\int -\frac{1-\sqrt{-4p^2+1}}{p\sqrt{-4p^2+1}\left(-2p^2+\sqrt{-4p^2+1}-1\right)}dp}$ results in

$$
x(p) = c_2\mathrm{e}^{-\left(\int \frac{-1+\sqrt{-4p^2+1}}{p\sqrt{-4p^2+1}\left(-2p^2+\sqrt{-4p^2+1}-1\right)}dp\right)}
$$

Since the solution $x(p)$ has unresolved integral, unable to continue.

Solving ode 2A Taking derivative of (*) w.r.t. x gives

$$
\begin{aligned}
p &= f + (xf' + g')\frac{dp}{dx} \\
p - f &= (xf' + g')\frac{dp}{dx} \qquad (2)
\end{aligned}
$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= \frac{-1-\sqrt{-4p^2+1}}{2p} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p - \frac{-1-\sqrt{-4p^2+1}}{2p} = x\left(-\frac{-1-\sqrt{-4p^2+1}}{2p^2}+\frac{2}{\sqrt{-4p^2+1}}\right)p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p - \frac{-1-\sqrt{-4p^2+1}}{2p} = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= i\sqrt{2} \\
p &= -i\sqrt{2}
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= -i\sqrt{2}\,x \\
y &= i\sqrt{2}\,x
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) - \frac{-1-\sqrt{-4p(x)^2+1}}{2p(x)}}{x\left(-\frac{-1-\sqrt{-4p(x)^2+1}}{2p(x)^2}+\frac{2}{\sqrt{-4p(x)^2+1}}\right)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp}x(p) = \frac{x(p)\left(-\frac{-1-\sqrt{-4p^2+1}}{2p^2}+\frac{2}{\sqrt{-4p^2+1}}\right)}{p-\frac{-1-\sqrt{-4p^2+1}}{2p}} \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
\frac{d}{dp}x(p) + p(p)x(p) = q(p)
$$

Where here

$$
\begin{aligned}
p(p) &= -\frac{1+\sqrt{-4p^2+1}}{p\sqrt{-4p^2+1}\,\left(2p^2+\sqrt{-4p^2+1}+1\right)} \\
q(p) &= 0
\end{aligned}
$$

Hence the ode is

$$
\frac{d}{dp}x(p) - \frac{x(p)\left(1+\sqrt{-4p^2+1}\right)}{p\sqrt{-4p^2+1}\,\left(2p^2+\sqrt{-4p^2+1}+1\right)} = 0
$$

The integrating factor μ is

$$
\mu = \mathrm{e}^{\int -\frac{1+\sqrt{-4p^2+1}}{p\sqrt{-4p^2+1}\,\left(2p^2+\sqrt{-4p^2+1}+1\right)}dp}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}p}\mu x &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}p}\left(\mathrm{e}^{\int -\frac{1+\sqrt{-4p^2+1}}{p\sqrt{-4p^2+1}\,\left(2p^2+\sqrt{-4p^2+1}+1\right)}dp}x\right) &= 0
\end{aligned}
$$

Integrating gives

$$
\mathrm{e}^{\int -\frac{1+\sqrt{-4p^2+1}}{p\sqrt{-4p^2+1}\,\left(2p^2+\sqrt{-4p^2+1}+1\right)}dp}x = c_4
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\int -\frac{1+\sqrt{-4p^2+1}}{p\sqrt{-4p^2+1}\,\left(2p^2+\sqrt{-4p^2+1}+1\right)}dp}$ results in

$$
x(p) = c_4\mathrm{e}^{\int \frac{1+\sqrt{-4p^2+1}}{p\sqrt{-4p^2+1}\,\left(2p^2+\sqrt{-4p^2+1}+1\right)}dp}
$$

Since the solution $x(p)$ has unresolved integral, unable to continue.

Solving ode 3A Taking derivative of (*) w.r.t. x gives

$$
\begin{aligned}
p &= f + (xf' + g')\frac{dp}{dx} \\
p - f &= (xf' + g')\frac{dp}{dx} \qquad (2)
\end{aligned}
$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= \frac{p}{2} + \frac{\sqrt{p^2 - 4}}{2} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
\frac{p}{2} - \frac{\sqrt{p^2 - 4}}{2} = x\left(\frac{1}{2} + \frac{p}{2\sqrt{p^2 - 4}}\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
\frac{p}{2} - \frac{\sqrt{p^2 - 4}}{2} = 0
$$

No singular solution are found

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{\frac{p(x)}{2} - \frac{\sqrt{p(x)^2-4}}{2}}{x\left(\frac{1}{2} + \frac{p(x)}{2\sqrt{p(x)^2-4}}\right)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp}x(p) = \frac{x(p)\left(\frac{1}{2} + \frac{p}{2\sqrt{p^2-4}}\right)}{\frac{p}{2} - \frac{\sqrt{p^2-4}}{2}} \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
\frac{d}{dp}x(p) + p(p)x(p) = q(p)
$$

Where here

$$
\begin{aligned}
p(p) &= -\frac{p + \sqrt{p^2 - 4}}{\sqrt{p^2 - 4}\,\left(p - \sqrt{p^2 - 4}\right)} \\
q(p) &= 0
\end{aligned}
$$

Hence the ode is

$$\frac{d}{dp}x(p) - \frac{\left(p + \sqrt{p^2-4}\right)x(p)}{\sqrt{p^2-4}\,\left(p - \sqrt{p^2-4}\right)} = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{p+\sqrt{p^2-4}}{\sqrt{p^2-4}\left(p-\sqrt{p^2-4}\right)}dp} \\ &= \mathrm{e}^{-\frac{\sqrt{p^2-4}\,p}{4}-\frac{p^2}{4}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}\mu x &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}p}\left(\mathrm{e}^{-\frac{\sqrt{p^2-4}\,p}{4}-\frac{p^2}{4}}x\right) &= 0\end{aligned}$$

Integrating gives

$$\mathrm{e}^{-\frac{\sqrt{p^2-4}\,p}{4}-\frac{p^2}{4}}x = c_6$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-\frac{\sqrt{p^2-4}\,p}{4}-\frac{p^2}{4}}$ results in

$$x(p) = c_6\mathrm{e}^{\frac{\left(p+\sqrt{p^2-4}\right)p}{4}}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = \frac{x^2+y^2}{xy}$$

Substituting the above in the solution for x found above gives

$$x = c_6\mathrm{e}^{\frac{\left(\sqrt{\frac{\left(x^2-y^2\right)^2}{y^2x^2}}\,xy+x^2+y^2\right)\left(x^2+y^2\right)}{4y^2x^2}}$$

Solving ode 4A Taking derivative of (*) w.r.t. x gives

$$\begin{aligned}p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \qquad (2)\end{aligned}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= \frac{p}{2} - \frac{\sqrt{p^2-4}}{2} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
\frac{p}{2} + \frac{\sqrt{p^2-4}}{2} = x\left(\frac{1}{2} - \frac{p}{2\sqrt{p^2-4}}\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
\frac{p}{2} + \frac{\sqrt{p^2-4}}{2} = 0
$$

No singular solution are found

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{\frac{p(x)}{2} + \frac{\sqrt{p(x)^2-4}}{2}}{x\left(\frac{1}{2} - \frac{p(x)}{2\sqrt{p(x)^2-4}}\right)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp}x(p) = \frac{x(p)\left(\frac{1}{2} - \frac{p}{2\sqrt{p^2-4}}\right)}{\frac{p}{2} + \frac{\sqrt{p^2-4}}{2}} \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
\frac{d}{dp}x(p) + p(p)x(p) = q(p)
$$

Where here

$$
\begin{aligned}
p(p) &= -\frac{-p+\sqrt{p^2-4}}{\sqrt{p^2-4}\,\left(p+\sqrt{p^2-4}\right)} \\
q(p) &= 0
\end{aligned}
$$

Hence the ode is

$$\frac{d}{dp}x(p) - \frac{x(p)\left(-p+\sqrt{p^2-4}\right)}{\sqrt{p^2-4}\,\left(p+\sqrt{p^2-4}\right)} = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{-p+\sqrt{p^2-4}}{\sqrt{p^2-4}\,\left(p+\sqrt{p^2-4}\right)}dp} \\ &= \mathrm{e}^{\frac{\sqrt{p^2-4}\,p}{4}-\frac{p^2}{4}}\end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}p}\mu x = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\mathrm{e}^{\frac{\sqrt{p^2-4}\,p}{4}-\frac{p^2}{4}}x\right) = 0$$

Integrating gives

$$\mathrm{e}^{\frac{\sqrt{p^2-4}\,p}{4}-\frac{p^2}{4}}x = c_8$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\frac{\sqrt{p^2-4}\,p}{4}-\frac{p^2}{4}}$ results in

$$x(p) = c_8\mathrm{e}^{-\frac{\left(-p+\sqrt{p^2-4}\right)p}{4}}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = \frac{x^2+y^2}{xy}$$

Substituting the above in the solution for x found above gives

$$x = c_8\mathrm{e}^{\frac{\left(-\sqrt{\frac{\left(x^2-y^2\right)^2}{y^2x^2}}\,xy+x^2+y^2\right)\left(x^2+y^2\right)}{4y^2x^2}}$$

Summary

The solution(s) found are the following

$$y = -i\sqrt{2}\,x \tag{1}$$

$$y = i\sqrt{2}\,x \tag{2}$$

$$x = c_6 \,\mathrm{e}^{\frac{\left(\sqrt{\frac{\left(x^2-y^2\right)^2}{y^2x^2}}\,xy+x^2+y^2\right)\left(x^2+y^2\right)}{4y^2x^2}} \tag{3}$$

$$x = c_8 \,\mathrm{e}^{\frac{\left(-\sqrt{\frac{\left(x^2-y^2\right)^2}{y^2x^2}}\,xy+x^2+y^2\right)\left(x^2+y^2\right)}{4y^2x^2}} \tag{4}$$

Verification of solutions

$$y = -i\sqrt{2}\,x$$

Verified OK.

$$y = i\sqrt{2}\,x$$

Verified OK.

$$x = c_6 \,\mathrm{e}^{\frac{\left(\sqrt{\frac{\left(x^2-y^2\right)^2}{y^2x^2}}\,xy+x^2+y^2\right)\left(x^2+y^2\right)}{4y^2x^2}}$$

Verified OK.

$$x = c_8 \,\mathrm{e}^{\frac{\left(-\sqrt{\frac{\left(x^2-y^2\right)^2}{y^2x^2}}\,xy+x^2+y^2\right)\left(x^2+y^2\right)}{4y^2x^2}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 1.015 (sec). Leaf size: 248

```
dsolve(x*y(x)*(x^2+y(x)^2)*(diff(y(x),x)^2-1)=diff(y(x),x)*(x^4+x^2*y(x)^2+y(x)^4),y(x), sin
```

$$y(x) = \frac{\sqrt{x^2 c_1 \left(c_1 x^2 - \sqrt{c_1^2 x^4 + 1}\right)}}{x \left(c_1 x^2 - \sqrt{c_1^2 x^4 + 1}\right) c_1}$$

$$y(x) = \frac{\sqrt{x^2 c_1 \left(c_1 x^2 + \sqrt{c_1^2 x^4 + 1}\right)}}{x \left(c_1 x^2 + \sqrt{c_1^2 x^4 + 1}\right) c_1}$$

$$y(x) = \frac{\sqrt{x^2 c_1 \left(c_1 x^2 - \sqrt{c_1^2 x^4 + 1}\right)}}{x \left(-c_1 x^2 + \sqrt{c_1^2 x^4 + 1}\right) c_1}$$

$$y(x) = -\frac{\sqrt{x^2 c_1 \left(c_1 x^2 + \sqrt{c_1^2 x^4 + 1}\right)}}{x \left(c_1 x^2 + \sqrt{c_1^2 x^4 + 1}\right) c_1}$$

$$y(x) = \sqrt{2 \ln (x) + c_1}\, x$$

$$y(x) = -\sqrt{2 \ln (x) + c_1}\, x$$

✓ Solution by Mathematica

Time used: 9.298 (sec). Leaf size: 248

```
DSolve[x*y[x]*(x^2+y[x]^2)*((y'[x])^2-1)==y'[x]*(x^4+x^2*y[x]^2+y[x]^4),y[x],x,IncludeSingul
```

$$y(x) \to -\sqrt{-x^2 - \sqrt{x^4 + e^{4c_1}}}$$
$$y(x) \to \sqrt{-x^2 - \sqrt{x^4 + e^{4c_1}}}$$
$$y(x) \to -\sqrt{-x^2 + \sqrt{x^4 + e^{4c_1}}}$$
$$y(x) \to \sqrt{-x^2 + \sqrt{x^4 + e^{4c_1}}}$$
$$y(x) \to -x\sqrt{2\log(x) + c_1}$$
$$y(x) \to x\sqrt{2\log(x) + c_1}$$
$$y(x) \to -\sqrt{-\sqrt{x^4} - x^2}$$
$$y(x) \to \sqrt{-\sqrt{x^4} - x^2}$$
$$y(x) \to -\sqrt{\sqrt{x^4} - x^2}$$
$$y(x) \to \sqrt{\sqrt{x^4} - x^2}$$

1.18 problem 18

Internal problem ID [6784]
Internal file name [`OUTPUT/6031_Tuesday_July_26_2022_05_04_44_AM_84731963/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 94. Factoring the left member. EXERCISES Page 309
Problem number: 18.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"exact", "linear", "quadrature", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{xy'^3 - (x^2 + x + y)\, y'^2 + (x^2 + xy + y)\, y' - xy = 0}$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = 1 \tag{1}$$

$$y' = x \tag{2}$$

$$y' = \frac{y}{x} \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int 1 \, \mathrm{d}x \\ &= x + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x + c_1 \tag{1}$$

Verification of solutions

$$y = x + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\begin{aligned} y &= \int x \, \mathrm{d}x \\ &= \frac{x^2}{2} + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{x^2}{2} + c_2 \tag{1}$$

Verification of solutions

$$y = \frac{x^2}{2} + c_2$$

Verified OK.

Solving equation (3)

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{y}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\,dy &= \frac{1}{x}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{1}{x}\,dx \\
\ln(y) &= \ln(x) + c_3 \\
y &= e^{\ln(x)+c_3} \\
&= c_3 x
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_3 x \tag{1}$$

Verification of solutions

$$y = c_3 x$$

Verified OK.

1.18.1 Maple step by step solution

Let's solve

$xy'^3 - (x^2 + x + y)\,y'^2 + (x^2 + xy + y)\,y' - xy = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \left(xy'^3 - (x^2 + x + y)\,y'^2 + (x^2 + xy + y)\,y' - xy\right) dx = \int 0 dx + c_1$

- Cannot compute integral

  $\int \left(xy'^3 - (x^2 + x + y)\,y'^2 + (x^2 + xy + y)\,y' - xy\right) dx = c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(x*diff(y(x),x)^3-(x^2+x+y(x))*diff(y(x),x)^2+(x^2+x*y(x)+y(x))*diff(y(x),x)-x*y(x)=0,
```

$$y(x) = c_1 x$$
$$y(x) = x + c_1$$
$$y(x) = \frac{x^2}{2} + c_1$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 36

```
DSolve[x*(y'[x])^3-(x^2+x+y[x])*(y'[x])^2+(x^2+x*y[x]+y[x])*y'[x]-x*y[x]==0,y[x],x,IncludeSi
```

$$y(x) \to c_1 x$$
$$y(x) \to x + c_1$$
$$y(x) \to \frac{x^2}{2} + c_1$$
$$y(x) \to 0$$

1.19 problem 19

Internal problem ID [6785]
Internal file name [`OUTPUT/6032_Tuesday_July_26_2022_05_04_44_AM_54275792/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 94. Factoring the left member. EXERCISES Page 309
Problem number: 19.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{xy{y'}^2+(x+y)\,y'=-1}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y'=-\frac{1}{y} \tag{1}$$
$$y'=-\frac{1}{x} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int -y dy = x+c_1$$
$$-\frac{y^2}{2} = x+c_1$$

Solving for y gives these solutions

$$y_1=\sqrt{-2c_1-2x}$$
$$y_2=-\sqrt{-2c_1-2x}$$

Summary
The solution(s) found are the following

$$y = \sqrt{-2c_1 - 2x} \tag{1}$$
$$y = -\sqrt{-2c_1 - 2x} \tag{2}$$

Verification of solutions

$$y = \sqrt{-2c_1 - 2x}$$

Verified OK.

$$y = -\sqrt{-2c_1 - 2x}$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{1}{x}\,\mathrm{d}x \\ &= -\ln(x) + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -\ln(x) + c_2 \tag{1}$$

Verification of solutions

$$y = -\ln(x) + c_2$$

Verified OK.

1.19.1 Maple step by step solution

Let's solve

$xyy'^2 + (x+y)\,y' = -1$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \left(xyy'^2 + (x+y)\,y'\right)dx = \int (-1)\,dx + c_1$

- Cannot compute integral

$$\int \left(x y {y'}^{2}+\left(x +y \right) y'\right)d x = -x +c_{1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 32

```
dsolve(x*y(x)*diff(y(x),x)^2+(x+y(x))*diff(y(x),x)+1=0,y(x), singsol=all)
```

$$y(x) = -\ln\left(x\right)+c_{1}$$
$$y(x) = \sqrt{c_{1}-2x}$$
$$y(x) = -\sqrt{c_{1}-2x}$$

✓ Solution by Mathematica

Time used: 0.062 (sec). Leaf size: 53

```
DSolve[x*y[x]*(y'[x])^2+(x+y[x])*y'[x]+1==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -\sqrt{2}\sqrt{-x+c_{1}}$$
$$y(x)\to \sqrt{2}\sqrt{-x+c_{1}}$$
$$y(x)\to -\log(x)+c_{1}$$

2 CHAPTER 16. Nonlinear equations. Section 97. The p-discriminant equation. EXERCISES Page 314

2.1 problem 8

Internal problem ID [6786]
Internal file name [`OUTPUT/6033_Tuesday_July_26_2022_05_04_45_AM_33678810/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 97. The p-discriminant equation. EXERCISES Page 314
Problem number: 8.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{xy'^2 - 2y'y = -4x}$$

2.1.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$x\,p^2 - 2py = -4x$$

Solving for y from the above results in

$$y = \frac{x(p^2+4)}{2p} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = \frac{p^2+4}{2p}$$
$$g = 0$$

Hence (2) becomes

$$p - \frac{p^2+4}{2p} = x\left(1 - \frac{p^2+4}{2p^2}\right)p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{p^2+4}{2p} = 0$$

Solving for p from the above gives

$$p = 2$$
$$p = -2$$

Substituting these in (1A) gives

$$y = -2x$$
$$y = 2x$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - \frac{p(x)^2+4}{2p(x)}}{x\left(1 - \frac{p(x)^2+4}{2p(x)^2}\right)} \tag{3}$$

This ODE is now solved for $p(x)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = 0$$

Hence the ode is

$$p'(x) - \frac{p(x)}{x} = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\mu p &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{p}{x}\right) &= 0\end{aligned}$$

Integrating gives

$$\frac{p}{x} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$p(x) = c_1 x$$

Substituing the above solution for p in (2A) gives

$$y = \frac{c_1^2 x^2 + 4}{2c_1}$$

Summary

The solution(s) found are the following

$$y = -2x \tag{1}$$

$$y = 2x \tag{2}$$

$$y = \frac{c_1^2 x^2 + 4}{2c_1} \tag{3}$$

Verification of solutions

$$y = -2x$$

Verified OK.

$$y = 2x$$

Verified OK.

$$y = \frac{c_1^2 x^2 + 4}{2c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 30

```
dsolve(x*diff(y(x),x)^2-2*y(x)*diff(y(x),x)+4*x=0,y(x), singsol=all)
```

$$y(x) = -2x$$
$$y(x) = 2x$$
$$y(x) = \frac{4c_1^2 + x^2}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.294 (sec). Leaf size: 43

```
DSolve[x*(y'[x])^2-2*y[x]*y'[x]+4*x==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -2x \cosh(-\log(x) + c_1)$$
$$y(x) \to -2x \cosh(\log(x) + c_1)$$
$$y(x) \to -2x$$
$$y(x) \to 2x$$

2.2 problem 9

Internal problem ID [6787]
Internal file name [`OUTPUT/6034_Tuesday_July_26_2022_05_04_47_AM_71368481/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 97. The p-discriminant equation. EXERCISES Page 314
Problem number: 9.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{3x^4{y'}^2 - xy' - y = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{1 + \sqrt{1 + 12yx^2}}{6x^3} \tag{1}$$

$$y' = -\frac{-1 + \sqrt{1 + 12yx^2}}{6x^3} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$\begin{aligned} y' &= \frac{1 + \sqrt{12y\,x^2 + 1}}{6x^3} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{\left(1+\sqrt{12y\,x^2+1}\right)(b_3-a_2)}{6x^3} - \frac{\left(1+\sqrt{12y\,x^2+1}\right)^2 a_3}{36x^6} \\ &- \left(-\frac{1+\sqrt{12y\,x^2+1}}{2x^4} + \frac{2y}{x^2\sqrt{12y\,x^2+1}}\right)(xa_2+ya_3+a_1) \\ &- \frac{xb_2+yb_3+b_1}{x\sqrt{12y\,x^2+1}} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-36b_2x^6\sqrt{12y\,x^2+1} + 36x^6b_2 - 72x^5ya_2 - 36x^5yb_3 - 144x^4y^2a_3 + 36x^5b_1 - 144x^4ya_1 - 12\sqrt{12y\,x^2+}}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &36b_2x^6\sqrt{12y\,x^2+1} - 36x^6b_2 + 72x^5ya_2 + 36x^5yb_3 + 144x^4y^2a_3 \\ &- 36x^5b_1 + 144x^4ya_1 + 12\sqrt{12y\,x^2+1}\,x^3a_2 + 6\sqrt{12y\,x^2+1}\,x^3b_3 \\ &+ 18\sqrt{12y\,x^2+1}\,x^2ya_3 - \left(12y\,x^2+1\right)^{\frac{3}{2}} a_3 + 18\sqrt{12y\,x^2+1}\,x^2a_1 \\ &+ 12x^3a_2 + 6x^3b_3 - 6x^2ya_3 + 18x^2a_1 - a_3\sqrt{12y\,x^2+1} - 2a_3 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &36b_2x^6\sqrt{12y\,x^2+1} - 36x^6b_2 - 72x^5ya_2 - 36x^5yb_3 - 72x^4y^2a_3 \\ &+ 12\left(12y\,x^2+1\right)x^3a_2 + 6\left(12y\,x^2+1\right)x^3b_3 + 18\left(12y\,x^2+1\right)x^2ya_3 \\ &- 36x^5b_1 - 72x^4ya_1 + 18\left(12y\,x^2+1\right)x^2a_1 + 12\sqrt{12y\,x^2+1}\,x^3a_2 \\ &+ 6\sqrt{12y\,x^2+1}\,x^3b_3 + 18\sqrt{12y\,x^2+1}\,x^2ya_3 - \left(12y\,x^2+1\right)^{\frac{3}{2}} a_3 \\ &+ 18\sqrt{12y\,x^2+1}\,x^2a_1 - 2\left(12y\,x^2+1\right)a_3 - a_3\sqrt{12y\,x^2+1} = 0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$
36b_2x^6\sqrt{12y\,x^2+1}-36x^6b_2+72x^5ya_2+36x^5yb_3+144x^4y^2a_3-36x^5b_1+144x^4ya_1 \\
+12\sqrt{12y\,x^2+1}\,x^3a_2+6\sqrt{12y\,x^2+1}\,x^3b_3+6\sqrt{12y\,x^2+1}\,x^2ya_3+12x^3a_2 \\
+6x^3b_3+18\sqrt{12y\,x^2+1}\,x^2a_1-6x^2ya_3+18x^2a_1-2a_3\sqrt{12y\,x^2+1}-2a_3=0
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{12y\,x^2+1}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x = v_1, y = v_2, \sqrt{12y\,x^2+1} = v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&36b_2v_1^6v_3+72v_1^5v_2a_2+144v_1^4v_2^2a_3-36v_1^6b_2+36v_1^5v_2b_3+144v_1^4v_2a_1 \\
&-36v_1^5b_1+12v_3v_1^3a_2+6v_3v_1^2v_2a_3+6v_3v_1^3b_3+18v_3v_1^2a_1 \\
&+12v_1^3a_2-6v_1^2v_2a_3+6v_1^3b_3+18v_1^2a_1-2a_3v_3-2a_3=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&36b_2v_1^6v_3-36v_1^6b_2+(72a_2+36b_3)\,v_1^5v_2-36v_1^5b_1+144v_1^4v_2^2a_3 \\
&+144v_1^4v_2a_1+(12a_2+6b_3)\,v_1^3v_3+(12a_2+6b_3)\,v_1^3+6v_3v_1^2v_2a_3 \\
&-6v_1^2v_2a_3+18v_3v_1^2a_1+18v_1^2a_1-2a_3v_3-2a_3=0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
18a_1 &= 0\\
144a_1 &= 0\\
-6a_3 &= 0\\
-2a_3 &= 0\\
6a_3 &= 0\\
144a_3 &= 0\\
-36b_1 &= 0\\
-36b_2 &= 0\\
36b_2 &= 0\\
12a_2 + 6b_3 &= 0\\
72a_2 + 36b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0\\
a_2 &= a_2\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= -2a_2
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x\\
\eta &= -2y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi\\
&= -2y - \left(\frac{1 + \sqrt{12y\,x^2 + 1}}{6x^3}\right)(x)\\
&= \frac{-12y\,x^2 - \sqrt{12y\,x^2 + 1} - 1}{6x^2}\\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-12y\,x^2 - \sqrt{12y\,x^2+1} - 1}{6x^2}} dy \end{aligned}$$

Which results in

$$S = -\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{12y\,x^2 + 1}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{1 + \sqrt{12y\,x^2 + 1}}{6x^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{x\sqrt{12y\,x^2 + 1}} \\ S_y &= \frac{-1 + \frac{1}{\sqrt{12y\,x^2+1}}}{2y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{1+12yx^2}\right) = c_1$$

Which simplifies to

$$-\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{1+12yx^2}\right) = c_1$$

Summary

The solution(s) found are the following

$$-\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{1+12yx^2}\right) = c_1 \tag{1}$$

Verification of solutions

$$-\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{1+12yx^2}\right) = c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = -\frac{-1+\sqrt{12y\,x^2+1}}{6x^3}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{\left(-1+\sqrt{12y\,x^2+1}\right)(b_3 - a_2)}{6x^3} - \frac{\left(-1+\sqrt{12y\,x^2+1}\right)^2 a_3}{36x^6} \\ &- \left(-\frac{2y}{x^2\sqrt{12y\,x^2+1}} + \frac{-1+\sqrt{12y\,x^2+1}}{2x^4}\right)(x a_2 + y a_3 + a_1) \\ &+ \frac{x b_2 + y b_3 + b_1}{x\sqrt{12y\,x^2+1}} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-36 b_2 x^6 \sqrt{12y\,x^2+1} - 36x^6 b_2 + 72x^5 y a_2 + 36x^5 y b_3 + 144x^4 y^2 a_3 - 36x^5 b_1 + 144x^4 y a_1 - 12\sqrt{12y\,x^2 + }}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &36 b_2 x^6 \sqrt{12y\,x^2+1} + 36x^6 b_2 - 72x^5 y a_2 - 36x^5 y b_3 - 144x^4 y^2 a_3 \\ &+ 36x^5 b_1 - 144x^4 y a_1 + 12\sqrt{12y\,x^2+1}\,x^3 a_2 + 6\sqrt{12y\,x^2+1}\,x^3 b_3 \\ &+ 18\sqrt{12y\,x^2+1}\,x^2 y a_3 - \left(12y\,x^2+1\right)^{\frac{3}{2}} a_3 + 18\sqrt{12y\,x^2+1}\,x^2 a_1 \\ &- 12x^3 a_2 - 6x^3 b_3 + 6x^2 y a_3 - 18x^2 a_1 - a_3\sqrt{12y\,x^2+1} + 2a_3 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$
\begin{aligned}
&36b_2x^6\sqrt{12y\,x^2+1}+36x^6b_2+72x^5ya_2+36x^5yb_3+72x^4y^2a_3\\
&-12\left(12y\,x^2+1\right)x^3a_2-6\left(12y\,x^2+1\right)x^3b_3-18\left(12y\,x^2+1\right)x^2ya_3\\
&+36x^5b_1+72x^4ya_1-18\left(12y\,x^2+1\right)x^2a_1+12\sqrt{12y\,x^2+1}\,x^3a_2\\
&+6\sqrt{12y\,x^2+1}\,x^3b_3+18\sqrt{12y\,x^2+1}\,x^2ya_3-\left(12y\,x^2+1\right)^{\frac{3}{2}}a_3\\
&+18\sqrt{12y\,x^2+1}\,x^2a_1+2\left(12y\,x^2+1\right)a_3-a_3\sqrt{12y\,x^2+1}=0
\end{aligned}
\tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&36b_2x^6\sqrt{12y\,x^2+1}+36x^6b_2-72x^5ya_2-36x^5yb_3-144x^4y^2a_3+36x^5b_1-144x^4ya_1\\
&+12\sqrt{12y\,x^2+1}\,x^3a_2+6\sqrt{12y\,x^2+1}\,x^3b_3+6\sqrt{12y\,x^2+1}\,x^2ya_3-12x^3a_2\\
&-6x^3b_3+18\sqrt{12y\,x^2+1}\,x^2a_1+6x^2ya_3-18x^2a_1-2a_3\sqrt{12y\,x^2+1}+2a_3=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$
\left\{x,y,\sqrt{12y\,x^2+1}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$
\left\{x=v_1,y=v_2,\sqrt{12y\,x^2+1}=v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&36b_2v_1^6v_3-72v_1^5v_2a_2-144v_1^4v_2^2a_3+36v_1^6b_2-36v_1^5v_2b_3-144v_1^4v_2a_1\\
&+36v_1^5b_1+12v_3v_1^3a_2+6v_3v_1^2v_2a_3+6v_3v_1^3b_3+18v_3v_1^2a_1\\
&-12v_1^3a_2+6v_1^2v_2a_3-6v_1^3b_3-18v_1^2a_1-2a_3v_3+2a_3=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1,v_2,v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&36b_2v_1^6v_3+36v_1^6b_2+(-72a_2-36b_3)\,v_1^5v_2+36v_1^5b_1-144v_1^4v_2^2a_3\\
&-144v_1^4v_2a_1+(12a_2+6b_3)\,v_1^3v_3+(-12a_2-6b_3)\,v_1^3\\
&+6v_3v_1^2v_2a_3+6v_1^2v_2a_3+18v_3v_1^2a_1-18v_1^2a_1-2a_3v_3+2a_3=0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-144a_1 &= 0 \\
-18a_1 &= 0 \\
18a_1 &= 0 \\
-144a_3 &= 0 \\
-2a_3 &= 0 \\
2a_3 &= 0 \\
6a_3 &= 0 \\
36b_1 &= 0 \\
36b_2 &= 0 \\
-72a_2 - 36b_3 &= 0 \\
-12a_2 - 6b_3 &= 0 \\
12a_2 + 6b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= -2a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= -2y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= -2y - \left(-\frac{-1 + \sqrt{12y\, x^2 + 1}}{6x^3}\right)(x) \\
&= \frac{-12y\, x^2 + \sqrt{12y\, x^2 + 1} - 1}{6x^2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-12y\,x^2 + \sqrt{12y\,x^2+1} - 1}{6x^2}} dy \end{aligned}$$

Which results in

$$S = -\frac{\ln(y)}{2} + \operatorname{arctanh}\left(\sqrt{12y\,x^2+1}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{-1 + \sqrt{12y\,x^2+1}}{6x^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{1}{x\sqrt{12y\,x^2+1}} \\ S_y &= \frac{-\frac{1}{\sqrt{12y\,x^2+1}} - 1}{2y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(y)}{2} + \operatorname{arctanh}\left(\sqrt{1 + 12yx^2}\right) = c_1$$

Which simplifies to

$$-\frac{\ln(y)}{2} + \operatorname{arctanh}\left(\sqrt{1 + 12yx^2}\right) = c_1$$

Summary

The solution(s) found are the following

$$-\frac{\ln(y)}{2} + \operatorname{arctanh}\left(\sqrt{1 + 12yx^2}\right) = c_1 \tag{1}$$

Verification of solutions

$$-\frac{\ln(y)}{2} + \operatorname{arctanh}\left(\sqrt{1 + 12yx^2}\right) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      <- homogeneous successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      <- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 97

```
dsolve(3*x^4*diff(y(x),x)^2-x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{1}{12x^2}$$
$$y(x) = \frac{-i\sqrt{3}\,c_1 - 3x}{3c_1^2 x}$$
$$y(x) = \frac{i\sqrt{3}\,c_1 - 3x}{3x\,c_1^2}$$
$$y(x) = \frac{i\sqrt{3}\,c_1 - 3x}{3x\,c_1^2}$$
$$y(x) = \frac{-i\sqrt{3}\,c_1 - 3x}{3c_1^2 x}$$

✓ Solution by Mathematica

Time used: 0.518 (sec). Leaf size: 123

```
DSolve[3*x^4*(y'[x])^2-x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[-\frac{x\sqrt{12x^2y(x)+1}\text{arctanh}\left(\sqrt{12x^2y(x)+1}\right)}{\sqrt{12x^4y(x)+x^2}} - \frac{1}{2}\log(y(x)) = c_1, y(x)\right]$$
$$\text{Solve}\left[\frac{x\sqrt{12x^2y(x)+1}\text{arctanh}\left(\sqrt{12x^2y(x)+1}\right)}{\sqrt{12x^4y(x)+x^2}} - \frac{1}{2}\log(y(x)) = c_1, y(x)\right]$$
$$y(x) \to 0$$

2.3 problem 10

Internal problem ID [6788]
Internal file name [`OUTPUT/6035_Tuesday_July_26_2022_05_04_48_AM_61588331/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 97. The p-discriminant equation. EXERCISES Page 314
Problem number: 10.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _dAlembert]`

$$\boxed{y'^2 - xy' - y = 0}$$

2.3.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$p^2 - xp - y = 0$$

Solving for y from the above results in

$$y = p^2 - xp \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= -p \\
g &= p^2
\end{aligned}
$$

Hence (2) becomes

$$
2p = (-x + 2p)\, p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
2p = 0
$$

Solving for p from the above gives

$$
p = 0
$$

Substituting these in (1A) gives

$$
y = 0
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{2p(x)}{-x + 2p\,(x)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp}x(p) = \frac{-x(p) + 2p}{2p} \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
\frac{d}{dp}x(p) + p(p)x(p) = q(p)
$$

Where here

$$
\begin{aligned}
p(p) &= \frac{1}{2p} \\
q(p) &= 1
\end{aligned}
$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{x(p)}{2p} = 1$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{2p}dp} \\ &= \sqrt{p}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}(\mu x) &= \mu \\ \frac{\mathrm{d}}{\mathrm{d}p}(\sqrt{p}\,x) &= \sqrt{p} \\ \mathrm{d}(\sqrt{p}\,x) &= \sqrt{p}\mathrm{d}p\end{aligned}$$

Integrating gives

$$\begin{aligned}\sqrt{p}\,x &= \int \sqrt{p}\,\mathrm{d}p \\ \sqrt{p}\,x &= \frac{2p^{\frac{3}{2}}}{3} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sqrt{p}$ results in

$$x(p) = \frac{2p}{3} + \frac{c_1}{\sqrt{p}}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$\begin{aligned}p &= \frac{x}{2} + \frac{\sqrt{x^2+4y}}{2} \\ p &= \frac{x}{2} - \frac{\sqrt{x^2+4y}}{2}\end{aligned}$$

Substituting the above in the solution for x found above gives

$$\begin{aligned}x &= \frac{x}{3} + \frac{\sqrt{x^2+4y}}{3} + \frac{2c_1}{\sqrt{2x+2\sqrt{x^2+4y}}} \\ x &= \frac{x}{3} - \frac{\sqrt{x^2+4y}}{3} + \frac{2c_1}{\sqrt{2x-2\sqrt{x^2+4y}}}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$

$$x = \frac{x}{3} + \frac{\sqrt{x^2 + 4y}}{3} + \frac{2c_1}{\sqrt{2x + 2\sqrt{x^2 + 4y}}} \tag{2}$$

$$x = \frac{x}{3} - \frac{\sqrt{x^2 + 4y}}{3} + \frac{2c_1}{\sqrt{2x - 2\sqrt{x^2 + 4y}}} \tag{3}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$x = \frac{x}{3} + \frac{\sqrt{x^2 + 4y}}{3} + \frac{2c_1}{\sqrt{2x + 2\sqrt{x^2 + 4y}}}$$

Verified OK.

$$x = \frac{x}{3} - \frac{\sqrt{x^2 + 4y}}{3} + \frac{2c_1}{\sqrt{2x - 2\sqrt{x^2 + 4y}}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 77

```
dsolve(diff(y(x),x)^2-x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$\frac{c_1}{\sqrt{2x-2\sqrt{x^2+4y(x)}}}+\frac{2x}{3}+\frac{\sqrt{x^2+4y(x)}}{3}=0$$
$$\frac{c_1}{\sqrt{2x+2\sqrt{x^2+4y(x)}}}+\frac{2x}{3}-\frac{\sqrt{x^2+4y(x)}}{3}=0$$

✓ Solution by Mathematica

Time used: 60.178 (sec). Leaf size: 1003

```
DSolve[(y'[x])^2-x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\left(x^2+\sqrt[3]{-x^6+20e^{3c_1}x^3+8\sqrt{e^{3c_1}\left(-x^3+e^{3c_1}\right)^3}+8e^{6c_1}}\right)^2+8e^{3c_1}x}{4\sqrt[3]{-x^6+20e^{3c_1}x^3+8\sqrt{e^{3c_1}\left(-x^3+e^{3c_1}\right)^3}+8e^{6c_1}}}$$

$$y(x) \to \frac{1}{8}\left(4x^2-\frac{i\left(\sqrt{3}-i\right)x\left(x^3+8e^{3c_1}\right)}{\sqrt[3]{-x^6+20e^{3c_1}x^3+8\sqrt{e^{3c_1}\left(-x^3+e^{3c_1}\right)^3}+8e^{6c_1}}}\right.$$
$$\left.+i\left(\sqrt{3}+i\right)\sqrt[3]{-x^6+20e^{3c_1}x^3+8\sqrt{e^{3c_1}\left(-x^3+e^{3c_1}\right)^3}+8e^{6c_1}}\right)$$

$$y(x) \to \frac{1}{8}\left(4x^2+\frac{i\left(\sqrt{3}+i\right)x\left(x^3+8e^{3c_1}\right)}{\sqrt[3]{-x^6+20e^{3c_1}x^3+8\sqrt{e^{3c_1}\left(-x^3+e^{3c_1}\right)^3}+8e^{6c_1}}}\right.$$
$$\left.-\left(1+i\sqrt{3}\right)\sqrt[3]{-x^6+20e^{3c_1}x^3+8\sqrt{e^{3c_1}\left(-x^3+e^{3c_1}\right)^3}+8e^{6c_1}}\right)$$

$$y(x)$$
$$\to \frac{2\sqrt[3]{2}x^4+2^{2/3}\left(-2x^6-10e^{3c_1}x^3+\sqrt{e^{3c_1}\left(4x^3+e^{3c_1}\right)^3}+e^{6c_1}\right)^{2/3}+4x^2\sqrt[3]{-2x^6-10e^{3c_1}x^3+\sqrt{e^{3c_1}\left(4x\right.}}}{8\sqrt[3]{-2x^6-10e^{3c_1}x^3+\sqrt{e^{3c_1}\left(4x^3+e^{3c_1}\right)^3}+e^{6c_1}}}$$

$$y(x) \to \frac{1}{16}\left(8x^2+\frac{2\sqrt[3]{2}\left(1+i\sqrt{3}\right)x\left(-x^3+2e^{3c_1}\right)}{\sqrt[3]{-2x^6-10e^{3c_1}x^3+\sqrt{e^{3c_1}\left(4x^3+e^{3c_1}\right)^3}+e^{6c_1}}}\right.$$
$$\left.+i2^{2/3}\left(\sqrt{3}+i\right)\sqrt[3]{-2x^6-10e^{3c_1}x^3+\sqrt{e^{3c_1}\left(4x^3+e^{3c_1}\right)^3}+e^{6c_1}}\right)$$

$$y(x) \to \frac{1}{16}\left(8x^2+\frac{2i\sqrt[3]{2}\left(\sqrt{3}+i\right)x\left(x^3-2e^{3c_1}\right)}{\sqrt[3]{-2x^6-10e^{3c_1}x^3+\sqrt{e^{3c_1}\left(4x^3+e^{3c_1}\right)^3}+e^{6c_1}}}\right.$$
$$\left.-2^{2/3}\left(1+i\sqrt{3}\right)\sqrt[3]{-2x^6-10e^{3c_1}x^3+\sqrt{e^{3c_1}\left(4x^3+e^{3c_1}\right)^3}+e^{6c_1}}\right)$$

2.4 problem 11

Internal problem ID [6789]
Internal file name [`OUTPUT/6036_Tuesday_July_26_2022_05_04_51_AM_70143130/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 97. The p-discriminant equation. EXERCISES Page 314
Problem number: 11.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _Clairaut]`

$$\boxed{{y'}^2 - xy' + y = 0}$$

2.4.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$p^2 - xp + y = 0$$

Solving for y from the above results in

$$y = -p^2 + xp \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= -p^2 + xp \\ &= -p^2 + xp \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = -p^2$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned} p &= \frac{d}{dx}(xp + g) \\ p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\ p &= p + (x + g')\frac{dp}{dx} \\ 0 &= (x + g')\frac{dp}{dx} \end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned} \frac{dp}{dx} &= 0 \\ p &= c_1 \end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = -c_1^2 + c_1 x$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = -p^2$, then the above equation becomes

$$\begin{aligned} x + g'(p) &= x - 2p \\ &= 0 \end{aligned}$$

Solving the above for p results in

$$p_1 = \frac{x}{2}$$

Substituting the above back in (1) results in

$$y_1 = \frac{x^2}{4}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -c_1^2 + c_1 x \tag{1}$$

$$y = \frac{x^2}{4} \tag{2}$$

<u>Verification of solutions</u>

$$y = -c_1^2 + c_1 x$$

Verified OK.

$$y = \frac{x^2}{4}$$

Verified OK.

<u>Maple trace</u>

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 19

```
dsolve(diff(y(x),x)^2-x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x^2}{4}$$
$$y(x) = c_1(x - c_1)$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 25

```
DSolve[(y'[x])^2-x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1(x - c_1)$$
$$y(x) \to \frac{x^2}{4}$$

2.5 problem 12

Internal problem ID [6790]
Internal file name [`OUTPUT/6037_Tuesday_July_26_2022_05_04_52_AM_5469299/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 97. The p-discriminant equation. EXERCISES Page 314
Problem number: 12.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries]]
```

$$\boxed{y'^2 + 4y'x^5 - 12yx^4 = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = 2\left(-x^3 + \sqrt{x^6 + 3y}\right) x^2 \tag{1}$$

$$y' = 2\left(-x^3 - \sqrt{x^6 + 3y}\right) x^2 \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = 2\left(-x^3 + \sqrt{x^6 + 3y}\right) x^2$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + 2\left(-x^3 + \sqrt{x^6 + 3y}\right) x^2 (b_3 - a_2) - 4\left(-x^3 + \sqrt{x^6 + 3y}\right)^2 x^4 a_3 \\ &- \left(2\left(-3x^2 + \frac{3x^5}{\sqrt{x^6 + 3y}}\right) x^2 + 4\left(-x^3 + \sqrt{x^6 + 3y}\right) x\right) (xa_2 + ya_3 + a_1) \\ &- \frac{3x^2 (xb_2 + yb_3 + b_1)}{\sqrt{x^6 + 3y}} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-8x^{13}a_3 + 4\sqrt{x^6 + 3y}\, x^{10}a_3 + 12x^8a_2 - 2x^8b_3 - 14x^7ya_3 + 4(x^6 + 3y)^{\frac{3}{2}} x^4a_3 + 10x^7a_1 - 12\sqrt{x^6 + 3y}\, x^5}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &8x^{13}a_3 - 4\sqrt{x^6 + 3y}\, x^{10}a_3 - 12x^8a_2 + 2x^8b_3 + 14x^7ya_3 \\ &- 4\left(x^6 + 3y\right)^{\frac{3}{2}} x^4a_3 - 10x^7a_1 + 12\sqrt{x^6 + 3y}\, x^5a_2 - 2\sqrt{x^6 + 3y}\, x^5b_3 \\ &+ 10\sqrt{x^6 + 3y}\, x^4ya_3 + 10\sqrt{x^6 + 3y}\, x^4a_1 - 3x^3b_2 - 18x^2ya_2 \\ &+ 3x^2yb_3 - 12x\, y^2a_3 - 3x^2b_1 - 12xya_1 + b_2\sqrt{x^6 + 3y} = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &-4\sqrt{x^6 + 3y}\, x^{10}a_3 + 8\left(x^6 + 3y\right) x^7a_3 - 6x^8a_2 - 6x^7ya_3 - 4\left(x^6 + 3y\right)^{\frac{3}{2}} x^4a_3 \\ &- 6x^7a_1 + 12\sqrt{x^6 + 3y}\, x^5a_2 - 2\sqrt{x^6 + 3y}\, x^5b_3 + 10\sqrt{x^6 + 3y}\, x^4ya_3 \\ &+ 10\sqrt{x^6 + 3y}\, x^4a_1 - 6\left(x^6 + 3y\right) x^2a_2 + 2\left(x^6 + 3y\right) x^2b_3 - 4\left(x^6 + 3y\right) xya_3 \\ &- 4\left(x^6 + 3y\right) xa_1 - 3x^3b_2 - 3x^2yb_3 - 3x^2b_1 + b_2\sqrt{x^6 + 3y} = 0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned}&8x^{13}a_3 - 8\sqrt{x^6+3y}\,x^{10}a_3 - 12x^8a_2 + 2x^8b_3 + 14x^7ya_3 - 10x^7a_1 \\ &+ 12\sqrt{x^6+3y}\,x^5a_2 - 2\sqrt{x^6+3y}\,x^5b_3 - 2\sqrt{x^6+3y}\,x^4ya_3 + 10\sqrt{x^6+3y}\,x^4a_1 \\ &- 3x^3b_2 - 18x^2ya_2 + 3x^2yb_3 - 12x\,y^2a_3 - 3x^2b_1 - 12xya_1 + b_2\sqrt{x^6+3y} = 0\end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{x^6+3y}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{x^6+3y} = v_3\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}&8v_1^{13}a_3 - 8v_3v_1^{10}a_3 - 12v_1^8a_2 + 14v_1^7v_2a_3 + 2v_1^8b_3 - 10v_1^7a_1 \\ &+ 12v_3v_1^5a_2 - 2v_3v_1^4v_2a_3 - 2v_3v_1^5b_3 + 10v_3v_1^4a_1 - 18v_1^2v_2a_2 \\ &- 12v_1v_2^2a_3 - 3v_1^3b_2 + 3v_1^2v_2b_3 - 12v_1v_2a_1 - 3v_1^2b_1 + b_2v_3 = 0\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned}&8v_1^{13}a_3 - 8v_3v_1^{10}a_3 + (-12a_2 + 2b_3)\,v_1^8 + 14v_1^7v_2a_3 - 10v_1^7a_1 \\ &+ (12a_2 - 2b_3)\,v_1^5v_3 - 2v_3v_1^4v_2a_3 + 10v_3v_1^4a_1 - 3v_1^3b_2 \\ &+ (-18a_2 + 3b_3)\,v_1^2v_2 - 3v_1^2b_1 - 12v_1v_2^2a_3 - 12v_1v_2a_1 + b_2v_3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_2 &= 0 \\
-12a_1 &= 0 \\
-10a_1 &= 0 \\
10a_1 &= 0 \\
-12a_3 &= 0 \\
-8a_3 &= 0 \\
-2a_3 &= 0 \\
8a_3 &= 0 \\
14a_3 &= 0 \\
-3b_1 &= 0 \\
-3b_2 &= 0 \\
-18a_2 + 3b_3 &= 0 \\
-12a_2 + 2b_3 &= 0 \\
12a_2 - 2b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 6a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= 6y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= 6y - \left(2\left(-x^3 + \sqrt{x^6 + 3y}\right)x^2\right)(x) \\
&= 2x^6 - 2\sqrt{x^6 + 3y}\,x^3 + 6y \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{2x^6 - 2\sqrt{x^6 + 3y}\, x^3 + 6y} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(y)}{6} + \frac{\ln\left(-x^3 + \sqrt{x^6 + 3y}\right)}{6} - \frac{\ln\left(x^3 + \sqrt{x^6 + 3y}\right)}{6}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = 2\left(-x^3 + \sqrt{x^6 + 3y}\right) x^2$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{x^2}{\sqrt{x^6 + 3y}} \\ S_y &= \frac{1}{\sqrt{x^6 + 3y}\, \left(-2x^3 + 2\sqrt{x^6 + 3y}\right)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y)}{6} + \frac{\ln\left(-x^3 + \sqrt{x^6 + 3y}\right)}{6} - \frac{\ln\left(x^3 + \sqrt{x^6 + 3y}\right)}{6} = c_1$$

Which simplifies to

$$\frac{\ln(y)}{6} + \frac{\ln\left(-x^3 + \sqrt{x^6 + 3y}\right)}{6} - \frac{\ln\left(x^3 + \sqrt{x^6 + 3y}\right)}{6} = c_1$$

Summary

The solution(s) found are the following

$$\frac{\ln(y)}{6} + \frac{\ln\left(-x^3 + \sqrt{x^6 + 3y}\right)}{6} - \frac{\ln\left(x^3 + \sqrt{x^6 + 3y}\right)}{6} = c_1 \tag{1}$$

Verification of solutions

$$\frac{\ln(y)}{6} + \frac{\ln\left(-x^3 + \sqrt{x^6 + 3y}\right)}{6} - \frac{\ln\left(x^3 + \sqrt{x^6 + 3y}\right)}{6} = c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = -2x^2\left(x^3 + \sqrt{x^6 + 3y}\right)$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - 2x^2\left(x^3 + \sqrt{x^6+3y}\right)(b_3 - a_2) - 4x^4\left(x^3 + \sqrt{x^6+3y}\right)^2 a_3 \\ &- \left(-4x\left(x^3 + \sqrt{x^6+3y}\right) - 2x^2\left(3x^2 + \frac{3x^5}{\sqrt{x^6+3y}}\right)\right)(xa_2 + ya_3 + a_1) \\ &+ \frac{3x^2(xb_2 + yb_3 + b_1)}{\sqrt{x^6+3y}} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{8x^{13}a_3 + 4\sqrt{x^6+3y}\,x^{10}a_3 - 12x^8a_2 + 2x^8b_3 + 14x^7ya_3 + 4(x^6+3y)^{\frac{3}{2}}\,x^4a_3 - 10x^7a_1 - 12\sqrt{x^6+3y}\,x^5a}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-8x^{13}a_3 - 4\sqrt{x^6+3y}\,x^{10}a_3 + 12x^8a_2 - 2x^8b_3 - 14x^7ya_3 \\ &- 4\left(x^6+3y\right)^{\frac{3}{2}}x^4a_3 + 10x^7a_1 + 12\sqrt{x^6+3y}\,x^5a_2 - 2\sqrt{x^6+3y}\,x^5b_3 \\ &+ 10\sqrt{x^6+3y}\,x^4ya_3 + 10\sqrt{x^6+3y}\,x^4a_1 + 3x^3b_2 + 18x^2a_2y \\ &- 3x^2yb_3 + 12x\,y^2a_3 + 3x^2b_1 + 12xa_1y + b_2\sqrt{x^6+3y} = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$
\begin{align*}
&-4\sqrt{x^{6}+3y}\,x^{10}a_{3}-8\left(x^{6}+3y\right)x^{7}a_{3}+6x^{8}a_{2}+6x^{7}ya_{3}-4\left(x^{6}+3y\right)^{\frac{3}{2}}x^{4}a_{3}\\
&+6x^{7}a_{1}+12\sqrt{x^{6}+3y}\,x^{5}a_{2}-2\sqrt{x^{6}+3y}\,x^{5}b_{3}+10\sqrt{x^{6}+3y}\,x^{4}ya_{3}\\
&+10\sqrt{x^{6}+3y}\,x^{4}a_{1}+6\left(x^{6}+3y\right)x^{2}a_{2}-2\left(x^{6}+3y\right)x^{2}b_{3}+4\left(x^{6}+3y\right)xya_{3}\\
&+4\left(x^{6}+3y\right)xa_{1}+3x^{3}b_{2}+3x^{2}yb_{3}+3x^{2}b_{1}+b_{2}\sqrt{x^{6}+3y}=0
\end{align*}
\tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{align*}
&-8x^{13}a_{3}-8\sqrt{x^{6}+3y}\,x^{10}a_{3}+12x^{8}a_{2}-2x^{8}b_{3}-14x^{7}ya_{3}+10x^{7}a_{1}\\
&+12\sqrt{x^{6}+3y}\,x^{5}a_{2}-2\sqrt{x^{6}+3y}\,x^{5}b_{3}-2\sqrt{x^{6}+3y}\,x^{4}ya_{3}+10\sqrt{x^{6}+3y}\,x^{4}a_{1}\\
&+3x^{3}b_{2}+18x^{2}a_{2}y-3x^{2}yb_{3}+12x\,y^{2}a_{3}+3x^{2}b_{1}+12xa_{1}y+b_{2}\sqrt{x^{6}+3y}=0
\end{align*}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{x^{6}+3y}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x=v_{1}, y=v_{2}, \sqrt{x^{6}+3y}=v_{3}\right\}
$$

The above PDE (6E) now becomes

$$
\begin{align*}
&-8v_{1}^{13}a_{3}-8v_{3}v_{1}^{10}a_{3}+12v_{1}^{8}a_{2}-14v_{1}^{7}v_{2}a_{3}-2v_{1}^{8}b_{3}+10v_{1}^{7}a_{1}\\
&+12v_{3}v_{1}^{5}a_{2}-2v_{3}v_{1}^{4}v_{2}a_{3}-2v_{3}v_{1}^{5}b_{3}+10v_{3}v_{1}^{4}a_{1}+18v_{1}^{2}a_{2}v_{2}\\
&+12v_{1}v_{2}^{2}a_{3}+3v_{1}^{3}b_{2}-3v_{1}^{2}v_{2}b_{3}+12v_{1}a_{1}v_{2}+3v_{1}^{2}b_{1}+b_{2}v_{3}=0
\end{align*}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_{1}, v_{2}, v_{3}\}
$$

Equation (7E) now becomes

$$
\begin{align*}
&-8v_{1}^{13}a_{3}-8v_{3}v_{1}^{10}a_{3}+(12a_{2}-2b_{3})\,v_{1}^{8}-14v_{1}^{7}v_{2}a_{3}+10v_{1}^{7}a_{1}\\
&+(12a_{2}-2b_{3})\,v_{1}^{5}v_{3}-2v_{3}v_{1}^{4}v_{2}a_{3}+10v_{3}v_{1}^{4}a_{1}+3v_{1}^{3}b_{2}\\
&+(18a_{2}-3b_{3})\,v_{1}^{2}v_{2}+3v_{1}^{2}b_{1}+12v_{1}v_{2}^{2}a_{3}+12v_{1}a_{1}v_{2}+b_{2}v_{3}=0
\end{align*}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_2 &= 0 \\
10a_1 &= 0 \\
12a_1 &= 0 \\
-14a_3 &= 0 \\
-8a_3 &= 0 \\
-2a_3 &= 0 \\
12a_3 &= 0 \\
3b_1 &= 0 \\
3b_2 &= 0 \\
12a_2 - 2b_3 &= 0 \\
18a_2 - 3b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 6a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= 6y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= 6y - \left(-2x^2 \left(x^3 + \sqrt{x^6 + 3y}\right)\right)(x) \\
&= 2x^6 + 2\sqrt{x^6 + 3y}\, x^3 + 6y \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{2x^6 + 2\sqrt{x^6 + 3y}\, x^3 + 6y} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(y)}{6} - \frac{\ln\left(-x^3 + \sqrt{x^6 + 3y}\right)}{6} + \frac{\ln\left(x^3 + \sqrt{x^6 + 3y}\right)}{6}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -2x^2 \left(x^3 + \sqrt{x^6 + 3y}\right)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x^2}{\sqrt{x^6 + 3y}} \\ S_y &= \frac{1}{\sqrt{x^6 + 3y}\, \left(2x^3 + 2\sqrt{x^6 + 3y}\right)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y)}{6} - \frac{\ln\left(-x^3 + \sqrt{x^6 + 3y}\right)}{6} + \frac{\ln\left(x^3 + \sqrt{x^6 + 3y}\right)}{6} = c_1$$

Which simplifies to

$$\frac{\ln(y)}{6} - \frac{\ln\left(-x^3 + \sqrt{x^6 + 3y}\right)}{6} + \frac{\ln\left(x^3 + \sqrt{x^6 + 3y}\right)}{6} = c_1$$

Summary

The solution(s) found are the following

$$\frac{\ln(y)}{6} - \frac{\ln\left(-x^3 + \sqrt{x^6 + 3y}\right)}{6} + \frac{\ln\left(x^3 + \sqrt{x^6 + 3y}\right)}{6} = c_1 \tag{1}$$

Verification of solutions

$$\frac{\ln(y)}{6} - \frac{\ln\left(-x^3 + \sqrt{x^6 + 3y}\right)}{6} + \frac{\ln\left(x^3 + \sqrt{x^6 + 3y}\right)}{6} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
      -> Calling odsolve with the ODE`, diff(diff(y(x), x), x)-2*(diff(y(x), x))/x, y(x)`
         Methods for second order ODEs:
         --- Trying classification methods ---
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         <- LODE of Euler type successful
      <- 1st order ODE linearizable_by_differentiation successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
      <- 1st order ODE linearizable_by_differentiation successful`
```

✓ Solution by Maple

Time used: 0.359 (sec). Leaf size: 23

```
dsolve(diff(y(x),x)^2+4*x^5*diff(y(x),x)-12*x^4*y(x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{x^6}{3}$$
$$y(x) = c_1 x^3 + \frac{3}{4} c_1^2$$

✓ Solution by Mathematica

Time used: 1.361 (sec). Leaf size: 217

```
DSolve[(y'[x])^2+4*x^5*y'[x]-12*x^4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{1}{6}\left(\log(y(x)) - \frac{x^2\sqrt{x^6+3y(x)}\log(y(x))}{\sqrt{x^4(x^6+3y(x))}}\right) + \frac{x^2\sqrt{x^6+3y(x)}\log\left(\sqrt{x^6+3y(x)}+x^3\right)}{3\sqrt{x^4(x^6+3y(x))}} = c_1, y(x)\right]$$
$$\text{Solve}\left[\frac{1}{6}\left(\frac{x^2\sqrt{x^6+3y(x)}\log(y(x))}{\sqrt{x^4(x^6+3y(x))}} + \log(y(x))\right) - \frac{x^2\sqrt{x^6+3y(x)}\log\left(\sqrt{x^6+3y(x)}+x^3\right)}{3\sqrt{x^4(x^6+3y(x))}} = c_1, y(x)\right]$$
$$y(x) \to -\frac{x^6}{3}$$

2.6 problem 13

Internal problem ID [6791]
Internal file name [`OUTPUT/6038_Tuesday_July_26_2022_05_04_55_AM_15542793/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 97. The p-discriminant equation. EXERCISES Page 314
Problem number: 13.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _rational]
```

$$\boxed{4y^3y'^2 - 4xy' + y = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{x + \sqrt{x^2 - y^4}}{2y^3} \tag{1}$$

$$y' = -\frac{-x + \sqrt{x^2 - y^4}}{2y^3} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = \frac{x + \sqrt{-y^4 + x^2}}{2y^3}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{\left(x+\sqrt{-y^4+x^2}\right)(b_3-a_2)}{2y^3} - \frac{\left(x+\sqrt{-y^4+x^2}\right)^2 a_3}{4y^6} \\ &- \frac{\left(1+\frac{x}{\sqrt{-y^4+x^2}}\right)(xa_2+ya_3+a_1)}{2y^3} \\ &- \left(-\frac{1}{\sqrt{-y^4+x^2}} - \frac{3\left(x+\sqrt{-y^4+x^2}\right)}{2y^4}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-4b_2\sqrt{-y^4+x^2}\,y^6 + 2x\,y^6b_2 - 2y^7a_2 + 4y^7b_3 + 2y^6b_1 - 6\sqrt{-y^4+x^2}\,x^2y^2b_2 + 4\sqrt{-y^4+x^2}\,x\,y^3a_2 - 8\sqrt{}}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &4b_2\sqrt{-y^4+x^2}\,y^6 - 2x\,y^6b_2 + 2y^7a_2 - 4y^7b_3 - 2y^6b_1 + 6\sqrt{-y^4+x^2}\,x^2y^2b_2 \\ &- 4\sqrt{-y^4+x^2}\,x\,y^3a_2 + 8\sqrt{-y^4+x^2}\,x\,y^3b_3 - 2\sqrt{-y^4+x^2}\,y^4a_3 \\ &+ 6x^3y^2b_2 - 4x^2y^3a_2 + 8x^2y^3b_3 + 6\sqrt{-y^4+x^2}\,x\,y^2b_1 - 2\sqrt{-y^4+x^2}\,y^3a_1 \\ &+ 6x^2y^2b_1 - 2x\,y^3a_1 - \left(-y^4+x^2\right)^{\frac{3}{2}} a_3 - \sqrt{-y^4+x^2}\,x^2a_3 - 2x^3a_3 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$
\begin{aligned}
&4b_2\sqrt{-y^4+x^2}\, y^6+4x\, y^6 b_2+4y^7 b_3+4y^6 b_1+6\left(-y^4+x^2\right) x\, y^2 b_2 \\
&-2\left(-y^4+x^2\right) y^3 a_2+8\left(-y^4+x^2\right) y^3 b_3+6\sqrt{-y^4+x^2}\, x^2 y^2 b_2 \\
&-4\sqrt{-y^4+x^2}\, x\, y^3 a_2+8\sqrt{-y^4+x^2}\, x\, y^3 b_3 \\
&-2\sqrt{-y^4+x^2}\, y^4 a_3-2x^2 y^3 a_2-2x\, y^4 a_3+6\left(-y^4+x^2\right) y^2 b_1 \\
&+6\sqrt{-y^4+x^2}\, x\, y^2 b_1-2\sqrt{-y^4+x^2}\, y^3 a_1-2x\, y^3 a_1 \\
&-\left(-y^4+x^2\right)^{\frac{3}{2}} a_3-2\left(-y^4+x^2\right) x a_3-\sqrt{-y^4+x^2}\, x^2 a_3=0
\end{aligned} \tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&-2x\, y^6 b_2+4b_2\sqrt{-y^4+x^2}\, y^6+2y^7 a_2-4y^7 b_3-2y^6 b_1+6x^3 y^2 b_2 \\
&+6\sqrt{-y^4+x^2}\, x^2 y^2 b_2-4x^2 y^3 a_2+8x^2 y^3 b_3-4\sqrt{-y^4+x^2}\, x\, y^3 a_2 \\
&+8\sqrt{-y^4+x^2}\, x\, y^3 b_3-\sqrt{-y^4+x^2}\, y^4 a_3+6x^2 y^2 b_1+6\sqrt{-y^4+x^2}\, x\, y^2 b_1 \\
&-2x\, y^3 a_1-2\sqrt{-y^4+x^2}\, y^3 a_1-2x^3 a_3-2\sqrt{-y^4+x^2}\, x^2 a_3=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{-y^4+x^2}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x=v_1, y=v_2, \sqrt{-y^4+x^2}=v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&2v_2^7 a_2-2v_1 v_2^6 b_2+4b_2 v_3 v_2^6-4v_2^7 b_3-2v_2^6 b_1-4v_1^2 v_2^3 a_2-4v_3 v_1 v_2^3 a_2 \\
&-v_3 v_2^4 a_3+6v_1^3 v_2^2 b_2+6v_3 v_1^2 v_2^2 b_2+8v_1^2 v_2^3 b_3+8v_3 v_1 v_2^3 b_3-2v_1 v_2^3 a_1 \\
&-2v_3 v_2^3 a_1+6v_1^2 v_2^2 b_1+6v_3 v_1 v_2^2 b_1-2v_1^3 a_3-2v_3 v_1^2 a_3=0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$6v_1^3v_2^2b_2 - 2v_1^3a_3 + (-4a_2 + 8b_3)\,v_1^2v_2^3 + 6v_3v_1^2v_2^2b_2 + 6v_1^2v_2^2b_1 \\ - 2v_3v_1^2a_3 - 2v_1v_2^6b_2 + (-4a_2 + 8b_3)\,v_1v_2^3v_3 - 2v_1v_2^3a_1 + 6v_3v_1v_2^2b_1 \\ + (2a_2 - 4b_3)\,v_2^7 + 4b_2v_3v_2^6 - 2v_2^6b_1 - v_3v_2^4a_3 - 2v_3v_2^3a_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -2a_1 &= 0 \\ -2a_3 &= 0 \\ -a_3 &= 0 \\ -2b_1 &= 0 \\ 6b_1 &= 0 \\ -2b_2 &= 0 \\ 4b_2 &= 0 \\ 6b_2 &= 0 \\ -4a_2 + 8b_3 &= 0 \\ 2a_2 - 4b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 2b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 2x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{x+\sqrt{-y^4+x^2}}{2y^3}\right)(2x) \\
&= \frac{y^4-\sqrt{-y^4+x^2}\,x-x^2}{y^3} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{y^4-\sqrt{-y^4+x^2}\,x-x^2}{y^3}} dy
\end{aligned}$$

Which results in

$$S = -\frac{\ln(y^2-x)}{4} + \ln(y) - \frac{\ln(y^2+x)}{4} + \frac{\ln(y^4-x^2)}{4} + \frac{x\ln\left(\frac{2x^2+2\sqrt{x^2}\,\sqrt{-y^4+x^2}}{y^2}\right)}{2\sqrt{x^2}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x + \sqrt{-y^4 + x^2}}{2y^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x + \sqrt{-y^4 + x^2}}{2\sqrt{-y^4 + x^2}\, x} \\ S_y &= -\frac{y^3}{\sqrt{-y^4 + x^2}\, \left(x + \sqrt{-y^4 + x^2}\right)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{2x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R)}{2} + c_1 \tag{4}$$

Summary

The solution(s) found are the following

$$\frac{\ln(2)}{2} + \frac{\ln(x)}{2} + \frac{\ln\left(x + \sqrt{x^2 - y^4}\right)}{2} = \frac{\ln(x)}{2} + c_1 \tag{1}$$

Verification of solutions

$$\frac{\ln(2)}{2} + \frac{\ln(x)}{2} + \frac{\ln\left(x + \sqrt{x^2 - y^4}\right)}{2} = \frac{\ln(x)}{2} + c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{\sqrt{-y^4+x^2}-x}{2y^3} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = xa_2 + ya_3 + a_1 \tag{1E}
$$

$$
\eta = xb_2 + yb_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 - \frac{\left(\sqrt{-y^4+x^2}-x\right)(b_3-a_2)}{2y^3} - \frac{\left(\sqrt{-y^4+x^2}-x\right)^2 a_3}{4y^6} \\
&+ \frac{\left(-1+\frac{x}{\sqrt{-y^4+x^2}}\right)(xa_2+ya_3+a_1)}{2y^3} \\
&- \left(\frac{1}{\sqrt{-y^4+x^2}} + \frac{\frac{3\sqrt{-y^4+x^2}}{2}-\frac{3x}{2}}{y^4}\right)(xb_2+yb_3+b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{-4b_2\sqrt{-y^4+x^2}\,y^6 - 2x\,y^6b_2 + 2y^7a_2 - 4y^7b_3 - 2y^6b_1 - 6\sqrt{-y^4+x^2}\,x^2y^2b_2 + 4\sqrt{-y^4+x^2}\,x\,y^3a_2 - 8\sqrt{}}{}
$$

$= 0$

Setting the numerator to zero gives

$$
\begin{aligned}
&4b_2\sqrt{-y^4+x^2}\,y^6+2x\,y^6b_2-2y^7a_2+4y^7b_3+2y^6b_1+6\sqrt{-y^4+x^2}\,x^2y^2b_2\\
&-4\sqrt{-y^4+x^2}\,x\,y^3a_2+8\sqrt{-y^4+x^2}\,x\,y^3b_3-2\sqrt{-y^4+x^2}\,y^4a_3\\
&-6x^3y^2b_2+4x^2y^3a_2-8x^2y^3b_3+6\sqrt{-y^4+x^2}\,x\,y^2b_1-2\sqrt{-y^4+x^2}\,y^3a_1\\
&-6x^2y^2b_1+2x\,y^3a_1-\left(-y^4+x^2\right)^{\frac{3}{2}}a_3-\sqrt{-y^4+x^2}\,x^2a_3+2x^3a_3=0
\end{aligned}
\tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
&4b_2\sqrt{-y^4+x^2}\,y^6-4x\,y^6b_2-4y^7b_3-4y^6b_1-6\left(-y^4+x^2\right)x\,y^2b_2\\
&+2\left(-y^4+x^2\right)y^3a_2-8\left(-y^4+x^2\right)y^3b_3+6\sqrt{-y^4+x^2}\,x^2y^2b_2\\
&-4\sqrt{-y^4+x^2}\,x\,y^3a_2+8\sqrt{-y^4+x^2}\,x\,y^3b_3\\
&-2\sqrt{-y^4+x^2}\,y^4a_3+2x^2y^3a_2+2x\,y^4a_3-6\left(-y^4+x^2\right)y^2b_1\\
&+6\sqrt{-y^4+x^2}\,x\,y^2b_1-2\sqrt{-y^4+x^2}\,y^3a_1+2x\,y^3a_1\\
&-\left(-y^4+x^2\right)^{\frac{3}{2}}a_3+2\left(-y^4+x^2\right)xa_3-\sqrt{-y^4+x^2}\,x^2a_3=0
\end{aligned}
\tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&2x\,y^6b_2+4b_2\sqrt{-y^4+x^2}\,y^6-2y^7a_2+4y^7b_3+2y^6b_1-6x^3y^2b_2\\
&+6\sqrt{-y^4+x^2}\,x^2y^2b_2+4x^2y^3a_2-8x^2y^3b_3-4\sqrt{-y^4+x^2}\,x\,y^3a_2\\
&+8\sqrt{-y^4+x^2}\,x\,y^3b_3-\sqrt{-y^4+x^2}\,y^4a_3-6x^2y^2b_1+6\sqrt{-y^4+x^2}\,x\,y^2b_1\\
&+2x\,y^3a_1-2\sqrt{-y^4+x^2}\,y^3a_1+2x^3a_3-2\sqrt{-y^4+x^2}\,x^2a_3=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$
\left\{x,y,\sqrt{-y^4+x^2}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$
\left\{x=v_1,y=v_2,\sqrt{-y^4+x^2}=v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-2v_2^7a_2+2v_1v_2^6b_2+4b_2v_3v_2^6+4v_2^7b_3+2v_2^6b_1+4v_1^2v_2^3a_2-4v_3v_1v_2^3a_2\\
&-v_3v_2^4a_3-6v_1^3v_2^2b_2+6v_3v_1^2v_2^2b_2-8v_1^2v_2^3b_3+8v_3v_1v_2^3b_3+2v_1v_2^3a_1\\
&-2v_3v_2^3a_1-6v_1^2v_2^2b_1+6v_3v_1v_2^2b_1+2v_1^3a_3-2v_3v_1^2a_3=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned} &-6v_1^3v_2^2b_2 + 2v_1^3a_3 + (4a_2 - 8b_3)\, v_1^2v_2^3 + 6v_3v_1^2v_2^2b_2 - 6v_1^2v_2^2b_1 \\ &- 2v_3v_1^2a_3 + 2v_1v_2^6b_2 + (-4a_2 + 8b_3)\, v_1v_2^3v_3 + 2v_1v_2^3a_1 + 6v_3v_1v_2^2b_1 \\ &+ (-2a_2 + 4b_3)\, v_2^7 + 4b_2v_3v_2^6 + 2v_2^6b_1 - v_3v_2^4a_3 - 2v_3v_2^3a_1 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -2a_1 &= 0 \\ 2a_1 &= 0 \\ -2a_3 &= 0 \\ -a_3 &= 0 \\ 2a_3 &= 0 \\ -6b_1 &= 0 \\ 2b_1 &= 0 \\ 6b_1 &= 0 \\ -6b_2 &= 0 \\ 2b_2 &= 0 \\ 4b_2 &= 0 \\ 6b_2 &= 0 \\ -4a_2 + 8b_3 &= 0 \\ -2a_2 + 4b_3 &= 0 \\ 4a_2 - 8b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 2b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = 2x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(-\frac{\sqrt{-y^4 + x^2} - x}{2y^3}\right)(2x) \\
&= \frac{y^4 + \sqrt{-y^4 + x^2}\, x - x^2}{y^3} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{y^4 + \sqrt{-y^4 + x^2}\, x - x^2}{y^3}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln(y^4 - x^2)}{4} - \frac{\ln(y^2 - x)}{4} + \ln(y) - \frac{\ln(y^2 + x)}{4} - \frac{x \ln\left(\frac{2x^2 + 2\sqrt{x^2}\, \sqrt{-y^4 + x^2}}{y^2}\right)}{2\sqrt{x^2}}$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR}=\frac{S_x+\omega(x,y)S_y}{R_x+\omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y)=-\frac{\sqrt{-y^4+x^2}-x}{2y^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1\\
R_y &= 0\\
S_x &= -\frac{x+\sqrt{-y^4+x^2}}{2\sqrt{-y^4+x^2}\,x}\\
S_y &= \frac{-y^4+2x^2+2\sqrt{-y^4+x^2}\,x}{y\sqrt{-y^4+x^2}\,\left(x+\sqrt{-y^4+x^2}\right)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR}=-\frac{-y^4+\sqrt{-y^4+x^2}\,x+x^2}{2\sqrt{-y^4+x^2}\,x\left(x+\sqrt{-y^4+x^2}\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x,y in terms of R,S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR}=-\frac{1}{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R,S. Integrating the above gives

$$S(R)=-\frac{\ln(R)}{2}+c_1 \tag{4}$$

Which gives

$$y=\mathrm{e}^{\frac{\ln(2)}{4}+\frac{\ln\left(-2\,\mathrm{e}^{4c_1}\mathrm{e}^{-2c_1}+2\,\mathrm{e}^{2c_1}\mathrm{e}^{-2c_1}x\right)}{4}+\frac{c_1}{2}}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\frac{\ln(2)}{4}+\frac{\ln\left(-2\,\mathrm{e}^{4c_1}\,\mathrm{e}^{-2c_1}+2\,\mathrm{e}^{2c_1}\,\mathrm{e}^{-2c_1}x\right)}{4}+\frac{c_1}{2}} \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{\frac{\ln(2)}{4}+\frac{\ln\left(-2\,\mathrm{e}^{4c_1}\,\mathrm{e}^{-2c_1}+2\,\mathrm{e}^{2c_1}\,\mathrm{e}^{-2c_1}x\right)}{4}+\frac{c_1}{2}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
   -> Solving 1st order ODE of high degree, Lie methods, 1st trial
   `, `-> Computing symmetries using: way = 3`[2*x, y]
```

✓ Solution by Maple

Time used: 0.281 (sec). Leaf size: 85

```
dsolve(4*y(x)^3*diff(y(x),x)^2-4*x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{-x}$$
$$y(x) = -\sqrt{-x}$$
$$y(x) = \sqrt{x}$$
$$y(x) = -\sqrt{x}$$
$$y(x) = 0$$
$$y(x) = \operatorname{RootOf}\left(-\ln(x) + 2\left(\int^{\_Z} -\frac{\_a^4 - \sqrt{-\_a^4+1} - 1}{\_a\,(\_a^4-1)} d\_a\right) + c_1\right)\sqrt{x}$$

✓ Solution by Mathematica

Time used: 0.587 (sec). Leaf size: 282

```
DSolve[4*y[x]^3*(y'[x])^2-4*x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -e^{\frac{c_1}{4}}\sqrt[4]{e^{c_1} - 2ix}$$
$$y(x) \to -ie^{\frac{c_1}{4}}\sqrt[4]{e^{c_1} - 2ix}$$
$$y(x) \to ie^{\frac{c_1}{4}}\sqrt[4]{e^{c_1} - 2ix}$$
$$y(x) \to e^{\frac{c_1}{4}}\sqrt[4]{e^{c_1} - 2ix}$$
$$y(x) \to -e^{\frac{c_1}{4}}\sqrt[4]{2ix + e^{c_1}}$$
$$y(x) \to -ie^{\frac{c_1}{4}}\sqrt[4]{2ix + e^{c_1}}$$
$$y(x) \to ie^{\frac{c_1}{4}}\sqrt[4]{2ix + e^{c_1}}$$
$$y(x) \to e^{\frac{c_1}{4}}\sqrt[4]{2ix + e^{c_1}}$$
$$y(x) \to 0$$
$$y(x) \to -\sqrt{x}$$
$$y(x) \to -i\sqrt{x}$$
$$y(x) \to i\sqrt{x}$$
$$y(x) \to \sqrt{x}$$

2.7 problem 14

Internal problem ID [6792]
Internal file name [`OUTPUT/6039_Tuesday_July_26_2022_05_04_58_AM_75081633/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 97. The p-discriminant equation. EXERCISES Page 314
Problem number: 14.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _rational]
```

$$\boxed{4y^3y'^2 + 4xy' + y = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{-x + \sqrt{x^2 - y^4}}{2y^3} \tag{1}$$

$$y' = -\frac{x + \sqrt{x^2 - y^4}}{2y^3} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = \frac{\sqrt{-y^4 + x^2} - x}{2y^3}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{\left(\sqrt{-y^4+x^2}-x\right)(b_3-a_2)}{2y^3} - \frac{\left(\sqrt{-y^4+x^2}-x\right)^2 a_3}{4y^6} \\ &- \frac{\left(-1+\frac{x}{\sqrt{-y^4+x^2}}\right)(xa_2+ya_3+a_1)}{2y^3} \\ &- \left(-\frac{1}{\sqrt{-y^4+x^2}} - \frac{3\left(\sqrt{-y^4+x^2}-x\right)}{2y^4}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-4b_2\sqrt{-y^4+x^2}\,y^6 + 2x\,y^6b_2 - 2y^7a_2 + 4y^7b_3 + 2y^6b_1 + 6\sqrt{-y^4+x^2}\,x^2y^2b_2 - 4\sqrt{-y^4+x^2}\,x\,y^3a_2 + 8\sqrt{}}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &4b_2\sqrt{-y^4+x^2}\,y^6 - 2x\,y^6b_2 + 2y^7a_2 - 4y^7b_3 - 2y^6b_1 - 6\sqrt{-y^4+x^2}\,x^2y^2b_2 \\ &+ 4\sqrt{-y^4+x^2}\,x\,y^3a_2 - 8\sqrt{-y^4+x^2}\,x\,y^3b_3 + 2\sqrt{-y^4+x^2}\,y^4a_3 + 6x^3y^2b_2 \\ &- 4x^2y^3a_2 + 8x^2y^3b_3 - 4x\,y^4a_3 - 6\sqrt{-y^4+x^2}\,x\,y^2b_1 + 2\sqrt{-y^4+x^2}\,y^3a_1 \\ &+ 6x^2y^2b_1 - 2x\,y^3a_1 - \left(-y^4+x^2\right)^{\frac{3}{2}} a_3 - \sqrt{-y^4+x^2}\,x^2a_3 + 2x^3a_3 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$
\begin{align*}
&4b_2\sqrt{-y^4+x^2}\,y^6+4x\,y^6b_2+4y^7b_3+4y^6b_1+6\left(-y^4+x^2\right)x\,y^2b_2\\
&-2\left(-y^4+x^2\right)y^3a_2+8\left(-y^4+x^2\right)y^3b_3-6\sqrt{-y^4+x^2}\,x^2y^2b_2\\
&+4\sqrt{-y^4+x^2}\,x\,y^3a_2-8\sqrt{-y^4+x^2}\,x\,y^3b_3+2\sqrt{-y^4+x^2}\,y^4a_3-2x^2y^3a_2\\
&-2x\,y^4a_3+6\left(-y^4+x^2\right)y^2b_1-6\sqrt{-y^4+x^2}\,x\,y^2b_1+2\sqrt{-y^4+x^2}\,y^3a_1\\
&-2x\,y^3a_1-\left(-y^4+x^2\right)^{\frac{3}{2}}a_3+2\left(-y^4+x^2\right)xa_3-\sqrt{-y^4+x^2}\,x^2a_3=0
\end{align*}
\tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{align*}
&-2x\,y^6b_2+4b_2\sqrt{-y^4+x^2}\,y^6+2y^7a_2-4y^7b_3-2y^6b_1\\
&+6x^3y^2b_2-6\sqrt{-y^4+x^2}\,x^2y^2b_2-4x^2y^3a_2+8x^2y^3b_3\\
&+4\sqrt{-y^4+x^2}\,x\,y^3a_2-8\sqrt{-y^4+x^2}\,x\,y^3b_3-4x\,y^4a_3\\
&+3\sqrt{-y^4+x^2}\,y^4a_3+6x^2y^2b_1-6\sqrt{-y^4+x^2}\,x\,y^2b_1-2x\,y^3a_1\\
&+2\sqrt{-y^4+x^2}\,y^3a_1+2x^3a_3-2\sqrt{-y^4+x^2}\,x^2a_3=0
\end{align*}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{-y^4+x^2}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x=v_1, y=v_2, \sqrt{-y^4+x^2}=v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{align*}
&2v_2^7a_2-2v_1v_2^6b_2+4b_2v_3v_2^6-4v_2^7b_3-2v_2^6b_1-4v_1^2v_2^3a_2+4v_3v_1v_2^3a_2\\
&-4v_1v_2^4a_3+3v_3v_2^4a_3+6v_1^3v_2^2b_2-6v_3v_1^2v_2^2b_2+8v_1^2v_2^3b_3-8v_3v_1v_2^3b_3\\
&-2v_1v_2^3a_1+2v_3v_2^3a_1+6v_1^2v_2^2b_1-6v_3v_1v_2^2b_1+2v_1^3a_3-2v_3v_1^2a_3=0
\end{align*}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&6v_1^3v_2^2b_2 + 2v_1^3a_3 + (-4a_2 + 8b_3)\, v_1^2v_2^3 - 6v_3v_1^2v_2^2b_2 + 6v_1^2v_2^2b_1 - 2v_3v_1^2a_3 \\
&\quad - 2v_1v_2^6b_2 - 4v_1v_2^4a_3 + (4a_2 - 8b_3)\, v_1v_2^3v_3 - 2v_1v_2^3a_1 - 6v_3v_1v_2^2b_1 \\
&\quad + (2a_2 - 4b_3)\, v_2^7 + 4b_2v_3v_2^6 - 2v_2^6b_1 + 3v_3v_2^4a_3 + 2v_3v_2^3a_1 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-2a_1 &= 0 \\
2a_1 &= 0 \\
-4a_3 &= 0 \\
-2a_3 &= 0 \\
2a_3 &= 0 \\
3a_3 &= 0 \\
-6b_1 &= 0 \\
-2b_1 &= 0 \\
6b_1 &= 0 \\
-6b_2 &= 0 \\
-2b_2 &= 0 \\
4b_2 &= 0 \\
6b_2 &= 0 \\
-4a_2 + 8b_3 &= 0 \\
2a_2 - 4b_3 &= 0 \\
4a_2 - 8b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 2b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 2x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{\sqrt{-y^4+x^2}-x}{2y^3}\right)(2x) \\
&= \frac{y^4+x^2-\sqrt{-y^4+x^2}\,x}{y^3} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta}dy \\
&= \int \frac{1}{\frac{y^4+x^2-\sqrt{-y^4+x^2}\,x}{y^3}}dy
\end{aligned}
$$

Which results in

$$
S = \frac{\ln(y^4+3x^2)}{6} + \frac{\ln(y)}{3} - \frac{x\ln\left(\frac{2x^2+2\sqrt{x^2}\sqrt{-y^4+x^2}}{y^2}\right)}{6\sqrt{x^2}} + \frac{x\ln\left(\frac{8x^2-2\sqrt{-3x^2}\left(y^2-\sqrt{-3x^2}\right)+4\sqrt{x^2}\sqrt{-\left(y^2-\sqrt{-3x^2}\right)^2-}}{y^2-\sqrt{-3x^2}}\right.}{6\sqrt{x^2}}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\sqrt{-y^4+x^2}-x}{2y^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1\\
R_y &= 0\\
S_x &= \frac{-\sqrt{-y^4+x^2}\,y^4-7y^4x+11\sqrt{-y^4+x^2}\,x^2+7x^3}{6\left(x+i\sqrt{3}\,y^2+2\sqrt{-y^4+x^2}\right)\left(x-i\sqrt{3}\,y^2+2\sqrt{-y^4+x^2}\right)\sqrt{-y^4+x^2}\,x}\\
S_y &= \frac{y^8-4x\,y^4\sqrt{-y^4+x^2}-7y^4x^2+6\sqrt{-y^4+x^2}\,x^3+6x^4}{\left(x+i\sqrt{3}\,y^2+2\sqrt{-y^4+x^2}\right)\left(x-i\sqrt{3}\,y^2+2\sqrt{-y^4+x^2}\right)\left(x+\sqrt{-y^4+x^2}\right)\sqrt{-y^4+x^2}\,y}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y^8-5x\,y^4\sqrt{-y^4+x^2}-10y^4x^2+9\sqrt{-y^4+x^2}\,x^3+9x^4}{6\left(x+i\sqrt{3}\,y^2+2\sqrt{-y^4+x^2}\right)\left(x-i\sqrt{3}\,y^2+2\sqrt{-y^4+x^2}\right)\sqrt{-y^4+x^2}\,x\left(x+\sqrt{-y^4+x^2}\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{6R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R)}{6}+c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y^4+3x^2)}{6}+\frac{2\ln(y)}{3}+\frac{\ln(2)}{6}+\frac{\ln(x)}{6}-\frac{\ln\left(x+\sqrt{x^2-y^4}\right)}{6}+\frac{\ln\left(ix+\sqrt{3}\,y^2+2i\sqrt{x^2-y^4}\right)}{6}-\frac{\ln\left(\sqrt{3}\right.}{}$$

Which simplifies to

$$\frac{\ln(y^4+3x^2)}{6}+\frac{2\ln(y)}{3}+\frac{\ln(2)}{6}-\frac{\ln\left(x+\sqrt{x^2-y^4}\right)}{6}+\frac{\ln\left(ix+\sqrt{3}\,y^2+2i\sqrt{x^2-y^4}\right)}{6}+\frac{\ln(3)}{6}-\frac{\ln\left(i\sqrt{\right.}}{}$$

Summary

The solution(s) found are the following

$$
\begin{align*}
&\frac{\ln(y^4+3x^2)}{6}+\frac{2\ln(y)}{3}+\frac{\ln(2)}{6}-\frac{\ln\left(x+\sqrt{x^2-y^4}\right)}{6} \\
&+\frac{\ln\left(ix+\sqrt{3}\,y^2+2i\sqrt{x^2-y^4}\right)}{6}+\frac{\ln(3)}{6}-\frac{\ln\left(i\sqrt{3}\,y^2+3x\right)}{6} \\
&+\frac{\ln\left(ix-\sqrt{3}\,y^2+2i\sqrt{x^2-y^4}\right)}{6}-\frac{\ln\left(i\sqrt{3}\,y^2-3x\right)}{6}-c_1=0 \tag{1}
\end{align*}
$$

Verification of solutions

$$
\begin{align*}
&\frac{\ln(y^4+3x^2)}{6}+\frac{2\ln(y)}{3}+\frac{\ln(2)}{6}-\frac{\ln\left(x+\sqrt{x^2-y^4}\right)}{6} \\
&+\frac{\ln\left(ix+\sqrt{3}\,y^2+2i\sqrt{x^2-y^4}\right)}{6}+\frac{\ln(3)}{6}-\frac{\ln\left(i\sqrt{3}\,y^2+3x\right)}{6} \\
&+\frac{\ln\left(ix-\sqrt{3}\,y^2+2i\sqrt{x^2-y^4}\right)}{6}-\frac{\ln\left(i\sqrt{3}\,y^2-3x\right)}{6}-c_1=0
\end{align*}
$$

Verified OK.

Solving equation (2)

Writing the ode as

$$
\begin{align*}
y' &= -\frac{x+\sqrt{-y^4+x^2}}{2y^3} \\
y' &= \omega(x,y)
\end{align*}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = xa_2+ya_3+a_1 \tag{1E}
$$

$$
\eta = xb_2+yb_3+b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1,a_2,a_3,b_1,b_2,b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 - \frac{\left(x+\sqrt{-y^4+x^2}\right)(b_3-a_2)}{2y^3} - \frac{\left(x+\sqrt{-y^4+x^2}\right)^2 a_3}{4y^6} \\
&+ \frac{\left(1+\frac{x}{\sqrt{-y^4+x^2}}\right)(xa_2+ya_3+a_1)}{2y^3} \\
&- \left(\frac{1}{\sqrt{-y^4+x^2}} + \frac{\frac{3x}{2}+\frac{3\sqrt{-y^4+x^2}}{2}}{y^4}\right)(xb_2+yb_3+b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{-4b_2\sqrt{-y^4+x^2}\,y^6 - 2x\,y^6b_2 + 2y^7a_2 - 4y^7b_3 - 2y^6b_1 + 6\sqrt{-y^4+x^2}\,x^2y^2b_2 - 4\sqrt{-y^4+x^2}\,x\,y^3a_2 + 8\sqrt{}}{}
$$

$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&4b_2\sqrt{-y^4+x^2}\,y^6 + 2x\,y^6b_2 - 2y^7a_2 + 4y^7b_3 + 2y^6b_1 - 6\sqrt{-y^4+x^2}\,x^2y^2b_2 \\
&+ 4\sqrt{-y^4+x^2}\,x\,y^3a_2 - 8\sqrt{-y^4+x^2}\,x\,y^3b_3 + 2\sqrt{-y^4+x^2}\,y^4a_3 - 6x^3y^2b_2 \\
&+ 4x^2y^3a_2 - 8y^3b_3x^2 + 4x\,y^4a_3 - 6\sqrt{-y^4+x^2}\,x\,y^2b_1 + 2\sqrt{-y^4+x^2}\,y^3a_1 \\
&- 6y^2b_1x^2 + 2x\,y^3a_1 - \left(-y^4+x^2\right)^{\frac{3}{2}}a_3 - \sqrt{-y^4+x^2}\,x^2a_3 - 2x^3a_3 = 0
\end{aligned} \tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
&4b_2\sqrt{-y^4+x^2}\,y^6 - 4x\,y^6b_2 - 4y^7b_3 - 4y^6b_1 - 6\left(-y^4+x^2\right)x\,y^2b_2 \\
&+ 2\left(-y^4+x^2\right)y^3a_2 - 8\left(-y^4+x^2\right)y^3b_3 - 6\sqrt{-y^4+x^2}\,x^2y^2b_2 \\
&+ 4\sqrt{-y^4+x^2}\,x\,y^3a_2 - 8\sqrt{-y^4+x^2}\,x\,y^3b_3 + 2\sqrt{-y^4+x^2}\,y^4a_3 + 2x^2y^3a_2 \\
&+ 2x\,y^4a_3 - 6\left(-y^4+x^2\right)y^2b_1 - 6\sqrt{-y^4+x^2}\,x\,y^2b_1 + 2\sqrt{-y^4+x^2}\,y^3a_1 \\
&+ 2x\,y^3a_1 - \left(-y^4+x^2\right)^{\frac{3}{2}}a_3 - 2\left(-y^4+x^2\right)xa_3 - \sqrt{-y^4+x^2}\,x^2a_3 = 0
\end{aligned} \tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&2x\,y^6b_2 + 4b_2\sqrt{-y^4+x^2}\,y^6 - 2y^7a_2 + 4y^7b_3 + 2y^6b_1 \\
&- 6x^3y^2b_2 - 6\sqrt{-y^4+x^2}\,x^2y^2b_2 + 4x^2y^3a_2 - 8y^3b_3x^2 \\
&+ 4\sqrt{-y^4+x^2}\,x\,y^3a_2 - 8\sqrt{-y^4+x^2}\,x\,y^3b_3 + 4x\,y^4a_3 \\
&+ 3\sqrt{-y^4+x^2}\,y^4a_3 - 6y^2b_1x^2 - 6\sqrt{-y^4+x^2}\,x\,y^2b_1 + 2x\,y^3a_1 \\
&+ 2\sqrt{-y^4+x^2}\,y^3a_1 - 2x^3a_3 - 2\sqrt{-y^4+x^2}\,x^2a_3 = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{-y^4+x^2}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{-y^4+x^2} = v_3\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-2v_2^7a_2 + 2v_1v_2^6b_2 + 4b_2v_3v_2^6 + 4v_2^7b_3 + 2v_2^6b_1 + 4v_1^2v_2^3a_2 + 4v_3v_1v_2^3a_2 \\
&+ 4v_1v_2^4a_3 + 3v_3v_2^4a_3 - 6v_1^3v_2^2b_2 - 6v_3v_1^2v_2^2b_2 - 8v_2^3b_3v_1^2 - 8v_3v_1v_2^3b_3 \\
&+ 2v_1v_2^3a_1 + 2v_3v_2^3a_1 - 6v_2^2b_1v_1^2 - 6v_3v_1v_2^2b_1 - 2v_1^3a_3 - 2v_3v_1^2a_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned}
&-6v_1^3v_2^2b_2 - 2v_1^3a_3 + (4a_2 - 8b_3)\, v_1^2v_2^3 - 6v_3v_1^2v_2^2b_2 - 6v_2^2b_1v_1^2 - 2v_3v_1^2a_3 \\
&+ 2v_1v_2^6b_2 + 4v_1v_2^4a_3 + (4a_2 - 8b_3)\, v_1v_2^3v_3 + 2v_1v_2^3a_1 - 6v_3v_1v_2^2b_1 \\
&+ (-2a_2 + 4b_3)\, v_2^7 + 4b_2v_3v_2^6 + 2v_2^6b_1 + 3v_3v_2^4a_3 + 2v_3v_2^3a_1 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
2a_1 &= 0 \\
-2a_3 &= 0 \\
3a_3 &= 0 \\
4a_3 &= 0 \\
-6b_1 &= 0 \\
2b_1 &= 0 \\
-6b_2 &= 0 \\
2b_2 &= 0 \\
4b_2 &= 0 \\
-2a_2 + 4b_3 &= 0 \\
4a_2 - 8b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 2b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 2x \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying simple symmetries for implicit equations
Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying exact
   Looking for potential symmetries
   trying an equivalence to an Abel ODE
   trying 1st order ODE linearizable_by_differentiation
-> Solving 1st order ODE of high degree, Lie methods, 1st trial
`, `-> Computing symmetries using: way = 3`[2*x, y]
```

✓ Solution by Maple

Time used: 0.313 (sec). Leaf size: 307

```
dsolve(4*y(x)^3*diff(y(x),x)^2+4*x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = 0$$

$$\frac{\left(\int_{\_b}^{x} \frac{-2\_a+\sqrt{-y(x)^4+\_a^2}}{y(x)^4+3\_a^2} d\_a\right)}{2} - \left(\int^{y(x)} \frac{\left(1+\left(\_f^4-\sqrt{-\_f^4+x^2}\, x+x^2\right)\left(\int_{\_b}^{x} \frac{\_f^4+4\sqrt{-\_f^4+\_a^2}\,\_a-5\_a^2}{\sqrt{-\_f^4+\_a^2}\left(\_f^4+3\_a^2\right)^2} d\_a\right)\right)\_f^3}{\_f^4-\sqrt{-\_f^4+x^2}\, x+x^2} d\_f\right) + c_1 = 0$$

$$-\frac{\left(\int_{\_b}^{x} \frac{2\_a+\sqrt{-y(x)^4+\_a^2}}{y(x)^4+3\_a^2} d\_a\right)}{2} - \left(\int^{y(x)} \frac{\left(1+\left(\_f^4+\sqrt{-\_f^4+x^2}\, x+x^2\right)\left(\int_{\_b}^{x} \frac{-\_f^4+5\_a^2+4\sqrt{-\_f^4+\_a^2}\,\_a}{\sqrt{-\_f^4+\_a^2}\left(\_f^4+3\_a^2\right)^2} d\_a\right)\right)\_f^3}{\_f^4+\sqrt{-\_f^4+x^2}\, x+x^2} d\_f\right) + c_1 = 0$$

✓ Solution by Mathematica

Time used: 60.284 (sec). Leaf size: 2815

```
DSolve[4*y[x]^3*(y'[x])^2+4*x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

2.8 problem 15

Internal problem ID [6793]
Internal file name [`OUTPUT/6040_Tuesday_July_26_2022_05_05_00_AM_657202/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 97. The p-discriminant equation. EXERCISES Page 314
Problem number: 15.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[_dAlembert]`

$$\boxed{y'^3 + xy'^2 - y = 0}$$

2.8.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$p^3 + x\,p^2 - y = 0$$

Solving for y from the above results in

$$y = p^3 + x\,p^2 \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= p^2 \\
g &= p^3
\end{aligned}
$$

Hence (2) becomes

$$
-p^2 + p = \left(3p^2 + 2xp\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
-p^2 + p = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= 0 \\
p &= 1
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= 0 \\
y &= 1 + x
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{-p(x)^2 + p(x)}{3p(x)^2 + 2p(x)\, x} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp} x(p) = \frac{3p^2 + 2x(p)\, p}{-p^2 + p} \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
\frac{d}{dp} x(p) + p(p) x(p) = q(p)
$$

Where here

$$
\begin{aligned}
p(p) &= \frac{2}{p - 1} \\
q(p) &= -\frac{3p}{p - 1}
\end{aligned}
$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{2x(p)}{p-1} = -\frac{3p}{p-1}$$

The integrating factor μ is

$$\mu = e^{\int \frac{2}{p-1}dp}$$
$$= (p-1)^2$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}p}(\mu x) = (\mu)\left(-\frac{3p}{p-1}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}p}\left((p-1)^2 x\right) = \left((p-1)^2\right)\left(-\frac{3p}{p-1}\right)$$
$$\mathrm{d}\left((p-1)^2 x\right) = (-3p(p-1))\ \mathrm{d}p$$

Integrating gives

$$(p-1)^2 x = \int -3p(p-1)\ \mathrm{d}p$$
$$(p-1)^2 x = -p^3 + \frac{3}{2}p^2 + c_1$$

Dividing both sides by the integrating factor $\mu = (p-1)^2$ results in

$$x(p) = \frac{-p^3 + \frac{3}{2}p^2}{(p-1)^2} + \frac{c_1}{(p-1)^2}$$

which simplifies to

$$x(p) = \frac{-2p^3 + 3p^2 + 2c_1}{2(p-1)^2}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = \frac{\left(108y - 8x^3 + 12\sqrt{81y^2 - 12yx^3}\right)^{\frac{1}{3}}}{6} + \frac{2x^2}{3\left(108y - 8x^3 + 12\sqrt{81y^2 - 12yx^3}\right)^{\frac{1}{3}}} - \frac{x}{3}$$

$$p = -\frac{\left(108y - 8x^3 + 12\sqrt{81y^2 - 12yx^3}\right)^{\frac{1}{3}}}{12} - \frac{x^2}{3\left(108y - 8x^3 + 12\sqrt{81y^2 - 12yx^3}\right)^{\frac{1}{3}}} - \frac{x}{3} + \frac{i\sqrt{3}\left(\frac{\left(108y-8x^3-\right.}{}\right.}{}$$

$$p = -\frac{\left(108y - 8x^3 + 12\sqrt{81y^2 - 12yx^3}\right)^{\frac{1}{3}}}{12} - \frac{x^2}{3\left(108y - 8x^3 + 12\sqrt{81y^2 - 12yx^3}\right)^{\frac{1}{3}}} - \frac{x}{3} - \frac{i\sqrt{3}\left(\frac{\left(108y-8x^3-\right.}{}\right.}{}$$

Substituting the above in the solution for x found above gives

$$x = \frac{24\left(x^3+\frac{3x^2}{2}-3y+3c_1\right)\left(x^3-\frac{3\sqrt{3}\,\sqrt{-4yx^3+27y^2}}{2}-\frac{27y}{2}\right)\left(108y-8x^3+12\sqrt{3}\,\sqrt{-4yx^3+27y^2}\right)^{\frac{2}{3}}+96\Big(\Big(\cdots}{\left(2x^3-3\sqrt{3}\,\sqrt{-4yx^3+27y^2}-27y\right)\Big(\big(108y-8x^3+12\sqrt{3}\,\sqrt{\cdots}}$$

$$x = \frac{96\left(x^3+\frac{3x^2}{2}-3y+3c_1\right)\left(x^3-\frac{3\sqrt{3}\,\sqrt{-4yx^3+27y^2}}{2}-\frac{27y}{2}\right)\left(108y-8x^3+12\sqrt{3}\,\sqrt{-4yx^3+27y^2}\right)^{\frac{2}{3}}+192\big(x\cdots}{\left(2x^3-3\sqrt{3}\,\sqrt{-4yx^3+27y^2}-27y\right)\Big(4i\sqrt{3}\,x^2-i\sqrt{3}\,\big(108y-8x^3+12\sqrt{\cdots}}$$

$$x = \frac{96\left(x^3+\frac{3x^2}{2}-3y+3c_1\right)\left(x^3-\frac{3\sqrt{3}\,\sqrt{-4yx^3+27y^2}}{2}-\frac{27y}{2}\right)\left(108y-8x^3+12\sqrt{3}\,\sqrt{-4yx^3+27y^2}\right)^{\frac{2}{3}}-192\big(x\cdots}{\left(2x^3-3\sqrt{3}\,\sqrt{-4yx^3+27y^2}-27y\right)\Big(4i\sqrt{3}\,x^2-i\sqrt{3}\,\big(108y-8x^3+12\sqrt{\cdots}}$$

<u>Summary</u>

The solution(s) found are the following

$$y = 0 \tag{1}$$

$$y = 1 + x \tag{2}$$

$$x = \frac{24\left(x^3+\frac{3x^2}{2}-3y+3c_1\right)\left(x^3-\frac{3\sqrt{3}\,\sqrt{-4yx^3+27y^2}}{2}-\frac{27y}{2}\right)\left(108y-8x^3+12\sqrt{3}\,\sqrt{-4yx^3+27y^2}\right)^{\frac{2}{3}}+96\Big(\Big(\cdots}{\left(2x^3-3\sqrt{3}\,\sqrt{-4yx^3+27y^2}-27y\right)\Big(\big(108y-8x^3+12\sqrt{3}\,\sqrt{\cdots}} \tag{3}$$

$$x = \frac{96\left(x^3+\frac{3x^2}{2}-3y+3c_1\right)\left(x^3-\frac{3\sqrt{3}\,\sqrt{-4yx^3+27y^2}}{2}-\frac{27y}{2}\right)\left(108y-8x^3+12\sqrt{3}\,\sqrt{-4yx^3+27y^2}\right)^{\frac{2}{3}}+192\big(x\cdots}{\left(2x^3-3\sqrt{3}\,\sqrt{-4yx^3+27y^2}-27y\right)\Big(4i\sqrt{3}\,x^2-i\sqrt{3}\,\big(108y-8x^3+12\sqrt{\cdots}} \tag{4}$$

$$x = \frac{96\left(x^3+\frac{3x^2}{2}-3y+3c_1\right)\left(x^3-\frac{3\sqrt{3}\,\sqrt{-4yx^3+27y^2}}{2}-\frac{27y}{2}\right)\left(108y-8x^3+12\sqrt{3}\,\sqrt{-4yx^3+27y^2}\right)^{\frac{2}{3}}-192\big(x\cdots}{\left(2x^3-3\sqrt{3}\,\sqrt{-4yx^3+27y^2}-27y\right)\Big(4i\sqrt{3}\,x^2-i\sqrt{3}\,\big(108y-8x^3+12\sqrt{\cdots}} \tag{5}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$y = 1 + x$$

Verified OK.

$$x = \frac{24\left(x^3 + \frac{3x^2}{2} - 3y + 3c_1\right)\left(x^3 - \frac{3\sqrt{3}\,\sqrt{-4yx^3+27y^2}}{2} - \frac{27y}{2}\right)\left(108y - 8x^3 + 12\sqrt{3}\,\sqrt{-4yx^3 + 27y^2}\right)^{\frac{2}{3}} + 96\left(\left(\right.\right.}{\left(2x^3 - 3\sqrt{3}\,\sqrt{-4yx^3 + 27y^2} - 27y\right)\left(\left(108y - 8x^3 + 12\sqrt{3}\,\sqrt{}\right.\right.}$$

Warning, solution could not be verified

$$x = \frac{96\left(x^3 + \frac{3x^2}{2} - 3y + 3c_1\right)\left(x^3 - \frac{3\sqrt{3}\,\sqrt{-4yx^3+27y^2}}{2} - \frac{27y}{2}\right)\left(108y - 8x^3 + 12\sqrt{3}\,\sqrt{-4yx^3 + 27y^2}\right)^{\frac{2}{3}} + 192(x}{\left(2x^3 - 3\sqrt{3}\,\sqrt{-4yx^3 + 27y^2} - 27y\right)\left(4i\sqrt{3}\,x^2 - i\sqrt{3}\,\left(108y - 8x^3 + 12\sqrt{}\right.\right.}$$

Warning, solution could not be verified

$$x = \frac{96\left(x^3 + \frac{3x^2}{2} - 3y + 3c_1\right)\left(x^3 - \frac{3\sqrt{3}\,\sqrt{-4yx^3+27y^2}}{2} - \frac{27y}{2}\right)\left(108y - 8x^3 + 12\sqrt{3}\,\sqrt{-4yx^3 + 27y^2}\right)^{\frac{2}{3}} - 192(x}{\left(2x^3 - 3\sqrt{3}\,\sqrt{-4yx^3 + 27y^2} - 27y\right)\left(4i\sqrt{3}\,x^2 - i\sqrt{3}\,\left(108y - 8x^3 + 12\sqrt{}\right.\right.}$$

Warning, solution could not be verified

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 994

```
dsolve(diff(y(x),x)^3+x*diff(y(x),x)^2-y(x)=0,y(x), singsol=all)
```

$$y(x) = 0$$

$$y(x) = \frac{\left(4x^2 - 2x\left(-36x^2 - 54x + 108c_1 - 8x^3 + 27 + 6\sqrt{-6\left(1+2c_1\right)\left(4x^3+18x^2-27c_1+27x\right)}\right)^{\frac{1}{3}} + 12x + \cdots}{\cdots}$$

$$y(x) = \frac{\left(\frac{\left(-i\sqrt{3}-1\right)\left(-36x^2-54x+108c_1-8x^3+27+6\sqrt{-6(1+2c_1)(4x^3+18x^2-27c_1+27x)}\right)^{\frac{2}{3}}}{4} + \left(2x+\frac{3}{2}\right)\left(-36x^2 - 54x + 108c_1 - \cdots\right.\right.}{\cdots}$$

$$y(x) = \frac{\left(\frac{\left(i\sqrt{3}-1\right)\left(-36x^2-54x+108c_1-8x^3+27+6\sqrt{-6(1+2c_1)(4x^3+18x^2-27c_1+27x)}\right)^{\frac{2}{3}}}{4} - \left(-2x-\frac{3}{2}\right)\left(-36x^2 - 54x + 108c_1 - \cdots\right.\right.}{\cdots}$$

✓ Solution by Mathematica

Time used: 84.497 (sec). Leaf size: 1516

```
DSolve[(y'[x])^3+x*(y'[x])^2-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{-16x^4 + 8\left(\sqrt[3]{-8x^3 - 36x^2 - 54x + 108c_1 + 6\sqrt{6}\sqrt{-\left(\left(4x^3 + 18x^2 + 27x - 27c_1\right)\left(2c_1 + 1\right)\right)} + 27 - 12}\right.}{}$$

$$y(x) \rightarrow \frac{1}{6}\left(-\frac{i\left(\sqrt{3} - i\right)x(2x+3)^2}{\sqrt[3]{-8x^3 - 36x^2 + 6\sqrt{6}\sqrt{-\left(\left(1+2c_1\right)\left(4x^3 + 18x^2 + 27x - 27c_1\right)\right)} - 54x + 27 + 108c_1}} + \frac{1}{16}\left(-\frac{i\left(\sqrt{3} - i\right)(2x+3)^2}{\sqrt[3]{-8x^3 - 36x^2 + 6\sqrt{6}\sqrt{-\left(\left(1+2c_1\right)\left(4x^3 + 18x^2 + 27x - 27c_1\right)\right)} - 54x + 27 + 108c_1}} + i\left(\sqrt{3}+i\right)\sqrt[3]{-8x^3 - 36x^2 + 6\sqrt{6}\sqrt{-\left(\left(1+2c_1\right)\left(4x^3 + 18x^2 + 27x - 27c_1\right)\right)} - 54x + 27 + 108c_1} - 4x + 6\right)^2 + i\left(\sqrt{3} + i\right)x\sqrt[3]{-8x^3 - 36x^2 + 6\sqrt{6}\sqrt{-\left(\left(1+2c_1\right)\left(4x^3 + 18x^2 + 27x - 27c_1\right)\right)} - 54x + 27 + 108c_1} + 2(3-2x)x - 6x + 6c_1\right)$$

$$y(x) \rightarrow \frac{1}{6}\left(\frac{i\left(\sqrt{3} + i\right)x(2x+3)^2}{\sqrt[3]{-8x^3 - 36x^2 + 6\sqrt{6}\sqrt{-\left(\left(1+2c_1\right)\left(4x^3 + 18x^2 + 27x - 27c_1\right)\right)} - 54x + 27 + 108c_1}} + \frac{1}{16}\left(\frac{\left(1 - i\sqrt{3}\right)(2x+3)^2}{\sqrt[3]{-8x^3 - 36x^2 + 6\sqrt{6}\sqrt{-\left(\left(1+2c_1\right)\left(4x^3 + 18x^2 + 27x - 27c_1\right)\right)} - 54x + 27 + 108c_1}} + \left(1+i\sqrt{3}\right)\sqrt[3]{-8x^3 - 36x^2 + 6\sqrt{6}\sqrt{-\left(\left(1+2c_1\right)\left(4x^3 + 18x^2 + 27x - 27c_1\right)\right)} - 54x + 27 + 108c_1} + 4x - 6\right)^2 - \left(1 + i\sqrt{3}\right)x\sqrt[3]{-8x^3 - 36x^2 + 6\sqrt{6}\sqrt{-\left(\left(1+2c_1\right)\left(4x^3 + 18x^2 + 27x - 27c_1\right)\right)} - 54x + 27 + 108c_1} + 2(3-2x)x - 6x + 6c_1\right)$$

2.9 problem 16

Internal problem ID [6794]
Internal file name [`OUTPUT/6041_Tuesday_July_26_2022_05_07_04_AM_18042897/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 97. The p-discriminant equation. EXERCISES Page 314
Problem number: 16.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries]]
```

$$\boxed{y^4 {y'}^3 - 6xy' + 2y = 0}$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}}{y^2} + \frac{2x}{y^2\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}} \tag{1}$$

$$y' = -\frac{\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}}{2y^2} - \frac{x}{y^2\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}}{y^2} - \frac{2x}{y^2\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}}\right)}{2} \tag{2}$$

$$y' = -\frac{\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}}{2y^2} - \frac{x}{y^2\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}}{y^2} - \frac{2x}{y^2\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}}\right)}{2} \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = \frac{\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}}+2x}{y^2\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$
$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1,a_2,a_3,b_1,b_2,b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2+\frac{\left(\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}}+2x\right)(b_3-a_2)}{y^2\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}} \\
&-\frac{\left(\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}}+2x\right)^2 a_3}{y^4\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}}}-\left(\frac{-\frac{8x^2}{\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}\sqrt{y^6-8x^3}}+2}{y^2\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}}\right. \\
&\left.+\frac{4\left(\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}}+2x\right)x^2}{y^2\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{4}{3}}\sqrt{y^6-8x^3}}\right)(xa_2+ya_3+a_1) \\
&-\left(\frac{-2y^2+\frac{2y^5}{\sqrt{y^6-8x^3}}}{y^2\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}}}-\frac{2\left(\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}}+2x\right)}{y^3\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}}\right. \\
&\left.-\frac{\left(\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}}+2x\right)\left(-3y^2+\frac{3y^5}{\sqrt{y^6-8x^3}}\right)}{3y^2\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{4}{3}}}\right)(xb_2+yb_3+b_1)=0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-4\sqrt{y^6-8x^3}\,x\,y^5a_2-\sqrt{y^6-8x^3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}}y^5b_3-2\sqrt{y^6-8x^3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{5}{3}}yb_1-\sqrt{}}{}$$

$$=0$$

Setting the numerator to zero gives

$$\begin{aligned}
&4\sqrt{y^6-8x^3}\,x\,y^5a_2+\sqrt{y^6-8x^3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}}y^5b_3\\
&+2\sqrt{y^6-8x^3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{5}{3}}yb_1\\
&+\sqrt{y^6-8x^3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}}y^4b_1-6\sqrt{y^6-8x^3}\,x^2y^4b_2\\
&-8\sqrt{y^6-8x^3}\,x\,y^5b_3+4\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}}x^3y^2a_2\\
&+4\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}}x^2y^3a_3\\
&-4\sqrt{y^6-8x^3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{4}{3}}xa_3\\
&-6\sqrt{y^6-8x^3}\,x\,y^4b_1+4\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}}x^2y^2a_1\\
&-4\sqrt{y^6-8x^3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}}x^2a_3\\
&+2\sqrt{y^6-8x^3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{5}{3}}xyb_2\\
&+\sqrt{y^6-8x^3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}}x\,y^4b_2\\
&-\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}}x\,y^7b_2\\
&+b_2y^4\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{4}{3}}\sqrt{y^6-8x^3}\\
&-\sqrt{y^6-8x^3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{5}{3}}y^2a_2\\
&+3\sqrt{y^6-8x^3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{5}{3}}y^2b_3\\
&-\left(y^6-8x^3\right)^{\frac{3}{2}}a_3-2y^8a_1+6x^2y^7b_2+8x\,y^8b_3\\
&+6x\,y^7b_1+24x^4y^2a_2-8x^3y^3a_3+8x^3y^2a_1\\
&-\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}}y^8b_3-\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}}y^7b_1\\
&+\sqrt{y^6-8x^3}\,y^6a_3+2\sqrt{y^6-8x^3}\,y^5a_1-4x\,y^8a_2\\
&-32x^5yb_2-48x^4y^2b_3-32x^4yb_1=0
\end{aligned}\tag{6E}$$

Simplifying the above gives

$$
\begin{aligned}
&\sqrt{y^6-8x^3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} y^5 b_3+2\sqrt{y^6-8x^3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{5}{3}} y b_1 \\
&-2\sqrt{y^6-8x^3}\left(-y^3+\sqrt{y^6-8x^3}\right) y^3 a_3+\sqrt{y^6-8x^3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} y^4 b_1 \\
&-2\sqrt{y^6-8x^3}\, x^2 y^4 b_2-2\sqrt{y^6-8x^3}\, x\, y^5 b_3+4\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} x^3 y^2 a_2 \\
&+4\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} x^2 y^3 a_3-4\sqrt{y^6-8x^3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{4}{3}} x a_3 \\
&-2\sqrt{y^6-8x^3}\left(-y^3+\sqrt{y^6-8x^3}\right) y^2 a_1-2\sqrt{y^6-8x^3}\, x\, y^4 b_1 \\
&+4\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} x^2 y^2 a_1-4\sqrt{y^6-8x^3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} x^2 a_3 \\
&+2\sqrt{y^6-8x^3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{5}{3}} x y b_2+\sqrt{y^6-8x^3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} x\, y^4 b_2 \\
&+4\sqrt{y^6-8x^3}\left(-y^3+\sqrt{y^6-8x^3}\right) x^2 y b_2-4\sqrt{y^6-8x^3}\left(-y^3+\sqrt{y^6-8x^3}\right) x\, y^2 a_2 \\
&+6\sqrt{y^6-8x^3}\left(-y^3+\sqrt{y^6-8x^3}\right) x\, y^2 b_3+4\sqrt{y^6-8x^3}\left(-y^3+\sqrt{y^6-8x^3}\right) x y b_1 \\
&-\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} x\, y^7 b_2+b_2 y^4\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{4}{3}} \sqrt{y^6-8x^3} \\
&-\sqrt{y^6-8x^3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{5}{3}} y^2 a_2+3\sqrt{y^6-8x^3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{5}{3}} y^2 b_3 \\
&+2x^2 y^7 b_2+2x\, y^8 b_3+2x\, y^7 b_1-8x^4 y^2 a_2-8x^3 y^3 a_3-8x^3 y^2 a_1-\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} y^8 b_3 \\
&-\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} y^7 b_1-\sqrt{y^6-8x^3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{2} a_3=0
\end{aligned}
\tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&4\sqrt{y^6-8x^3}\,x\,y^5 a_2 - 2\sqrt{y^6-8x^3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} y^5 b_3 \\
&- \sqrt{y^6-8x^3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} y^4 b_1 \\
&- 6\sqrt{y^6-8x^3}\,x^2 y^4 b_2 - 8\sqrt{y^6-8x^3}\,x\,y^5 b_3 \\
&+ 12\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} x^3 y^2 a_2 + 4\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} x^2 y^3 a_3 \\
&- 6\sqrt{y^6-8x^3}\,x\,y^4 b_1 + 4\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} x^2 y^2 a_1 \\
&- 4\sqrt{y^6-8x^3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} x^2 a_3 \\
&+ 32x^4\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}} a_3 + 8x^3\sqrt{y^6-8x^3}\,a_3 \\
&+ \left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}} y^{10} b_2 - \left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} y^8 a_2 \\
&- \sqrt{y^6-8x^3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}} y^7 b_2 \\
&+ \sqrt{y^6-8x^3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} y^5 a_2 \\
&- 24x^3\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} y^2 b_3 \\
&- 16x^3\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} y b_1 - 8x^3\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}} y^4 b_2 \\
&- 4x\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}} y^6 a_3 - 16x^4\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} y b_2 \\
&- \sqrt{y^6-8x^3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} x\,y^4 b_2 \\
&+ 4x\sqrt{y^6-8x^3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}} y^3 a_3 \\
&+ \left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} x\,y^7 b_2 - 2y^8 a_1 + 6x^2 y^7 b_2 \\
&+ 8x\,y^8 b_3 + 6x\,y^7 b_1 + 24x^4 y^2 a_2 - 8x^3 y^3 a_3 + 8x^3 y^2 a_1 \\
&+ 2\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} y^8 b_3 + \left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} y^7 b_1 \\
&+ 2\sqrt{y^6-8x^3}\,y^5 a_1 - 4x\,y^8 a_2 - 32x^5 y b_2 - 48x^4 y^2 b_3 - 32x^4 y b_1 = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}, \left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}}, \sqrt{y^6-8x^3}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}} = v_3, \left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} = v_4, \sqrt{y^6-8x^3} = v_5\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&v_3 v_2^{10} b_2 - 4v_1 v_2^8 a_2 - v_4 v_2^8 a_2 + 6v_1^2 v_2^7 b_2 + v_4 v_1 v_2^7 b_2 - v_5 v_3 v_2^7 b_2 \\
&+ 8v_1 v_2^8 b_3 + 2v_4 v_2^8 b_3 - 2v_2^8 a_1 - 4v_1 v_3 v_2^6 a_3 + 6v_1 v_2^7 b_1 + v_4 v_2^7 b_1 \\
&- 8v_1^3 v_3 v_2^4 b_2 + 4v_5 v_1 v_2^5 a_2 + v_5 v_4 v_2^5 a_2 - 6v_5 v_1^2 v_2^4 b_2 - v_5 v_4 v_1 v_2^4 b_2 \\
&- 8v_5 v_1 v_2^5 b_3 - 2v_5 v_4 v_2^5 b_3 + 2v_5 v_2^5 a_1 + 24v_1^4 v_2^2 a_2 + 12v_4 v_1^3 v_2^2 a_2 - 8v_1^3 v_2^3 a_3 \\
&+ 4v_4 v_1^2 v_2^3 a_3 + 4v_1 v_5 v_3 v_2^3 a_3 - 6v_5 v_1 v_2^4 b_1 - v_5 v_4 v_2^4 b_1 - 32v_1^5 v_2 b_2 \\
&- 16v_1^4 v_4 v_2 b_2 - 48v_1^4 v_2^2 b_3 - 24v_1^3 v_4 v_2^2 b_3 + 8v_1^3 v_2^2 a_1 + 4v_4 v_1^2 v_2^2 a_1 \\
&+ 32v_1^4 v_3 a_3 - 32v_1^4 v_2 b_1 - 16v_1^3 v_4 v_2 b_1 + 8v_1^3 v_5 a_3 - 4v_5 v_4 v_1^2 a_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$\begin{aligned}
&32v_1^4 v_3 a_3 + 8v_1^3 v_5 a_3 + v_3 v_2^{10} b_2 + 6v_1^2 v_2^7 b_2 + 6v_1 v_2^7 b_1 - 8v_1^3 v_2^3 a_3 \\
&+ 8v_1^3 v_2^2 a_1 + v_4 v_2^7 b_1 + 2v_5 v_2^5 a_1 - 32v_1^5 v_2 b_2 - 32v_1^4 v_2 b_1 - v_5 v_4 v_2^4 b_1 \\
&- 6v_5 v_1^2 v_2^4 b_2 + 4v_4 v_1^2 v_2^3 a_3 - 6v_5 v_1 v_2^4 b_1 + 4v_4 v_1^2 v_2^2 a_1 - 4v_5 v_4 v_1^2 a_3 \\
&- v_5 v_3 v_2^7 b_2 - 16v_1^3 v_4 v_2 b_1 - 8v_1^3 v_3 v_2^4 b_2 - 4v_1 v_3 v_2^6 a_3 - 16v_1^4 v_4 v_2 b_2 \\
&+ v_4 v_1 v_2^7 b_2 - 2v_2^8 a_1 + (-4a_2 + 8b_3) v_1 v_2^8 + (-a_2 + 2b_3) v_2^8 v_4 \\
&+ (12a_2 - 24b_3) v_1^3 v_2^2 v_4 + (4a_2 - 8b_3) v_1 v_2^5 v_5 + (a_2 - 2b_3) v_2^5 v_4 v_5 \\
&+ (24a_2 - 48b_3) v_1^4 v_2^2 - v_5 v_4 v_1 v_2^4 b_2 + 4v_1 v_5 v_3 v_2^3 a_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_1 &= 0 \\
b_2 &= 0 \\
-2a_1 &= 0 \\
2a_1 &= 0 \\
4a_1 &= 0 \\
8a_1 &= 0 \\
-8a_3 &= 0 \\
-4a_3 &= 0 \\
4a_3 &= 0 \\
8a_3 &= 0 \\
32a_3 &= 0 \\
-32b_1 &= 0 \\
-16b_1 &= 0 \\
-6b_1 &= 0 \\
-b_1 &= 0 \\
6b_1 &= 0 \\
-32b_2 &= 0 \\
-16b_2 &= 0 \\
-8b_2 &= 0 \\
-6b_2 &= 0 \\
-b_2 &= 0 \\
6b_2 &= 0 \\
-4a_2 + 8b_3 &= 0 \\
-a_2 + 2b_3 &= 0 \\
a_2 - 2b_3 &= 0 \\
4a_2 - 8b_3 &= 0 \\
12a_2 - 24b_3 &= 0 \\
24a_2 - 48b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0\\
a_2 &= 2b_3\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 2x\\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi}\\
&= \frac{y}{2x}\\
&= \frac{y}{2x}
\end{aligned}
$$

This is easily solved to give

$$
y = c_1 \sqrt{x}
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = \frac{y}{\sqrt{x}}
$$

And S is found from

$$\begin{aligned} dS &= \frac{dx}{\xi} \\ &= \frac{dx}{2x} \end{aligned}$$

Integrating gives

$$\begin{aligned} S &= \int \frac{dx}{T} \\ &= \frac{\ln(x)}{2} \end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\left(-y^3 + \sqrt{y^6 - 8x^3}\right)^{\frac{2}{3}} + 2x}{y^2 \left(-y^3 + \sqrt{y^6 - 8x^3}\right)^{\frac{1}{3}}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= -\frac{y}{2x^{\frac{3}{2}}} \\ R_y &= \frac{1}{\sqrt{x}} \\ S_x &= \frac{1}{2x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\sqrt{x}\, y^2 \left(-y^3 + \sqrt{y^6 - 8x^3}\right)^{\frac{1}{3}}}{-y^3 \left(-y^3 + \sqrt{y^6 - 8x^3}\right)^{\frac{1}{3}} + 2\left(-y^3 + \sqrt{y^6 - 8x^3}\right)^{\frac{2}{3}} x + 4x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{R^2 \left(-R^3 + \sqrt{R^6 - 8}\right)^{\frac{1}{3}}}{\left(-R^3 + \sqrt{R^6 - 8}\right)^{\frac{1}{3}} R^3 - 2\left(-R^3 + \sqrt{R^6 - 8}\right)^{\frac{2}{3}} - 4}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int \frac{R^2 \left(-R^3+\sqrt{R^6-8}\right)^{\frac{1}{3}}}{-\left(-R^3+\sqrt{R^6-8}\right)^{\frac{1}{3}} R^3+2 \left(-R^3+\sqrt{R^6-8}\right)^{\frac{2}{3}}+4} dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(x)}{2} = \int^{\frac{y}{\sqrt{x}}} \frac{\_a^2 \left(-\_a^3+\sqrt{\_a^6-8}\right)^{\frac{1}{3}}}{-\left(-\_a^3+\sqrt{\_a^6-8}\right)^{\frac{1}{3}} \_a^3+2 \left(-\_a^3+\sqrt{\_a^6-8}\right)^{\frac{2}{3}}+4} d\_a + c_1$$

Which simplifies to

$$\frac{\ln(x)}{2} = \int^{\frac{y}{\sqrt{x}}} \frac{\_a^2 \left(-\_a^3+\sqrt{\_a^6-8}\right)^{\frac{1}{3}}}{-\left(-\_a^3+\sqrt{\_a^6-8}\right)^{\frac{1}{3}} \_a^3+2 \left(-\_a^3+\sqrt{\_a^6-8}\right)^{\frac{2}{3}}+4} d\_a + c_1$$

Summary

The solution(s) found are the following

$$\frac{\ln(x)}{2} = \int^{\frac{y}{\sqrt{x}}} \frac{\_a^2 \left(-\_a^3+\sqrt{\_a^6-8}\right)^{\frac{1}{3}}}{-\left(-\_a^3+\sqrt{\_a^6-8}\right)^{\frac{1}{3}} \_a^3+2 \left(-\_a^3+\sqrt{\_a^6-8}\right)^{\frac{2}{3}}+4} d\_a + c_1 \tag{1}$$

Verification of solutions

$$\frac{\ln(x)}{2} = \int^{\frac{y}{\sqrt{x}}} \frac{\_a^2 \left(-\_a^3+\sqrt{\_a^6-8}\right)^{\frac{1}{3}}}{-\left(-\_a^3+\sqrt{\_a^6-8}\right)^{\frac{1}{3}} \_a^3+2 \left(-\_a^3+\sqrt{\_a^6-8}\right)^{\frac{2}{3}}+4} d\_a + c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = \frac{i\sqrt{3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} - 2i\sqrt{3}\,x - \left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} - 2x}{2y^2\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 + \frac{\left(i\sqrt{3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} - 2i\sqrt{3}\,x - \left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} - 2x\right)(b_3-a_2)}{2y^2\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}} \\
&- \frac{\left(i\sqrt{3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} - 2i\sqrt{3}\,x - \left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} - 2x\right)^2 a_3}{4y^4\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}}} \\
&- \left(\frac{-\frac{8i\sqrt{3}\,x^2}{\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}\sqrt{y^6-8x^3}} - 2i\sqrt{3} + \frac{8x^2}{\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}\sqrt{y^6-8x^3}} - 2}{2y^2\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}}\right. \\
&\left. + \frac{2\left(i\sqrt{3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} - 2i\sqrt{3}\,x - \left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} - 2x\right)x^2}{y^2\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{4}{3}}\sqrt{y^6-8x^3}}\right)(x a_2 \\
&+ y a_3 + a_1) - \left(\frac{\frac{2i\sqrt{3}\left(-3y^2+\frac{3y^5}{\sqrt{y^6-8x^3}}\right)}{3\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}} - \frac{2\left(-3y^2+\frac{3y^5}{\sqrt{y^6-8x^3}}\right)}{3\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}}}{2y^2\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}}\right. \\
&- \frac{i\sqrt{3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} - 2i\sqrt{3}\,x - \left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} - 2x}{y^3\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}} \\
&\left. - \frac{\left(i\sqrt{3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} - 2i\sqrt{3}\,x - \left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} - 2x\right)\left(-3y^2+\frac{3y^5}{\sqrt{y^6-8x^3}}\right)}{6y^2\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{4}{3}}}\right)(x b_2 \\
&+ y b_3 + b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Simplifying the above gives

Expression too large to display (6E)

Since the PDE has radicals, simplifying gives

Expression too large to display

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}, \left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}}, \sqrt{y^6-8x^3}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x=v_1, y=v_2, \left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}=v_3, \left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}}=v_4, \sqrt{y^6-8x^3}=v_5\right\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&4v_2^8a_1 - 4v_5v_3v_2^7b_2 - 2v_5v_4v_2^5a_2 - 32v_1^3v_3v_2^4b_2 - 16v_1v_3v_2^6a_3\\
&+ 32v_1^4v_4v_2b_2 + 48v_1^3v_4v_2^2b_3 + 32v_1^3v_4v_2b_1 - 2v_4v_1v_2^7b_2\\
&+ 8v_5v_4v_1^2a_3 + 4v_5v_4v_2^5b_3 + 2v_5v_4v_2^4b_1 + 12v_5v_1^2v_2^4b_2\\
&+ 16v_5v_1v_2^5b_3 - 24v_4v_1^3v_2^2a_2 - 8v_4v_1^2v_2^3a_3 + 2v_5v_4v_1v_2^4b_2\\
&+ 16v_1v_5v_3v_2^3a_3 - 2i\sqrt{3}\,v_5v_4v_1v_2^4b_2 + 16i\sqrt{3}\,v_1^3v_2^3a_3\\
&- 4i\sqrt{3}\,v_5v_2^5a_1 - 12i\sqrt{3}\,v_1^2v_2^7b_2 + 8i\sqrt{3}\,v_1v_2^8a_2 - 16i\sqrt{3}\,v_1v_2^8b_3\\
&+ 2i\sqrt{3}\,v_4v_2^7b_1 - 16i\sqrt{3}\,v_1^3v_5a_3 - 2i\sqrt{3}\,v_4v_2^8a_2 + 4i\sqrt{3}\,v_4v_2^8b_3\\
&- 16i\sqrt{3}\,v_1^3v_2^2a_1 - 12i\sqrt{3}\,v_1v_2^7b_1 + 64i\sqrt{3}\,v_1^5v_2b_2\\
&- 48i\sqrt{3}\,v_1^4v_2^2a_2 + 96i\sqrt{3}\,v_1^4v_2^2b_3 + 64i\sqrt{3}\,v_1^4v_2b_1 + 12v_5v_1v_2^4b_1\\
&- 8v_4v_1^2v_2^2a_1 - 8v_5v_1v_2^5a_2 + 4i\sqrt{3}\,v_2^8a_1 - 4v_5v_2^5a_1 - 16v_1^3v_5a_3\\
&+ 8v_1v_2^8a_2 + 64v_1^5v_2b_2 + 96v_1^4v_2^2b_3 + 64v_1^4v_2b_1 - 48v_1^4v_2^2a_2\\
&+ 16v_1^3v_2^3a_3 - 16v_1^3v_2^2a_1 - 12v_1^2v_2^7b_2 - 16v_1v_2^8b_3 - 12v_1v_2^7b_1\\
&- 4v_4v_2^8b_3 - 2v_4v_2^7b_1 + 4v_3v_2^{10}b_2 + 2v_4v_2^8a_2 + 128v_1^4v_3a_3\\
&+ 2i\sqrt{3}\,v_4v_1v_2^7b_2 + 8i\sqrt{3}\,v_4v_1^2v_2^3a_3 + 8i\sqrt{3}\,v_4v_1^2v_2^2a_1\\
&- 8i\sqrt{3}\,v_5v_4v_1^2a_3 + 12i\sqrt{3}\,v_5v_1^2v_2^4b_2 - 8i\sqrt{3}\,v_5v_1v_2^5a_2\\
&+ 16i\sqrt{3}\,v_5v_1v_2^5b_3 + 12i\sqrt{3}\,v_5v_1v_2^4b_1 + 2i\sqrt{3}\,v_5v_4v_2^5a_2\\
&- 4i\sqrt{3}\,v_5v_4v_2^5b_3 - 2i\sqrt{3}\,v_5v_4v_2^4b_1 - 32i\sqrt{3}\,v_1^4v_4v_2b_2\\
&+ 24i\sqrt{3}\,v_4v_1^3v_2^2a_2 - 48i\sqrt{3}\,v_1^3v_4v_2^2b_3 - 32i\sqrt{3}\,v_1^3v_4v_2b_1 = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&-4v_5v_3v_2^7b_2 + \left(4i\sqrt{3}\,a_1 + 4a_1\right)v_2^8 - 32v_1^3v_3v_2^4b_2 \\
&\quad - 16v_1v_3v_2^6a_3 + \left(-32i\sqrt{3}\,b_2 + 32b_2\right)v_1^4v_2v_4 \\
&\quad + \left(24i\sqrt{3}\,a_2 - 48i\sqrt{3}\,b_3 - 24a_2 + 48b_3\right)v_1^3v_2^2v_4 \\
&\quad + \left(-32i\sqrt{3}\,b_1 + 32b_1\right)v_1^3v_2v_4 + \left(12i\sqrt{3}\,b_2 + 12b_2\right)v_1^2v_2^4v_5 \\
&\quad + \left(8i\sqrt{3}\,a_3 - 8a_3\right)v_1^2v_2^3v_4 + \left(8i\sqrt{3}\,a_1 - 8a_1\right)v_1^2v_2^2v_4 \\
&\quad + \left(-8i\sqrt{3}\,a_3 + 8a_3\right)v_1^2v_4v_5 + \left(2i\sqrt{3}\,b_2 - 2b_2\right)v_1v_2^7v_4 \\
&\quad + \left(-8i\sqrt{3}\,a_2 + 16i\sqrt{3}\,b_3 - 8a_2 + 16b_3\right)v_1v_2^5v_5 \\
&\quad + \left(12i\sqrt{3}\,b_1 + 12b_1\right)v_1v_2^4v_5 \\
&\quad + \left(2i\sqrt{3}\,a_2 - 4i\sqrt{3}\,b_3 - 2a_2 + 4b_3\right)v_2^5v_4v_5 \\
&\quad + \left(-2i\sqrt{3}\,b_1 + 2b_1\right)v_2^4v_4v_5 + 16v_1v_5v_3v_2^3a_3 \\
&\quad + \left(-48i\sqrt{3}\,a_2 + 96i\sqrt{3}\,b_3 - 48a_2 + 96b_3\right)v_1^4v_2^2 \\
&\quad + \left(64i\sqrt{3}\,b_1 + 64b_1\right)v_1^4v_2 + \left(16i\sqrt{3}\,a_3 + 16a_3\right)v_1^3v_2^3 \\
&\quad + \left(-16i\sqrt{3}\,a_1 - 16a_1\right)v_1^3v_2^2 \\
&\quad + \left(-16i\sqrt{3}\,a_3 - 16a_3\right)v_1^3v_5 + \left(-12i\sqrt{3}\,b_2 - 12b_2\right)v_1^2v_2^7 \\
&\quad + \left(8i\sqrt{3}\,a_2 - 16i\sqrt{3}\,b_3 + 8a_2 - 16b_3\right)v_1v_2^8 \\
&\quad + \left(-12i\sqrt{3}\,b_1 - 12b_1\right)v_1v_2^7 \\
&\quad + \left(-2i\sqrt{3}\,a_2 + 4i\sqrt{3}\,b_3 + 2a_2 - 4b_3\right)v_2^8v_4 + \left(2i\sqrt{3}\,b_1 - 2b_1\right)v_2^7v_4 \\
&\quad + \left(-4i\sqrt{3}\,a_1 - 4a_1\right)v_2^5v_5 + 4v_3v_2^{10}b_2 + 128v_1^4v_3a_3 \\
&\quad + \left(64i\sqrt{3}\,b_2 + 64b_2\right)v_1^5v_2 + \left(-2i\sqrt{3}\,b_2 + 2b_2\right)v_1v_2^4v_4v_5 = 0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-16a_3 &= 0 \\
16a_3 &= 0 \\
128a_3 &= 0 \\
-32b_2 &= 0 \\
-4b_2 &= 0 \\
4b_2 &= 0 \\
-32i\sqrt{3}\, b_1 + 32b_1 &= 0 \\
-32i\sqrt{3}\, b_2 + 32b_2 &= 0 \\
-16i\sqrt{3}\, a_1 - 16a_1 &= 0 \\
-16i\sqrt{3}\, a_3 - 16a_3 &= 0 \\
-12i\sqrt{3}\, b_1 - 12b_1 &= 0 \\
-12i\sqrt{3}\, b_2 - 12b_2 &= 0 \\
-8i\sqrt{3}\, a_3 + 8a_3 &= 0 \\
-4i\sqrt{3}\, a_1 - 4a_1 &= 0 \\
-2i\sqrt{3}\, b_1 + 2b_1 &= 0 \\
-2i\sqrt{3}\, b_2 + 2b_2 &= 0 \\
2i\sqrt{3}\, b_1 - 2b_1 &= 0 \\
2i\sqrt{3}\, b_2 - 2b_2 &= 0 \\
4i\sqrt{3}\, a_1 + 4a_1 &= 0 \\
8i\sqrt{3}\, a_1 - 8a_1 &= 0 \\
8i\sqrt{3}\, a_3 - 8a_3 &= 0 \\
12i\sqrt{3}\, b_1 + 12b_1 &= 0 \\
12i\sqrt{3}\, b_2 + 12b_2 &= 0 \\
16i\sqrt{3}\, a_3 + 16a_3 &= 0 \\
64i\sqrt{3}\, b_1 + 64b_1 &= 0 \\
64i\sqrt{3}\, b_2 + 64b_2 &= 0 \\
-48i\sqrt{3}\, a_2 + 96i\sqrt{3}\, b_3 - 48a_2 + 96b_3 &= 0 \\
-8i\sqrt{3}\, a_2 + 16i\sqrt{3}\, b_3 - 8a_2 + 16b_3 &= 0 \\
-2i\sqrt{3}\, a_2 + 4i\sqrt{3}\, b_3 + 2a_2 - 4b_3 &= 0 \\
2i\sqrt{3}\, a_2 - 4i\sqrt{3}\, b_3 - 2a_2 + 4b_3 &= 0 \\
8i\sqrt{3}\, a_2 - 16i\sqrt{3}\, b_3 + 8a_2 - 16b_3 &= 0 \\
24i\sqrt{3}\, a_2 - 48i\sqrt{3}\, b_3 - 24a_2 + 48b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 2b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 2x \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Solving equation (3)

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{i\sqrt{3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} - 2i\sqrt{3}\,x + \left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} + 2x}{2y^2\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 - \frac{\left(i\sqrt{3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} - 2i\sqrt{3}\,x + \left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} + 2x\right)(b_3-a_2)}{2y^2\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}} \\
&- \frac{\left(i\sqrt{3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} - 2i\sqrt{3}\,x + \left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} + 2x\right)^2 a_3}{4y^4\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}}} \\
&- \left(-\frac{-\frac{8i\sqrt{3}\,x^2}{\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}\sqrt{y^6-8x^3}} - 2i\sqrt{3} - \frac{8x^2}{\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}\sqrt{y^6-8x^3}} + 2}{2y^2\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}}\right. \\
&\left. - \frac{2\left(i\sqrt{3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} - 2i\sqrt{3}\,x + \left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} + 2x\right)x^2}{y^2\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{4}{3}}\sqrt{y^6-8x^3}}\right)(xa_2 \\
&+ ya_3 + a_1) - \left(-\frac{\frac{2i\sqrt{3}\left(-3y^2+\frac{3y^5}{\sqrt{y^6-8x^3}}\right)}{3\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}} + \frac{-2y^2+\frac{2y^5}{\sqrt{y^6-8x^3}}}{\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}}}{2y^2\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}}\right. \\
&+ \frac{i\sqrt{3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} - 2i\sqrt{3}\,x + \left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} + 2x}{y^3\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}} \\
&\left. + \frac{\left(i\sqrt{3}\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} - 2i\sqrt{3}\,x + \left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} + 2x\right)\left(-3y^2+\frac{3y^5}{\sqrt{y^6-8x^3}}\right)}{6y^2\left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{4}{3}}}\right)(xb_2 \\
&+ yb_3 + b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Simplifying the above gives

Expression too large to display (6E)

Since the PDE has radicals, simplifying gives

Expression too large to display

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}}, \left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}}, \sqrt{y^6-8x^3}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{1}{3}} = v_3, \left(-y^3+\sqrt{y^6-8x^3}\right)^{\frac{2}{3}} = v_4, \sqrt{y^6-8x^3} = v_5\right\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&4v_2^8 a_1 - 2iv_4\sqrt{3}\,v_1 v_2^7 b_2 + 2iv_5 v_4\sqrt{3}\,v_2^4 b_1 - 8iv_4\sqrt{3}\,v_1^2 v_2^3 a_3\\
&- 8iv_4\sqrt{3}\,v_1^2 v_2^2 a_1 + 8iv_5 v_4\sqrt{3}\,v_1^2 a_3 - 2i\sqrt{3}\,v_5 v_4 v_2^5 a_2\\
&+ 32i\sqrt{3}\,v_1^4 v_4 v_2 b_2 + 16i\sqrt{3}\,v_1^3 v_5 a_3 + 12i\sqrt{3}\,v_1^2 v_2^7 b_2\\
&- 8i\sqrt{3}\,v_1 v_2^8 a_2 + 16i\sqrt{3}\,v_1 v_2^8 b_3 + 12i\sqrt{3}\,v_1 v_2^7 b_1 - 64i\sqrt{3}\,v_1^5 v_2 b_2\\
&+ 48i\sqrt{3}\,v_1^4 v_2^2 a_2 - 96i\sqrt{3}\,v_1^4 v_2^2 b_3 - 24iv_4\sqrt{3}\,v_1^3 v_2^2 a_2\\
&+ 48i\sqrt{3}\,v_1^3 v_4 v_2^2 b_3 + 32i\sqrt{3}\,v_1^3 v_4 v_2 b_1 + 4iv_5 v_4\sqrt{3}\,v_2^5 b_3\\
&- 12iv_5\sqrt{3}\,v_1^2 v_2^4 b_2 + 8iv_5\sqrt{3}\,v_1 v_2^5 a_2 - 16iv_5\sqrt{3}\,v_1 v_2^5 b_3\\
&- 12iv_5\sqrt{3}\,v_1 v_2^4 b_1 + 2iv_5 v_4\sqrt{3}\,v_1 v_2^4 b_2 + 2v_5 v_4 v_1 v_2^4 b_2\\
&+ 16v_1 v_5 v_3 v_2^3 a_3 - 64i\sqrt{3}\,v_1^4 v_2 b_1 - 2iv_4\sqrt{3}\,v_2^7 b_1 + 16i\sqrt{3}\,v_1^3 v_2^2 a_1\\
&+ 4iv_5\sqrt{3}\,v_2^5 a_1 - 16i\sqrt{3}\,v_1^3 v_2^3 a_3 + 2i\sqrt{3}\,v_4 v_2^8 a_2 - 4iv_4\sqrt{3}\,v_2^8 b_3\\
&- 4i\sqrt{3}\,v_2^8 a_1 - 8v_4 v_1^2 v_2^3 a_3 + 12v_5 v_1 v_2^4 b_1 - 8v_4 v_1^2 v_2^2 a_1\\
&- 4v_5 v_3 v_2^7 b_2 - 2v_5 v_4 v_2^5 a_2 + 32v_1^3 v_4 v_2 b_1 - 32v_1^3 v_3 v_2^4 b_2\\
&- 16v_1 v_3 v_2^6 a_3 + 32v_1^4 v_4 v_2 b_2 + 48v_1^3 v_4 v_2^2 b_3 - 2v_4 v_1 v_2^7 b_2\\
&+ 8v_5 v_4 v_1^2 a_3 + 4v_5 v_4 v_2^5 b_3 + 2v_5 v_4 v_2^4 b_1 + 12v_5 v_1^2 v_2^4 b_2\\
&+ 16v_5 v_1 v_2^5 b_3 - 8v_5 v_1 v_2^5 a_2 - 24v_4 v_1^3 v_2^2 a_2 + 8v_1 v_2^8 a_2 + 64v_1^5 v_2 b_2\\
&+ 96v_1^4 v_2^2 b_3 + 64v_1^4 v_2 b_1 - 48v_1^4 v_2^2 a_2 + 16v_1^3 v_2^3 a_3 - 16v_1^3 v_2^2 a_1\\
&- 12v_1^2 v_2^7 b_2 - 16v_1 v_2^8 b_3 - 12v_1 v_2^7 b_1 - 4v_4 v_2^8 b_3 - 2v_4 v_2^7 b_1\\
&+ 2v_4 v_2^8 a_2 + 128v_1^4 v_3 a_3 - 16v_1^3 v_5 a_3 - 4v_5 v_2^5 a_1 + 4v_3 v_2^{10} b_2 = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&\left(-4i\sqrt{3}\,a_1+4a_1\right)v_2^8+\left(-64i\sqrt{3}\,b_2+64b_2\right)v_1^5v_2\\
&+\left(48i\sqrt{3}\,a_2-96i\sqrt{3}\,b_3-48a_2+96b_3\right)v_1^4v_2^2\\
&+\left(-64i\sqrt{3}\,b_1+64b_1\right)v_1^4v_2\\
&+\left(-16i\sqrt{3}\,a_3+16a_3\right)v_1^3v_2^3+\left(16i\sqrt{3}\,a_1-16a_1\right)v_1^3v_2^2\\
&+\left(16i\sqrt{3}\,a_3-16a_3\right)v_1^3v_5+\left(12i\sqrt{3}\,b_2-12b_2\right)v_1^2v_2^7\\
&+\left(-8i\sqrt{3}\,a_2+16i\sqrt{3}\,b_3+8a_2-16b_3\right)v_1v_2^8\\
&+\left(12i\sqrt{3}\,b_1-12b_1\right)v_1v_2^7+\left(2i\sqrt{3}\,a_2-4i\sqrt{3}\,b_3+2a_2-4b_3\right)v_2^8v_4\\
&+\left(32i\sqrt{3}\,b_2+32b_2\right)v_1^4v_2v_4\\
&+\left(-24i\sqrt{3}\,a_2+48i\sqrt{3}\,b_3-24a_2+48b_3\right)v_1^3v_2^2v_4\\
&+\left(32i\sqrt{3}\,b_1+32b_1\right)v_1^3v_2v_4+\left(-12i\sqrt{3}\,b_2+12b_2\right)v_1^2v_2^4v_5\\
&+\left(-8i\sqrt{3}\,a_3-8a_3\right)v_1^2v_2^3v_4+\left(-8i\sqrt{3}\,a_1-8a_1\right)v_1^2v_2^2v_4\\
&+\left(8i\sqrt{3}\,a_3+8a_3\right)v_1^2v_4v_5+\left(-2i\sqrt{3}\,b_2-2b_2\right)v_1v_2^7v_4\\
&+16v_1v_5v_3v_2^3a_3-4v_5v_3v_2^7b_2-32v_1^3v_3v_2^4b_2-16v_1v_3v_2^6a_3\\
&+\left(-2i\sqrt{3}\,b_1-2b_1\right)v_2^7v_4+\left(4i\sqrt{3}\,a_1-4a_1\right)v_2^5v_5\\
&+\left(8i\sqrt{3}\,a_2-16i\sqrt{3}\,b_3-8a_2+16b_3\right)v_1v_2^5v_5\\
&+\left(-12i\sqrt{3}\,b_1+12b_1\right)v_1v_2^4v_5\\
&+\left(-2i\sqrt{3}\,a_2+4i\sqrt{3}\,b_3-2a_2+4b_3\right)v_2^5v_4v_5\\
&+\left(2i\sqrt{3}\,b_1+2b_1\right)v_2^4v_4v_5+128v_1^4v_3a_3\\
&+4v_3v_2^{10}b_2+\left(2i\sqrt{3}\,b_2+2b_2\right)v_1v_2^4v_4v_5=0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-16a_3 &= 0 \\
16a_3 &= 0 \\
128a_3 &= 0 \\
-32b_2 &= 0 \\
-4b_2 &= 0 \\
4b_2 &= 0 \\
-64i\sqrt{3}\, b_1 + 64b_1 &= 0 \\
-64i\sqrt{3}\, b_2 + 64b_2 &= 0 \\
-16i\sqrt{3}\, a_3 + 16a_3 &= 0 \\
-12i\sqrt{3}\, b_1 + 12b_1 &= 0 \\
-12i\sqrt{3}\, b_2 + 12b_2 &= 0 \\
-8i\sqrt{3}\, a_1 - 8a_1 &= 0 \\
-8i\sqrt{3}\, a_3 - 8a_3 &= 0 \\
-4i\sqrt{3}\, a_1 + 4a_1 &= 0 \\
-2i\sqrt{3}\, b_1 - 2b_1 &= 0 \\
-2i\sqrt{3}\, b_2 - 2b_2 &= 0 \\
2i\sqrt{3}\, b_1 + 2b_1 &= 0 \\
2i\sqrt{3}\, b_2 + 2b_2 &= 0 \\
4i\sqrt{3}\, a_1 - 4a_1 &= 0 \\
8i\sqrt{3}\, a_3 + 8a_3 &= 0 \\
12i\sqrt{3}\, b_1 - 12b_1 &= 0 \\
12i\sqrt{3}\, b_2 - 12b_2 &= 0 \\
16i\sqrt{3}\, a_1 - 16a_1 &= 0 \\
16i\sqrt{3}\, a_3 - 16a_3 &= 0 \\
32i\sqrt{3}\, b_1 + 32b_1 &= 0 \\
32i\sqrt{3}\, b_2 + 32b_2 &= 0 \\
-24i\sqrt{3}\, a_2 + 48i\sqrt{3}\, b_3 - 24a_2 + 48b_3 &= 0 \\
-8i\sqrt{3}\, a_2 + 16i\sqrt{3}\, b_3 + 8a_2 - 16b_3 &= 0 \\
-2i\sqrt{3}\, a_2 + 4i\sqrt{3}\, b_3 - 2a_2 + 4b_3 &= 0 \\
2i\sqrt{3}\, a_2 - 4i\sqrt{3}\, b_3 + 2a_2 - 4b_3 &= 0 \\
8i\sqrt{3}\, a_2 - 16i\sqrt{3}\, b_3 - 8a_2 + 16b_3 &= 0 \\
48i\sqrt{3}\, a_2 - 96i\sqrt{3}\, b_3 - 48a_2 + 96b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 2b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 2x \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 3 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
   -> Solving 1st order ODE of high degree, Lie methods, 1st trial
   `, `-> Computing symmetries using: way = 2
   `, `-> Computing symmetries using: way = 2
   -> Solving 1st order ODE of high degree, 2nd attempt. Trying parametric methods
   trying dAlembert
   -> Calling odsolve with the ODE`, diff(y(x), x) = (-y(x)^4*x^3+y(x))/(2*y(x)^3*x^4-2*x),
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   <- 1st order, parametric methods successful`
```

✓ Solution by Maple

Time used: 0.266 (sec). Leaf size: 167

```
dsolve(y(x)^4*diff(y(x),x)^3-6*x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{x\left(-i\sqrt{3}-1\right)}$$
$$y(x) = \sqrt{\left(i\sqrt{3}-1\right)x}$$
$$y(x) = -\sqrt{-\left(1+i\sqrt{3}\right)x}$$
$$y(x) = -\sqrt{\left(i\sqrt{3}-1\right)x}$$
$$y(x) = \sqrt{x}\,\sqrt{2}$$
$$y(x) = -\sqrt{x}\,\sqrt{2}$$
$$y(x) = 0$$
$$y(x) = \frac{2^{\frac{2}{3}}(-c_1^3+6c_1x)^{\frac{1}{3}}}{2}$$
$$y(x) = -\frac{2^{\frac{2}{3}}(-c_1^3+6c_1x)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)}{4}$$
$$y(x) = \frac{2^{\frac{2}{3}}(-c_1^3+6c_1x)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)}{4}$$

✓ Solution by Mathematica

Time used: 70.054 (sec). Leaf size: 22649

```
DSolve[y[x]^4*(y'[x])^3-6*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

3 CHAPTER 16. Nonlinear equations. Section 99. Clairaut's equation. EXERCISES Page 320

3.1 problem 3

Internal problem ID [6795]
Internal file name [`OUTPUT/6042_Tuesday_July_26_2022_11_23_37_PM_9550685/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 99. Clairaut's equation. EXERCISES Page 320
Problem number: 3.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries]]`

$$\boxed{y'^2 + x^3 y' - 2x^2 y = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \left(-\frac{x^2}{2} + \frac{\sqrt{x^4 + 8y}}{2}\right) x \tag{1}$$

$$y' = \left(-\frac{x^2}{2} - \frac{\sqrt{x^4 + 8y}}{2}\right) x \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = \frac{\left(-x^2 + \sqrt{x^4 + 8y}\right) x}{2}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{\left(-x^2 + \sqrt{x^4 + 8y}\right) x(b_3 - a_2)}{2} - \frac{\left(-x^2 + \sqrt{x^4 + 8y}\right)^2 x^2 a_3}{4} - \left(\frac{\left(-2x + \frac{2x^3}{\sqrt{x^4+8y}}\right) x}{2} - \frac{x^2}{2} + \frac{\sqrt{x^4 + 8y}}{2}\right)(xa_2 + ya_3 + a_1) - \frac{2x(xb_2 + yb_3 + b_1)}{\sqrt{x^4 + 8y}} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-2x^8 a_3 + \sqrt{x^4 + 8y}\, x^6 a_3 + (x^4 + 8y)^{\frac{3}{2}} x^2 a_3 + 8x^5 a_2 - 2x^5 b_3 - 10x^4 y a_3 - 8\sqrt{x^4 + 8y}\, x^3 a_2 + 2\sqrt{x^4 + 8y}}{} = 0$$

Setting the numerator to zero gives

$$2x^8 a_3 - \sqrt{x^4 + 8y}\, x^6 a_3 - \left(x^4 + 8y\right)^{\frac{3}{2}} x^2 a_3 - 8x^5 a_2 + 2x^5 b_3 + 10x^4 y a_3 + 8\sqrt{x^4 + 8y}\, x^3 a_2 - 2\sqrt{x^4 + 8y}\, x^3 b_3 + 6\sqrt{x^4 + 8y}\, x^2 y a_3 - 6x^4 a_1 + 6\sqrt{x^4 + 8y}\, x^2 a_1 - 8x^2 b_2 - 32xya_2 + 8xyb_3 - 16y^2 a_3 + 4b_2\sqrt{x^4 + 8y} - 8xb_1 - 16ya_1 = 0 \tag{6E}$$

Simplifying the above gives

$$-\sqrt{x^4 + 8y}\, x^6 a_3 + 2\left(x^4 + 8y\right) x^4 a_3 - \left(x^4 + 8y\right)^{\frac{3}{2}} x^2 a_3 - 4x^5 a_2 - 4x^4 y a_3 + 8\sqrt{x^4 + 8y}\, x^3 a_2 - 2\sqrt{x^4 + 8y}\, x^3 b_3 + 6\sqrt{x^4 + 8y}\, x^2 y a_3 - 4x^4 a_1 - 4\left(x^4 + 8y\right) x a_2 + 2\left(x^4 + 8y\right) x b_3 - 2\left(x^4 + 8y\right) y a_3 + 6\sqrt{x^4 + 8y}\, x^2 a_1 - 2\left(x^4 + 8y\right) a_1 - 8x^2 b_2 - 8xyb_3 + 4b_2\sqrt{x^4 + 8y} - 8xb_1 = 0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&2x^8a_3 - 2\sqrt{x^4+8y}\,x^6a_3 - 8x^5a_2 + 2x^5b_3 + 10x^4ya_3 - 6x^4a_1 + 8\sqrt{x^4+8y}\,x^3a_2 \\
&- 2\sqrt{x^4+8y}\,x^3b_3 - 2\sqrt{x^4+8y}\,x^2ya_3 + 6\sqrt{x^4+8y}\,x^2a_1 - 8x^2b_2 \\
&- 32xya_2 + 8xyb_3 - 16y^2a_3 - 8xb_1 + 4b_2\sqrt{x^4+8y} - 16ya_1 = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{x^4+8y}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x = v_1, y = v_2, \sqrt{x^4+8y} = v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&2v_1^8a_3 - 2v_3v_1^6a_3 - 8v_1^5a_2 + 10v_1^4v_2a_3 + 2v_1^5b_3 - 6v_1^4a_1 \\
&+ 8v_3v_1^3a_2 - 2v_3v_1^2v_2a_3 - 2v_3v_1^3b_3 + 6v_3v_1^2a_1 - 32v_1v_2a_2 \\
&- 16v_2^2a_3 - 8v_1^2b_2 + 8v_1v_2b_3 - 16v_2a_1 - 8v_1b_1 + 4b_2v_3 = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&2v_1^8a_3 - 2v_3v_1^6a_3 + (-8a_2 + 2b_3)\,v_1^5 + 10v_1^4v_2a_3 - 6v_1^4a_1 \\
&+ (8a_2 - 2b_3)\,v_1^3v_3 - 2v_3v_1^2v_2a_3 + 6v_3v_1^2a_1 - 8v_1^2b_2 \\
&+ (-32a_2 + 8b_3)\,v_1v_2 - 8v_1b_1 - 16v_2^2a_3 - 16v_2a_1 + 4b_2v_3 = 0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-16a_1 &= 0 \\
-6a_1 &= 0 \\
6a_1 &= 0 \\
-16a_3 &= 0 \\
-2a_3 &= 0 \\
2a_3 &= 0 \\
10a_3 &= 0 \\
-8b_1 &= 0 \\
-8b_2 &= 0 \\
4b_2 &= 0 \\
-32a_2 + 8b_3 &= 0 \\
-8a_2 + 2b_3 &= 0 \\
8a_2 - 2b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 4a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= 4y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= 4y - \left(\frac{\left(-x^2 + \sqrt{x^4 + 8y}\right) x}{2}\right)(x) \\
&= \frac{x^4}{2} - \frac{\sqrt{x^4 + 8y}\,x^2}{2} + 4y \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^4}{2} - \frac{\sqrt{x^4+8y}\, x^2}{2} + 4y} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(y)}{4} - \frac{\ln\left(x^2 + \sqrt{x^4 + 8y}\right)}{4} + \frac{\ln\left(-x^2 + \sqrt{x^4 + 8y}\right)}{4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{\left(-x^2 + \sqrt{x^4 + 8y}\right) x}{2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{x}{\sqrt{x^4 + 8y}} \\ S_y &= \frac{2}{\left(-x^2 + \sqrt{x^4 + 8y}\right) \sqrt{x^4 + 8y}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y)}{4} - \frac{\ln\left(x^2 + \sqrt{x^4 + 8y}\right)}{4} + \frac{\ln\left(-x^2 + \sqrt{x^4 + 8y}\right)}{4} = c_1$$

Which simplifies to

$$\frac{\ln(y)}{4} - \frac{\ln\left(x^2 + \sqrt{x^4 + 8y}\right)}{4} + \frac{\ln\left(-x^2 + \sqrt{x^4 + 8y}\right)}{4} = c_1$$

Summary
The solution(s) found are the following

$$\frac{\ln(y)}{4} - \frac{\ln\left(x^2 + \sqrt{x^4 + 8y}\right)}{4} + \frac{\ln\left(-x^2 + \sqrt{x^4 + 8y}\right)}{4} = c_1 \tag{1}$$

Verification of solutions

$$\frac{\ln(y)}{4} - \frac{\ln\left(x^2 + \sqrt{x^4 + 8y}\right)}{4} + \frac{\ln\left(-x^2 + \sqrt{x^4 + 8y}\right)}{4} = c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = -\frac{x\left(x^2 + \sqrt{x^4 + 8y}\right)}{2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{x\left(x^2 + \sqrt{x^4 + 8y}\right)(b_3 - a_2)}{2} - \frac{x^2\left(x^2 + \sqrt{x^4 + 8y}\right)^2 a_3}{4} \\ & - \left(-\frac{x^2}{2} - \frac{\sqrt{x^4 + 8y}}{2} - \frac{x\left(2x + \frac{2x^3}{\sqrt{x^4+8y}}\right)}{2}\right)(xa_2 + ya_3 + a_1) \\ & + \frac{2x(xb_2 + yb_3 + b_1)}{\sqrt{x^4 + 8y}} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2x^8a_3 + \sqrt{x^4 + 8y}\, x^6a_3 + (x^4 + 8y)^{\frac{3}{2}}\, x^2a_3 - 8x^5a_2 + 2x^5b_3 + 10x^4ya_3 - 8\sqrt{x^4 + 8y}\, x^3a_2 + 2\sqrt{x^4 + 8y}\, x}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & -2x^8a_3 - \sqrt{x^4 + 8y}\, x^6a_3 - \left(x^4 + 8y\right)^{\frac{3}{2}} x^2a_3 + 8x^5a_2 \\ & - 2x^5b_3 - 10x^4ya_3 + 8\sqrt{x^4 + 8y}\, x^3a_2 - 2\sqrt{x^4 + 8y}\, x^3b_3 \\ & + 6\sqrt{x^4 + 8y}\, x^2ya_3 + 6x^4a_1 + 6\sqrt{x^4 + 8y}\, x^2a_1 + 8x^2b_2 \\ & + 32xa_2y - 8xyb_3 + 16y^2a_3 + 4b_2\sqrt{x^4 + 8y} + 8xb_1 + 16a_1y = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$
\begin{aligned}
&-\sqrt{x^4+8y}\,x^6a_3-2\left(x^4+8y\right)x^4a_3-\left(x^4+8y\right)^{\frac{3}{2}}x^2a_3+4x^5a_2+4x^4ya_3\\
&+8\sqrt{x^4+8y}\,x^3a_2-2\sqrt{x^4+8y}\,x^3b_3+6\sqrt{x^4+8y}\,x^2ya_3+4x^4a_1\\
&+4\left(x^4+8y\right)xa_2-2\left(x^4+8y\right)xb_3+2\left(x^4+8y\right)ya_3+6\sqrt{x^4+8y}\,x^2a_1\\
&+2\left(x^4+8y\right)a_1+8x^2b_2+8xyb_3+4b_2\sqrt{x^4+8y}+8xb_1=0
\end{aligned}
\tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&-2x^8a_3-2\sqrt{x^4+8y}\,x^6a_3+8x^5a_2-2x^5b_3-10x^4ya_3+6x^4a_1\\
&+8\sqrt{x^4+8y}\,x^3a_2-2\sqrt{x^4+8y}\,x^3b_3-2\sqrt{x^4+8y}\,x^2ya_3+6\sqrt{x^4+8y}\,x^2a_1\\
&+8x^2b_2+32xa_2y-8xyb_3+16y^2a_3+8xb_1+4b_2\sqrt{x^4+8y}+16a_1y=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$
\left\{x,y,\sqrt{x^4+8y}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$
\left\{x=v_1,y=v_2,\sqrt{x^4+8y}=v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-2v_1^8a_3-2v_3v_1^6a_3+8v_1^5a_2-10v_1^4v_2a_3-2v_1^5b_3+6v_1^4a_1\\
&+8v_3v_1^3a_2-2v_3v_1^2v_2a_3-2v_3v_1^3b_3+6v_3v_1^2a_1+32v_1a_2v_2\\
&+16v_2^2a_3+8v_1^2b_2-8v_1v_2b_3+16a_1v_2+8v_1b_1+4b_2v_3=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1,v_2,v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&-2v_1^8a_3-2v_3v_1^6a_3+(8a_2-2b_3)\,v_1^5-10v_1^4v_2a_3+6v_1^4a_1\\
&+(8a_2-2b_3)\,v_1^3v_3-2v_3v_1^2v_2a_3+6v_3v_1^2a_1+8v_1^2b_2\\
&+(32a_2-8b_3)\,v_1v_2+8v_1b_1+16v_2^2a_3+16a_1v_2+4b_2v_3=0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
6a_1 &= 0\\
16a_1 &= 0\\
-10a_3 &= 0\\
-2a_3 &= 0\\
16a_3 &= 0\\
8b_1 &= 0\\
4b_2 &= 0\\
8b_2 &= 0\\
8a_2 - 2b_3 &= 0\\
32a_2 - 8b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0\\
a_2 &= a_2\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= 4a_2
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x\\
\eta &= 4y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi\\
&= 4y - \left(-\frac{x\left(x^2+\sqrt{x^4+8y}\right)}{2}\right)(x)\\
&= \frac{x^4}{2} + \frac{\sqrt{x^4+8y}\,x^2}{2} + 4y\\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^4}{2} + \frac{\sqrt{x^4+8y}\,x^2}{2} + 4y} dy \end{aligned}$$

Which results in

$$S = \frac{\ln (y)}{4} + \frac{\ln \left(x^2 + \sqrt{x^4 + 8y}\right)}{4} - \frac{\ln \left(-x^2 + \sqrt{x^4 + 8y}\right)}{4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x\left(x^2 + \sqrt{x^4 + 8y}\right)}{2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x}{\sqrt{x^4 + 8y}} \\ S_y &= \frac{2}{\sqrt{x^4 + 8y}\,\left(x^2 + \sqrt{x^4 + 8y}\right)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y)}{4} + \frac{\ln\left(x^2+\sqrt{x^4+8y}\right)}{4} - \frac{\ln\left(-x^2+\sqrt{x^4+8y}\right)}{4} = c_1$$

Which simplifies to

$$\frac{\ln(y)}{4} + \frac{\ln\left(x^2+\sqrt{x^4+8y}\right)}{4} - \frac{\ln\left(-x^2+\sqrt{x^4+8y}\right)}{4} = c_1$$

<u>Summary</u>
The solution(s) found are the following

$$\frac{\ln(y)}{4} + \frac{\ln\left(x^2+\sqrt{x^4+8y}\right)}{4} - \frac{\ln\left(-x^2+\sqrt{x^4+8y}\right)}{4} = c_1 \tag{1}$$

<u>Verification of solutions</u>

$$\frac{\ln(y)}{4} + \frac{\ln\left(x^2+\sqrt{x^4+8y}\right)}{4} - \frac{\ln\left(-x^2+\sqrt{x^4+8y}\right)}{4} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
      -> Calling odsolve with the ODE`, diff(diff(y(x), x), x)-(diff(y(x), x))/x, y(x)`
         Methods for second order ODEs:
         --- Trying classification methods ---
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         <- LODE of Euler type successful
      <- 1st order ODE linearizable_by_differentiation successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
      <- 1st order ODE linearizable_by_differentiation successful`
```

✓ Solution by Maple

Time used: 0.375 (sec). Leaf size: 21

```
dsolve(diff(y(x),x)^2+x^3*diff(y(x),x)-2*x^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{x^4}{8}$$
$$y(x) = c_1(x^2 + 2c_1)$$

✓ Solution by Mathematica

Time used: 1.255 (sec). Leaf size: 209

```
DSolve[(y'[x])^2+x^3*y'[x]-2*x^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{\sqrt{x^6+8x^2y(x)}\log\left(\sqrt{x^4+8y(x)}+x^2\right)}{2x\sqrt{x^4+8y(x)}} + \frac{1}{4}\left(1-\frac{\sqrt{x^6+8x^2y(x)}}{x\sqrt{x^4+8y(x)}}\right)\log(y(x)) = c_1, y(x)\right]$$
$$\text{Solve}\left[\frac{1}{4}\left(\frac{\sqrt{x^6+8x^2y(x)}}{x\sqrt{x^4+8y(x)}}+1\right)\log(y(x)) - \frac{\sqrt{x^6+8x^2y(x)}\log\left(\sqrt{x^4+8y(x)}+x^2\right)}{2x\sqrt{x^4+8y(x)}} = c_1, y(x)\right]$$
$$y(x) \to -\frac{x^4}{8}$$

3.2 problem 4

Internal problem ID [6796]
Internal file name [`OUTPUT/6043_Tuesday_July_26_2022_11_23_40_PM_37504089/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 99. Clairaut's equation. EXERCISES Page 320
Problem number: 4.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries]]`

$$\boxed{y'^2 + 4x^5y' - 12yx^4 = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = 2\left(-x^3 + \sqrt{x^6 + 3y}\right)x^2 \tag{1}$$

$$y' = 2\left(-x^3 - \sqrt{x^6 + 3y}\right)x^2 \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = 2\left(-x^3 + \sqrt{x^6 + 3y}\right)x^2$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + 2\left(-x^3 + \sqrt{x^6 + 3y}\right) x^2 (b_3 - a_2) - 4\left(-x^3 + \sqrt{x^6 + 3y}\right)^2 x^4 a_3 \\ &- \left(2\left(-3x^2 + \frac{3x^5}{\sqrt{x^6 + 3y}}\right) x^2 + 4\left(-x^3 + \sqrt{x^6 + 3y}\right) x\right) (xa_2 + ya_3 + a_1) \\ &- \frac{3x^2 (xb_2 + yb_3 + b_1)}{\sqrt{x^6 + 3y}} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-8x^{13}a_3 + 4\sqrt{x^6 + 3y}\, x^{10} a_3 + 12x^8 a_2 - 2x^8 b_3 - 14x^7 y a_3 + 4(x^6 + 3y)^{\frac{3}{2}} x^4 a_3 + 10x^7 a_1 - 12\sqrt{x^6 + 3y}\, x^5}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &8x^{13}a_3 - 4\sqrt{x^6 + 3y}\, x^{10} a_3 - 12x^8 a_2 + 2x^8 b_3 + 14x^7 y a_3 \\ &- 4\left(x^6 + 3y\right)^{\frac{3}{2}} x^4 a_3 - 10x^7 a_1 + 12\sqrt{x^6 + 3y}\, x^5 a_2 - 2\sqrt{x^6 + 3y}\, x^5 b_3 \\ &+ 10\sqrt{x^6 + 3y}\, x^4 y a_3 + 10\sqrt{x^6 + 3y}\, x^4 a_1 - 3x^3 b_2 - 18x^2 y a_2 \\ &+ 3x^2 y b_3 - 12x\, y^2 a_3 - 3x^2 b_1 - 12xy a_1 + b_2\sqrt{x^6 + 3y} = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &-4\sqrt{x^6 + 3y}\, x^{10} a_3 + 8\left(x^6 + 3y\right) x^7 a_3 - 6x^8 a_2 - 6x^7 y a_3 - 4\left(x^6 + 3y\right)^{\frac{3}{2}} x^4 a_3 \\ &- 6x^7 a_1 + 12\sqrt{x^6 + 3y}\, x^5 a_2 - 2\sqrt{x^6 + 3y}\, x^5 b_3 + 10\sqrt{x^6 + 3y}\, x^4 y a_3 \\ &+ 10\sqrt{x^6 + 3y}\, x^4 a_1 - 6\left(x^6 + 3y\right) x^2 a_2 + 2\left(x^6 + 3y\right) x^2 b_3 - 4\left(x^6 + 3y\right) xy a_3 \\ &- 4\left(x^6 + 3y\right) x a_1 - 3x^3 b_2 - 3x^2 y b_3 - 3x^2 b_1 + b_2\sqrt{x^6 + 3y} = 0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&8x^{13}a_3 - 8\sqrt{x^6+3y}\,x^{10}a_3 - 12x^8a_2 + 2x^8b_3 + 14x^7ya_3 - 10x^7a_1 \\
&+ 12\sqrt{x^6+3y}\,x^5a_2 - 2\sqrt{x^6+3y}\,x^5b_3 - 2\sqrt{x^6+3y}\,x^4ya_3 + 10\sqrt{x^6+3y}\,x^4a_1 \\
&- 3x^3b_2 - 18x^2ya_2 + 3x^2yb_3 - 12x\,y^2a_3 - 3x^2b_1 - 12xya_1 + b_2\sqrt{x^6+3y} = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{x^6+3y}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x = v_1, y = v_2, \sqrt{x^6+3y} = v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&8v_1^{13}a_3 - 8v_3v_1^{10}a_3 - 12v_1^8a_2 + 14v_1^7v_2a_3 + 2v_1^8b_3 - 10v_1^7a_1 \\
&+ 12v_3v_1^5a_2 - 2v_3v_1^4v_2a_3 - 2v_3v_1^5b_3 + 10v_3v_1^4a_1 - 18v_1^2v_2a_2 \\
&- 12v_1v_2^2a_3 - 3v_1^3b_2 + 3v_1^2v_2b_3 - 12v_1v_2a_1 - 3v_1^2b_1 + b_2v_3 = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&8v_1^{13}a_3 - 8v_3v_1^{10}a_3 + (-12a_2 + 2b_3)\,v_1^8 + 14v_1^7v_2a_3 - 10v_1^7a_1 \\
&+ (12a_2 - 2b_3)\,v_1^5v_3 - 2v_3v_1^4v_2a_3 + 10v_3v_1^4a_1 - 3v_1^3b_2 \\
&+ (-18a_2 + 3b_3)\,v_1^2v_2 - 3v_1^2b_1 - 12v_1v_2^2a_3 - 12v_1v_2a_1 + b_2v_3 = 0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_2 &= 0 \\
-12a_1 &= 0 \\
-10a_1 &= 0 \\
10a_1 &= 0 \\
-12a_3 &= 0 \\
-8a_3 &= 0 \\
-2a_3 &= 0 \\
8a_3 &= 0 \\
14a_3 &= 0 \\
-3b_1 &= 0 \\
-3b_2 &= 0 \\
-18a_2 + 3b_3 &= 0 \\
-12a_2 + 2b_3 &= 0 \\
12a_2 - 2b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 6a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= 6y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= 6y - \left(2\left(-x^3 + \sqrt{x^6 + 3y}\right) x^2\right)(x) \\
&= 2x^6 - 2\sqrt{x^6 + 3y}\, x^3 + 6y \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{2x^6 - 2\sqrt{x^6 + 3y}\, x^3 + 6y} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(y)}{6} - \frac{\ln\left(x^3 + \sqrt{x^6 + 3y}\right)}{6} + \frac{\ln\left(-x^3 + \sqrt{x^6 + 3y}\right)}{6}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = 2\left(-x^3 + \sqrt{x^6 + 3y}\right) x^2$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{x^2}{\sqrt{x^6 + 3y}} \\ S_y &= \frac{1}{\sqrt{x^6 + 3y}\, \left(-2x^3 + 2\sqrt{x^6 + 3y}\right)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y)}{6} - \frac{\ln\left(x^3 + \sqrt{x^6 + 3y}\right)}{6} + \frac{\ln\left(-x^3 + \sqrt{x^6 + 3y}\right)}{6} = c_1$$

Which simplifies to

$$\frac{\ln(y)}{6} - \frac{\ln\left(x^3 + \sqrt{x^6 + 3y}\right)}{6} + \frac{\ln\left(-x^3 + \sqrt{x^6 + 3y}\right)}{6} = c_1$$

Summary

The solution(s) found are the following

$$\frac{\ln(y)}{6} - \frac{\ln\left(x^3 + \sqrt{x^6 + 3y}\right)}{6} + \frac{\ln\left(-x^3 + \sqrt{x^6 + 3y}\right)}{6} = c_1 \tag{1}$$

Verification of solutions

$$\frac{\ln(y)}{6} - \frac{\ln\left(x^3 + \sqrt{x^6 + 3y}\right)}{6} + \frac{\ln\left(-x^3 + \sqrt{x^6 + 3y}\right)}{6} = c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = -2x^2\left(x^3 + \sqrt{x^6 + 3y}\right)$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - 2x^2 \left(x^3 + \sqrt{x^6 + 3y}\right)(b_3 - a_2) - 4x^4 \left(x^3 + \sqrt{x^6 + 3y}\right)^2 a_3 \\ & - \left(-4x\left(x^3 + \sqrt{x^6 + 3y}\right) - 2x^2\left(3x^2 + \frac{3x^5}{\sqrt{x^6 + 3y}}\right)\right)(x a_2 + y a_3 + a_1) \\ & + \frac{3x^2 (x b_2 + y b_3 + b_1)}{\sqrt{x^6 + 3y}} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{8x^{13} a_3 + 4\sqrt{x^6 + 3y}\, x^{10} a_3 - 12x^8 a_2 + 2x^8 b_3 + 14x^7 y a_3 + 4(x^6 + 3y)^{\frac{3}{2}} x^4 a_3 - 10x^7 a_1 - 12\sqrt{x^6 + 3y}\, x^5 a_2}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & -8x^{13} a_3 - 4\sqrt{x^6 + 3y}\, x^{10} a_3 + 12x^8 a_2 - 2x^8 b_3 - 14x^7 y a_3 \\ & - 4\left(x^6 + 3y\right)^{\frac{3}{2}} x^4 a_3 + 10x^7 a_1 + 12\sqrt{x^6 + 3y}\, x^5 a_2 - 2\sqrt{x^6 + 3y}\, x^5 b_3 \\ & + 10\sqrt{x^6 + 3y}\, x^4 y a_3 + 10\sqrt{x^6 + 3y}\, x^4 a_1 + 3x^3 b_2 + 18x^2 a_2 y \\ & - 3x^2 y b_3 + 12x\, y^2 a_3 + 3x^2 b_1 + 12x a_1 y + b_2 \sqrt{x^6 + 3y} = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &-4\sqrt{x^6+3y}\,x^{10}a_3-8\left(x^6+3y\right)x^7a_3+6x^8a_2+6x^7ya_3-4\left(x^6+3y\right)^{\frac{3}{2}}x^4a_3\\ &+6x^7a_1+12\sqrt{x^6+3y}\,x^5a_2-2\sqrt{x^6+3y}\,x^5b_3+10\sqrt{x^6+3y}\,x^4ya_3\\ &+10\sqrt{x^6+3y}\,x^4a_1+6\left(x^6+3y\right)x^2a_2-2\left(x^6+3y\right)x^2b_3+4\left(x^6+3y\right)xya_3\\ &+4\left(x^6+3y\right)xa_1+3x^3b_2+3x^2yb_3+3x^2b_1+b_2\sqrt{x^6+3y}=0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned} &-8x^{13}a_3-8\sqrt{x^6+3y}\,x^{10}a_3+12x^8a_2-2x^8b_3-14x^7ya_3+10x^7a_1\\ &+12\sqrt{x^6+3y}\,x^5a_2-2\sqrt{x^6+3y}\,x^5b_3-2\sqrt{x^6+3y}\,x^4ya_3+10\sqrt{x^6+3y}\,x^4a_1\\ &+3x^3b_2+18x^2a_2y-3x^2yb_3+12x\,y^2a_3+3x^2b_1+12xa_1y+b_2\sqrt{x^6+3y}=0 \end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$\left\{x,y,\sqrt{x^6+3y}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$\left\{x=v_1,y=v_2,\sqrt{x^6+3y}=v_3\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-8v_1^{13}a_3-8v_3v_1^{10}a_3+12v_1^8a_2-14v_1^7v_2a_3-2v_1^8b_3+10v_1^7a_1\\ &+12v_3v_1^5a_2-2v_3v_1^4v_2a_3-2v_3v_1^5b_3+10v_3v_1^4a_1+18v_1^2a_2v_2\\ &+12v_1v_2^2a_3+3v_1^3b_2-3v_1^2v_2b_3+12v_1a_1v_2+3v_1^2b_1+b_2v_3=0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1,v_2,v_3\}$$

Equation (7E) now becomes

$$\begin{aligned} &-8v_1^{13}a_3-8v_3v_1^{10}a_3+(12a_2-2b_3)\,v_1^8-14v_1^7v_2a_3+10v_1^7a_1\\ &+(12a_2-2b_3)\,v_1^5v_3-2v_3v_1^4v_2a_3+10v_3v_1^4a_1+3v_1^3b_2\\ &+(18a_2-3b_3)\,v_1^2v_2+3v_1^2b_1+12v_1v_2^2a_3+12v_1a_1v_2+b_2v_3=0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_2 &= 0 \\
10a_1 &= 0 \\
12a_1 &= 0 \\
-14a_3 &= 0 \\
-8a_3 &= 0 \\
-2a_3 &= 0 \\
12a_3 &= 0 \\
3b_1 &= 0 \\
3b_2 &= 0 \\
12a_2 - 2b_3 &= 0 \\
18a_2 - 3b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 6a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= 6y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= 6y - \left(-2x^2\left(x^3 + \sqrt{x^6 + 3y}\right)\right)(x) \\
&= 2x^6 + 2\sqrt{x^6 + 3y}\,x^3 + 6y \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi}=\frac{dy}{\eta}=dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R=x$$

S is found from

$$\begin{aligned}S&=\int\frac{1}{\eta}dy\\&=\int\frac{1}{2x^6+2\sqrt{x^6+3y}\,x^3+6y}dy\end{aligned}$$

Which results in

$$S=\frac{\ln(y)}{6}+\frac{\ln\left(x^3+\sqrt{x^6+3y}\right)}{6}-\frac{\ln\left(-x^3+\sqrt{x^6+3y}\right)}{6}$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR}=\frac{S_x+\omega(x,y)S_y}{R_x+\omega(x,y)R_y} \tag{2}$$

Where in the above R_x,R_y,S_x,S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y)=-2x^2\left(x^3+\sqrt{x^6+3y}\right)$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x&=1\\R_y&=0\\S_x&=\frac{x^2}{\sqrt{x^6+3y}}\\S_y&=\frac{1}{\sqrt{x^6+3y}\,\left(2x^3+2\sqrt{x^6+3y}\right)}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR}=0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y)}{6} + \frac{\ln\left(x^3 + \sqrt{x^6 + 3y}\right)}{6} - \frac{\ln\left(-x^3 + \sqrt{x^6 + 3y}\right)}{6} = c_1$$

Which simplifies to

$$\frac{\ln(y)}{6} + \frac{\ln\left(x^3 + \sqrt{x^6 + 3y}\right)}{6} - \frac{\ln\left(-x^3 + \sqrt{x^6 + 3y}\right)}{6} = c_1$$

Summary

The solution(s) found are the following

$$\frac{\ln(y)}{6} + \frac{\ln\left(x^3 + \sqrt{x^6 + 3y}\right)}{6} - \frac{\ln\left(-x^3 + \sqrt{x^6 + 3y}\right)}{6} = c_1 \tag{1}$$

Verification of solutions

$$\frac{\ln(y)}{6} + \frac{\ln\left(x^3 + \sqrt{x^6 + 3y}\right)}{6} - \frac{\ln\left(-x^3 + \sqrt{x^6 + 3y}\right)}{6} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
      <- 1st order ODE linearizable_by_differentiation successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
      <- 1st order ODE linearizable_by_differentiation successful`
```

✓ Solution by Maple

Time used: 0.359 (sec). Leaf size: 23

```
dsolve(diff(y(x),x)^2+4*x^5*diff(y(x),x)-12*x^4*y(x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{x^6}{3}$$
$$y(x) = c_1 x^3 + \frac{3}{4} c_1^2$$

✓ Solution by Mathematica

Time used: 0.603 (sec). Leaf size: 217

```
DSolve[(y'[x])^2+4*x^5*y'[x]-12*x^4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{1}{6}\left(\log(y(x)) - \frac{x^2\sqrt{x^6+3y(x)}\log(y(x))}{\sqrt{x^4(x^6+3y(x))}}\right) + \frac{x^2\sqrt{x^6+3y(x)}\log\left(\sqrt{x^6+3y(x)}+x^3\right)}{3\sqrt{x^4(x^6+3y(x))}} = c_1, y(x)\right]$$
$$\text{Solve}\left[\frac{1}{6}\left(\frac{x^2\sqrt{x^6+3y(x)}\log(y(x))}{\sqrt{x^4(x^6+3y(x))}} + \log(y(x))\right) - \frac{x^2\sqrt{x^6+3y(x)}\log\left(\sqrt{x^6+3y(x)}+x^3\right)}{3\sqrt{x^4(x^6+3y(x))}} = c_1, y(x)\right]$$
$$y(x) \to -\frac{x^6}{3}$$

3.3 problem 5

Internal problem ID [6797]
Internal file name [`OUTPUT/6044_Tuesday_July_26_2022_11_23_43_PM_92405610/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 99. Clairaut's equation. EXERCISES Page 320
Problem number: 5.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries]]
```

$$\boxed{2xy'^3 - 6yy'^2 = -x^4}$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\left(-2x^6 + 2\sqrt{x^6 - 8y^3}\,x^3 + 8y^3\right)^{\frac{1}{3}}}{2x} + \frac{2y^2}{x\left(-2x^6 + 2\sqrt{x^6 - 8y^3}\,x^3 + 8y^3\right)^{\frac{1}{3}}} + \frac{y}{x} \tag{1}$$

$$y' = -\frac{\left(-2x^6 + 2\sqrt{x^6 - 8y^3}\,x^3 + 8y^3\right)^{\frac{1}{3}}}{4x} - \frac{y^2}{x\left(-2x^6 + 2\sqrt{x^6 - 8y^3}\,x^3 + 8y^3\right)^{\frac{1}{3}}} + \frac{y}{x} + \frac{i\sqrt{3}\left(\frac{\left(-2x^6+2\sqrt{x^6-8y\right.}}{2x}\right.}{} \tag{2}$$

$$y' = -\frac{\left(-2x^6 + 2\sqrt{x^6 - 8y^3}\,x^3 + 8y^3\right)^{\frac{1}{3}}}{4x} - \frac{y^2}{x\left(-2x^6 + 2\sqrt{x^6 - 8y^3}\,x^3 + 8y^3\right)^{\frac{1}{3}}} + \frac{y}{x} - \frac{i\sqrt{3}\left(\frac{\left(-2x^6+2\sqrt{x^6-8y\right.}}{2x}\right.}{} \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$
\begin{aligned}
y' &= \frac{\left(-2x^6+2\sqrt{x^6-8y^3}\,x^3+8y^3\right)^{\frac{2}{3}}+2y\left(-2x^6+2\sqrt{x^6-8y^3}\,x^3+8y^3\right)^{\frac{1}{3}}+4y^2}{2x\left(-2x^6+2\sqrt{x^6-8y^3}\,x^3+8y^3\right)^{\frac{1}{3}}}\\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = xa_2+ya_3+a_1 \tag{1E}
$$

$$
\eta = xb_2+yb_3+b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1,a_2,a_3,b_1,b_2,b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 \\
&+ \frac{\left(\left(-2x^6+2\sqrt{x^6-8y^3}\,x^3+8y^3\right)^{\frac{2}{3}}+2y\left(-2x^6+2\sqrt{x^6-8y^3}\,x^3+8y^3\right)^{\frac{1}{3}}+4y^2\right)(b_3-a_2)}{2x\left(-2x^6+2\sqrt{x^6-8y^3}\,x^3+8y^3\right)^{\frac{1}{3}}} \\
&- \frac{\left(\left(-2x^6+2\sqrt{x^6-8y^3}\,x^3+8y^3\right)^{\frac{2}{3}}+2y\left(-2x^6+2\sqrt{x^6-8y^3}\,x^3+8y^3\right)^{\frac{1}{3}}+4y^2\right)^2 a_3}{4x^2\left(-2x^6+2\sqrt{x^6-8y^3}\,x^3+8y^3\right)^{\frac{2}{3}}} \\
&- \left(\frac{\frac{-8x^5+\frac{4x^8}{\sqrt{x^6-8y^3}}+4\sqrt{x^6-8y^3}\,x^2}{\left(-2x^6+2\sqrt{x^6-8y^3}\,x^3+8y^3\right)^{\frac{1}{3}}}+\frac{2y\left(-12x^5+\frac{6x^8}{\sqrt{x^6-8y^3}}+6\sqrt{x^6-8y^3}\,x^2\right)}{3\left(-2x^6+2\sqrt{x^6-8y^3}\,x^3+8y^3\right)^{\frac{2}{3}}}}{2x\left(-2x^6+2\sqrt{x^6-8y^3}\,x^3+8y^3\right)^{\frac{1}{3}}}\right. \\
&- \frac{\left(-2x^6+2\sqrt{x^6-8y^3}\,x^3+8y^3\right)^{\frac{2}{3}}+2y\left(-2x^6+2\sqrt{x^6-8y^3}\,x^3+8y^3\right)^{\frac{1}{3}}+4y^2}{2x^2\left(-2x^6+2\sqrt{x^6-8y^3}\,x^3+8y^3\right)^{\frac{1}{3}}} \\
&- \frac{\left(\left(-2x^6+2\sqrt{x^6-8y^3}\,x^3+8y^3\right)^{\frac{2}{3}}+2y\left(-2x^6+2\sqrt{x^6-8y^3}\,x^3+8y^3\right)^{\frac{1}{3}}+4y^2\right)\left(-12x^5+\frac{6x^8}{\sqrt{x^6-8y^3}}+6\sqrt{\right.}{6x\left(-2x^6+2\sqrt{x^6-8y^3}\,x^3+8y^3\right)^{\frac{4}{3}}} \\
&+ ya_3 + a_1) \\
&- \left(\frac{\frac{-\frac{16x^3y^2}{\sqrt{x^6-8y^3}}+16y^2}{\left(-2x^6+2\sqrt{x^6-8y^3}\,x^3+8y^3\right)^{\frac{1}{3}}}+2\left(-2x^6+2\sqrt{x^6-8y^3}\,x^3+8y^3\right)^{\frac{1}{3}}+\frac{2y\left(-\frac{24x^3y^2}{\sqrt{x^6-8y^3}}+24y^2\right)}{3\left(-2x^6+2\sqrt{x^6-8y^3}\,x^3+8y^3\right)^{\frac{2}{3}}}+8y}{2x\left(-2x^6+2\sqrt{x^6-8y^3}\,x^3+8y^3\right)^{\frac{1}{3}}}\right. \\
&\left. - \frac{\left(\left(-2x^6+2\sqrt{x^6-8y^3}\,x^3+8y^3\right)^{\frac{2}{3}}+2y\left(-2x^6+2\sqrt{x^6-8y^3}\,x^3+8y^3\right)^{\frac{1}{3}}+4y^2\right)\left(-\frac{24x^3y^2}{\sqrt{x^6-8y^3}}+24y^2\right)}{6x\left(-2x^6+2\sqrt{x^6-8y^3}\,x^3+8y^3\right)^{\frac{4}{3}}}\right)(\\
&+ yb_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

$$\text{Expression too large to display} \tag{6E}$$

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(-2x^6+2\sqrt{x^6-8y^3}\,x^3+8y^3\right)^{\frac{1}{3}}, \left(-2x^6+2\sqrt{x^6-8y^3}\,x^3+8y^3\right)^{\frac{2}{3}}, \sqrt{x^6-8y^3}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \left(-2x^6+2\sqrt{x^6-8y^3}\,x^3+8y^3\right)^{\frac{1}{3}} = v_3, \left(-2x^6+2\sqrt{x^6-8y^3}\,x^3+8y^3\right)^{\frac{2}{3}} = v_4, \sqrt{x^6-8y^3} = v_5\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&8v_1^{15}a_3 - 8v_5v_1^{12}a_3 + 32v_1^{10}v_2^2a_2 - 80v_1^9v_2^3a_3 - 16v_1^9v_3v_2^2a_3 \\
&- 32v_1^{11}v_2b_2 - 16v_1^{10}v_2^2b_3 + 48v_1^9v_2^2a_1 + 8v_1^9v_3v_2a_1 - 8v_4v_1^{10}a_2 \\
&- 12v_4v_1^9v_2a_3 - 32v_1^{10}v_2b_1 - 8v_1^{10}v_3b_1 + 4v_1^{10}v_4b_3 - 4v_4v_1^9a_1 \\
&- 32v_5v_1^7v_2^2a_2 + 48v_5v_1^6v_2^3a_3 + 16v_1^6v_5v_3v_2^2a_3 + 32v_5v_1^8v_2b_2 \\
&+ 16v_5v_1^7v_2^2b_3 - 48v_5v_1^6v_2^2a_1 - 8v_1^6v_5v_3v_2a_1 + 8v_5v_4v_1^7a_2 - 128v_1^4v_2^5a_2 \\
&+ 12v_5v_4v_1^6v_2a_3 + 256v_1^3v_2^6a_3 + 128v_1^3v_3v_2^5a_3 + 32v_5v_1^7v_2b_1 + 8v_1^7v_5v_3b_1 \\
&+ 192v_1^5v_2^4b_2 - 4v_1^7v_5v_4b_3 + 64v_1^4v_2^5b_3 + 4v_5v_4v_1^6a_1 - 256v_1^3v_2^5a_1 \\
&- 64v_1^3v_3v_2^4a_1 + 32v_4v_1^4v_2^3a_2 + 64v_4v_1^3v_2^4a_3 + 192v_1^4v_2^4b_1 + 64v_1^4v_3v_2^3b_1 \\
&+ 16v_4v_1^5v_2^2b_2 - 16v_4v_1^4v_2^3b_3 - 128v_5v_2^6a_3 - 64v_5v_3v_2^5a_3 + 16v_4v_1^4v_2^2b_1 \\
&- 64v_5v_1^2v_2^4b_2 + 64v_5v_2^5a_1 + 32v_5v_3v_2^4a_1 - 32v_5v_4v_2^4a_3 - 64v_5v_1v_2^4b_1 \\
&- 32v_1v_5v_3v_2^3b_1 - 16v_5v_4v_1^2v_2^2b_2 + 16v_5v_4v_2^3a_1 - 16v_5v_4v_1v_2^2b_1 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$\begin{aligned}
&8v_1^{15}a_3 + 16v_5v_4v_2^3a_1 + 32v_5v_3v_2^4a_1 + 32v_5v_1^8v_2b_2 + 32v_5v_1^7v_2b_1 \\
&+ 64v_4v_1^3v_2^4a_3 + (-32a_2 + 16b_3)\, v_1^7v_2^2v_5 + (8a_2 - 4b_3)\, v_1^7v_4v_5 \\
&+ (32a_2 - 16b_3)\, v_1^4v_2^3v_4 + 12v_5v_4v_1^6v_2a_3 - 16v_5v_4v_1^2v_2^2b_2 \\
&- 16v_5v_4v_1v_2^2b_1 + 16v_1^6v_5v_3v_2^2a_3 - 8v_1^6v_5v_3v_2a_1 - 32v_1v_5v_3v_2^3b_1 \\
&- 16v_1^9v_3v_2^2a_3 + 8v_1^9v_3v_2a_1 + 128v_1^3v_3v_2^5a_3 + 64v_1^4v_3v_2^3b_1 - 64v_1^3v_3v_2^4a_1 \\
&+ 8v_1^7v_5v_3b_1 - 12v_4v_1^9v_2a_3 + 48v_5v_1^6v_2^3a_3 + 4v_5v_4v_1^6a_1 - 48v_5v_1^6v_2^2a_1 \\
&+ 16v_4v_1^5v_2^2b_2 + 16v_4v_1^4v_2^2b_1 - 32v_5v_4v_2^4a_3 - 64v_5v_1^2v_2^4b_2 - 64v_5v_1v_2^4b_1 \\
&+ (32a_2 - 16b_3)\, v_1^{10}v_2^2 + (-8a_2 + 4b_3)\, v_1^{10}v_4 + (-128a_2 + 64b_3)\, v_1^4v_2^5 \\
&- 8v_1^{10}v_3b_1 - 80v_1^9v_2^3a_3 + 48v_1^9v_2^2a_1 + 192v_1^5v_2^4b_2 + 192v_1^4v_2^4b_1 \\
&- 4v_4v_1^9a_1 + 256v_1^3v_2^6a_3 - 256v_1^3v_2^5a_1 - 8v_5v_1^{12}a_3 - 128v_5v_2^6a_3 \\
&+ 64v_5v_2^5a_1 - 32v_1^{11}v_2b_2 - 32v_1^{10}v_2b_1 - 64v_5v_3v_2^5a_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-256a_1 &= 0\\
-64a_1 &= 0\\
-48a_1 &= 0\\
-8a_1 &= 0\\
-4a_1 &= 0\\
4a_1 &= 0\\
8a_1 &= 0\\
16a_1 &= 0\\
32a_1 &= 0\\
48a_1 &= 0\\
64a_1 &= 0\\
-128a_3 &= 0\\
-80a_3 &= 0\\
-64a_3 &= 0\\
-32a_3 &= 0\\
-16a_3 &= 0\\
-12a_3 &= 0\\
-8a_3 &= 0\\
8a_3 &= 0\\
12a_3 &= 0\\
16a_3 &= 0\\
48a_3 &= 0\\
64a_3 &= 0\\
128a_3 &= 0\\
256a_3 &= 0\\
-64b_1 &= 0\\
-32b_1 &= 0\\
-16b_1 &= 0\\
-8b_1 &= 0\\
8b_1 &= 0\\
16b_1 &= 0\\
32b_1 &= 0\\
64b_1 &= 0\\
192b_1 &= 0\\
-64b_2 &= 0\\
-32b_2 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 2a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= 2y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{2y}{x} \\
&= \frac{2y}{x}
\end{aligned}
$$

This is easily solved to give

$$
y = c_1 x^2
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = \frac{y}{x^2}
$$

And S is found from

$$
\begin{aligned}
dS &= \frac{dx}{\xi} \\
&= \frac{dx}{x}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
S &= \int \frac{dx}{T} \\
&= \ln(x)
\end{aligned}
$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{\left(-2x^6 + 2\sqrt{x^6 - 8y^3}\,x^3 + 8y^3\right)^{\frac{2}{3}} + 2y\left(-2x^6 + 2\sqrt{x^6 - 8y^3}\,x^3 + 8y^3\right)^{\frac{1}{3}} + 4y^2}{2x\left(-2x^6 + 2\sqrt{x^6 - 8y^3}\,x^3 + 8y^3\right)^{\frac{1}{3}}}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= -\frac{2y}{x^3} \\
R_y &= \frac{1}{x^2} \\
S_x &= \frac{1}{x} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{2x^2\left(-2x^6 + 2\sqrt{x^6 - 8y^3}\,x^3 + 8y^3\right)^{\frac{1}{3}}}{\left(-2x^6 + 2\sqrt{x^6 - 8y^3}\,x^3 + 8y^3\right)^{\frac{2}{3}} - 2y\left(-2x^6 + 2\sqrt{x^6 - 8y^3}\,x^3 + 8y^3\right)^{\frac{1}{3}} + 4y^2} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{2\,2^{\frac{1}{3}}\left(4R^3 + \sqrt{-8R^3 + 1} - 1\right)^{\frac{1}{3}}}{2^{\frac{2}{3}}\left(4R^3 + \sqrt{-8R^3 + 1} - 1\right)^{\frac{2}{3}} - 2\,2^{\frac{1}{3}}\left(4R^3 + \sqrt{-8R^3 + 1} - 1\right)^{\frac{1}{3}} R + 4R^2}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int \frac{2\left(8R^3+2\sqrt{-8R^3+1}-2\right)^{\frac{1}{3}}}{4^{\frac{1}{3}}\left(\left(4R^3+\sqrt{-8R^3+1}-1\right)^2\right)^{\frac{1}{3}}-2R\left(8R^3+2\sqrt{-8R^3+1}-2\right)^{\frac{1}{3}}+4R^2}dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(x) = \int^{\frac{y}{x^2}} \frac{2\left(8\_a^3+2\sqrt{-8\_a^3+1}-2\right)^{\frac{1}{3}}}{4^{\frac{1}{3}}\left(\left(4\_a^3+\sqrt{-8\_a^3+1}-1\right)^2\right)^{\frac{1}{3}}-2\_a\left(8\_a^3+2\sqrt{-8\_a^3+1}-2\right)^{\frac{1}{3}}+4\_a^2}d\_a + c_1$$

Which simplifies to

$$\ln(x) = \int^{\frac{y}{x^2}} \frac{2\left(8\_a^3+2\sqrt{-8\_a^3+1}-2\right)^{\frac{1}{3}}}{4^{\frac{1}{3}}\left(\left(4\_a^3+\sqrt{-8\_a^3+1}-1\right)^2\right)^{\frac{1}{3}}-2\_a\left(8\_a^3+2\sqrt{-8\_a^3+1}-2\right)^{\frac{1}{3}}+4\_a^2}d\_a + c_1$$

Summary

The solution(s) found are the following

$$\ln(x) = \int^{\frac{y}{x^2}} \frac{2\left(8\_a^3+2\sqrt{-8\_a^3+1}-2\right)^{\frac{1}{3}}}{4^{\frac{1}{3}}\left(\left(4\_a^3+\sqrt{-8\_a^3+1}-1\right)^2\right)^{\frac{1}{3}}-2\_a\left(8\_a^3+2\sqrt{-8\_a^3+1}-2\right)^{\frac{1}{3}}+4\_a^2}d\_a + c_1 \tag{1}$$

Verification of solutions

$$\ln(x) = \int^{\frac{y}{x^2}} \frac{2\left(8\_a^3+2\sqrt{-8\_a^3+1}-2\right)^{\frac{1}{3}}}{4^{\frac{1}{3}}\left(\left(4\_a^3+\sqrt{-8\_a^3+1}-1\right)^2\right)^{\frac{1}{3}}-2\_a\left(8\_a^3+2\sqrt{-8\_a^3+1}-2\right)^{\frac{1}{3}}+4\_a^2}d\_a + c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = \frac{i\left(-2x^6+2\sqrt{x^6-8y^3}\,x^3+8y^3\right)^{\frac{2}{3}}\sqrt{3}-\left(-2x^6+2\sqrt{x^6-8y^3}\,x^3+8y^3\right)^{\frac{2}{3}}+4y\left(-2x^6+2\sqrt{x^6-8y^3}\,x}{4x\left(-2x^6+2\sqrt{x^6-8y^3}\,x^3+8y^3\right)^{\frac{1}{3}}}$$

$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1,a_2,a_3,b_1,b_2,b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

Expression too large to display (5E)

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Simplifying the above gives

Expression too large to display (6E)

Since the PDE has radicals, simplifying gives

Expression too large to display

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(-2x^{6}+2\sqrt{x^{6}-8y^{3}}\, x^{3}+8y^{3}\right)^{\frac{1}{3}}, \left(-2x^{6}+2\sqrt{x^{6}-8y^{3}}\, x^{3}+8y^{3}\right)^{\frac{2}{3}}, \sqrt{x^{6}-8y^{3}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_{1}, y = v_{2}, \left(-2x^{6}+2\sqrt{x^{6}-8y^{3}}\, x^{3}+8y^{3}\right)^{\frac{1}{3}} = v_{3}, \left(-2x^{6}+2\sqrt{x^{6}-8y^{3}}\, x^{3}+8y^{3}\right)^{\frac{2}{3}} = v_{4}, \sqrt{x^{6}-8y^{3}} = v_{5}\right\}$$

The above PDE (6E) now becomes

$$\text{Expression too large to display} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_{1}, v_{2}, v_{3}, v_{4}, v_{5}\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&\left(16i\sqrt{3}\,a_3+16a_3\right)v_1^{12}v_5+\left(64i\sqrt{3}\,b_2+64b_2\right)v_1^{11}v_2\\
&+\left(-64i\sqrt{3}\,a_2+32i\sqrt{3}\,b_3-64a_2+32b_3\right)v_1^{10}v_2^2\\
&+\left(64i\sqrt{3}\,b_1+64b_1\right)v_1^{10}v_2\\
&+\left(-16i\sqrt{3}\,a_2+8i\sqrt{3}\,b_3+16a_2-8b_3\right)v_1^{10}v_4\\
&+\left(160i\sqrt{3}\,a_3+160a_3\right)v_1^{9}v_2^3+\left(-96i\sqrt{3}\,a_1-96a_1\right)v_1^{9}v_2^2\\
&-32v_1^{10}v_3b_1+\left(-16i\sqrt{3}\,a_3-16a_3\right)v_1^{15}+32v_1^{7}v_3v_5b_1\\
&-256v_3v_5v_2^5a_3+128v_3v_5v_2^4a_1-64v_1^9v_3v_2^2a_3\\
&+32v_1^9v_3v_2a_1+512v_1^3v_3v_2^5a_3+256v_1^4v_3v_2^3b_1\\
&-256v_1^3v_3v_2^4a_1+64v_1^6v_3v_5v_2^2a_3-32v_1^6v_3v_5v_2a_1\\
&-128v_1v_3v_5v_2^3b_1+\left(24i\sqrt{3}\,a_3-24a_3\right)v_1^6v_2v_4v_5\\
&+\left(-32i\sqrt{3}\,b_2+32b_2\right)v_1^2v_2^2v_4v_5\\
&+\left(-32i\sqrt{3}\,b_1+32b_1\right)v_1v_2^2v_4v_5\\
&+\left(-64i\sqrt{3}\,b_2-64b_2\right)v_1^8v_2v_5\\
&+\left(64i\sqrt{3}\,a_2-32i\sqrt{3}\,b_3+64a_2-32b_3\right)v_1^7v_2^2v_5\\
&+\left(-64i\sqrt{3}\,b_1-64b_1\right)v_1^7v_2v_5\\
&+\left(16i\sqrt{3}\,a_2-8i\sqrt{3}\,b_3-16a_2+8b_3\right)v_1^7v_4v_5\\
&+\left(-96i\sqrt{3}\,a_3-96a_3\right)v_1^6v_2^3v_5 \qquad\qquad (8E)\\
&+\left(96i\sqrt{3}\,a_1+96a_1\right)v_1^6v_2^2v_5\\
&+\left(8i\sqrt{3}\,a_1-8a_1\right)v_1^6v_4v_5+\left(32i\sqrt{3}\,b_2-32b_2\right)v_1^5v_2^2v_4\\
&+\left(64i\sqrt{3}\,a_2-32i\sqrt{3}\,b_3-64a_2+32b_3\right)v_1^4v_2^3v_4\\
&+\left(32i\sqrt{3}\,b_1-32b_1\right)v_1^4v_2^2v_4\\
&+\left(128i\sqrt{3}\,a_3-128a_3\right)v_1^3v_2^4v_4\\
&+\left(128i\sqrt{3}\,b_2+128b_2\right)v_1^2v_2^4v_5\\
&+\left(128i\sqrt{3}\,b_1+128b_1\right)v_1v_2^4v_5\\
&+\left(-64i\sqrt{3}\,a_3+64a_3\right)v_2^4v_4v_5\\
&+\left(32i\sqrt{3}\,a_1-32a_1\right)v_2^3v_4v_5\\
&+\left(-24i\sqrt{3}\,a_3+24a_3\right)v_1^9v_2v_4\\
&+\left(-8i\sqrt{3}\,a_1+8a_1\right)v_1^9v_4+\left(-384i\sqrt{3}\,b_2-384b_2\right)v_1^5v_2^4\\
&+\left(256i\sqrt{3}\,a_2-128i\sqrt{3}\,b_3+256a_2-128b_3\right)v_1^4v_2^5
\end{aligned}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-256a_1 &= 0 \\
-32a_1 &= 0 \\
32a_1 &= 0 \\
128a_1 &= 0 \\
-256a_3 &= 0 \\
-64a_3 &= 0 \\
64a_3 &= 0 \\
512a_3 &= 0 \\
-128b_1 &= 0 \\
-32b_1 &= 0 \\
32b_1 &= 0 \\
256b_1 &= 0 \\
-512i\sqrt{3}\, a_3 - 512a_3 &= 0 \\
-384i\sqrt{3}\, b_1 - 384b_1 &= 0 \\
-384i\sqrt{3}\, b_2 - 384b_2 &= 0 \\
-128i\sqrt{3}\, a_1 - 128a_1 &= 0 \\
-96i\sqrt{3}\, a_1 - 96a_1 &= 0 \\
-96i\sqrt{3}\, a_3 - 96a_3 &= 0 \\
-64i\sqrt{3}\, a_3 + 64a_3 &= 0 \\
-64i\sqrt{3}\, b_1 - 64b_1 &= 0 \\
-64i\sqrt{3}\, b_2 - 64b_2 &= 0 \\
-32i\sqrt{3}\, b_1 + 32b_1 &= 0 \\
-32i\sqrt{3}\, b_2 + 32b_2 &= 0 \\
-24i\sqrt{3}\, a_3 + 24a_3 &= 0 \\
-16i\sqrt{3}\, a_3 - 16a_3 &= 0 \\
-8i\sqrt{3}\, a_1 + 8a_1 &= 0 \\
8i\sqrt{3}\, a_1 - 8a_1 &= 0 \\
16i\sqrt{3}\, a_3 + 16a_3 &= 0 \\
24i\sqrt{3}\, a_3 - 24a_3 &= 0 \\
32i\sqrt{3}\, a_1 - 32a_1 &= 0 \\
32i\sqrt{3}\, b_1 - 32b_1 &= 0 \\
32i\sqrt{3}\, b_2 - 32b_2 &= 0 \\
64i\sqrt{3}\, b_1 + 64b_1 &= 0 \\
64i\sqrt{3}\, b_2 + 64b_2 &= 0 \\
96i\sqrt{3}\, a_1 + 96a_1 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 2a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= 2y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Solving equation (3)

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{i\left(-2x^6 + 2\sqrt{x^6 - 8y^3}\, x^3 + 8y^3\right)^{\frac{2}{3}} \sqrt{3} + \left(-2x^6 + 2\sqrt{x^6 - 8y^3}\, x^3 + 8y^3\right)^{\frac{2}{3}} - 4y\left(-2x^6 + 2\sqrt{x^6 - 8y^3}\right.}{4x\left(-2x^6 + 2\sqrt{x^6 - 8y^3}\, x^3 + 8y^3\right)^{\frac{1}{3}}} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\text{Expression too large to display} \tag{5E}$$

Putting the above in normal form gives

$$\text{Expression too large to display}$$

Setting the numerator to zero gives

$$\text{Expression too large to display} \tag{6E}$$

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(-2x^6 + 2\sqrt{x^6 - 8y^3}\, x^3 + 8y^3\right)^{\frac{1}{3}}, \left(-2x^6 + 2\sqrt{x^6 - 8y^3}\, x^3 + 8y^3\right)^{\frac{2}{3}}, \sqrt{x^6 - 8y^3}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \left(-2x^6 + 2\sqrt{x^6 - 8y^3}\, x^3 + 8y^3\right)^{\frac{1}{3}} = v_3, \left(-2x^6 + 2\sqrt{x^6 - 8y^3}\, x^3 + 8y^3\right)^{\frac{2}{3}} = v_4, \sqrt{x^6 - 8y^3} = v_5\right\}$$

The above PDE (6E) now becomes

$$\text{Expression too large to display} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&\left(-24i\sqrt{3}\,a_3 - 24a_3\right) v_1^6 v_2 v_4 v_5 \\
&+ \left(32i\sqrt{3}\,b_2 + 32b_2\right) v_1^2 v_2^2 v_4 v_5 \\
&- 32v_1^{10} v_3 b_1 + \left(32i\sqrt{3}\,b_1 + 32b_1\right) v_1 v_2^2 v_4 v_5 \\
&- 64v_1^9 v_3 v_2^2 a_3 + \left(128i\sqrt{3}\,a_1 - 128a_1\right) v_2^5 v_5 \\
&+ \left(-16i\sqrt{3}\,a_3 + 16a_3\right) v_1^{12} v_5 \\
&+ \left(-64i\sqrt{3}\,b_2 + 64b_2\right) v_1^{11} v_2 \\
&+ \left(64i\sqrt{3}\,a_2 - 32i\sqrt{3}\,b_3 - 64a_2 + 32b_3\right) v_1^{10} v_2^2 \\
&+ \left(-64i\sqrt{3}\,b_1 + 64b_1\right) v_1^{10} v_2 \\
&+ \left(16i\sqrt{3}\,a_2 - 8i\sqrt{3}\,b_3 + 16a_2 - 8b_3\right) v_1^{10} v_4 \\
&+ \left(-160i\sqrt{3}\,a_3 + 160a_3\right) v_1^9 v_2^3 + \left(96i\sqrt{3}\,a_1 - 96a_1\right) v_1^9 v_2^2 \\
&+ \left(8i\sqrt{3}\,a_1 + 8a_1\right) v_1^9 v_4 + \left(384i\sqrt{3}\,b_2 - 384b_2\right) v_1^5 v_2^4 \\
&+ \left(-256i\sqrt{3}\,a_2 + 128i\sqrt{3}\,b_3 + 256a_2 - 128b_3\right) v_1^4 v_2^5 \\
&+ \left(384i\sqrt{3}\,b_1 - 384b_1\right) v_1^4 v_2^4 + \left(512i\sqrt{3}\,a_3 - 512a_3\right) v_1^3 v_2^6 \\
&+ \left(-512i\sqrt{3}\,a_1 + 512a_1\right) v_1^3 v_2^5 \\
&+ \left(-256i\sqrt{3}\,a_3 + 256a_3\right) v_2^6 v_5 + 32v_1^9 v_3 v_2 a_1 \\
&+ 512v_1^3 v_3 v_2^5 a_3 + 256v_1^4 v_3 v_2^3 b_1 - 256v_1^3 v_3 v_2^4 a_1 \\
&+ 128v_3 v_5 v_2^4 a_1 + 32v_1^7 v_3 v_5 b_1 - 256v_3 v_5 v_2^5 a_3 \\
&+ \left(16i\sqrt{3}\,a_3 - 16a_3\right) v_1^{15} + \left(24i\sqrt{3}\,a_3 + 24a_3\right) v_1^9 v_2 v_4 \\
&+ \left(64i\sqrt{3}\,b_2 - 64b_2\right) v_1^8 v_2 v_5 \\
&+ \left(-64i\sqrt{3}\,a_2 + 32i\sqrt{3}\,b_3 + 64a_2 - 32b_3\right) v_1^7 v_2^2 v_5 \\
&+ \left(64i\sqrt{3}\,b_1 - 64b_1\right) v_1^7 v_2 v_5 \\
&+ \left(-16i\sqrt{3}\,a_2 + 8i\sqrt{3}\,b_3 - 16a_2 + 8b_3\right) v_1^7 v_4 v_5 \\
&+ \left(96i\sqrt{3}\,a_3 - 96a_3\right) v_1^6 v_2^3 v_5 \\
&+ \left(-96i\sqrt{3}\,a_1 + 96a_1\right) v_1^6 v_2^2 v_5 \\
&+ \left(-8i\sqrt{3}\,a_1 - 8a_1\right) v_1^6 v_4 v_5 \\
&+ \left(-32i\sqrt{3}\,b_2 - 32b_2\right) v_1^5 v_2^2 v_4 \\
&+ \left(-64i\sqrt{3}\,a_2 + 32i\sqrt{3}\,b_3 - 64a_2 + 32b_3\right) v_1^4 v_2^3 v_4 \\
&+ \left(-32i\sqrt{3}\,b_1 - 32b_1\right) v_1^4 v_2^2 v_4
\end{aligned}
\qquad (8E)
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-256a_1 &= 0\\
-32a_1 &= 0\\
32a_1 &= 0\\
128a_1 &= 0\\
-256a_3 &= 0\\
-64a_3 &= 0\\
64a_3 &= 0\\
512a_3 &= 0\\
-128b_1 &= 0\\
-32b_1 &= 0\\
32b_1 &= 0\\
256b_1 &= 0\\
-512i\sqrt{3}\,a_1 + 512a_1 &= 0\\
-256i\sqrt{3}\,a_3 + 256a_3 &= 0\\
-160i\sqrt{3}\,a_3 + 160a_3 &= 0\\
-128i\sqrt{3}\,a_3 - 128a_3 &= 0\\
-128i\sqrt{3}\,b_1 + 128b_1 &= 0\\
-128i\sqrt{3}\,b_2 + 128b_2 &= 0\\
-96i\sqrt{3}\,a_1 + 96a_1 &= 0\\
-64i\sqrt{3}\,b_1 + 64b_1 &= 0\\
-64i\sqrt{3}\,b_2 + 64b_2 &= 0\\
-32i\sqrt{3}\,a_1 - 32a_1 &= 0\\
-32i\sqrt{3}\,b_1 - 32b_1 &= 0\\
-32i\sqrt{3}\,b_2 - 32b_2 &= 0\\
-24i\sqrt{3}\,a_3 - 24a_3 &= 0\\
-16i\sqrt{3}\,a_3 + 16a_3 &= 0\\
-8i\sqrt{3}\,a_1 - 8a_1 &= 0\\
8i\sqrt{3}\,a_1 + 8a_1 &= 0\\
16i\sqrt{3}\,a_3 - 16a_3 &= 0\\
24i\sqrt{3}\,a_3 + 24a_3 &= 0\\
32i\sqrt{3}\,b_1 + 32b_1 &= 0\\
32i\sqrt{3}\,b_2 + 32b_2 &= 0\\
64i\sqrt{3}\,a_3 + 64a_3 &= 0\\
64i\sqrt{3}\,b_1 - 64b_1 &= 0\\
64i\sqrt{3}\,b_2 - 64b_2 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 2a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= 2y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 3 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
   -> Solving 1st order ODE of high degree, Lie methods, 1st trial
   `, `-> Computing symmetries using: way = 2
   `, `-> Computing symmetries using: way = 2
   -> Solving 1st order ODE of high degree, 2nd attempt. Trying parametric methods
   -> Calling odsolve with the ODE`, diff(y(x), x) = ((2*x^3+y(x)^3)*y(x)/x-3*y(x)*x^2)/(2*y
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   <- 1st order, parametric methods successful`
```

✓ Solution by Maple

Time used: 0.328 (sec). Leaf size: 56

```
dsolve(2*x*diff(y(x),x)^3-6*y(x)*diff(y(x),x)^2+x^4=0,y(x), singsol=all)
```

$$y(x) = -\frac{(1+i\sqrt{3})\,x^2}{4}$$
$$y(x) = \frac{(i\sqrt{3}-1)\,x^2}{4}$$
$$y(x) = \frac{x^2}{2}$$
$$y(x) = \frac{1}{6c_1^2} + \frac{c_1 x^3}{3}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[2*x*(y'[x])^3-6*y[x]*(y'[x])^2+x^4==0,y[x],x,IncludeSingularSolutions -> True]
```

Timed out

3.4 problem 6

Internal problem ID [6798]
Internal file name [`OUTPUT/6045_Tuesday_July_26_2022_11_23_48_PM_44109262/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 99. Clairaut's equation. EXERCISES Page 320
Problem number: 6.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _Clairaut]`

$$\boxed{{y'}^{2}-xy'+y=0}$$

3.4.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$p^2 - xp + y = 0$$

Solving for y from the above results in

$$y = -p^2 + xp \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= -p^2 + xp \\ &= -p^2 + xp \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = -p^2$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned} p &= \frac{d}{dx}(xp + g) \\ p &= \left(p + x\frac{dp}{dx}\right) + \left(g' \frac{dp}{dx}\right) \\ p &= p + (x + g')\frac{dp}{dx} \\ 0 &= (x + g')\frac{dp}{dx} \end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned} \frac{dp}{dx} &= 0 \\ p &= c_1 \end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = -c_1^2 + c_1 x$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = -p^2$, then the above equation becomes

$$\begin{aligned} x + g'(p) &= x - 2p \\ &= 0 \end{aligned}$$

Solving the above for p results in

$$p_1 = \frac{x}{2}$$

Substituting the above back in (1) results in

$$y_1 = \frac{x^2}{4}$$

Summary

The solution(s) found are the following

$$y = -c_1^2 + c_1 x \tag{1}$$

$$y = \frac{x^2}{4} \tag{2}$$

Verification of solutions

$$y = -c_1^2 + c_1 x$$

Verified OK.

$$y = \frac{x^2}{4}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 19

```
dsolve(diff(y(x),x)^2-x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x^2}{4}$$
$$y(x) = c_1(x - c_1)$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 25

```
DSolve[(y'[x])^2-x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1(x - c_1)$$
$$y(x) \to \frac{x^2}{4}$$

3.5 problem 7

Internal problem ID [6799]
Internal file name [`OUTPUT/6046_Tuesday_July_26_2022_11_23_49_PM_95069466/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 99. Clairaut's equation. EXERCISES Page 320
Problem number: 7.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _Clairaut]`

$$\boxed{y - xy' - ky'^2 = 0}$$

3.5.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$-k\,p^2 - xp + y = 0$$

Solving for y from the above results in

$$y = k\,p^2 + xp \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= k\,p^2 + xp \\ &= k\,p^2 + xp \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = k\,p^2$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\,\frac{dp}{dx} \\
0 &= (x + g')\,\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1^2 k + c_1 x$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = k\,p^2$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= 2kp + x \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$p_1 = -\frac{x}{2k}$$

Substituting the above back in (1) results in

$$y_1 = -\frac{x^2}{4k}$$

Summary

The solution(s) found are the following

$$y = c_1^2 k + c_1 x \tag{1}$$

$$y = -\frac{x^2}{4k} \tag{2}$$

Verification of solutions

$$y = c_1^2 k + c_1 x$$

Verified OK.

$$y = -\frac{x^2}{4k}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 22

```
dsolve(y(x)=diff(y(x),x)*x+k*diff(y(x),x)^2,y(x), singsol=all)
```

$$y(x) = -\frac{x^2}{4k}$$
$$y(x) = c_1(c_1k + x)$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 28

```
DSolve[y[x]==y'[x]*x+k*(y'[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1(x + c_1k)$$
$$y(x) \to -\frac{x^2}{4k}$$

3.6 problem 8

Internal problem ID [6800]
Internal file name [`OUTPUT/6047_Tuesday_July_26_2022_11_23_50_PM_50530793/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 99. Clairaut's equation. EXERCISES Page 320
Problem number: 8.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`]]
```

$$\boxed{x^8 {y'}^2 + 3xy' + 9y = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{-\frac{3}{2} + \frac{3\sqrt{1-4yx^6}}{2}}{x^7} \tag{1}$$

$$y' = -\frac{3\left(1 + \sqrt{1 - 4yx^6}\right)}{2x^7} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = \frac{-\frac{3}{2} + \frac{3\sqrt{-4x^6y+1}}{2}}{x^7}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{3\left(-1+\sqrt{-4x^6y+1}\right)(b_3-a_2)}{2x^7} - \frac{9\left(-1+\sqrt{-4x^6y+1}\right)^2 a_3}{4x^{14}} - \left(-\frac{21\left(-1+\sqrt{-4x^6y+1}\right)}{2x^8} - \frac{18y}{x^2\sqrt{-4x^6y+1}}\right)(xa_2+ya_3+a_1) + \frac{3xb_2+3yb_3+3b_1}{\sqrt{-4x^6y+1}\,x} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-4b_2x^{14}\sqrt{-4x^6y+1} - 12x^{14}b_2 + 72x^{13}ya_2 + 12x^{13}yb_3 + 96x^{12}y^2a_3 - 12x^{13}b_1 + 96x^{12}ya_1 + 36\sqrt{-4x^6y}}{} = 0$$

Setting the numerator to zero gives

$$4b_2x^{14}\sqrt{-4x^6y+1} + 12x^{14}b_2 - 72x^{13}ya_2 - 12x^{13}yb_3 - 96x^{12}y^2a_3 + 12x^{13}b_1 - 96x^{12}ya_1 - 36\sqrt{-4x^6y+1}\,x^7a_2 - 6\sqrt{-4x^6y+1}\,x^7b_3 - 42\sqrt{-4x^6y+1}\,x^6ya_3 - 42\sqrt{-4x^6y+1}\,x^6a_1 + 36x^7a_2 + 6x^7b_3 - 30x^6ya_3 + 42x^6a_1 - 9\left(-4x^6y+1\right)^{\frac{3}{2}}a_3 - 9a_3\sqrt{-4x^6y+1} + 18a_3 = 0 \tag{6E}$$

Simplifying the above gives

$$4b_2x^{14}\sqrt{-4x^6y+1} + 12x^{14}b_2 + 72x^{13}ya_2 + 12x^{13}yb_3 + 72x^{12}y^2a_3 + 12x^{13}b_1 + 72x^{12}ya_1 + 36\left(-4x^6y+1\right)x^7a_2 + 6\left(-4x^6y+1\right)x^7b_3 + 42\left(-4x^6y+1\right)x^6ya_3 + 42\left(-4x^6y+1\right)x^6a_1 - 36\sqrt{-4x^6y+1}\,x^7a_2 - 6\sqrt{-4x^6y+1}\,x^7b_3 - 42\sqrt{-4x^6y+1}\,x^6ya_3 - 42\sqrt{-4x^6y+1}\,x^6a_1 - 9\left(-4x^6y+1\right)^{\frac{3}{2}}a_3 + 18\left(-4x^6y+1\right)a_3 - 9a_3\sqrt{-4x^6y+1} = 0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&4b_2x^{14}\sqrt{-4x^6y+1}+12x^{14}b_2-72x^{13}ya_2-12x^{13}yb_3-96x^{12}y^2a_3 \\
&+12x^{13}b_1-96x^{12}ya_1-36\sqrt{-4x^6y+1}\,x^7a_2-6\sqrt{-4x^6y+1}\,x^7b_3 \\
&-6\sqrt{-4x^6y+1}\,x^6ya_3+36x^7a_2+6x^7b_3-42\sqrt{-4x^6y+1}\,x^6a_1 \\
&-30x^6ya_3+42x^6a_1-18a_3\sqrt{-4x^6y+1}+18a_3=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$
\left\{x,y,\sqrt{-4x^6y+1}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$
\left\{x=v_1,y=v_2,\sqrt{-4x^6y+1}=v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&4b_2v_1^{14}v_3-72v_1^{13}v_2a_2-96v_1^{12}v_2^2a_3+12v_1^{14}b_2-12v_1^{13}v_2b_3-96v_1^{12}v_2a_1 \\
&+12v_1^{13}b_1-36v_3v_1^7a_2-6v_3v_1^6v_2a_3-6v_3v_1^7b_3-42v_3v_1^6a_1 \\
&+36v_1^7a_2-30v_1^6v_2a_3+6v_1^7b_3+42v_1^6a_1-18a_3v_3+18a_3=0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1,v_2,v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&4b_2v_1^{14}v_3+12v_1^{14}b_2+(-72a_2-12b_3)\,v_1^{13}v_2+12v_1^{13}b_1-96v_1^{12}v_2^2a_3 \\
&-96v_1^{12}v_2a_1+(-36a_2-6b_3)\,v_1^7v_3+(36a_2+6b_3)\,v_1^7-6v_3v_1^6v_2a_3 \\
&-30v_1^6v_2a_3-42v_3v_1^6a_1+42v_1^6a_1-18a_3v_3+18a_3=0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-96a_1 &= 0 \\
-42a_1 &= 0 \\
42a_1 &= 0 \\
-96a_3 &= 0 \\
-30a_3 &= 0 \\
-18a_3 &= 0 \\
-6a_3 &= 0 \\
18a_3 &= 0 \\
12b_1 &= 0 \\
4b_2 &= 0 \\
12b_2 &= 0 \\
-72a_2 - 12b_3 &= 0 \\
-36a_2 - 6b_3 &= 0 \\
36a_2 + 6b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= -6a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= -6y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= -6y - \left(\frac{-\frac{3}{2} + \frac{3\sqrt{-4x^6y+1}}{2}}{x^7}\right)(x) \\
&= \frac{-12x^6y - 3\sqrt{-4x^6y+1} + 3}{2x^6} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-12x^6y - 3\sqrt{-4x^6y+1}+3}{2x^6}} dy
\end{aligned}$$

Which results in

$$S = -\frac{\ln(y)}{6} + \frac{\operatorname{arctanh}\left(\sqrt{-4x^6y+1}\right)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-\frac{3}{2} + \frac{3\sqrt{-4x^6y+1}}{2}}{x^7}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{1}{x\sqrt{-4x^6y+1}} \\
S_y &= \frac{-1-\frac{1}{\sqrt{-4x^6y+1}}}{6y}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\frac{\ln(y)}{6} + \frac{\operatorname{arctanh}\left(\sqrt{1-4yx^6}\right)}{3} = c_1
$$

Which simplifies to

$$
-\frac{\ln(y)}{6} + \frac{\operatorname{arctanh}\left(\sqrt{1-4yx^6}\right)}{3} = c_1
$$

<u>Summary</u>

The solution(s) found are the following

$$
-\frac{\ln(y)}{6} + \frac{\operatorname{arctanh}\left(\sqrt{1-4yx^6}\right)}{3} = c_1 \tag{1}
$$

<u>Verification of solutions</u>

$$
-\frac{\ln(y)}{6} + \frac{\operatorname{arctanh}\left(\sqrt{1-4yx^6}\right)}{3} = c_1
$$

Verified OK.

Solving equation (2)

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{3\left(\sqrt{-4x^6y+1}+1\right)}{2x^7} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = xa_2 + ya_3 + a_1 \tag{1E}
$$

$$
\eta = xb_2 + yb_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 - \frac{3\left(\sqrt{-4x^6y+1}+1\right)(b_3-a_2)}{2x^7} - \frac{9\left(\sqrt{-4x^6y+1}+1\right)^2 a_3}{4x^{14}} \\
&- \left(\frac{18y}{x^2\sqrt{-4x^6y+1}} + \frac{\frac{21\sqrt{-4x^6y+1}}{2}+\frac{21}{2}}{x^8}\right)(xa_2+ya_3+a_1) \\
&- \frac{3(xb_2+yb_3+b_1)}{x\sqrt{-4x^6y+1}} = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{-4b_2x^{14}\sqrt{-4x^6y+1} + 12x^{14}b_2 - 72x^{13}ya_2 - 12x^{13}yb_3 - 96x^{12}y^2a_3 + 12x^{13}b_1 - 96x^{12}ya_1 + 36\sqrt{-4x^6y}}{}
$$

$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&4b_2x^{14}\sqrt{-4x^6y+1} - 12x^{14}b_2 + 72x^{13}ya_2 + 12x^{13}yb_3 + 96x^{12}y^2a_3 \\
&- 12x^{13}b_1 + 96x^{12}ya_1 - 36\sqrt{-4x^6y+1}\,x^7a_2 - 6\sqrt{-4x^6y+1}\,x^7b_3 \\
&- 42\sqrt{-4x^6y+1}\,x^6ya_3 - 42\sqrt{-4x^6y+1}\,x^6a_1 - 36x^7a_2 - 6x^7b_3 \\
&+ 30x^6ya_3 - 42x^6a_1 - 9\left(-4x^6y+1\right)^{\frac{3}{2}}a_3 - 9a_3\sqrt{-4x^6y+1} - 18a_3 = 0
\end{aligned} \tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
&4b_2x^{14}\sqrt{-4x^6y+1}-12x^{14}b_2-72x^{13}ya_2-12x^{13}yb_3-72x^{12}y^2a_3\\
&-12x^{13}b_1-72x^{12}ya_1-36\left(-4x^6y+1\right)x^7a_2-6\left(-4x^6y+1\right)x^7b_3\\
&-42\left(-4x^6y+1\right)x^6ya_3-42\left(-4x^6y+1\right)x^6a_1-36\sqrt{-4x^6y+1}\,x^7a_2\\
&-6\sqrt{-4x^6y+1}\,x^7b_3-42\sqrt{-4x^6y+1}\,x^6ya_3-42\sqrt{-4x^6y+1}\,x^6a_1\\
&-9\left(-4x^6y+1\right)^{\frac{3}{2}}a_3-18\left(-4x^6y+1\right)a_3-9a_3\sqrt{-4x^6y+1}=0
\end{aligned}
\tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&4b_2x^{14}\sqrt{-4x^6y+1}-12x^{14}b_2+72x^{13}ya_2+12x^{13}yb_3+96x^{12}y^2a_3\\
&-12x^{13}b_1+96x^{12}ya_1-36\sqrt{-4x^6y+1}\,x^7a_2-6\sqrt{-4x^6y+1}\,x^7b_3\\
&-6\sqrt{-4x^6y+1}\,x^6ya_3-36x^7a_2-6x^7b_3-42\sqrt{-4x^6y+1}\,x^6a_1\\
&+30x^6ya_3-42x^6a_1-18a_3\sqrt{-4x^6y+1}-18a_3=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{-4x^6y+1}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x=v_1, y=v_2, \sqrt{-4x^6y+1}=v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&4b_2v_1^{14}v_3+72v_1^{13}v_2a_2+96v_1^{12}v_2^2a_3-12v_1^{14}b_2+12v_1^{13}v_2b_3+96v_1^{12}v_2a_1\\
&-12v_1^{13}b_1-36v_3v_1^7a_2-6v_3v_1^6v_2a_3-6v_3v_1^7b_3-42v_3v_1^6a_1\\
&-36v_1^7a_2+30v_1^6v_2a_3-6v_1^7b_3-42v_1^6a_1-18a_3v_3-18a_3=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&4b_2v_1^{14}v_3-12v_1^{14}b_2+(72a_2+12b_3)\,v_1^{13}v_2-12v_1^{13}b_1+96v_1^{12}v_2^2a_3\\
&+96v_1^{12}v_2a_1+(-36a_2-6b_3)\,v_1^7v_3+(-36a_2-6b_3)\,v_1^7-6v_3v_1^6v_2a_3\\
&+30v_1^6v_2a_3-42v_3v_1^6a_1-42v_1^6a_1-18a_3v_3-18a_3=0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-42a_1 &= 0 \\
96a_1 &= 0 \\
-18a_3 &= 0 \\
-6a_3 &= 0 \\
30a_3 &= 0 \\
96a_3 &= 0 \\
-12b_1 &= 0 \\
-12b_2 &= 0 \\
4b_2 &= 0 \\
-36a_2 - 6b_3 &= 0 \\
72a_2 + 12b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= -6a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= -6y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= -6y - \left(-\frac{3\left(\sqrt{-4x^6y+1}+1\right)}{2x^7}\right)(x) \\
&= \frac{-12x^6y + 3\sqrt{-4x^6y+1} + 3}{2x^6} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-12x^6y+3\sqrt{-4x^6y+1}+3}{2x^6}} dy \end{aligned}$$

Which results in

$$S = -\frac{\ln(y)}{6} - \frac{\operatorname{arctanh}\left(\sqrt{-4x^6y+1}\right)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{3\left(\sqrt{-4x^6y+1}+1\right)}{2x^7}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{x\sqrt{-4x^6y+1}} \\ S_y &= \frac{-1 + \frac{1}{\sqrt{-4x^6y+1}}}{6y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(y)}{6} - \frac{\operatorname{arctanh}\left(\sqrt{1-4yx^6}\right)}{3} = c_1$$

Which simplifies to

$$-\frac{\ln(y)}{6} - \frac{\operatorname{arctanh}\left(\sqrt{1-4yx^6}\right)}{3} = c_1$$

Summary

The solution(s) found are the following

$$-\frac{\ln(y)}{6} - \frac{\operatorname{arctanh}\left(\sqrt{1-4yx^6}\right)}{3} = c_1 \tag{1}$$

Verification of solutions

$$-\frac{\ln(y)}{6} - \frac{\operatorname{arctanh}\left(\sqrt{1-4yx^6}\right)}{3} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      <- homogeneous successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      <- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 45

```
dsolve(x^8*diff(y(x),x)^2+3*x*diff(y(x),x)+9*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{1}{4x^6}$$
$$y(x) = \frac{-x^3 + c_1}{x^3 c_1^2}$$
$$y(x) = \frac{-x^3 - c_1}{x^3 c_1^2}$$

✓ Solution by Mathematica

Time used: 0.583 (sec). Leaf size: 130

```
DSolve[x^8*(y'[x])^2+3*x*y'[x]+9*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{x\sqrt{4x^6y(x)-1}\arctan\left(\sqrt{4x^6y(x)-1}\right)}{3\sqrt{x^2-4x^8y(x)}}-\frac{1}{6}\log(y(x))=c_1,y(x)\right]$$
$$\text{Solve}\left[\frac{\sqrt{x^2-4x^8y(x)}\arctan\left(\sqrt{4x^6y(x)-1}\right)}{3x\sqrt{4x^6y(x)-1}}-\frac{1}{6}\log(y(x))=c_1,y(x)\right]$$
$$y(x)\to 0$$

3.7 problem 9

Internal problem ID [6801]
Internal file name [`OUTPUT/6048_Tuesday_July_26_2022_11_23_52_PM_64795675/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 99. Clairaut's equation. EXERCISES Page 320
Problem number: 9.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"exact", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{x^4{y'}^2 + 2yy'x^3 = 4}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{-yx + \sqrt{y^2x^2 + 4}}{x^2} \tag{1}$$

$$y' = \frac{-yx - \sqrt{y^2x^2 + 4}}{x^2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$\begin{aligned} y' &= -\frac{xy - \sqrt{x^2y^2 + 4}}{x^2} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{\left(xy - \sqrt{x^2y^2+4}\right)(b_3 - a_2)}{x^2} - \frac{\left(xy - \sqrt{x^2y^2+4}\right)^2 a_3}{x^4} \\ &- \left(-\frac{y - \frac{x\,y^2}{\sqrt{x^2y^2+4}}}{x^2} + \frac{2xy - 2\sqrt{x^2y^2+4}}{x^3}\right)(xa_2 + ya_3 + a_1) \\ &+ \frac{\left(x - \frac{y\,x^2}{\sqrt{x^2y^2+4}}\right)(xb_2 + yb_3 + b_1)}{x^2} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^5yb_2 - 3x^3y^3a_3 - 2b_2x^4\sqrt{x^2y^2+4} + 2\sqrt{x^2y^2+4}\,x^2y^2a_3 + x^4yb_1 - x^3y^2a_1 - \sqrt{x^2y^2+4}\,x^3b_1 + \sqrt{x^2y^2 -}}{x^4\sqrt{x^2y^2+4}}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-x^5yb_2 + 3x^3y^3a_3 + 2b_2x^4\sqrt{x^2y^2+4} - 2\sqrt{x^2y^2+4}\,x^2y^2a_3 \\ &- x^4yb_1 + x^3y^2a_1 + \sqrt{x^2y^2+4}\,x^3b_1 - \sqrt{x^2y^2+4}\,x^2ya_1 \\ &- \left(x^2y^2+4\right)^{\frac{3}{2}} a_3 + 4x^2a_2 + 4x^2b_3 + 16xya_3 + 8xa_1 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &-x^5yb_2 - x^4y^2a_2 - x^4y^2b_3 - x^3y^3a_3 + 2b_2x^4\sqrt{x^2y^2+4} - 2\sqrt{x^2y^2+4}\,x^2y^2a_3 \\ &- x^4yb_1 - x^3y^2a_1 + \left(x^2y^2+4\right)x^2a_2 + \left(x^2y^2+4\right)x^2b_3 + 4\left(x^2y^2+4\right)xya_3 \\ &+ \sqrt{x^2y^2+4}\,x^3b_1 - \sqrt{x^2y^2+4}\,x^2ya_1 - \left(x^2y^2+4\right)^{\frac{3}{2}} a_3 + 2\left(x^2y^2+4\right)xa_1 = 0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&-x^5yb_2+3x^3y^3a_3+2b_2x^4\sqrt{x^2y^2+4}-x^4yb_1+x^3y^2a_1\\
&-3\sqrt{x^2y^2+4}\,x^2y^2a_3+\sqrt{x^2y^2+4}\,x^3b_1-\sqrt{x^2y^2+4}\,x^2ya_1\\
&+4x^2a_2+4x^2b_3+16xya_3+8xa_1-4\sqrt{x^2y^2+4}\,a_3=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$
\left\{x,y,\sqrt{x^2y^2+4}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$
\left\{x=v_1,y=v_2,\sqrt{x^2y^2+4}=v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&3v_1^3v_2^3a_3-v_1^5v_2b_2+v_1^3v_2^2a_1-3v_3v_1^2v_2^2a_3-v_1^4v_2b_1+2b_2v_1^4v_3-v_3v_1^2v_2a_1\\
&+v_3v_1^3b_1+4v_1^2a_2+16v_1v_2a_3+4v_1^2b_3+8v_1a_1-4v_3a_3=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1,v_2,v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&-v_1^5v_2b_2-v_1^4v_2b_1+2b_2v_1^4v_3+3v_1^3v_2^3a_3+v_1^3v_2^2a_1+v_3v_1^3b_1-3v_3v_1^2v_2^2a_3\\
&-v_3v_1^2v_2a_1+(4a_2+4b_3)\,v_1^2+16v_1v_2a_3+8v_1a_1-4v_3a_3=0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_1 &= 0 \\
b_1 &= 0 \\
-a_1 &= 0 \\
8a_1 &= 0 \\
-4a_3 &= 0 \\
-3a_3 &= 0 \\
3a_3 &= 0 \\
16a_3 &= 0 \\
-b_1 &= 0 \\
-b_2 &= 0 \\
2b_2 &= 0 \\
4a_2 + 4b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(-\frac{xy - \sqrt{x^2y^2 + 4}}{x^2}\right)(-x) \\
&= \frac{\sqrt{x^2y^2 + 4}}{x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{\sqrt{x^2y^2+4}}{x}} dy \end{aligned}$$

Which results in

$$S = \frac{x \ln\left(\frac{x^2 y}{\sqrt{x^2}} + \sqrt{x^2y^2+4}\right)}{\sqrt{x^2}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{xy - \sqrt{x^2y^2+4}}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y}{\sqrt{x^2y^2+4}} \\ S_y &= \frac{x}{\sqrt{x^2y^2+4}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln\left(yx + \sqrt{y^2x^2 + 4}\right) = \ln(x) + c_1$$

Which simplifies to

$$\ln\left(yx + \sqrt{y^2x^2 + 4}\right) = \ln(x) + c_1$$

Which gives

$$y = \frac{(e^{2c_1}x^2 - 4)\,e^{-c_1}}{2x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{(e^{2c_1}x^2 - 4)\,e^{-c_1}}{2x^2} \tag{1}$$

Verification of solutions

$$y = \frac{(e^{2c_1}x^2 - 4)\,e^{-c_1}}{2x^2}$$

Verified OK.

Solving equation (2)

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{xy+\sqrt{x^2y^2+4}}{x^2} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = xa_2+ya_3+a_1 \tag{1E}
$$

$$
\eta = xb_2+yb_3+b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1,a_2,a_3,b_1,b_2,b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2-\frac{\left(xy+\sqrt{x^2y^2+4}\right)(b_3-a_2)}{x^2}-\frac{\left(xy+\sqrt{x^2y^2+4}\right)^2 a_3}{x^4} \\
&-\left(-\frac{y+\frac{x\,y^2}{\sqrt{x^2y^2+4}}}{x^2}+\frac{2xy+2\sqrt{x^2y^2+4}}{x^3}\right)(xa_2+ya_3+a_1) \\
&+\frac{\left(x+\frac{y\,x^2}{\sqrt{x^2y^2+4}}\right)(xb_2+yb_3+b_1)}{x^2}=0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{-x^5yb_2+3x^3y^3a_3-2b_2x^4\sqrt{x^2y^2+4}+2\sqrt{x^2y^2+4}\,x^2y^2a_3-x^4yb_1+x^3y^2a_1-\sqrt{x^2y^2+4}\,x^3b_1+\sqrt{x^2y^2}}{x^4\sqrt{x^2y^2+4}}
$$

$$
=0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&x^5yb_2-3x^3y^3a_3+2b_2x^4\sqrt{x^2y^2+4}-2\sqrt{x^2y^2+4}\,x^2y^2a_3 \\
&+x^4yb_1-x^3y^2a_1+\sqrt{x^2y^2+4}\,x^3b_1-\sqrt{x^2y^2+4}\,x^2ya_1 \\
&-\left(x^2y^2+4\right)^{\frac{3}{2}}a_3-4x^2a_2-4x^2b_3-16xya_3-8xa_1=0
\end{aligned} \tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
&x^5 y b_2 + x^4 y^2 a_2 + x^4 y^2 b_3 + x^3 y^3 a_3 + 2 b_2 x^4 \sqrt{x^2 y^2 + 4} - 2\sqrt{x^2 y^2 + 4}\, x^2 y^2 a_3 \\
&+ x^4 y b_1 + x^3 y^2 a_1 - \left(x^2 y^2 + 4\right) x^2 a_2 - \left(x^2 y^2 + 4\right) x^2 b_3 - 4\left(x^2 y^2 + 4\right) x y a_3 \\
&+ \sqrt{x^2 y^2 + 4}\, x^3 b_1 - \sqrt{x^2 y^2 + 4}\, x^2 y a_1 - \left(x^2 y^2 + 4\right)^{\frac{3}{2}} a_3 - 2\left(x^2 y^2 + 4\right) x a_1 = 0
\end{aligned} \tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&x^5 y b_2 - 3 x^3 y^3 a_3 + 2 b_2 x^4 \sqrt{x^2 y^2 + 4} + x^4 y b_1 - x^3 y^2 a_1 \\
&- 3\sqrt{x^2 y^2 + 4}\, x^2 y^2 a_3 + \sqrt{x^2 y^2 + 4}\, x^3 b_1 - \sqrt{x^2 y^2 + 4}\, x^2 y a_1 \\
&- 4 x^2 a_2 - 4 x^2 b_3 - 16 x y a_3 - 8 x a_1 - 4\sqrt{x^2 y^2 + 4}\, a_3 = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{x^2 y^2 + 4}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x = v_1, y = v_2, \sqrt{x^2 y^2 + 4} = v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-3 v_1^3 v_2^3 a_3 + v_1^5 v_2 b_2 - v_1^3 v_2^2 a_1 - 3 v_3 v_1^2 v_2^2 a_3 + v_1^4 v_2 b_1 + 2 b_2 v_1^4 v_3 \\
&- v_3 v_1^2 v_2 a_1 + v_3 v_1^3 b_1 - 4 v_1^2 a_2 - 16 v_1 v_2 a_3 - 4 v_1^2 b_3 - 8 v_1 a_1 - 4 v_3 a_3 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&v_1^5 v_2 b_2 + v_1^4 v_2 b_1 + 2 b_2 v_1^4 v_3 - 3 v_1^3 v_2^3 a_3 - v_1^3 v_2^2 a_1 + v_3 v_1^3 b_1 - 3 v_3 v_1^2 v_2^2 a_3 \\
&- v_3 v_1^2 v_2 a_1 + \left(-4 a_2 - 4 b_3\right) v_1^2 - 16 v_1 v_2 a_3 - 8 v_1 a_1 - 4 v_3 a_3 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_1 &= 0 \\
b_2 &= 0 \\
-8a_1 &= 0 \\
-a_1 &= 0 \\
-16a_3 &= 0 \\
-4a_3 &= 0 \\
-3a_3 &= 0 \\
2b_2 &= 0 \\
-4a_2 - 4b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{xy + \sqrt{x^2y^2+4}}{x^2}\right)(-x) \\
&= -\frac{\sqrt{x^2y^2+4}}{x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\frac{\sqrt{x^2y^2+4}}{x}} dy \end{aligned}$$

Which results in

$$S = -\frac{x \ln\left(\frac{x^2y}{\sqrt{x^2}} + \sqrt{x^2y^2+4}\right)}{\sqrt{x^2}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{xy + \sqrt{x^2y^2+4}}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{\sqrt{x^2y^2+4}} \\ S_y &= -\frac{x}{\sqrt{x^2y^2+4}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln\left(yx + \sqrt{y^2x^2 + 4}\right) = \ln(x) + c_1$$

Which simplifies to

$$-\ln\left(yx + \sqrt{y^2x^2 + 4}\right) = \ln(x) + c_1$$

Which gives

$$y = -\frac{(4\,e^{2c_1}x^2 - 1)\,e^{-c_1}}{2x^2}$$

Summary

The solution(s) found are the following

$$y = -\frac{(4\,e^{2c_1}x^2 - 1)\,e^{-c_1}}{2x^2} \tag{1}$$

Verification of solutions

$$y = -\frac{(4\,e^{2c_1}x^2 - 1)\,e^{-c_1}}{2x^2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      <- homogeneous successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      <- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 49

```
dsolve(x^4*diff(y(x),x)^2+2*x^3*y(x)*diff(y(x),x)-4=0,y(x), singsol=all)
```

$$y(x) = -\frac{2i}{x}$$
$$y(x) = \frac{2i}{x}$$
$$y(x) = \frac{2\sinh(-\ln(x) + c_1)}{x}$$
$$y(x) = -\frac{2\sinh(-\ln(x) + c_1)}{x}$$

✓ Solution by Mathematica

Time used: 0.688 (sec). Leaf size: 71

```
DSolve[x^4*(y'[x])^2+2*x^3*y[x]*y'[x]-4==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{4e^{c_1}}{x^2} - \frac{e^{-c_1}}{4}$$
$$y(x) \to \frac{e^{-c_1}}{4} - \frac{4e^{c_1}}{x^2}$$
$$y(x) \to -\frac{2i}{x}$$
$$y(x) \to \frac{2i}{x}$$

3.8 problem 10

Internal problem ID [6802]
Internal file name [`OUTPUT/6049_Tuesday_July_26_2022_11_23_53_PM_99466071/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 99. Clairaut's equation. EXERCISES Page 320
Problem number: 10.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{xy'^2 - 2yy' = -4x}$$

3.8.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$x\,p^2 - 2yp = -4x$$

Solving for y from the above results in

$$y = \frac{x(p^2+4)}{2p} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = \frac{p^2 + 4}{2p}$$
$$g = 0$$

Hence (2) becomes

$$p - \frac{p^2 + 4}{2p} = x\left(1 - \frac{p^2 + 4}{2p^2}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{p^2 + 4}{2p} = 0$$

Solving for p from the above gives

$$p = 2$$
$$p = -2$$

Substituting these in (1A) gives

$$y = -2x$$
$$y = 2x$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - \frac{p(x)^2+4}{2p(x)}}{x\left(1 - \frac{p(x)^2+4}{2p(x)^2}\right)} \tag{3}$$

This ODE is now solved for $p(x)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = 0$$

Hence the ode is

$$p'(x) - \frac{p(x)}{x} = 0$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x}dx} = \frac{1}{x}$$

The ode becomes

$$\frac{d}{dx}\mu p = 0$$
$$\frac{d}{dx}\left(\frac{p}{x}\right) = 0$$

Integrating gives

$$\frac{p}{x} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$p(x) = c_1 x$$

Substituing the above solution for p in (2A) gives

$$y = \frac{c_1^2 x^2 + 4}{2c_1}$$

Summary

The solution(s) found are the following

$$y = -2x \tag{1}$$
$$y = 2x \tag{2}$$
$$y = \frac{c_1^2 x^2 + 4}{2c_1} \tag{3}$$

Verification of solutions

$$y = -2x$$

Verified OK.

$$y = 2x$$

Verified OK.

$$y = \frac{c_1^2 x^2 + 4}{2c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 30

```
dsolve(x*diff(y(x),x)^2-2*y(x)*diff(y(x),x)+4*x=0,y(x), singsol=all)
```

$$y(x) = -2x$$
$$y(x) = 2x$$
$$y(x) = \frac{4c_1^2 + x^2}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.202 (sec). Leaf size: 43

```
DSolve[x*(y'[x])^2-2*y[x]*y'[x]+4*x==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -2x\cosh(-\log(x) + c_1)$$
$$y(x) \to -2x\cosh(\log(x) + c_1)$$
$$y(x) \to -2x$$
$$y(x) \to 2x$$

3.9 problem 11

Internal problem ID [6803]
Internal file name [OUTPUT/6050_Tuesday_July_26_2022_11_23_55_PM_24219364/index.tex]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 99. Clairaut's equation. EXERCISES Page 320
Problem number: 11.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{3x^4y'^2 - xy' - y = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{1 + \sqrt{1 + 12x^2y}}{6x^3} \tag{1}$$

$$y' = -\frac{-1 + \sqrt{1 + 12x^2y}}{6x^3} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = \frac{1 + \sqrt{12y\,x^2 + 1}}{6x^3}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{\left(1+\sqrt{12y\,x^2+1}\right)(b_3-a_2)}{6x^3} - \frac{\left(1+\sqrt{12y\,x^2+1}\right)^2 a_3}{36x^6} - \left(-\frac{1+\sqrt{12y\,x^2+1}}{2x^4} + \frac{2y}{x^2\sqrt{12y\,x^2+1}}\right)(xa_2+ya_3+a_1) - \frac{xb_2+yb_3+b_1}{x\sqrt{12y\,x^2+1}} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-36b_2x^6\sqrt{12y\,x^2+1} + 36x^6b_2 - 72x^5ya_2 - 36x^5yb_3 - 144x^4y^2a_3 + 36x^5b_1 - 144x^4ya_1 - 12\sqrt{12y\,x^2+}}{}$$

$$= 0$$

Setting the numerator to zero gives

$$36b_2x^6\sqrt{12y\,x^2+1} - 36x^6b_2 + 72x^5ya_2 + 36x^5yb_3 + 144x^4y^2a_3 - 36x^5b_1 + 144x^4ya_1 + 12\sqrt{12y\,x^2+1}\,x^3a_2 + 6\sqrt{12y\,x^2+1}\,x^3b_3 + 18\sqrt{12y\,x^2+1}\,x^2ya_3 - \left(12y\,x^2+1\right)^{\frac{3}{2}}a_3 + 18\sqrt{12y\,x^2+1}\,x^2a_1 + 12x^3a_2 + 6x^3b_3 - 6x^2ya_3 + 18x^2a_1 - a_3\sqrt{12y\,x^2+1} - 2a_3 = 0 \tag{6E}$$

Simplifying the above gives

$$36b_2x^6\sqrt{12y\,x^2+1} - 36x^6b_2 - 72x^5ya_2 - 36x^5yb_3 - 72x^4y^2a_3 + 12\left(12y\,x^2+1\right)x^3a_2 + 6\left(12y\,x^2+1\right)x^3b_3 + 18\left(12y\,x^2+1\right)x^2ya_3 - 36x^5b_1 - 72x^4ya_1 + 18\left(12y\,x^2+1\right)x^2a_1 + 12\sqrt{12y\,x^2+1}\,x^3a_2 + 6\sqrt{12y\,x^2+1}\,x^3b_3 + 18\sqrt{12y\,x^2+1}\,x^2ya_3 - \left(12y\,x^2+1\right)^{\frac{3}{2}}a_3 + 18\sqrt{12y\,x^2+1}\,x^2a_1 - 2\left(12y\,x^2+1\right)a_3 - a_3\sqrt{12y\,x^2+1} = 0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&36 b_2 x^6 \sqrt{12 y\, x^2+1}-36 x^6 b_2+72 x^5 y a_2+36 x^5 y b_3+144 x^4 y^2 a_3-36 x^5 b_1+144 x^4 y a_1 \\
&+12 \sqrt{12 y\, x^2+1}\, x^3 a_2+6 \sqrt{12 y\, x^2+1}\, x^3 b_3+6 \sqrt{12 y\, x^2+1}\, x^2 y a_3+12 x^3 a_2 \\
&+6 x^3 b_3+18 \sqrt{12 y\, x^2+1}\, x^2 a_1-6 x^2 y a_3+18 x^2 a_1-2 a_3 \sqrt{12 y\, x^2+1}-2 a_3=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{12 y\, x^2+1}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x=v_1, y=v_2, \sqrt{12 y\, x^2+1}=v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&36 b_2 v_1^6 v_3+72 v_1^5 v_2 a_2+144 v_1^4 v_2^2 a_3-36 v_1^6 b_2+36 v_1^5 v_2 b_3+144 v_1^4 v_2 a_1 \\
&-36 v_1^5 b_1+12 v_3 v_1^3 a_2+6 v_3 v_1^2 v_2 a_3+6 v_3 v_1^3 b_3+18 v_3 v_1^2 a_1 \\
&+12 v_1^3 a_2-6 v_1^2 v_2 a_3+6 v_1^3 b_3+18 v_1^2 a_1-2 a_3 v_3-2 a_3=0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&36 b_2 v_1^6 v_3-36 v_1^6 b_2+(72 a_2+36 b_3) v_1^5 v_2-36 v_1^5 b_1+144 v_1^4 v_2^2 a_3 \\
&+144 v_1^4 v_2 a_1+(12 a_2+6 b_3) v_1^3 v_3+(12 a_2+6 b_3) v_1^3+6 v_3 v_1^2 v_2 a_3 \\
&-6 v_1^2 v_2 a_3+18 v_3 v_1^2 a_1+18 v_1^2 a_1-2 a_3 v_3-2 a_3=0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
18a_1 &= 0\\
144a_1 &= 0\\
-6a_3 &= 0\\
-2a_3 &= 0\\
6a_3 &= 0\\
144a_3 &= 0\\
-36b_1 &= 0\\
-36b_2 &= 0\\
36b_2 &= 0\\
12a_2 + 6b_3 &= 0\\
72a_2 + 36b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0\\
a_2 &= a_2\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= -2a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x\\
\eta &= -2y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi\\
&= -2y - \left(\frac{1+\sqrt{12y\,x^2+1}}{6x^3}\right)(x)\\
&= \frac{-12y\,x^2 - \sqrt{12y\,x^2+1} - 1}{6x^2}\\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-12y\,x^2 - \sqrt{12y\,x^2+1} - 1}{6x^2}} dy \end{aligned}$$

Which results in

$$S = -\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{12y\,x^2+1}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{1 + \sqrt{12y\,x^2 + 1}}{6x^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{x\sqrt{12y\,x^2+1}} \\ S_y &= \frac{-1 + \frac{1}{\sqrt{12y\,x^2+1}}}{2y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{1 + 12x^2 y}\right) = c_1$$

Which simplifies to

$$-\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{1 + 12x^2 y}\right) = c_1$$

Summary

The solution(s) found are the following

$$-\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{1 + 12x^2 y}\right) = c_1 \tag{1}$$

Verification of solutions

$$-\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{1 + 12x^2 y}\right) = c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = -\frac{-1 + \sqrt{12y\,x^2 + 1}}{6x^3}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{\left(-1+\sqrt{12 y\, x^2+1}\right)(b_3 - a_2)}{6x^3} - \frac{\left(-1+\sqrt{12 y\, x^2+1}\right)^2 a_3}{36 x^6} - \left(-\frac{2y}{x^2\sqrt{12 y\, x^2+1}} + \frac{-1+\sqrt{12 y\, x^2+1}}{2x^4}\right)(x a_2 + y a_3 + a_1) + \frac{x b_2 + y b_3 + b_1}{x\sqrt{12 y\, x^2+1}} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-36 b_2 x^6 \sqrt{12 y\, x^2+1} - 36 x^6 b_2 + 72 x^5 y a_2 + 36 x^5 y b_3 + 144 x^4 y^2 a_3 - 36 x^5 b_1 + 144 x^4 y a_1 - 12\sqrt{12 y\, x^2+}}{}$$

$$= 0$$

Setting the numerator to zero gives

$$36 b_2 x^6 \sqrt{12 y\, x^2+1} + 36 x^6 b_2 - 72 x^5 y a_2 - 36 x^5 y b_3 - 144 x^4 y^2 a_3 + 36 x^5 b_1 - 144 x^4 y a_1 + 12\sqrt{12 y\, x^2+1}\, x^3 a_2 + 6\sqrt{12 y\, x^2+1}\, x^3 b_3 + 18\sqrt{12 y\, x^2+1}\, x^2 y a_3 - \left(12 y\, x^2+1\right)^{\frac{3}{2}} a_3 + 18\sqrt{12 y\, x^2+1}\, x^2 a_1 - 12 x^3 a_2 - 6 x^3 b_3 + 6 x^2 y a_3 - 18 x^2 a_1 - a_3\sqrt{12 y\, x^2+1} + 2 a_3 = 0 \tag{6E}$$

Simplifying the above gives

$$
\begin{aligned}
&36b_2x^6\sqrt{12y\,x^2+1}+36x^6b_2+72x^5ya_2+36x^5yb_3+72x^4y^2a_3 \\
&-12\left(12y\,x^2+1\right)x^3a_2-6\left(12y\,x^2+1\right)x^3b_3-18\left(12y\,x^2+1\right)x^2ya_3 \\
&+36x^5b_1+72x^4ya_1-18\left(12y\,x^2+1\right)x^2a_1+12\sqrt{12y\,x^2+1}\,x^3a_2 \\
&+6\sqrt{12y\,x^2+1}\,x^3b_3+18\sqrt{12y\,x^2+1}\,x^2ya_3-\left(12y\,x^2+1\right)^{\frac{3}{2}}a_3 \\
&+18\sqrt{12y\,x^2+1}\,x^2a_1+2\left(12y\,x^2+1\right)a_3-a_3\sqrt{12y\,x^2+1}=0
\end{aligned}
\tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&36b_2x^6\sqrt{12y\,x^2+1}+36x^6b_2-72x^5ya_2-36x^5yb_3-144x^4y^2a_3+36x^5b_1-144x^4ya_1 \\
&+12\sqrt{12y\,x^2+1}\,x^3a_2+6\sqrt{12y\,x^2+1}\,x^3b_3+6\sqrt{12y\,x^2+1}\,x^2ya_3-12x^3a_2 \\
&-6x^3b_3+18\sqrt{12y\,x^2+1}\,x^2a_1+6x^2ya_3-18x^2a_1-2a_3\sqrt{12y\,x^2+1}+2a_3=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{12y\,x^2+1}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x=v_1, y=v_2, \sqrt{12y\,x^2+1}=v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&36b_2v_1^6v_3-72v_1^5v_2a_2-144v_1^4v_2^2a_3+36v_1^6b_2-36v_1^5v_2b_3-144v_1^4v_2a_1 \\
&+36v_1^5b_1+12v_3v_1^3a_2+6v_3v_1^2v_2a_3+6v_3v_1^3b_3+18v_3v_1^2a_1 \\
&-12v_1^3a_2+6v_1^2v_2a_3-6v_1^3b_3-18v_1^2a_1-2a_3v_3+2a_3=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&36b_2v_1^6v_3+36v_1^6b_2+(-72a_2-36b_3)\,v_1^5v_2+36v_1^5b_1-144v_1^4v_2^2a_3 \\
&-144v_1^4v_2a_1+(12a_2+6b_3)\,v_1^3v_3+(-12a_2-6b_3)\,v_1^3 \\
&+6v_3v_1^2v_2a_3+6v_1^2v_2a_3+18v_3v_1^2a_1-18v_1^2a_1-2a_3v_3+2a_3=0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-144a_1 &= 0\\
-18a_1 &= 0\\
18a_1 &= 0\\
-144a_3 &= 0\\
-2a_3 &= 0\\
2a_3 &= 0\\
6a_3 &= 0\\
36b_1 &= 0\\
36b_2 &= 0\\
-72a_2 - 36b_3 &= 0\\
-12a_2 - 6b_3 &= 0\\
12a_2 + 6b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0\\
a_2 &= a_2\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= -2a_2
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x\\
\eta &= -2y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi\\
&= -2y - \left(-\frac{-1 + \sqrt{12y\,x^2 + 1}}{6x^3}\right)(x)\\
&= \frac{-12y\,x^2 + \sqrt{12y\,x^2 + 1} - 1}{6x^2}\\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-12y\,x^2+\sqrt{12y\,x^2+1}-1}{6x^2}} dy \end{aligned}$$

Which results in

$$S = -\frac{\ln(y)}{2} + \operatorname{arctanh}\left(\sqrt{12y\,x^2+1}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{-1 + \sqrt{12y\,x^2+1}}{6x^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{1}{x\sqrt{12y\,x^2+1}} \\ S_y &= \frac{-\frac{1}{\sqrt{12y\,x^2+1}} - 1}{2y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(y)}{2} + \operatorname{arctanh}\left(\sqrt{1 + 12x^2y}\right) = c_1$$

Which simplifies to

$$-\frac{\ln(y)}{2} + \operatorname{arctanh}\left(\sqrt{1 + 12x^2y}\right) = c_1$$

Summary

The solution(s) found are the following

$$-\frac{\ln(y)}{2} + \operatorname{arctanh}\left(\sqrt{1 + 12x^2y}\right) = c_1 \tag{1}$$

Verification of solutions

$$-\frac{\ln(y)}{2} + \operatorname{arctanh}\left(\sqrt{1 + 12x^2y}\right) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      <- homogeneous successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      <- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 97

```
dsolve(3*x^4*diff(y(x),x)^2-x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{1}{12x^2}$$
$$y(x) = \frac{-i\sqrt{3}\,c_1 - 3x}{3c_1^2 x}$$
$$y(x) = \frac{i\sqrt{3}\,c_1 - 3x}{3x\,c_1^2}$$
$$y(x) = \frac{i\sqrt{3}\,c_1 - 3x}{3x\,c_1^2}$$
$$y(x) = \frac{-i\sqrt{3}\,c_1 - 3x}{3c_1^2 x}$$

✓ Solution by Mathematica

Time used: 0.512 (sec). Leaf size: 123

```
DSolve[3*x^4*(y'[x])^2-x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[-\frac{x\sqrt{12x^2y(x)+1}\text{arctanh}\left(\sqrt{12x^2y(x)+1}\right)}{\sqrt{12x^4y(x)+x^2}} - \frac{1}{2}\log(y(x)) = c_1, y(x)\right]$$
$$\text{Solve}\left[\frac{x\sqrt{12x^2y(x)+1}\text{arctanh}\left(\sqrt{12x^2y(x)+1}\right)}{\sqrt{12x^4y(x)+x^2}} - \frac{1}{2}\log(y(x)) = c_1, y(x)\right]$$
$$y(x) \to 0$$

3.10 problem 12

Internal problem ID [6804]
Internal file name [`OUTPUT/6051_Tuesday_July_26_2022_11_23_57_PM_14832249/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 99. Clairaut's equation. EXERCISES Page 320
Problem number: 12.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _rational, _dAlembert]`

$$\boxed{xy'^2 + (x - y)\, y' - y = -1}$$

3.10.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$x\, p^2 + (x - y)\, p - y = -1$$

Solving for y from the above results in

$$y = \frac{x\, p^2 + xp + 1}{p + 1} \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= xp + \frac{1}{p+1} \\ &= xp + \frac{1}{p+1} \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = \frac{1}{p+1}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1x + \frac{1}{c_1 + 1}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = \frac{1}{p+1}$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x - \frac{1}{(p+1)^2} \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$
\begin{aligned}
p_1 &= -\frac{\sqrt{x}-1}{\sqrt{x}} \\
p_2 &= -\frac{\sqrt{x}+1}{\sqrt{x}}
\end{aligned}
$$

Substituting the above back in (1) results in

$$
\begin{aligned}
y_1 &= 2\sqrt{x} - x \\
y_2 &= -x - 2\sqrt{x}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1 x + \frac{1}{c_1+1} \tag{1}$$

$$y = 2\sqrt{x} - x \tag{2}$$

$$y = -x - 2\sqrt{x} \tag{3}$$

Verification of solutions

$$y = c_1 x + \frac{1}{c_1+1}$$

Verified OK.

$$y = 2\sqrt{x} - x$$

Verified OK.

$$y = -x - 2\sqrt{x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple
Time used: 0.078 (sec). Leaf size: 44

```
dsolve(x*diff(y(x),x)^2+(x-y(x))*diff(y(x),x)+1-y(x)=0,y(x), singsol=all)
```

$$y(x) = -x - 2\sqrt{x}$$
$$y(x) = -x + 2\sqrt{x}$$
$$y(x) = \frac{c_1^2 x + c_1 x + 1}{c_1 + 1}$$

✓ Solution by Mathematica
Time used: 0.012 (sec). Leaf size: 46

```
DSolve[x*(y'[x])^2+(x-y[x])*y'[x]+1-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x + \frac{1}{1 + c_1}$$
$$y(x) \to -x - 2\sqrt{x}$$
$$y(x) \to 2\sqrt{x} - x$$

3.11 problem 13

Internal problem ID [6805]
Internal file name [OUTPUT/6052_Tuesday_July_26_2022_11_23_59_PM_12534222/index.tex]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 99. Clairaut's equation. EXERCISES Page 320
Problem number: 13.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _rational, _Clairaut]`

$$\boxed{y'(xy' - y + k) = -a}$$

3.11.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$p(xp + k - y) = -a$$

Solving for y from the above results in

$$y = \frac{p^2x + kp + a}{p} \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= xp + \frac{kp + a}{p} \\ &= xp + \frac{kp + a}{p} \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = \frac{kp + a}{p}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1 x + \frac{c_1 k + a}{c_1}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = \frac{kp+a}{p}$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x + \frac{k}{p} - \frac{kp + a}{p^2} \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$p_1 = \frac{\sqrt{xa}}{x}$$
$$p_2 = -\frac{\sqrt{xa}}{x}$$

Substituting the above back in (1) results in

$$y_1 = \frac{2xa + \sqrt{xa}\, k}{\sqrt{xa}}$$
$$y_2 = \frac{\sqrt{xa}\, k - 2xa}{\sqrt{xa}}$$

Summary

The solution(s) found are the following

$$y = c_1 x + \frac{c_1 k + a}{c_1} \tag{1}$$
$$y = \frac{2xa + \sqrt{xa}\, k}{\sqrt{xa}} \tag{2}$$
$$y = \frac{\sqrt{xa}\, k - 2xa}{\sqrt{xa}} \tag{3}$$

Verification of solutions

$$y = c_1 x + \frac{c_1 k + a}{c_1}$$

Verified OK.

$$y = \frac{2xa + \sqrt{xa}\, k}{\sqrt{xa}}$$

Verified OK.

$$y = \frac{\sqrt{xa}\, k - 2xa}{\sqrt{xa}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 42

```
dsolve(diff(y(x),x)*( x*diff(y(x),x)-y(x)+k )+a=0,y(x), singsol=all)
```

$$y(x) = k - 2\sqrt{ax}$$
$$y(x) = k + 2\sqrt{ax}$$
$$y(x) = \frac{c_1^2 x + c_1 k + a}{c_1}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 58

```
DSolve[y'[x]*( x*y'[x]-y[x]+k )+a==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{a}{c_1} + k + c_1 x$$
$$y(x) \to \text{Indeterminate}$$
$$y(x) \to k - 2\sqrt{a}\sqrt{x}$$
$$y(x) \to 2\sqrt{a}\sqrt{x} + k$$

3.12 problem 14

Internal problem ID [6806]
Internal file name [`OUTPUT/6053_Tuesday_July_26_2022_11_24_01_PM_41120136/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 99. Clairaut's equation. EXERCISES Page 320
Problem number: 14.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries]]
```

$$\boxed{x^6 {y'}^3 - 3xy' - 3y = 0}$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\frac{\left(\left(12yx+4\sqrt{\frac{9y^2x^3-4}{x}}\right)x^2\right)^{\frac{1}{3}}}{2x^2} + \frac{2}{x\left(\left(12yx+4\sqrt{\frac{9y^2x^3-4}{x}}\right)x^2\right)^{\frac{1}{3}}}}{x} \tag{1}$$

$$y' = \frac{-\frac{\left(\left(12yx+4\sqrt{\frac{9y^2x^3-4}{x}}\right)x^2\right)^{\frac{1}{3}}}{4x^2} - \frac{1}{x\left(\left(12yx+4\sqrt{\frac{9y^2x^3-4}{x}}\right)x^2\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(\left(12yx+4\sqrt{\frac{9y^2x^3-4}{x}}\right)x^2\right)^{\frac{1}{3}}}{2x^2} - \frac{2}{x\left(\left(12yx+4\sqrt{\frac{9y^2x^3-4}{x}}\right)x^2\right)}\right)}{2}}{x} \tag{2}$$

$$y' = \frac{-\frac{\left(\left(12yx+4\sqrt{\frac{9y^2x^3-4}{x}}\right)x^2\right)^{\frac{1}{3}}}{4x^2} - \frac{1}{x\left(\left(12yx+4\sqrt{\frac{9y^2x^3-4}{x}}\right)x^2\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(\left(12yx+4\sqrt{\frac{9y^2x^3-4}{x}}\right)x^2\right)^{\frac{1}{3}}}{2x^2} - \frac{2}{x\left(\left(12yx+4\sqrt{\frac{9y^2x^3-4}{x}}\right)x^2\right)}\right)}{2}}{x} \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = \frac{\left(\left(12xy+4\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{2}{3}} + 4x}{2x^3\left(\left(12xy+4\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{1}{3}}}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{multlined}
b_2+\frac{\left(\left(\left(12xy+4\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{2}{3}}+4x\right)(b_3-a_2)}{2x^3\left(\left(12xy+4\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{1}{3}}} \\
-\frac{\left(\left(\left(12xy+4\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{2}{3}}+4x\right)^2 a_3}{4x^6\left(\left(12xy+4\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{2}{3}}} \\
-\left(\frac{\frac{\frac{2\left(12y+\frac{54x\,y^2-\frac{2\left(9x^3y^2-4\right)}{x^2}}{\sqrt{\frac{9x^3y^2-4}{x}}}\right)x^2}{3}+\frac{4\left(12xy+4\sqrt{\frac{9x^3y^2-4}{x}}\right)x}{3}}{\left(\left(12xy+4\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{1}{3}}}+4}{2x^3\left(\left(12xy+4\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{1}{3}}}\right. \\
-\frac{3\left(\left(\left(12xy+4\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{2}{3}}+4x\right)}{2x^4\left(\left(12xy+4\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{1}{3}}} \\
\left.-\frac{\left(\left(\left(12xy+4\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{2}{3}}+4x\right)\left(\left(12y+\frac{54x\,y^2-\frac{2\left(9x^3y^2-4\right)}{x^2}}{\sqrt{\frac{9x^3y^2-4}{x}}}\right)x^2+2\left(12xy+4\sqrt{\frac{9x^3y^2-4}{x}}\right)x\right)}{6x^3\left(\left(12xy+4\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{4}{3}}}\right)(x \\
+ya_3+a_1)-\left(\frac{12x+\frac{36y\,x^2}{\sqrt{\frac{9x^3y^2-4}{x}}}}{3\left(\left(12xy+4\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{2}{3}}x}\right. \\
\left.-\frac{\left(\left(\left(12xy+4\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{2}{3}}+4x\right)\left(12x+\frac{36y\,x^2}{\sqrt{\frac{9x^3y^2-4}{x}}}\right)}{6x\left(\left(12xy+4\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{4}{3}}}\right)(xb_2+yb_3+b_1)=0
\end{multlined}
\tag{5E}
$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Simplifying the above gives

Expression too large to display (6E)

Since the PDE has radicals, simplifying gives

Expression too large to display

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(\left(3xy+\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{1}{3}}, \left(\left(3xy+\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{2}{3}}, \sqrt{\frac{9x^3y^2-4}{x}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x=v_1, y=v_2, \left(\left(3xy+\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{1}{3}}=v_3, \left(\left(3xy+\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{2}{3}}=v_4, \sqrt{\frac{9x^3y^2-4}{x}}=v\right.$$

The above PDE (6E) now becomes

$$\begin{aligned} -48v_1^2\Big(&4v_3 2^{\frac{2}{3}}v_1^5b_2-3v_5v_3 2^{\frac{2}{3}}v_1^7v_2b_2-3v_5v_4 2^{\frac{1}{3}}v_1^4v_2a_2-2v_5v_4 2^{\frac{1}{3}}v_1^4v_2b_3\\ &-6v_5v_4 2^{\frac{1}{3}}v_1^3v_2^2a_3-6v_5v_4 2^{\frac{1}{3}}v_1^3v_2a_1+6v_5v_3 2^{\frac{2}{3}}v_1^2v_2a_3-12v_5v_1^5v_2a_2\\ &-8v_5v_1^5v_2b_3-18v_5v_1^4v_2a_1+6v_4 2^{\frac{1}{3}}v_1^2a_2+4v_4 2^{\frac{1}{3}}v_1^2b_3+2v_5v_4 2^{\frac{1}{3}}a_3\\ &+10v_4 2^{\frac{1}{3}}v_1a_1-6v_1^7v_2b_2-36v_1^6v_2^2a_2-24v_1^6v_2^2b_3-6v_1^6v_2b_1-54v_1^5v_2^2a_1\\ &-4v_1^2v_2a_3-2v_5v_1^6b_2-2v_5v_1^5b_1-8v_3 2^{\frac{2}{3}}a_3-4v_5v_1a_3-6v_4 2^{\frac{1}{3}}v_1^5v_2^2b_3\\ &-18v_4 2^{\frac{1}{3}}v_1^4v_2^3a_3+v_5v_4 2^{\frac{1}{3}}v_1^5b_2+3v_4 2^{\frac{1}{3}}v_1^5v_2b_1-18v_4 2^{\frac{1}{3}}v_1^4v_2^2a_1\\ &+v_5v_4 2^{\frac{1}{3}}v_1^4b_1+12v_1^3a_2+8v_1^3b_3+20v_1^2a_1+18v_3 2^{\frac{2}{3}}v_1^3v_2^2a_3\\ &+10v_4 2^{\frac{1}{3}}v_1v_2a_3-9v_3 2^{\frac{2}{3}}v_1^8v_2^2b_2+3v_4 2^{\frac{1}{3}}v_1^6v_2b_2-9v_4 2^{\frac{1}{3}}v_1^5v_2^2a_2\Big)=0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$\begin{aligned}
&-480\,2^{\frac{1}{3}}a_1v_4v_1^3+384\,2^{\frac{2}{3}}a_3v_3v_1^2-960a_1v_1^4+864\,2^{\frac{1}{3}}a_3v_2^3v_4v_1^6\\
&+864\,2^{\frac{1}{3}}a_1v_2^2v_4v_1^6+\left(144\,2^{\frac{1}{3}}a_2+96\,2^{\frac{1}{3}}b_3\right)v_2v_4v_5v_1^6-48\,2^{\frac{1}{3}}b_1v_4v_5v_1^6\\
&-864\,2^{\frac{2}{3}}a_3v_2^2v_3v_1^5-480\,2^{\frac{1}{3}}a_3v_2v_4v_1^3-96\,2^{\frac{1}{3}}a_3v_4v_5v_1^2\\
&+432\,2^{\frac{2}{3}}b_2v_2^2v_3v_1^{10}-144\,2^{\frac{1}{3}}b_2v_2v_4v_1^8-144\,2^{\frac{1}{3}}b_1v_2v_4v_1^7\\
&-48\,2^{\frac{1}{3}}b_2v_4v_5v_1^7+288\,2^{\frac{1}{3}}a_1v_2v_4v_5v_1^5-288\,2^{\frac{2}{3}}a_3v_2v_3v_5v_1^4\\
&+\left(432\,2^{\frac{1}{3}}a_2+288\,2^{\frac{1}{3}}b_3\right)v_2^2v_4v_1^7+(576a_2+384b_3)\,v_2v_5v_1^7\\
&-192\,2^{\frac{2}{3}}b_2v_3v_1^7+864a_1v_2v_5v_1^6+144\,2^{\frac{2}{3}}b_2v_2v_3v_5v_1^9+288\,2^{\frac{1}{3}}a_3v_2^2v_4v_5v_1^5\\
&+(-576a_2-384b_3)\,v_1^5+192a_3v_2v_1^4+\left(-288\,2^{\frac{1}{3}}a_2-192\,2^{\frac{1}{3}}b_3\right)v_4v_1^4\\
&+192a_3v_5v_1^3+288b_2v_2v_1^9+(1728a_2+1152b_3)\,v_2^2v_1^8\\
&+288b_1v_2v_1^8+96b_2v_5v_1^8+2592a_1v_2^2v_1^7+96b_1v_5v_1^7=0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-960a_1 &= 0\\
864a_1 &= 0\\
2592a_1 &= 0\\
192a_3 &= 0\\
96b_1 &= 0\\
288b_1 &= 0\\
96b_2 &= 0\\
288b_2 &= 0\\
-480\,2^{\frac{1}{3}}a_1 &= 0\\
288\,2^{\frac{1}{3}}a_1 &= 0\\
864\,2^{\frac{1}{3}}a_1 &= 0\\
-480\,2^{\frac{1}{3}}a_3 &= 0\\
-96\,2^{\frac{1}{3}}a_3 &= 0\\
288\,2^{\frac{1}{3}}a_3 &= 0\\
864\,2^{\frac{1}{3}}a_3 &= 0\\
-144\,2^{\frac{1}{3}}b_1 &= 0\\
-48\,2^{\frac{1}{3}}b_1 &= 0\\
-144\,2^{\frac{1}{3}}b_2 &= 0\\
-48\,2^{\frac{1}{3}}b_2 &= 0\\
-864\,2^{\frac{2}{3}}a_3 &= 0\\
-288\,2^{\frac{2}{3}}a_3 &= 0\\
384\,2^{\frac{2}{3}}a_3 &= 0\\
-192\,2^{\frac{2}{3}}b_2 &= 0\\
144\,2^{\frac{2}{3}}b_2 &= 0\\
432\,2^{\frac{2}{3}}b_2 &= 0\\
-576a_2 - 384b_3 &= 0\\
576a_2 + 384b_3 &= 0\\
1728a_2 + 1152b_3 &= 0\\
-288\,2^{\frac{1}{3}}a_2 - 192\,2^{\frac{1}{3}}b_3 &= 0\\
144\,2^{\frac{1}{3}}a_2 + 96\,2^{\frac{1}{3}}b_3 &= 0\\
432\,2^{\frac{1}{3}}a_2 + 288\,2^{\frac{1}{3}}b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -\frac{2b_3}{3} \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -\frac{2x}{3} \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{y}{-\frac{2x}{3}} \\
&= -\frac{3y}{2x}
\end{aligned}
$$

This is easily solved to give

$$
y = \frac{c_1}{x^{\frac{3}{2}}}
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = y\, x^{\frac{3}{2}}
$$

And S is found from

$$\begin{aligned} dS &= \frac{dx}{\xi} \\ &= \frac{dx}{-\frac{2x}{3}} \end{aligned}$$

Integrating gives

$$\begin{aligned} S &= \int \frac{dx}{T} \\ &= -\frac{3\ln(x)}{2} \end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\left(\left(12xy + 4\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{2}{3}} + 4x}{2x^3\left(\left(12xy + 4\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{1}{3}}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= \frac{3y\sqrt{x}}{2} \\ R_y &= x^{\frac{3}{2}} \\ S_x &= -\frac{3}{2x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{3\sqrt{x}\left(3y\,x^3 + x^2\sqrt{\frac{9x^3y^2-4}{x}}\right)^{\frac{1}{3}}}{3y\,x^2\left(3y\,x^3 + x^2\sqrt{\frac{9x^3y^2-4}{x}}\right)^{\frac{1}{3}} + 2\,2^{\frac{1}{3}}x + \left(3y\,x^3 + x^2\sqrt{\frac{9x^3y^2-4}{x}}\right)^{\frac{2}{3}}2^{\frac{2}{3}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{3\left(\sqrt{9R^2-4}+3R\right)^{\frac{1}{3}}}{\left(\sqrt{9R^2-4}+3R\right)^{\frac{2}{3}} 2^{\frac{2}{3}}+3\left(\sqrt{9R^2-4}+3R\right)^{\frac{1}{3}} R+2\,2^{\frac{1}{3}}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int -\frac{3\left(\sqrt{9R^2-4}+3R\right)^{\frac{1}{3}}}{\left(\sqrt{9R^2-4}+3R\right)^{\frac{2}{3}} 2^{\frac{2}{3}}+3\left(\sqrt{9R^2-4}+3R\right)^{\frac{1}{3}} R+2\,2^{\frac{1}{3}}}dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{3\ln(x)}{2} = \int^{y x^{\frac{3}{2}}} -\frac{3\left(\sqrt{9\_a^2-4}+3\_a\right)^{\frac{1}{3}}}{\left(\sqrt{9\_a^2-4}+3\_a\right)^{\frac{2}{3}} 2^{\frac{2}{3}}+3\left(\sqrt{9\_a^2-4}+3\_a\right)^{\frac{1}{3}} \_a+2\,2^{\frac{1}{3}}}d\_a + c_1$$

Which simplifies to

$$-\frac{3\ln(x)}{2} = \int^{y x^{\frac{3}{2}}} -\frac{3\left(\sqrt{9\_a^2-4}+3\_a\right)^{\frac{1}{3}}}{\left(\sqrt{9\_a^2-4}+3\_a\right)^{\frac{2}{3}} 2^{\frac{2}{3}}+3\left(\sqrt{9\_a^2-4}+3\_a\right)^{\frac{1}{3}} \_a+2\,2^{\frac{1}{3}}}d\_a + c_1$$

Summary
The solution(s) found are the following

$$-\frac{3\ln(x)}{2} = \int^{y x^{\frac{3}{2}}} -\frac{3\left(\sqrt{9\_a^2-4}+3\_a\right)^{\frac{1}{3}}}{\left(\sqrt{9\_a^2-4}+3\_a\right)^{\frac{2}{3}} 2^{\frac{2}{3}}+3\left(\sqrt{9\_a^2-4}+3\_a\right)^{\frac{1}{3}} \_a+2\,2^{\frac{1}{3}}}d\_a + c_1 \tag{1}$$

Verification of solutions

$$-\frac{3\ln(x)}{2} = \int^{y x^{\frac{3}{2}}} -\frac{3\left(\sqrt{9\_a^2-4}+3\_a\right)^{\frac{1}{3}}}{\left(\sqrt{9\_a^2-4}+3\_a\right)^{\frac{2}{3}} 2^{\frac{2}{3}}+3\left(\sqrt{9\_a^2-4}+3\_a\right)^{\frac{1}{3}} \_a+2\,2^{\frac{1}{3}}}d\_a + c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = \frac{i\sqrt{3}\left(\left(12xy+4\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{2}{3}} - 4i\sqrt{3}\,x - \left(\left(12xy+4\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{2}{3}} - 4x}{4x^3\left(\left(12xy+4\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{1}{3}}}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

Expression too large to display (5E)

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Simplifying the above gives

Expression too large to display (6E)

Since the PDE has radicals, simplifying gives

Expression too large to display

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(\left(3xy+\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{1}{3}}, \left(\left(3xy+\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{2}{3}}, \sqrt{\frac{9x^3y^2-4}{x}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \left(\left(3xy+\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{1}{3}} = v_3, \left(\left(3xy+\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{2}{3}} = v_4, \sqrt{\frac{9x^3y^2-4}{x}} = v\right.$$

The above PDE (6E) now becomes

$$\text{Expression too large to display} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&576\,2^{\frac{2}{3}}b_2v_2v_3v_5v_1^9+\left(1920i\sqrt{3}\,a_1+1920a_1\right)v_1^4\\
&+\left(1152i\sqrt{3}\,a_2+768i\sqrt{3}\,b_3+1152a_2+768b_3\right)v_1^5\\
&-1152\,2^{\frac{2}{3}}a_3v_2v_3v_5v_1^4+1728\,2^{\frac{2}{3}}b_2v_2^2v_3v_1^{10}\\
&-3456\,2^{\frac{2}{3}}a_3v_2^2v_3v_1^5+\Big(288i2^{\frac{1}{3}}\sqrt{3}\,a_2\\
&+192i2^{\frac{1}{3}}\sqrt{3}\,b_3-288\,2^{\frac{1}{3}}a_2-192\,2^{\frac{1}{3}}b_3\Big)v_2v_4v_5v_1^6\\
&+\left(576i2^{\frac{1}{3}}\sqrt{3}\,a_3-576\,2^{\frac{1}{3}}a_3\right)v_2^2v_4v_5v_1^5\\
&+\left(576i2^{\frac{1}{3}}\sqrt{3}\,a_1-576\,2^{\frac{1}{3}}a_1\right)v_2v_4v_5v_1^5\\
&+\left(1728i2^{\frac{1}{3}}\sqrt{3}\,a_3-1728\,2^{\frac{1}{3}}a_3\right)v_2^3v_4v_1^6\\
&+\left(1728i2^{\frac{1}{3}}\sqrt{3}\,a_1-1728\,2^{\frac{1}{3}}a_1\right)v_2^2v_4v_1^6\\
&+\left(-1728i\sqrt{3}\,a_1-1728a_1\right)v_2v_5v_1^6\\
&+\left(-96i2^{\frac{1}{3}}\sqrt{3}\,b_1+96\,2^{\frac{1}{3}}b_1\right)v_4v_5v_1^6\\
&+\left(-288i2^{\frac{1}{3}}\sqrt{3}\,b_2+288\,2^{\frac{1}{3}}b_2\right)v_2v_4v_1^8\\
&+\left(-960i2^{\frac{1}{3}}\sqrt{3}\,a_3+960\,2^{\frac{1}{3}}a_3\right)v_2v_4v_1^3\\
&+\left(-192i2^{\frac{1}{3}}\sqrt{3}\,a_3+192\,2^{\frac{1}{3}}a_3\right)v_4v_5v_1^2+\Big(864i2^{\frac{1}{3}}\sqrt{3}\,a_2\\
&+576i2^{\frac{1}{3}}\sqrt{3}\,b_3-864\,2^{\frac{1}{3}}a_2-576\,2^{\frac{1}{3}}b_3\Big)v_2^2v_4v_1^7\\
&+\left(-288i2^{\frac{1}{3}}\sqrt{3}\,b_1+288\,2^{\frac{1}{3}}b_1\right)v_2v_4v_1^7\\
&+\left(-1152i\sqrt{3}\,a_2-768i\sqrt{3}\,b_3-1152a_2-768b_3\right)v_2v_5v_1^7\\
&+\left(-96i2^{\frac{1}{3}}\sqrt{3}\,b_2+96\,2^{\frac{1}{3}}b_2\right)v_4v_5v_1^7+1536\,2^{\frac{2}{3}}a_3v_3v_1^2\\
&-768\,2^{\frac{2}{3}}b_2v_3v_1^7+\left(-576i\sqrt{3}\,b_2-576b_2\right)v_2v_1^9\\
&+\left(-5184i\sqrt{3}\,a_1-5184a_1\right)v_2^2v_1^7\\
&+\left(-192i\sqrt{3}\,b_1-192b_1\right)v_5v_1^7\\
&+\left(-3456i\sqrt{3}\,a_2-2304i\sqrt{3}\,b_3-3456a_2-2304b_3\right)v_2^2v_1^8\\
&+\left(-576i\sqrt{3}\,b_1-576b_1\right)v_2v_1^8+\left(-192i\sqrt{3}\,b_2-192b_2\right)v_5v_1^8\\
&+\left(-384i\sqrt{3}\,a_3-384a_3\right)v_2v_1^4+\Big(-576i2^{\frac{1}{3}}\sqrt{3}\,a_2\\
&-384i2^{\frac{1}{3}}\sqrt{3}\,b_3+576\,2^{\frac{1}{3}}a_2+384\,2^{\frac{1}{3}}b_3\Big)v_4v_1^4\\
&+\left(-960i2^{\frac{1}{3}}\sqrt{3}\,a_1+960\,2^{\frac{1}{3}}a_1\right)v_4v_1^3\\
&+\left(-384i\sqrt{3}\,a_3-384a_3\right)v_5v_1^3=0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$-3456\, 2^{\frac{2}{3}} a_3 = 0$$
$$-1152\, 2^{\frac{2}{3}} a_3 = 0$$
$$1536\, 2^{\frac{2}{3}} a_3 = 0$$
$$-768\, 2^{\frac{2}{3}} b_2 = 0$$
$$576\, 2^{\frac{2}{3}} b_2 = 0$$
$$1728\, 2^{\frac{2}{3}} b_2 = 0$$
$$-5184 i\sqrt{3}\, a_1 - 5184 a_1 = 0$$
$$-1728 i\sqrt{3}\, a_1 - 1728 a_1 = 0$$
$$-576 i\sqrt{3}\, b_1 - 576 b_1 = 0$$
$$-576 i\sqrt{3}\, b_2 - 576 b_2 = 0$$
$$-384 i\sqrt{3}\, a_3 - 384 a_3 = 0$$
$$-192 i\sqrt{3}\, b_1 - 192 b_1 = 0$$
$$-192 i\sqrt{3}\, b_2 - 192 b_2 = 0$$
$$1920 i\sqrt{3}\, a_1 + 1920 a_1 = 0$$
$$-960 i 2^{\frac{1}{3}} \sqrt{3}\, a_1 + 960\, 2^{\frac{1}{3}} a_1 = 0$$
$$-960 i 2^{\frac{1}{3}} \sqrt{3}\, a_3 + 960\, 2^{\frac{1}{3}} a_3 = 0$$
$$-288 i 2^{\frac{1}{3}} \sqrt{3}\, b_1 + 288\, 2^{\frac{1}{3}} b_1 = 0$$
$$-288 i 2^{\frac{1}{3}} \sqrt{3}\, b_2 + 288\, 2^{\frac{1}{3}} b_2 = 0$$
$$-192 i 2^{\frac{1}{3}} \sqrt{3}\, a_3 + 192\, 2^{\frac{1}{3}} a_3 = 0$$
$$-96 i 2^{\frac{1}{3}} \sqrt{3}\, b_1 + 96\, 2^{\frac{1}{3}} b_1 = 0$$
$$-96 i 2^{\frac{1}{3}} \sqrt{3}\, b_2 + 96\, 2^{\frac{1}{3}} b_2 = 0$$
$$576 i 2^{\frac{1}{3}} \sqrt{3}\, a_1 - 576\, 2^{\frac{1}{3}} a_1 = 0$$
$$576 i 2^{\frac{1}{3}} \sqrt{3}\, a_3 - 576\, 2^{\frac{1}{3}} a_3 = 0$$
$$1728 i 2^{\frac{1}{3}} \sqrt{3}\, a_1 - 1728\, 2^{\frac{1}{3}} a_1 = 0$$
$$1728 i 2^{\frac{1}{3}} \sqrt{3}\, a_3 - 1728\, 2^{\frac{1}{3}} a_3 = 0$$
$$-3456 i\sqrt{3}\, a_2 - 2304 i\sqrt{3}\, b_3 - 3456 a_2 - 2304 b_3 = 0$$
$$-1152 i\sqrt{3}\, a_2 - 768 i\sqrt{3}\, b_3 - 1152 a_2 - 768 b_3 = 0$$
$$1152 i\sqrt{3}\, a_2 + 768 i\sqrt{3}\, b_3 + 1152 a_2 + 768 b_3 = 0$$
$$-576 i 2^{\frac{1}{3}} \sqrt{3}\, a_2 - 384 i 2^{\frac{1}{3}} \sqrt{3}\, b_3 + 576\, 2^{\frac{1}{3}} a_2 + 384\, 2^{\frac{1}{3}} b_3 = 0$$
$$288 i 2^{\frac{1}{3}} \sqrt{3}\, a_2 + 192 i 2^{\frac{1}{3}} \sqrt{3}\, b_3 - 288\, 2^{\frac{1}{3}} a_2 - 192\, 2^{\frac{1}{3}} b_3 = 0$$
$$864 i 2^{\frac{1}{3}} \sqrt{3}\, a_2 + 576 i 2^{\frac{1}{3}} \sqrt{3}\, b_3 - 864\, 2^{\frac{1}{3}} a_2 - 576\, 2^{\frac{1}{3}} b_3 = 0$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= -\frac{3a_2}{2}
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= -\frac{3y}{2}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Solving equation (3)

Writing the ode as

$$
y' = -\frac{i\sqrt{3}\left(\left(12xy + 4\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{2}{3}} - 4i\sqrt{3}\,x + \left(\left(12xy + 4\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{2}{3}} + 4x}{4x^3\left(\left(12xy + 4\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{1}{3}}}
$$

$$
y' = \omega(x, y)
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

Expression too large to display (5E)

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Simplifying the above gives

Expression too large to display (6E)

Since the PDE has radicals, simplifying gives

Expression too large to display

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(\left(3xy+\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{1}{3}}, \left(\left(3xy+\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{2}{3}}, \sqrt{\frac{9x^3y^2-4}{x}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \left(\left(3xy+\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{1}{3}} = v_3, \left(\left(3xy+\sqrt{\frac{9x^3y^2-4}{x}}\right)x^2\right)^{\frac{2}{3}} = v_4, \sqrt{\frac{9x^3y^2-4}{x}} = v$$

The above PDE (6E) now becomes

$$\text{Expression too large to display} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$\begin{aligned}
&-768\,2^{\frac{2}{3}}b_2v_3v_1^7+\left(96i2^{\frac{1}{3}}\sqrt{3}\,b_2+96\,2^{\frac{1}{3}}b_2\right)v_4v_5v_1^7\\
&+\left(960i2^{\frac{1}{3}}\sqrt{3}\,a_3+960\,2^{\frac{1}{3}}a_3\right)v_2v_4v_1^3\\
&+\left(192i2^{\frac{1}{3}}\sqrt{3}\,a_3+192\,2^{\frac{1}{3}}a_3\right)v_4v_5v_1^2\\
&+\left(-1728i2^{\frac{1}{3}}\sqrt{3}\,a_3-1728\,2^{\frac{1}{3}}a_3\right)v_2^3v_4v_1^6\\
&+\left(-1728i2^{\frac{1}{3}}\sqrt{3}\,a_1-1728\,2^{\frac{1}{3}}a_1\right)v_2^2v_4v_1^6\\
&+\left(1728i\sqrt{3}\,a_1-1728a_1\right)v_2v_5v_1^6\\
&+\left(96i2^{\frac{1}{3}}\sqrt{3}\,b_1+96\,2^{\frac{1}{3}}b_1\right)v_4v_5v_1^6+576\,2^{\frac{2}{3}}b_2v_2v_3v_5v_1^9\\
&+1536\,2^{\frac{2}{3}}a_3v_3v_1^2+\left(288i2^{\frac{1}{3}}\sqrt{3}\,b_2+288\,2^{\frac{1}{3}}b_2\right)v_2v_4v_1^8\\
&+\left(-864i2^{\frac{1}{3}}\sqrt{3}\,a_2-576i2^{\frac{1}{3}}\sqrt{3}\,b_3-864\,2^{\frac{1}{3}}a_2\right.\\
&\left.-576\,2^{\frac{1}{3}}b_3\right)v_2^2v_4v_1^7+\left(288i2^{\frac{1}{3}}\sqrt{3}\,b_1+288\,2^{\frac{1}{3}}b_1\right)v_2v_4v_1^7\\
&+\left(1152i\sqrt{3}\,a_2+768i\sqrt{3}\,b_3-1152a_2-768b_3\right)v_2v_5v_1^7\\
&+\left(576i\sqrt{3}\,b_2-576b_2\right)v_2v_1^9\\
&+\left(3456i\sqrt{3}\,a_2+2304i\sqrt{3}\,b_3-3456a_2-2304b_3\right)v_2^2v_1^8\\
&+\left(576i\sqrt{3}\,b_1-576b_1\right)v_2v_1^8+\left(192i\sqrt{3}\,b_2-192b_2\right)v_5v_1^8\\
&+\left(5184i\sqrt{3}\,a_1-5184a_1\right)v_2^2v_1^7\\
&+\left(192i\sqrt{3}\,b_1-192b_1\right)v_5v_1^7+\left(384i\sqrt{3}\,a_3-384a_3\right)v_2v_1^4\\
&+\left(576i2^{\frac{1}{3}}\sqrt{3}\,a_2+384i2^{\frac{1}{3}}\sqrt{3}\,b_3+576\,2^{\frac{1}{3}}a_2+384\,2^{\frac{1}{3}}b_3\right)v_4v_1^4\\
&+\left(960i2^{\frac{1}{3}}\sqrt{3}\,a_1+960\,2^{\frac{1}{3}}a_1\right)v_4v_1^3\\
&+\left(384i\sqrt{3}\,a_3-384a_3\right)v_5v_1^3\\
&-3456\,2^{\frac{2}{3}}a_3v_2^2v_3v_1^5-1152\,2^{\frac{2}{3}}a_3v_2v_3v_5v_1^4\\
&+\left(-576i2^{\frac{1}{3}}\sqrt{3}\,a_1-576\,2^{\frac{1}{3}}a_1\right)v_2v_4v_5v_1^5\\
&+\left(-1152i\sqrt{3}\,a_2-768i\sqrt{3}\,b_3+1152a_2+768b_3\right)v_1^5\\
&+\left(-1920i\sqrt{3}\,a_1+1920a_1\right)v_1^4+\left(-288i2^{\frac{1}{3}}\sqrt{3}\,a_2\right.\\
&\left.-192i2^{\frac{1}{3}}\sqrt{3}\,b_3-288\,2^{\frac{1}{3}}a_2-192\,2^{\frac{1}{3}}b_3\right)v_2v_4v_5v_1^6\\
&+\left(-576i2^{\frac{1}{3}}\sqrt{3}\,a_3-576\,2^{\frac{1}{3}}a_3\right)v_2^2v_4v_5v_1^5\\
&+1728\,2^{\frac{2}{3}}b_2v_2^2v_3v_1^{10}=0
\end{aligned}\tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$-3456\,2^{\frac{2}{3}} a_3 = 0$$
$$-1152\,2^{\frac{2}{3}} a_3 = 0$$
$$1536\,2^{\frac{2}{3}} a_3 = 0$$
$$-768\,2^{\frac{2}{3}} b_2 = 0$$
$$576\,2^{\frac{2}{3}} b_2 = 0$$
$$1728\,2^{\frac{2}{3}} b_2 = 0$$
$$-1920i\sqrt{3}\,a_1 + 1920a_1 = 0$$
$$192i\sqrt{3}\,b_1 - 192b_1 = 0$$
$$192i\sqrt{3}\,b_2 - 192b_2 = 0$$
$$384i\sqrt{3}\,a_3 - 384a_3 = 0$$
$$576i\sqrt{3}\,b_1 - 576b_1 = 0$$
$$576i\sqrt{3}\,b_2 - 576b_2 = 0$$
$$1728i\sqrt{3}\,a_1 - 1728a_1 = 0$$
$$5184i\sqrt{3}\,a_1 - 5184a_1 = 0$$
$$-1728i2^{\frac{1}{3}}\sqrt{3}\,a_1 - 1728\,2^{\frac{1}{3}} a_1 = 0$$
$$-1728i2^{\frac{1}{3}}\sqrt{3}\,a_3 - 1728\,2^{\frac{1}{3}} a_3 = 0$$
$$-576i2^{\frac{1}{3}}\sqrt{3}\,a_1 - 576\,2^{\frac{1}{3}} a_1 = 0$$
$$-576i2^{\frac{1}{3}}\sqrt{3}\,a_3 - 576\,2^{\frac{1}{3}} a_3 = 0$$
$$96i2^{\frac{1}{3}}\sqrt{3}\,b_1 + 96\,2^{\frac{1}{3}} b_1 = 0$$
$$96i2^{\frac{1}{3}}\sqrt{3}\,b_2 + 96\,2^{\frac{1}{3}} b_2 = 0$$
$$192i2^{\frac{1}{3}}\sqrt{3}\,a_3 + 192\,2^{\frac{1}{3}} a_3 = 0$$
$$288i2^{\frac{1}{3}}\sqrt{3}\,b_1 + 288\,2^{\frac{1}{3}} b_1 = 0$$
$$288i2^{\frac{1}{3}}\sqrt{3}\,b_2 + 288\,2^{\frac{1}{3}} b_2 = 0$$
$$960i2^{\frac{1}{3}}\sqrt{3}\,a_1 + 960\,2^{\frac{1}{3}} a_1 = 0$$
$$960i2^{\frac{1}{3}}\sqrt{3}\,a_3 + 960\,2^{\frac{1}{3}} a_3 = 0$$
$$-1152i\sqrt{3}\,a_2 - 768i\sqrt{3}\,b_3 + 1152a_2 + 768b_3 = 0$$
$$1152i\sqrt{3}\,a_2 + 768i\sqrt{3}\,b_3 - 1152a_2 - 768b_3 = 0$$
$$3456i\sqrt{3}\,a_2 + 2304i\sqrt{3}\,b_3 - 3456a_2 - 2304b_3 = 0$$
$$-864i2^{\frac{1}{3}}\sqrt{3}\,a_2 - 576i2^{\frac{1}{3}}\sqrt{3}\,b_3 - 864\,2^{\frac{1}{3}} a_2 - 576\,2^{\frac{1}{3}} b_3 = 0$$
$$-288i2^{\frac{1}{3}}\sqrt{3}\,a_2 - 192i2^{\frac{1}{3}}\sqrt{3}\,b_3 - 288\,2^{\frac{1}{3}} a_2 - 192\,2^{\frac{1}{3}} b_3 = 0$$
$$576i2^{\frac{1}{3}}\sqrt{3}\,a_2 + 384i2^{\frac{1}{3}}\sqrt{3}\,b_3 + 576\,2^{\frac{1}{3}} a_2 + 384\,2^{\frac{1}{3}} b_3 = 0$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= -\frac{3a_2}{2}
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= -\frac{3y}{2}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 3 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
   -> Solving 1st order ODE of high degree, Lie methods, 1st trial
   `, `-> Computing symmetries using: way = 2
   `, `-> Computing symmetries using: way = 2
   -> Solving 1st order ODE of high degree, 2nd attempt. Trying parametric methods
   -> Calling odsolve with the ODE`, diff(y(x), x) = ((1/3)*(x^2*y(x)^5-3)*y(x)+(2/3)*y(x)^6
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   <- 1st order, parametric methods successful`
```

✓ Solution by Maple

Time used: 0.093 (sec). Leaf size: 32

```
dsolve(x^6*diff(y(x),x)^3-3*x*diff(y(x),x)-3*y(x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{2}{3x^{\frac{3}{2}}}$$
$$y(x) = \frac{2}{3x^{\frac{3}{2}}}$$
$$y(x) = \frac{c_1^3}{3} - \frac{c_1}{x}$$

✓ Solution by Mathematica

Time used: 136.42 (sec). Leaf size: 24834

```
DSolve[x^6*(y'[x])^3-3*x*y'[x]-3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

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3.13 problem 15

Internal problem ID [6807]
Internal file name [`OUTPUT/6054_Tuesday_July_26_2022_11_24_06_PM_71082803/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 99. Clairaut's equation. EXERCISES Page 320
Problem number: 15.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries]]
```

$$\boxed{y - x^6 {y'}^3 + x y' = 0}$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\frac{\left(\left(108yx+12\sqrt{3}\,\sqrt{\frac{27y^2x^3-4}{x}}\right)x^2\right)^{\frac{1}{3}}}{6x^2} + \frac{2}{x\left(\left(108yx+12\sqrt{3}\,\sqrt{\frac{27y^2x^3-4}{x}}\right)x^2\right)^{\frac{1}{3}}}}{x} \tag{1}$$

$$y' = \frac{-\frac{\left(\left(108yx+12\sqrt{3}\,\sqrt{\frac{27y^2x^3-4}{x}}\right)x^2\right)^{\frac{1}{3}}}{12x^2} - \frac{1}{x\left(\left(108yx+12\sqrt{3}\,\sqrt{\frac{27y^2x^3-4}{x}}\right)x^2\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(\left(108yx+12\sqrt{3}\,\sqrt{\frac{27y^2x^3-4}{x}}\right)x^2\right)^{\frac{1}{3}}}{6x^2} - \frac{\cdots}{x\left(\left(108\cdots\right.\right.}\right.}{2}}{x} \tag{2}$$

$$y' = \frac{-\frac{\left(\left(108yx+12\sqrt{3}\,\sqrt{\frac{27y^2x^3-4}{x}}\right)x^2\right)^{\frac{1}{3}}}{12x^2} - \frac{1}{x\left(\left(108yx+12\sqrt{3}\,\sqrt{\frac{27y^2x^3-4}{x}}\right)x^2\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(\left(108yx+12\sqrt{3}\,\sqrt{\frac{27y^2x^3-4}{x}}\right)x^2\right)^{\frac{1}{3}}}{6x^2} - \frac{\cdots}{x\left(\left(108\cdots\right.\right.}\right.}{2}}{x} \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = \frac{\left(\left(108xy+12\sqrt{3}\,\sqrt{\frac{27x^3y^2-4}{x}}\right)x^2\right)^{\frac{2}{3}} + 12x}{6x^3\left(\left(108xy+12\sqrt{3}\,\sqrt{\frac{27x^3y^2-4}{x}}\right)x^2\right)^{\frac{1}{3}}}$$

$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2+\frac{\left(\left(\left(108xy+12\sqrt{3}\,\sqrt{\frac{27x^3y^2-4}{x}}\right)x^2\right)^{\frac{2}{3}}+12x\right)(b_3-a_2)}{6x^3\left(\left(108xy+12\sqrt{3}\,\sqrt{\frac{27x^3y^2-4}{x}}\right)x^2\right)^{\frac{1}{3}}} \tag{5E}$$

$$-\frac{\left(\left(\left(108xy+12\sqrt{3}\,\sqrt{\frac{27x^3y^2-4}{x}}\right)x^2\right)^{\frac{2}{3}}+12x\right)^2 a_3}{36x^6\left(\left(108xy+12\sqrt{3}\,\sqrt{\frac{27x^3y^2-4}{x}}\right)x^2\right)^{\frac{2}{3}}}$$

$$-\left(\frac{\frac{\frac{2\left(108y+\frac{6\sqrt{3}\left(81x\,y^2-\frac{27x^3y^2-4}{x^2}\right)}{\sqrt{\frac{27x^3y^2-4}{x}}}\right)x^2}{3}+\frac{4\left(108xy+12\sqrt{3}\,\sqrt{\frac{27x^3y^2-4}{x}}\right)x}{3}}{\left(\left(108xy+12\sqrt{3}\,\sqrt{\frac{27x^3y^2-4}{x}}\right)x^2\right)^{\frac{1}{3}}}+12}{6x^3\left(\left(108xy+12\sqrt{3}\,\sqrt{\frac{27x^3y^2-4}{x}}\right)x^2\right)^{\frac{1}{3}}}\right.$$

$$-\frac{\left(\left(108xy+12\sqrt{3}\,\sqrt{\frac{27x^3y^2-4}{x}}\right)x^2\right)^{\frac{2}{3}}+12x}{2x^4\left(\left(108xy+12\sqrt{3}\,\sqrt{\frac{27x^3y^2-4}{x}}\right)x^2\right)^{\frac{1}{3}}}$$

$$-\frac{\left(\left(\left(108xy+12\sqrt{3}\,\sqrt{\frac{27x^3y^2-4}{x}}\right)x^2\right)^{\frac{2}{3}}+12x\right)\left(\left(108y+\frac{6\sqrt{3}\left(81x\,y^2-\frac{27x^3y^2-4}{x^2}\right)}{\sqrt{\frac{27x^3y^2-4}{x}}}\right)x^2+2\left(108xy+12\sqrt{3}\right.\right.}{18x^3\left(\left(108xy+12\sqrt{3}\,\sqrt{\frac{27x^3y^2-4}{x}}\right)x^2\right)^{\frac{4}{3}}}$$

$$+ya_3+a_1)-\left(\frac{108x+\frac{324\sqrt{3}\,y\,x^2}{\sqrt{\frac{27x^3y^2-4}{x}}}}{9\left(\left(108xy+12\sqrt{3}\,\sqrt{\frac{27x^3y^2-4}{x}}\right)x^2\right)^{\frac{2}{3}}x}\right.$$

$$\left.-\frac{\left(\left(\left(108xy+12\sqrt{3}\,\sqrt{\frac{27x^3y^2-4}{x}}\right)x^2\right)^{\frac{2}{3}}+12x\right)\left(108x+\frac{324\sqrt{3}\,y\,x^2}{\sqrt{\frac{27x^3y^2-4}{x}}}\right)}{18x\left(\left(108xy+12\sqrt{3}\,\sqrt{\frac{27x^3y^2-4}{x}}\right)x^2\right)^{\frac{4}{3}}}\right)(xb_2$$

$$+yb_3+b_1)=0$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

$$\text{Expression too large to display} \tag{6E}$$

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

Expression too large to display

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(\left(\sqrt{3}\,\sqrt{\frac{27x^{3}y^{2}-4}{x}}+9xy\right)x^{2}\right)^{\frac{1}{3}}, \left(\left(\sqrt{3}\,\sqrt{\frac{27x^{3}y^{2}-4}{x}}+9xy\right)x^{2}\right)^{\frac{2}{3}}, \sqrt{\frac{27x^{3}y^{2}-4}{x}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x=v_{1}, y=v_{2}, \left(\left(\sqrt{3}\,\sqrt{\frac{27x^{3}y^{2}-4}{x}}+9xy\right)x^{2}\right)^{\frac{1}{3}}=v_{3}, \left(\left(\sqrt{3}\,\sqrt{\frac{27x^{3}y^{2}-4}{x}}+9xy\right)x^{2}\right)^{\frac{2}{3}}=v_{4}, \sqrt{\frac{27x\cdots}{}}\right.$$

The above PDE (6E) now becomes

$$\begin{aligned}
-72v_{1}^{2}\Big(&3v_{5}12^{\frac{2}{3}}v_{4}v_{1}^{5}b_{2}+3v_{5}12^{\frac{2}{3}}v_{4}v_{1}^{4}b_{1}+24\sqrt{3}\,12^{\frac{1}{3}}v_{3}v_{1}^{5}b_{2} \\
&+6\sqrt{3}\,12^{\frac{2}{3}}v_{4}v_{1}^{2}a_{2}+4\sqrt{3}\,12^{\frac{2}{3}}v_{4}v_{1}^{2}b_{3}+10\sqrt{3}\,12^{\frac{2}{3}}v_{4}v_{1}a_{1}-36v_{5}v_{1}^{5}b_{1} \\
&-36v_{5}v_{1}^{6}b_{2}+72\sqrt{3}\,v_{1}^{3}a_{2}+48\sqrt{3}\,v_{1}^{3}b_{3}+120\sqrt{3}\,v_{1}^{2}a_{1}-24v_{5}v_{1}a_{3} \\
&-648\sqrt{3}\,v_{1}^{6}v_{2}^{2}a_{2}-432\sqrt{3}\,v_{1}^{6}v_{2}^{2}b_{3}-108\sqrt{3}\,v_{1}^{6}v_{2}b_{1}-972\sqrt{3}\,v_{1}^{5}v_{2}^{2}a_{1} \\
&-216v_{5}v_{1}^{5}v_{2}a_{2}-144v_{5}v_{1}^{5}v_{2}b_{3}-324v_{5}v_{1}^{4}v_{2}a_{1}+2v_{5}12^{\frac{2}{3}}v_{4}a_{3} \\
&-24\sqrt{3}\,v_{1}^{2}v_{2}a_{3}-16\sqrt{3}\,12^{\frac{1}{3}}v_{3}a_{3}-162\sqrt{3}\,12^{\frac{1}{3}}v_{3}v_{1}^{8}v_{2}^{2}b_{2} \\
&+9\sqrt{3}\,12^{\frac{2}{3}}v_{4}v_{1}^{6}v_{2}b_{2}-27\sqrt{3}\,12^{\frac{2}{3}}v_{4}v_{1}^{5}v_{2}^{2}a_{2}-18\sqrt{3}\,12^{\frac{2}{3}}v_{4}v_{1}^{5}v_{2}^{2}b_{3} \\
&-54\sqrt{3}\,12^{\frac{2}{3}}v_{4}v_{1}^{4}v_{2}^{3}a_{3}+9\sqrt{3}\,12^{\frac{2}{3}}v_{4}v_{1}^{5}v_{2}b_{1}-54\sqrt{3}\,12^{\frac{2}{3}}v_{4}v_{1}^{4}v_{2}^{2}a_{1} \\
&-54v_{5}12^{\frac{1}{3}}v_{3}v_{1}^{7}v_{2}b_{2}-9v_{5}12^{\frac{2}{3}}v_{4}v_{1}^{4}v_{2}a_{2}-6v_{5}12^{\frac{2}{3}}v_{4}v_{1}^{4}v_{2}b_{3} \\
&-18v_{5}12^{\frac{2}{3}}v_{4}v_{1}^{3}v_{2}^{2}a_{3}-18v_{5}12^{\frac{2}{3}}v_{4}v_{1}^{3}v_{2}a_{1}+108\sqrt{3}\,12^{\frac{1}{3}}v_{3}v_{1}^{3}v_{2}^{2}a_{3} \\
&+10\sqrt{3}\,12^{\frac{2}{3}}v_{4}v_{1}v_{2}a_{3}+36v_{5}12^{\frac{1}{3}}v_{3}v_{1}^{2}v_{2}a_{3}-108\sqrt{3}\,v_{1}^{7}v_{2}b_{2}\Big)=0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$\begin{aligned}
&\left(46656\sqrt{3}\,a_2 + 31104\sqrt{3}\,b_3\right) v_2^2 v_1^8 + 2592 b_2 v_5 v_1^8 \\
&+ 2592 b_1 v_5 v_1^7 + \left(-5184\sqrt{3}\,a_2 - 3456\sqrt{3}\,b_3\right) v_1^5 \\
&+ \left(-432\sqrt{3}\,12^{\frac{2}{3}} a_2 - 288\sqrt{3}\,12^{\frac{2}{3}} b_3\right) v_4 v_1^4 \\
&- 8640\sqrt{3}\,a_1 v_1^4 - 1728\sqrt{3}\,12^{\frac{1}{3}} b_2 v_3 v_1^7 - 216\,12^{\frac{2}{3}} b_2 v_4 v_5 v_1^7 \\
&+ \left(648\,12^{\frac{2}{3}} a_2 + 432\,12^{\frac{2}{3}} b_3\right) v_2 v_4 v_5 v_1^6 - 216\,12^{\frac{2}{3}} b_1 v_4 v_5 v_1^6 \\
&- 720\sqrt{3}\,12^{\frac{2}{3}} a_1 v_4 v_1^3 + 1152\sqrt{3}\,12^{\frac{1}{3}} a_3 v_3 v_1^2 - 144\,12^{\frac{2}{3}} a_3 v_4 v_5 v_1^2 \\
&+ 23328 a_1 v_2 v_5 v_1^6 + 1728\sqrt{3}\,a_3 v_2 v_1^4 + 7776\sqrt{3}\,b_2 v_2 v_1^9 \\
&+ 7776\sqrt{3}\,b_1 v_2 v_1^8 + \left(1944\sqrt{3}\,12^{\frac{2}{3}} a_2 + 1296\sqrt{3}\,12^{\frac{2}{3}} b_3\right) v_2^2 v_4 v_1^7 \\
&+ 69984\sqrt{3}\,a_1 v_2^2 v_1^7 + \left(15552 a_2 + 10368 b_3\right) v_2 v_5 v_1^7 \\
&+ 1728 a_3 v_5 v_1^3 + 11664\sqrt{3}\,12^{\frac{1}{3}} b_2 v_2^2 v_3 v_1^{10} + 3888\,12^{\frac{1}{3}} b_2 v_2 v_3 v_5 v_1^9 \\
&- 648\sqrt{3}\,12^{\frac{2}{3}} b_2 v_2 v_4 v_1^8 - 648\sqrt{3}\,12^{\frac{2}{3}} b_1 v_2 v_4 v_1^7 + 3888\sqrt{3}\,12^{\frac{2}{3}} a_3 v_2^3 v_4 v_1^6 \\
&+ 3888\sqrt{3}\,12^{\frac{2}{3}} a_1 v_2^2 v_4 v_1^6 - 7776\sqrt{3}\,12^{\frac{1}{3}} a_3 v_2^2 v_3 v_1^5 \\
&+ 1296\,12^{\frac{2}{3}} a_3 v_2^2 v_4 v_5 v_1^5 + 1296\,12^{\frac{2}{3}} a_1 v_2 v_4 v_5 v_1^5 \\
&- 2592\,12^{\frac{1}{3}} a_3 v_2 v_3 v_5 v_1^4 - 720\sqrt{3}\,12^{\frac{2}{3}} a_3 v_2 v_4 v_1^3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
23328a_{1} &= 0\\
1728a_{3} &= 0\\
2592b_{1} &= 0\\
2592b_{2} &= 0\\
-8640\sqrt{3}\, a_{1} &= 0\\
69984\sqrt{3}\, a_{1} &= 0\\
1728\sqrt{3}\, a_{3} &= 0\\
7776\sqrt{3}\, b_{1} &= 0\\
7776\sqrt{3}\, b_{2} &= 0\\
-2592\, 12^{\frac{1}{3}} a_{3} &= 0\\
3888\, 12^{\frac{1}{3}} b_{2} &= 0\\
1296\, 12^{\frac{2}{3}} a_{1} &= 0\\
-144\, 12^{\frac{2}{3}} a_{3} &= 0\\
1296\, 12^{\frac{2}{3}} a_{3} &= 0\\
-216\, 12^{\frac{2}{3}} b_{1} &= 0\\
-216\, 12^{\frac{2}{3}} b_{2} &= 0\\
-7776\sqrt{3}\, 12^{\frac{1}{3}} a_{3} &= 0\\
1152\sqrt{3}\, 12^{\frac{1}{3}} a_{3} &= 0\\
-1728\sqrt{3}\, 12^{\frac{1}{3}} b_{2} &= 0\\
11664\sqrt{3}\, 12^{\frac{1}{3}} b_{2} &= 0\\
-720\sqrt{3}\, 12^{\frac{2}{3}} a_{1} &= 0\\
3888\sqrt{3}\, 12^{\frac{2}{3}} a_{1} &= 0\\
-720\sqrt{3}\, 12^{\frac{2}{3}} a_{3} &= 0\\
3888\sqrt{3}\, 12^{\frac{2}{3}} a_{3} &= 0\\
-648\sqrt{3}\, 12^{\frac{2}{3}} b_{1} &= 0\\
-648\sqrt{3}\, 12^{\frac{2}{3}} b_{2} &= 0\\
15552a_{2} + 10368b_{3} &= 0\\
-5184\sqrt{3}\, a_{2} - 3456\sqrt{3}\, b_{3} &= 0\\
46656\sqrt{3}\, a_{2} + 31104\sqrt{3}\, b_{3} &= 0\\
648\, 12^{\frac{2}{3}} a_{2} + 432\, 12^{\frac{2}{3}} b_{3} &= 0\\
-432\sqrt{3}\, 12^{\frac{2}{3}} a_{2} - 288\sqrt{3}\, 12^{\frac{2}{3}} b_{3} &= 0\\
1944\sqrt{3}\, 12^{\frac{2}{3}} a_{2} + 1296\sqrt{3}\, 12^{\frac{2}{3}} b_{3} &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0\\
a_2 &= -\frac{2b_3}{3}\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -\frac{2x}{3}\\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi}\\
&= \frac{y}{-\frac{2x}{3}}\\
&= -\frac{3y}{2x}
\end{aligned}
$$

This is easily solved to give

$$
y = \frac{c_1}{x^{\frac{3}{2}}}
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = y\, x^{\frac{3}{2}}
$$

And S is found from

$$\begin{aligned} dS &= \frac{dx}{\xi} \\ &= \frac{dx}{-\frac{2x}{3}} \end{aligned}$$

Integrating gives

$$\begin{aligned} S &= \int \frac{dx}{T} \\ &= -\frac{3\ln(x)}{2} \end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\left(\left(108xy + 12\sqrt{3}\,\sqrt{\frac{27x^3y^2-4}{x}}\right)x^2\right)^{\frac{2}{3}} + 12x}{6x^3\left(\left(108xy + 12\sqrt{3}\,\sqrt{\frac{27x^3y^2-4}{x}}\right)x^2\right)^{\frac{1}{3}}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= \frac{3y\sqrt{x}}{2} \\ R_y &= x^{\frac{3}{2}} \\ S_x &= -\frac{3}{2x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{9\sqrt{x}\left(x^2\sqrt{3}\,\sqrt{\frac{27x^3y^2-4}{x}} + 9y\,x^3\right)^{\frac{1}{3}}}{9y\,x^2\left(x^2\sqrt{3}\,\sqrt{\frac{27x^3y^2-4}{x}} + 9y\,x^3\right)^{\frac{1}{3}} + 12^{\frac{2}{3}}x + 12^{\frac{1}{3}}\left(x^2\sqrt{3}\,\sqrt{\frac{27x^3y^2-4}{x}} + 9y\,x^3\right)^{\frac{2}{3}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{9\left(\sqrt{3}\,\sqrt{27R^2-4}+9R\right)^{\frac{1}{3}}}{12^{\frac{1}{3}}\left(\sqrt{3}\,\sqrt{27R^2-4}+9R\right)^{\frac{2}{3}}+12^{\frac{2}{3}}+9\left(\sqrt{3}\,\sqrt{27R^2-4}+9R\right)^{\frac{1}{3}}R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int -\frac{9\left(\sqrt{81R^2-12}+9R\right)^{\frac{1}{3}}}{12^{\frac{1}{3}}\left(\left(\sqrt{81R^2-12}+9R\right)^2\right)^{\frac{1}{3}}+12^{\frac{2}{3}}+9\left(\sqrt{81R^2-12}+9R\right)^{\frac{1}{3}}R}dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{3\ln(x)}{2} = \int^{yx^{\frac{3}{2}}} -\frac{9\left(\sqrt{81\_a^2-12}+9\_a\right)^{\frac{1}{3}}}{12^{\frac{1}{3}}\left(\left(\sqrt{81\_a^2-12}+9\_a\right)^2\right)^{\frac{1}{3}}+12^{\frac{2}{3}}+9\left(\sqrt{81\_a^2-12}+9\_a\right)^{\frac{1}{3}}\_a}d\_a + c_1$$

Which simplifies to

$$-\frac{3\ln(x)}{2} = \int^{yx^{\frac{3}{2}}} -\frac{9\left(\sqrt{81\_a^2-12}+9\_a\right)^{\frac{1}{3}}}{12^{\frac{1}{3}}\left(\left(\sqrt{81\_a^2-12}+9\_a\right)^2\right)^{\frac{1}{3}}+12^{\frac{2}{3}}+9\left(\sqrt{81\_a^2-12}+9\_a\right)^{\frac{1}{3}}\_a}d\_a + c_1$$

Summary

The solution(s) found are the following

$$-\frac{3\ln(x)}{2} = \int^{yx^{\frac{3}{2}}} -\frac{9\left(\sqrt{81\_a^2-12}+9\_a\right)^{\frac{1}{3}}}{12^{\frac{1}{3}}\left(\left(\sqrt{81\_a^2-12}+9\_a\right)^2\right)^{\frac{1}{3}}+12^{\frac{2}{3}}+9\left(\sqrt{81\_a^2-12}+9\_a\right)^{\frac{1}{3}}\_a}d\_a + c_1 \tag{1}$$

Verification of solutions

$$-\frac{3\ln(x)}{2} = \int^{yx^{\frac{3}{2}}} -\frac{9\left(\sqrt{81\_a^2-12}+9\_a\right)^{\frac{1}{3}}}{12^{\frac{1}{3}}\left(\left(\sqrt{81\_a^2-12}+9\_a\right)^2\right)^{\frac{1}{3}}+12^{\frac{2}{3}}+9\left(\sqrt{81\_a^2-12}+9\_a\right)^{\frac{1}{3}}\_a} d\_a + c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = \frac{i\sqrt{3}\left(\left(108xy+12\sqrt{3}\sqrt{\frac{27x^3y^2-4}{x}}\right)x^2\right)^{\frac{2}{3}} - 12i\sqrt{3}\,x - \left(\left(108xy+12\sqrt{3}\sqrt{\frac{27x^3y^2-4}{x}}\right)x^2\right)^{\frac{2}{3}} - 12x}{12x^3\left(\left(108xy+12\sqrt{3}\sqrt{\frac{27x^3y^2-4}{x}}\right)x^2\right)^{\frac{1}{3}}}$$

$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\text{Expression too large to display} \tag{5E}$$

Putting the above in normal form gives

$$\text{Expression too large to display}$$

Setting the numerator to zero gives

$$\text{Expression too large to display} \tag{6E}$$

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(\left(\sqrt{3}\,\sqrt{\frac{27x^3y^2-4}{x}}+9xy\right)x^2\right)^{\frac{1}{3}}, \left(\left(\sqrt{3}\,\sqrt{\frac{27x^3y^2-4}{x}}+9xy\right)x^2\right)^{\frac{2}{3}}, \sqrt{\frac{27x^3y^2-4}{x}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x=v_1, y=v_2, \left(\left(\sqrt{3}\,\sqrt{\frac{27x^3y^2-4}{x}}+9xy\right)x^2\right)^{\frac{1}{3}}=v_3, \left(\left(\sqrt{3}\,\sqrt{\frac{27x^3y^2-4}{x}}+9xy\right)x^2\right)^{\frac{2}{3}}=v_4, \sqrt{\frac{27x^3\ldots}{}}\right.$$

The above PDE (6E) now becomes

$$\text{Expression too large to display} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&46656\sqrt{3}\,12^{\frac{1}{3}}b_2v_2^2v_3v_1^{10} + 15552\,12^{\frac{1}{3}}b_2v_2v_3v_5v_1^9\\
&- 31104\sqrt{3}\,12^{\frac{1}{3}}a_3v_2^2v_3v_1^5 - 10368\,12^{\frac{1}{3}}a_3v_2v_3v_5v_1^4\\
&+ \Big(-31104i\sqrt{3}\,a_2 - 20736i\sqrt{3}\,b_3\\
&- 31104a_2 - 20736b_3\Big)\,v_2v_5v_1^7\\
&+ \Big(-432i\sqrt{3}\,12^{\frac{2}{3}}b_2 + 432\,12^{\frac{2}{3}}b_2\Big)\,v_4v_5v_1^7\\
&+ \Big(-4320i12^{\frac{2}{3}}a_3 + 1440\sqrt{3}\,12^{\frac{2}{3}}a_3\Big)\,v_2v_4v_1^3\\
&+ \Big(-288i\sqrt{3}\,12^{\frac{2}{3}}a_3 + 288\,12^{\frac{2}{3}}a_3\Big)\,v_4v_5v_1^2\\
&+ \Big(23328i12^{\frac{2}{3}}a_3 - 7776\sqrt{3}\,12^{\frac{2}{3}}a_3\Big)\,v_2^3v_4v_1^6\\
&+ \Big(23328i12^{\frac{2}{3}}a_1 - 7776\sqrt{3}\,12^{\frac{2}{3}}a_1\Big)\,v_2^2v_4v_1^6\\
&+ \Big(-46656i\sqrt{3}\,a_1 - 46656a_1\Big)\,v_2v_5v_1^6\\
&+ \Big(-432i\sqrt{3}\,12^{\frac{2}{3}}b_1 + 432\,12^{\frac{2}{3}}b_1\Big)\,v_4v_5v_1^6\\
&+ \Big(-3888i12^{\frac{2}{3}}b_2 + 1296\sqrt{3}\,12^{\frac{2}{3}}b_2\Big)\,v_2v_4v_1^8\\
&+ \Big(11664i12^{\frac{2}{3}}a_2 + 7776i12^{\frac{2}{3}}b_3\\
&- 3888\sqrt{3}\,12^{\frac{2}{3}}a_2 - 2592\sqrt{3}\,12^{\frac{2}{3}}b_3\Big)\,v_2^2v_4v_1^7\\
&+ \Big(-3888i12^{\frac{2}{3}}b_1 + 1296\sqrt{3}\,12^{\frac{2}{3}}b_1\Big)\,v_2v_4v_1^7\\
&+ \Big(-4320i12^{\frac{2}{3}}a_1 + 1440\sqrt{3}\,12^{\frac{2}{3}}a_1\Big)\,v_4v_1^3\\
&+ \Big(-3456i\sqrt{3}\,a_3 - 3456a_3\Big)\,v_5v_1^3\\
&+ \Big(-419904ia_1 - 139968\sqrt{3}\,a_1\Big)\,v_2^2v_1^7\\
&+ \Big(-5184i\sqrt{3}\,b_1 - 5184b_1\Big)\,v_5v_1^7\\
&+ \Big(-10368ia_3 - 3456\sqrt{3}\,a_3\Big)\,v_2v_1^4 + \Big(-2592i12^{\frac{2}{3}}a_2\\
&- 1728i12^{\frac{2}{3}}b_3 + 864\sqrt{3}\,12^{\frac{2}{3}}a_2 + 576\sqrt{3}\,12^{\frac{2}{3}}b_3\Big)\,v_4v_1^4\\
&+ \Big(-46656ib_2 - 15552\sqrt{3}\,b_2\Big)\,v_2v_1^9 + \Big(-279936ia_2\\
&- 186624ib_3 - 93312\sqrt{3}\,a_2 - 62208\sqrt{3}\,b_3\Big)\,v_2^2v_1^8\\
&+ \Big(-46656ib_1 - 15552\sqrt{3}\,b_1\Big)\,v_2v_1^8\\
&+ \Big(-5184i\sqrt{3}\,b_2 - 5184b_2\Big)\,v_5v_1^8 - 6912\sqrt{3}\,12^{\frac{1}{3}}b_2v_3v_1^7\\
&+ 4608\sqrt{3}\,12^{\frac{1}{3}}a_3v_3v_1^2 + \Big(1296i\sqrt{3}\,12^{\frac{2}{3}}a_2\\
&+ 864i\sqrt{3}\,12^{\frac{2}{3}}b_3 - 1296\,12^{\frac{2}{3}}a_2 - 864\,12^{\frac{2}{3}}b_3\Big)\,v_2v_4v_5v_1^6\\
&+ \Big(2592i\sqrt{3}\,12^{\frac{2}{3}}a_3 - 2592\,12^{\frac{2}{3}}a_3\Big)\,v_2^2v_4v_5v_1^5
\end{aligned}
\qquad (8E)
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-10368\, 12^{\frac{1}{3}} a_3 &= 0 \\
15552\, 12^{\frac{1}{3}} b_2 &= 0 \\
-31104\sqrt{3}\, 12^{\frac{1}{3}} a_3 &= 0 \\
4608\sqrt{3}\, 12^{\frac{1}{3}} a_3 &= 0 \\
-6912\sqrt{3}\, 12^{\frac{1}{3}} b_2 &= 0 \\
46656\sqrt{3}\, 12^{\frac{1}{3}} b_2 &= 0 \\
-419904 i a_1 - 139968\sqrt{3}\, a_1 &= 0 \\
-46656 i b_1 - 15552\sqrt{3}\, b_1 &= 0 \\
-46656 i b_2 - 15552\sqrt{3}\, b_2 &= 0 \\
-10368 i a_3 - 3456\sqrt{3}\, a_3 &= 0 \\
51840 i a_1 + 17280\sqrt{3}\, a_1 &= 0 \\
-46656 i\sqrt{3}\, a_1 - 46656 a_1 &= 0 \\
-5184 i\sqrt{3}\, b_1 - 5184 b_1 &= 0 \\
-5184 i\sqrt{3}\, b_2 - 5184 b_2 &= 0 \\
-4320 i 12^{\frac{2}{3}} a_1 + 1440\sqrt{3}\, 12^{\frac{2}{3}} a_1 &= 0 \\
-4320 i 12^{\frac{2}{3}} a_3 + 1440\sqrt{3}\, 12^{\frac{2}{3}} a_3 &= 0 \\
-3888 i 12^{\frac{2}{3}} b_1 + 1296\sqrt{3}\, 12^{\frac{2}{3}} b_1 &= 0 \\
-3888 i 12^{\frac{2}{3}} b_2 + 1296\sqrt{3}\, 12^{\frac{2}{3}} b_2 &= 0 \\
-3456 i\sqrt{3}\, a_3 - 3456 a_3 &= 0 \\
23328 i 12^{\frac{2}{3}} a_1 - 7776\sqrt{3}\, 12^{\frac{2}{3}} a_1 &= 0 \\
23328 i 12^{\frac{2}{3}} a_3 - 7776\sqrt{3}\, 12^{\frac{2}{3}} a_3 &= 0 \\
-432 i\sqrt{3}\, 12^{\frac{2}{3}} b_1 + 432\, 12^{\frac{2}{3}} b_1 &= 0 \\
-432 i\sqrt{3}\, 12^{\frac{2}{3}} b_2 + 432\, 12^{\frac{2}{3}} b_2 &= 0 \\
-288 i\sqrt{3}\, 12^{\frac{2}{3}} a_3 + 288\, 12^{\frac{2}{3}} a_3 &= 0 \\
2592 i\sqrt{3}\, 12^{\frac{2}{3}} a_1 - 2592\, 12^{\frac{2}{3}} a_1 &= 0 \\
2592 i\sqrt{3}\, 12^{\frac{2}{3}} a_3 - 2592\, 12^{\frac{2}{3}} a_3 &= 0 \\
-279936 i a_2 - 186624 i b_3 - 93312\sqrt{3}\, a_2 - 62208\sqrt{3}\, b_3 &= 0 \\
31104 i a_2 + 20736 i b_3 + 10368\sqrt{3}\, a_2 + 6912\sqrt{3}\, b_3 &= 0 \\
-31104 i\sqrt{3}\, a_2 - 20736 i\sqrt{3}\, b_3 - 31104 a_2 - 20736 b_3 &= 0 \\
-2592 i 12^{\frac{2}{3}} a_2 - 1728 i 12^{\frac{2}{3}} b_3 + 864\sqrt{3}\, 12^{\frac{2}{3}} a_2 + 576\sqrt{3}\, 12^{\frac{2}{3}} b_3 &= 0 \\
11664 i 12^{\frac{2}{3}} a_2 + 7776 i 12^{\frac{2}{3}} b_3 - 3888\sqrt{3}\, 12^{\frac{2}{3}} a_2 - 2592\sqrt{3}\, 12^{\frac{2}{3}} b_3 &= 0 \\
1296 i\sqrt{3}\, 12^{\frac{2}{3}} a_2 + 864 i\sqrt{3}\, 12^{\frac{2}{3}} b_3 - 1296\, 12^{\frac{2}{3}} a_2 - 864\, 12^{\frac{2}{3}} b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= -\frac{2b_3}{3} \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= -\frac{2x}{3} \\
\eta &= y
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Solving equation (3)

Writing the ode as

$$y' = -\frac{i\sqrt{3}\left(\left(108xy + 12\sqrt{3}\sqrt{\frac{27x^3y^2-4}{x}}\right)x^2\right)^{\frac{2}{3}} - 12i\sqrt{3}\,x + \left(\left(108xy + 12\sqrt{3}\sqrt{\frac{27x^3y^2-4}{x}}\right)x^2\right)^{\frac{2}{3}} + 12x}{12x^3\left(\left(108xy + 12\sqrt{3}\sqrt{\frac{27x^3y^2-4}{x}}\right)x^2\right)^{\frac{1}{3}}}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\text{Expression too large to display} \tag{5E}$$

Putting the above in normal form gives

$$\text{Expression too large to display}$$

Setting the numerator to zero gives

$$\text{Expression too large to display} \tag{6E}$$

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(\left(\sqrt{3}\,\sqrt{\frac{27x^3y^2-4}{x}}+9xy\right)x^2\right)^{\frac{1}{3}}, \left(\left(\sqrt{3}\,\sqrt{\frac{27x^3y^2-4}{x}}+9xy\right)x^2\right)^{\frac{2}{3}}, \sqrt{\frac{27x^3y^2-4}{x}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x=v_1, y=v_2, \left(\left(\sqrt{3}\,\sqrt{\frac{27x^3y^2-4}{x}}+9xy\right)x^2\right)^{\frac{1}{3}}=v_3, \left(\left(\sqrt{3}\,\sqrt{\frac{27x^3y^2-4}{x}}+9xy\right)x^2\right)^{\frac{2}{3}}=v_4, \sqrt{\frac{27x^3}{}}\right.$$

The above PDE (6E) now becomes

$$\text{Expression too large to display} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&\left(419904ia_1 - 139968\sqrt{3}\,a_1\right) v_2^2 v_1^7\\
&+ \left(5184i\sqrt{3}\,b_1 - 5184b_1\right) v_5 v_1^7\\
&+ \left(46656ib_2 - 15552\sqrt{3}\,b_2\right) v_2 v_1^9 + \left(279936ia_2\right.\\
&\left.+ 186624ib_3 - 93312\sqrt{3}\,a_2 - 62208\sqrt{3}\,b_3\right) v_2^2 v_1^8\\
&+ \left(46656ib_1 - 15552\sqrt{3}\,b_1\right) v_2 v_1^8\\
&+ \left(5184i\sqrt{3}\,b_2 - 5184b_2\right) v_5 v_1^8\\
&+ \left(10368ia_3 - 3456\sqrt{3}\,a_3\right) v_2 v_1^4\\
&+ 46656\sqrt{3}\,12^{\frac{1}{3}} b_2 v_2^2 v_3 v_1^{10} + 15552\,12^{\frac{1}{3}} b_2 v_2 v_3 v_5 v_1^9\\
&- 31104\sqrt{3}\,12^{\frac{1}{3}} a_3 v_2^2 v_3 v_1^5 - 10368\,12^{\frac{1}{3}} a_3 v_2 v_3 v_5 v_1^4\\
&+ \left(4320i12^{\frac{2}{3}} a_3 + 1440\sqrt{3}\,12^{\frac{2}{3}} a_3\right) v_2 v_4 v_1^3\\
&+ \left(288i\sqrt{3}\,12^{\frac{2}{3}} a_3 + 288\,12^{\frac{2}{3}} a_3\right) v_4 v_5 v_1^2\\
&+ \left(-23328i12^{\frac{2}{3}} a_3 - 7776\sqrt{3}\,12^{\frac{2}{3}} a_3\right) v_2^3 v_4 v_1^6\\
&+ \left(-23328i12^{\frac{2}{3}} a_1 - 7776\sqrt{3}\,12^{\frac{2}{3}} a_1\right) v_2^2 v_4 v_1^6\\
&+ \left(46656i\sqrt{3}\,a_1 - 46656a_1\right) v_2 v_5 v_1^6\\
&+ \left(432i\sqrt{3}\,12^{\frac{2}{3}} b_1 + 432\,12^{\frac{2}{3}} b_1\right) v_4 v_5 v_1^6\\
&+ \left(3888i12^{\frac{2}{3}} b_2 + 1296\sqrt{3}\,12^{\frac{2}{3}} b_2\right) v_2 v_4 v_1^8\\
&+ \left(-11664i12^{\frac{2}{3}} a_2 - 7776i12^{\frac{2}{3}} b_3\right.\\
&\left.- 3888\sqrt{3}\,12^{\frac{2}{3}} a_2 - 2592\sqrt{3}\,12^{\frac{2}{3}} b_3\right) v_2^2 v_4 v_1^7\\
&+ \left(3888i12^{\frac{2}{3}} b_1 + 1296\sqrt{3}\,12^{\frac{2}{3}} b_1\right) v_2 v_4 v_1^7 + \left(31104i\sqrt{3}\,a_2\right.\\
&\left.+ 20736i\sqrt{3}\,b_3 - 31104a_2 - 20736b_3\right) v_2 v_5 v_1^7\\
&+ \left(432i\sqrt{3}\,12^{\frac{2}{3}} b_2 + 432\,12^{\frac{2}{3}} b_2\right) v_4 v_5 v_1^7 + \left(2592i12^{\frac{2}{3}} a_2\right.\\
&\left.+ 1728i12^{\frac{2}{3}} b_3 + 864\sqrt{3}\,12^{\frac{2}{3}} a_2 + 576\sqrt{3}\,12^{\frac{2}{3}} b_3\right) v_4 v_1^4\\
&+ \left(4320i12^{\frac{2}{3}} a_1 + 1440\sqrt{3}\,12^{\frac{2}{3}} a_1\right) v_4 v_1^3\\
&+ \left(3456i\sqrt{3}\,a_3 - 3456a_3\right) v_5 v_1^3 + \left(-31104ia_2\right.\\
&\left.- 20736ib_3 + 10368\sqrt{3}\,a_2 + 6912\sqrt{3}\,b_3\right) v_1^5\\
&+ \left(-51840ia_1 + 17280\sqrt{3}\,a_1\right) v_1^4 + \left(-1296i\sqrt{3}\,12^{\frac{2}{3}} a_2\right.\\
&\left.- 864i\sqrt{3}\,12^{\frac{2}{3}} b_3 - 1296\,12^{\frac{2}{3}} a_2 - 864\,12^{\frac{2}{3}} b_3\right) v_2 v_4 v_5 v_1^6\\
&+ \left(-2592i\sqrt{3}\,12^{\frac{2}{3}} a_3 - 2592\,12^{\frac{2}{3}} a_3\right) v_2^2 v_4 v_5 v_1^5
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$-10368\, 12^{\frac{1}{3}} a_3 = 0$$
$$15552\, 12^{\frac{1}{3}} b_2 = 0$$
$$-31104\sqrt{3}\, 12^{\frac{1}{3}} a_3 = 0$$
$$4608\sqrt{3}\, 12^{\frac{1}{3}} a_3 = 0$$
$$-6912\sqrt{3}\, 12^{\frac{1}{3}} b_2 = 0$$
$$46656\sqrt{3}\, 12^{\frac{1}{3}} b_2 = 0$$
$$-51840 i a_1 + 17280\sqrt{3}\, a_1 = 0$$
$$10368 i a_3 - 3456\sqrt{3}\, a_3 = 0$$
$$46656 i b_1 - 15552\sqrt{3}\, b_1 = 0$$
$$46656 i b_2 - 15552\sqrt{3}\, b_2 = 0$$
$$419904 i a_1 - 139968\sqrt{3}\, a_1 = 0$$
$$-23328 i 12^{\frac{2}{3}} a_1 - 7776\sqrt{3}\, 12^{\frac{2}{3}} a_1 = 0$$
$$-23328 i 12^{\frac{2}{3}} a_3 - 7776\sqrt{3}\, 12^{\frac{2}{3}} a_3 = 0$$
$$3456 i\sqrt{3}\, a_3 - 3456 a_3 = 0$$
$$3888 i 12^{\frac{2}{3}} b_1 + 1296\sqrt{3}\, 12^{\frac{2}{3}} b_1 = 0$$
$$3888 i 12^{\frac{2}{3}} b_2 + 1296\sqrt{3}\, 12^{\frac{2}{3}} b_2 = 0$$
$$4320 i 12^{\frac{2}{3}} a_1 + 1440\sqrt{3}\, 12^{\frac{2}{3}} a_1 = 0$$
$$4320 i 12^{\frac{2}{3}} a_3 + 1440\sqrt{3}\, 12^{\frac{2}{3}} a_3 = 0$$
$$5184 i\sqrt{3}\, b_1 - 5184 b_1 = 0$$
$$5184 i\sqrt{3}\, b_2 - 5184 b_2 = 0$$
$$46656 i\sqrt{3}\, a_1 - 46656 a_1 = 0$$
$$-2592 i\sqrt{3}\, 12^{\frac{2}{3}} a_1 - 2592\, 12^{\frac{2}{3}} a_1 = 0$$
$$-2592 i\sqrt{3}\, 12^{\frac{2}{3}} a_3 - 2592\, 12^{\frac{2}{3}} a_3 = 0$$
$$288 i\sqrt{3}\, 12^{\frac{2}{3}} a_3 + 288\, 12^{\frac{2}{3}} a_3 = 0$$
$$432 i\sqrt{3}\, 12^{\frac{2}{3}} b_1 + 432\, 12^{\frac{2}{3}} b_1 = 0$$
$$432 i\sqrt{3}\, 12^{\frac{2}{3}} b_2 + 432\, 12^{\frac{2}{3}} b_2 = 0$$
$$-31104 i a_2 - 20736 i b_3 + 10368\sqrt{3}\, a_2 + 6912\sqrt{3}\, b_3 = 0$$
$$279936 i a_2 + 186624 i b_3 - 93312\sqrt{3}\, a_2 - 62208\sqrt{3}\, b_3 = 0$$
$$-11664 i 12^{\frac{2}{3}} a_2 - 7776 i 12^{\frac{2}{3}} b_3 - 3888\sqrt{3}\, 12^{\frac{2}{3}} a_2 - 2592\sqrt{3}\, 12^{\frac{2}{3}} b_3 = 0$$
$$2592 i 12^{\frac{2}{3}} a_2 + 1728 i 12^{\frac{2}{3}} b_3 + 864\sqrt{3}\, 12^{\frac{2}{3}} a_2 + 576\sqrt{3}\, 12^{\frac{2}{3}} b_3 = 0$$
$$31104 i\sqrt{3}\, a_2 + 20736 i\sqrt{3}\, b_3 - 31104 a_2 - 20736 b_3 = 0$$
$$-1296 i\sqrt{3}\, 12^{\frac{2}{3}} a_2 - 864 i\sqrt{3}\, 12^{\frac{2}{3}} b_3 - 1296\, 12^{\frac{2}{3}} a_2 - 864\, 12^{\frac{2}{3}} b_3 = 0$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -\frac{2b_3}{3} \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -\frac{2x}{3} \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 3 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
   -> Solving 1st order ODE of high degree, Lie methods, 1st trial
   `, `-> Computing symmetries using: way = 2
   `, `-> Computing symmetries using: way = 2
   -> Solving 1st order ODE of high degree, 2nd attempt. Trying parametric methods
   -> Calling odsolve with the ODE`, diff(y(x), x) = ((x^2*y(x)^5-1)*y(x)+2*y(x)^6*x^2)/(-6*
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   <- 1st order, parametric methods successful`
```

✓ Solution by Maple

Time used: 0.187 (sec). Leaf size: 36

```
dsolve(y(x)=x^6*diff(y(x),x)^3-x*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = -\frac{2\sqrt{3}}{9x^{\frac{3}{2}}}$$
$$y(x) = \frac{2\sqrt{3}}{9x^{\frac{3}{2}}}$$
$$y(x) = c_1^3 - \frac{c_1}{x}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]==x^6*(y'[x])^3-x*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

Timed out

3.14 problem 16

Internal problem ID [6808]
Internal file name [`OUTPUT/6055_Tuesday_July_26_2022_11_24_10_PM_54110075/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 99. Clairaut's equation. EXERCISES Page 320
Problem number: 16.
ODE order: 1.
ODE degree: 4.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{xy'^4 - 2yy'^3 = -12x^3}$$

Solving the given ode for y' results in 1 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \text{RootOf}\left(x\_Z^4 - 2y\_Z^3 + 12x^3\right) \tag{1}$$

Now each one of the above ODE is solved.

Solving equation (1)

Unable to determine ODE type.

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   trying simple symmetries for implicit equations
   --- Trying classification methods ---
   trying homogeneous types:
   trying exact
   Looking for potential symmetries
   trying an equivalence to an Abel ODE
   trying 1st order ODE linearizable_by_differentiation
   -> Solving 1st order ODE of high degree, Lie methods, 1st trial
   `, `-> Computing symmetries using: way = 2
   `, `-> Computing symmetries using: way = 2
   -> Solving 1st order ODE of high degree, 2nd attempt. Trying parametric methods
   -> Calling odsolve with the ODE`, diff(y(x), x) = ((18*y(x)+(x^6+324*y(x)^2)^(1/2))^(2/3)
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
   -> Calling odsolve with the ODE`, diff(y(x), x) = (3*(x^4+12*y(x)^2)*y(x)/x-4*y(x)*x^3)/(
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   <- 1st order, parametric methods successful`
```

✓ Solution by Maple

Time used: 0.203 (sec). Leaf size: 66

```
dsolve(x*diff(y(x),x)^4-2*y(x)*diff(y(x),x)^3+12*x^3=0,y(x), singsol=all)
```

$$y(x) = \frac{2\sqrt{6}\,(-x)^{\frac{3}{2}}}{3}$$
$$y(x) = -\frac{2\sqrt{6}\,(-x)^{\frac{3}{2}}}{3}$$
$$y(x) = -\frac{2\sqrt{6}\,x^{\frac{3}{2}}}{3}$$
$$y(x) = \frac{2\sqrt{6}\,x^{\frac{3}{2}}}{3}$$
$$y(x) = \frac{12c_1^4 + x^2}{2c_1}$$

✓ Solution by Mathematica

Time used: 37.824 (sec). Leaf size: 30947

```
DSolve[x*(y'[x])^4-2*y[x]*(y'[x])^3+12*x^3==0,y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

3.15 problem 17

Internal problem ID [6809]
Internal file name [OUTPUT/6056_Tuesday_July_26_2022_11_39_14_PM_76761105/index.tex]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 99. Clairaut's equation. EXERCISES Page 320
Problem number: 17.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _Clairaut]`

$$\boxed{xy'^3 - yy'^2 = -1}$$

3.15.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$x\,p^3 - y\,p^2 = -1$$

Solving for y from the above results in

$$y = \frac{x\,p^3 + 1}{p^2} \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= px + \frac{1}{p^2} \\ &= px + \frac{1}{p^2} \end{aligned}$$

Writing the ode as

$$y = px + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = px + g \tag{1}$$

Then we see that

$$g = \frac{1}{p^2}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1 x + \frac{1}{c_1^2}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = \frac{1}{p^2}$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x - \frac{2}{p^3} \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$
\begin{aligned}
p_1 &= \frac{2^{\frac{1}{3}} \left(x^2\right)^{\frac{1}{3}}}{x} \\
p_2 &= -\frac{2^{\frac{1}{3}} \left(x^2\right)^{\frac{1}{3}}}{2x} + \frac{i\sqrt{3}\, 2^{\frac{1}{3}} \left(x^2\right)^{\frac{1}{3}}}{2x} \\
p_3 &= -\frac{2^{\frac{1}{3}} \left(x^2\right)^{\frac{1}{3}}}{2x} - \frac{i\sqrt{3}\, 2^{\frac{1}{3}} \left(x^2\right)^{\frac{1}{3}}}{2x}
\end{aligned}
$$

Substituting the above back in (1) results in

$$
\begin{aligned}
y_1 &= \frac{3x^2 2^{\frac{1}{3}}}{2 \left(x^2\right)^{\frac{2}{3}}} \\
y_2 &= -\frac{3x^2 2^{\frac{1}{3}}}{\left(x^2\right)^{\frac{2}{3}} \left(1 + i\sqrt{3}\right)} \\
y_3 &= \frac{3x^2 2^{\frac{1}{3}}}{\left(x^2\right)^{\frac{2}{3}} \left(i\sqrt{3} - 1\right)}
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = c_1 x + \frac{1}{c_1^2} \tag{1}
$$

$$
y = \frac{3x^2 2^{\frac{1}{3}}}{2 \left(x^2\right)^{\frac{2}{3}}} \tag{2}
$$

$$
y = -\frac{3x^2 2^{\frac{1}{3}}}{\left(x^2\right)^{\frac{2}{3}} \left(1 + i\sqrt{3}\right)} \tag{3}
$$

$$
y = \frac{3x^2 2^{\frac{1}{3}}}{\left(x^2\right)^{\frac{2}{3}} \left(i\sqrt{3} - 1\right)} \tag{4}
$$

Verification of solutions

$$y = c_1 x + \frac{1}{c_1^2}$$

Verified OK.

$$y = \frac{3x^2 2^{\frac{1}{3}}}{2 (x^2)^{\frac{2}{3}}}$$

Verified OK.

$$y = -\frac{3x^2 2^{\frac{1}{3}}}{(x^2)^{\frac{2}{3}} \left(1 + i\sqrt{3}\right)}$$

Verified OK.

$$y = \frac{3x^2 2^{\frac{1}{3}}}{(x^2)^{\frac{2}{3}} \left(i\sqrt{3} - 1\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.11 (sec). Leaf size: 66

```
dsolve(x*diff(y(x),x)^3-y(x)*diff(y(x),x)^2+1=0,y(x), singsol=all)
```

$$y(x) = \frac{3\,2^{\frac{1}{3}}(x^2)^{\frac{1}{3}}}{2}$$
$$y(x) = -\frac{3\,2^{\frac{1}{3}}(x^2)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)}{4}$$
$$y(x) = \frac{3\,2^{\frac{1}{3}}(x^2)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)}{4}$$
$$y(x) = c_1x + \frac{1}{c_1^2}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 69

```
DSolve[x*(y'[x])^3-y[x]*(y'[x])^2+1==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1x + \frac{1}{c_1{}^2}$$
$$y(x) \to 3\left(-\frac{1}{2}\right)^{2/3}x^{2/3}$$
$$y(x) \to \frac{3x^{2/3}}{2^{2/3}}$$
$$y(x) \to -\frac{3\sqrt[3]{-1}x^{2/3}}{2^{2/3}}$$

3.16 problem 19

Internal problem ID [6810]
Internal file name [`OUTPUT/6057_Tuesday_July_26_2022_11_39_17_PM_13635991/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 99. Clairaut's equation. EXERCISES Page 320
Problem number: 19.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _dAlembert]`

$$\boxed{y'^2 - xy' - y = 0}$$

3.16.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$p^2 - xp - y = 0$$

Solving for y from the above results in

$$y = p^2 - xp \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= -p \\
g &= p^2
\end{aligned}
$$

Hence (2) becomes

$$
2p = (-x + 2p)\, p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
2p = 0
$$

Solving for p from the above gives

$$
p = 0
$$

Substituting these in (1A) gives

$$
y = 0
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{2p(x)}{-x + 2p\,(x)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp}x(p) = \frac{-x(p) + 2p}{2p} \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
\frac{d}{dp}x(p) + p(p)x(p) = q(p)
$$

Where here

$$
\begin{aligned}
p(p) &= \frac{1}{2p} \\
q(p) &= 1
\end{aligned}
$$

Hence the ode is

$$\frac{d}{dp}x(p)+\frac{x(p)}{2p}=1$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{2p}dp}\\ &= \sqrt{p}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}(\mu x) &= \mu\\ \frac{\mathrm{d}}{\mathrm{d}p}(\sqrt{p}\,x) &= \sqrt{p}\\ \mathrm{d}(\sqrt{p}\,x) &= \sqrt{p}\mathrm{d}p\end{aligned}$$

Integrating gives

$$\begin{aligned}\sqrt{p}\,x &= \int \sqrt{p}\,\mathrm{d}p\\ \sqrt{p}\,x &= \frac{2p^{\frac{3}{2}}}{3}+c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu=\sqrt{p}$ results in

$$x(p)=\frac{2p}{3}+\frac{c_1}{\sqrt{p}}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$\begin{aligned}p &= \frac{x}{2}+\frac{\sqrt{x^2+4y}}{2}\\ p &= \frac{x}{2}-\frac{\sqrt{x^2+4y}}{2}\end{aligned}$$

Substituting the above in the solution for x found above gives

$$\begin{aligned}x &= \frac{x}{3}+\frac{\sqrt{x^2+4y}}{3}+\frac{2c_1}{\sqrt{2x+2\sqrt{x^2+4y}}}\\ x &= \frac{x}{3}-\frac{\sqrt{x^2+4y}}{3}+\frac{2c_1}{\sqrt{2x-2\sqrt{x^2+4y}}}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$

$$x = \frac{x}{3} + \frac{\sqrt{x^2+4y}}{3} + \frac{2c_1}{\sqrt{2x+2\sqrt{x^2+4y}}} \tag{2}$$

$$x = \frac{x}{3} - \frac{\sqrt{x^2+4y}}{3} + \frac{2c_1}{\sqrt{2x-2\sqrt{x^2+4y}}} \tag{3}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$x = \frac{x}{3} + \frac{\sqrt{x^2+4y}}{3} + \frac{2c_1}{\sqrt{2x+2\sqrt{x^2+4y}}}$$

Verified OK.

$$x = \frac{x}{3} - \frac{\sqrt{x^2+4y}}{3} + \frac{2c_1}{\sqrt{2x-2\sqrt{x^2+4y}}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 77

```
dsolve(diff(y(x),x)^2-x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$\frac{c_1}{\sqrt{2x-2\sqrt{x^2+4y(x)}}}+\frac{2x}{3}+\frac{\sqrt{x^2+4y(x)}}{3}=0$$
$$\frac{c_1}{\sqrt{2x+2\sqrt{x^2+4y(x)}}}+\frac{2x}{3}-\frac{\sqrt{x^2+4y(x)}}{3}=0$$

✓ Solution by Mathematica

Time used: 60.129 (sec). Leaf size: 1003

```
DSolve[(y'[x])^2-x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to \frac{\left(x^2+\sqrt[3]{-x^6+20e^{3c_1}x^3+8\sqrt{e^{3c_1}\left(-x^3+e^{3c_1}\right)^3}+8e^{6c_1}}\right)^2+8e^{3c_1}x}{4\sqrt[3]{-x^6+20e^{3c_1}x^3+8\sqrt{e^{3c_1}\left(-x^3+e^{3c_1}\right)^3}+8e^{6c_1}}}$$

$$y(x)\to \frac{1}{8}\left(4x^2-\frac{i\left(\sqrt{3}-i\right)x\left(x^3+8e^{3c_1}\right)}{\sqrt[3]{-x^6+20e^{3c_1}x^3+8\sqrt{e^{3c_1}\left(-x^3+e^{3c_1}\right)^3}+8e^{6c_1}}}+i\left(\sqrt{3}+i\right)\sqrt[3]{-x^6+20e^{3c_1}x^3+8\sqrt{e^{3c_1}\left(-x^3+e^{3c_1}\right)^3}+8e^{6c_1}}\right)$$

$$y(x)\to \frac{1}{8}\left(4x^2+\frac{i\left(\sqrt{3}+i\right)x\left(x^3+8e^{3c_1}\right)}{\sqrt[3]{-x^6+20e^{3c_1}x^3+8\sqrt{e^{3c_1}\left(-x^3+e^{3c_1}\right)^3}+8e^{6c_1}}}-\left(1+i\sqrt{3}\right)\sqrt[3]{-x^6+20e^{3c_1}x^3+8\sqrt{e^{3c_1}\left(-x^3+e^{3c_1}\right)^3}+8e^{6c_1}}\right)$$

$$y(x)\to \frac{2\sqrt[3]{2}x^4+2^{2/3}\left(-2x^6-10e^{3c_1}x^3+\sqrt{e^{3c_1}\left(4x^3+e^{3c_1}\right)^3}+e^{6c_1}\right)^{2/3}+4x^2\sqrt[3]{-2x^6-10e^{3c_1}x^3+\sqrt{e^{3c_1}(4x}}}{8\sqrt[3]{-2x^6-10e^{3c_1}x^3+\sqrt{e^{3c_1}\left(4x^3+e^{3c_1}\right)^3}+e^{6c_1}}}$$

$$y(x)\to \frac{1}{16}\left(8x^2+\frac{2\sqrt[3]{2}\left(1+i\sqrt{3}\right)x\left(-x^3+2e^{3c_1}\right)}{\sqrt[3]{-2x^6-10e^{3c_1}x^3+\sqrt{e^{3c_1}\left(4x^3+e^{3c_1}\right)^3}+e^{6c_1}}}+i2^{2/3}\left(\sqrt{3}+i\right)\sqrt[3]{-2x^6-10e^{3c_1}x^3+\sqrt{e^{3c_1}\left(4x^3+e^{3c_1}\right)^3}+e^{6c_1}}\right)$$

$$y(x)\to \frac{1}{16}\left(8x^2+\frac{2i\sqrt[3]{2}\left(\sqrt{3}+i\right)x\left(x^3-2e^{3c_1}\right)}{\sqrt[3]{-2x^6-10e^{3c_1}x^3+\sqrt{e^{3c_1}\left(4x^3+e^{3c_1}\right)^3}+e^{6c_1}}}-2^{2/3}\left(1+i\sqrt{3}\right)\sqrt[3]{-2x^6-10e^{3c_1}x^3+\sqrt{e^{3c_1}\left(4x^3+e^{3c_1}\right)^3}+e^{6c_1}}\right)$$

3.17 problem 20

Internal problem ID [6811]
Internal file name [`OUTPUT/6058_Thursday_July_28_2022_04_28_20_AM_9550685/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 99. Clairaut's equation. EXERCISES Page 320
Problem number: 20.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _dAlembert]`

$$\boxed{2{y'}^3 + xy' - 2y = 0}$$

3.17.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$2p^3 + xp - 2y = 0$$

Solving for y from the above results in

$$y = p^3 + \frac{1}{2}xp \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$

$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = \frac{p}{2}$$
$$g = p^3$$

Hence (2) becomes

$$\frac{p}{2} = \left(\frac{x}{2} + 3p^2\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$\frac{p}{2} = 0$$

Solving for p from the above gives

$$p = 0$$

Substituting these in (1A) gives

$$y = 0$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x)}{x + 6p(x)^2} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{x(p) + 6p^2}{p} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = -\frac{1}{p}$$
$$q(p) = 6p$$

Hence the ode is

$$\frac{d}{dp}x(p) - \frac{x(p)}{p} = 6p$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{p}dp} \\ &= \frac{1}{p}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}(\mu x) &= (\mu)(6p) \\ \frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{x}{p}\right) &= \left(\frac{1}{p}\right)(6p) \\ \mathrm{d}\left(\frac{x}{p}\right) &= 6\,\mathrm{d}p\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{x}{p} &= \int 6\,\mathrm{d}p \\ \frac{x}{p} &= 6p + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{p}$ results in

$$x(p) = c_1 p + 6p^2$$

which simplifies to

$$x(p) = p(c_1 + 6p)$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = \frac{\left(108y + 6\sqrt{6x^3 + 324y^2}\right)^{\frac{1}{3}}}{6} - \frac{x}{\left(108y + 6\sqrt{6x^3 + 324y^2}\right)^{\frac{1}{3}}}$$

$$p = -\frac{\left(108y + 6\sqrt{6x^3 + 324y^2}\right)^{\frac{1}{3}}}{12} + \frac{x}{2\left(108y + 6\sqrt{6x^3 + 324y^2}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(108y+6\sqrt{6x^3+324y^2}\right)^{\frac{1}{3}}}{6} + \frac{x}{\left(108y+6\sqrt{6x^3+324y^2}\right)^{\frac{1}{3}}}\right)}{2}$$

$$p = -\frac{\left(108y + 6\sqrt{6x^3 + 324y^2}\right)^{\frac{1}{3}}}{12} + \frac{x}{2\left(108y + 6\sqrt{6x^3 + 324y^2}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(108y+6\sqrt{6x^3+324y^2}\right)^{\frac{1}{3}}}{6} + \frac{x}{\left(108y+6\sqrt{6x^3+324y^2}\right)^{\frac{1}{3}}}\right)}{2}$$

Substituting the above in the solution for x found above gives

$$x = \frac{\left(\left(108y+6\sqrt{6x^3+324y^2}\right)^{\frac{2}{3}}-6x\right)\left(\left(108y+6\sqrt{6x^3+324y^2}\right)^{\frac{2}{3}}+c_1\left(108y+6\sqrt{6x^3+324y^2}\right)^{\frac{1}{3}}-6x\right)}{6\left(108y+6\sqrt{6x^3+324y^2}\right)^{\frac{2}{3}}}$$

$$x = \frac{\left(i\left(108y+6\sqrt{6x^3+324y^2}\right)^{\frac{2}{3}}\sqrt{3}+6i\sqrt{3}\,x-\left(108y+6\sqrt{6x^3+324y^2}\right)^{\frac{2}{3}}+6x\right)\left(i\left(108y+6\sqrt{6x^3+324y\right.\right.}{24\left(108y+6\sqrt{6x^3+32\right.}$$

$$x = \frac{\left(i\left(108y+6\sqrt{6x^3+324y^2}\right)^{\frac{2}{3}}\sqrt{3}+6i\sqrt{3}\,x+\left(108y+6\sqrt{6x^3+324y^2}\right)^{\frac{2}{3}}-6x\right)\left(i\left(108y+6\sqrt{6x^3+324y\right.\right.}{24\left(108y+6\sqrt{6x^3+32\right.}$$

<u>Summary</u>

The solution(s) found are the following

$$y = 0 \tag{1}$$

$$x = \frac{\left(\left(108y+6\sqrt{6x^3+324y^2}\right)^{\frac{2}{3}}-6x\right)\left(\left(108y+6\sqrt{6x^3+324y^2}\right)^{\frac{2}{3}}+c_1\left(108y+6\sqrt{6x^3+324y^2}\right)^{\frac{1}{3}}-6x\right)}{6\left(108y+6\sqrt{6x^3+324y^2}\right)^{\frac{2}{3}}} \tag{2}$$

$$x = \frac{\left(i\left(108y+6\sqrt{6x^3+324y^2}\right)^{\frac{2}{3}}\sqrt{3}+6i\sqrt{3}\,x-\left(108y+6\sqrt{6x^3+324y^2}\right)^{\frac{2}{3}}+6x\right)\left(i\left(108y+6\sqrt{6x^3+324y\right.\right.}{24\left(108y+6\sqrt{6x^3+32\right.} \tag{3}$$

$$x = \frac{\left(i\left(108y+6\sqrt{6x^3+324y^2}\right)^{\frac{2}{3}}\sqrt{3}+6i\sqrt{3}\,x+\left(108y+6\sqrt{6x^3+324y^2}\right)^{\frac{2}{3}}-6x\right)\left(i\left(108y+6\sqrt{6x^3+324y\right.\right.}{24\left(108y+6\sqrt{6x^3+32\right.} \tag{4}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$x = \frac{\left(\left(108y+6\sqrt{6x^3+324y^2}\right)^{\frac{2}{3}}-6x\right)\left(\left(108y+6\sqrt{6x^3+324y^2}\right)^{\frac{2}{3}}+c_1\left(108y+6\sqrt{6x^3+324y^2}\right)^{\frac{1}{3}}-6x\right)}{6\left(108y+6\sqrt{6x^3+324y^2}\right)^{\frac{2}{3}}}$$

Verified OK.

$$x = \frac{\left(i\left(108y+6\sqrt{6x^3+324y^2}\right)^{\frac{2}{3}}\sqrt{3}+6i\sqrt{3}\,x-\left(108y+6\sqrt{6x^3+324y^2}\right)^{\frac{2}{3}}+6x\right)\left(i\left(108y+6\sqrt{6x^3+324y\right.\right.}{24\left(108y+6\sqrt{6x^3+32\right.}$$

Warning, solution could not be verified

$$x = \frac{\left(i\left(108y+6\sqrt{6x^3+324y^2}\right)^{\frac{2}{3}}\sqrt{3}+6i\sqrt{3}\,x+\left(108y+6\sqrt{6x^3+324y^2}\right)^{\frac{2}{3}}-6x\right)\left(i\left(108y+6\sqrt{6x^3+324y\right.\right.}{24\left(108y+6\sqrt{6x^3+32\right.}$$

Warning, solution could not be verified

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 58

```
dsolve(2*diff(y(x),x)^3+x*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{(-c_1^2 - 24x)\sqrt{c_1^2 + 24x}}{432} - \frac{c_1^3}{432} - \frac{c_1 x}{12}$$
$$y(x) = \frac{(c_1^2 + 24x)^{\frac{3}{2}}}{432} - \frac{c_1^3}{432} - \frac{c_1 x}{12}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[2*(y'[x])^3+x*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Timed out

3.18 problem 21

Internal problem ID [6812]
Internal file name [`OUTPUT/6059_Thursday_July_28_2022_04_28_33_AM_759705/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 99. Clairaut's equation. EXERCISES Page 320
Problem number: 21.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _dAlembert]`

$$\boxed{2y'^2 + xy' - 2y = 0}$$

3.18.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$2p^2 + xp - 2y = 0$$

Solving for y from the above results in

$$y = p^2 + \frac{1}{2}xp \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= \frac{p}{2} \\
g &= p^2
\end{aligned}
$$

Hence (2) becomes

$$
\frac{p}{2} = \left(\frac{x}{2} + 2p\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
\frac{p}{2} = 0
$$

Solving for p from the above gives

$$
p = 0
$$

Substituting these in (1A) gives

$$
y = 0
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x)}{x + 4p(x)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp}x(p) = \frac{x(p) + 4p}{p} \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
\frac{d}{dp}x(p) + p(p)x(p) = q(p)
$$

Where here

$$
\begin{aligned}
p(p) &= -\frac{1}{p} \\
q(p) &= 4
\end{aligned}
$$

Hence the ode is

$$\frac{d}{dp}x(p) - \frac{x(p)}{p} = 4$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{p}dp} \\ &= \frac{1}{p}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}(\mu x) &= (\mu)(4) \\ \frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{x}{p}\right) &= \left(\frac{1}{p}\right)(4) \\ \mathrm{d}\left(\frac{x}{p}\right) &= \left(\frac{4}{p}\right)\mathrm{d}p\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{x}{p} &= \int \frac{4}{p}\,\mathrm{d}p \\ \frac{x}{p} &= 4\ln(p) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{p}$ results in

$$x(p) = 4p\ln(p) + c_1 p$$

which simplifies to

$$x(p) = p(4\ln(p) + c_1)$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$\begin{aligned}p &= -\frac{x}{4} + \frac{\sqrt{x^2+16y}}{4} \\ p &= -\frac{x}{4} - \frac{\sqrt{x^2+16y}}{4}\end{aligned}$$

Substituting the above in the solution for x found above gives

$$\begin{aligned}x &= -\frac{\left(x-\sqrt{x^2+16y}\right)\left(-8\ln(2)+4\ln\left(-x+\sqrt{x^2+16y}\right)+c_1\right)}{4} \\ x &= -\frac{\left(x+\sqrt{x^2+16y}\right)\left(-8\ln(2)+4\ln\left(-x-\sqrt{x^2+16y}\right)+c_1\right)}{4}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = 0 \tag{1}$$

$$x = -\frac{\left(x-\sqrt{x^2+16y}\right)\left(-8\ln\left(2\right)+4\ln\left(-x+\sqrt{x^2+16y}\right)+c_1\right)}{4} \tag{2}$$

$$x = -\frac{\left(x+\sqrt{x^2+16y}\right)\left(-8\ln\left(2\right)+4\ln\left(-x-\sqrt{x^2+16y}\right)+c_1\right)}{4} \tag{3}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$x = -\frac{\left(x-\sqrt{x^2+16y}\right)\left(-8\ln\left(2\right)+4\ln\left(-x+\sqrt{x^2+16y}\right)+c_1\right)}{4}$$

Verified OK.

$$x = -\frac{\left(x+\sqrt{x^2+16y}\right)\left(-8\ln\left(2\right)+4\ln\left(-x-\sqrt{x^2+16y}\right)+c_1\right)}{4}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 31

```
dsolve(2*diff(y(x),x)^2+x*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x^2\left(1 + 2\,\mathrm{LambertW}\left(\frac{x\,\mathrm{e}^{\frac{c_1}{4}}}{4}\right)\right)}{16\,\mathrm{LambertW}\left(\frac{x\,\mathrm{e}^{\frac{c_1}{4}}}{4}\right)^2}$$

✓ Solution by Mathematica

Time used: 1.194 (sec). Leaf size: 130

```
DSolve[2*(y'[x])^2+x*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[-\frac{\frac{1}{2}x\sqrt{x^2+16y(x)} - 8y(x)\log\left(\sqrt{x^2+16y(x)} - x\right) + \frac{x^2}{2}}{8y(x)} = c_1, y(x)\right]$$

$$\text{Solve}\left[\frac{\frac{1}{2}x\sqrt{x^2+16y(x)} - 8y(x)\log\left(\sqrt{x^2+16y(x)} - x\right) - \frac{x^2}{2}}{8y(x)} + \log(y(x)) = c_1, y(x)\right]$$

$$y(x) \to 0$$

3.19 problem 22

Internal problem ID [6813]
Internal file name [`OUTPUT/6060_Thursday_July_28_2022_04_28_35_AM_13746180/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 99. Clairaut's equation. EXERCISES Page 320
Problem number: 22.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _dAlembert]`

$$\boxed{y'^3 + 2xy' - y = 0}$$

3.19.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$p^3 + 2xp - y = 0$$

Solving for y from the above results in

$$y = p^3 + 2xp \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= 2p \\
g &= p^3
\end{aligned}
$$

Hence (2) becomes

$$
-p = (3p^2 + 2x)\, p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
-p = 0
$$

Solving for p from the above gives

$$
p = 0
$$

Substituting these in (1A) gives

$$
y = 0
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = -\frac{p(x)}{3p\,(x)^2 + 2x} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp}x(p) = -\frac{3p^2 + 2x(p)}{p} \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
\frac{d}{dp}x(p) + p(p)x(p) = q(p)
$$

Where here

$$
\begin{aligned}
p(p) &= \frac{2}{p} \\
q(p) &= -3p
\end{aligned}
$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{2x(p)}{p} = -3p$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{p}dp} \\ &= p^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}(\mu x) &= (\mu)(-3p) \\ \frac{\mathrm{d}}{\mathrm{d}p}(p^2 x) &= (p^2)(-3p) \\ \mathrm{d}(p^2 x) &= (-3p^3)\ \mathrm{d}p\end{aligned}$$

Integrating gives

$$\begin{aligned}p^2 x &= \int -3p^3\,\mathrm{d}p \\ p^2 x &= -\frac{3p^4}{4} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = p^2$ results in

$$x(p) = -\frac{3p^2}{4} + \frac{c_1}{p^2}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = \frac{\left(108y + 12\sqrt{96x^3 + 81y^2}\right)^{\frac{1}{3}}}{6} - \frac{4x}{\left(108y + 12\sqrt{96x^3 + 81y^2}\right)^{\frac{1}{3}}}$$

$$p = -\frac{\left(108y + 12\sqrt{96x^3 + 81y^2}\right)^{\frac{1}{3}}}{12} + \frac{2x}{\left(108y + 12\sqrt{96x^3 + 81y^2}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(108y+12\sqrt{96x^3+81y^2}\right)^{\frac{1}{3}}}{6} + \frac{\cdots}{\left(108y+12\cdots\right.}\right.}{2}$$

$$p = -\frac{\left(108y + 12\sqrt{96x^3 + 81y^2}\right)^{\frac{1}{3}}}{12} + \frac{2x}{\left(108y + 12\sqrt{96x^3 + 81y^2}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(108y+12\sqrt{96x^3+81y^2}\right)^{\frac{1}{3}}}{6} + \frac{\cdots}{\left(108y+12\cdots\right.}\right.}{2}$$

Substituting the above in the solution for x found above gives

$$x = -\frac{\left(\left(108y+12\sqrt{96x^{3}+81y^{2}}\right)^{\frac{2}{3}}-24x\right)^{2}}{48\left(108y+12\sqrt{96x^{3}+81y^{2}}\right)^{\frac{2}{3}}}+\frac{36c_{1}\left(108y+12\sqrt{96x^{3}+81y^{2}}\right)^{\frac{2}{3}}}{\left(\left(108y+12\sqrt{96x^{3}+81y^{2}}\right)^{\frac{2}{3}}-24x\right)^{2}}$$

$$x = \frac{3\left(\frac{\left(\sqrt{3}+i\right)\left(108y+12\sqrt{96x^{3}+81y^{2}}\right)^{\frac{2}{3}}}{24}+x\left(-i+\sqrt{3}\right)\right)^{2}}{\left(108y+12\sqrt{96x^{3}+81y^{2}}\right)^{\frac{2}{3}}} + \frac{144c_{1}\left(108y+12\sqrt{96x^{3}+81y^{2}}\right)^{\frac{2}{3}}}{\left(i\sqrt{3}\,\left(108y+12\sqrt{96x^{3}+81y^{2}}\right)^{\frac{2}{3}}+24i\sqrt{3}\,x-\left(108y+12\sqrt{96x^{3}+81y^{2}}\right)^{\frac{2}{3}}+24x\right)^{2}}$$

$$x = \frac{3\left(\frac{\left(108y+12\sqrt{96x^{3}+81y^{2}}\right)^{\frac{2}{3}}\left(-i+\sqrt{3}\right)}{24}+x\left(\sqrt{3}+i\right)\right)^{2}}{\left(108y+12\sqrt{96x^{3}+81y^{2}}\right)^{\frac{2}{3}}} + \frac{144c_{1}\left(108y+12\sqrt{96x^{3}+81y^{2}}\right)^{\frac{2}{3}}}{\left(i\sqrt{3}\,\left(108y+12\sqrt{96x^{3}+81y^{2}}\right)^{\frac{2}{3}}+24i\sqrt{3}\,x+\left(108y+12\sqrt{96x^{3}+81y^{2}}\right)^{\frac{2}{3}}-24x\right)^{2}}$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$

$$x = -\frac{\left(\left(108y+12\sqrt{96x^3+81y^2}\right)^{\frac{2}{3}}-24x\right)^2}{48\left(108y+12\sqrt{96x^3+81y^2}\right)^{\frac{2}{3}}}+\frac{36c_1\left(108y+12\sqrt{96x^3+81y^2}\right)^{\frac{2}{3}}}{\left(\left(108y+12\sqrt{96x^3+81y^2}\right)^{\frac{2}{3}}-24x\right)^2} \tag{2}$$

$$x = \frac{3\left(\frac{\left(\sqrt{3}+i\right)\left(108y+12\sqrt{96x^3+81y^2}\right)^{\frac{2}{3}}}{24}+x\left(-i+\sqrt{3}\right)\right)^2}{\left(108y+12\sqrt{96x^3+81y^2}\right)^{\frac{2}{3}}} + \frac{144c_1\left(108y+12\sqrt{96x^3+81y^2}\right)^{\frac{2}{3}}}{\left(i\sqrt{3}\,\left(108y+12\sqrt{96x^3+81y^2}\right)^{\frac{2}{3}}+24i\sqrt{3}\,x-\left(108y+12\sqrt{96x^3+81y^2}\right)^{\frac{2}{3}}+24x\right)^2} \tag{3}$$

$$x = \frac{3\left(\frac{\left(108y+12\sqrt{96x^3+81y^2}\right)^{\frac{2}{3}}\left(-i+\sqrt{3}\right)}{24}+x\left(\sqrt{3}+i\right)\right)^2}{\left(108y+12\sqrt{96x^3+81y^2}\right)^{\frac{2}{3}}} + \frac{144c_1\left(108y+12\sqrt{96x^3+81y^2}\right)^{\frac{2}{3}}}{\left(i\sqrt{3}\,\left(108y+12\sqrt{96x^3+81y^2}\right)^{\frac{2}{3}}+24i\sqrt{3}\,x+\left(108y+12\sqrt{96x^3+81y^2}\right)^{\frac{2}{3}}-24x\right)^2} \tag{4}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$x = -\frac{\left(\left(108y+12\sqrt{96x^3+81y^2}\right)^{\frac{2}{3}}-24x\right)^2}{48\left(108y+12\sqrt{96x^3+81y^2}\right)^{\frac{2}{3}}} + \frac{36c_1\left(108y+12\sqrt{96x^3+81y^2}\right)^{\frac{2}{3}}}{\left(\left(108y+12\sqrt{96x^3+81y^2}\right)^{\frac{2}{3}}-24x\right)^2}$$

Verified OK.

$$x = \frac{3\left(\frac{\left(\sqrt{3}+i\right)\left(108y+12\sqrt{96x^3+81y^2}\right)^{\frac{2}{3}}}{24}+x\left(-i+\sqrt{3}\right)\right)^2}{\left(108y+12\sqrt{96x^3+81y^2}\right)^{\frac{2}{3}}} + \frac{144c_1\left(108y+12\sqrt{96x^3+81y^2}\right)^{\frac{2}{3}}}{\left(i\sqrt{3}\,\left(108y+12\sqrt{96x^3+81y^2}\right)^{\frac{2}{3}}+24i\sqrt{3}\,x-\left(108y+12\sqrt{96x^3+81y^2}\right)^{\frac{2}{3}}+24x\right)^2}$$

Verified OK.

$$x = \frac{3\left(\frac{\left(108y+12\sqrt{96x^3+81y^2}\right)^{\frac{2}{3}}\left(-i+\sqrt{3}\right)}{24}+x\left(\sqrt{3}+i\right)\right)^2}{\left(108y+12\sqrt{96x^3+81y^2}\right)^{\frac{2}{3}}} + \frac{144c_1\left(108y+12\sqrt{96x^3+81y^2}\right)^{\frac{2}{3}}}{\left(i\sqrt{3}\,\left(108y+12\sqrt{96x^3+81y^2}\right)^{\frac{2}{3}}+24i\sqrt{3}\,x+\left(108y+12\sqrt{96x^3+81y^2}\right)^{\frac{2}{3}}-24x\right)^2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 141

```
dsolve(diff(y(x),x)^3+2*x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{2\left(-2x+\sqrt{x^2+3c_1}\right)\sqrt{-6\sqrt{x^2+3c_1}-6x}}{9}$$
$$y(x) = -\frac{2\left(-2x+\sqrt{x^2+3c_1}\right)\sqrt{-6\sqrt{x^2+3c_1}-6x}}{9}$$
$$y(x) = -\frac{2\left(2x+\sqrt{x^2+3c_1}\right)\sqrt{6\sqrt{x^2+3c_1}-6x}}{9}$$
$$y(x) = \frac{2\left(2x+\sqrt{x^2+3c_1}\right)\sqrt{6\sqrt{x^2+3c_1}-6x}}{9}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(y'[x])^3+2*x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Timed out

3.20 problem 23

Internal problem ID [6814]
Internal file name [`OUTPUT/6061_Thursday_July_28_2022_04_28_39_AM_98227792/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 99. Clairaut's equation. EXERCISES Page 320
Problem number: 23.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _dAlembert]
```

$$\boxed{4x{y'}^2 - 3yy' = -3}$$

3.20.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$4x\,p^2 - 3yp = -3$$

Solving for y from the above results in

$$y = \frac{4px}{3} + \frac{1}{p} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$

$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = \frac{4p}{3}$$
$$g = \frac{1}{p}$$

Hence (2) becomes

$$-\frac{p}{3} = \left(\frac{4x}{3} - \frac{1}{p^2}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$-\frac{p}{3} = 0$$

Solving for p from the above gives

$$p = 0$$

None of these values lead to defined solutions. Hence no singular solutions exist

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = -\frac{p(x)}{3\left(\frac{4x}{3} - \frac{1}{p(x)^2}\right)} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = -\frac{3\left(\frac{4x(p)}{3} - \frac{1}{p^2}\right)}{p} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = \frac{4}{p}$$
$$q(p) = \frac{3}{p^3}$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{4x(p)}{p} = \frac{3}{p^3}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{4}{p}dp} \\ &= p^4\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}(\mu x) &= (\mu)\left(\frac{3}{p^3}\right) \\ \frac{\mathrm{d}}{\mathrm{d}p}\left(p^4 x\right) &= \left(p^4\right)\left(\frac{3}{p^3}\right) \\ \mathrm{d}\left(p^4 x\right) &= (3p)\,\mathrm{d}p\end{aligned}$$

Integrating gives

$$\begin{aligned}p^4 x &= \int 3p\,\mathrm{d}p \\ p^4 x &= \frac{3p^2}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = p^4$ results in

$$x(p) = \frac{3}{2p^2} + \frac{c_1}{p^4}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$\begin{aligned}p &= \frac{3y + \sqrt{9y^2 - 48x}}{8x} \\ p &= -\frac{-3y + \sqrt{9y^2 - 48x}}{8x}\end{aligned}$$

Substituting the above in the solution for x found above gives

$$\begin{aligned}x &= \frac{64x^2\left(64c_1x^2 + 9y\sqrt{9y^2 - 48x} + 27y^2 - 72x\right)}{\left(3y + \sqrt{9y^2 - 48x}\right)^4} \\ x &= -\frac{64x^2\left(-64c_1x^2 + 9y\sqrt{9y^2 - 48x} - 27y^2 + 72x\right)}{\left(-3y + \sqrt{9y^2 - 48x}\right)^4}\end{aligned}$$

Summary

The solution(s) found are the following

$$x = \frac{64x^2(64c_1x^2 + 9y\sqrt{9y^2 - 48x} + 27y^2 - 72x)}{(3y + \sqrt{9y^2 - 48x})^4} \tag{1}$$

$$x = -\frac{64x^2(-64c_1x^2 + 9y\sqrt{9y^2 - 48x} - 27y^2 + 72x)}{(-3y + \sqrt{9y^2 - 48x})^4} \tag{2}$$

Verification of solutions

$$x = \frac{64x^2(64c_1x^2 + 9y\sqrt{9y^2 - 48x} + 27y^2 - 72x)}{(3y + \sqrt{9y^2 - 48x})^4}$$

Verified OK.

$$x = -\frac{64x^2(-64c_1x^2 + 9y\sqrt{9y^2 - 48x} - 27y^2 + 72x)}{(-3y + \sqrt{9y^2 - 48x})^4}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.079 (sec). Leaf size: 123

```
dsolve(4*x*diff(y(x),x)^2-3*y(x)*diff(y(x),x)+3=0,y(x), singsol=all)
```

$$y(x) = -\frac{2x\left(6+\sqrt{16c_1x+9}\right)}{3\sqrt{x\left(3+\sqrt{16c_1x+9}\right)}}$$
$$y(x) = \frac{2x\left(6+\sqrt{16c_1x+9}\right)}{3\sqrt{x\left(3+\sqrt{16c_1x+9}\right)}}$$
$$y(x) = \frac{2x\left(-6+\sqrt{16c_1x+9}\right)}{3\sqrt{-x\left(-3+\sqrt{16c_1x+9}\right)}}$$
$$y(x) = -\frac{2x\left(-6+\sqrt{16c_1x+9}\right)}{3\sqrt{-x\left(-3+\sqrt{16c_1x+9}\right)}}$$

✓ Solution by Mathematica

Time used: 23.695 (sec). Leaf size: 187

```
DSolve[4*x*(y'[x])^2-3*y[x]*y'[x]+3==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{432x - e^{-\frac{c_1}{2}}\left(-144x+e^{c_1}\right)^{3/2}+e^{c_1}}}{6\sqrt{3}}$$
$$y(x) \to \frac{\sqrt{432x - e^{-\frac{c_1}{2}}\left(-144x+e^{c_1}\right)^{3/2}+e^{c_1}}}{6\sqrt{3}}$$
$$y(x) \to -\frac{\sqrt{432x + e^{-\frac{c_1}{2}}\left(-144x+e^{c_1}\right)^{3/2}+e^{c_1}}}{6\sqrt{3}}$$
$$y(x) \to \frac{\sqrt{432x + e^{-\frac{c_1}{2}}\left(-144x+e^{c_1}\right)^{3/2}+e^{c_1}}}{6\sqrt{3}}$$

3.21 problem 24

Internal problem ID [6815]
Internal file name [`OUTPUT/6062_Thursday_July_28_2022_04_28_41_AM_47236068/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 99. Clairaut's equation. EXERCISES Page 320
Problem number: 24.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _dAlembert]`

$$\boxed{y'^3 - xy' + 2y = 0}$$

3.21.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$p^3 - xp + 2y = 0$$

Solving for y from the above results in

$$y = -\frac{1}{2}p^3 + \frac{1}{2}xp \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = \frac{p}{2}$$
$$g = -\frac{p^3}{2}$$

Hence (2) becomes

$$\frac{p}{2} = \left(\frac{x}{2} - \frac{3p^2}{2}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$\frac{p}{2} = 0$$

Solving for p from the above gives

$$p = 0$$

Substituting these in (1A) gives

$$y = 0$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x)}{x - 3p\left(x\right)^2} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{x(p) - 3p^2}{p} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = -\frac{1}{p}$$
$$q(p) = -3p$$

Hence the ode is

$$\frac{d}{dp}x(p) - \frac{x(p)}{p} = -3p$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{p}dp} \\ &= \frac{1}{p}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}(\mu x) &= (\mu)(-3p) \\ \frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{x}{p}\right) &= \left(\frac{1}{p}\right)(-3p) \\ \mathrm{d}\left(\frac{x}{p}\right) &= -3\,\mathrm{d}p\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{x}{p} &= \int -3\,\mathrm{d}p \\ \frac{x}{p} &= -3p + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{p}$ results in

$$x(p) = c_1 p - 3p^2$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = \frac{\left(-27y + 3\sqrt{-3x^3 + 81y^2}\right)^{\frac{1}{3}}}{3} + \frac{x}{\left(-27y + 3\sqrt{-3x^3 + 81y^2}\right)^{\frac{1}{3}}}$$

$$p = -\frac{\left(-27y + 3\sqrt{-3x^3 + 81y^2}\right)^{\frac{1}{3}}}{6} - \frac{x}{2\left(-27y + 3\sqrt{-3x^3 + 81y^2}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(-27y+3\sqrt{-3x^3+81y^2}\right)^{\frac{1}{3}}}{3} - \frac{\cdots}{\left(-27y\cdots\right.}\right.}{2}\cdots$$

$$p = -\frac{\left(-27y + 3\sqrt{-3x^3 + 81y^2}\right)^{\frac{1}{3}}}{6} - \frac{x}{2\left(-27y + 3\sqrt{-3x^3 + 81y^2}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(-27y+3\sqrt{-3x^3+81y^2}\right)^{\frac{1}{3}}}{3} - \frac{\cdots}{\left(-27y\cdots\right.}\right.}{2}\cdots$$

Substituting the above in the solution for x found above gives

$$x = -\frac{\left(\left(-27y+3\sqrt{-3x^3+81y^2}\right)^{\frac{2}{3}}+3x\right)\left(\left(-27y+3\sqrt{-3x^3+81y^2}\right)^{\frac{2}{3}}-c_1\left(-27y+3\sqrt{-3x^3+81y^2}\right)^{\frac{1}{3}}+\right.}{3\left(-27y+3\sqrt{-3x^3+81y^2}\right)^{\frac{2}{3}}}$$

$$x = -\frac{\left(-i\left(-27y+3\sqrt{-3x^3+81y^2}\right)^{\frac{2}{3}}\sqrt{3}+3i\sqrt{3}\,x+\left(-27y+3\sqrt{-3x^3+81y^2}\right)^{\frac{2}{3}}+3x\right)\left(-i\left(-27y+3\sqrt{-\right.\right.}{12\left(-27y+3\sqrt{-\right.}}$$

$$x = -\frac{\left(i\left(-27y+3\sqrt{-3x^3+81y^2}\right)^{\frac{2}{3}}\sqrt{3}-3i\sqrt{3}\,x+\left(-27y+3\sqrt{-3x^3+81y^2}\right)^{\frac{2}{3}}+3x\right)\left(i\left(-27y+3\sqrt{-3x\right.\right.}{12\left(-27y+3\sqrt{-3\right.}}$$

<u>Summary</u>
The solution(s) found are the following

$$y = 0 \tag{1}$$

$$x = -\frac{\left(\left(-27y+3\sqrt{-3x^3+81y^2}\right)^{\frac{2}{3}}+3x\right)\left(\left(-27y+3\sqrt{-3x^3+81y^2}\right)^{\frac{2}{3}}-c_1\left(-27y+3\sqrt{-3x^3+81y^2}\right)^{\frac{1}{3}}+\right.}{3\left(-27y+3\sqrt{-3x^3+81y^2}\right)^{\frac{2}{3}}} \tag{2}$$

$$x = -\frac{\left(-i\left(-27y+3\sqrt{-3x^3+81y^2}\right)^{\frac{2}{3}}\sqrt{3}+3i\sqrt{3}\,x+\left(-27y+3\sqrt{-3x^3+81y^2}\right)^{\frac{2}{3}}+3x\right)\left(-i\left(-27y+3\sqrt{-\right.\right.}{12\left(-27y+3\sqrt{-\right.}} \tag{3}$$

$$x = -\frac{\left(i\left(-27y+3\sqrt{-3x^3+81y^2}\right)^{\frac{2}{3}}\sqrt{3}-3i\sqrt{3}\,x+\left(-27y+3\sqrt{-3x^3+81y^2}\right)^{\frac{2}{3}}+3x\right)\left(i\left(-27y+3\sqrt{-3x\right.\right.}{12\left(-27y+3\sqrt{-3\right.}} \tag{4}$$

Verification of solutions

$$y = 0$$

Verified OK.

$x =$

$$-\frac{\left(\left(-27y+3\sqrt{-3x^3+81y^2}\right)^{\frac{2}{3}}+3x\right)\left(\left(-27y+3\sqrt{-3x^3+81y^2}\right)^{\frac{2}{3}}-c_1\left(-27y+3\sqrt{-3x^3+81y^2}\right)^{\frac{1}{3}}+\right.}{3\left(-27y+3\sqrt{-3x^3+81y^2}\right)^{\frac{2}{3}}}$$

Verified OK.

$x =$

$$-\frac{\left(-i\left(-27y+3\sqrt{-3x^3+81y^2}\right)^{\frac{2}{3}}\sqrt{3}+3i\sqrt{3}\,x+\left(-27y+3\sqrt{-3x^3+81y^2}\right)^{\frac{2}{3}}+3x\right)\left(-i\left(-27y+3\sqrt{-\right.\right.}{12\left(-27y+3\sqrt{-\right.}}$$

Warning, solution could not be verified

$x =$

$$-\frac{\left(i\left(-27y+3\sqrt{-3x^3+81y^2}\right)^{\frac{2}{3}}\sqrt{3}-3i\sqrt{3}\,x+\left(-27y+3\sqrt{-3x^3+81y^2}\right)^{\frac{2}{3}}+3x\right)\left(i\left(-27y+3\sqrt{-3x\right.\right.}{12\left(-27y+3\sqrt{-3\right.}}$$

Warning, solution could not be verified

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 58

```
dsolve(diff(y(x),x)^3-x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{(c_1^2 - 12x)^{\frac{3}{2}}}{108} - \frac{c_1^3}{108} + \frac{c_1 x}{6}$$
$$y(x) = \frac{(-c_1^2 + 12x)\sqrt{c_1^2 - 12x}}{108} - \frac{c_1^3}{108} + \frac{c_1 x}{6}$$

✓ Solution by Mathematica

Time used: 29.375 (sec). Leaf size: 10134

```
DSolve[(y'[x])^3-x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

3.22 problem 25

Internal problem ID [6816]
Internal file name [`OUTPUT/6063_Thursday_July_28_2022_04_28_51_AM_64077138/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 99. Clairaut's equation. EXERCISES Page 320
Problem number: 25.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _dAlembert]`

$$\boxed{5y'^2 + 6xy' - 2y = 0}$$

3.22.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$5p^2 + 6xp - 2y = 0$$

Solving for y from the above results in

$$y = \frac{5}{2}p^2 + 3xp \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = 3p$$
$$g = \frac{5p^2}{2}$$

Hence (2) becomes

$$-2p = (3x + 5p)\, p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$-2p = 0$$

Solving for p from the above gives

$$p = 0$$

Substituting these in (1A) gives

$$y = 0$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = -\frac{2p(x)}{3x + 5p\,(x)} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = -\frac{3x(p) + 5p}{2p} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = \frac{3}{2p}$$
$$q(p) = -\frac{5}{2}$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{3x(p)}{2p} = -\frac{5}{2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{3}{2p}dp} \\ &= p^{\frac{3}{2}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}(\mu x) &= (\mu)\left(-\frac{5}{2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}p}\left(p^{\frac{3}{2}}x\right) &= \left(p^{\frac{3}{2}}\right)\left(-\frac{5}{2}\right) \\ \mathrm{d}\left(p^{\frac{3}{2}}x\right) &= \left(-\frac{5p^{\frac{3}{2}}}{2}\right)\mathrm{d}p\end{aligned}$$

Integrating gives

$$\begin{aligned}p^{\frac{3}{2}}x &= \int -\frac{5p^{\frac{3}{2}}}{2}\,\mathrm{d}p \\ p^{\frac{3}{2}}x &= -p^{\frac{5}{2}} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = p^{\frac{3}{2}}$ results in

$$x(p) = -p + \frac{c_1}{p^{\frac{3}{2}}}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$\begin{aligned}p &= -\frac{3x}{5} + \frac{\sqrt{9x^2+10y}}{5} \\ p &= -\frac{3x}{5} - \frac{\sqrt{9x^2+10y}}{5}\end{aligned}$$

Substituting the above in the solution for x found above gives

$$\begin{aligned}x &= \frac{3x}{5} - \frac{\sqrt{9x^2+10y}}{5} + \frac{125c_1}{\left(-15x+5\sqrt{9x^2+10y}\right)^{\frac{3}{2}}} \\ x &= \frac{3x}{5} + \frac{\sqrt{9x^2+10y}}{5} + \frac{125c_1}{\left(-15x-5\sqrt{9x^2+10y}\right)^{\frac{3}{2}}}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$

$$x = \frac{3x}{5} - \frac{\sqrt{9x^2 + 10y}}{5} + \frac{125c_1}{\left(-15x + 5\sqrt{9x^2 + 10y}\right)^{\frac{3}{2}}} \tag{2}$$

$$x = \frac{3x}{5} + \frac{\sqrt{9x^2 + 10y}}{5} + \frac{125c_1}{\left(-15x - 5\sqrt{9x^2 + 10y}\right)^{\frac{3}{2}}} \tag{3}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$x = \frac{3x}{5} - \frac{\sqrt{9x^2 + 10y}}{5} + \frac{125c_1}{\left(-15x + 5\sqrt{9x^2 + 10y}\right)^{\frac{3}{2}}}$$

Verified OK.

$$x = \frac{3x}{5} + \frac{\sqrt{9x^2 + 10y}}{5} + \frac{125c_1}{\left(-15x - 5\sqrt{9x^2 + 10y}\right)^{\frac{3}{2}}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 85

```
dsolve(5*diff(y(x),x)^2+6*x*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$\frac{c_1}{\left(-15x-5\sqrt{9x^2+10y(x)}\right)^{\frac{3}{2}}}+\frac{2x}{5}-\frac{\sqrt{9x^2+10y(x)}}{5}=0$$

$$\frac{c_1}{\left(-15x+5\sqrt{9x^2+10y(x)}\right)^{\frac{3}{2}}}+\frac{2x}{5}+\frac{\sqrt{9x^2+10y(x)}}{5}=0$$

✓ Solution by Mathematica

Time used: 14.31 (sec). Leaf size: 771

```
DSolve[5*(y'[x])^2+6*x*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{Root}\left[4\#1^5 + 4\#1^4x^2 + \#1^3x^4 + 1000\#1^2e^{5c_1}x + 900\#1e^{5c_1}x^3 + 216e^{5c_1}x^5 - 25000e^{10c_1}\&, 1\right]$$

$$y(x) \to \text{Root}\left[4\#1^5 + 4\#1^4x^2 + \#1^3x^4 + 1000\#1^2e^{5c_1}x + 900\#1e^{5c_1}x^3 + 216e^{5c_1}x^5 - 25000e^{10c_1}\&, 2\right]$$

$$y(x) \to \text{Root}\left[4\#1^5 + 4\#1^4x^2 + \#1^3x^4 + 1000\#1^2e^{5c_1}x + 900\#1e^{5c_1}x^3 + 216e^{5c_1}x^5 - 25000e^{10c_1}\&, 3\right]$$

$$y(x) \to \text{Root}\left[4\#1^5 + 4\#1^4x^2 + \#1^3x^4 + 1000\#1^2e^{5c_1}x + 900\#1e^{5c_1}x^3 + 216e^{5c_1}x^5 - 25000e^{10c_1}\&, 4\right]$$

$$y(x) \to \text{Root}\left[4\#1^5 + 4\#1^4x^2 + \#1^3x^4 + 1000\#1^2e^{5c_1}x + 900\#1e^{5c_1}x^3 + 216e^{5c_1}x^5 - 25000e^{10c_1}\&, 5\right]$$

$$y(x) \to \text{Root}\left[100000\#1^5 + 100000\#1^4x^2 + 25000\#1^3x^4 - 1000\#1^2e^{5c_1}x - 900\#1e^{5c_1}x^3 - 216e^{5c_1}x^5 - e^{10c_1}\&, 1\right]$$

$$y(x) \to \text{Root}\left[100000\#1^5 + 100000\#1^4x^2 + 25000\#1^3x^4 - 1000\#1^2e^{5c_1}x - 900\#1e^{5c_1}x^3 - 216e^{5c_1}x^5 - e^{10c_1}\&, 2\right]$$

$$y(x) \to \text{Root}\left[100000\#1^5 + 100000\#1^4x^2 + 25000\#1^3x^4 - 1000\#1^2e^{5c_1}x - 900\#1e^{5c_1}x^3 - 216e^{5c_1}x^5 - e^{10c_1}\&, 3\right]$$

$$y(x) \to \text{Root}\left[100000\#1^5 + 100000\#1^4x^2 + 25000\#1^3x^4 - 1000\#1^2e^{5c_1}x - 900\#1e^{5c_1}x^3 - 216e^{5c_1}x^5 - e^{10c_1}\&, 4\right]$$

$$y(x) \to \text{Root}\left[100000\#1^5 + 100000\#1^4x^2 + 25000\#1^3x^4 - 1000\#1^2e^{5c_1}x - 900\#1e^{5c_1}x^3 - 216e^{5c_1}x^5 - e^{10c_1}\&, 5\right]$$

$$y(x) \to 0$$

3.23 problem 26

Internal problem ID [6817]
Internal file name [`OUTPUT/6064_Thursday_July_28_2022_04_28_53_AM_25660040/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 99. Clairaut's equation. EXERCISES Page 320
Problem number: 26.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[_rational, _dAlembert]`

$$\boxed{2xy'^2 + (2x - y)\, y' - y = -1}$$

3.23.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$2x\,p^2 + (2x - y)\,p - y = -1$$

Solving for y from the above results in

$$y = \frac{(2p^2 + 2p)\,x}{p+1} + \frac{1}{p+1} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = 2p$$
$$g = \frac{1}{p+1}$$

Hence (2) becomes

$$-p = \left(2x - \frac{1}{(p+1)^2}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$-p = 0$$

Solving for p from the above gives

$$p = 0$$

Substituting these in (1A) gives

$$y = 1$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = -\frac{p(x)}{2x - \frac{1}{(p(x)+1)^2}} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = -\frac{2x(p) - \frac{1}{(p+1)^2}}{p} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = \frac{2}{p}$$
$$q(p) = \frac{1}{p\,(p+1)^2}$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{2x(p)}{p} = \frac{1}{p\left(p+1\right)^{2}}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \frac{2}{p}dp} \\ &= p^{2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}(\mu x) &= (\mu)\left(\frac{1}{p\left(p+1\right)^{2}}\right) \\ \frac{\mathrm{d}}{\mathrm{d}p}\left(x\,p^{2}\right) &= \left(p^{2}\right)\left(\frac{1}{p\left(p+1\right)^{2}}\right) \\ \mathrm{d}\left(x\,p^{2}\right) &= \left(\frac{p}{\left(p+1\right)^{2}}\right)\,\mathrm{d}p\end{aligned}$$

Integrating gives

$$\begin{aligned}x\,p^{2} &= \int \frac{p}{\left(p+1\right)^{2}}\,\mathrm{d}p \\ x\,p^{2} &= \ln\left(p+1\right) + \frac{1}{p+1} + c_{1}\end{aligned}$$

Dividing both sides by the integrating factor $\mu = p^{2}$ results in

$$x(p) = \frac{\ln\left(p+1\right) + \frac{1}{p+1}}{p^{2}} + \frac{c_{1}}{p^{2}}$$

which simplifies to

$$x(p) = \frac{\ln\left(p+1\right) + \frac{1}{p+1} + c_{1}}{p^{2}}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$\begin{aligned}p &= \frac{-2x + y + \sqrt{y^{2} + 4yx + 4x^{2} - 8x}}{4x} \\ p &= -\frac{2x - y + \sqrt{y^{2} + 4yx + 4x^{2} - 8x}}{4x}\end{aligned}$$

Substituting the above in the solution for x found above gives

$$x = \frac{32\left(\left(x+\frac{y}{2}+\frac{\sqrt{4x^2+(4y-8)x+y^2}}{2}\right)\ln\left(\frac{2x+y+\sqrt{4x^2+(4y-8)x+y^2}}{x}\right)+\left(\frac{c_1}{2}-\ln(2)\right)\sqrt{4x^2+(4y-8)x+y^2}+(-2}{\left(2x+y+\sqrt{4x^2+(4y-8)x+y^2}\right)\left(2x-y-\sqrt{4x^2+(4y-8)x+y^2}}$$

$$x = \frac{32x^2\left(\left(x+\frac{y}{2}-\frac{\sqrt{4x^2+(4y-8)x+y^2}}{2}\right)\ln\left(\frac{2x+y-\sqrt{4x^2+(4y-8)x+y^2}}{x}\right)+\left(-\frac{c_1}{2}+\ln(2)\right)\sqrt{4x^2+(4y-8)x+y^2}+}{\left(2x+y-\sqrt{4x^2+(4y-8)x+y^2}\right)\left(2x-y+\sqrt{4x^2+(4y-8)x+y}\right.}$$

Summary

The solution(s) found are the following

$$y = 1 \tag{1}$$

$$x = \frac{32\left(\left(x+\frac{y}{2}+\frac{\sqrt{4x^2+(4y-8)x+y^2}}{2}\right)\ln\left(\frac{2x+y+\sqrt{4x^2+(4y-8)x+y^2}}{x}\right)+\left(\frac{c_1}{2}-\ln(2)\right)\sqrt{4x^2+(4y-8)x+y^2}+(-2}{\left(2x+y+\sqrt{4x^2+(4y-8)x+y^2}\right)\left(2x-y-\sqrt{4x^2+(4y-8)x+y^2}}\tag{2}$$

$$x = \frac{32x^2\left(\left(x+\frac{y}{2}-\frac{\sqrt{4x^2+(4y-8)x+y^2}}{2}\right)\ln\left(\frac{2x+y-\sqrt{4x^2+(4y-8)x+y^2}}{x}\right)+\left(-\frac{c_1}{2}+\ln(2)\right)\sqrt{4x^2+(4y-8)x+y^2}+}{\left(2x+y-\sqrt{4x^2+(4y-8)x+y^2}\right)\left(2x-y+\sqrt{4x^2+(4y-8)x+y}\right.}\tag{3}$$

Verification of solutions

$$y = 1$$

Verified OK.

$$x = \frac{32\left(\left(x+\frac{y}{2}+\frac{\sqrt{4x^2+(4y-8)x+y^2}}{2}\right)\ln\left(\frac{2x+y+\sqrt{4x^2+(4y-8)x+y^2}}{x}\right)+\left(\frac{c_1}{2}-\ln(2)\right)\sqrt{4x^2+(4y-8)x+y^2}+(-2}{\left(2x+y+\sqrt{4x^2+(4y-8)x+y^2}\right)\left(2x-y-\sqrt{4x^2+(4y-8)x+y^2}}$$

Verified OK.

$$x = \frac{32x^2\left(\left(x+\frac{y}{2}-\frac{\sqrt{4x^2+(4y-8)x+y^2}}{2}\right)\ln\left(\frac{2x+y-\sqrt{4x^2+(4y-8)x+y^2}}{x}\right)+\left(-\frac{c_1}{2}+\ln(2)\right)\sqrt{4x^2+(4y-8)x+y^2}+}{\left(2x+y-\sqrt{4x^2+(4y-8)x+y^2}\right)\left(2x-y+\sqrt{4x^2+(4y-8)x+y}\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 110

```
dsolve(2*x*diff(y(x),x)^2+(2*x-y(x))*diff(y(x),x)+1-y(x)=0,y(x), singsol=all)
```

$$y(x) = -2\left(x\,\mathrm{e}^{\mathrm{RootOf}\left(-\mathrm{e}^{3\_Z}x+2x\,\mathrm{e}^{2\_Z}+c_1\mathrm{e}^{\_Z}+\mathrm{e}^{\_Z}\_Z-x\,\mathrm{e}^{\_Z}+1\right)} - \mathrm{e}^{2\,\mathrm{RootOf}\left(-\mathrm{e}^{3\_Z}x+2x\,\mathrm{e}^{2\_Z}+c_1\mathrm{e}^{\_Z}+\mathrm{e}^{\_Z}\_Z-x\,\mathrm{e}^{\_Z}+1\right)}x - \frac{1}{2}\right)\mathrm{e}^{-\,\mathrm{RootOf}\left(-\mathrm{e}^{3\_Z}x+2x\,\mathrm{e}^{2\_Z}+c_1\mathrm{e}^{\_Z}+\mathrm{e}^{\_Z}\_Z-x\,\mathrm{e}^{\_Z}+1\right)}$$

✓ Solution by Mathematica

Time used: 1.438 (sec). Leaf size: 49

```
DSolve[2*x*(y'[x])^2+(2*x-y[x])*y'[x]+1-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[\left\{x = \frac{\frac{1}{K[1]+1} + \log(K[1]+1)}{K[1]^2} + \frac{c_1}{K[1]^2}, y(x) = 2xK[1] + \frac{1}{K[1]+1}\right\}, \{y(x), K[1]\}\right]$$

3.24 problem 27

Internal problem ID [6818]
Internal file name [`OUTPUT/6065_Thursday_July_28_2022_04_28_56_AM_97945755/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 99. Clairaut's equation. EXERCISES Page 320
Problem number: 27.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _dAlembert]`

$$\boxed{5y'^2 + 3xy' - y = 0}$$

3.24.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$5p^2 + 3xp - y = 0$$

Solving for y from the above results in

$$y = 5p^2 + 3xp \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$

$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= 3p \\
g &= 5p^2
\end{aligned}
$$

Hence (2) becomes

$$
-2p = (3x + 10p)\, p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
-2p = 0
$$

Solving for p from the above gives

$$
p = 0
$$

Substituting these in (1A) gives

$$
y = 0
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = -\frac{2p(x)}{3x + 10p\,(x)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp}x(p) = -\frac{3x(p) + 10p}{2p} \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
\frac{d}{dp}x(p) + p(p)x(p) = q(p)
$$

Where here

$$
\begin{aligned}
p(p) &= \frac{3}{2p} \\
q(p) &= -5
\end{aligned}
$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{3x(p)}{2p} = -5$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{3}{2p}dp} \\ &= p^{\frac{3}{2}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}(\mu x) &= (\mu)(-5) \\ \frac{\mathrm{d}}{\mathrm{d}p}\left(p^{\frac{3}{2}}x\right) &= \left(p^{\frac{3}{2}}\right)(-5) \\ \mathrm{d}\left(p^{\frac{3}{2}}x\right) &= \left(-5p^{\frac{3}{2}}\right)\,\mathrm{d}p\end{aligned}$$

Integrating gives

$$\begin{aligned}p^{\frac{3}{2}}x &= \int -5p^{\frac{3}{2}}\,\mathrm{d}p \\ p^{\frac{3}{2}}x &= -2p^{\frac{5}{2}} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = p^{\frac{3}{2}}$ results in

$$x(p) = -2p + \frac{c_1}{p^{\frac{3}{2}}}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$\begin{aligned}p &= -\frac{3x}{10} + \frac{\sqrt{9x^2+20y}}{10} \\ p &= -\frac{3x}{10} - \frac{\sqrt{9x^2+20y}}{10}\end{aligned}$$

Substituting the above in the solution for x found above gives

$$\begin{aligned}x &= \frac{3x}{5} - \frac{\sqrt{9x^2+20y}}{5} + \frac{1000c_1}{\left(-30x+10\sqrt{9x^2+20y}\right)^{\frac{3}{2}}} \\ x &= \frac{3x}{5} + \frac{\sqrt{9x^2+20y}}{5} + \frac{1000c_1}{\left(-30x-10\sqrt{9x^2+20y}\right)^{\frac{3}{2}}}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = 0 \tag{1}$$

$$x = \frac{3x}{5} - \frac{\sqrt{9x^2 + 20y}}{5} + \frac{1000c_1}{\left(-30x + 10\sqrt{9x^2 + 20y}\right)^{\frac{3}{2}}} \tag{2}$$

$$x = \frac{3x}{5} + \frac{\sqrt{9x^2 + 20y}}{5} + \frac{1000c_1}{\left(-30x - 10\sqrt{9x^2 + 20y}\right)^{\frac{3}{2}}} \tag{3}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$x = \frac{3x}{5} - \frac{\sqrt{9x^2 + 20y}}{5} + \frac{1000c_1}{\left(-30x + 10\sqrt{9x^2 + 20y}\right)^{\frac{3}{2}}}$$

Verified OK.

$$x = \frac{3x}{5} + \frac{\sqrt{9x^2 + 20y}}{5} + \frac{1000c_1}{\left(-30x - 10\sqrt{9x^2 + 20y}\right)^{\frac{3}{2}}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 85

```
dsolve(5*diff(y(x),x)^2+3*x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$\frac{c_1}{\left(-30x-10\sqrt{9x^2+20y(x)}\right)^{\frac{3}{2}}}+\frac{2x}{5}-\frac{\sqrt{9x^2+20y(x)}}{5}=0$$

$$\frac{c_1}{\left(-30x+10\sqrt{9x^2+20y(x)}\right)^{\frac{3}{2}}}+\frac{2x}{5}+\frac{\sqrt{9x^2+20y(x)}}{5}=0$$

✓ Solution by Mathematica

Time used: 14.529 (sec). Leaf size: 771

```
DSolve[5*(y'[x])^2+3*x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{Root}\left[16\#1^5 + 8\#1^4x^2 + \#1^3x^4 + 4000\#1^2e^{5c_1}x + 1800\#1e^{5c_1}x^3 + 216e^{5c_1}x^5 - 200000e^{10c_1}\&, 1\right]$$
$$y(x) \to \text{Root}\left[16\#1^5 + 8\#1^4x^2 + \#1^3x^4 + 4000\#1^2e^{5c_1}x + 1800\#1e^{5c_1}x^3 + 216e^{5c_1}x^5 - 200000e^{10c_1}\&, 2\right]$$
$$y(x) \to \text{Root}\left[16\#1^5 + 8\#1^4x^2 + \#1^3x^4 + 4000\#1^2e^{5c_1}x + 1800\#1e^{5c_1}x^3 + 216e^{5c_1}x^5 - 200000e^{10c_1}\&, 3\right]$$
$$y(x) \to \text{Root}\left[16\#1^5 + 8\#1^4x^2 + \#1^3x^4 + 4000\#1^2e^{5c_1}x + 1800\#1e^{5c_1}x^3 + 216e^{5c_1}x^5 - 200000e^{10c_1}\&, 4\right]$$
$$y(x) \to \text{Root}\left[16\#1^5 + 8\#1^4x^2 + \#1^3x^4 + 4000\#1^2e^{5c_1}x + 1800\#1e^{5c_1}x^3 + 216e^{5c_1}x^5 - 200000e^{10c_1}\&, 5\right]$$
$$y(x) \to \text{Root}\left[3200000\#1^5 + 1600000\#1^4x^2 + 200000\#1^3x^4 - 4000\#1^2e^{5c_1}x - 1800\#1e^{5c_1}x^3 - 216e^{5c_1}x^5 - e^{10c_1}\&, 1\right]$$
$$y(x) \to \text{Root}\left[3200000\#1^5 + 1600000\#1^4x^2 + 200000\#1^3x^4 - 4000\#1^2e^{5c_1}x - 1800\#1e^{5c_1}x^3 - 216e^{5c_1}x^5 - e^{10c_1}\&, 2\right]$$
$$y(x) \to \text{Root}\left[3200000\#1^5 + 1600000\#1^4x^2 + 200000\#1^3x^4 - 4000\#1^2e^{5c_1}x - 1800\#1e^{5c_1}x^3 - 216e^{5c_1}x^5 - e^{10c_1}\&, 3\right]$$
$$y(x) \to \text{Root}\left[3200000\#1^5 + 1600000\#1^4x^2 + 200000\#1^3x^4 - 4000\#1^2e^{5c_1}x - 1800\#1e^{5c_1}x^3 - 216e^{5c_1}x^5 - e^{10c_1}\&, 4\right]$$
$$y(x) \to \text{Root}\left[3200000\#1^5 + 1600000\#1^4x^2 + 200000\#1^3x^4 - 4000\#1^2e^{5c_1}x - 1800\#1e^{5c_1}x^3 - 216e^{5c_1}x^5 - e^{10c_1}\&, 5\right]$$
$$y(x) \to 0$$

3.25 problem 28

Internal problem ID [6819]
Internal file name [`OUTPUT/6066_Thursday_July_28_2022_04_28_58_AM_71850312/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 99. Clairaut's equation. EXERCISES Page 320
Problem number: 28.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _dAlembert]`

$$\boxed{y'^2 + 3xy' - y = 0}$$

3.25.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$p^2 + 3xp - y = 0$$

Solving for y from the above results in

$$y = p^2 + 3xp \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= 3p \\
g &= p^2
\end{aligned}
$$

Hence (2) becomes

$$
-2p = (3x + 2p)\, p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
-2p = 0
$$

Solving for p from the above gives

$$
p = 0
$$

Substituting these in (1A) gives

$$
y = 0
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = -\frac{2p(x)}{3x + 2p(x)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp}x(p) = -\frac{3x(p) + 2p}{2p} \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
\frac{d}{dp}x(p) + p(p)x(p) = q(p)
$$

Where here

$$
\begin{aligned}
p(p) &= \frac{3}{2p} \\
q(p) &= -1
\end{aligned}
$$

Hence the ode is

$$\frac{d}{dp}x(p)+\frac{3x(p)}{2p}=-1$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \frac{3}{2p}dp}\\ &= p^{\frac{3}{2}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}(\mu x) &= (\mu)(-1)\\ \frac{\mathrm{d}}{\mathrm{d}p}\left(p^{\frac{3}{2}}x\right) &= \left(p^{\frac{3}{2}}\right)(-1)\\ \mathrm{d}\left(p^{\frac{3}{2}}x\right) &= \left(-p^{\frac{3}{2}}\right)\,\mathrm{d}p\end{aligned}$$

Integrating gives

$$\begin{aligned}p^{\frac{3}{2}}x &= \int -p^{\frac{3}{2}}\,\mathrm{d}p\\ p^{\frac{3}{2}}x &= -\frac{2p^{\frac{5}{2}}}{5}+c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu=p^{\frac{3}{2}}$ results in

$$x(p)=-\frac{2p}{5}+\frac{c_1}{p^{\frac{3}{2}}}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$\begin{aligned}p &= -\frac{3x}{2}+\frac{\sqrt{9x^2+4y}}{2}\\ p &= -\frac{3x}{2}-\frac{\sqrt{9x^2+4y}}{2}\end{aligned}$$

Substituting the above in the solution for x found above gives

$$\begin{aligned}x &= \frac{3x}{5}-\frac{\sqrt{9x^2+4y}}{5}+\frac{8c_1}{\left(-6x+2\sqrt{9x^2+4y}\right)^{\frac{3}{2}}}\\ x &= \frac{3x}{5}+\frac{\sqrt{9x^2+4y}}{5}+\frac{8c_1}{\left(-6x-2\sqrt{9x^2+4y}\right)^{\frac{3}{2}}}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = 0 \tag{1}$$

$$x = \frac{3x}{5} - \frac{\sqrt{9x^2+4y}}{5} + \frac{8c_1}{\left(-6x+2\sqrt{9x^2+4y}\right)^{\frac{3}{2}}} \tag{2}$$

$$x = \frac{3x}{5} + \frac{\sqrt{9x^2+4y}}{5} + \frac{8c_1}{\left(-6x-2\sqrt{9x^2+4y}\right)^{\frac{3}{2}}} \tag{3}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$x = \frac{3x}{5} - \frac{\sqrt{9x^2+4y}}{5} + \frac{8c_1}{\left(-6x+2\sqrt{9x^2+4y}\right)^{\frac{3}{2}}}$$

Verified OK.

$$x = \frac{3x}{5} + \frac{\sqrt{9x^2+4y}}{5} + \frac{8c_1}{\left(-6x-2\sqrt{9x^2+4y}\right)^{\frac{3}{2}}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 85

```
dsolve(diff(y(x),x)^2+3*x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$\frac{c_1}{\left(-6x-2\sqrt{9x^2+4y(x)}\right)^{\frac{3}{2}}}+\frac{2x}{5}-\frac{\sqrt{9x^2+4y(x)}}{5}=0$$

$$\frac{c_1}{\left(-6x+2\sqrt{9x^2+4y(x)}\right)^{\frac{3}{2}}}+\frac{2x}{5}+\frac{\sqrt{9x^2+4y(x)}}{5}=0$$

✓ Solution by Mathematica

Time used: 14.495 (sec). Leaf size: 776

```
DSolve[(y'[x])^2+3*x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{Root}\left[16\#1^5 + 40\#1^4x^2 + 25\#1^3x^4 + 160\#1^2e^{5c_1}x + 360\#1e^{5c_1}x^3 + 216e^{5c_1}x^5 - 64e^{10c_1}\&, 1\right]$$

$$y(x) \to \text{Root}\left[16\#1^5 + 40\#1^4x^2 + 25\#1^3x^4 + 160\#1^2e^{5c_1}x + 360\#1e^{5c_1}x^3 + 216e^{5c_1}x^5 - 64e^{10c_1}\&, 2\right]$$

$$y(x) \to \text{Root}\left[16\#1^5 + 40\#1^4x^2 + 25\#1^3x^4 + 160\#1^2e^{5c_1}x + 360\#1e^{5c_1}x^3 + 216e^{5c_1}x^5 - 64e^{10c_1}\&, 3\right]$$

$$y(x) \to \text{Root}\left[16\#1^5 + 40\#1^4x^2 + 25\#1^3x^4 + 160\#1^2e^{5c_1}x + 360\#1e^{5c_1}x^3 + 216e^{5c_1}x^5 - 64e^{10c_1}\&, 4\right]$$

$$y(x) \to \text{Root}\left[16\#1^5 + 40\#1^4x^2 + 25\#1^3x^4 + 160\#1^2e^{5c_1}x + 360\#1e^{5c_1}x^3 + 216e^{5c_1}x^5 - 64e^{10c_1}\&, 5\right]$$

$$y(x) \to \text{Root}\left[1024\#1^5 + 2560\#1^4x^2 + 1600\#1^3x^4 - 160\#1^2e^{5c_1}x - 360\#1e^{5c_1}x^3 - 216e^{5c_1}x^5 - e^{10c_1}\&, 1\right]$$

$$y(x) \to \text{Root}\left[1024\#1^5 + 2560\#1^4x^2 + 1600\#1^3x^4 - 160\#1^2e^{5c_1}x - 360\#1e^{5c_1}x^3 - 216e^{5c_1}x^5 - e^{10c_1}\&, 2\right]$$

$$y(x) \to \text{Root}\left[1024\#1^5 + 2560\#1^4x^2 + 1600\#1^3x^4 - 160\#1^2e^{5c_1}x - 360\#1e^{5c_1}x^3 - 216e^{5c_1}x^5 - e^{10c_1}\&, 3\right]$$

$$y(x) \to \text{Root}\left[1024\#1^5 + 2560\#1^4x^2 + 1600\#1^3x^4 - 160\#1^2e^{5c_1}x - 360\#1e^{5c_1}x^3 - 216e^{5c_1}x^5 - e^{10c_1}\&, 4\right]$$

$$y(x) \to \text{Root}\left[1024\#1^5 + 2560\#1^4x^2 + 1600\#1^3x^4 - 160\#1^2e^{5c_1}x - 360\#1e^{5c_1}x^3 - 216e^{5c_1}x^5 - e^{10c_1}\&, 5\right]$$

$$y(x) \to 0$$

3.26 problem 29

Internal problem ID [6820]
Internal file name [`OUTPUT/6067_Thursday_July_28_2022_04_29_00_AM_25586871/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 99. Clairaut's equation. EXERCISES Page 320
Problem number: 29.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"exact", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{y - xy' - x^3 {y'}^2 = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{-1 + \sqrt{1 + 4yx}}{2x^2} \tag{1}$$
$$y' = -\frac{1 + \sqrt{1 + 4yx}}{2x^2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = \frac{-1 + \sqrt{4xy + 1}}{2x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{\left(-1+\sqrt{4xy+1}\right)(b_3 - a_2)}{2x^2} - \frac{\left(-1+\sqrt{4xy+1}\right)^2 a_3}{4x^4} - \left(-\frac{-1+\sqrt{4xy+1}}{x^3} + \frac{y}{x^2\sqrt{4xy+1}}\right)(xa_2 + ya_3 + a_1) - \frac{xb_2 + yb_3 + b_1}{x\sqrt{4xy+1}} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-4b_2x^4\sqrt{4xy+1} + 4x^4b_2 - 4x^3ya_2 - 4x^3yb_3 - 12x^2y^2a_3 + (4xy+1)^{\frac{3}{2}} a_3 + 2\sqrt{4xy+1}\, x^2a_2 + 2\sqrt{4xy+}}{}$$

$$= 0$$

Setting the numerator to zero gives

$$4b_2x^4\sqrt{4xy+1} - 4x^4b_2 + 4x^3ya_2 + 4x^3yb_3 + 12x^2y^2a_3 - (4xy+1)^{\frac{3}{2}} a_3 - 2\sqrt{4xy+1}\, x^2a_2 - 2\sqrt{4xy+1}\, x^2b_3 - 4\sqrt{4xy+1}\, xya_3 - 4x^3b_1 + 12x^2ya_1 - 4\sqrt{4xy+1}\, xa_1 + 2x^2a_2 + 2x^2b_3 + 12xya_3 - a_3\sqrt{4xy+1} + 4xa_1 + 2a_3 = 0 \tag{6E}$$

Simplifying the above gives

$$4b_2x^4\sqrt{4xy+1} + 2(4xy+1)\, x^2a_2 + 2(4xy+1)\, x^2b_3 + 4(4xy+1)\, xya_3 - 4x^4b_2 - 4x^3ya_2 - 4x^3yb_3 - 4x^2y^2a_3 - (4xy+1)^{\frac{3}{2}} a_3 + 4(4xy+1)\, xa_1 - 2\sqrt{4xy+1}\, x^2a_2 - 2\sqrt{4xy+1}\, x^2b_3 - 4\sqrt{4xy+1}\, xya_3 - 4x^3b_1 - 4x^2ya_1 + 2(4xy+1)\, a_3 - 4\sqrt{4xy+1}\, xa_1 - a_3\sqrt{4xy+1} = 0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&4b_2x^4\sqrt{4xy+1}-4x^4b_2+4x^3ya_2+4x^3yb_3+12x^2y^2a_3-4x^3b_1\\
&-2\sqrt{4xy+1}\,x^2a_2-2\sqrt{4xy+1}\,x^2b_3+12x^2ya_1-8\sqrt{4xy+1}\,xya_3+2x^2a_2\\
&+2x^2b_3-4\sqrt{4xy+1}\,xa_1+12xya_3+4xa_1-2a_3\sqrt{4xy+1}+2a_3=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{4xy+1}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x=v_1, y=v_2, \sqrt{4xy+1}=v_3\right\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&4b_2v_1^4v_3+4v_1^3v_2a_2+12v_1^2v_2^2a_3-4v_1^4b_2+4v_1^3v_2b_3+12v_1^2v_2a_1-2v_3v_1^2a_2\\
&-8v_3v_1v_2a_3-4v_1^3b_1-2v_3v_1^2b_3-4v_3v_1a_1+2v_1^2a_2+12v_1v_2a_3+2v_1^2b_3+4v_1a_1\\
&-2a_3v_3+2a_3=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&4b_2v_1^4v_3-4v_1^4b_2+(4a_2+4b_3)\,v_1^3v_2-4v_1^3b_1+12v_1^2v_2^2a_3\\
&+12v_1^2v_2a_1+(-2a_2-2b_3)\,v_1^2v_3+(2a_2+2b_3)\,v_1^2-8v_3v_1v_2a_3\\
&+12v_1v_2a_3-4v_3v_1a_1+4v_1a_1-2a_3v_3+2a_3=0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-4a_1 &= 0 \\
4a_1 &= 0 \\
12a_1 &= 0 \\
-8a_3 &= 0 \\
-2a_3 &= 0 \\
2a_3 &= 0 \\
12a_3 &= 0 \\
-4b_1 &= 0 \\
-4b_2 &= 0 \\
4b_2 &= 0 \\
-2a_2 - 2b_3 &= 0 \\
2a_2 + 2b_3 &= 0 \\
4a_2 + 4b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{-1 + \sqrt{4xy + 1}}{2x^2}\right)(-x) \\
&= \frac{2xy + \sqrt{4xy + 1} - 1}{2x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{2xy+\sqrt{4xy+1}-1}{2x}} dy \end{aligned}$$

Which results in

$$S = \frac{3 \ln (xy - 2)}{4} + \frac{\ln (y)}{4} - \frac{3 \ln \left(\sqrt{4xy + 1} - 3\right)}{4} - \frac{\ln \left(\sqrt{4xy + 1} + 1\right)}{4} + \frac{\ln \left(-1 + \sqrt{4xy + 1}\right)}{4} + \frac{3 \ln (\sqrt{}}{}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-1 + \sqrt{4xy + 1}}{2x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{-3xy + 2\sqrt{4xy + 1}}{(4xy - 8)\, x} \\ S_y &= -\frac{2\left(-2xy + \sqrt{4xy + 1} + 1\right)}{(4xy - 8)\, y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{3}{4x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{3}{4R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{3\ln(R)}{4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{3\ln(yx-2)}{4} + \frac{\ln(y)}{4} - \frac{3\ln\left(\sqrt{1+4yx}-3\right)}{4} - \frac{\ln\left(1+\sqrt{1+4yx}\right)}{4} + \frac{\ln\left(-1+\sqrt{1+4yx}\right)}{4} + \frac{3\ln\left(\sqrt{1+4\right.}{4}$$

Which simplifies to

$$\frac{3\ln(yx-2)}{4} + \frac{\ln(y)}{4} - \frac{3\ln\left(\sqrt{1+4yx}-3\right)}{4} - \frac{\ln\left(1+\sqrt{1+4yx}\right)}{4} + \frac{\ln\left(-1+\sqrt{1+4yx}\right)}{4} + \frac{3\ln\left(\sqrt{1+4\right.}{4}$$

Summary

The solution(s) found are the following

$$\begin{aligned} &\frac{3\ln(yx-2)}{4} + \frac{\ln(y)}{4} - \frac{3\ln\left(\sqrt{1+4yx}-3\right)}{4} - \frac{\ln\left(1+\sqrt{1+4yx}\right)}{4} \\ &+ \frac{\ln\left(-1+\sqrt{1+4yx}\right)}{4} + \frac{3\ln\left(\sqrt{1+4yx}+3\right)}{4} = \frac{3\ln(x)}{4} + c_1 \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} &\frac{3\ln(yx-2)}{4} + \frac{\ln(y)}{4} - \frac{3\ln\left(\sqrt{1+4yx}-3\right)}{4} - \frac{\ln\left(1+\sqrt{1+4yx}\right)}{4} \\ &+ \frac{\ln\left(-1+\sqrt{1+4yx}\right)}{4} + \frac{3\ln\left(\sqrt{1+4yx}+3\right)}{4} = \frac{3\ln(x)}{4} + c_1 \end{aligned}$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = -\frac{\sqrt{4xy+1}+1}{2x^2}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{\left(\sqrt{4xy+1}+1\right)(b_3-a_2)}{2x^2} - \frac{\left(\sqrt{4xy+1}+1\right)^2 a_3}{4x^4} - \left(-\frac{y}{x^2\sqrt{4xy+1}} + \frac{\sqrt{4xy+1}+1}{x^3}\right)(xa_2+ya_3+a_1) + \frac{xb_2+yb_3+b_1}{x\sqrt{4xy+1}} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-4b_2x^4\sqrt{4xy+1} - 4x^4b_2 + 4x^3ya_2 + 4x^3yb_3 + 12x^2y^2a_3 + (4xy+1)^{\frac{3}{2}}a_3 + 2\sqrt{4xy+1}\,x^2a_2 + 2\sqrt{4xy+}}{} = 0$$

Setting the numerator to zero gives

$$4b_2x^4\sqrt{4xy+1} + 4x^4b_2 - 4x^3ya_2 - 4x^3yb_3 - 12x^2y^2a_3 - (4xy+1)^{\frac{3}{2}}a_3 - 2\sqrt{4xy+1}\,x^2a_2 - 2\sqrt{4xy+1}\,x^2b_3 - 4\sqrt{4xy+1}\,xya_3 + 4x^3b_1 - 12x^2ya_1 - 4\sqrt{4xy+1}\,xa_1 - 2x^2a_2 - 2x^2b_3 - 12xya_3 - a_3\sqrt{4xy+1} - 4xa_1 - 2a_3 = 0 \tag{6E}$$

Simplifying the above gives

$$
\begin{aligned}
&4b_2x^4\sqrt{4xy+1}-2(4xy+1)\,x^2a_2-2(4xy+1)\,x^2b_3-4(4xy+1)\,xya_3\\
&+4x^4b_2+4x^3ya_2+4x^3yb_3+4x^2y^2a_3-(4xy+1)^{\frac{3}{2}}\,a_3-4(4xy+1)\,xa_1\\
&-2\sqrt{4xy+1}\,x^2a_2-2\sqrt{4xy+1}\,x^2b_3-4\sqrt{4xy+1}\,xya_3+4x^3b_1\\
&+4x^2ya_1-2(4xy+1)\,a_3-4\sqrt{4xy+1}\,xa_1-a_3\sqrt{4xy+1}=0
\end{aligned} \tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&4b_2x^4\sqrt{4xy+1}+4x^4b_2-4x^3ya_2-4x^3yb_3-12x^2y^2a_3+4x^3b_1\\
&-2\sqrt{4xy+1}\,x^2a_2-2\sqrt{4xy+1}\,x^2b_3-12x^2ya_1-8\sqrt{4xy+1}\,xya_3-2x^2a_2\\
&-2x^2b_3-4\sqrt{4xy+1}\,xa_1-12xya_3-4xa_1-2a_3\sqrt{4xy+1}-2a_3=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{4xy+1}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x=v_1, y=v_2, \sqrt{4xy+1}=v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&4b_2v_1^4v_3-4v_1^3v_2a_2-12v_1^2v_2^2a_3+4v_1^4b_2-4v_1^3v_2b_3-12v_1^2v_2a_1-2v_3v_1^2a_2\\
&-8v_3v_1v_2a_3+4v_1^3b_1-2v_3v_1^2b_3-4v_3v_1a_1-2v_1^2a_2-12v_1v_2a_3-2v_1^2b_3-4v_1a_1\\
&-2a_3v_3-2a_3=0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&4b_2v_1^4v_3+4v_1^4b_2+(-4a_2-4b_3)\,v_1^3v_2+4v_1^3b_1-12v_1^2v_2^2a_3\\
&-12v_1^2v_2a_1+(-2a_2-2b_3)\,v_1^2v_3+(-2a_2-2b_3)\,v_1^2\\
&-8v_3v_1v_2a_3-12v_1v_2a_3-4v_3v_1a_1-4v_1a_1-2a_3v_3-2a_3=0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-12a_1 &= 0\\
-4a_1 &= 0\\
-12a_3 &= 0\\
-8a_3 &= 0\\
-2a_3 &= 0\\
4b_1 &= 0\\
4b_2 &= 0\\
-4a_2 - 4b_3 &= 0\\
-2a_2 - 2b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0\\
a_2 &= -b_3\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -x\\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi\\
&= y - \left(-\frac{\sqrt{4xy+1}+1}{2x^2}\right)(-x)\\
&= \frac{2xy - \sqrt{4xy+1} - 1}{2x}\\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{2xy-\sqrt{4xy+1}-1}{2x}} dy \end{aligned}$$

Which results in

$$S = \frac{3\ln(xy-2)}{4} + \frac{\ln(y)}{4} + \frac{3\ln\left(\sqrt{4xy+1}-3\right)}{4} + \frac{\ln\left(\sqrt{4xy+1}+1\right)}{4} - \frac{\ln\left(-1+\sqrt{4xy+1}\right)}{4} - \frac{3\ln\left(\sqrt{}\right.}{}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{\sqrt{4xy+1}+1}{2x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{3xy + 2\sqrt{4xy+1}}{(4xy-8)\,x} \\ S_y &= \frac{4xy + 2\sqrt{4xy+1} - 2}{(4xy-8)\,y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{3}{4x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{3}{4R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{3\ln(R)}{4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{3\ln(yx-2)}{4} + \frac{\ln(y)}{4} + \frac{3\ln\left(\sqrt{1+4yx}-3\right)}{4} + \frac{\ln\left(1+\sqrt{1+4yx}\right)}{4} - \frac{\ln\left(-1+\sqrt{1+4yx}\right)}{4} - \frac{3\ln\left(\sqrt{1+4\right.}}{4}$$

Which simplifies to

$$\frac{3\ln(yx-2)}{4} + \frac{\ln(y)}{4} + \frac{3\ln\left(\sqrt{1+4yx}-3\right)}{4} + \frac{\ln\left(1+\sqrt{1+4yx}\right)}{4} - \frac{\ln\left(-1+\sqrt{1+4yx}\right)}{4} - \frac{3\ln\left(\sqrt{1+4\right.}}{4}$$

Summary

The solution(s) found are the following

$$\frac{3\ln(yx-2)}{4} + \frac{\ln(y)}{4} + \frac{3\ln\left(\sqrt{1+4yx}-3\right)}{4} + \frac{\ln\left(1+\sqrt{1+4yx}\right)}{4} - \frac{\ln\left(-1+\sqrt{1+4yx}\right)}{4} - \frac{3\ln\left(\sqrt{1+4yx}+3\right)}{4} = \frac{3\ln(x)}{4} + c_1 \tag{1}$$

Verification of solutions

$$\frac{3\ln(yx-2)}{4} + \frac{\ln(y)}{4} + \frac{3\ln\left(\sqrt{1+4yx}-3\right)}{4} + \frac{\ln\left(1+\sqrt{1+4yx}\right)}{4} - \frac{\ln\left(-1+\sqrt{1+4yx}\right)}{4} - \frac{3\ln\left(\sqrt{1+4yx}+3\right)}{4} = \frac{3\ln(x)}{4} + c_1$$

Verified OK.

✗ Solution by Maple

```
dsolve(y(x)=x*diff(y(x),x)+x^3*diff(y(x),x)^2,y(x), singsol=all)
```

No solution found

✓ Solution by Mathematica

Time used: 105.562 (sec). Leaf size: 7052

```
DSolve[y[x]==x*y'[x]+x^3*(y'[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

4 CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324

4.1 problem 1

Internal problem ID [6821]
Internal file name [`OUTPUT/6068_Thursday_July_28_2022_04_29_02_AM_22805052/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y], [_2nd_order, _reducible, _mu_y_y1]]
```

$$\boxed{y'' - xy'^3 = 0}$$

4.1.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - xp(x)^3 = 0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned} p' &= F(x,p) \\ &= f(x)g(p) \\ &= x\,p^3 \end{aligned}$$

Where $f(x) = x$ and $g(p) = p^3$. Integrating both sides gives

$$\frac{1}{p^3}\,dp = x\,dx$$
$$\int \frac{1}{p^3}\,dp = \int x\,dx$$
$$-\frac{1}{2p^2} = \frac{x^2}{2} + c_1$$

The solution is

$$-\frac{1}{2p\left(x\right)^2} - \frac{x^2}{2} - c_1 = 0$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$-\frac{1}{2y'^2} - \frac{x^2}{2} - c_1 = 0$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = -\frac{1}{\sqrt{-x^2 - 2c_1}} \tag{1}$$
$$y' = \frac{1}{\sqrt{-x^2 - 2c_1}} \tag{2}$$

Now each one of the above ODE is solved.

<u>Solving equation (1)</u>

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{1}{\sqrt{-x^2 - 2c_1}}\,\mathrm{d}x \\ &= -\arctan\left(\frac{x}{\sqrt{-x^2 - 2c_1}}\right) + c_2 \end{aligned}$$

<u>Solving equation (2)</u>

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{1}{\sqrt{-x^2 - 2c_1}}\,\mathrm{d}x \\ &= \arctan\left(\frac{x}{\sqrt{-x^2 - 2c_1}}\right) + c_3 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\arctan\left(\frac{x}{\sqrt{-x^2-2c_1}}\right)+c_2 \tag{1}$$

$$y = \arctan\left(\frac{x}{\sqrt{-x^2-2c_1}}\right)+c_3 \tag{2}$$

Verification of solutions

$$y = -\arctan\left(\frac{x}{\sqrt{-x^2-2c_1}}\right)+c_2$$

Verified OK.

$$y = \arctan\left(\frac{x}{\sqrt{-x^2-2c_1}}\right)+c_3$$

Verified OK.

4.1.2 Maple step by step solution

Let's solve

$y''-xy'^3=0$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u=y'$ to reduce order of ODE

  $u'(x)-xu(x)^3=0$

- Separate variables

  $\frac{u'(x)}{u(x)^3}=x$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{u(x)^3}dx=\int xdx+c_1$

- Evaluate integral

  $-\frac{1}{2u(x)^2}=\frac{x^2}{2}+c_1$

- Solve for $u(x)$

  $\left\{u(x)=\frac{1}{\sqrt{-x^2-2c_1}}, u(x)=-\frac{1}{\sqrt{-x^2-2c_1}}\right\}$

- Solve 1st ODE for $u(x)$

$u(x) = \frac{1}{\sqrt{-x^2-2c_1}}$

- Make substitution $u = y'$

  $y' = \frac{1}{\sqrt{-x^2-2c_1}}$

- Integrate both sides to solve for y

  $\int y' dx = \int \frac{1}{\sqrt{-x^2-2c_1}} dx + c_2$

- Compute integrals

  $y = \arctan\left(\frac{x}{\sqrt{-x^2-2c_1}}\right) + c_2$

- Solve 2nd ODE for $u(x)$

  $u(x) = -\frac{1}{\sqrt{-x^2-2c_1}}$

- Make substitution $u = y'$

  $y' = -\frac{1}{\sqrt{-x^2-2c_1}}$

- Integrate both sides to solve for y

  $\int y' dx = \int -\frac{1}{\sqrt{-x^2-2c_1}} dx + c_2$

- Compute integrals

  $y = -\arctan\left(\frac{x}{\sqrt{-x^2-2c_1}}\right) + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = _a*_b(_a)^3, _b(_a), HINT = [[_a, -_b]]
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, -_b]
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 37

```
dsolve(diff(y(x),x$2)=x*(diff(y(x),x))^3,y(x), singsol=all)
```

$$y(x) = \arctan\left(\frac{x}{\sqrt{-x^2+c_1}}\right)+c_2$$
$$y(x) = -\arctan\left(\frac{x}{\sqrt{-x^2+c_1}}\right)+c_2$$

✓ Solution by Mathematica

Time used: 10.922 (sec). Leaf size: 57

```
DSolve[y''[x]==x*(y'[x])^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - \arctan\left(\frac{x}{\sqrt{-x^2-2c_1}}\right)$$
$$y(x) \to \arctan\left(\frac{x}{\sqrt{-x^2-2c_1}}\right)+c_2$$
$$y(x) \to c_2$$

4.2 problem 2

Internal problem ID [6822]
Internal file name [`OUTPUT/6069_Thursday_July_28_2022_04_29_03_AM_89762227/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{x^2y'' + y'^2 - 2xy' = 0}$$

With initial conditions

$$[y(2) = 5, y'(2) = -4]$$

4.2.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$x^2p'(x) + (p(x) - 2x)\,p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Using the change of variables $p(x) = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x^2(u'(x)\,x + u(x)) + (u(x)\,x - 2x)\,u(x)\,x = 0$$

In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{u(u-1)}{x}
\end{aligned}
$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u(u-1)$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u\left(u-1\right)}\,du &= -\frac{1}{x}\,dx \\
\int \frac{1}{u\left(u-1\right)}\,du &= \int -\frac{1}{x}\,dx \\
-\ln\left(u\right) + \ln\left(u-1\right) &= -\ln\left(x\right) + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
e^{-\ln(u)+\ln(u-1)} = e^{-\ln(x)+c_2}
$$

Which simplifies to

$$
\frac{u-1}{u} = \frac{c_3}{x}
$$

Therefore the solution $p(x)$ is

$$
\begin{aligned}
p(x) &= ux \\
&= -\frac{x^2}{c_3 - x}
\end{aligned}
$$

Initial conditions are used to solve for c_3. Substituting $x = 2$ and $p = -4$ in the above solution gives an equation to solve for the constant of integration.

$$
-4 = -\frac{4}{c_3 - 2}
$$

$$
c_3 = 3
$$

Substituting c_3 found above in the general solution gives

$$
p(x) = \frac{x^2}{-3 + x}
$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{x^2}{-3 + x}$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{x^2}{-3 + x} \, \mathrm{d}x \\ &= \frac{x^2}{2} + 3x + 9 \ln(-3 + x) + c_4 \end{aligned}$$

Initial conditions are used to solve for c_4. Substituting $x = 2$ and $y = 5$ in the above solution gives an equation to solve for the constant of integration.

$$5 = 9i\pi + c_4 + 8$$

$$c_4 = -9i\pi - 3$$

Substituting c_4 found above in the general solution gives

$$y = \frac{x^2}{2} + 3x + 9 \ln(-3 + x) - 9i\pi - 3$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = \frac{x^2}{2} + 3x + 9 \ln(-3 + x) - 9i\pi - 3 \tag{1}$$

Verification of solutions

$$y = \frac{x^2}{2} + 3x + 9 \ln(-3 + x) - 9i\pi - 3$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -_b(_a)*(_b(_a)-2*_a)/_a^2, _b(_a), HIN
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, _b]
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 24

```
dsolve([x^2*diff(y(x),x$2)+diff(y(x),x)^2-2*x*diff(y(x),x)=0,y(2) = 5, D(y)(2) = -4],y(x), s
```

$$y(x)=\frac{x^2}{2}+3x+9\ln(x-3)-3-9i\pi$$

✓ Solution by Mathematica

Time used: 0.478 (sec). Leaf size: 28

```
DSolve[{x^2*y''[x]+(y'[x])^2-2*x*y'[x]==0,{y[2]==5,y'[2]==-4}},y[x],x,IncludeSingularSolutio
```

$$y(x)\to\frac{x^2}{2}+3x+9\log(x-3)-9i\pi-3$$

4.3 problem 3

Internal problem ID [6823]
Internal file name [`OUTPUT/6070_Thursday_July_28_2022_04_29_06_AM_15447379/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{x^2y'' + y'^2 - 2xy' = 0}$$

With initial conditions

$$[y(2) = 5, y'(2) = 2]$$

4.3.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$x^2p'(x) + (p(x) - 2x)\,p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Using the change of variables $p(x) = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x^2(u'(x)\,x + u(x)) + (u(x)\,x - 2x)\,u(x)\,x = 0$$

In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{u(u-1)}{x}
\end{aligned}
$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u(u-1)$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u\left(u-1\right)}\, du &= -\frac{1}{x}\, dx \\
\int \frac{1}{u\left(u-1\right)}\, du &= \int -\frac{1}{x}\, dx \\
-\ln\left(u\right) + \ln\left(u-1\right) &= -\ln\left(x\right) + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
\mathrm{e}^{-\ln(u)+\ln(u-1)} = \mathrm{e}^{-\ln(x)+c_2}
$$

Which simplifies to

$$
\frac{u-1}{u} = \frac{c_3}{x}
$$

Therefore the solution $p(x)$ is

$$
\begin{aligned}
p(x) &= ux \\
&= -\frac{x^2}{c_3 - x}
\end{aligned}
$$

Initial conditions are used to solve for c_3. Substituting $x = 2$ and $p = 2$ in the above solution gives an equation to solve for the constant of integration.

$$
2 = -\frac{4}{c_3 - 2}
$$

$$
c_3 = 0
$$

Substituting c_3 found above in the general solution gives

$$
p(x) = x
$$

Since $p = y'$ then the new first order ode to solve is

$$y' = x$$

Integrating both sides gives

$$\begin{aligned} y &= \int x \,\mathrm{d}x \\ &= \frac{x^2}{2} + c_4 \end{aligned}$$

Initial conditions are used to solve for c_4. Substituting $x = 2$ and $y = 5$ in the above solution gives an equation to solve for the constant of integration.

$$5 = 2 + c_4$$

$$c_4 = 3$$

Substituting c_4 found above in the general solution gives

$$y = \frac{x^2}{2} + 3$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = \frac{x^2}{2} + 3 \tag{1}$$

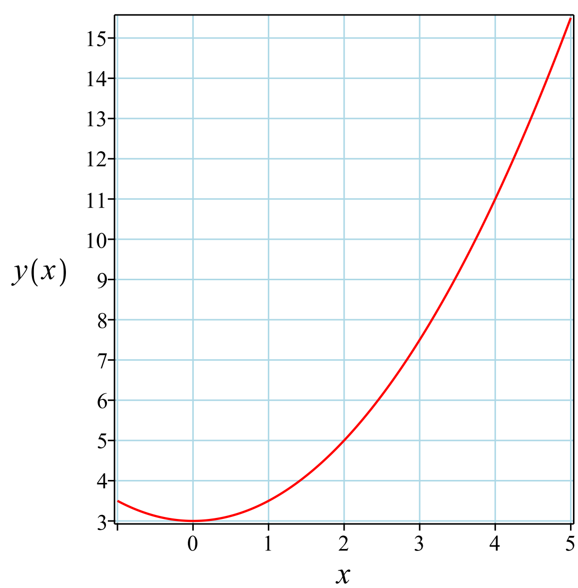

Figure 1: Solution plot

Verification of solutions

$$y = \frac{x^2}{2} + 3$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -_b(_a)*(_b(_a)-2*_a)/_a^2, _b(_a), HIN
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, _b]
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 11

```
dsolve([x^2*diff(y(x),x$2)+diff(y(x),x)^2-2*x*diff(y(x),x)=0,y(2) = 5, D(y)(2) = 2],y(x), si
```

$$y(x) = \frac{x^2}{2} + 3$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 14

```
DSolve[{x^2*y''[x]+(y'[x])^2-2*x*y'[x]==0,{y[2]==5,y'[2]==2}},y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{1}{2}(x^2 + 6)$$

4.4 problem 4

Internal problem ID [6824]
Internal file name [`OUTPUT/6071_Thursday_July_28_2022_04_29_09_AM_95148677/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is", "second_order_ode_missing_x", "exact nonlinear second order ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _exact, _nonlinear],
   _Liouville, [_2nd_order, _reducible, _mu_x_y1], [_2nd_order,
   _reducible, _mu_xy]]
```

$$\boxed{yy'' + y'^2 = 0}$$

4.4.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left(yy'' + y'^2\right) dx = 0$$
$$yy' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{y}{c_1} dy = x + c_2$$
$$\frac{y^2}{2c_1} = x + c_2$$

Solving for y gives these solutions

$$y_1 = \sqrt{2c_1c_2 + 2c_1x}$$
$$y_2 = -\sqrt{2c_1c_2 + 2c_1x}$$

Summary

The solution(s) found are the following

$$y = \sqrt{2c_1c_2 + 2c_1x} \tag{1}$$
$$y = -\sqrt{2c_1c_2 + 2c_1x} \tag{2}$$

Verification of solutions

$$y = \sqrt{2c_1c_2 + 2c_1x}$$

Verified OK.

$$y = -\sqrt{2c_1c_2 + 2c_1x}$$

Verified OK.

4.4.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y)\left(\frac{d}{dy}p(y)\right) + p(y)^2 = 0$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned}
p' &= F(y,p) \\
&= f(y)g(p) \\
&= -\frac{p}{y}
\end{aligned}$$

Where $f(y) = -\frac{1}{y}$ and $g(p) = p$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{p}\, dp &= -\frac{1}{y}\, dy \\
\int \frac{1}{p}\, dp &= \int -\frac{1}{y}\, dy \\
\ln(p) &= -\ln(y) + c_1 \\
p &= \mathrm{e}^{-\ln(y)+c_1} \\
&= \frac{c_1}{y}
\end{aligned}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{c_1}{y}$$

Integrating both sides gives

$$\begin{aligned}
\int \frac{y}{c_1} dy &= x + c_2 \\
\frac{y^2}{2c_1} &= x + c_2
\end{aligned}$$

Solving for y gives these solutions

$$\begin{aligned}
y_1 &= \sqrt{2c_1c_2 + 2c_1x} \\
y_2 &= -\sqrt{2c_1c_2 + 2c_1x}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sqrt{2c_1c_2 + 2c_1x} \tag{1}$$
$$y = -\sqrt{2c_1c_2 + 2c_1x} \tag{2}$$

Verification of solutions

$$y = \sqrt{2c_1c_2 + 2c_1x}$$

Verified OK.

$$y = -\sqrt{2c_1c_2 + 2c_1x}$$

Verified OK.

4.4.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$yy'' + y'^2 = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left(yy'' + y'^2\right) dx = 0$$
$$yy' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{y}{c_1} dy = x + c_2$$
$$\frac{y^2}{2c_1} = x + c_2$$

Solving for y gives these solutions

$$y_1 = \sqrt{2c_1c_2 + 2c_1x}$$
$$y_2 = -\sqrt{2c_1c_2 + 2c_1x}$$

Summary
The solution(s) found are the following

$$y = \sqrt{2c_1c_2 + 2c_1x} \tag{1}$$
$$y = -\sqrt{2c_1c_2 + 2c_1x} \tag{2}$$

Verification of solutions

$$y = \sqrt{2c_1c_2 + 2c_1x}$$

Verified OK.

$$y = -\sqrt{2c_1c_2 + 2c_1x}$$

Verified OK.

4.4.4 Solving as exact nonlinear second order ode ode

An exact non-linear second order ode has the form

$$a_2(x,y,y')\,y'' + a_1(x,y,y')\,y' + a_0(x,y,y') = 0$$

Where the following conditions are satisfied

$$\begin{aligned}
\frac{\partial a_2}{\partial y} &= \frac{\partial a_1}{\partial y'} \\
\frac{\partial a_2}{\partial x} &= \frac{\partial a_0}{\partial y'} \\
\frac{\partial a_1}{\partial x} &= \frac{\partial a_0}{\partial y}
\end{aligned}$$

Looking at the the ode given we see that

$$\begin{aligned}
a_2 &= y \\
a_1 &= y' \\
a_0 &= 0
\end{aligned}$$

Applying the conditions to the above shows this is a nonlinear exact second order ode. Therefore it can be reduced to first order ode given by

$$\begin{aligned}
\int a_2\, dy' + \int a_1\, dy + \int a_0\, dx &= c_1 \\
\int y\, dy' + \int y'\, dy + \int 0\, dx &= c_1
\end{aligned}$$

Which results in

$$2yy' = c_1$$

Which is now solved Integrating both sides gives

$$\begin{aligned}
\int \frac{2y}{c_1} dy &= x + c_2 \\
\frac{y^2}{c_1} &= x + c_2
\end{aligned}$$

Solving for y gives these solutions

$$\begin{aligned}
y_1 &= \sqrt{c_1 c_2 + c_1 x} \\
y_2 &= -\sqrt{c_1 c_2 + c_1 x}
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \sqrt{c_1 c_2 + c_1 x} \tag{1}$$
$$y = -\sqrt{c_1 c_2 + c_1 x} \tag{2}$$

Verification of solutions

$$y = \sqrt{c_1 c_2 + c_1 x}$$

Verified OK.

$$y = -\sqrt{c_1 c_2 + c_1 x}$$

Verified OK.

4.4.5 Maple step by step solution

Let's solve

$y y'' + {y'}^2 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y' \left(\frac{d}{dy} u(y)\right) = y''$

- Substitute in the definition of u

  $u(y) \left(\frac{d}{dy} u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y) \left(\frac{d}{dy} u(y)\right)$ to reduce order of ODE

  $y u(y) \left(\frac{d}{dy} u(y)\right) + u(y)^2 = 0$

- Separate variables

  $\frac{\frac{d}{dy} u(y)}{u(y)} = -\frac{1}{y}$

- Integrate both sides with respect to y

$\int \frac{\frac{d}{dy}u(y)}{u(y)}dy = \int -\frac{1}{y}dy + c_1$

- Evaluate integral

  $\ln(u(y)) = -\ln(y) + c_1$

- Solve for $u(y)$

  $u(y) = \frac{e^{c_1}}{y}$

- Solve 1st ODE for $u(y)$

  $u(y) = \frac{e^{c_1}}{y}$

- Revert to original variables with substitution $u(y) = y', y = y$

  $y' = \frac{e^{c_1}}{y}$

- Separate variables

  $yy' = e^{c_1}$

- Integrate both sides with respect to x

  $\int yy'dx = \int e^{c_1}dx + c_2$

- Evaluate integral

  $\frac{y^2}{2} = e^{c_1}x + c_2$

- Solve for y

  $\left\{y = \sqrt{2\,e^{c_1}x + 2c_2}, y = -\sqrt{2\,e^{c_1}x + 2c_2}\right\}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 33

```
dsolve(y(x)*diff(y(x),x$2)+diff(y(x),x)^2=0,y(x), singsol=all)
```

$$
\begin{aligned}
y(x) &= 0 \\
y(x) &= \sqrt{2c_1x + 2c_2} \\
y(x) &= -\sqrt{2c_1x + 2c_2}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.172 (sec). Leaf size: 20

```
DSolve[y[x]*y''[x]+(y'[x])^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2\sqrt{2x - c_1}$$

4.5 problem 5

Internal problem ID [6825]
Internal file name [`OUTPUT/6072_Thursday_July_28_2022_04_29_13_AM_10730317/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1],
   [_2nd_order, _reducible, _mu_y_y1]]
```

$$\boxed{y^2y'' + y'^3 = 0}$$

4.5.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$y^2 p(y) \left(\frac{d}{dy} p(y)\right) + p(y)^3 = 0$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned} p' &= F(y, p) \\ &= f(y) g(p) \\ &= -\frac{p^2}{y^2} \end{aligned}$$

Where $f(y) = -\frac{1}{y^2}$ and $g(p) = p^2$. Integrating both sides gives

$$\begin{aligned} \frac{1}{p^2}\, dp &= -\frac{1}{y^2}\, dy \\ \int \frac{1}{p^2}\, dp &= \int -\frac{1}{y^2}\, dy \\ -\frac{1}{p} &= \frac{1}{y} + c_1 \end{aligned}$$

The solution is

$$-\frac{1}{p(y)} - \frac{1}{y} - c_1 = 0$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$-\frac{1}{y'} - \frac{1}{y} - c_1 = 0$$

Integrating both sides gives

$$\begin{aligned} \int -\frac{c_1 y + 1}{y} dy &= x + c_2 \\ -c_1 y - \ln(y) &= x + c_2 \end{aligned}$$

Solving for y gives these solutions

Summary

The solution(s) found are the following

$$y = e^{-\operatorname{LambertW}\left(c_1 e^{-x-c_2}\right) - x - c_2} \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{-\operatorname{LambertW}\left(c_1 \mathrm{e}^{-x-c_2}\right)-x-c_2}$$

Verified OK.

4.5.2 Maple step by step solution

Let's solve

$y^2 y'' + {y'}^3 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y' \left(\frac{d}{dy} u(y)\right) = y''$

- Substitute in the definition of u

  $u(y) \left(\frac{d}{dy} u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y) \left(\frac{d}{dy} u(y)\right)$ to reduce order of ODE

  $y^2 u(y) \left(\frac{d}{dy} u(y)\right) + u(y)^3 = 0$

- Separate variables

  $\frac{\frac{d}{dy} u(y)}{u(y)^2} = -\frac{1}{y^2}$

- Integrate both sides with respect to y

  $\int \frac{\frac{d}{dy} u(y)}{u(y)^2} dy = \int -\frac{1}{y^2} dy + c_1$

- Evaluate integral

  $-\frac{1}{u(y)} = \frac{1}{y} + c_1$

- Solve for $u(y)$

  $u(y) = -\frac{y}{c_1 y + 1}$

- Solve 1st ODE for $u(y)$

  $u(y) = -\frac{y}{c_1 y+1}$
- Revert to original variables with substitution $u(y) = y', y = y$

  $y' = -\frac{y}{1+c_1 y}$
- Separate variables

  $\frac{y'(1+c_1 y)}{y} = -1$
- Integrate both sides with respect to x

  $\int \frac{y'(1+c_1 y)}{y} dx = \int (-1) \, dx + c_2$
- Evaluate integral

  $c_1 y + \ln(y) = -x + c_2$
- Solve for y

  $y = \mathrm{e}^{-\mathit{LambertW}\left(c_1 \mathrm{e}^{-x+c_2}\right)-x+c_2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+_b(_a)^3/_a^2 = 0, _b(_a)`   ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   <- Bernoulli successful
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 29

```
dsolve(y(x)^2*diff(y(x),x$2)+diff(y(x),x)^3=0,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = c_1$$
$$y(x) = -\frac{\mathrm{LambertW}\left(-c_1 \mathrm{e}^{-x-c_2}\right)}{c_1}$$

✓ Solution by Mathematica

Time used: 0.609 (sec). Leaf size: 37

```
DSolve[y[x]^2*y''[x]+(y'[x])^3==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 \left(1 + \frac{1}{\text{InverseFunction}\left[-\frac{1}{\#1} - \log(\#1) + \log(\#1+1)\&\right][-x + c_1]}\right)$$

4.6 problem 6

Internal problem ID [6826]
Internal file name [`OUTPUT/6073_Thursday_July_28_2022_04_29_16_AM_33134786/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], _Liouville, [_2nd_order, _reducible,
    _mu_x_y1], [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{(y+1)\, y'' - {y'}^2 = 0}$$

4.6.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$(y+1)\,p(y)\left(\frac{d}{dy}p(y)\right)-p(y)^2=0$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned}
p' &= F(y,p)\\
&= f(y)g(p)\\
&= \frac{p}{y+1}
\end{aligned}$$

Where $f(y)=\frac{1}{y+1}$ and $g(p)=p$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{p}\,dp &= \frac{1}{y+1}\,dy\\
\int \frac{1}{p}\,dp &= \int \frac{1}{y+1}\,dy\\
\ln(p) &= \ln(y+1)+c_1\\
p &= \mathrm{e}^{\ln(y+1)+c_1}\\
&= c_1(y+1)
\end{aligned}$$

For solution (1) found earlier, since $p=y'$ then we now have a new first order ode to solve which is

$$y'=c_1(y+1)$$

Integrating both sides gives

$$\begin{aligned}
\int \frac{1}{c_1(y+1)}dy &= \int dx\\
\frac{\ln(y+1)}{c_1} &= x+c_2
\end{aligned}$$

Raising both side to exponential gives

$$\mathrm{e}^{\frac{\ln(y+1)}{c_1}}=\mathrm{e}^{x+c_2}$$

Which simplifies to

$$(y+1)^{\frac{1}{c_1}}=c_3\mathrm{e}^{x}$$

<u>Summary</u>
The solution(s) found are the following

$$y=(c_3\mathrm{e}^{x})^{c_1}-1 \tag{1}$$

Verification of solutions

$$y = (c_3 e^x)^{c_1} - 1$$

Verified OK.

4.6.2 Maple step by step solution

Let's solve

$(y+1)\, y'' - y'^2 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y' \left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y) \left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $(y+1)\, u(y) \left(\frac{d}{dy}u(y)\right) - u(y)^2 = 0$

- Separate variables

  $\frac{\frac{d}{dy}u(y)}{u(y)} = \frac{1}{y+1}$

- Integrate both sides with respect to y

  $\int \frac{\frac{d}{dy}u(y)}{u(y)} dy = \int \frac{1}{y+1} dy + c_1$

- Evaluate integral

  $\ln(u(y)) = \ln(y+1) + c_1$

- Solve for $u(y)$

  $u(y) = e^{c_1}(y+1)$

- Solve 1st ODE for $u(y)$

$u(y) = e^{c_1}(y+1)$

- Revert to original variables with substitution $u(y) = y', y = y$

  $y' = e^{c_1}(y+1)$

- Separate variables

  $\frac{y'}{y+1} = e^{c_1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y+1}dx = \int e^{c_1}dx + c_2$

- Evaluate integral

  $\ln(y+1) = e^{c_1}x + c_2$

- Solve for y

  $y = e^{e^{c_1}x+c_2} - 1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 16

```
dsolve((y(x)+1)*diff(y(x),x$2)=diff(y(x),x)^2,y(x), singsol=all)
```

$$y(x) = -1$$
$$y(x) = e^{c_1 x}c_2 - 1$$

✓ Solution by Mathematica

Time used: 1.193 (sec). Leaf size: 26

```
DSolve[(y[x]+1)*y''[x]==(y'[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -1 + \frac{e^{c_1(x+c_2)}}{c_1}$$
$$y(x) \to \text{Indeterminate}$$

4.7 problem 7

Internal problem ID [6827]
Internal file name [`OUTPUT/6074_Thursday_July_28_2022_04_29_17_AM_48951552/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_y_y1]]
```

$$\boxed{2ay'' + y'^3 = 0}$$

4.7.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$2ap'(x) + p(x)^3 = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int -\frac{2a}{p^3}dp = x + c_1$$
$$\frac{a}{p^2} = x + c_1$$

Solving for p gives these solutions

$$
\begin{aligned}
p_1 &= \frac{\sqrt{(x+c_1)\,a}}{x+c_1} \\
p_2 &= -\frac{\sqrt{(x+c_1)\,a}}{x+c_1}
\end{aligned}
$$

For solution (1) found earlier, since $p = y'$ then the new first order ode to solve is

$$
y' = \frac{\sqrt{(x+c_1)\,a}}{x+c_1}
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int \frac{\sqrt{(x+c_1)\,a}}{x+c_1}\,\mathrm{d}x \\
&= 2\sqrt{(x+c_1)\,a} + c_2
\end{aligned}
$$

Since $p = y'$ then the new first order ode to solve is

$$
y' = -\frac{\sqrt{(x+c_1)\,a}}{x+c_1}
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int -\frac{\sqrt{(x+c_1)\,a}}{x+c_1}\,\mathrm{d}x \\
&= -2\sqrt{(x+c_1)\,a} + c_3
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = 2\sqrt{(x+c_1)\,a} + c_2 \tag{1}
$$

$$
y = -2\sqrt{(x+c_1)\,a} + c_3 \tag{2}
$$

Verification of solutions

$$
y = 2\sqrt{(x+c_1)\,a} + c_2
$$

Verified OK.

$$
y = -2\sqrt{(x+c_1)\,a} + c_3
$$

Verified OK.

4.7.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$2ap(y)\left(\frac{d}{dy}p(y)\right) + p(y)^3 = 0$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$\begin{aligned} \int -\frac{2a}{p^2}dp &= y + c_1 \\ \frac{2a}{p} &= y + c_1 \end{aligned}$$

Solving for p gives these solutions

$$p_1 = \frac{2a}{y + c_1}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{2a}{y + c_1}$$

Integrating both sides gives

$$\begin{aligned} \int \frac{y + c_1}{2a}dy &= x + c_2 \\ \frac{\frac{1}{2}y^2 + c_1 y}{2a} &= x + c_2 \end{aligned}$$

Solving for y gives these solutions

$$
\begin{aligned}
y_1 &= -c_1 - \sqrt{4ac_2 + 4xa + c_1^2} \\
y_2 &= -c_1 + \sqrt{4ac_2 + 4xa + c_1^2}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = -c_1 - \sqrt{4ac_2 + 4xa + c_1^2} \tag{1}
$$

$$
y = -c_1 + \sqrt{4ac_2 + 4xa + c_1^2} \tag{2}
$$

Verification of solutions

$$
y = -c_1 - \sqrt{4ac_2 + 4xa + c_1^2}
$$

Verified OK.

$$
y = -c_1 + \sqrt{4ac_2 + 4xa + c_1^2}
$$

Verified OK.

4.7.3 Maple step by step solution

Let's solve

$2ay'' + y'^3 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $2au'(x) + u(x)^3 = 0$

- Separate variables

  $\frac{u'(x)}{u(x)^3} = -\frac{1}{2a}$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{u(x)^3} dx = \int -\frac{1}{2a} dx + c_1$

- Evaluate integral

  $-\frac{1}{2u(x)^2} = -\frac{x}{2a} + c_1$

- Solve for $u(x)$

  $\left\{u(x) = \frac{\sqrt{-(2c_1a-x)a}}{2c_1a-x}, u(x) = -\frac{\sqrt{-(2c_1a-x)a}}{2c_1a-x}\right\}$

- Solve 1st ODE for $u(x)$

  $u(x) = \frac{\sqrt{-(2c_1a-x)a}}{2c_1a-x}$

- Make substitution $u = y'$

  $y' = \frac{\sqrt{-(2c_1a-x)a}}{2c_1a-x}$

- Integrate both sides to solve for y

  $\int y'dx = \int \frac{\sqrt{-(2c_1a-x)a}}{2c_1a-x}dx + c_2$

- Compute integrals

  $y = -2\sqrt{-(2c_1a-x)\,a} + c_2$

- Solve 2nd ODE for $u(x)$

  $u(x) = -\frac{\sqrt{-(2c_1a-x)a}}{2c_1a-x}$

- Make substitution $u = y'$

  $y' = -\frac{\sqrt{-(2c_1a-x)a}}{2c_1a-x}$

- Integrate both sides to solve for y

  $\int y'dx = \int -\frac{\sqrt{-(2c_1a-x)a}}{2c_1a-x}dx + c_2$

- Compute integrals

  $y = 2\sqrt{-(2c_1a-x)\,a} + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -(1/2)*_b(_a)^3/a, _b(_a), HINT = [[1,
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[1, 0], [y, -_b^2], [_a, -1/
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 29

```
dsolve(2*a*diff(y(x),x$2)+diff(y(x),x)^3=0,y(x), singsol=all)
```

$$y(x) = 2\sqrt{(x + c_1)\, a} + c_2$$
$$y(x) = -2\sqrt{(x + c_1)\, a} + c_2$$

✓ Solution by Mathematica

Time used: 0.33 (sec). Leaf size: 51

```
DSolve[2*a*y''[x]+(y'[x])^3==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - 2\sqrt{a}\sqrt{x - 2ac_1}$$
$$y(x) \to 2\sqrt{a}\sqrt{x - 2ac_1} + c_2$$

4.8 problem 9

Internal problem ID [6828]
Internal file name [`OUTPUT/6075_Thursday_July_28_2022_04_29_19_AM_15098074/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{xy'' - y' = x^5}$$

With initial conditions

$$\left[y(1) = \frac{1}{2}, y'(1) = 1\right]$$

4.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= 0 \\ F &= x^4 \end{aligned}$$

Hence the ode is

$$y'' - \frac{y'}{x} = x^4$$

The domain of $p(x) = -\frac{1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $F = x^4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

4.8.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\begin{aligned} \int (xy'' - y')\, dx &= \int x^5 dx \\ xy' - 2y &= \frac{x^6}{6} + c_1 \end{aligned}$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= \frac{x^6 + 6c_1}{6x} \end{aligned}$$

Hence the ode is

$$y' - \frac{2y}{x} = \frac{x^6 + 6c_1}{6x}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{2}{x}dx} \\ &= \frac{1}{x^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{x^6 + 6c_1}{6x}\right) \\ \frac{d}{dx}\left(\frac{y}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(\frac{x^6 + 6c_1}{6x}\right) \\ d\left(\frac{y}{x^2}\right) &= \left(\frac{x^6 + 6c_1}{6x^3}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x^2} &= \int \frac{x^6 + 6c_1}{6x^3} dx \\ \frac{y}{x^2} &= \frac{x^4}{24} - \frac{c_1}{2x^2} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = x^2\left(\frac{x^4}{24} - \frac{c_1}{2x^2}\right) + c_2 x^2$$

which simplifies to

$$y = \frac{1}{24}x^6 - \frac{1}{2}c_1 + c_2 x^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{1}{24}x^6 - \frac{1}{2}c_1 + c_2 x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{1}{2}$ and $x = 1$ in the above gives

$$\frac{1}{2} = \frac{1}{24} - \frac{c_1}{2} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{1}{4}x^5 + 2c_2x$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$1 = \frac{1}{4} + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{1}{6}$$
$$c_2 = \frac{3}{8}$$

Substituting these values back in above solution results in

$$y = \frac{1}{24}x^6 + \frac{1}{12} + \frac{3}{8}x^2$$

Summary

The solution(s) found are the following

$$y = \frac{1}{24}x^6 + \frac{1}{12} + \frac{3}{8}x^2 \tag{1}$$

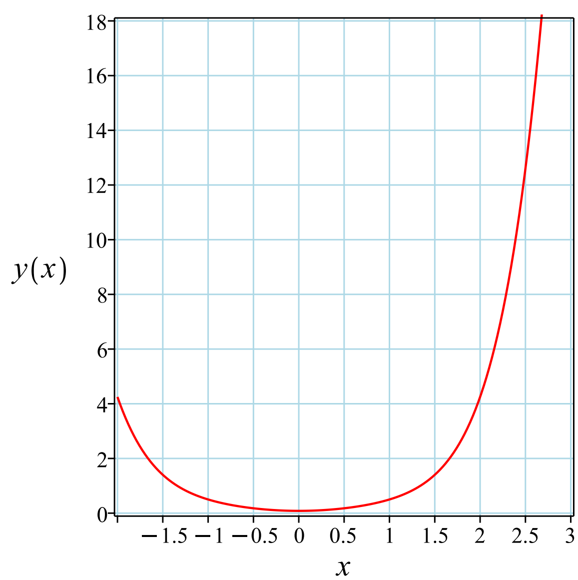

Figure 2: Solution plot

Verification of solutions

$$y = \frac{1}{24}x^6 + \frac{1}{12} + \frac{3}{8}x^2$$

Verified OK.

4.8.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$xp'(x) - p(x) - x^5 = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu p) &= (\mu)\left(x^4\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{p}{x}\right) &= \left(\frac{1}{x}\right)\left(x^4\right) \\ \mathrm{d}\left(\frac{p}{x}\right) &= x^3\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{p}{x} &= \int x^3\,\mathrm{d}x \\ \frac{p}{x} &= \frac{x^4}{4} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$p(x) = \frac{1}{4}x^5 + c_1 x$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $p = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{4} + c_1$$

$$c_1 = \frac{3}{4}$$

Substituting c_1 found above in the general solution gives

$$p(x) = \frac{1}{4}x^5 + \frac{3}{4}x$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{1}{4}x^5 + \frac{3}{4}x$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{1}{4}x^5 + \frac{3}{4}x \, \mathrm{d}x \\ &= \frac{1}{24}x^6 + \frac{3}{8}x^2 + c_2 \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 1$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = \frac{5}{12} + c_2$$

$$c_2 = \frac{1}{12}$$

Substituting c_2 found above in the general solution gives

$$y = \frac{1}{24}x^6 + \frac{1}{12} + \frac{3}{8}x^2$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = \frac{1}{24}x^6 + \frac{1}{12} + \frac{3}{8}x^2 \tag{1}$$

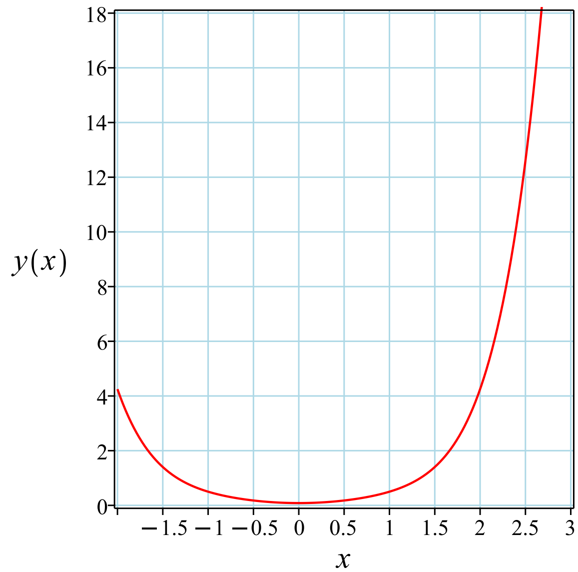

Figure 3: Solution plot

Verification of solutions

$$y = \frac{1}{24}x^6 + \frac{1}{12} + \frac{3}{8}x^2$$

Verified OK.

4.8.4 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x \\ B &= -1 \\ C &= 0 \\ F &= x^5 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= (x)(0) + (-1)(0) + (0)(-1) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-xv'' + (1) v' = 0$$

Now by applying $v' = u$ the above becomes

$$-xu'(x) + u(x) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{u}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= \frac{1}{x}\, dx \\ \int \frac{1}{u}\, du &= \int \frac{1}{x}\, dx \\ \ln(u) &= \ln(x) + c_1 \\ u &= e^{\ln(x) + c_1} \\ &= c_1 x \end{aligned}$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= c_1 x
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int c_1 x \, \mathrm{d}x \\
&= \frac{c_1 x^2}{2} + c_2
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= Bv \\
&= (-1)\left(\frac{c_1 x^2}{2} + c_2\right) \\
&= -\frac{c_1 x^2}{2} - c_2
\end{aligned}
$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= -1 \\
y_2 &= x^2
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} -1 & x^2 \\ \frac{d}{dx}(-1) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} -1 & x^2 \\ 0 & 2x \end{vmatrix}$$

Therefore

$$W = (-1)(2x) - (x^2)(0)$$

Which simplifies to

$$W = -2x$$

Which simplifies to

$$W = -2x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^7}{-2x^2}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{x^5}{2} dx$$

Hence

$$u_1 = \frac{x^6}{12}$$

And Eq. (3) becomes

$$u_2 = \int \frac{-x^5}{-2x^2}\, dx$$

Which simplifies to

$$u_2 = \int \frac{x^3}{2} dx$$

Hence

$$u_2 = \frac{x^4}{8}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x^6}{24}$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= \left(-\frac{c_1 x^2}{2} - c_2\right) + \left(\frac{x^6}{24}\right) \\ &= -\frac{1}{2}c_1 x^2 - c_2 + \frac{1}{24}x^6 \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -\frac{1}{2}c_1 x^2 - c_2 + \frac{1}{24}x^6 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{1}{2}$ and $x = 1$ in the above gives

$$\frac{1}{2} = -\frac{c_1}{2} - c_2 + \frac{1}{24} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 x + \frac{1}{4}x^5$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$1 = -c_1 + \frac{1}{4} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -\frac{3}{4} \\ c_2 &= -\frac{1}{12} \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{1}{24}x^6 + \frac{1}{12} + \frac{3}{8}x^2$$

Summary

The solution(s) found are the following

$$y = \frac{1}{24}x^6 + \frac{1}{12} + \frac{3}{8}x^2 \tag{1}$$

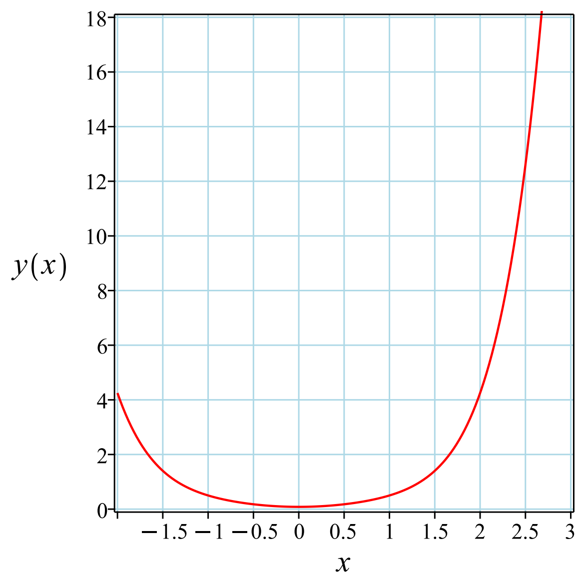

Figure 4: Solution plot

Verification of solutions

$$y = \frac{1}{24}x^6 + \frac{1}{12} + \frac{3}{8}x^2$$

Verified OK.

4.8.5 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$xy'' - y' = x^5$$

Integrating both sides of the ODE w.r.t x gives

$$\int (xy'' - y')\, dx = \int x^5 dx$$
$$xy' - 2y = \frac{x^6}{6} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2}{x}$$
$$q(x) = \frac{x^6 + 6c_1}{6x}$$

Hence the ode is

$$y' - \frac{2y}{x} = \frac{x^6 + 6c_1}{6x}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{2}{x} dx}$$
$$= \frac{1}{x^2}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{x^6 + 6c_1}{6x}\right)$$
$$\frac{d}{dx}\left(\frac{y}{x^2}\right) = \left(\frac{1}{x^2}\right)\left(\frac{x^6 + 6c_1}{6x}\right)$$
$$d\left(\frac{y}{x^2}\right) = \left(\frac{x^6 + 6c_1}{6x^3}\right) dx$$

Integrating gives

$$\frac{y}{x^2} = \int \frac{x^6 + 6c_1}{6x^3}\, dx$$
$$\frac{y}{x^2} = \frac{x^4}{24} - \frac{c_1}{2x^2} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = x^2\left(\frac{x^4}{24} - \frac{c_1}{2x^2}\right) + c_2x^2$$

which simplifies to

$$y = \frac{1}{24}x^6 - \frac{1}{2}c_1 + c_2x^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{1}{24}x^6 - \frac{1}{2}c_1 + c_2x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{1}{2}$ and $x = 1$ in the above gives

$$\frac{1}{2} = \frac{1}{24} - \frac{c_1}{2} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{1}{4}x^5 + 2c_2x$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$1 = \frac{1}{4} + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{1}{6}$$
$$c_2 = \frac{3}{8}$$

Substituting these values back in above solution results in

$$y = \frac{1}{24}x^6 + \frac{1}{12} + \frac{3}{8}x^2$$

Summary

The solution(s) found are the following

$$y = \frac{1}{24}x^6 + \frac{1}{12} + \frac{3}{8}x^2 \tag{1}$$

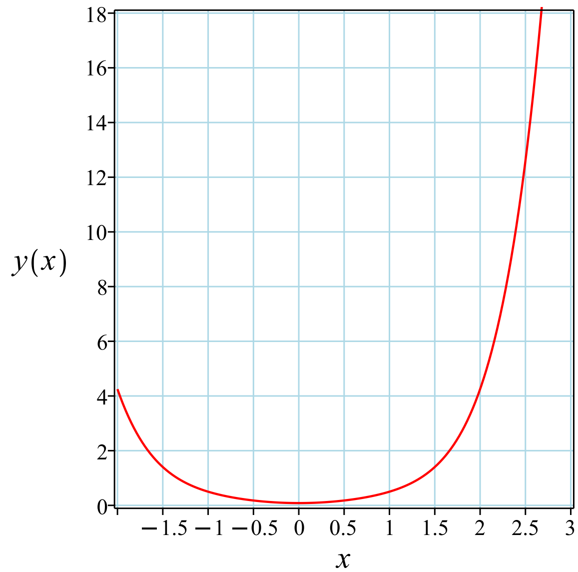

Figure 5: Solution plot

Verification of solutions

$$y = \frac{1}{24}x^6 + \frac{1}{12} + \frac{3}{8}x^2$$

Verified OK.

4.8.6 Solving using Kovacic algorithm

Writing the ode as

$$xy'' - y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 20: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)\,(0) \\ &= -\frac{1}{2x} \\ &= -\frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered}
(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) = 0 \\
0 = 0
\end{gathered}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{1}{2x} dx} \\
&= \frac{1}{\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-1}{x} \, dx} \\
&= z_1 e^{\frac{\ln(x)}{2}} \\
&= z_1 \left(\sqrt{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{x} dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2} dx \\
&= y_1 \left(\frac{x^2}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2 \left(1 \left(\frac{x^2}{2}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$xy'' - y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 + \frac{c_2 x^2}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = 1$$

$$y_2 = \frac{x^2}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \frac{x^2}{2} \\ \frac{d}{dx}(1) & \frac{d}{dx}\left(\frac{x^2}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \frac{x^2}{2} \\ 0 & x \end{vmatrix}$$

Therefore

$$W = (1)(x) - \left(\frac{x^2}{2}\right)(0)$$

Which simplifies to

$$W = x$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^7}{2}}{x^2}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{x^5}{2} dx$$

Hence

$$u_1 = -\frac{x^6}{12}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^5}{x^2} dx$$

Which simplifies to

$$u_2 = \int x^3 dx$$

Hence

$$u_2 = \frac{x^4}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x^6}{24}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + \frac{c_2 x^2}{2}\right) + \left(\frac{x^6}{24}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 + \frac{1}{2} c_2 x^2 + \frac{1}{24} x^6 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{1}{2}$ and $x = 1$ in the above gives

$$\frac{1}{2} = c_1 + \frac{c_2}{2} + \frac{1}{24} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_2 x + \frac{1}{4}x^5$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$1 = c_2 + \frac{1}{4} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{1}{12}$$
$$c_2 = \frac{3}{4}$$

Substituting these values back in above solution results in

$$y = \frac{1}{24}x^6 + \frac{1}{12} + \frac{3}{8}x^2$$

Summary

The solution(s) found are the following

$$y = \frac{1}{24}x^6 + \frac{1}{12} + \frac{3}{8}x^2 \tag{1}$$

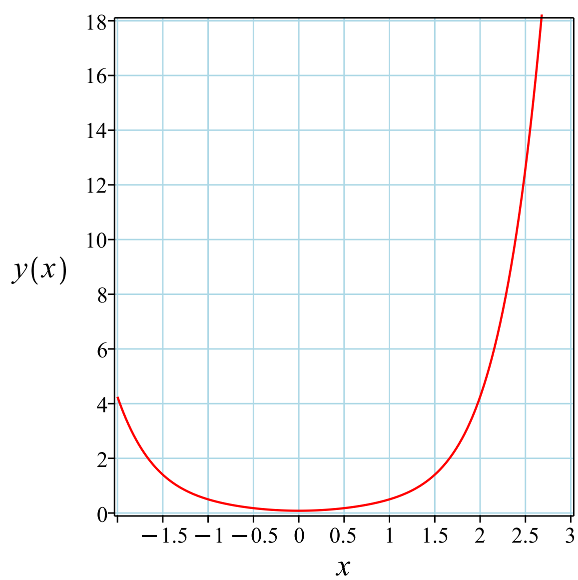

Figure 6: Solution plot

Verification of solutions

$$y = \frac{1}{24}x^6 + \frac{1}{12} + \frac{3}{8}x^2$$

Verified OK.

4.8.7 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x \\ q(x) &= -1 \\ r(x) &= 0 \\ s(x) &= x^5 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$xy' - 2y = \int x^5\, dx$$

We now have a first order ode to solve which is

$$xy' - 2y = \frac{x^6}{6} + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}
p(x) &= -\frac{2}{x} \\
q(x) &= \frac{x^6 + 6c_1}{6x}
\end{aligned}$$

Hence the ode is

$$y' - \frac{2y}{x} = \frac{x^6 + 6c_1}{6x}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{2}{x}dx} \\
&= \frac{1}{x^2}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{x^6 + 6c_1}{6x}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(\frac{x^6 + 6c_1}{6x}\right) \\
\mathrm{d}\left(\frac{y}{x^2}\right) &= \left(\frac{x^6 + 6c_1}{6x^3}\right)\mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{y}{x^2} &= \int \frac{x^6 + 6c_1}{6x^3}\,\mathrm{d}x \\
\frac{y}{x^2} &= \frac{x^4}{24} - \frac{c_1}{2x^2} + c_2
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = x^2\left(\frac{x^4}{24} - \frac{c_1}{2x^2}\right) + c_2x^2$$

which simplifies to

$$y = \frac{1}{24}x^6 - \frac{1}{2}c_1 + c_2 x^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{1}{24}x^6 - \frac{1}{2}c_1 + c_2 x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{1}{2}$ and $x = 1$ in the above gives

$$\frac{1}{2} = \frac{1}{24} - \frac{c_1}{2} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{1}{4}x^5 + 2c_2 x$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$1 = \frac{1}{4} + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{1}{6}$$
$$c_2 = \frac{3}{8}$$

Substituting these values back in above solution results in

$$y = \frac{1}{24}x^6 + \frac{1}{12} + \frac{3}{8}x^2$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{1}{24}x^6 + \frac{1}{12} + \frac{3}{8}x^2 \tag{1}$$

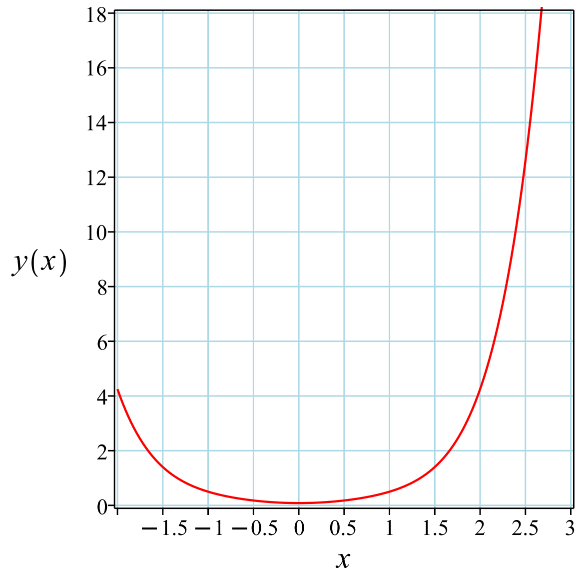

Figure 7: Solution plot

Verification of solutions

$$y = \frac{1}{24}x^6 + \frac{1}{12} + \frac{3}{8}x^2$$

Verified OK.

4.8.8 Maple step by step solution

Let's solve

$\left[xy'' - y' = x^5, y(1) = \frac{1}{2}, y'\Big|_{\{x=1\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $xu'(x) - u(x) = x^5$

- Isolate the derivative

  $u'(x) = \frac{u(x)}{x} + x^4$

- Group terms with $u(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $u'(x) - \frac{u(x)}{x} = x^4$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(u'(x)-\frac{u(x)}{x}\right)=\mu(x)\,x^4$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,u(x))$

$$\mu(x)\left(u'(x)-\frac{u(x)}{x}\right)=\mu'(x)\,u(x)+\mu(x)\,u'(x)$$

- Isolate $\mu'(x)$

$$\mu'(x)=-\frac{\mu(x)}{x}$$

- Solve to find the integrating factor

$$\mu(x)=\frac{1}{x}$$

- Integrate both sides with respect to x

$$\int\left(\frac{d}{dx}(\mu(x)\,u(x))\right)dx=\int \mu(x)\,x^4dx+c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\,u(x)=\int \mu(x)\,x^4dx+c_1$$

- Solve for $u(x)$

$$u(x)=\frac{\int \mu(x)x^4dx+c_1}{\mu(x)}$$

- Substitute $\mu(x)=\frac{1}{x}$

$$u(x)=x\left(\int x^3dx+c_1\right)$$

- Evaluate the integrals on the rhs

$$u(x)=x\left(\frac{x^4}{4}+c_1\right)$$

- Simplify

$$u(x)=\frac{x(x^4+4c_1)}{4}$$

- Solve 1st ODE for $u(x)$

$$u(x)=\frac{x(x^4+4c_1)}{4}$$

- Make substitution $u=y'$

$$y'=\frac{x(x^4+4c_1)}{4}$$

- Integrate both sides to solve for y

$$\int y'dx=\int\frac{x(x^4+4c_1)}{4}dx+c_2$$

- Compute integrals

$$y=\frac{1}{24}x^6+\frac{1}{2}c_1x^2+c_2$$

□ Check validity of solution $y=\frac{1}{24}x^6+\frac{1}{2}c_1x^2+c_2$

- Use initial condition $y(1) = \frac{1}{2}$

  $\frac{1}{2} = \frac{1}{24} + \frac{c_1}{2} + c_2$
- Compute derivative of the solution

  $y' = \frac{1}{4}x^5 + c_1 x$
- Use the initial condition $y'\Big|_{\{x=1\}} = 1$

  $1 = \frac{1}{4} + c_1$
- Solve for c_1 and c_2

  $\{c_1 = \frac{3}{4}, c_2 = \frac{1}{12}\}$
- Substitute constant values into general solution and simplify

  $y = \frac{1}{24}x^6 + \frac{1}{12} + \frac{3}{8}x^2$

• Solution to the IVP

  $y = \frac{1}{24}x^6 + \frac{1}{12} + \frac{3}{8}x^2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = (_a^5+_b(_a))/_a, _b(_a)`   *** Subleve
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 16

```
dsolve([x*diff(y(x),x$2)=diff(y(x),x)+x^5,y(1) = 1/2, D(y)(1) = 1],y(x), singsol=all)
```

$$y(x) = \frac{1}{24}x^6 + \frac{3}{8}x^2 + \frac{1}{12}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 19

```
DSolve[{x*y''[x]==y'[x]+x^5,{y[1]==1/2,y'[1]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{24}(x^6 + 9x^2 + 2)$$

4.9 problem 10

Internal problem ID [6829]
Internal file name [`OUTPUT/6076_Thursday_July_28_2022_04_29_21_AM_8781136/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{xy'' + y' = -x}$$

With initial conditions

$$\left[y(2) = -1, y'(2) = -\frac{1}{2}\right]$$

4.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= 0 \\ F &= -1 \end{aligned}$$

Hence the ode is

$$y'' + \frac{y'}{x} = -1$$

The domain of $p(x) = \frac{1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 2$ is inside this domain. The domain of $F = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 2$ is also inside this domain. Hence solution exists and is unique.

4.9.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\begin{aligned} \int (xy'' + y')\, dx &= \int -x dx \\ xy' &= -\frac{x^2}{2} + c_1 \end{aligned}$$

Which is now solved for y. Integrating both sides gives

$$\begin{aligned} y &= \int \frac{-x^2 + 2c_1}{2x}\, \mathrm{d}x \\ &= -\frac{x^2}{4} + c_1 \ln(x) + c_2 \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -\frac{x^2}{4} + c_1 \ln(x) + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -1$ and $x = 2$ in the above gives

$$-1 = -1 + c_1 \ln(2) + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{x}{2} + \frac{c_1}{x}$$

substituting $y' = -\frac{1}{2}$ and $x = 2$ in the above gives

$$-\frac{1}{2} = \frac{c_1}{2} - 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 1 \\ c_2 &= -\ln(2) \end{aligned}$$

Substituting these values back in above solution results in

$$y = -\frac{x^2}{4} + \ln(x) - \ln(2)$$

Summary

The solution(s) found are the following

$$y = -\frac{x^2}{4} + \ln(x) - \ln(2) \tag{1}$$

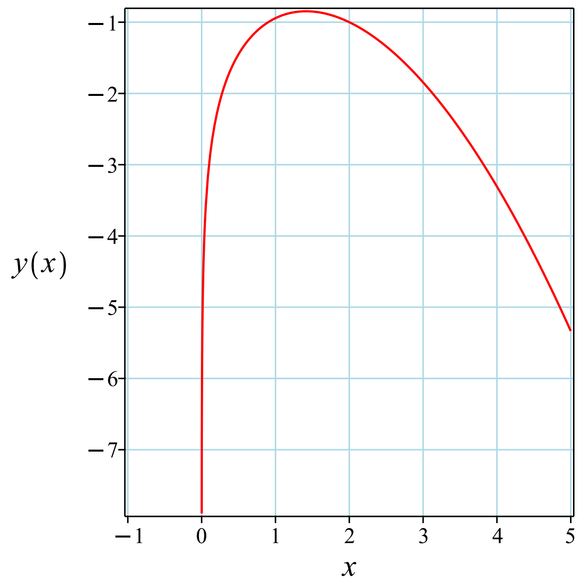

Figure 8: Solution plot

<u>Verification of solutions</u>

$$
y = -\frac{x^2}{4} + \ln(x) - \ln(2)
$$

Verified OK.

4.9.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$
p(x) = y'
$$

Then

$$
p'(x) = y''
$$

Hence the ode becomes

$$
xp'(x) + p(x) + x = 0
$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{1}{x}dx} \\
&= x
\end{aligned}
$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu p) &= (\mu)(-1) \\
\frac{\mathrm{d}}{\mathrm{d}x}(xp) &= (x)(-1) \\
\mathrm{d}(xp) &= (-x)\ \mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
xp &= \int -x\,\mathrm{d}x \\
xp &= -\frac{x^2}{2} + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$p(x) = -\frac{x}{2} + \frac{c_1}{x}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $p = -\frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{1}{2} = \frac{c_1}{2} - 1$$

$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

$$p(x) = -\frac{x^2 - 2}{2x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -\frac{x^2 - 2}{2x}$$

Integrating both sides gives

$$\begin{aligned}
y &= \int -\frac{x^2 - 2}{2x}\,\mathrm{d}x \\
&= -\frac{x^2}{4} + \ln(x) + c_2
\end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 2$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -1 + \ln(2) + c_2$$

$$c_2 = -\ln(2)$$

Substituting c_2 found above in the general solution gives

$$y = -\frac{x^2}{4} + \ln(x) - \ln(2)$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = -\frac{x^2}{4} + \ln(x) - \ln(2) \tag{1}$$

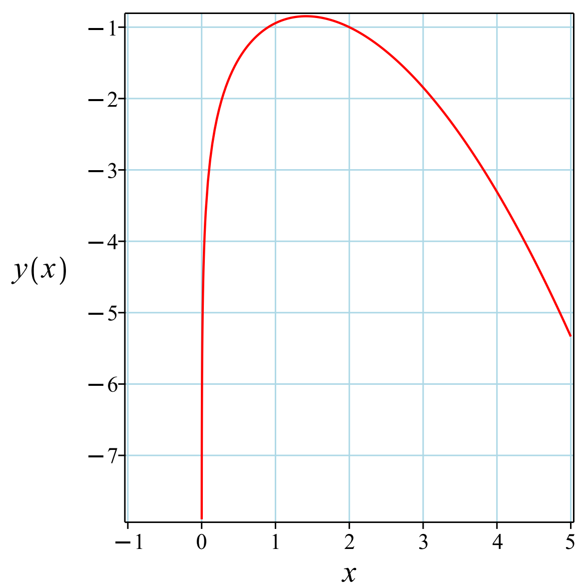

Figure 9: Solution plot

Verification of solutions

$$y = -\frac{x^2}{4} + \ln(x) - \ln(2)$$

Verified OK.

4.9.4 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x \\ B &= 1 \\ C &= 0 \\ F &= -x \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= (x)(0) + (1)(0) + (0)(1) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$xv'' + (1)\, v' = 0$$

Now by applying $v' = u$ the above becomes

$$xu'(x) + u(x) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{1}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{1}{x}\, dx \\ \ln(u) &= -\ln(x) + c_1 \\ u &= \mathrm{e}^{-\ln(x)+c_1} \\ &= \frac{c_1}{x} \end{aligned}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= \frac{c_1}{x} \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int \frac{c_1}{x}\, \mathrm{d}x \\ &= c_1 \ln(x) + c_2 \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= Bv \\ &= (1)\,(c_1 \ln(x) + c_2) \\ &= c_1 \ln(x) + c_2 \end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= 1 \\ y_2 &= \ln(x) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \ln(x) \\ \frac{d}{dx}(1) & \frac{d}{dx}(\ln(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \ln(x) \\ 0 & \frac{1}{x} \end{vmatrix}$$

Therefore

$$W = (1)\left(\frac{1}{x}\right) - (\ln(x))(0)$$

Which simplifies to

$$W = \frac{1}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\ln(x)\, x}{1} dx$$

Which simplifies to

$$u_1 = -\int -\ln(x)\, x dx$$

Hence

$$u_1 = \frac{\ln(x)\, x^2}{2} - \frac{x^2}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{-x}{1} dx$$

Which simplifies to

$$u_2 = \int -x dx$$

Hence

$$u_2 = -\frac{x^2}{2}$$

Which simplifies to

$$u_1 = \frac{x^2(-1 + 2\ln(x))}{4}$$

$$u_2 = -\frac{x^2}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x^2(-1 + 2\ln(x))}{4} - \frac{\ln(x)\, x^2}{2}$$

Which simplifies to

$$y_p(x) = -\frac{x^2}{4}$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= (c_1 \ln(x) + c_2) + \left(-\frac{x^2}{4}\right) \\ &= -\frac{x^2}{4} + c_1 \ln(x) + c_2 \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -\frac{x^2}{4} + c_1 \ln(x) + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -1$ and $x = 2$ in the above gives

$$-1 = -1 + c_1 \ln(2) + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{x}{2} + \frac{c_1}{x}$$

substituting $y' = -\frac{1}{2}$ and $x = 2$ in the above gives

$$-\frac{1}{2} = \frac{c_1}{2} - 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 1 \\ c_2 &= -\ln(2) \end{aligned}$$

Substituting these values back in above solution results in

$$y = -\frac{x^2}{4} + \ln(x) - \ln(2)$$

Summary

The solution(s) found are the following

$$y = -\frac{x^2}{4} + \ln(x) - \ln(2) \tag{1}$$

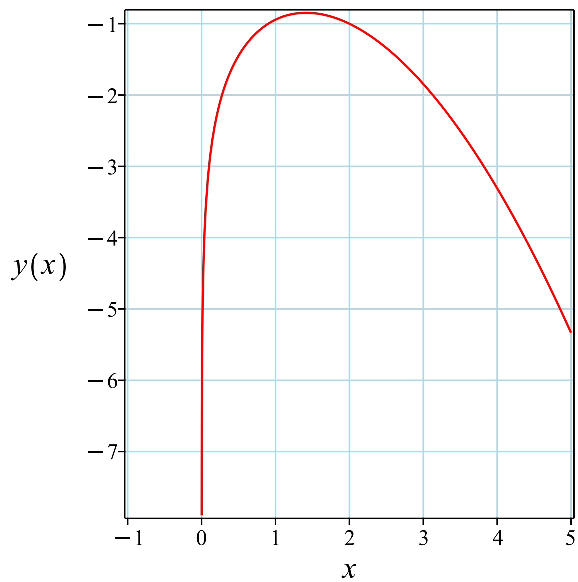

Figure 10: Solution plot

Verification of solutions

$$y = -\frac{x^2}{4} + \ln(x) - \ln(2)$$

Verified OK.

4.9.5 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$xy'' + y' = -x$$

Integrating both sides of the ODE w.r.t x gives

$$\int (xy'' + y')\,dx = \int -x\,dx$$

$$xy' = -\frac{x^2}{2} + c_1$$

Which is now solved for y. Integrating both sides gives

$$y = \int \frac{-x^2 + 2c_1}{2x} \, dx$$
$$= -\frac{x^2}{4} + c_1 \ln(x) + c_2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -\frac{x^2}{4} + c_1 \ln(x) + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -1$ and $x = 2$ in the above gives

$$-1 = -1 + c_1 \ln(2) + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{x}{2} + \frac{c_1}{x}$$

substituting $y' = -\frac{1}{2}$ and $x = 2$ in the above gives

$$-\frac{1}{2} = \frac{c_1}{2} - 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = -\ln(2)$$

Substituting these values back in above solution results in

$$y = -\frac{x^2}{4} + \ln(x) - \ln(2)$$

Summary
The solution(s) found are the following

$$y = -\frac{x^2}{4} + \ln(x) - \ln(2) \tag{1}$$

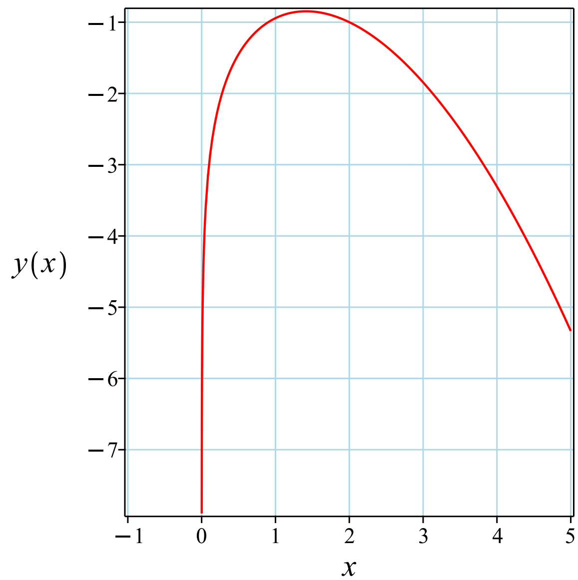

Figure 11: Solution plot

Verification of solutions

$$y = -\frac{x^2}{4} + \ln(x) - \ln(2)$$

Verified OK.

4.9.6 Solving using Kovacic algorithm

Writing the ode as

$$xy'' + y' = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 22: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{x}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2(1(\ln(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
xy'' + y' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 + c_2 \ln(x)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= \ln(x)
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \ln(x) \\ \frac{d}{dx}(1) & \frac{d}{dx}(\ln(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \ln(x) \\ 0 & \frac{1}{x} \end{vmatrix}$$

Therefore

$$W = (1)\left(\frac{1}{x}\right) - (\ln(x))(0)$$

Which simplifies to

$$W = \frac{1}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\ln(x)\, x}{1}\, dx$$

Which simplifies to

$$u_1 = -\int -\ln(x)\, x dx$$

Hence

$$u_1 = \frac{\ln(x)\, x^2}{2} - \frac{x^2}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{-x}{1}\, dx$$

Which simplifies to

$$u_2 = \int -x dx$$

Hence

$$u_2 = -\frac{x^2}{2}$$

Which simplifies to

$$u_1 = \frac{x^2(-1 + 2\ln(x))}{4}$$
$$u_2 = -\frac{x^2}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x^2(-1 + 2\ln(x))}{4} - \frac{\ln(x)\, x^2}{2}$$

Which simplifies to

$$y_p(x) = -\frac{x^2}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 + c_2 \ln(x)) + \left(-\frac{x^2}{4}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 + c_2 \ln(x) - \frac{x^2}{4} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -1$ and $x = 2$ in the above gives

$$-1 = c_1 + c_2 \ln(2) - 1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_2}{x} - \frac{x}{2}$$

substituting $y' = -\frac{1}{2}$ and $x = 2$ in the above gives

$$-\frac{1}{2} = \frac{c_2}{2} - 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\ln(2)$$
$$c_2 = 1$$

Substituting these values back in above solution results in

$$y = -\frac{x^2}{4} + \ln(x) - \ln(2)$$

Summary

The solution(s) found are the following

$$y = -\frac{x^2}{4} + \ln(x) - \ln(2) \tag{1}$$

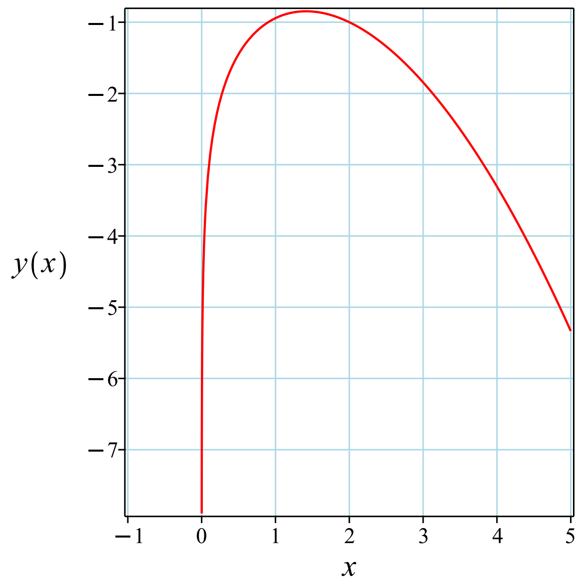

Figure 12: Solution plot

<u>Verification of solutions</u>

$$y = -\frac{x^2}{4} + \ln(x) - \ln(2)$$

Verified OK.

4.9.7 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x \\ q(x) &= 1 \\ r(x) &= 0 \\ s(x) &= -x \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$xy' = \int -x\, dx$$

We now have a first order ode to solve which is

$$xy' = -\frac{x^2}{2} + c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{-x^2 + 2c_1}{2x}\, \mathrm{d}x \\ &= -\frac{x^2}{4} + c_1 \ln(x) + c_2 \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -\frac{x^2}{4} + c_1 \ln(x) + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -1$ and $x = 2$ in the above gives

$$-1 = -1 + c_1 \ln(2) + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{x}{2} + \frac{c_1}{x}$$

substituting $y' = -\frac{1}{2}$ and $x = 2$ in the above gives

$$-\frac{1}{2} = \frac{c_1}{2} - 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = -\ln(2)$$

Substituting these values back in above solution results in

$$y = -\frac{x^2}{4} + \ln(x) - \ln(2)$$

Summary

The solution(s) found are the following

$$y = -\frac{x^2}{4} + \ln(x) - \ln(2) \tag{1}$$

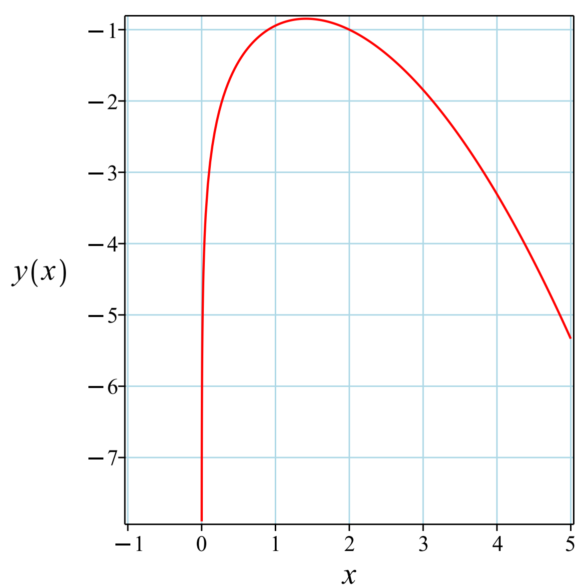

Figure 13: Solution plot

Verification of solutions

$$y = -\frac{x^2}{4} + \ln(x) - \ln(2)$$

Verified OK.

4.9.8 Maple step by step solution

Let's solve

$\left[x y''+y'=-x, y(2)=-1, y'\Big|_{\{x=2\}}=-\frac{1}{2}\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u=y'$ to reduce order of ODE

  $x u'(x)+u(x)=-x$

- Isolate the derivative

  $u'(x)=-1-\frac{u(x)}{x}$

- Group terms with $u(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $u'(x)+\frac{u(x)}{x}=-1$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(u'(x)+\frac{u(x)}{x}\right)=-\mu(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,u(x))$

  $\mu(x)\left(u'(x)+\frac{u(x)}{x}\right)=\mu'(x)\,u(x)+\mu(x)\,u'(x)$

- Isolate $\mu'(x)$

  $\mu'(x)=\frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x)=x$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,u(x))\right)dx=\int-\mu(x)\,dx+c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,u(x)=\int-\mu(x)\,dx+c_1$

- Solve for $u(x)$

  $u(x)=\frac{\int-\mu(x)dx+c_1}{\mu(x)}$

- Substitute $\mu(x)=x$

  $u(x)=\frac{\int-x dx+c_1}{x}$

- Evaluate the integrals on the rhs

  $u(x) = \frac{-\frac{x^2}{2}+c_1}{x}$

- Solve 1st ODE for $u(x)$

  $u(x) = \frac{-\frac{x^2}{2}+c_1}{x}$

- Make substitution $u = y'$

  $y' = \frac{-\frac{x^2}{2}+c_1}{x}$

- Integrate both sides to solve for y

  $\int y'dx = \int \frac{-\frac{x^2}{2}+c_1}{x}dx + c_2$

- Compute integrals

  $y = -\frac{x^2}{4} + c_1 \ln(x) + c_2$

☐ Check validity of solution $y = -\frac{x^2}{4} + c_1 \ln(x) + c_2$

  ◦ Use initial condition $y(2) = -1$

    $-1 = -1 + c_1 \ln(2) + c_2$

  ◦ Compute derivative of the solution

    $y' = -\frac{x}{2} + \frac{c_1}{x}$

  ◦ Use the initial condition $y'\Big|_{\{x=2\}} = -\frac{1}{2}$

    $-\frac{1}{2} = \frac{c_1}{2} - 1$

  ◦ Solve for c_1 and c_2

    $\{c_1 = 1, c_2 = -\ln(2)\}$

  ◦ Substitute constant values into general solution and simplify

    $y = -\frac{x^2}{4} + \ln(x) - \ln(2)$

- Solution to the IVP

  $y = -\frac{x^2}{4} + \ln(x) - \ln(2)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -(_b(_a)+_a)/_a, _b(_a)`   *** Sublevel
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 16

```
dsolve([x*diff(y(x),x$2)+diff(y(x),x)+x=0,y(2) = -1, D(y)(2) = -1/2],y(x), singsol=all)
```

$$y(x) = -\frac{x^2}{4} + \ln(x) - \ln(2)$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 19

```
DSolve[{x*y''[x]+y'[x]+x==0,{y[2]==-1,y'[2]==-1/2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \log\left(\frac{x}{2}\right) - \frac{x^2}{4}$$

4.10 problem 11

Internal problem ID [6830]
Internal file name [`OUTPUT/6077_Thursday_July_28_2022_04_29_23_AM_4198183/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1],
   [_2nd_order, _reducible, _mu_y_y1]]
```

$$\boxed{y'' - 2yy'^3 = 0}$$

4.10.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right)-2yp(y)^3=0$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned}p' &= F(y,p)\\ &= f(y)g(p)\\ &= 2p^2y\end{aligned}$$

Where $f(y)=2y$ and $g(p)=p^2$. Integrating both sides gives

$$\begin{aligned}\frac{1}{p^2}\,dp &= 2y\,dy\\ \int\frac{1}{p^2}\,dp &= \int 2y\,dy\\ -\frac{1}{p} &= y^2+c_1\end{aligned}$$

The solution is

$$-\frac{1}{p(y)}-y^2-c_1=0$$

For solution (1) found earlier, since $p=y'$ then we now have a new first order ode to solve which is

$$-\frac{1}{y'}-y^2-c_1=0$$

Integrating both sides gives

$$\begin{aligned}\int\left(-y^2-c_1\right)dy &= x+c_2\\ -\frac{1}{3}y^3-c_1y &= x+c_2\end{aligned}$$

Solving for y gives these solutions

$$y_1 = \frac{\left(-12c_2 - 12x + 4\sqrt{4c_1^3 + 9c_2^2 + 18c_2x + 9x^2}\right)^{\frac{1}{3}}}{2} - \frac{2c_1}{\left(-12c_2 - 12x + 4\sqrt{4c_1^3 + 9c_2^2 + 18c_2x + 9x^2}\right)^{\frac{1}{3}}}$$

$$y_2 = -\frac{\left(-12c_2 - 12x + 4\sqrt{4c_1^3 + 9c_2^2 + 18c_2x + 9x^2}\right)^{\frac{1}{3}}}{4} + \frac{c_1}{\left(-12c_2 - 12x + 4\sqrt{4c_1^3 + 9c_2^2 + 18c_2x + 9x^2}\right)^{\frac{1}{3}}}$$

$$y_3 = -\frac{\left(-12c_2 - 12x + 4\sqrt{4c_1^3 + 9c_2^2 + 18c_2x + 9x^2}\right)^{\frac{1}{3}}}{4} + \frac{c_1}{\left(-12c_2 - 12x + 4\sqrt{4c_1^3 + 9c_2^2 + 18c_2x + 9x^2}\right)^{\frac{1}{3}}}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-12c_2 - 12x + 4\sqrt{4c_1^3 + 9c_2^2 + 18c_2x + 9x^2}\right)^{\frac{1}{3}}}{2} - \frac{2c_1}{\left(-12c_2 - 12x + 4\sqrt{4c_1^3 + 9c_2^2 + 18c_2x + 9x^2}\right)^{\frac{1}{3}}} \tag{1}$$

$$y = -\frac{\left(-12c_2 - 12x + 4\sqrt{4c_1^3 + 9c_2^2 + 18c_2x + 9x^2}\right)^{\frac{1}{3}}}{4} + \frac{c_1}{\left(-12c_2 - 12x + 4\sqrt{4c_1^3 + 9c_2^2 + 18c_2x + 9x^2}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(-12c_2 - 12x + 4\sqrt{4c_1^3 + 9c_2^2 + 18c_2x + 9x^2}\right)^{\frac{1}{3}}}{2} + \frac{2c_1}{\left(-12c_2 - 12x + 4\sqrt{4c_1^3 + 9c_2^2 + 18c_2x + 9x^2}\right)^{\frac{1}{3}}}\right)}{2} \tag{2}$$

$$y = -\frac{\left(-12c_2 - 12x + 4\sqrt{4c_1^3 + 9c_2^2 + 18c_2x + 9x^2}\right)^{\frac{1}{3}}}{4} + \frac{c_1}{\left(-12c_2 - 12x + 4\sqrt{4c_1^3 + 9c_2^2 + 18c_2x + 9x^2}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(-12c_2 - 12x + 4\sqrt{4c_1^3 + 9c_2^2 + 18c_2x + 9x^2}\right)^{\frac{1}{3}}}{2} + \frac{2c_1}{\left(-12c_2 - 12x + 4\sqrt{4c_1^3 + 9c_2^2 + 18c_2x + 9x^2}\right)^{\frac{1}{3}}}\right)}{2} \tag{3}$$

Verification of solutions

$$y = \frac{\left(-12c_2 - 12x + 4\sqrt{4c_1^3 + 9c_2^2 + 18c_2x + 9x^2}\right)^{\frac{1}{3}}}{2} - \frac{2c_1}{\left(-12c_2 - 12x + 4\sqrt{4c_1^3 + 9c_2^2 + 18c_2x + 9x^2}\right)^{\frac{1}{3}}}$$

Verified OK.

$$y = -\frac{\left(-12c_2 - 12x + 4\sqrt{4c_1^3 + 9c_2^2 + 18c_2x + 9x^2}\right)^{\frac{1}{3}}}{4} + \frac{c_1}{\left(-12c_2 - 12x + 4\sqrt{4c_1^3 + 9c_2^2 + 18c_2x + 9x^2}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(-12c_2-12x+4\sqrt{4c_1^3+9c_2^2+18c_2x+9x^2}\right)^{\frac{1}{3}}}{2} + \frac{2c_1}{\left(-12c_2-12x+4\sqrt{4c_1^3+9c_2^2+18c_2x+9x^2}\right)^{\frac{1}{3}}}\right)}{2}$$

Verified OK.

$$y = -\frac{\left(-12c_2 - 12x + 4\sqrt{4c_1^3 + 9c_2^2 + 18c_2x + 9x^2}\right)^{\frac{1}{3}}}{4} + \frac{c_1}{\left(-12c_2 - 12x + 4\sqrt{4c_1^3 + 9c_2^2 + 18c_2x + 9x^2}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(-12c_2-12x+4\sqrt{4c_1^3+9c_2^2+18c_2x+9x^2}\right)^{\frac{1}{3}}}{2} + \frac{2c_1}{\left(-12c_2-12x+4\sqrt{4c_1^3+9c_2^2+18c_2x+9x^2}\right)^{\frac{1}{3}}}\right)}{2}$$

Verified OK.

4.10.2 Maple step by step solution

Let's solve

$y'' - 2yy'^3 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

$u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $u(y)\left(\frac{d}{dy}u(y)\right) - 2yu(y)^3 = 0$

- Separate variables

  $\frac{\frac{d}{dy}u(y)}{u(y)^2} = 2y$

- Integrate both sides with respect to y

  $\int \frac{\frac{d}{dy}u(y)}{u(y)^2}dy = \int 2ydy + c_1$

- Evaluate integral

  $-\frac{1}{u(y)} = y^2 + c_1$

- Solve for $u(y)$

  $u(y) = -\frac{1}{y^2+c_1}$

- Solve 1st ODE for $u(y)$

  $u(y) = -\frac{1}{y^2+c_1}$

- Revert to original variables with substitution $u(y) = y', y = y$

  $y' = -\frac{1}{y^2+c_1}$

- Separate variables

  $y'(y^2 + c_1) = -1$

- Integrate both sides with respect to x

  $\int y'(y^2 + c_1)\,dx = \int (-1)\,dx + c_2$

- Evaluate integral

  $\frac{y^3}{3} + c_1y = -x + c_2$

- Solve for y

$$y = \frac{\left(12c_2-12x+4\sqrt{4c_1^3+9c_2^2-18c_2x+9x^2}\right)^{\frac{1}{3}}}{2} - \frac{2c_1}{\left(12c_2-12x+4\sqrt{4c_1^3+9c_2^2-18c_2x+9x^2}\right)^{\frac{1}{3}}}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-2*_a*_b(_a)^3 = 0, _b(_a)`   ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   <- Bernoulli successful
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 324

```
dsolve(diff(y(x),x$2)=2*y(x)*diff(y(x),x)^3,y(x), singsol=all)
```

$$y(x) = c_1$$

$$y(x) = \frac{\left(-12c_2 - 12x + 4\sqrt{-4c_1^3 + 9c_2^2 + 18c_2x + 9x^2}\right)^{\frac{2}{3}} + 4c_1}{2\left(-12c_2 - 12x + 4\sqrt{-4c_1^3 + 9c_2^2 + 18c_2x + 9x^2}\right)^{\frac{1}{3}}}$$

$$y(x) = \frac{-i\sqrt{3}\left(-12c_2 - 12x + 4\sqrt{-4c_1^3 + 9c_2^2 + 18c_2x + 9x^2}\right)^{\frac{2}{3}} + 4i\sqrt{3}\,c_1 - \left(-12c_2 - 12x + 4\sqrt{-4c_1^3 + 9c_2^2 +}\right.}{4\left(-12c_2 - 12x + 4\sqrt{-4c_1^3 + 9c_2^2 + 18c_2x + 9x^2}\right)^{\frac{1}{3}}}$$

$$y(x) = -\frac{-i\sqrt{3}\left(-12c_2 - 12x + 4\sqrt{-4c_1^3 + 9c_2^2 + 18c_2x + 9x^2}\right)^{\frac{2}{3}} + 4i\sqrt{3}\,c_1 + \left(-12c_2 - 12x + 4\sqrt{-4c_1^3 + 9c_2^2}\right.}{4\left(-12c_2 - 12x + 4\sqrt{-4c_1^3 + 9c_2^2 + 18c_2x + 9x^2}\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 7.768 (sec). Leaf size: 351

```
DSolve[y''[x]==2*y[x]*(y'[x])^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt[3]{2}c_1}{\sqrt[3]{\sqrt{9x^2 + 18c_2x + 4c_1{}^3 + 9c_2{}^2} + 3x + 3c_2}} - \frac{\sqrt[3]{\sqrt{9x^2 + 18c_2x + 4c_1{}^3 + 9c_2{}^2} + 3x + 3c_2}}{\sqrt[3]{2}}$$

$$y(x) \to \frac{2^{2/3}\left(1 - i\sqrt{3}\right)\left(\sqrt{9x^2 + 18c_2x + 4c_1{}^3 + 9c_2{}^2} + 3x + 3c_2\right)^{2/3} + \sqrt[3]{2}\left(-2 - 2i\sqrt{3}\right)c_1}{4\sqrt[3]{\sqrt{9x^2 + 18c_2x + 4c_1{}^3 + 9c_2{}^2} + 3x + 3c_2}}$$

$$y(x) \to \frac{2^{2/3}\left(1 + i\sqrt{3}\right)\left(\sqrt{9x^2 + 18c_2x + 4c_1{}^3 + 9c_2{}^2} + 3x + 3c_2\right)^{2/3} + 2i\sqrt[3]{2}\left(\sqrt{3} + i\right)c_1}{4\sqrt[3]{\sqrt{9x^2 + 18c_2x + 4c_1{}^3 + 9c_2{}^2} + 3x + 3c_2}}$$

$$y(x) \to 0$$

4.11 problem 12

Internal problem ID [6831]
Internal file name [`OUTPUT/6078_Thursday_July_28_2022_04_29_27_AM_20246841/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1],
   [_2nd_order, _reducible, _mu_y_y1]]
```

$$\boxed{yy'' + {y'}^3 - {y'}^2 = 0}$$

4.11.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y)\left(\frac{d}{dy}p(y)\right)+\left(p(y)^2-p(y)\right)p(y)=0$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned}p' &= F(y,p)\\ &= f(y)g(p)\\ &= -\frac{p(p-1)}{y}\end{aligned}$$

Where $f(y)=-\frac{1}{y}$ and $g(p)=p(p-1)$. Integrating both sides gives

$$\begin{aligned}\frac{1}{p\left(p-1\right)}\,dp &= -\frac{1}{y}\,dy\\ \int\frac{1}{p\left(p-1\right)}\,dp &= \int-\frac{1}{y}\,dy\\ -\ln\left(p\right)+\ln\left(p-1\right) &= -\ln\left(y\right)+c_1\end{aligned}$$

Raising both side to exponential gives

$$e^{-\ln(p)+\ln(p-1)}=e^{-\ln(y)+c_1}$$

Which simplifies to

$$\frac{p-1}{p}=\frac{c_2}{y}$$

For solution (1) found earlier, since $p=y'$ then we now have a new first order ode to solve which is

$$y'=-\frac{y}{c_2-y}$$

Integrating both sides gives

$$\begin{aligned}\int\frac{-c_2+y}{y}dy &= x+c_3\\ y-c_2\ln\left(y\right) &= x+c_3\end{aligned}$$

Solving for y gives these solutions

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{c_2 \operatorname{LambertW}\left(-\frac{\mathrm{e}^{-\frac{x+c_3}{c_2}}}{c_2}\right)+c_3+x}{c_2}} \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{-\frac{c_2 \operatorname{LambertW}\left(-\frac{\mathrm{e}^{-\frac{x+c_3}{c_2}}}{c_2}\right)+c_3+x}{c_2}}$$

Verified OK.

4.11.2 Maple step by step solution

Let's solve

$yy'' + \left(y'^2 - y'\right) y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $yu(y)\left(\frac{d}{dy}u(y)\right) + \left(u(y)^2 - u(y)\right)u(y) = 0$

- Separate variables

  $\frac{\frac{d}{dy}u(y)}{u(y)^2-u(y)} = -\frac{1}{y}$

- Integrate both sides with respect to y

$\int \frac{\frac{d}{dy}u(y)}{u(y)^2-u(y)}dy = \int -\frac{1}{y}dy + c_1$

- Evaluate integral

  $-\ln(u(y)) + \ln(u(y)-1) = -\ln(y) + c_1$

- Solve for $u(y)$

  $u(y) = -\frac{y}{e^{c_1}-y}$

- Solve 1st ODE for $u(y)$

  $u(y) = -\frac{y}{e^{c_1}-y}$

- Revert to original variables with substitution $u(y) = y', y = y$

  $y' = -\frac{y}{e^{c_1}-y}$

- Separate variables

  $\frac{y'(e^{c_1}-y)}{y} = -1$

- Integrate both sides with respect to x

  $\int \frac{y'(e^{c_1}-y)}{y}dx = \int (-1)\,dx + c_2$

- Evaluate integral

  $-y + e^{c_1}\ln(y) = -x + c_2$

- Solve for y

  $y = e^{-\frac{LambertW\left(-e^{-\frac{c_1 e^{c_1}-c_2+x}{e^{c_1}}}\right)e^{c_1}-c_2+x}{e^{c_1}}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+_b(_a)^2*(_b(_a)-1)/_a = 0, _b(_
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   <- Bernoulli successful
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 36

```
dsolve(y(x)*diff(y(x),x$2)+diff(y(x),x)^3-diff(y(x),x)^2=0,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = c_1$$
$$y(x) = \mathrm{e}^{\frac{-c_1 \operatorname{LambertW}\left(\frac{\mathrm{e}^{\frac{x+c_2}{c_1}}}{c_1}\right)+c_2+x}{c_1}}$$

✓ Solution by Mathematica

Time used: 22.067 (sec). Leaf size: 32

```
DSolve[y[x]*y''[x]+(y'[x])^3-(y'[x])^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{c_1} W\left(e^{e^{-c_1}(x - e^{c_1} c_1 + c_2)}\right)$$

4.12 problem 13

Internal problem ID [6832]
Internal file name [`OUTPUT/6079_Thursday_July_28_2022_04_29_28_AM_82848687/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + \beta^2 y = 0}$$

4.12.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = \beta^2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \beta^2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\beta^2 + \lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = \beta^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(\beta^2)} \\ &= \pm\sqrt{-\beta^2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{-\beta^2} \\ \lambda_2 &= -\sqrt{-\beta^2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \sqrt{-\beta^2} \\ \lambda_2 &= -\sqrt{-\beta^2}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{\left(\sqrt{-\beta^2}\right)x} + c_2 e^{\left(-\sqrt{-\beta^2}\right)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{\sqrt{-\beta^2}\, x} + c_2 \mathrm{e}^{-\sqrt{-\beta^2}\, x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\sqrt{-\beta^2}\, x} + c_2 \mathrm{e}^{-\sqrt{-\beta^2}\, x} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{\sqrt{-\beta^2}\, x} + c_2 \mathrm{e}^{-\sqrt{-\beta^2}\, x}$$

Verified OK.

4.12.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + \beta^2 y' y = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' + \beta^2 y' y)\, dx = 0$$
$$\frac{y'^2}{2} + \frac{\beta^2 y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-\beta^2 y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{-\beta^2 y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-\beta^2 y^2 + 2c_1}} dy = \int dx$$
$$\frac{\arctan\left(\frac{\sqrt{\beta^2}\, y}{\sqrt{-\beta^2 y^2 + 2c_1}}\right)}{\sqrt{\beta^2}} = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-\beta^2 y^2 + 2c_1}} dy = \int dx$$
$$-\frac{\arctan\left(\frac{\sqrt{\beta^2}\, y}{\sqrt{-\beta^2 y^2 + 2c_1}}\right)}{\sqrt{\beta^2}} = x + c_3$$

Summary

The solution(s) found are the following

$$\frac{\arctan\left(\frac{\sqrt{\beta^2}\, y}{\sqrt{-\beta^2 y^2+2c_1}}\right)}{\sqrt{\beta^2}} = x + c_2 \tag{1}$$

$$-\frac{\arctan\left(\frac{\sqrt{\beta^2}\, y}{\sqrt{-\beta^2 y^2+2c_1}}\right)}{\sqrt{\beta^2}} = x + c_3 \tag{2}$$

Verification of solutions

$$\frac{\arctan\left(\frac{\sqrt{\beta^2}\, y}{\sqrt{-\beta^2 y^2+2c_1}}\right)}{\sqrt{\beta^2}} = x + c_2$$

Verified OK.

$$-\frac{\arctan\left(\frac{\sqrt{\beta^2}\, y}{\sqrt{-\beta^2 y^2+2c_1}}\right)}{\sqrt{\beta^2}} = x + c_3$$

Verified OK.

4.12.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + \beta^2 y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= \beta^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-\beta^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -\beta^2$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\beta^2\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 26: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\beta^2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{\sqrt{-\beta^2}\, x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{\sqrt{-\beta^2}\, x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\sqrt{-\beta^2}\, x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= \mathrm{e}^{\sqrt{-\beta^2}\, x} \int \frac{1}{\mathrm{e}^{2\sqrt{-\beta^2}\, x}} \, dx \\ &= \mathrm{e}^{\sqrt{-\beta^2}\, x} \left(\frac{\sqrt{-\beta^2}\, \mathrm{e}^{-2\sqrt{-\beta^2}\, x}}{2\beta^2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{\sqrt{-\beta^2}\, x}\right) + c_2\left(\mathrm{e}^{\sqrt{-\beta^2}\, x}\left(\frac{\sqrt{-\beta^2}\, \mathrm{e}^{-2\sqrt{-\beta^2}\, x}}{2\beta^2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{\sqrt{-\beta^2}\, x} + \frac{c_2 \sqrt{-\beta^2}\, \mathrm{e}^{-\sqrt{-\beta^2}\, x}}{2\beta^2} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{\sqrt{-\beta^2}\, x} + \frac{c_2 \sqrt{-\beta^2}\, \mathrm{e}^{-\sqrt{-\beta^2}\, x}}{2\beta^2}
$$

Verified OK.

4.12.4 Maple step by step solution

Let's solve

$y'' + \beta^2 y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $\beta^2 + r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm \left(\sqrt{-4\beta^2}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(\sqrt{-\beta^2}, -\sqrt{-\beta^2}\right)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{\sqrt{-\beta^2}\, x}$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{-\sqrt{-\beta^2}\, x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{\sqrt{-\beta^2}\, x} + c_2 e^{-\sqrt{-\beta^2}\, x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)+beta^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sin(\beta x) + c_2 \cos(\beta x)$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 20

```
DSolve[y''[x]+\[Beta]^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos(\beta x) + c_2 \sin(\beta x)$$

4.13 problem 14

Internal problem ID [6833]
Internal file name [`OUTPUT/6080_Thursday_July_28_2022_04_29_29_AM_353270/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1],
   [_2nd_order, _reducible, _mu_y_y1]]
```

$$\boxed{yy'' + {y'}^3 = 0}$$

4.13.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$
yp(y) \left(\frac{d}{dy}p(y)\right) + p(y)^3 = 0
$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$
\begin{aligned}
p' &= F(y, p) \\
&= f(y)g(p) \\
&= -\frac{p^2}{y}
\end{aligned}
$$

Where $f(y) = -\frac{1}{y}$ and $g(p) = p^2$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{p^2}\,dp &= -\frac{1}{y}\,dy \\
\int \frac{1}{p^2}\,dp &= \int -\frac{1}{y}\,dy \\
-\frac{1}{p} &= -\ln(y) + c_1
\end{aligned}
$$

The solution is

$$
-\frac{1}{p(y)} + \ln(y) - c_1 = 0
$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
-\frac{1}{y'} + \ln(y) - c_1 = 0
$$

Integrating both sides gives

$$
\begin{aligned}
\int (\ln(y) - c_1)\,dy &= x + c_2 \\
-c_1 y + y\ln(y) - y &= x + c_2
\end{aligned}
$$

Solving for y gives these solutions

<u>Summary</u>
The solution(s) found are the following

$$
y = e^{\operatorname{LambertW}\left((x+c_2)e^{-c_1-1}\right)+c_1+1} \tag{1}
$$

Verification of solutions

$$y = \mathrm{e}^{\mathrm{LambertW}\left((x+c_2)\,\mathrm{e}^{-c_1-1}\right)+c_1+1}$$

Verified OK.

4.13.2 Maple step by step solution

Let's solve

$y y'' + y'^3 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $y u(y)\left(\frac{d}{dy}u(y)\right) + u(y)^3 = 0$

- Separate variables

  $\frac{\frac{d}{dy}u(y)}{u(y)^2} = -\frac{1}{y}$

- Integrate both sides with respect to y

  $\int \frac{\frac{d}{dy}u(y)}{u(y)^2} dy = \int -\frac{1}{y} dy + c_1$

- Evaluate integral

  $-\frac{1}{u(y)} = -\ln(y) + c_1$

- Solve for $u(y)$

  $u(y) = \frac{1}{\ln(y) - c_1}$

- Solve 1st ODE for $u(y)$

  $u(y)=\frac{1}{\ln(y)-c_1}$

- Revert to original variables with substitution $u(y)=y', y=y$

  $y'=\frac{1}{\ln(y)-c_1}$

- Separate variables

  $y'(\ln(y)-c_1)=1$

- Integrate both sides with respect to x

  $\int y'(\ln(y)-c_1)\,dx=\int 1dx+c_2$

- Evaluate integral

  $-c_1y+y\ln(y)-y=x+c_2$

- Solve for y

  $y=\mathrm{e}^{\mathit{LambertW}\left((x+c_2)\mathrm{e}^{-c_1-1}\right)+c_1+1}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 27

```
dsolve(y(x)*diff(y(x),x$2)+diff(y(x),x)^3=0,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = c_1$$
$$y(x) = \frac{x + c_2}{\text{LambertW}\left((x + c_2)\,\mathrm{e}^{c_1 - 1}\right)}$$

✓ Solution by Mathematica

Time used: 60.095 (sec). Leaf size: 26

```
DSolve[y[x]*y''[x]+(y'[x])^3==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x + c_2}{W\left(e^{-1-c_1}(x + c_2)\right)}$$

4.14 problem 15

Internal problem ID [6834]
Internal file name [`OUTPUT/6081_Thursday_July_28_2022_04_29_32_AM_67325963/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' \cos(x) - y' = 0}$$

4.14.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x)\cos(x) - p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned}
p' &= F(x,p) \\
&= f(x)g(p) \\
&= \frac{p}{\cos(x)}
\end{aligned}$$

Where $f(x) = \frac{1}{\cos(x)}$ and $g(p) = p$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{p}\, dp &= \frac{1}{\cos(x)}\, dx \\
\int \frac{1}{p}\, dp &= \int \frac{1}{\cos(x)}\, dx \\
\ln(p) &= \ln(\sec(x) + \tan(x)) + c_1 \\
p &= e^{\ln(\sec(x)+\tan(x))+c_1} \\
&= c_1(\sec(x) + \tan(x))
\end{aligned}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = c_1(\sec(x) + \tan(x))$$

Integrating both sides gives

$$\begin{aligned}
y &= \int c_1(\sec(x) + \tan(x))\, \mathrm{d}x \\
&= c_1(\ln(\sec(x) + \tan(x)) - \ln(\cos(x))) + c_2
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1(\ln(\sec(x) + \tan(x)) - \ln(\cos(x))) + c_2 \tag{1}$$

Verification of solutions

$$y = c_1(\ln(\sec(x) + \tan(x)) - \ln(\cos(x))) + c_2$$

Verified OK.

4.14.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right)v' + \left(AB'' + BB' + CB\right)v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right)v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right)u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= \cos(x) \\ B &= -1 \\ C &= 0 \\ F &= 0 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= (\cos(x))(0) + (-1)(0) + (0)(-1) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-\cos(x)\, v'' + (1)\, v' = 0$$

Now by applying $v' = u$ the above becomes

$$-\cos(x)\, u'(x) + u(x) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u}{\cos(x)} \end{aligned}$$

Where $f(x) = \frac{1}{\cos(x)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= \frac{1}{\cos(x)}\, dx \\ \int \frac{1}{u}\, du &= \int \frac{1}{\cos(x)}\, dx \\ \ln(u) &= \ln(\sec(x) + \tan(x)) + c_1 \\ u &= e^{\ln(\sec(x)+\tan(x))+c_1} \\ &= c_1(\sec(x) + \tan(x)) \end{aligned}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= c_1(\sec(x) + \tan(x)) \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int c_1(\sec(x) + \tan(x))\, \mathrm{d}x \\ &= c_1(\ln(\sec(x) + \tan(x)) - \ln(\cos(x))) + c_2 \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y(x) &= Bv \\ &= (-1)\,(c_1(\ln(\sec(x) + \tan(x)) - \ln(\cos(x))) + c_2) \\ &= c_1(-\ln(\sec(x) + \tan(x)) + \ln(\cos(x))) - c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1(-\ln(\sec(x) + \tan(x)) + \ln(\cos(x))) - c_2 \tag{1}$$

Verification of solutions

$$y = c_1(-\ln(\sec(x) + \tan(x)) + \ln(\cos(x))) - c_2$$

Verified OK.

4.14.3 Maple step by step solution

Let's solve

$y'' \cos(x) - y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x)\cos(x) - u(x) = 0$

- Separate variables

  $\frac{u'(x)}{u(x)} = \frac{1}{\cos(x)}$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{u(x)} dx = \int \frac{1}{\cos(x)} dx + c_1$

- Evaluate integral

  $\ln(u(x)) = \ln(\sec(x) + \tan(x)) + c_1$

- Solve for $u(x)$

  $u(x) = -\frac{e^{c_1}\cos(x)}{\sin(x)-1}$

- Solve 1st ODE for $u(x)$

  $u(x) = -\frac{e^{c_1}\cos(x)}{\sin(x)-1}$

- Make substitution $u = y'$

  $y' = -\frac{e^{c_1}\cos(x)}{\sin(x)-1}$

- Integrate both sides to solve for y

  $\int y' dx = \int -\frac{e^{c_1}\cos(x)}{\sin(x)-1} dx + c_2$

- Compute integrals

$$y = -e^{c_1} \ln (\sin (x) - 1) + c_2$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
<- LODE missing y successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x$2)*cos(x)=diff(y(x),x),y(x), singsol=all)
```

$$y(x) = c_1 + (\ln (\sec (x) + \tan (x)) - \ln (\cos (x))) c_2$$

✓ Solution by Mathematica

Time used: 0.181 (sec). Leaf size: 25

```
DSolve[y''[x]*Cos[x]==y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \log \left(e^{4\operatorname{arctanh}\left(\tan\left(\frac{x}{2}\right)\right)} + 1\right) + c_2$$

4.15 problem 16

Internal problem ID [6835]
Internal file name [`OUTPUT/6082_Thursday_July_28_2022_04_29_34_AM_6396199/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y], [_2nd_order, _reducible, _mu_y_y1]]
```

$$\boxed{y'' - x{y'}^2 = 0}$$

With initial conditions

$$\left[y(2) = \frac{\pi}{4}, y'(2) = -\frac{1}{4}\right]$$

4.15.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - xp(x)^2 = 0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned} p' &= F(x,p) \\ &= f(x)g(p) \\ &= x\,p^2 \end{aligned}$$

Where $f(x)=x$ and $g(p)=p^2$. Integrating both sides gives

$$\frac{1}{p^2}\,dp = x\,dx$$

$$\int \frac{1}{p^2}\,dp = \int x\,dx$$

$$-\frac{1}{p} = \frac{x^2}{2}+c_1$$

The solution is

$$-\frac{1}{p(x)}-\frac{x^2}{2}-c_1 = 0$$

Initial conditions are used to solve for c_1. Substituting $x=2$ and $p=-\frac{1}{4}$ in the above solution gives an equation to solve for the constant of integration.

$$2-c_1 = 0$$

$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$-\frac{x^2p+4p+2}{2p} = 0$$

The above simplifies to

$$-x^2p-4p-2 = 0$$

For solution (1) found earlier, since $p=y'$ then we now have a new first order ode to solve which is

$$-x^2y'-4y'-2 = 0$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int -\frac{2}{x^2+4}\,\mathrm{d}x \\
&= -\arctan\left(\frac{x}{2}\right) + c_2
\end{aligned}
$$

Initial conditions are used to solve for c_2. Substituting $x = 2$ and $y = \frac{\pi}{4}$ in the above solution gives an equation to solve for the constant of integration.

$$
\frac{\pi}{4} = -\frac{\pi}{4} + c_2
$$

$$
c_2 = \frac{\pi}{2}
$$

Substituting c_2 found above in the general solution gives

$$
y = -\arctan\left(\frac{x}{2}\right) + \frac{\pi}{2}
$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$
y = -\arctan\left(\frac{x}{2}\right) + \frac{\pi}{2} \tag{1}
$$

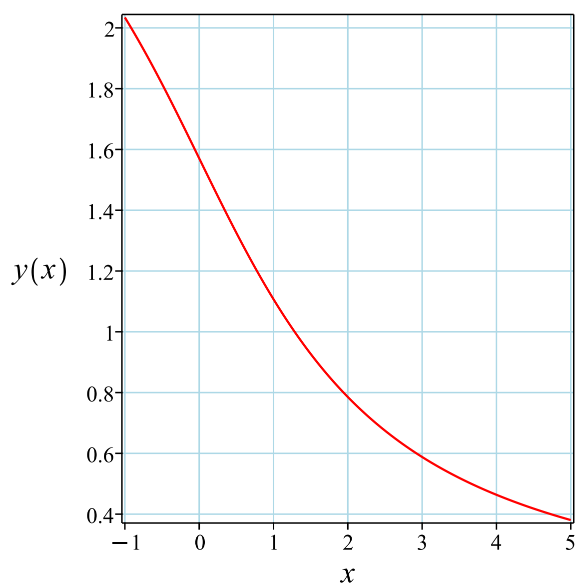

Figure 14: Solution plot

Verification of solutions

$$y = -\arctan\left(\frac{x}{2}\right) + \frac{\pi}{2}$$

Verified OK.

4.15.2 Maple step by step solution

Let's solve

$\left[y'' - xy'^2 = 0, y(2) = \frac{\pi}{4}, y'\Big|_{\{x=2\}} = -\frac{1}{4}\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x) - xu(x)^2 = 0$

- Separate variables

  $\frac{u'(x)}{u(x)^2} = x$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{u(x)^2} dx = \int x dx + c_1$

- Evaluate integral

  $-\frac{1}{u(x)} = \frac{x^2}{2} + c_1$

- Solve for $u(x)$

  $u(x) = -\frac{2}{x^2+2c_1}$

- Solve 1st ODE for $u(x)$

  $u(x) = -\frac{2}{x^2+2c_1}$

- Make substitution $u = y'$

  $y' = -\frac{2}{x^2+2c_1}$

- Integrate both sides to solve for y

  $\int y' dx = \int -\frac{2}{x^2+2c_1} dx + c_2$

- Compute integrals

  $y = -\frac{\sqrt{2}\arctan\left(\frac{x\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} + c_2$

□ Check validity of solution $y = -\frac{\sqrt{2}\,\arctan\left(\frac{x\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} + c_2$

- Use initial condition $y(2) = \frac{\pi}{4}$

  $\frac{\pi}{4} = -\frac{\sqrt{2}\,\arctan\left(\frac{\sqrt{2}}{\sqrt{c_1}}\right)}{\sqrt{c_1}} + c_2$

- Compute derivative of the solution

  $y' = -\frac{1}{c_1\left(\frac{x^2}{2c_1}+1\right)}$

- Use the initial condition $y'\Big|_{\{x=2\}} = -\frac{1}{4}$

  $-\frac{1}{4} = -\frac{1}{c_1\left(\frac{2}{c_1}+1\right)}$

- Solve for c_1 and c_2

  $\left\{c_1 = 2, c_2 = \frac{\pi}{2}\right\}$

- Substitute constant values into general solution and simplify

  $y = \operatorname{arccot}\left(\frac{x}{2}\right)$

• Solution to the IVP

  $y = \operatorname{arccot}\left(\frac{x}{2}\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = _a*_b(_a)^2, _b(_a), HINT = [[_a, -2*_b
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, -2*_b]
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 8

```
dsolve([diff(y(x),x$2)=x*diff(y(x),x)^2,y(2) = 1/4*Pi, D(y)(2) = -1/4],y(x), singsol=all)
```

$$y(x) = \operatorname{arccot}\left(\frac{x}{2}\right)$$

✓ Solution by Mathematica

Time used: 1.241 (sec). Leaf size: 19

```
DSolve[{y''[x]==x*(y'[x])^2,{y[2]==1/4*Pi,y'[2]==-1/4}},y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \frac{1}{2}\left(\pi - 2\arctan\left(\frac{x}{2}\right)\right)$$

4.16 problem 17

Internal problem ID [6836]
Internal file name [`OUTPUT/6083_Thursday_July_28_2022_04_29_37_AM_72415050/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y], [_2nd_order, _reducible, _mu_y_y1]]
```

$$\boxed{y'' - x{y'}^2 = 0}$$

With initial conditions

$$\left[y(0) = 1, y'(0) = \frac{1}{2}\right]$$

4.16.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - xp(x)^2 = 0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned} p' &= F(x,p) \\ &= f(x)g(p) \\ &= x\,p^2 \end{aligned}$$

Where $f(x) = x$ and $g(p) = p^2$. Integrating both sides gives

$$\frac{1}{p^2}\,dp = x\,dx$$

$$\int \frac{1}{p^2}\,dp = \int x\,dx$$

$$-\frac{1}{p} = \frac{x^2}{2} + c_1$$

The solution is

$$-\frac{1}{p(x)} - \frac{x^2}{2} - c_1 = 0$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $p = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$-2 - c_1 = 0$$

$$c_1 = -2$$

Substituting c_1 found above in the general solution gives

$$-\frac{x^2p - 4p + 2}{2p} = 0$$

The above simplifies to

$$-x^2p + 4p - 2 = 0$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$-x^2y' + 4y' - 2 = 0$$

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{2}{x^2-4}\,\mathrm{d}x \\ &= \frac{\ln(x+2)}{2} - \frac{\ln(x-2)}{2} + c_2 \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{i\pi}{2} + c_2$$

$$c_2 = \frac{i\pi}{2} + 1$$

Substituting c_2 found above in the general solution gives

$$y = \frac{\ln(x+2)}{2} - \frac{\ln(x-2)}{2} + \frac{i\pi}{2} + 1$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = \frac{\ln(x+2)}{2} - \frac{\ln(x-2)}{2} + \frac{i\pi}{2} + 1 \tag{1}$$

Verification of solutions

$$y = \frac{\ln(x+2)}{2} - \frac{\ln(x-2)}{2} + \frac{i\pi}{2} + 1$$

Verified OK.

4.16.2 Maple step by step solution

Let's solve

$$\left[y'' - xy'^2 = 0, y(0) = 1, y'\Big|_{\{x=0\}} = \frac{1}{2}\right]$$

- Highest derivative means the order of the ODE is 2

  y''
- Make substitution $u = y'$ to reduce order of ODE

$u'(x) - x u(x)^2 = 0$

- Separate variables

  $\frac{u'(x)}{u(x)^2} = x$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{u(x)^2} dx = \int x dx + c_1$

- Evaluate integral

  $-\frac{1}{u(x)} = \frac{x^2}{2} + c_1$

- Solve for $u(x)$

  $u(x) = -\frac{2}{x^2 + 2c_1}$

- Solve 1st ODE for $u(x)$

  $u(x) = -\frac{2}{x^2 + 2c_1}$

- Make substitution $u = y'$

  $y' = -\frac{2}{x^2 + 2c_1}$

- Integrate both sides to solve for y

  $\int y' dx = \int -\frac{2}{x^2 + 2c_1} dx + c_2$

- Compute integrals

  $y = -\frac{\sqrt{2}\,\arctan\left(\frac{x\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} + c_2$

□ Check validity of solution $y = -\frac{\sqrt{2}\,\arctan\left(\frac{x\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} + c_2$

- ○ Use initial condition $y(0) = 1$

  $1 = c_2$

- ○ Compute derivative of the solution

  $y' = -\frac{1}{c_1\left(\frac{x^2}{2c_1} + 1\right)}$

- ○ Use the initial condition $y'\Big|_{\{x=0\}} = \frac{1}{2}$

  $\frac{1}{2} = -\frac{1}{c_1}$

- ○ Solve for c_1 and c_2

  $\{c_1 = -2, c_2 = 1\}$

- ○ Substitute constant values into general solution and simplify

$y = \operatorname{arctanh}\left(\frac{x}{2}\right) + 1$

- Solution to the IVP

$y = \operatorname{arctanh}\left(\frac{x}{2}\right) + 1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = _a*_b(_a)^2, _b(_a), HINT = [[_a, -2*_b
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, -2*_b]
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 10

```
dsolve([diff(y(x),x$2)=x*diff(y(x),x)^2,y(0) = 1, D(y)(0) = 1/2],y(x), singsol=all)
```

$$y(x) = \operatorname{arctanh}\left(\frac{x}{2}\right) + 1$$

✓ Solution by Mathematica

Time used: 0.229 (sec). Leaf size: 13

```
DSolve[{y''[x]==x*(y'[x])^2,{y[0]==1,y'[0]==1/2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \operatorname{arctanh}\left(\frac{x}{2}\right) + 1$$

4.17 problem 18

Internal problem ID [6837]
Internal file name [`OUTPUT/6084_Thursday_July_28_2022_04_29_39_AM_54278317/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{y'' + \mathrm{e}^{-2y} = 0}$$

With initial conditions

$$[y(3) = 0, y'(3) = 1]$$

4.17.1 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + y'\mathrm{e}^{-2y} = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' + y'\mathrm{e}^{-2y})\, dx = 0$$
$$\frac{y'^2}{2} - \frac{\mathrm{e}^{-2y}}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{e^{-2y} + 2c_1} \tag{1}$$

$$y' = -\sqrt{e^{-2y} + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{e^{-2y} + 2c_1}} dy = \int dx$$

$$\frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{e^{-2y}+2c_1}\,\sqrt{2}}{2\sqrt{c_1}}\right)}{2\sqrt{c_1}} = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{e^{-2y} + 2c_1}} dy = \int dx$$

$$-\frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{e^{-2y}+2c_1}\,\sqrt{2}}{2\sqrt{c_1}}\right)}{2\sqrt{c_1}} = x + c_3$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$\frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{e^{-2y}+2c_1}\,\sqrt{2}}{2\sqrt{c_1}}\right)}{2\sqrt{c_1}} = x + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 3$ in the above gives

$$\frac{\operatorname{arctanh}\left(\frac{\sqrt{1+2c_1}\,\sqrt{2}}{2\sqrt{c_1}}\right)\sqrt{2}}{2\sqrt{c_1}} = 3 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{2\tanh\left(\sqrt{c_1}\,(x+c_2)\,\sqrt{2}\right) c_1^{\frac{3}{2}}\sqrt{2}\left(1-\tanh\left(\sqrt{c_1}\,(x+c_2)\,\sqrt{2}\right)^2\right)}{2\tanh\left(\sqrt{c_1}\,(x+c_2)\,\sqrt{2}\right)^2 c_1 - 2c_1}$$

substituting $y' = 1$ and $x = 3$ in the above gives

$$1 = \frac{\left(\mathrm{e}^{2(3+c_2)\sqrt{c_1}\,\sqrt{2}} - 1\right)\sqrt{c_1}\,\sqrt{2}}{\mathrm{e}^{2(3+c_2)\sqrt{c_1}\,\sqrt{2}} + 1} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Looking at the Second solution

$$-\frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{\mathrm{e}^{-2y}+2c_1}\,\sqrt{2}}{2\sqrt{c_1}}\right)}{2\sqrt{c_1}} = x + c_3 \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 3$ in the above gives

$$-\frac{\operatorname{arctanh}\left(\frac{\sqrt{1+2c_1}\,\sqrt{2}}{2\sqrt{c_1}}\right)\sqrt{2}}{2\sqrt{c_1}} = 3 + c_3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{2\tanh\left(\sqrt{c_1}\,(x + c_3)\,\sqrt{2}\right)c_1^{\frac{3}{2}}\sqrt{2}\left(1 - \tanh\left(\sqrt{c_1}\,(x + c_3)\,\sqrt{2}\right)^2\right)}{2\tanh\left(\sqrt{c_1}\,(x + c_3)\,\sqrt{2}\right)^2 c_1 - 2c_1}$$

substituting $y' = 1$ and $x = 3$ in the above gives

$$1 = \frac{\left(\mathrm{e}^{2(3+c_3)\sqrt{c_1}\,\sqrt{2}} - 1\right)\sqrt{c_1}\,\sqrt{2}}{\mathrm{e}^{2(3+c_3)\sqrt{c_1}\,\sqrt{2}} + 1} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

4.17.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$
\begin{aligned}
y'' &= \frac{dp}{dx} \\
&= \frac{dy}{dx}\frac{dp}{dy} \\
&= p\frac{dp}{dy}
\end{aligned}
$$

Hence the ode becomes

$$
p(y)\left(\frac{d}{dy}p(y)\right) = -\mathrm{e}^{-2y}
$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$
\begin{aligned}
p' &= F(y,p) \\
&= f(y)g(p) \\
&= -\frac{\mathrm{e}^{-2y}}{p}
\end{aligned}
$$

Where $f(y) = -\mathrm{e}^{-2y}$ and $g(p) = \frac{1}{p}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{1}{p}}\,dp &= -\mathrm{e}^{-2y}\,dy \\
\int \frac{1}{\frac{1}{p}}\,dp &= \int -\mathrm{e}^{-2y}\,dy \\
\frac{p^2}{2} &= \frac{\mathrm{e}^{-2y}}{2} + c_1
\end{aligned}
$$

The solution is

$$
\frac{p(y)^2}{2} - \frac{\mathrm{e}^{-2y}}{2} - c_1 = 0
$$

Initial conditions are used to solve for c_1. Substituting $y = 0$ and $p = 1$ in the above solution gives an equation to solve for the constant of integration.

$$
-c_1 = 0
$$

$$
c_1 = 0
$$

Substituting c_1 found above in the general solution gives

$$\frac{p^2}{2} - \frac{e^{-2y}}{2} = 0$$

Solving for $p(y)$ from the above gives

$$p(y) = e^{-y}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = e^{-y}$$

Integrating both sides gives

$$\int e^{y} dy = x + c_2$$
$$e^{y} = x + c_2$$

Solving for y gives these solutions

$$y_1 = \ln(x + c_2)$$

Initial conditions are used to solve for c_2. Substituting $x = 3$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \ln(3 + c_2)$$

$$c_2 = -2$$

Substituting c_2 found above in the general solution gives

$$y = \ln(x - 2)$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = \ln(x - 2) \tag{1}$$

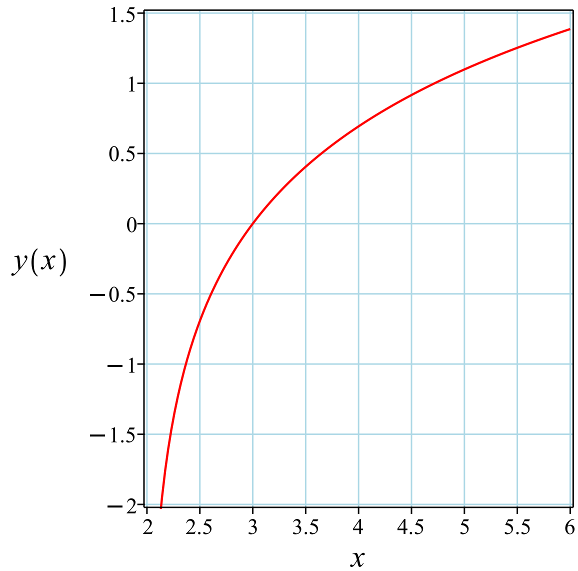

Figure 15: Solution plot

Verification of solutions

$$y = \ln (x-2)$$

Verified OK.

4.17.3 Maple step by step solution

Let's solve

$$\left[y'' = -\mathrm{e}^{-2y}, y(3) = 0, y'\Big|_{\{x=3\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

$u(y)\left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

$u(y)\left(\frac{d}{dy}u(y)\right) = -\mathrm{e}^{-2y}$

- Integrate both sides with respect to y

$\int u(y)\left(\frac{d}{dy}u(y)\right)dy = \int -\mathrm{e}^{-2y}dy + c_1$

- Evaluate integral

$\frac{u(y)^2}{2} = \frac{\mathrm{e}^{-2y}}{2} + c_1$

- Solve for $u(y)$

$\left\{u(y) = \sqrt{\mathrm{e}^{-2y} + 2c_1}, u(y) = -\sqrt{\mathrm{e}^{-2y} + 2c_1}\right\}$

- Solve 1st ODE for $u(y)$

$u(y) = \sqrt{\mathrm{e}^{-2y} + 2c_1}$

- Revert to original variables with substitution $u(y) = y', y = y$

$y' = \sqrt{\mathrm{e}^{-2y} + 2c_1}$

- Separate variables

$\frac{y'}{\sqrt{\mathrm{e}^{-2y}+2c_1}} = 1$

- Integrate both sides with respect to x

$\int \frac{y'}{\sqrt{\mathrm{e}^{-2y}+2c_1}}dx = \int 1dx + c_2$

- Evaluate integral

$\frac{\sqrt{2}\,\mathrm{arctanh}\left(\frac{\sqrt{\mathrm{e}^{-2y}+2c_1}\,\sqrt{2}}{2\sqrt{c_1}}\right)}{2\sqrt{c_1}} = x + c_2$

- Solve for y

$y = -\frac{\ln\left(2\tanh\left(\sqrt{c_1}\,(x+c_2)\sqrt{2}\right)^2 c_1 - 2c_1\right)}{2}$

- Solve 2nd ODE for $u(y)$

$u(y) = -\sqrt{\mathrm{e}^{-2y} + 2c_1}$

- Revert to original variables with substitution $u(y) = y', y = y$

$y' = -\sqrt{\mathrm{e}^{-2y} + 2c_1}$

- Separate variables

$$\frac{y'}{\sqrt{e^{-2y}+2c_1}} = -1$$

- Integrate both sides with respect to x

$$\int \frac{y'}{\sqrt{e^{-2y}+2c_1}}dx = \int (-1)\, dx + c_2$$

- Evaluate integral

$$\frac{\sqrt{2}\,\mathrm{arctanh}\left(\frac{\sqrt{e^{-2y}+2c_1}\,\sqrt{2}}{2\sqrt{c_1}}\right)}{2\sqrt{c_1}} = -x + c_2$$

- Solve for y

$$y = -\frac{\ln\left(2\tanh\left(\sqrt{c_1}\,(-x+c_2)\sqrt{2}\right)^2 c_1 - 2c_1\right)}{2}$$

☐ Check validity of solution $y = -\frac{\ln\left(2\tanh\left(\sqrt{c_1}\,(-x+c_2)\sqrt{2}\right)^2 c_1 - 2c_1\right)}{2}$

- ○ Use initial condition $y(3) = 0$

$$0 = -\frac{\ln\left(2\tanh\left(\sqrt{c_1}\,(-3+c_2)\sqrt{2}\right)^2 c_1 - 2c_1\right)}{2}$$

- ○ Compute derivative of the solution

$$y' = \frac{2\tanh\left(\sqrt{c_1}\,(-x+c_2)\sqrt{2}\right)c_1^{\frac{3}{2}}\sqrt{2}\left(1-\tanh\left(\sqrt{c_1}\,(-x+c_2)\sqrt{2}\right)^2\right)}{2\tanh\left(\sqrt{c_1}\,(-x+c_2)\sqrt{2}\right)^2 c_1 - 2c_1}$$

- ○ Use the initial condition $y'\Big|_{\{x=3\}} = 1$

$$1 = \frac{2\tanh\left(\sqrt{c_1}\,(-3+c_2)\sqrt{2}\right)c_1^{\frac{3}{2}}\sqrt{2}\left(1-\tanh\left(\sqrt{c_1}\,(-3+c_2)\sqrt{2}\right)^2\right)}{2\tanh\left(\sqrt{c_1}\,(-3+c_2)\sqrt{2}\right)^2 c_1 - 2c_1}$$

- ○ Solve for c_1 and c_2
- ○ The solution does not satisfy the initial conditions

☐ Check validity of solution $y = -\frac{\ln\left(2\tanh\left(\sqrt{c_1}\,(x+c_2)\sqrt{2}\right)^2 c_1 - 2c_1\right)}{2}$

- ○ Use initial condition $y(3) = 0$

$$0 = -\frac{\ln\left(2\tanh\left((3+c_2)\sqrt{c_1}\,\sqrt{2}\right)^2 c_1 - 2c_1\right)}{2}$$

- ○ Compute derivative of the solution

$$y' = -\frac{2\tanh\left(\sqrt{c_1}\,(x+c_2)\sqrt{2}\right)c_1^{\frac{3}{2}}\sqrt{2}\left(1-\tanh\left(\sqrt{c_1}\,(x+c_2)\sqrt{2}\right)^2\right)}{2\tanh\left(\sqrt{c_1}\,(x+c_2)\sqrt{2}\right)^2 c_1 - 2c_1}$$

- Use the initial condition $y'\Big|_{\{x=3\}} = 1$

$$1 = -\frac{2\tanh\left((3+c_2)\sqrt{c_1}\,\sqrt{2}\right)c_1^{\frac{3}{2}}\sqrt{2}\left(1-\tanh\left((3+c_2)\sqrt{c_1}\,\sqrt{2}\right)^2\right)}{2\tanh\left((3+c_2)\sqrt{c_1}\,\sqrt{2}\right)^2 c_1 - 2c_1}$$

- Solve for c_1 and c_2
- The solution does not satisfy the initial conditions

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+exp(-2*_a) = 0, _b(_a), HINT = [
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[1, -_b]
```

✓ Solution by Maple

Time used: 0.093 (sec). Leaf size: 12

```
dsolve([diff(y(x),x$2)=-exp(-2*y(x)),y(3) = 0, D(y)(3) = 1],y(x), singsol=all)
```

$$y(x) = \frac{\ln\left((-2+x)^2\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.157 (sec). Leaf size: 9

```
DSolve[{y''[x]==-Exp[-2*y[x]],{y[3]==0,y'[3]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \log(x-2)$$

4.18 problem 19

Internal problem ID [6838]
Internal file name [`OUTPUT/6085_Thursday_July_28_2022_04_29_41_AM_88428700/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{y''+\mathrm{e}^{-2y}=0}$$

With initial conditions

$$[y(3)=0, y'(3)=-1]$$

4.18.1 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y''+y'\mathrm{e}^{-2y}=0$$

Integrating the above w.r.t x gives

$$\int \left(y'y''+y'\mathrm{e}^{-2y}\right) dx=0$$
$$\frac{y'^2}{2}-\frac{\mathrm{e}^{-2y}}{2}=c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{e^{-2y} + 2c_1} \tag{1}$$
$$y' = -\sqrt{e^{-2y} + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{e^{-2y} + 2c_1}} dy = \int dx$$
$$\frac{\sqrt{2}\, \operatorname{arctanh}\left(\frac{\sqrt{e^{-2y}+2c_1}\, \sqrt{2}}{2\sqrt{c_1}}\right)}{2\sqrt{c_1}} = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{e^{-2y} + 2c_1}} dy = \int dx$$
$$-\frac{\sqrt{2}\, \operatorname{arctanh}\left(\frac{\sqrt{e^{-2y}+2c_1}\, \sqrt{2}}{2\sqrt{c_1}}\right)}{2\sqrt{c_1}} = x + c_3$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$\frac{\sqrt{2}\, \operatorname{arctanh}\left(\frac{\sqrt{e^{-2y}+2c_1}\, \sqrt{2}}{2\sqrt{c_1}}\right)}{2\sqrt{c_1}} = x + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 3$ in the above gives

$$\frac{\operatorname{arctanh}\left(\frac{\sqrt{1+2c_1}\, \sqrt{2}}{2\sqrt{c_1}}\right) \sqrt{2}}{2\sqrt{c_1}} = 3 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{2\tanh\left(\sqrt{c_1}\,(x + c_2)\,\sqrt{2}\right) c_1^{\frac{3}{2}} \sqrt{2} \left(1 - \tanh\left(\sqrt{c_1}\,(x + c_2)\,\sqrt{2}\right)^2\right)}{2\tanh\left(\sqrt{c_1}\ (x + c_2)\,\sqrt{2}\right)^2 c_1 - 2c_1}$$

substituting $y' = -1$ and $x = 3$ in the above gives

$$-1 = \frac{\left(e^{2(3+c_2)\sqrt{c_1}\,\sqrt{2}} - 1\right)\sqrt{c_1}\,\sqrt{2}}{e^{2(3+c_2)\sqrt{c_1}\,\sqrt{2}} + 1} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Looking at the Second solution

$$-\frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{e^{-2y}+2c_1}\,\sqrt{2}}{2\sqrt{c_1}}\right)}{2\sqrt{c_1}} = x + c_3 \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 3$ in the above gives

$$-\frac{\operatorname{arctanh}\left(\frac{\sqrt{1+2c_1}\,\sqrt{2}}{2\sqrt{c_1}}\right)\sqrt{2}}{2\sqrt{c_1}} = 3 + c_3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{2\tanh\left(\sqrt{c_1}\,(x+c_3)\,\sqrt{2}\right) c_1^{\frac{3}{2}}\sqrt{2}\left(1 - \tanh\left(\sqrt{c_1}\,(x+c_3)\,\sqrt{2}\right)^2\right)}{2\tanh\left(\sqrt{c_1}\,(x+c_3)\,\sqrt{2}\right)^2 c_1 - 2c_1}$$

substituting $y' = -1$ and $x = 3$ in the above gives

$$-1 = \frac{\left(e^{2(3+c_3)\sqrt{c_1}\,\sqrt{2}} - 1\right)\sqrt{c_1}\,\sqrt{2}}{e^{2(3+c_3)\sqrt{c_1}\,\sqrt{2}} + 1} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

4.18.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$
\begin{aligned}
y'' &= \frac{dp}{dx} \\
&= \frac{dy}{dx}\frac{dp}{dy} \\
&= p\frac{dp}{dy}
\end{aligned}
$$

Hence the ode becomes

$$
p(y)\left(\frac{d}{dy}p(y)\right) = -\mathrm{e}^{-2y}
$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$
\begin{aligned}
p' &= F(y,p) \\
&= f(y)g(p) \\
&= -\frac{\mathrm{e}^{-2y}}{p}
\end{aligned}
$$

Where $f(y) = -\mathrm{e}^{-2y}$ and $g(p) = \frac{1}{p}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{1}{p}}\,dp &= -\mathrm{e}^{-2y}\,dy \\
\int \frac{1}{\frac{1}{p}}\,dp &= \int -\mathrm{e}^{-2y}\,dy \\
\frac{p^2}{2} &= \frac{\mathrm{e}^{-2y}}{2} + c_1
\end{aligned}
$$

The solution is

$$
\frac{p(y)^2}{2} - \frac{\mathrm{e}^{-2y}}{2} - c_1 = 0
$$

Initial conditions are used to solve for c_1. Substituting $y = 0$ and $p = -1$ in the above solution gives an equation to solve for the constant of integration.

$$
-c_1 = 0
$$

$$
c_1 = 0
$$

Substituting c_1 found above in the general solution gives

$$\frac{p^2}{2} - \frac{\mathrm{e}^{-2y}}{2} = 0$$

Solving for $p(y)$ from the above gives

$$p(y) = -\mathrm{e}^{-y}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\mathrm{e}^{-y}$$

Integrating both sides gives

$$\int -\mathrm{e}^{y} dy = x + c_2$$
$$-\mathrm{e}^{y} = x + c_2$$

Solving for y gives these solutions

$$y_1 = -\ln\left(-\frac{1}{x + c_2}\right)$$

Initial conditions are used to solve for c_2. Substituting $x = 3$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\ln\left(-\frac{1}{3 + c_2}\right)$$

$$c_2 = -4$$

Substituting c_2 found above in the general solution gives

$$y = -\ln\left(-\frac{1}{x - 4}\right)$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = -\ln\left(-\frac{1}{x - 4}\right) \tag{1}$$

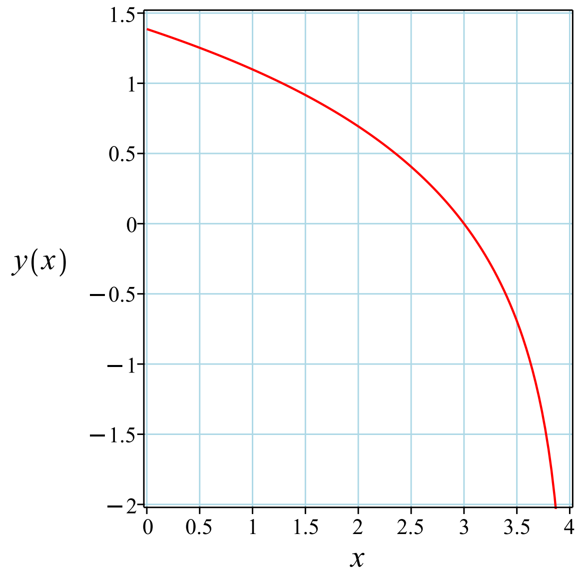

Figure 16: Solution plot

Verification of solutions

$$y = -\ln\left(-\frac{1}{x-4}\right)$$

Verified OK.

4.18.3 Maple step by step solution

Let's solve

$$\left[y'' = -e^{-2y}, y(3) = 0, y'\Big|_{\{x=3\}} = -1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right)=y''$
- Make substitutions $y'=u(y)\,, y''=u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $u(y)\left(\frac{d}{dy}u(y)\right)=-\mathrm{e}^{-2y}$
- Integrate both sides with respect to y

  $\int u(y)\left(\frac{d}{dy}u(y)\right)dy=\int -\mathrm{e}^{-2y}dy+c_1$
- Evaluate integral

  $\frac{u(y)^2}{2}=\frac{\mathrm{e}^{-2y}}{2}+c_1$
- Solve for $u(y)$

  $\left\{u(y)=\sqrt{\mathrm{e}^{-2y}+2c_1}, u(y)=-\sqrt{\mathrm{e}^{-2y}+2c_1}\right\}$
- Solve 1st ODE for $u(y)$

  $u(y)=\sqrt{\mathrm{e}^{-2y}+2c_1}$
- Revert to original variables with substitution $u(y)=y', y=y$

  $y'=\sqrt{\mathrm{e}^{-2y}+2c_1}$
- Separate variables

  $\frac{y'}{\sqrt{\mathrm{e}^{-2y}+2c_1}}=1$
- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{\mathrm{e}^{-2y}+2c_1}}dx=\int 1dx+c_2$
- Evaluate integral

  $\frac{\sqrt{2}\,\mathrm{arctanh}\left(\frac{\sqrt{\mathrm{e}^{-2y}+2c_1}\,\sqrt{2}}{2\sqrt{c_1}}\right)}{2\sqrt{c_1}}=x+c_2$
- Solve for y

  $y=-\frac{\ln\left(2\tanh\left(\sqrt{c_1}\,(x+c_2)\sqrt{2}\right)^2 c_1-2c_1\right)}{2}$
- Solve 2nd ODE for $u(y)$

  $u(y)=-\sqrt{\mathrm{e}^{-2y}+2c_1}$
- Revert to original variables with substitution $u(y)=y', y=y$

  $y'=-\sqrt{\mathrm{e}^{-2y}+2c_1}$

- Separate variables

$$\frac{y'}{\sqrt{\mathrm{e}^{-2y}+2c_1}} = -1$$

- Integrate both sides with respect to x

$$\int \frac{y'}{\sqrt{\mathrm{e}^{-2y}+2c_1}}dx = \int (-1)\, dx + c_2$$

- Evaluate integral

$$\frac{\sqrt{2}\,\mathrm{arctanh}\left(\frac{\sqrt{\mathrm{e}^{-2y}+2c_1}\,\sqrt{2}}{2\sqrt{c_1}}\right)}{2\sqrt{c_1}} = -x + c_2$$

- Solve for y

$$y = -\frac{\ln\left(2\tanh\left(\sqrt{c_1}\,(-x+c_2)\sqrt{2}\right)^2 c_1-2c_1\right)}{2}$$

☐ Check validity of solution $y = -\frac{\ln\left(2\tanh\left(\sqrt{c_1}\,(-x+c_2)\sqrt{2}\right)^2 c_1-2c_1\right)}{2}$

- ○ Use initial condition $y(3) = 0$

$$0 = -\frac{\ln\left(2\tanh\left(\sqrt{c_1}\,(-3+c_2)\sqrt{2}\right)^2 c_1-2c_1\right)}{2}$$

- ○ Compute derivative of the solution

$$y' = \frac{2\tanh\left(\sqrt{c_1}\,(-x+c_2)\sqrt{2}\right)c_1^{\frac{3}{2}}\sqrt{2}\,\left(1-\tanh\left(\sqrt{c_1}\,(-x+c_2)\sqrt{2}\right)^2\right)}{2\tanh\left(\sqrt{c_1}\,(-x+c_2)\sqrt{2}\right)^2 c_1-2c_1}$$

- ○ Use the initial condition $y'\Big|_{\{x=3\}} = -1$

$$-1 = \frac{2\tanh\left(\sqrt{c_1}\,(-3+c_2)\sqrt{2}\right)c_1^{\frac{3}{2}}\sqrt{2}\,\left(1-\tanh\left(\sqrt{c_1}\,(-3+c_2)\sqrt{2}\right)^2\right)}{2\tanh\left(\sqrt{c_1}\,(-3+c_2)\sqrt{2}\right)^2 c_1-2c_1}$$

- ○ Solve for c_1 and c_2
- ○ The solution does not satisfy the initial conditions

☐ Check validity of solution $y = -\frac{\ln\left(2\tanh\left(\sqrt{c_1}\,(x+c_2)\sqrt{2}\right)^2 c_1-2c_1\right)}{2}$

- ○ Use initial condition $y(3) = 0$

$$0 = -\frac{\ln\left(2\tanh\left((3+c_2)\sqrt{c_1}\,\sqrt{2}\right)^2 c_1-2c_1\right)}{2}$$

- ○ Compute derivative of the solution

$$y' = -\frac{2\tanh\left(\sqrt{c_1}\,(x+c_2)\sqrt{2}\right)c_1^{\frac{3}{2}}\sqrt{2}\,\left(1-\tanh\left(\sqrt{c_1}\,(x+c_2)\sqrt{2}\right)^2\right)}{2\tanh\left(\sqrt{c_1}\,(x+c_2)\sqrt{2}\right)^2 c_1-2c_1}$$

- Use the initial condition $y'\Big|_{\{x=3\}} = -1$

$$-1 = -\frac{2\tanh\left((3+c_2)\sqrt{c_1}\,\sqrt{2}\right)c_1^{\frac{3}{2}}\sqrt{2}\left(1-\tanh\left((3+c_2)\sqrt{c_1}\,\sqrt{2}\right)^2\right)}{2\tanh\left((3+c_2)\sqrt{c_1}\,\sqrt{2}\right)^2 c_1-2c_1}$$

- Solve for c_1 and c_2
- The solution does not satisfy the initial conditions

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 12

```
dsolve([diff(y(x),x$2)=-exp(-2*y(x)),y(3) = 0, D(y)(3) = -1],y(x), singsol=all)
```

$$y(x) = \frac{\ln\left((x-4)^2\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.146 (sec). Leaf size: 11

```
DSolve[{y''[x]==-Exp[-2*y[x]],{y[3]==0,y'[3]==-1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \log(4-x)$$

4.19 problem 20

Internal problem ID [6839]
Internal file name [`OUTPUT/6086_Thursday_July_28_2022_04_29_43_AM_89875315/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{2y'' - \sin(2y) = 0}$$

With initial conditions

$$\left[y(0) = \frac{\pi}{2}, y'(0) = 1\right]$$

4.19.1 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$2y'y'' - y'\sin(2y) = 0$$

Integrating the above w.r.t x gives

$$\int (2y'y'' - y'\sin(2y))\,dx = 0$$
$$y'^2 + \frac{\cos(2y)}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\sqrt{-2\cos(2y) + 4c_1}}{2} \tag{1}$$

$$y' = -\frac{\sqrt{-2\cos(2y) + 4c_1}}{2} \tag{2}$$

Now each one of the above ODE is solved.

<u>Solving equation (1)</u>

Integrating both sides gives

$$\int \frac{2}{\sqrt{-2\cos(2y) + 4c_1}} dy = \int dx$$

$$\frac{2\sqrt{-\frac{\cos(2y)-2c_1}{2c_1-1}}\,\text{InverseJacobiAM}\left(y, \sqrt{2}\,\sqrt{-\frac{1}{2c_1-1}}\right)}{\sqrt{-2\cos(2y) + 4c_1}} = x + c_2$$

<u>Solving equation (2)</u>

Integrating both sides gives

$$\int -\frac{2}{\sqrt{-2\cos(2y) + 4c_1}} dy = \int dx$$

$$-\frac{2\sqrt{-\frac{\cos(2y)-2c_1}{2c_1-1}}\,\text{InverseJacobiAM}\left(y, \sqrt{2}\,\sqrt{-\frac{1}{2c_1-1}}\right)}{\sqrt{-2\cos(2y) + 4c_1}} = x + c_3$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$\frac{2\sqrt{-\frac{\cos(2y)-2c_1}{2c_1-1}}\,\text{InverseJacobiAM}\left(y, \sqrt{2}\,\sqrt{-\frac{1}{2c_1-1}}\right)}{\sqrt{-2\cos(2y) + 4c_1}} = x + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{\pi}{2}$ and $x = 0$ in the above gives

$$\frac{2\sqrt{\frac{1+2c_1}{2c_1-1}}\,\text{EllipticK}\left(\sqrt{2}\,\sqrt{-\frac{1}{2c_1-1}}\right)}{\sqrt{2 + 4c_1}} = c_2 \tag{1A}$$

Unable to solve for y to solve for constant of integration

Looking at the Second solution

$$-\frac{2\sqrt{-\frac{\cos(2y)-2c_1}{2c_1-1}}\ \text{InverseJacobiAM}\left(y, \sqrt{2}\sqrt{-\frac{1}{2c_1-1}}\right)}{\sqrt{-2\cos(2y)+4c_1}} = x + c_3 \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{\pi}{2}$ and $x = 0$ in the above gives

$$-\frac{2\sqrt{\frac{1+2c_1}{2c_1-1}}\ \text{EllipticK}\left(\sqrt{2}\sqrt{-\frac{1}{2c_1-1}}\right)}{\sqrt{2+4c_1}} = c_3 \tag{1A}$$

Unable to solve for y to solve for constant of integration

Verification of solutions N/A

4.19.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$2p(y)\left(\frac{d}{dy}p(y)\right) = \sin(2y)$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned} p' &= F(y,p) \\ &= f(y)g(p) \\ &= \frac{\sin(2y)}{2p} \end{aligned}$$

Where $f(y) = \frac{\sin(2y)}{2}$ and $g(p) = \frac{1}{p}$. Integrating both sides gives

$$\frac{1}{\frac{1}{p}}\,dp = \frac{\sin(2y)}{2}\,dy$$

$$\int \frac{1}{\frac{1}{p}}\,dp = \int \frac{\sin(2y)}{2}\,dy$$

$$\frac{p^2}{2} = -\frac{\cos(2y)}{4} + c_1$$

The solution is

$$\frac{p(y)^2}{2} + \frac{\cos(2y)}{4} - c_1 = 0$$

Initial conditions are used to solve for c_1. Substituting $y = \frac{\pi}{2}$ and $p = 1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{4} - c_1 = 0$$

$$c_1 = \frac{1}{4}$$

Substituting c_1 found above in the general solution gives

$$\frac{p^2}{2} + \frac{\cos(2y)}{4} - \frac{1}{4} = 0$$

Solving for $p(y)$ from the above gives

$$p(y) = \frac{\sqrt{2 - 2\cos(2y)}}{2}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{\sqrt{2 - 2\cos(2y)}}{2}$$

Integrating both sides gives

$$\int \frac{2}{\sqrt{2 - 2\cos(2y)}}dy = \int dx$$

$$-\frac{\sin(y)\operatorname{arctanh}(\cos(y))}{\sqrt{\frac{1}{2} - \frac{\cos(2y)}{2}}} = x + c_2$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = \frac{\pi}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_2$$

$$c_2 = 0$$

Substituting c_2 found above in the general solution gives

$$-\frac{2\sin(y)\operatorname{arctanh}(\cos(y))}{\sqrt{2-2\cos(2y)}} = x$$

The above simplifies to

$$-\sin(y)\operatorname{arctanh}(\cos(y)) - x\sqrt{\frac{1}{2}-\frac{\cos(2y)}{2}} = 0$$

Simplifying the solution $\sin(y)(-\operatorname{arctanh}(\cos(y)) - \operatorname{csgn}(\sin(y))x) = 0$ to $\sin(y)(-\operatorname{arctanh}(\cos(y)) -$ 0 Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$\sin(y)(-\operatorname{arctanh}(\cos(y)) - x) = 0 \tag{1}$$

Verification of solutions

$$\sin(y)(-\operatorname{arctanh}(\cos(y)) - x) = 0$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-(1/2)*sin(2*_a) = 0, _b(_a)`   *
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   <- Bernoulli successful
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 140.984 (sec). Leaf size: 1495

```
dsolve([2*diff(y(x),x$2)=sin(2*y(x)),y(0) = 1/2*Pi, D(y)(0) = 1],y(x), singsol=all)
```

Expression too large to display

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{2*y''[x]==Sin[2*y[x]],{y[0]==Pi/2,y'[0]==1}},y[x],x,IncludeSingularSolutions -> True
```

{}

4.20 problem 21

Internal problem ID [6840]
Internal file name [`OUTPUT/6087_Thursday_July_28_2022_04_29_58_AM_68133921/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{2y'' - \sin(2y) = 0}$$

With initial conditions

$$\left[y(0) = -\frac{\pi}{2}, y'(0) = 1\right]$$

4.20.1 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$2y'y'' - y'\sin(2y) = 0$$

Integrating the above w.r.t x gives

$$\int (2y'y'' - y'\sin(2y))\,dx = 0$$
$$y'^2 + \frac{\cos(2y)}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\sqrt{-2\cos(2y) + 4c_1}}{2} \tag{1}$$

$$y' = -\frac{\sqrt{-2\cos(2y) + 4c_1}}{2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{2}{\sqrt{-2\cos(2y) + 4c_1}} dy = \int dx$$

$$\frac{2\sqrt{-\frac{\cos(2y)-2c_1}{2c_1-1}}\ \text{InverseJacobiAM}\left(y, \sqrt{2}\,\sqrt{-\frac{1}{2c_1-1}}\right)}{\sqrt{-2\cos(2y) + 4c_1}} = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{2}{\sqrt{-2\cos(2y) + 4c_1}} dy = \int dx$$

$$-\frac{2\sqrt{-\frac{\cos(2y)-2c_1}{2c_1-1}}\ \text{InverseJacobiAM}\left(y, \sqrt{2}\,\sqrt{-\frac{1}{2c_1-1}}\right)}{\sqrt{-2\cos(2y) + 4c_1}} = x + c_3$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$\frac{2\sqrt{-\frac{\cos(2y)-2c_1}{2c_1-1}}\ \text{InverseJacobiAM}\left(y, \sqrt{2}\,\sqrt{-\frac{1}{2c_1-1}}\right)}{\sqrt{-2\cos(2y) + 4c_1}} = x + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -\frac{\pi}{2}$ and $x = 0$ in the above gives

$$-\frac{2\sqrt{\frac{1+2c_1}{2c_1-1}}\ \text{EllipticK}\left(\sqrt{2}\,\sqrt{-\frac{1}{2c_1-1}}\right)}{\sqrt{2+4c_1}} = c_2 \tag{1A}$$

Unable to solve for y to solve for constant of integration

Looking at the Second solution

$$-\frac{2\sqrt{-\frac{\cos(2y)-2c_1}{2c_1-1}}\ \text{InverseJacobiAM}\left(y, \sqrt{2}\sqrt{-\frac{1}{2c_1-1}}\right)}{\sqrt{-2\cos(2y)+4c_1}} = x + c_3 \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -\frac{\pi}{2}$ and $x = 0$ in the above gives

$$\frac{2\sqrt{\frac{1+2c_1}{2c_1-1}}\ \text{EllipticK}\left(\sqrt{2}\sqrt{-\frac{1}{2c_1-1}}\right)}{\sqrt{2+4c_1}} = c_3 \tag{1A}$$

Unable to solve for y to solve for constant of integration

Verification of solutions N/A

4.20.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$2p(y)\left(\frac{d}{dy}p(y)\right) = \sin(2y)$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned} p' &= F(y,p) \\ &= f(y)g(p) \\ &= \frac{\sin(2y)}{2p} \end{aligned}$$

Where $f(y) = \frac{\sin(2y)}{2}$ and $g(p) = \frac{1}{p}$. Integrating both sides gives

$$\frac{1}{\frac{1}{p}}\,dp = \frac{\sin(2y)}{2}\,dy$$

$$\int \frac{1}{\frac{1}{p}}\,dp = \int \frac{\sin(2y)}{2}\,dy$$

$$\frac{p^2}{2} = -\frac{\cos(2y)}{4} + c_1$$

The solution is

$$\frac{p(y)^2}{2} + \frac{\cos(2y)}{4} - c_1 = 0$$

Initial conditions are used to solve for c_1. Substituting $y = -\frac{\pi}{2}$ and $p = 1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{4} - c_1 = 0$$

$$c_1 = \frac{1}{4}$$

Substituting c_1 found above in the general solution gives

$$\frac{p^2}{2} + \frac{\cos(2y)}{4} - \frac{1}{4} = 0$$

Solving for $p(y)$ from the above gives

$$p(y) = \frac{\sqrt{2-2\cos(2y)}}{2}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{\sqrt{2-2\cos(2y)}}{2}$$

Integrating both sides gives

$$\int \frac{2}{\sqrt{2-2\cos(2y)}}\,dy = \int dx$$

$$-\frac{\sin(y)\operatorname{arctanh}(\cos(y))}{\sqrt{\frac{1}{2}-\frac{\cos(2y)}{2}}} = x + c_2$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = -\frac{\pi}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_2$$

$$c_2 = 0$$

Substituting c_2 found above in the general solution gives

$$-\frac{2 \sin (y) \operatorname{arctanh} (\cos (y))}{\sqrt{2-2 \cos (2 y)}} = x$$

The above simplifies to

$$-\sin (y) \operatorname{arctanh} (\cos (y)) - x \sqrt{\frac{1}{2}-\frac{\cos (2 y)}{2}} = 0$$

Simplifying the solution $\sin (y) (-\operatorname{arctanh} (\cos (y)) - \operatorname{csgn} (\sin (y)) x) = 0$ to $\sin (y) (-\operatorname{arctanh} (\cos (y)) -$ 0 Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$\sin (y) (-\operatorname{arctanh} (\cos (y)) - x) = 0 \tag{1}$$

Verification of solutions

$$\sin (y) (-\operatorname{arctanh} (\cos (y)) - x) = 0$$

Warning, solution could not be verified

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-(1/2)*sin(2*_a) = 0, _b(_a)`   *
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   <- Bernoulli successful
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 107.406 (sec). Leaf size: 1490

```
dsolve([2*diff(y(x),x$2)=sin(2*y(x)),y(0) = -1/2*Pi, D(y)(0) = 1],y(x), singsol=all)
```

Expression too large to display

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{2*y''[x]==Sin[2*y[x]],{y[0]==-Pi/2,y'[0]==1}},y[x],x,IncludeSingularSolutions -> Tru
```

{}

4.21 problem 23

Internal problem ID [6841]
Internal file name [`OUTPUT/6088_Thursday_July_28_2022_04_30_05_AM_46510715/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_missing_y", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{x^3y'' - x^2y' = -x^2 + 3}$$

4.21.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$x^3p'(x) - p(x)\, x^2 + x^2 - 3 = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{-x^2 + 3}{x^3}$$

Hence the ode is

$$p'(x) - \frac{p(x)}{x} = \frac{-x^2 + 3}{x^3}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x}dx}$$
$$= \frac{1}{x}$$

The ode becomes

$$\frac{d}{dx}(\mu p) = (\mu)\left(\frac{-x^2+3}{x^3}\right)$$
$$\frac{d}{dx}\left(\frac{p}{x}\right) = \left(\frac{1}{x}\right)\left(\frac{-x^2+3}{x^3}\right)$$
$$d\left(\frac{p}{x}\right) = \left(\frac{-x^2+3}{x^4}\right) dx$$

Integrating gives

$$\frac{p}{x} = \int \frac{-x^2+3}{x^4} dx$$
$$\frac{p}{x} = \frac{1}{x} - \frac{1}{x^3} + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$p(x) = x\left(\frac{1}{x} - \frac{1}{x^3}\right) + c_1 x$$

which simplifies to

$$p(x) = \frac{c_1 x^3 + x^2 - 1}{x^2}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{c_1 x^3 + x^2 - 1}{x^2}$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{c_1 x^3 + x^2 - 1}{x^2}\, \mathrm{d}x \\ &= \frac{c_1 x^2}{2} + x + \frac{1}{x} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^2}{2} + x + \frac{1}{x} + c_2 \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^2}{2} + x + \frac{1}{x} + c_2$$

Verified OK.

4.21.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}
A &= x^3 \\
B &= -x^2 \\
C &= 0 \\
F &= -x^2 + 3
\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}
AB'' + BB' + CB &= \left(x^3\right)(-2) + \left(-x^2\right)(-2x) + (0)\left(-x^2\right) \\
&= 0
\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-x^5 v'' + \left(-3x^4\right) v' = 0$$

Now by applying $v' = u$ the above becomes

$$-x^4 (u'(x)\, x + 3u(x)) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x) g(u) \\
&= -\frac{3u}{x}
\end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= -\frac{3}{x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{3}{x}\, dx \\
\ln(u) &= -3\ln(x) + c_1 \\
u &= \mathrm{e}^{-3\ln(x) + c_1} \\
&= \frac{c_1}{x^3}
\end{aligned}$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= \frac{c_1}{x^3}
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int \frac{c_1}{x^3}\, \mathrm{d}x \\
&= -\frac{c_1}{2x^2} + c_2
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= Bv \\
&= \left(-x^2\right)\left(-\frac{c_1}{2x^2} + c_2\right) \\
&= -c_2 x^2 + \frac{c_1}{2}
\end{aligned}
$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \frac{1}{2} \\
y_2 &= x^2
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{2} & x^2 \\ \frac{d}{dx}\left(\frac{1}{2}\right) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{2} & x^2 \\ 0 & 2x \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{2}\right)(2x) - \left(x^2\right)(0)$$

Which simplifies to

$$W = x$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^2(-x^2+3)}{x^4}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{-x^2+3}{x^2}dx$$

Hence

$$u_1 = x + \frac{3}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{-\frac{x^2}{2}+\frac{3}{2}}{x^4}\,dx$$

Which simplifies to

$$u_2 = \int \frac{-x^2+3}{2x^4} dx$$

Hence

$$u_2 = \frac{1}{2x} - \frac{1}{2x^3}$$

Which simplifies to

$$u_1 = x + \frac{3}{x}$$
$$u_2 = \frac{x^2-1}{2x^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x}{2} + \frac{3}{2x} + \frac{x^2-1}{2x}$$

Which simplifies to

$$y_p(x) = \frac{x^2+1}{x}$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= \left(-c_2x^2 + \frac{c_1}{2}\right) + \left(\frac{x^2+1}{x}\right) \\ &= -c_2x^2 + \frac{c_1}{2} + \frac{x^2+1}{x} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -c_2x^2 + \frac{c_1}{2} + \frac{x^2+1}{x} \tag{1}$$

Verification of solutions

$$y = -c_2x^2 + \frac{c_1}{2} + \frac{x^2+1}{x}$$

Verified OK.

4.21.3 Solving using Kovacic algorithm

Writing the ode as

$$x^3y'' - x^2y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^3 \\ B &= -x^2 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 34: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{3}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2x} + (-)\,(0) \\
&= -\frac{1}{2x} \\
&= -\frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int -\frac{1}{2x} dx} \\
&= \frac{1}{\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x^2}{x^3}\,dx} \\ &= z_1 e^{\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{x^3}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{x^2}{2}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1(1) + c_2\left(1\left(\frac{x^2}{2}\right)\right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^3y'' - x^2y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 + \frac{c_2 x^2}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = 1$$
$$y_2 = \frac{x^2}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \frac{x^2}{2} \\ \frac{d}{dx}(1) & \frac{d}{dx}\left(\frac{x^2}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \frac{x^2}{2} \\ 0 & x \end{vmatrix}$$

Therefore

$$W = (1)(x) - \left(\frac{x^2}{2}\right)(0)$$

Which simplifies to

$$W = x$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^2(-x^2+3)}{2}}{x^4}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{-x^2+3}{2x^2}dx$$

Hence

$$u_1 = \frac{x}{2} + \frac{3}{2x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{-x^2+3}{x^4}\,dx$$

Which simplifies to

$$u_2 = \int \frac{-x^2+3}{x^4}dx$$

Hence

$$u_2 = \frac{1}{x} - \frac{1}{x^3}$$

Which simplifies to

$$u_1 = \frac{x}{2} + \frac{3}{2x}$$
$$u_2 = \frac{x^2-1}{x^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x}{2} + \frac{3}{2x} + \frac{x^2 - 1}{2x}$$

Which simplifies to

$$y_p(x) = \frac{x^2 + 1}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + \frac{c_2 x^2}{2}\right) + \left(\frac{x^2 + 1}{x}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 + \frac{c_2 x^2}{2} + \frac{x^2 + 1}{x} \tag{1}$$

Verification of solutions

$$y = c_1 + \frac{c_2 x^2}{2} + \frac{x^2 + 1}{x}$$

Verified OK.

4.21.4 Maple step by step solution

Let's solve

$x^3 y'' - x^2 y' = -x^2 + 3$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $x^3 u'(x) - x^2 u(x) = -x^2 + 3$

- Isolate the derivative

  $u'(x) = \frac{u(x)}{x} - \frac{x^2 - 3}{x^3}$

- Group terms with $u(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

$u'(x) - \frac{u(x)}{x} = -\frac{x^2-3}{x^3}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x) \left(u'(x) - \frac{u(x)}{x}\right) = -\frac{\mu(x)(x^2-3)}{x^3}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, u(x))$

  $\mu(x) \left(u'(x) - \frac{u(x)}{x}\right) = \mu'(x)\, u(x) + \mu(x)\, u'(x)$
- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x}$
- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, u(x))\right) dx = \int -\frac{\mu(x)(x^2-3)}{x^3} dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\, u(x) = \int -\frac{\mu(x)(x^2-3)}{x^3} dx + c_1$
- Solve for $u(x)$

  $u(x) = \frac{\int -\frac{\mu(x)\left(x^2-3\right)}{x^3} dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \frac{1}{x}$

  $u(x) = x\left(\int -\frac{x^2-3}{x^4} dx + c_1\right)$
- Evaluate the integrals on the rhs

  $u(x) = x\left(\frac{1}{x} - \frac{1}{x^3} + c_1\right)$
- Solve 1st ODE for $u(x)$

  $u(x) = x\left(\frac{1}{x} - \frac{1}{x^3} + c_1\right)$
- Make substitution $u = y'$

  $y' = x\left(\frac{1}{x} - \frac{1}{x^3} + c_1\right)$
- Integrate both sides to solve for y

  $\int y' dx = \int x\left(\frac{1}{x} - \frac{1}{x^3} + c_1\right) dx + c_2$
- Compute integrals

  $y = \frac{c_1 x^2}{2} + x + \frac{1}{x} + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = (_a^2*_b(_a)-_a^2+3)/_a^3, _b(_a)`   **
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(x^3*diff(y(x),x$2)-x^2*diff(y(x),x)=3-x^2,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^2}{2} + \frac{1}{x} + x + c_2$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 21

```
DSolve[x^3*y''[x]-x^2*y'[x]==3-x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1 x^2}{2} + x + \frac{1}{x} + c_2$$

4.22 problem 24

Internal problem ID [6842]
Internal file name [`OUTPUT/6089_Thursday_July_28_2022_04_30_08_AM_81941235/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], _Liouville, [_2nd_order, _reducible,
    _mu_xy]]
```

$$\boxed{y'' - y'^2 = 0}$$

4.22.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - p(x)^2 = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{p^2} dp = x + c_1$$
$$-\frac{1}{p} = x + c_1$$

Solving for p gives these solutions

$$p_1 = -\frac{1}{x + c_1}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -\frac{1}{x + c_1}$$

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{1}{x + c_1} \,\mathrm{d}x \\ &= -\ln(x + c_1) + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -\ln(x + c_1) + c_2 \tag{1}$$

Verification of solutions

$$y = -\ln(x + c_1) + c_2$$

Verified OK.

4.22.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y) \left(\frac{d}{dy}p(y)\right) - p(y)^2 = 0$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$\begin{aligned}
\int \frac{1}{p}dp &= y + c_1 \\
\ln(p) &= y + c_1 \\
p &= e^{y+c_1} \\
p &= c_1 e^y
\end{aligned}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = c_1 e^y$$

Integrating both sides gives

$$\begin{aligned}
\int \frac{e^{-y}}{c_1}dy &= x + c_2 \\
-\frac{e^{-y}}{c_1} &= x + c_2
\end{aligned}$$

Solving for y gives these solutions

$$y_1 = \ln\left(-\frac{1}{c_1(x+c_2)}\right)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \ln\left(-\frac{1}{c_1(x+c_2)}\right) \tag{1}$$

<u>Verification of solutions</u>

$$y = \ln\left(-\frac{1}{c_1(x+c_2)}\right)$$

Verified OK.

4.22.3 Maple step by step solution

Let's solve

$y'' - y'^2 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x) - u(x)^2 = 0$

- Separate variables

  $\frac{u'(x)}{u(x)^2} = 1$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{u(x)^2}dx = \int 1dx + c_1$

- Evaluate integral

  $-\frac{1}{u(x)} = x + c_1$

- Solve for $u(x)$

  $u(x) = -\frac{1}{x+c_1}$

- Solve 1st ODE for $u(x)$

  $u(x) = -\frac{1}{x+c_1}$

- Make substitution $u = y'$

  $y' = -\frac{1}{x+c_1}$

- Integrate both sides to solve for y

  $\int y'dx = \int -\frac{1}{x+c_1}dx + c_2$

- Compute integrals

  $y = -\ln(x + c_1) + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)=diff(y(x),x)^2,y(x), singsol=all)
```

$$y(x) = -\ln\left(-c_1 x - c_2\right)$$

✓ Solution by Mathematica

Time used: 0.197 (sec). Leaf size: 15

```
DSolve[y''[x]==(y'[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow c_2 - \log(x + c_1)$$

4.23 problem 25

Internal problem ID [6843]
Internal file name [`OUTPUT/6090_Thursday_July_28_2022_04_30_10_AM_70078846/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' - e^{x} {y'}^{2} = 0}$$

4.23.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - e^{x} p(x)^{2} = 0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned} p' &= F(x, p) \\ &= f(x) g(p) \\ &= e^{x} p^{2} \end{aligned}$$

Where $f(x) = e^x$ and $g(p) = p^2$. Integrating both sides gives

$$\frac{1}{p^2}\, dp = e^x\, dx$$

$$\int \frac{1}{p^2}\, dp = \int e^x\, dx$$

$$-\frac{1}{p} = e^x + c_1$$

The solution is

$$-\frac{1}{p\left(x\right)} - e^x - c_1 = 0$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$-\frac{1}{y'} - e^x - c_1 = 0$$

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{1}{e^x + c_1}\, \mathrm{d}x \\ &= -\frac{\ln\left(e^x\right)}{c_1} + \frac{\ln\left(e^x + c_1\right)}{c_1} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{\ln\left(e^x\right)}{c_1} + \frac{\ln\left(e^x + c_1\right)}{c_1} + c_2 \tag{1}$$

Verification of solutions

$$y = -\frac{\ln\left(e^x\right)}{c_1} + \frac{\ln\left(e^x + c_1\right)}{c_1} + c_2$$

Verified OK.

4.23.2 Maple step by step solution

Let's solve

$y'' - e^x y'^2 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x) - e^x u(x)^2 = 0$

- Separate variables

  $\frac{u'(x)}{u(x)^2} = e^x$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{u(x)^2} dx = \int e^x dx + c_1$

- Evaluate integral

  $-\frac{1}{u(x)} = e^x + c_1$

- Solve for $u(x)$

  $u(x) = -\frac{1}{e^x + c_1}$

- Solve 1st ODE for $u(x)$

  $u(x) = -\frac{1}{e^x + c_1}$

- Make substitution $u = y'$

  $y' = -\frac{1}{e^x + c_1}$

- Integrate both sides to solve for y

  $\int y' dx = \int -\frac{1}{e^x + c_1} dx + c_2$

- Compute integrals

  $y = -\frac{\ln(e^x)}{c_1} + \frac{\ln(e^x + c_1)}{c_1} + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = exp(_a)*_b(_a)^2, _b(_a), HINT = [[1, -
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[1, -_b]
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)=exp(x)*diff(y(x),x)^2,y(x), singsol=all)
```

$$y(x) = \frac{c_2 c_1 - \ln\left(\mathrm{e}^x - c_1\right) + \ln\left(\mathrm{e}^x\right)}{c_1}$$

✓ Solution by Mathematica

Time used: 0.985 (sec). Leaf size: 37

```
DSolve[y''[x]==Exp[x](y'[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-x + \log\left(e^x + c_1\right) + c_1 c_2}{c_1}$$
$$y(x) \to \text{Indeterminate}$$
$$y(x) \to c_2$$

4.24 problem 26

Internal problem ID [6844]
Internal file name [`OUTPUT/6091_Thursday_July_28_2022_04_30_12_AM_80846733/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y], [_2nd_order, _reducible, _mu_y_y1]]
```

$$\boxed{2y'' - y'^3 \sin(2x) = 0}$$

4.24.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$2p'(x) - p(x)^3 \sin(2x) = 0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned} p' &= F(x,p) \\ &= f(x)g(p) \\ &= \frac{p^3 \sin(2x)}{2} \end{aligned}$$

Where $f(x) = \frac{\sin(2x)}{2}$ and $g(p) = p^3$. Integrating both sides gives

$$\frac{1}{p^3}\, dp = \frac{\sin(2x)}{2}\, dx$$

$$\int \frac{1}{p^3}\, dp = \int \frac{\sin(2x)}{2}\, dx$$

$$-\frac{1}{2p^2} = -\frac{\cos(2x)}{4} + c_1$$

The solution is

$$-\frac{1}{2p(x)^2} + \frac{\cos(2x)}{4} - c_1 = 0$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$-\frac{1}{2y'^2} + \frac{\cos(2x)}{4} - c_1 = 0$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = -\frac{2}{\sqrt{-8c_1 + 2\cos(2x)}} \tag{1}$$

$$y' = \frac{2}{\sqrt{-8c_1 + 2\cos(2x)}} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{2}{\sqrt{-8c_1 + 2\cos(2x)}}\, \mathrm{d}x \\ &= -\frac{2\sqrt{-\frac{-4c_1+\cos(2x)}{4c_1-1}}\, \mathrm{InverseJacobiAM}\left(x, \sqrt{2}\sqrt{-\frac{1}{4c_1-1}}\right)}{\sqrt{-8c_1 + 2\cos(2x)}} + c_2 \end{aligned}$$

Solving equation (2)

Integrating both sides gives

$$
\begin{aligned}
y &= \int \frac{2}{\sqrt{-8c_1 + 2\cos(2x)}}\, \mathrm{d}x \\
&= \frac{2\sqrt{-\frac{-4c_1+\cos(2x)}{4c_1-1}}\ \text{InverseJacobiAM}\left(x, \sqrt{2}\,\sqrt{-\frac{1}{4c_1-1}}\right)}{\sqrt{-8c_1 + 2\cos(2x)}} + c_3
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{2\sqrt{-\frac{-4c_1+\cos(2x)}{4c_1-1}}\ \text{InverseJacobiAM}\left(x, \sqrt{2}\,\sqrt{-\frac{1}{4c_1-1}}\right)}{\sqrt{-8c_1 + 2\cos(2x)}} + c_2 \tag{1}
$$

$$
y = \frac{2\sqrt{-\frac{-4c_1+\cos(2x)}{4c_1-1}}\ \text{InverseJacobiAM}\left(x, \sqrt{2}\,\sqrt{-\frac{1}{4c_1-1}}\right)}{\sqrt{-8c_1 + 2\cos(2x)}} + c_3 \tag{2}
$$

Verification of solutions

$$
y = -\frac{2\sqrt{-\frac{-4c_1+\cos(2x)}{4c_1-1}}\ \text{InverseJacobiAM}\left(x, \sqrt{2}\,\sqrt{-\frac{1}{4c_1-1}}\right)}{\sqrt{-8c_1 + 2\cos(2x)}} + c_2
$$

Verified OK.

$$
y = \frac{2\sqrt{-\frac{-4c_1+\cos(2x)}{4c_1-1}}\ \text{InverseJacobiAM}\left(x, \sqrt{2}\,\sqrt{-\frac{1}{4c_1-1}}\right)}{\sqrt{-8c_1 + 2\cos(2x)}} + c_3
$$

Verified OK.

4.24.2 Maple step by step solution

Let's solve

$2y'' - y'^3 \sin(2x) = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $2u'(x) - u(x)^3 \sin(2x) = 0$

- Separate variables

  $\frac{u'(x)}{u(x)^3} = \frac{\sin(2x)}{2}$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{u(x)^3}dx = \int \frac{\sin(2x)}{2}dx + c_1$

- Evaluate integral

  $-\frac{1}{2u(x)^2} = -\frac{\cos(2x)}{4} + c_1$

- Solve for $u(x)$

  $\left\{u(x) = -\frac{2}{\sqrt{-8c_1+2\cos(2x)}}, u(x) = \frac{2}{\sqrt{-8c_1+2\cos(2x)}}\right\}$

- Solve 1st ODE for $u(x)$

  $u(x) = -\frac{2}{\sqrt{-8c_1+2\cos(2x)}}$

- Make substitution $u = y'$

  $y' = -\frac{2}{\sqrt{-8c_1+2\cos(2x)}}$

- Integrate both sides to solve for y

  $\int y'dx = \int -\frac{2}{\sqrt{-8c_1+2\cos(2x)}}dx + c_2$

- Compute integrals

  $y = -\frac{2\sqrt{-\frac{-4c_1+\cos(2x)}{4c_1-1}}\,\mathrm{InverseJacobiAM}\left(x, \sqrt{2}\,\sqrt{-\frac{1}{4c_1-1}}\right)}{\sqrt{-8c_1+2\cos(2x)}} + c_2$

- Solve 2nd ODE for $u(x)$

  $u(x) = \frac{2}{\sqrt{-8c_1+2\cos(2x)}}$

- Make substitution $u = y'$

  $y' = \frac{2}{\sqrt{-8c_1+2\cos(2x)}}$

- Integrate both sides to solve for y

  $\int y'dx = \int \frac{2}{\sqrt{-8c_1+2\cos(2x)}}dx + c_2$

- Compute integrals

  $y = \frac{2\sqrt{-\frac{-4c_1+\cos(2x)}{4c_1-1}}\,\mathrm{InverseJacobiAM}\left(x, \sqrt{2}\,\sqrt{-\frac{1}{4c_1-1}}\right)}{\sqrt{-8c_1+2\cos(2x)}} + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = (1/2)*_b(_a)^3*sin(2*_a), _b(_a)`   ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   <- Bernoulli successful
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 80

```
dsolve(2*diff(y(x),x$2)=diff(y(x),x)^3*sin(2*x),y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{-\sin(x)^2 c_1^2+1}\ \mathrm{InverseJacobiAM}(x, c_1)}{\sqrt{\frac{-\sin(x)^2 c_1^2+1}{c_1^2}}} + c_2$$

$$y(x) = -\frac{\sqrt{-\sin(x)^2 c_1^2+1}\ \mathrm{InverseJacobiAM}(x, c_1)}{\sqrt{\frac{-\sin(x)^2 c_1^2+1}{c_1^2}}} + c_2$$

✓ Solution by Mathematica

Time used: 6.102 (sec). Leaf size: 120

```
DSolve[2*y''[x]==(y'[x])^3*Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - \frac{\sqrt{-\frac{\cos(2x)+1-4c_1}{-1+2c_1}}\,\text{EllipticF}\left(x, \frac{1}{1-2c_1}\right)}{\sqrt{\cos(2x)+1-4c_1}}$$
$$y(x) \to \frac{\sqrt{-\frac{\cos(2x)+1-4c_1}{-1+2c_1}}\,\text{EllipticF}\left(x, \frac{1}{1-2c_1}\right)}{\sqrt{\cos(2x)+1-4c_1}} + c_2$$
$$y(x) \to c_2$$

4.25 problem 27

Internal problem ID [6845]
Internal file name [OUTPUT/6092_Thursday_July_28_2022_04_30_14_AM_58312407/index.tex]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y], [_2nd_order, _reducible, _mu_y_y1]]
```

$$\boxed{x^2y'' + y'^2 = 0}$$

4.25.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$x^2p'(x) + p(x)^2 = 0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned} p' &= F(x,p) \\ &= f(x)g(p) \\ &= -\frac{p^2}{x^2} \end{aligned}$$

Where $f(x) = -\frac{1}{x^2}$ and $g(p) = p^2$. Integrating both sides gives

$$\begin{aligned}\frac{1}{p^2}\, dp &= -\frac{1}{x^2}\, dx\\ \int \frac{1}{p^2}\, dp &= \int -\frac{1}{x^2}\, dx\\ -\frac{1}{p} &= \frac{1}{x} + c_1\end{aligned}$$

The solution is

$$-\frac{1}{p(x)} - \frac{1}{x} - c_1 = 0$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$-\frac{1}{y'} - \frac{1}{x} - c_1 = 0$$

Integrating both sides gives

$$\begin{aligned}y &= \int -\frac{x}{c_1 x + 1}\, \mathrm{d}x\\ &= -\frac{x}{c_1} + \frac{\ln(c_1 x + 1)}{c_1^2} + c_2\end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{c_1} + \frac{\ln(c_1 x + 1)}{c_1^2} + c_2 \tag{1}$$

Verification of solutions

$$y = -\frac{x}{c_1} + \frac{\ln(c_1 x + 1)}{c_1^2} + c_2$$

Verified OK.

4.25.2 Maple step by step solution

Let's solve

$x^2y'' + y'^2 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $x^2u'(x) + u(x)^2 = 0$

- Separate variables

  $\frac{u'(x)}{u(x)^2} = -\frac{1}{x^2}$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{u(x)^2}dx = \int -\frac{1}{x^2}dx + c_1$

- Evaluate integral

  $-\frac{1}{u(x)} = \frac{1}{x} + c_1$

- Solve for $u(x)$

  $u(x) = -\frac{x}{c_1x+1}$

- Solve 1st ODE for $u(x)$

  $u(x) = -\frac{x}{c_1x+1}$

- Make substitution $u = y'$

  $y' = -\frac{x}{c_1x+1}$

- Integrate both sides to solve for y

  $\int y'dx = \int -\frac{x}{c_1x+1}dx + c_2$

- Compute integrals

  $y = -\frac{x}{c_1} + \frac{\ln(c_1x+1)}{c_1^2} + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -_b(_a)^2/_a^2, _b(_a), HINT = [[_a, _b
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, _b]
```

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 21

```
dsolve(x^2*diff(y(x),x$2)+diff(y(x),x)^2=0,y(x), singsol=all)
```

$$y(x) = \frac{x}{c_1} + \frac{\ln(c_1 x - 1)}{c_1^2} + c_2$$

✓ Solution by Mathematica

Time used: 0.57 (sec). Leaf size: 47

```
DSolve[x^2*y''[x]+(y'[x])^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x}{c_1} + \frac{\log(1 + c_1 x)}{c_1{}^2} + c_2$$
$$y(x) \to c_2$$
$$y(x) \to -\frac{x^2}{2} + c_2$$

4.26 problem 28

Internal problem ID [6846]
Internal file name [`OUTPUT/6093_Thursday_July_28_2022_04_30_16_AM_7866207/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second__order__ode__missing__x", "second__order__ode__missing__y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{y'' - {y'}^2 = 1}$$

4.26.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - 1 - p(x)^2 = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{p^2+1} dp = x + c_1$$
$$\arctan(p) = x + c_1$$

Solving for p gives these solutions

$$p_1 = \tan(x + c_1)$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \tan(x + c_1)$$

Integrating both sides gives

$$\begin{aligned} y &= \int \tan(x + c_1)\, \mathrm{d}x \\ &= \frac{\ln\left(1 + \tan(x + c_1)^2\right)}{2} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\ln\left(1 + \tan(x + c_1)^2\right)}{2} + c_2 \tag{1}$$

Verification of solutions

$$y = \frac{\ln\left(1 + \tan(x + c_1)^2\right)}{2} + c_2$$

Verified OK.

4.26.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right) - p(y)^2 = 1$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$\int \frac{p}{p^2+1}dp = \int dy$$
$$\frac{\ln(p^2+1)}{2} = y + c_1$$

Raising both side to exponential gives

$$\sqrt{p^2+1} = e^{y+c_1}$$

Which simplifies to

$$\sqrt{p^2+1} = c_2 e^{y}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \text{RootOf}\left(\_Z^2 - c_2^2 e^{2y} + 1\right)$$

Integrating both sides gives

$$\int \frac{1}{\text{RootOf}\left(\_Z^2 - c_2^2 e^{2y} + 1\right)}dy = \int dx$$
$$\int^y \frac{1}{\text{RootOf}\left(\_Z^2 - c_2^2 e^{2\_a} + 1\right)}d\_a = x + c_3$$

Summary
The solution(s) found are the following

$$\int^y \frac{1}{\text{RootOf}\left(\_Z^2 - c_2^2 e^{2\_a} + 1\right)}d\_a = x + c_3 \tag{1}$$

Verification of solutions

$$\int^y \frac{1}{\text{RootOf}\left(\_Z^2 - c_2^2 e^{2\_a} + 1\right)}d\_a = x + c_3$$

Verified OK.

4.26.3 Maple step by step solution

Let's solve

$y'' - y'^2 = 1$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x) - u(x)^2 = 1$

- Separate variables

  $\frac{u'(x)}{u(x)^2+1} = 1$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{u(x)^2+1} dx = \int 1dx + c_1$

- Evaluate integral

  $\arctan(u(x)) = x + c_1$

- Solve for $u(x)$

  $u(x) = \tan(x + c_1)$

- Solve 1st ODE for $u(x)$

  $u(x) = \tan(x + c_1)$

- Make substitution $u = y'$

  $y' = \tan(x + c_1)$

- Integrate both sides to solve for y

  $\int y' dx = \int \tan(x + c_1) dx + c_2$

- Compute integrals

  $y = \frac{\ln(1+\tan(x+c_1)^2)}{2} + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)=1+diff(y(x),x)^2,y(x), singsol=all)
```

$$y(x) = -\ln\left(-\cos\left(x\right) c_2 + c_1 \sin\left(x\right)\right)$$

✓ Solution by Mathematica

Time used: 1.97 (sec). Leaf size: 16

```
DSolve[y''[x]==1+(y'[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - \log(\cos(x + c_1))$$

4.27 problem 30

Internal problem ID [6847]
Internal file name [`OUTPUT/6094_Thursday_July_28_2022_04_30_18_AM_11524310/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 30.
ODE order: 2.
ODE degree: 2.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - \left(1 + {y'}^2\right)^{\frac{3}{2}} = 0}$$

4.27.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - \left(1 + p(x)^2\right)^{\frac{3}{2}} = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{(p^2+1)^{\frac{3}{2}}} dp = \int dx$$
$$\frac{p(x)}{\sqrt{1+p(x)^2}} = x + c_1$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\frac{y'}{\sqrt{1+y'^2}} = x + c_1$$

Integrating both sides gives

$$y = \int c_1 \sqrt{-\frac{1}{c_1^2 + 2c_1 x + x^2 - 1}} + x\sqrt{-\frac{1}{c_1^2 + 2c_1 x + x^2 - 1}} \, \mathrm{d}x$$
$$= \sqrt{-\frac{1}{c_1^2 + 2c_1 x + x^2 - 1}} \, (c_1 + x + 1)\,(c_1 + x - 1) + c_2$$

Summary

The solution(s) found are the following

$$y = \sqrt{-\frac{1}{c_1^2 + 2c_1 x + x^2 - 1}} \, (c_1 + x + 1)\,(c_1 + x - 1) + c_2 \tag{1}$$

Verification of solutions

$$y = \sqrt{-\frac{1}{c_1^2 + 2c_1 x + x^2 - 1}} \, (c_1 + x + 1)\,(c_1 + x - 1) + c_2$$

Verified OK.

4.27.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$
\begin{aligned}
y'' &= \frac{dp}{dx} \\
&= \frac{dy}{dx}\frac{dp}{dy} \\
&= p\frac{dp}{dy}
\end{aligned}
$$

Hence the ode becomes

$$
p(y)\left(\frac{d}{dy}p(y)\right) = \left(1+p(y)^2\right)^{\frac{3}{2}}
$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$
\begin{aligned}
\int \frac{p}{(p^2+1)^{\frac{3}{2}}}dp &= \int dy \\
-\frac{1}{\sqrt{1+p(y)^2}} &= y + c_1
\end{aligned}
$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
-\frac{1}{\sqrt{1+y'^2}} = y + c_1
$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$
y' = \frac{\sqrt{-y^2-2c_1y-c_1^2+1}}{y+c_1} \tag{1}
$$

$$
y' = -\frac{\sqrt{-y^2-2c_1y-c_1^2+1}}{y+c_1} \tag{2}
$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$
\begin{aligned}
\int \frac{y+c_1}{\sqrt{-c_1^2-2c_1y-y^2+1}}dy &= \int dx \\
\frac{(y+c_1+1)(y+c_1-1)}{\sqrt{-y^2-2c_1y-c_1^2+1}} &= x + c_2
\end{aligned}
$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{y+c_1}{\sqrt{-c_1^2-2c_1y-y^2+1}}dy = \int dx$$
$$-\frac{(y+c_1+1)(y+c_1-1)}{\sqrt{-y^2-2c_1y-c_1^2+1}} = x+c_3$$

Summary

The solution(s) found are the following

$$\frac{(y+c_1+1)(y+c_1-1)}{\sqrt{-y^2-2c_1y-c_1^2+1}} = x+c_2 \quad (1)$$
$$-\frac{(y+c_1+1)(y+c_1-1)}{\sqrt{-y^2-2c_1y-c_1^2+1}} = x+c_3 \quad (2)$$

Verification of solutions

$$\frac{(y+c_1+1)(y+c_1-1)}{\sqrt{-y^2-2c_1y-c_1^2+1}} = x+c_2$$

Verified OK.

$$-\frac{(y+c_1+1)(y+c_1-1)}{\sqrt{-y^2-2c_1y-c_1^2+1}} = x+c_3$$

Verified OK.

4.27.3 Maple step by step solution

Let's solve

$y'' = \left(1+y'^2\right)^{\frac{3}{2}}$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x) = \left(u(x)^2+1\right)^{\frac{3}{2}}$

- Separate variables

$$\frac{u'(x)}{\left(u(x)^2+1\right)^{\frac{3}{2}}} = 1$$

- Integrate both sides with respect to x

$$\int \frac{u'(x)}{\left(u(x)^2+1\right)^{\frac{3}{2}}} dx = \int 1 dx + c_1$$

- Evaluate integral

$$\frac{u(x)}{\sqrt{u(x)^2+1}} = x + c_1$$

- Solve for $u(x)$

$$u(x) = c_1\sqrt{-\frac{1}{c_1^2+2c_1x+x^2-1}} + x\sqrt{-\frac{1}{c_1^2+2c_1x+x^2-1}}$$

- Solve 1st ODE for $u(x)$

$$u(x) = c_1\sqrt{-\frac{1}{c_1^2+2c_1x+x^2-1}} + x\sqrt{-\frac{1}{c_1^2+2c_1x+x^2-1}}$$

- Make substitution $u = y'$

$$y' = c_1\sqrt{-\frac{1}{c_1^2+2c_1x+x^2-1}} + x\sqrt{-\frac{1}{c_1^2+2c_1x+x^2-1}}$$

- Integrate both sides to solve for y

$$\int y' dx = \int \left(c_1\sqrt{-\frac{1}{c_1^2+2c_1x+x^2-1}} + x\sqrt{-\frac{1}{c_1^2+2c_1x+x^2-1}}\right) dx + c_2$$

- Compute integrals

$$y = \sqrt{-\frac{1}{c_1^2+2c_1x+x^2-1}}\,(c_1+x+1)\,(c_1+x-1) + c_2$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = (1+_b(_a)^2)^(3/2), _b(_a), HINT = [[1,
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[1, 0], [y, -_b^2-1]
```

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 49

```
dsolve(diff(y(x),x$2)=(1+diff(y(x),x)^2)^(3/2),y(x), singsol=all)
```

$$y(x) = -ix + c_1$$
$$y(x) = ix + c_1$$
$$y(x) = (c_1 + x + 1)(x - 1 + c_1)\sqrt{-\frac{1}{(c_1 + x + 1)(x - 1 + c_1)}} + c_2$$

✓ Solution by Mathematica

Time used: 0.269 (sec). Leaf size: 59

```
DSolve[y''[x]==(1+(y'[x])^2)^(3/2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - i\sqrt{x^2 + 2c_1x - 1 + c_1^2}$$
$$y(x) \to i\sqrt{x^2 + 2c_1x - 1 + c_1^2} + c_2$$

4.28 problem 31

Internal problem ID [6848]
Internal file name [`OUTPUT/6095_Thursday_July_28_2022_04_30_21_AM_39831181/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 31.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_y_y1]]
```

$$\boxed{yy'' - y'^2(1 - y'\sin(y) - yy'\cos(y)) = 0}$$

4.28.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y)\left(\frac{d}{dy}p(y)\right) + \left(\cos(y)\,yp(y)^2 + \sin(y)\,p(y)^2 - p(y)\right)p(y) = 0$$

Which is now solved as first order ode for $p(y)$. Using the change of variables $p(y) = u(y)\, y$ on the above ode results in new ode in $u(y)$

$$y^2 u(y) \left(\left(\frac{d}{dy}u(y)\right) y + u(y)\right) + \left(\cos(y)\, y^3 u(y)^2 + \sin(y)\, u(y)^2\, y^2 - u(y)\, y\right) u(y)\, y = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(y, u) \\ &= f(y)g(u) \\ &= u^2(-\cos(y)\, y - \sin(y)) \end{aligned}$$

Where $f(y) = -\cos(y)\, y - \sin(y)$ and $g(u) = u^2$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u^2}\, du &= -\cos(y)\, y - \sin(y)\ dy \\ \int \frac{1}{u^2}\, du &= \int -\cos(y)\, y - \sin(y)\ dy \\ -\frac{1}{u} &= -\sin(y)\, y + c_2 \end{aligned}$$

The solution is

$$-\frac{1}{u(y)} + \sin(y)\, y - c_2 = 0$$

Replacing $u(y)$ in the above solution by $\frac{p(y)}{y}$ results in the solution for $p(y)$ in implicit form

$$\begin{aligned} -\frac{y}{p(y)} + \sin(y)\, y - c_2 &= 0 \\ -\frac{y}{p(y)} + \sin(y)\, y - c_2 &= 0 \end{aligned}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$-\frac{y}{y'} + y\sin(y) - c_2 = 0$$

Integrating both sides gives

$$\begin{aligned} \int \frac{\sin(y)\, y - c_2}{y} dy &= \int dx \\ \int^y \frac{\sin(\_a)\_a - c_2}{\_a} d\_a &= x + c_3 \end{aligned}$$

Summary

The solution(s) found are the following

$$\int^{y} \frac{\sin(\_a)\_a - c_2}{\_a} d\_a = x + c_3 \tag{1}$$

Verification of solutions

$$\int^{y} \frac{\sin(\_a)\_a - c_2}{\_a} d\_a = x + c_3$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+_b(_a)^2*(_b(_a)*cos(_a)*_a+_b(_
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   <- Bernoulli successful
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 0.14 (sec). Leaf size: 24

```
dsolve(y(x)*diff(y(x),x$2)=diff(y(x),x)^2*(1-diff(y(x),x)*sin(y(x))-y(x)*diff(y(x),x)*cos(y(
```

$$y(x) = c_1$$
$$-\cos(y(x)) + c_1 \ln(y(x)) - x - c_2 = 0$$

✓ Solution by Mathematica

Time used: 0.489 (sec). Leaf size: 69

```
DSolve[y[x]*y''[x]==(y'[x])^2*(1-y'[x]*Sin[y[x]]-y[x]*y'[x]*Cos[y[x]] ),y[x],x,IncludeSingul
```

$$y(x) \to \text{InverseFunction}[-\cos(\#1) + c_1 \log(\#1)\&][x + c_2]$$
$$y(x) \to \text{InverseFunction}[-\cos(\#1) - c_1 \log(\#1)\&][x + c_2]$$
$$y(x) \to \text{InverseFunction}[-\cos(\#1) + c_1 \log(\#1)\&][x + c_2]$$

4.29 problem 32

Internal problem ID [6849]
Internal file name [`OUTPUT/6096_Thursday_July_28_2022_04_30_22_AM_21244975/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 32.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{(1+y^2)\, y''+y'^3+y'=0}$$

4.29.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$(y^2+1)\, p(y) \left(\frac{d}{dy}p(y)\right) + (1+p(y)^2)\, p(y) = 0$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned} p' &= F(y,p) \\ &= f(y)g(p) \\ &= \frac{-p^2-1}{y^2+1} \end{aligned}$$

Where $f(y) = \frac{1}{y^2+1}$ and $g(p) = -p^2 - 1$. Integrating both sides gives

$$\begin{aligned} \frac{1}{-p^2-1}\,dp &= \frac{1}{y^2+1}\,dy \\ \int \frac{1}{-p^2-1}\,dp &= \int \frac{1}{y^2+1}\,dy \\ -\arctan(p) &= \arctan(y) + c_1 \end{aligned}$$

The solution is

$$-\arctan(p(y)) - \arctan(y) - c_1 = 0$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$-\arctan(y') - \arctan(y) - c_1 = 0$$

Integrating both sides gives

$$\begin{aligned} \int -\frac{1}{\tan(\arctan(y) + c_1)}dy &= \int dx \\ \int^y -\frac{1}{\tan(\arctan(\_a) + c_1)}d\_a &= x + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$\int^y -\frac{1}{\tan(\arctan(\_a) + c_1)}d\_a = x + c_2 \tag{1}$$

Verification of solutions

$$\int^y -\frac{1}{\tan(\arctan(\_a) + c_1)}d\_a = x + c_2$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+_b(_a)*(1+_b(_a)^2)/(_a^2+1) = 0
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   trying separable
   <- separable successful
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 0.187 (sec). Leaf size: 118

```
dsolve((1+y(x)^2)*diff(y(x),x$2)+diff(y(x),x)^3+diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -i$$
$$y(x) = i$$
$$y(x) = c_1$$
$$y(x) = \frac{i c_1 - i - \mathrm{e}^{\frac{-4\,\mathrm{LambertW}\left(-\frac{i\,\mathrm{e}^{\frac{(-c_2-x+1)c_1^2+(-2c_2-2x-2)c_1-x-c_2+1}{4c_1}}(c_1-1)}{4c_1}\right)c_1+(-c_2-x+1)c_1^2+(-2c_2-2x-2)c_1-x-c_2+1}{4c_1}}}{c_1+1}$$

✓ Solution by Mathematica

Time used: 57.998 (sec). Leaf size: 56

```
DSolve[(1+y[x]^2)*y''[x]+(y'[x])^3+y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \csc(c_1)\sec(c_1)W\left(\sin(c_1)e^{-((x+c_2)\cos^2(c_1))-\sin^2(c_1)}\right)+\tan(c_1)$$
$$y(x) \to e^{-x-c_2}$$

4.30 problem 33

Internal problem ID [6850]
Internal file name [`OUTPUT/6097_Thursday_July_28_2022_04_30_22_AM_28511657/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 33.
ODE order: 2.
ODE degree: 2.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{\left(yy''+1+y'^2\right)^2-\left(1+y'^2\right)^3=0}$$

4.30.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y'=p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$\left(y^2p(y)\left(\frac{d}{dy}p(y)\right)+2yp(y)^2+2y\right)p(y)\left(\frac{d}{dy}p(y)\right)+\left(-p(y)^5-2p(y)^3-p(y)\right)p(y)=0$$

Which is now solved as first order ode for $p(y)$. Solving the given ode for $\frac{d}{dy}p(y)$ results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$\frac{d}{dy}p(y) = \frac{\left(-1+\sqrt{p\left(y\right)^{2}+1}\right)\left(p(y)^{2}+1\right)}{p\left(y\right)y} \tag{1}$$

$$\frac{d}{dy}p(y) = -\frac{\left(1+\sqrt{p\left(y\right)^{2}+1}\right)\left(p(y)^{2}+1\right)}{p\left(y\right)y} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

In canonical form the ODE is

$$\begin{aligned} p' &= F(y,p) \\ &= f(y)g(p) \\ &= \frac{\left(-1+\sqrt{p^{2}+1}\right)\left(p^{2}+1\right)}{py} \end{aligned}$$

Where $f(y)=\frac{1}{y}$ and $g(p)=\frac{\left(-1+\sqrt{p^{2}+1}\right)\left(p^{2}+1\right)}{p}$. Integrating both sides gives

$$\frac{1}{\frac{\left(-1+\sqrt{p^{2}+1}\right)\left(p^{2}+1\right)}{p}}\,dp = \frac{1}{y}\,dy$$

$$\int \frac{1}{\frac{\left(-1+\sqrt{p^{2}+1}\right)\left(p^{2}+1\right)}{p}}\,dp = \int \frac{1}{y}\,dy$$

$$-\operatorname{arctanh}\left(\frac{1}{\sqrt{p^{2}+1}}\right)+\ln\left(p\right)-\frac{\ln\left(p^{2}+1\right)}{2} = \ln\left(y\right)+c_{1}$$

The solution is

$$-\operatorname{arctanh}\left(\frac{1}{\sqrt{p\left(y\right)^{2}+1}}\right)+\ln\left(p(y)\right)-\frac{\ln\left(p(y)^{2}+1\right)}{2}-\ln\left(y\right)-c_{1}=0$$

Solving equation (2)

In canonical form the ODE is

$$
\begin{aligned}
p' &= F(y,p) \\
&= f(y)g(p) \\
&= -\frac{\left(\sqrt{p^2+1}+1\right)\left(p^2+1\right)}{py}
\end{aligned}
$$

Where $f(y) = -\frac{1}{y}$ and $g(p) = \frac{\left(\sqrt{p^2+1}+1\right)\left(p^2+1\right)}{p}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{\left(\sqrt{p^2+1}+1\right)\left(p^2+1\right)}{p}}\, dp &= -\frac{1}{y}\, dy \\
\int \frac{1}{\frac{\left(\sqrt{p^2+1}+1\right)\left(p^2+1\right)}{p}}\, dp &= \int -\frac{1}{y}\, dy \\
-\operatorname{arctanh}\left(\frac{1}{\sqrt{p^2+1}}\right) - \ln(p) + \frac{\ln\left(p^2+1\right)}{2} &= -\ln(y) + c_2
\end{aligned}
$$

The solution is

$$
-\operatorname{arctanh}\left(\frac{1}{\sqrt{p(y)^2+1}}\right) - \ln(p(y)) + \frac{\ln\left(p(y)^2+1\right)}{2} + \ln(y) - c_2 = 0
$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
-\operatorname{arctanh}\left(\frac{1}{\sqrt{1+y'^2}}\right) + \ln(y') - \frac{\ln\left(1+y'^2\right)}{2} - \ln(y) - c_1 = 0
$$

Integrating both sides gives

$$
\begin{aligned}
\int \frac{1}{\sqrt{-1+\mathrm{e}^{\operatorname{RootOf}\left(\mathrm{e}^{\_Z}\tanh\left(\frac{\_Z}{2}+c_1-\frac{\ln\left(\frac{\mathrm{e}^{\_Z}-1}{y^2}\right)}{2}\right)^2-1\right)}}} dy &= \int dx \\
\int^y \frac{1}{\sqrt{-1+\mathrm{e}^{\operatorname{RootOf}\left(\mathrm{e}^{\_Z}\tanh\left(\frac{\_Z}{2}+c_1-\frac{\ln\left(\frac{\mathrm{e}^{\_Z}-1}{\_a^2}\right)}{2}\right)^2-1\right)}}} d\_a &= x + c_3
\end{aligned}
$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$-\operatorname{arctanh}\left(\frac{1}{\sqrt{1+y'^2}}\right) - \ln(y') + \frac{\ln(1+y'^2)}{2} + \ln(y) - c_2 = 0$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{e^{\operatorname{RootOf}\left(-\_Z+2\operatorname{arctanh}\left(e^{-\frac{\_Z}{2}}\right)+2c_2-\ln\left(\frac{y^2}{e^{\_Z}-1}\right)\right)} - 1} \tag{1}$$

$$y' = \sqrt{e^{\operatorname{RootOf}\left(-\_Z-2\operatorname{arctanh}\left(e^{-\frac{\_Z}{2}}\right)+2c_2-\ln\left(\frac{y^2}{e^{\_Z}-1}\right)\right)} - 1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{e^{\operatorname{RootOf}\left(-\_Z+2\operatorname{arctanh}\left(e^{-\frac{\_Z}{2}}\right)+2c_2-\ln\left(\frac{y^2}{e^{\_Z}-1}\right)\right)} - 1}} dy = \int dx$$

$$\int^y \frac{1}{\sqrt{e^{\operatorname{RootOf}\left(-\_Z+2\operatorname{arctanh}\left(e^{-\frac{\_Z}{2}}\right)+2c_2-\ln\left(\frac{\_a^2}{e^{\_Z}-1}\right)\right)} - 1}} d\_a = x + c_4$$

Solving equation (2)

Integrating both sides gives

$$\int \frac{1}{\sqrt{e^{\operatorname{RootOf}\left(-\_Z-2\operatorname{arctanh}\left(e^{-\frac{\_Z}{2}}\right)+2c_2-\ln\left(\frac{y^2}{e^{\_Z}-1}\right)\right)} - 1}} dy = \int dx$$

$$\int^y \frac{1}{\sqrt{e^{\operatorname{RootOf}\left(-\_Z-2\operatorname{arctanh}\left(e^{-\frac{\_Z}{2}}\right)+2c_2-\ln\left(\frac{\_a^2}{e^{\_Z}-1}\right)\right)} - 1}} d\_a = x + c_5$$

Summary
The solution(s) found are the following

$$\int^{y} \frac{1}{\sqrt{-1+\mathrm{e}^{\mathrm{RootOf}\left(\mathrm{e}^{\_Z}\tanh\left(\frac{\_Z}{2}+c_1-\frac{\ln\left(\frac{\mathrm{e}^{\_Z}-1}{\_a^2}\right)}{2}\right)^2-1\right)}}}d\_a = x + c_3 \tag{1}$$

$$\int^{y} \frac{1}{\sqrt{\mathrm{e}^{\mathrm{RootOf}\left(-\_Z+2\,\mathrm{arctanh}\left(\mathrm{e}^{-\frac{\_Z}{2}}\right)+2c_2-\ln\left(\frac{\_a^2}{\mathrm{e}^{\_Z}-1}\right)\right)}-1}}d\_a = x + c_4 \tag{2}$$

$$\int^{y} \frac{1}{\sqrt{\mathrm{e}^{\mathrm{RootOf}\left(-\_Z-2\,\mathrm{arctanh}\left(\mathrm{e}^{-\frac{\_Z}{2}}\right)+2c_2-\ln\left(\frac{\_a^2}{\mathrm{e}^{\_Z}-1}\right)\right)}-1}}d\_a = x + c_5 \tag{3}$$

Verification of solutions

$$\int^{y} \frac{1}{\sqrt{-1+\mathrm{e}^{\mathrm{RootOf}\left(\mathrm{e}^{\_Z}\tanh\left(\frac{\_Z}{2}+c_1-\frac{\ln\left(\frac{\mathrm{e}^{\_Z}-1}{\_a^2}\right)}{2}\right)^2-1\right)}}}d\_a = x + c_3$$

Verified OK.

$$\int^{y} \frac{1}{\sqrt{\mathrm{e}^{\mathrm{RootOf}\left(-\_Z+2\,\mathrm{arctanh}\left(\mathrm{e}^{-\frac{\_Z}{2}}\right)+2c_2-\ln\left(\frac{\_a^2}{\mathrm{e}^{\_Z}-1}\right)\right)}-1}}d\_a = x + c_4$$

Verified OK.

$$\int^{y} \frac{1}{\sqrt{\mathrm{e}^{\mathrm{RootOf}\left(-\_Z-2\,\mathrm{arctanh}\left(\mathrm{e}^{-\frac{\_Z}{2}}\right)+2c_2-\ln\left(\frac{\_a^2}{\mathrm{e}^{\_Z}-1}\right)\right)}-1}}d\_a = x + c_5$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
   *** Sublevel 2 ***
   Methods for second order ODEs:
   Successful isolation of d^2y/dx^2: 2 solutions were found. Trying to solve each resulting
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      trying 2nd order, 2 integrating factors of the form mu(x,y)
      trying differential order: 2; missing variables
      `, `-> Computing symmetries using: way = 3
      -> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+(1+(1+_b(_a)^2)^(1/2))*(1+
         symmetry methods on request
      `, `1st order, trying reduction of order with given symmetries:`[_a, 0]
```

✓ Solution by Maple

Time used: 0.141 (sec). Leaf size: 107

```
dsolve((y(x)*diff(y(x),x$2)+1+diff(y(x),x)^2)^2=(1+diff(y(x),x)^2)^3,y(x), singsol=all)
```

$$y(x) = -ix + c_1$$
$$y(x) = ix + c_1$$
$$y(x) = 0$$
$$y(x) = -c_1 - \sqrt{-(x + c_1 + c_2)(x - c_1 + c_2)}$$
$$y(x) = -c_1 + \sqrt{-(x + c_1 + c_2)(x - c_1 + c_2)}$$
$$y(x) = c_1 - \sqrt{-(x + c_1 + c_2)(x - c_1 + c_2)}$$
$$y(x) = c_1 + \sqrt{-(x + c_1 + c_2)(x - c_1 + c_2)}$$

✓ Solution by Mathematica

Time used: 45.659 (sec). Leaf size: 155

```
DSolve[(y[x]*y''[x]+1+(y'[x])^2)^2==(1+(y'[x])^2)^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{e^{2c_1} - (x + c_2)^2} - e^{c_1}$$
$$y(x) \to e^{c_1} - \sqrt{e^{2c_1} - (x + c_2)^2}$$
$$y(x) \to \sqrt{e^{2c_1} - (x + c_2)^2} - e^{c_1}$$
$$y(x) \to \sqrt{e^{2c_1} - (x + c_2)^2} + e^{c_1}$$
$$y(x) \to -\sqrt{-(x + c_2)^2}$$
$$y(x) \to \sqrt{-(x + c_2)^2}$$

4.31 problem 34

Internal problem ID [6851]
Internal file name [`OUTPUT/6098_Friday_July_29_2022_02_05_35_AM_9550685/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 34.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{x^2y'' - y'(2x - y') = 0}$$

With initial conditions

$$[y(-1) = 5, y'(-1) = 1]$$

4.31.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$x^2p'(x) + (p(x) - 2x)\,p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Using the change of variables $p(x) = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x^2(u'(x)\,x + u(x)) + (u(x)\,x - 2x)\,u(x)\,x = 0$$

In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{u(u-1)}{x}
\end{aligned}
$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u(u-1)$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u\left(u-1\right)}\, du &= -\frac{1}{x}\, dx \\
\int \frac{1}{u\left(u-1\right)}\, du &= \int -\frac{1}{x}\, dx \\
-\ln\left(u\right) + \ln\left(u-1\right) &= -\ln\left(x\right) + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
\mathrm{e}^{-\ln(u)+\ln(u-1)} = \mathrm{e}^{-\ln(x)+c_2}
$$

Which simplifies to

$$
\frac{u-1}{u} = \frac{c_3}{x}
$$

Therefore the solution $p(x)$ is

$$
\begin{aligned}
p(x) &= ux \\
&= -\frac{x^2}{c_3 - x}
\end{aligned}
$$

Initial conditions are used to solve for c_3. Substituting $x = -1$ and $p = 1$ in the above solution gives an equation to solve for the constant of integration.

$$
1 = -\frac{1}{c_3 + 1}
$$

$$
c_3 = -2
$$

Substituting c_3 found above in the general solution gives

$$
p(x) = \frac{x^2}{x+2}
$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{x^2}{x+2}$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{x^2}{x+2}\, \mathrm{d}x \\ &= \frac{x^2}{2} - 2x + 4\ln(x+2) + c_4 \end{aligned}$$

Initial conditions are used to solve for c_4. Substituting $x = -1$ and $y = 5$ in the above solution gives an equation to solve for the constant of integration.

$$5 = \frac{5}{2} + c_4$$

$$c_4 = \frac{5}{2}$$

Substituting c_4 found above in the general solution gives

$$y = \frac{x^2}{2} - 2x + 4\ln(x+2) + \frac{5}{2}$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = \frac{x^2}{2} - 2x + 4\ln(x+2) + \frac{5}{2} \tag{1}$$

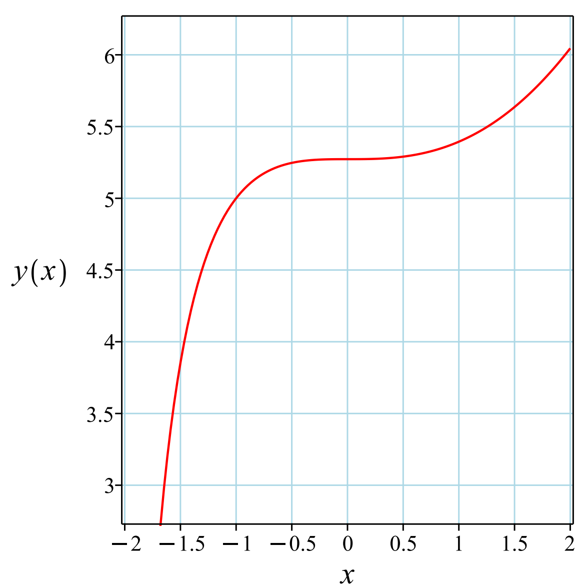

Figure 17: Solution plot

Verification of solutions

$$y = \frac{x^2}{2} - 2x + 4\ln(x+2) + \frac{5}{2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 20

```
dsolve([x^2*diff(y(x),x$2)=diff(y(x),x)*(2*x-diff(y(x),x)),y(-1) = 5, D(y)(-1) = 1],y(x), si
```

$$y(x) = \frac{x^2}{2} - 2x + 4\ln(x+2) + \frac{5}{2}$$

✓ Solution by Mathematica

Time used: 0.52 (sec). Leaf size: 23

```
DSolve[{x^2*y''[x]==y'[x]*(2*x-y'[x]),{y[-1]==5,y'[-1]==1}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{1}{2}\left(x^2 - 4x + 8\log(x+2) + 5\right)$$

4.32 problem 35

Internal problem ID [6852]
Internal file name [`OUTPUT/6099_Friday_July_29_2022_03_09_12_AM_9550685/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 35.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{x^2y'' - y'(3x - 2y') = 0}$$

4.32.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$x^2p'(x) + (2p(x) - 3x)\,p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Using the change of variables $p(x) = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x^2(u'(x)\,x + u(x)) + (2u(x)\,x - 3x)\,u(x)\,x = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2u(u-1)}{x} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u(u-1)$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u\left(u-1\right)}\,du &= -\frac{2}{x}\,dx \\ \int \frac{1}{u\left(u-1\right)}\,du &= \int -\frac{2}{x}\,dx \\ -\ln\left(u\right)+\ln\left(u-1\right) &= -2\ln\left(x\right)+c_2 \end{aligned}$$

Raising both side to exponential gives

$$\mathrm{e}^{-\ln(u)+\ln(u-1)} = \mathrm{e}^{-2\ln(x)+c_2}$$

Which simplifies to

$$\frac{u-1}{u} = \frac{c_3}{x^2}$$

Therefore the solution $p(x)$ is

$$\begin{aligned} p(x) &= xu \\ &= -\frac{x^3}{-x^2+c_3} \end{aligned}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -\frac{x^3}{-x^2+c_3}$$

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{x^3}{-x^2+c_3}\,\mathrm{d}x \\ &= \frac{x^2}{2}+\frac{c_3\ln\left(x^2-c_3\right)}{2}+c_4 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{x^2}{2} + \frac{c_3 \ln (x^2 - c_3)}{2} + c_4 \tag{1}$$

Verification of solutions

$$y = \frac{x^2}{2} + \frac{c_3 \ln (x^2 - c_3)}{2} + c_4$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -_b(_a)*(-3*_a+2*_b(_a))/_a^2, _b(_a),
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, _b]
```

✓ Solution by Maple
Time used: 0.062 (sec). Leaf size: 22

```
dsolve(x^2*diff(y(x),x$2)=diff(y(x),x)*(3*x-2*diff(y(x),x)),y(x), singsol=all)
```

$$y(x) = \frac{x^2}{2} + \frac{c_1 \ln (x^2 - c_1)}{2} + c_2$$

✓ Solution by Mathematica

Time used: 0.333 (sec). Leaf size: 28

```
DSolve[x^2*y''[x]==y'[x]*(3*x-2*y'[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(x^2 - c_1 \log\left(x^2 + c_1\right) + 2c_2\right)$$

4.33 problem 36

Internal problem ID [6853]
Internal file name [`OUTPUT/6100_Friday_July_29_2022_03_09_14_AM_38707999/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 36.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y", "second_order_nonlinear_solved_by_mainardi_lioville_method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y], _Liouville, [_2nd_order, _reducible,
    _mu_xy]]
```

$$\boxed{xy'' - y'(2 - 3xy') = 0}$$

4.33.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$xp'(x) + (3p(x)\, x - 2)\, p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Writing the ode as

$$
\begin{aligned}
p'(x) &= -\frac{(3px-2)\,p}{x} \\
p'(x) &= \omega(x,p)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_p - \xi_x) - \omega^2 \xi_p - \omega_x \xi - \omega_p \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 42: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,p) &= 0 \\ \eta(x,p) &= \frac{p^2}{x^2} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,p) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dp}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial p}\right) S(x,p) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{p^2}{x^2}} dy \end{aligned}$$

Which results in

$$S = -\frac{x^2}{p}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,p)S_p}{R_x + \omega(x,p)R_p} \tag{2}$$

Where in the above R_x, R_p, S_x, S_p are all partial derivatives and $\omega(x,p)$ is the right hand side of the original ode given by

$$\omega(x,p) = -\frac{(3px-2)\,p}{x}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_p &= 0 \\
S_x &= -\frac{2x}{p} \\
S_p &= \frac{x^2}{p^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -3x^2 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, p in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -3R^2
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -R^3 + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, p coordinates. This results in

$$
-\frac{x^2}{p(x)} = -x^3 + c_1
$$

Which simplifies to

$$
-\frac{x^2}{p(x)} = -x^3 + c_1
$$

Which gives

$$
p(x) = -\frac{x^2}{-x^3 + c_1}
$$

Since $p = y'$ then the new first order ode to solve is

$$
y' = -\frac{x^2}{-x^3 + c_1}
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int -\frac{x^2}{-x^3+c_1}\,\mathrm{d}x \\
&= \frac{\ln(x^3-c_1)}{3}+c_2
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{\ln(x^3-c_1)}{3}+c_2 \tag{1}
$$

Verification of solutions

$$
y = \frac{\ln(x^3-c_1)}{3}+c_2
$$

Verified OK.

4.33.2 Solving as second order nonlinear solved by mainardi lioville method ode

The ode has the Liouville form given by

$$
y'' + f(x)y' + g(y){y'}^2 = 0 \tag{1A}
$$

Where in this problem

$$
\begin{aligned}
f(x) &= -\frac{2}{x} \\
g(y) &= 3
\end{aligned}
$$

Dividing through by y' then Eq (1A) becomes

$$
\frac{y''}{y'} + f + gy' = 0 \tag{2A}
$$

But the first term in Eq (2A) can be written as

$$
\frac{y''}{y'} = \frac{d}{dx}\ln(y') \tag{3A}
$$

And the last term in Eq (2A) can be written as

$$
\begin{aligned}
g\frac{dy}{dx} &= \left(\frac{d}{dy}\int g dy\right)\frac{dy}{dx} \\
&= \frac{d}{dx}\int g dy
\end{aligned} \tag{4A}
$$

Substituting (3A,4A) back into (2A) gives

$$\frac{d}{dx}\ln\left(y'\right)+\frac{d}{dx}\int g dy=-f \tag{5A}$$

Integrating the above w.r.t. x gives

$$\ln\left(y'\right)+\int g dy=-\int f dx+c_1$$

Where c_1 is arbitrary constant. Taking the exponential of the above gives

$$y'=c_2 e^{\int -g dy}\, e^{\int -f dx} \tag{6A}$$

Where c_2 is a new arbitrary constant. But since $g=3$ and $f=-\frac{2}{x}$, then

$$\begin{aligned}\int -g dy &= \int (-3)\, dy\\ &= -3y\\ \int -f dx &= \int \frac{2}{x} dx\\ &= 2\ln(x)\end{aligned}$$

Substituting the above into Eq(6A) gives

$$y'=c_2 \mathrm{e}^{-3y} x^2$$

Which is now solved as first order separable ode. In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= f(x)g(y)\\ &= c_2 \mathrm{e}^{-3y} x^2\end{aligned}$$

Where $f(x)=c_2 x^2$ and $g(y)=\mathrm{e}^{-3y}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\mathrm{e}^{-3y}}\, dy &= c_2 x^2\, dx\\ \int \frac{1}{\mathrm{e}^{-3y}}\, dy &= \int c_2 x^2\, dx\\ \frac{\mathrm{e}^{3y}}{3} &= \frac{c_2 x^3}{3}+c_3\end{aligned}$$

The solution is

$$\frac{e^{3y}}{3}-\frac{c_2x^3}{3}-c_3=0$$

Summary

The solution(s) found are the following

$$\frac{e^{3y}}{3}-\frac{c_2x^3}{3}-c_3=0 \tag{1}$$

Verification of solutions

$$\frac{e^{3y}}{3}-\frac{c_2x^3}{3}-c_3=0$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 16

```
dsolve(x*diff(y(x),x$2)=diff(y(x),x)*(2-3*x*diff(y(x),x)),y(x), singsol=all)
```

$$y(x)=\frac{\ln(c_1x^3+3c_2)}{3}$$

✓ Solution by Mathematica

Time used: 0.267 (sec). Leaf size: 19

```
DSolve[x*y''[x]==y'[x]*(2-3*x*y'[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{1}{3}\log\left(x^3+c_1\right)+c_2$$

4.34 problem 37

Internal problem ID [6854]
Internal file name [OUTPUT/6101_Friday_July_29_2022_03_09_17_AM_29950073/index.tex]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 37.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{x^4y'' - y'(y' + x^3) = 0}$$

With initial conditions

$$[y(1) = 2, y'(1) = 1]$$

4.34.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$x^4p'(x) + (-x^3 - p(x))\,p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Using the change of variables $p(x) = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x^4(u'(x)\,x + u(x)) + \left(-x^3 - u(x)\,x\right) u(x)\,x = 0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{u^2}{x^3}
\end{aligned}$$

Where $f(x) = \frac{1}{x^3}$ and $g(u) = u^2$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u^2}\,du &= \frac{1}{x^3}\,dx \\
\int \frac{1}{u^2}\,du &= \int \frac{1}{x^3}\,dx \\
-\frac{1}{u} &= -\frac{1}{2x^2} + c_2
\end{aligned}$$

The solution is

$$-\frac{1}{u\left(x\right)} + \frac{1}{2x^2} - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{p(x)}{x}$ results in the solution for $p(x)$ in implicit form

$$\begin{aligned}
-\frac{x}{p\left(x\right)} + \frac{1}{2x^2} - c_2 = 0 \\
-\frac{x}{p\left(x\right)} + \frac{1}{2x^2} - c_2 = 0
\end{aligned}$$

Substituting initial conditions and solving for c_2 gives $c_2 = -\frac{1}{2}$. Hence the solution becomes Solving for $p(x)$ from the above gives

$$p(x) = \frac{2x^3}{x^2+1}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{2x^3}{x^2+1}$$

Integrating both sides gives

$$y = \int \frac{2x^3}{x^2+1}\, \mathrm{d}x$$
$$= x^2 - \ln\left(x^2+1\right) + c_3$$

Initial conditions are used to solve for c_3. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = 1 - \ln(2) + c_3$$

$$c_3 = 1 + \ln(2)$$

Substituting c_3 found above in the general solution gives

$$y = x^2 - \ln\left(x^2+1\right) + 1 + \ln(2)$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = x^2 - \ln\left(x^2+1\right) + 1 + \ln(2) \tag{1}$$

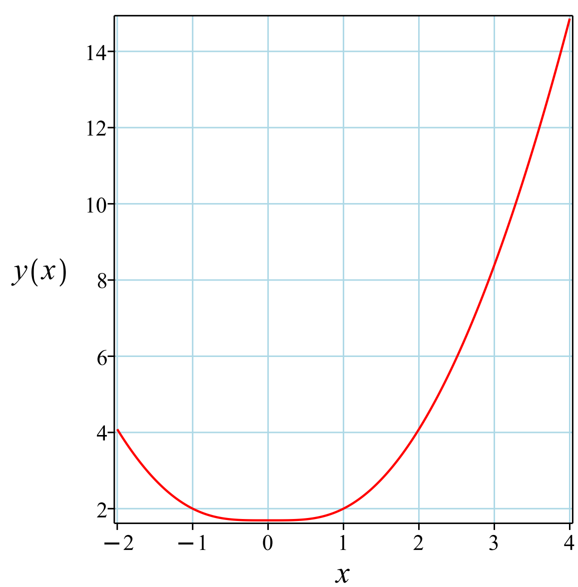

Figure 18: Solution plot

Verification of solutions

$$y = x^2 - \ln\left(x^2+1\right) + 1 + \ln(2)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = _b(_a)*(_a^3+_b(_a))/_a^4, _b(_a), HINT
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, 3*_b]
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 25

```
dsolve([x^4*diff(y(x),x$2)=diff(y(x),x)*(diff(y(x),x)+x^3),y(1) = 2, D(y)(1) = 1],y(x), sing
```

$$y(x) = x^2 - \ln\left(-x^2 - 1\right) + 1 + \ln\left(2\right) + i\pi$$

✓ Solution by Mathematica

Time used: 0.929 (sec). Leaf size: 20

```
DSolve[{x^4*y''[x]==y'[x]*(y'[x]+x^3),{y[1]==2,y'[1]==1}},y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to x^2 - \log\left(x^2 + 1\right) + 1 + \log(2)$$

4.35 problem 38

Internal problem ID [6855]
Internal file name [OUTPUT/6102_Friday_July_29_2022_03_09_20_AM_12648957/index.tex]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 38.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y], [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{y'' - \left(x^2 - y'\right)^2 = 2x}$$

4.35.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + \left(2x^2 - p(x)\right) p(x) - x^4 - 2x = 0$$

Which is now solve for $p(x)$ as first order ode. Writing the ode as

$$\begin{aligned} p'(x) &= x^4 - 2p\,x^2 + p^2 + 2x \\ p'(x) &= \omega(x, p) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_p - \xi_x) - \omega^2 \xi_p - \omega_x \xi - \omega_p \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = pa_3 + xa_2 + a_1 \tag{1E}$$

$$\eta = pb_3 + xb_2 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \left(x^4 - 2p\,x^2 + p^2 + 2x\right)(b_3 - a_2) - \left(x^4 - 2p\,x^2 + p^2 + 2x\right)^2 a_3 - \left(4x^3 - 4xp + 2\right)(pa_3 + xa_2 + a_1) - \left(-2x^2 + 2p\right)(pb_3 + xb_2 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-x^8 a_3 + 4p\,x^6 a_3 - 6p^2 x^4 a_3 + 4p^3 x^2 a_3 - 4x^5 a_3 - p^4 a_3 + 4p\,x^3 a_3 - 5x^4 a_2 + x^4 b_3 + 6p\,x^2 a_2 - 4x^3 a_1 + 2x^3 b_2 - p^2 a_2 - p^2 b_3 + 4pxa_1 - 2pxb_2 - 4x^2 a_3 + 2x^2 b_1 - 2pa_3 - 2pb_1 - 4xa_2 + 2xb_3 - 2a_1 + b_2 = 0$$

Setting the numerator to zero gives

$$-x^8 a_3 + 4p\,x^6 a_3 - 6p^2 x^4 a_3 + 4p^3 x^2 a_3 - 4x^5 a_3 - p^4 a_3 + 4p\,x^3 a_3 - 5x^4 a_2 + x^4 b_3 + 6p\,x^2 a_2 - 4x^3 a_1 + 2x^3 b_2 - p^2 a_2 - p^2 b_3 + 4pxa_1 - 2pxb_2 - 4x^2 a_3 + 2x^2 b_1 - 2pa_3 - 2pb_1 - 4xa_2 + 2xb_3 - 2a_1 + b_2 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{p, x\}$ in them.

$$\{p, x\}$$

The following substitution is now made to be able to collect on all terms with $\{p, x\}$ in them

$$\{p = v_1, x = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-a_3v_2^8 + 4a_3v_1v_2^6 - 6a_3v_1^2v_2^4 + 4a_3v_1^3v_2^2 - 4a_3v_2^5 - 5a_2v_2^4 - a_3v_1^4 + 4a_3v_1v_2^3 \\
&+ b_3v_2^4 - 4a_1v_2^3 + 6a_2v_1v_2^2 + 2b_2v_2^3 + 4a_1v_1v_2 - a_2v_1^2 - 4a_3v_2^2 + 2b_1v_2^2 \\
&- 2b_2v_1v_2 - b_3v_1^2 - 4a_2v_2 - 2a_3v_1 - 2b_1v_1 + 2b_3v_2 - 2a_1 + b_2 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&-a_3v_1^4 + 4a_3v_1^3v_2^2 - 6a_3v_1^2v_2^4 + (-a_2 - b_3)\,v_1^2 + 4a_3v_1v_2^6 + 4a_3v_1v_2^3 + 6a_2v_1v_2^2 \\
&+ (4a_1 - 2b_2)\,v_1v_2 + (-2a_3 - 2b_1)\,v_1 - a_3v_2^8 - 4a_3v_2^5 + (-5a_2 + b_3)\,v_2^4 \\
&+ (-4a_1 + 2b_2)\,v_2^3 + (-4a_3 + 2b_1)\,v_2^2 + (-4a_2 + 2b_3)\,v_2 - 2a_1 + b_2 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
6a_2 &= 0 \\
-6a_3 &= 0 \\
-4a_3 &= 0 \\
-a_3 &= 0 \\
4a_3 &= 0 \\
-4a_1 + 2b_2 &= 0 \\
-2a_1 + b_2 &= 0 \\
4a_1 - 2b_2 &= 0 \\
-5a_2 + b_3 &= 0 \\
-4a_2 + 2b_3 &= 0 \\
-a_2 - b_3 &= 0 \\
-4a_3 + 2b_1 &= 0 \\
-2a_3 - 2b_1 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= a_1 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 2a_1 \\ b_3 &= 0 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 1 \\ \eta &= 2x \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x,p)\,\xi \\ &= 2x - \left(x^4 - 2p\,x^2 + p^2 + 2x\right)(1) \\ &= -x^4 + 2p\,x^2 - p^2 \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,p) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dp}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial p}\right) S(x,p) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-x^4 + 2p\,x^2 - p^2} dy \end{aligned}$$

Which results in

$$S = \frac{1}{-x^2 + p}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,p)S_p}{R_x + \omega(x,p)R_p} \tag{2}$$

Where in the above R_x, R_p, S_x, S_p are all partial derivatives and $\omega(x,p)$ is the right hand side of the original ode given by

$$\omega(x,p) = x^4 - 2p\,x^2 + p^2 + 2x$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_p &= 0 \\ S_x &= \frac{2x}{(-x^2+p)^2} \\ S_p &= -\frac{1}{(-x^2+p)^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, p in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, p coordinates. This results in

$$\frac{1}{-x^2 + p(x)} = -x + c_1$$

Which simplifies to

$$\frac{1}{-x^2 + p(x)} = -x + c_1$$

Which gives

$$p(x) = \frac{c_1 x^2 - x^3 + 1}{-x + c_1}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{c_1 x^2 - x^3 + 1}{-x + c_1}$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{c_1 x^2 - x^3 + 1}{-x + c_1} \, \mathrm{d}x \\ &= \frac{x^3}{3} - \ln(-c_1 + x) + c_2 \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{x^3}{3} - \ln(-c_1 + x) + c_2 \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{x^3}{3} - \ln(-c_1 + x) + c_2$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 20

```
dsolve(diff(y(x),x$2)=2*x+(x^2-diff(y(x),x))^2,y(x), singsol=all)
```

$$y(x) = \frac{x^3}{3} - \ln\left(c_2 x - c_1\right)$$

✓ Solution by Mathematica

Time used: 0.298 (sec). Leaf size: 24

```
DSolve[y''[x]==2*x+(x^2-y'[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^3}{3} - \log(-x + c_1) + c_2$$

4.36 problem 39

Internal problem ID [6856]
Internal file name [`OUTPUT/6103_Friday_July_29_2022_03_09_22_AM_41838316/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 39.
ODE order: 2.
ODE degree: 2.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y''^2 - 2y'' + y'^2 - 2xy' = -x^2}$$

With initial conditions

$$\left[y(0) = \frac{1}{2}, y'(0) = 1\right]$$

4.36.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$(p'(x) - 2)\, p'(x) + (p(x) - 2x)\, p(x) + x^2 = 0$$

Which is now solve for $p(x)$ as first order ode. Let $p = p'(x)$ the ode becomes

$$(p-2)\,p + (p-2x)\,p = -x^2$$

Solving for $p(x)$ from the above results in

$$p(x) = x + \sqrt{-p^2 + 2p} \tag{1A}$$
$$p(x) = x - \sqrt{-p^2 + 2p} \tag{2A}$$

This has the form

$$p = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = p'(x)$. Each of the above ode's is dAlembert ode which is now solved. Solving ode 1A Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $p(x) = xf + g$ to (1A) shows that

$$f = 1$$
$$g = \sqrt{-(p-2)\,p}$$

Hence (2) becomes

$$p - 1 = \frac{(-2p+2)\,p'(x)}{2\sqrt{-(p-2)\,p}} \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - 1 = 0$$

Solving for p from the above gives

$$p = 1$$

Substituting these in (1A) gives

$$p(x) = x + 1$$

The general solution is found when $\frac{\mathrm{dp}}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{2(p(x)-1)\sqrt{-\left(p\left(x\right)-2\right)p\left(x\right)}}{-2p\left(x\right)+2} \tag{3}$$

This ODE is now solved for $p(x)$. Integrating both sides gives

$$\int -\frac{1}{\sqrt{-\left(p-2\right)p}}dp = x + c_1$$
$$-\arcsin\left(p-1\right) = x + c_1$$

Solving for p gives these solutions

$$p_1 = 1 - \sin\left(x + c_1\right)$$

Substituing the above solution for p in (2A) gives

$$p(x) = x + \sqrt{-\left(-1-\sin\left(x+c_1\right)\right)\left(1-\sin\left(x+c_1\right)\right)}$$

Solving ode 2A Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $p(x) = xf + g$ to (1A) shows that

$$f = 1$$
$$g = -\sqrt{-\left(p-2\right)p}$$

Hence (2) becomes

$$p - 1 = -\frac{\left(-2p+2\right)p'(x)}{2\sqrt{-\left(p-2\right)p}} \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - 1 = 0$$

Solving for p from the above gives

$$p = 1$$

Substituting these in (1A) gives

$$p(x) = x - 1$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = -\frac{2(p(x) - 1)\sqrt{-(p(x) - 2)\,p(x)}}{-2p(x) + 2} \tag{3}$$

This ODE is now solved for $p(x)$. Integrating both sides gives

$$\int \frac{1}{\sqrt{-(p-2)\,p}} dp = x + c_2$$
$$\arcsin(p - 1) = x + c_2$$

Solving for p gives these solutions

$$p_1 = 1 + \sin(x + c_2)$$

Substituing the above solution for p in (2A) gives

$$p(x) = x - \sqrt{-(-1 + \sin(x + c_2))(1 + \sin(x + c_2))}$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $p = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\sqrt{-(-1 + \sin(c_2))(1 + \sin(c_2))}$$

$$c_2 = \pi$$

Substituting c_2 found above in the general solution gives

$$p(x) = x - \sqrt{-(1 + \sin(x))(\sin(x) - 1)}$$

But this does not satisfy the initial conditions. Hence no solution can be found. Initial conditions are used to solve for c_1. Substituting $x = 0$ and $p = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \sqrt{-(1 + \sin(c_1))(-1 + \sin(c_1))}$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$p(x) = x + \sqrt{-(1+\sin(x))(\sin(x)-1)}$$

For solution (1) found earlier, since $p = y'$ then the new first order ode to solve is

$$y' = x + 1$$

Integrating both sides gives

$$\begin{aligned} y &= \int x + 1 \, \mathrm{d}x \\ &= \frac{1}{2}x^2 + x + c_3 \end{aligned}$$

Initial conditions are used to solve for c_3. Substituting $x = 0$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = c_3$$

$$c_3 = \frac{1}{2}$$

Substituting c_3 found above in the general solution gives

$$y = \frac{1}{2}x^2 + x + \frac{1}{2}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = x + \sqrt{-(1+\sin(x))(\sin(x)-1)}$$

Integrating both sides gives

$$\begin{aligned} y &= \int x + \sqrt{-(1+\sin(x))(\sin(x)-1)} \, \mathrm{d}x \\ &= \frac{x^2}{2} - \frac{2(\sin(x)-1)^2(1+\sin(x))}{3\cos(x)\sqrt{-(1+\sin(x))(\sin(x)-1)}} + c_4 \end{aligned}$$

Initial conditions are used to solve for c_4. Substituting $x = 0$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = c_4 - \frac{2}{3}$$

$$
c_4 = \frac{7}{6}
$$

Substituting c_4 found above in the general solution gives

$$
y = \frac{2\sin(x)}{3} + \frac{x^2}{2} + \frac{1}{2}
$$

Since $p = y'$ then the new first order ode to solve is

$$
y' = x - 1
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int x - 1 \,\mathrm{d}x \\
&= \frac{1}{2}x^2 - x + c_5
\end{aligned}
$$

Initial conditions are used to solve for c_5. Substituting $x = 0$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$
\frac{1}{2} = c_5
$$

$$
c_5 = \frac{1}{2}
$$

Substituting c_5 found above in the general solution gives

$$
y = \frac{1}{2}x^2 - x + \frac{1}{2}
$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$
y = \frac{1}{2}x^2 + x + \frac{1}{2} \tag{1}
$$

$$
y = \frac{2\sin(x)}{3} + \frac{x^2}{2} + \frac{1}{2} \tag{2}
$$

$$
y = \frac{1}{2}x^2 - x + \frac{1}{2} \tag{3}
$$

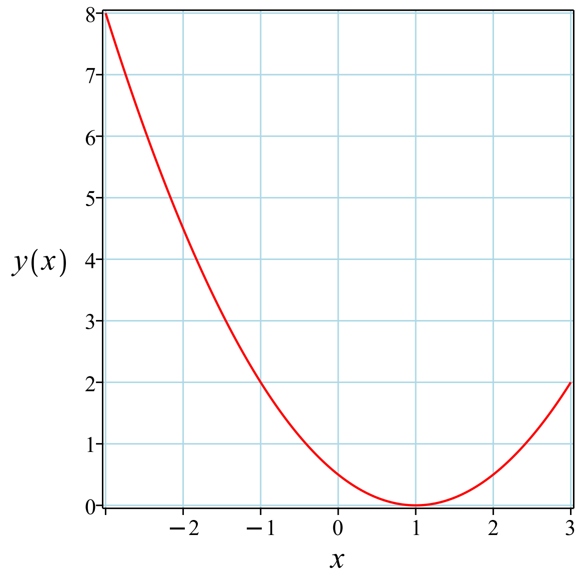

Figure 19: Solution plot

Verification of solutions

$$y = \frac{1}{2}x^2 + x + \frac{1}{2}$$

Verified OK.

$$y = \frac{2\sin(x)}{3} + \frac{x^2}{2} + \frac{1}{2}$$

Warning, solution could not be verified

$$y = \frac{1}{2}x^2 - x + \frac{1}{2}$$

Warning, solution could not be verified

Maple trace

```
`Methods for second order ODEs:
   *** Sublevel 2 ***
   Methods for second order ODEs:
   Successful isolation of d^2y/dx^2: 2 solutions were found. Trying to solve each resulting
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      -> Calling odsolve with the ODE`, diff(diff(diff(y(x), x), x), x)+diff(y(x), x)-x, y(x
         Methods for third order ODEs:
         --- Trying classification methods ---
         trying a quadrature
         trying high order exact linear fully integrable
         trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
         -> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = -_b(_a)+_a, _b(_a)`
            Methods for second order ODEs:
            --- Trying classification methods ---
            trying a quadrature
            trying high order exact linear fully integrable
            trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
            trying a double symmetry of the form [xi=0, eta=F(x)]
            -> Try solving first the homogeneous part of the ODE
               checking if the LODE has constant coefficients
               <- constant coefficients successful
            <- solving first the homogeneous part of the ODE successful
         <- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful
      <- 2nd order ODE linearizable_by_differentiation successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      <- 2nd order ODE linearizable_by_differentiation successful
-> Calling odsolve with the ODE`, diff(y(x), x) = x+1, y(x), singsol = none`   *** Sublevel
   Methods for first order ODEs:
```

✓ Solution by Maple

Time used: 0.391 (sec). Leaf size: 23

```
dsolve([diff(y(x),x$2)^2-2*diff(y(x),x$2)+diff(y(x),x)^2-2*x*diff(y(x),x)+x^2=0,y(0) = 1/2,
```

$$y(x) = \frac{(x+1)^2}{2}$$
$$y(x) = \frac{x^2}{2} + \sin(x) + \frac{1}{2}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{(y''[x])^2-2*y''[x]+(y'[x])^2-2*x*y'[x]+x^2==0,{y[0]==1/2,y'[0]==1}},y[x],x,IncludeS
```

Not solved

4.37 problem 40

Internal problem ID [6857]
Internal file name [`OUTPUT/6104_Friday_July_29_2022_03_09_36_AM_52345976/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 40.
ODE order: 2.
ODE degree: 2.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y''^2 - xy'' + y' = 0}$$

4.37.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$(p'(x) - x)\, p'(x) + p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. This is Clairaut ODE. It has the form

$$p = p'(x)\, x + g(p'(x))$$

Where g is function of $p'(x)$. Let $p = p'(x)$ the ode becomes

$$(p - x)\, p + p = 0$$

Solving for $p(x)$ from the above results in

$$p(x) = -(p - x)\,p \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing $p'(x)$ by p which gives

$$\begin{aligned} p(x) &= -p^2 + px \\ &= -p^2 + px \end{aligned}$$

Writing the ode as

$$p(x) = px + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$p = px + g \tag{1}$$

Then we see that

$$g = -p^2$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned} p &= \frac{d}{dx}(xp + g) \\ p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\ p &= p + (x + g')\frac{dp}{dx} \\ 0 &= (x + g')\frac{dp}{dx} \end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned} \frac{dp}{dx} &= 0 \\ p &= c_1 \end{aligned}$$

Substituting this in (1) gives the general solution as

$$p(x) = -c_1^2 + c_1 x$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = -p^2$, then the above equation becomes

$$\begin{aligned} x + g'(p) &= x - 2p \\ &= 0 \end{aligned}$$

Solving the above for p results in

$$p_1 = \frac{x}{2}$$

Substituting the above back in (1) results in

$$p(x)_1 = \frac{x^2}{4}$$

For solution (1) found earlier, since $p = y'$ then the new first order ode to solve is

$$y' = -c_1^2 + c_1 x$$

Integrating both sides gives

$$\begin{aligned} y &= \int -c_1^2 + c_1 x \, \mathrm{d}x \\ &= c_1 \left(\frac{1}{2} x^2 - c_1 x\right) + c_2 \end{aligned}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{x^2}{4}$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{x^2}{4} \, \mathrm{d}x \\ &= \frac{x^3}{12} + c_3 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \left(\frac{1}{2} x^2 - c_1 x\right) + c_2 \tag{1}$$

$$y = \frac{x^3}{12} + c_3 \tag{2}$$

Verification of solutions

$$
y = c_1\left(\frac{1}{2}x^2 - c_1x\right) + c_2
$$

Verified OK.

$$
y = \frac{x^3}{12} + c_3
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
   *** Sublevel 2 ***
   Methods for second order ODEs:
   Successful isolation of d^2y/dx^2: 2 solutions were found. Trying to solve each resulting
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      <- 2nd order ODE linearizable_by_differentiation successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      <- 2nd order ODE linearizable_by_differentiation successful
-> Calling odsolve with the ODE`, diff(y(x), x) = (1/4)*x^2, y(x), singsol = none`   *** Sub
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   <- quadrature successful`
```

✓ Solution by Maple

Time used: 0.219 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$2)^2-x*diff(y(x),x$2)+diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x^3}{12} + c_1$$
$$y(x) = \frac{1}{2}c_1x^2 - c_1^2x + c_2$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 24

```
DSolve[(y''[x])^2-x*y''[x]+y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1x^2}{2} - c_1{}^2x + c_2$$

4.38 problem 41

Internal problem ID [6858]
Internal file name [`OUTPUT/6105_Friday_July_29_2022_03_09_39_AM_93402856/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 41.
ODE order: 2.
ODE degree: 3.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y''^3 - 12y'(xy'' - 2y') = 0}$$

4.38.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$\left(p'(x)^2 - 12p(x)\, x\right) p'(x) + 24p(x)^2 = 0$$

Which is now solve for $p(x)$ as first order ode. Solving the given ode for $p'(x)$ results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$p'(x) = \left(-12p(x)^2 + 4\sqrt{-4p(x)^3 x^3 + 9p(x)^4}\right)^{\frac{1}{3}} + \frac{4p(x)\, x}{\left(-12p(x)^2 + 4\sqrt{-4p(x)^3 x^3 + 9p(x)^4}\right)^{\frac{1}{3}}} \tag{1}$$

$$p'(x) = -\frac{\left(-12p(x)^2 + 4\sqrt{-4p(x)^3 x^3 + 9p(x)^4}\right)^{\frac{1}{3}}}{2} - \frac{2p(x)\, x}{\left(-12p(x)^2 + 4\sqrt{-4p(x)^3 x^3 + 9p(x)^4}\right)^{\frac{1}{3}}} + i\sqrt{3} \tag{2}$$

$$p'(x) = -\frac{\left(-12p(x)^2 + 4\sqrt{-4p(x)^3 x^3 + 9p(x)^4}\right)^{\frac{1}{3}}}{2} - \frac{2p(x)\, x}{\left(-12p(x)^2 + 4\sqrt{-4p(x)^3 x^3 + 9p(x)^4}\right)^{\frac{1}{3}}} - i\sqrt{3} \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$p'(x) = \frac{\left(-12p^2 + 4\sqrt{-4p^3x^3 + 9p^4}\right)^{\frac{2}{3}} + 4px}{\left(-12p^2 + 4\sqrt{-4p^3x^3 + 9p^4}\right)^{\frac{1}{3}}}$$
$$p'(x) = \omega(x, p)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_p - \xi_x) - \omega^2\xi_p - \omega_x\xi - \omega_p\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = pa_3 + xa_2 + a_1 \tag{1E}$$
$$\eta = pb_3 + xb_2 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{\left(\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}}+4px\right)(b_3-a_2)}{\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{1}{3}}} - \frac{\left(\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}}+4px\right)^2 a_3}{\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}}} - \left(\frac{-\frac{16p^3x^2}{\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{1}{3}}\sqrt{-4p^3x^3+9p^4}}+4p}{\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{1}{3}}} + \frac{8\left(\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}}+4px\right)p^3x^2}{\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{4}{3}}\sqrt{-4p^3x^3+9p^4}}\right)(pa_3+xa_2+a_1) - \left(\frac{\frac{-16p+\frac{2\left(-24p^2x^3+72p^3\right)}{3\sqrt{-4p^3x^3+9p^4}}}{\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{1}{3}}}+4x}{\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{1}{3}}} - \frac{\left(\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}}+4px\right)\left(-24p+\frac{-24p^2x^3+72p^3}{\sqrt{-4p^3x^3+9p^4}}\right)}{3\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{4}{3}}}\right)(pb_3 + xb_2 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{8\sqrt{-4p^3x^3+9p^4}\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{4}{3}}pxa_3 + 16\sqrt{-4p^3x^3+9p^4}\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}}}{} = 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-8\sqrt{-4p^3x^3+9p^4}\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{4}{3}} p x a_3 \\
&- 16\sqrt{-4p^3x^3+9p^4}\left(-12p^2 \right. \\
&\left. + 4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}} p^2x^2a_3 \\
&+ 8\sqrt{-4p^3x^3+9p^4}\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}} p x b_2 \\
&- 16\left(-4p^3x^3+9p^4\right)^{\frac{3}{2}} a_3 + 720p^6a_3 - 144p^5a_1 \\
&- 352p^5x^3a_3 + 96p^4x^4a_2 - 32p^4x^4b_3 + 32p^3x^5b_2 \\
&+ 32p^4x^3a_1 + 32p^3x^4b_1 + 96p^5xb_3 - 48p^4x^2b_2 - 48p^4xb_1 \\
&- \sqrt{-4p^3x^3+9p^4}\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{5}{3}} a_2 \\
&+ \sqrt{-4p^3x^3+9p^4}\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{5}{3}} b_3 \\
&- 24\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}} p^4b_3 \\
&+ b_2\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{4}{3}} \sqrt{-4p^3x^3+9p^4} \\
&- 24\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}} p^3b_1 \\
&- 96\sqrt{-4p^3x^3+9p^4}\, p^4a_3 + 48\sqrt{-4p^3x^3+9p^4}\, p^3a_1 \\
&+ 8\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}} p^4x^2a_3 \\
&+ 8\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}} p^3x^3a_2 \\
&+ 8\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}} p^3x^3b_3 \\
&+ 8\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}} p^2x^4b_2 \\
&+ 8\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}} p^3x^2a_1 \\
&+ 8\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}} p^2x^3b_1 \\
&- 24\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}} p^3xb_2 \\
&+ 8\sqrt{-4p^3x^3+9p^4}\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}} p^2b_3 \\
&- 32\sqrt{-4p^3x^3+9p^4}\, p^3xb_3 + 16\sqrt{-4p^3x^3+9p^4}\, p^2x^2b_2 \\
&+ 8\sqrt{-4p^3x^3+9p^4}\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}} p b_1 \\
&+ 16\sqrt{-4p^3x^3+9p^4}\, p^2xb_1 \\
&+ 96\sqrt{-4p^3x^3+9p^4}\, p^3xa_2 - 288p^5xa_2 = 0
\end{aligned}
\tag{6E}
$$

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{p, x\}$ in them.

$$\left\{p, x, \sqrt{p^3 (-4x^3 + 9p)}, \left(-12p^2 + 4\sqrt{p^3 (-4x^3 + 9p)}\right)^{\frac{1}{3}}, \left(-12p^2 + 4\sqrt{p^3 (-4x^3 + 9p)}\right)^{\frac{2}{3}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{p, x\}$ in them

$$\left\{p = v_1, x = v_2, \sqrt{p^3 (-4x^3 + 9p)} = v_3, \left(-12p^2 + 4\sqrt{p^3 (-4x^3 + 9p)}\right)^{\frac{1}{3}} = v_4, \left(-12p^2 + 4\sqrt{p^3 (-4x^3 + 9p)}\right)^{\frac{2}{3}} = v_5\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-4v_1\big(-32v_4v_1^3v_2^4a_3 - 24v_1^3v_2^4a_2 + 88v_1^4v_2^3a_3 - 8v_1^2v_2^5b_2 + 8v_1^3v_2^4b_3 \\ &- 8v_1^3v_2^3a_1 - 6v_5v_1^2v_2^3a_2 + 72v_4v_1^4v_2a_3 - 2v_5v_1^3v_2^2a_3 - 16v_3v_1^2v_2^3a_3 - 8v_1^2v_2^4b_1 \\ &+ 4v_4v_1^2v_2^3b_2 - 2v_5v_1v_2^4b_2 + 2v_5v_1^2v_2^3b_3 - 2v_5v_1^2v_2^2a_1 + 72v_1^4v_2a_2 - 180v_1^5a_3 \\ &- 24v_4v_3v_1^2v_2a_3 + 4v_5v_3v_1v_2^2a_3 - 2v_5v_1v_2^3b_1 + 12v_1^3v_2^2b_2 - 24v_1^4v_2b_3 \\ &+ 36v_1^4a_1 + 9v_5v_1^3a_2 - 24v_3v_1^2v_2a_2 + 60v_3v_1^3a_3 + 12v_1^3v_2b_1 - 9v_4v_1^3b_2 \\ &+ 6v_5v_1^2v_2b_2 - 4v_3v_1v_2^2b_2 - 3v_5v_1^3b_3 + 8v_3v_1^2v_2b_3 - 12v_3v_1^2a_1 - 3v_5v_3v_1a_2 \\ &+ 6v_5v_1^2b_1 - 4v_3v_1v_2b_1 + 3v_4v_3v_1b_2 - 2v_5v_3v_2b_2 + v_5v_3v_1b_3 - 2v_5v_3b_1\big) = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$\begin{aligned}
&720a_3v_1^6 - 144a_1v_1^5 - 288a_3v_2v_4v_1^5 + 128a_3v_2^4v_4v_1^4 + 8a_3v_2^2v_5v_1^4 \\
&+ 64a_3v_2^3v_3v_1^3 - 16b_2v_2^3v_4v_1^3 + (24a_2 - 8b_3)\,v_2^3v_5v_1^3 + 8a_1v_2^2v_5v_1^3 \\
&+ (96a_2 - 32b_3)\,v_2v_3v_1^3 - 24b_2v_2v_5v_1^3 + 8b_2v_2^4v_5v_1^2 + 8b_1v_2^3v_5v_1^2 \\
&+ 16b_2v_2^2v_3v_1^2 + 16b_1v_2v_3v_1^2 - 12b_2v_3v_4v_1^2 + (12a_2 - 4b_3)\,v_3v_5v_1^2 \\
&+ 8b_1v_3v_5v_1 + 96a_3v_2v_3v_4v_1^3 - 16a_3v_2^2v_3v_5v_1^2 + 8b_2v_2v_3v_5v_1 \\
&- 352a_3v_2^3v_1^5 + (-288a_2 + 96b_3)\,v_2v_1^5 + (96a_2 - 32b_3)\,v_2^4v_1^4 \\
&+ 32a_1v_2^3v_1^4 - 48b_2v_2^2v_1^4 - 48b_1v_2v_1^4 - 240a_3v_3v_1^4 + 36b_2v_4v_1^4 \\
&+ (-36a_2 + 12b_3)\,v_5v_1^4 + 32b_2v_2^5v_1^3 + 32b_1v_2^4v_1^3 + 48a_1v_3v_1^3 - 24b_1v_5v_1^3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-144a_1 &= 0 \\
8a_1 &= 0 \\
32a_1 &= 0 \\
48a_1 &= 0 \\
-352a_3 &= 0 \\
-288a_3 &= 0 \\
-240a_3 &= 0 \\
-16a_3 &= 0 \\
8a_3 &= 0 \\
64a_3 &= 0 \\
96a_3 &= 0 \\
128a_3 &= 0 \\
720a_3 &= 0 \\
-48b_1 &= 0 \\
-24b_1 &= 0 \\
8b_1 &= 0 \\
16b_1 &= 0 \\
32b_1 &= 0 \\
-48b_2 &= 0 \\
-24b_2 &= 0 \\
-16b_2 &= 0 \\
-12b_2 &= 0 \\
8b_2 &= 0 \\
16b_2 &= 0 \\
32b_2 &= 0 \\
36b_2 &= 0 \\
-288a_2 + 96b_3 &= 0 \\
-36a_2 + 12b_3 &= 0 \\
12a_2 - 4b_3 &= 0 \\
24a_2 - 8b_3 &= 0 \\
96a_2 - 32b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= a_2 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= 3a_2 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= 3p \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, p) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dp}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial p}\right) S(x, p) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned} \frac{dp}{dx} &= \frac{\eta}{\xi} \\ &= \frac{3p}{x} \\ &= \frac{3p}{x} \end{aligned}$$

This is easily solved to give

$$p(x) = c_1 x^3$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{p}{x^3}$$

And S is found from

$$\begin{aligned} dS &= \frac{dx}{\xi} \\ &= \frac{dx}{x} \end{aligned}$$

Integrating gives

$$\begin{aligned} S &= \int \frac{dx}{T} \\ &= \ln(x) \end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,p)S_p}{R_x + \omega(x,p)R_p} \tag{2}$$

Where in the above R_x, R_p, S_x, S_p are all partial derivatives and $\omega(x,p)$ is the right hand side of the original ode given by

$$\omega(x,p) = \frac{\left(-12p^2 + 4\sqrt{-4p^3x^3 + 9p^4}\right)^{\frac{2}{3}} + 4px}{\left(-12p^2 + 4\sqrt{-4p^3x^3 + 9p^4}\right)^{\frac{1}{3}}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= -\frac{3p}{x^4} \\ R_p &= \frac{1}{x^3} \\ S_x &= \frac{1}{x} \\ S_p &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{x^3\left(-12p^2 + 4\sqrt{-4p^3x^3 + 9p^4}\right)^{\frac{1}{3}}}{\left(-12p^2 + 4\sqrt{-4p^3x^3 + 9p^4}\right)^{\frac{2}{3}} x + 4\left(x^2 - \frac{3\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{1}{3}}}{4}\right)p} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, p in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\left(1+i\sqrt{3}\right)2^{\frac{2}{3}}\left(-\sqrt{9R-4}+3\sqrt{R}\right)^{\frac{1}{3}}}{\sqrt{R}\left(8+2\left(i\sqrt{3}-1\right)2^{\frac{1}{3}}\left(-\sqrt{9R-4}+3\sqrt{R}\right)^{\frac{2}{3}}+3\left(-i\sqrt{3}-1\right)\sqrt{R}\,2^{\frac{2}{3}}\left(-\sqrt{9R-4}+3\sqrt{R}\right)^{\frac{1}{3}}\right)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int \frac{\left(1+i\sqrt{3}\right)\left(-4\sqrt{9R-4}+12\sqrt{R}\right.}{\left(-3i\sqrt{R}\,\sqrt{3}\,\left(-4\sqrt{9R-4}+12\sqrt{R}\right)^{\frac{1}{3}}+2i\sqrt{3}\,2^{\frac{1}{3}}\left(\left(-\sqrt{9R-4}+3\sqrt{R}\right)^{2}\right)^{\frac{1}{3}}-3\sqrt{R}\,\left(-4\sqrt{9}\right.\right.} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, p coordinates. This results in

$$\ln(x) = \int^{\frac{p(x)}{x^3}} \frac{\left(1+i\sqrt{3}\right)\left(-4\sqrt{9\_a-4}+\right.}{\left(-3i\sqrt{\_a}\,\sqrt{3}\,\left(-4\sqrt{9\_a-4}+12\sqrt{\_a}\right)^{\frac{1}{3}}+2i\sqrt{3}\,2^{\frac{1}{3}}\left(\left(-\sqrt{9\_a-4}+3\sqrt{\_a}\right)^{2}\right)^{\frac{1}{3}}-3\sqrt{\_a}\right.}$$

Which simplifies to

$$2^{\frac{2}{3}}\left(1+i\sqrt{3}\right)\left(\int^{\frac{p(x)}{x^3}} -\frac{\left(-\sqrt{9\_a-4}+3\sqrt{\_a}\right)^{\frac{1}{3}}}{\sqrt{\_a}\,\left(\left(2i\sqrt{3}-2\right)2^{\frac{1}{3}}\left(\sqrt{9\_a-4}-3\sqrt{\_a}\right)^{\frac{2}{3}}+8-3\sqrt{\_a}\,\left(1+i\sqrt{3}\right)2^{\frac{2}{3}}\left(-\sqrt{9\_a}\right.\right.}\right.$$

Solving equation (2)

Writing the ode as

$$p'(x) = \frac{i\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}}\sqrt{3}-4i\sqrt{3}\,px-\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}}-4px}{2\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{1}{3}}}$$
$$p'(x) = \omega(x, p)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_p - \xi_x) - \omega^2\xi_p - \omega_x\xi - \omega_p\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = pa_3 + xa_2 + a_1 \tag{1E}$$

$$\eta = pb_3 + xb_2 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 \\
&+ \frac{\left(i\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}}\sqrt{3}-4i\sqrt{3}\,px-\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}}-4px\right)(b_3-a_2)}{2\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{1}{3}}} \\
&- \frac{\left(i\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}}\sqrt{3}-4i\sqrt{3}\,px-\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}}-4px\right)^2 a_3}{4\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}}} \\
&- \left(\frac{-\frac{16i\sqrt{3}\,p^3x^2}{\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{1}{3}}\sqrt{-4p^3x^3+9p^4}}-4i\sqrt{3}\,p+\frac{16p^3x^2}{\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{1}{3}}\sqrt{-4p^3x^3+9p^4}}-4p}{2\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{1}{3}}}\right. \\
&\left.+ \frac{4\left(i\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}}\sqrt{3}-4i\sqrt{3}\,px-\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}}-4px\right)p^3x^2}{\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{4}{3}}\sqrt{-4p^3x^3+9p^4}}\right)(pa_3 \\
&+ xa_2 + a_1) - \left(\frac{\frac{2i\sqrt{3}\left(-24p+\frac{-24p^2x^3+72p^3}{\sqrt{-4p^3x^3+9p^4}}\right)}{3\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{1}{3}}}-4i\sqrt{3}\,x-\frac{2\left(-24p+\frac{-24p^2x^3+72p^3}{\sqrt{-4p^3x^3+9p^4}}\right)}{3\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{1}{3}}}-4x}{2\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{1}{3}}}\right. \\
&- \frac{\left(i\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}}\sqrt{3}-4i\sqrt{3}\,px-\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}}-4px\right)\left(-24p+\frac{-24p^2x^{3}}{\sqrt{-4p^3x}}\right.}{6\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{4}{3}}} \\
&+ xb_2 + b_1) = 0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

$$\text{Expression too large to display} \tag{6E}$$

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{p, x\}$ in them.

$$\left\{p, x, \sqrt{p^3 (-4x^3 + 9p)}, \left(-12p^2+4\sqrt{p^3 (-4x^3 + 9p)}\right)^{\frac{1}{3}}, \left(-12p^2+4\sqrt{p^3 (-4x^3 + 9p)}\right)^{\frac{2}{3}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{p, x\}$ in them

$$\left\{p = v_1, x = v_2, \sqrt{p^3 (-4x^3 + 9p)} = v_3, \left(-12p^2 + 4\sqrt{p^3 (-4x^3 + 9p)}\right)^{\frac{1}{3}} = v_4, \left(-12p^2 + 4\sqrt{p^3 (-4x^3 + 9p)}\right)^{\frac{2}{3}} = v_5\right\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
-8v_1\Big(&-36v_1^4a_1+180v_1^5a_3+2v_5v_1^3v_2^2a_3+6v_5v_1^2v_2^3a_2\\
&-2v_5v_1^2v_2^3b_3+2v_5v_1v_2^4b_2+2v_5v_1^2v_2^2a_1-8i\sqrt{3}\,v_1^3v_2^4b_3\\
&+8i\sqrt{3}\,v_1^2v_2^5b_2+8i\sqrt{3}\,v_1^3v_2^3a_1+8i\sqrt{3}\,v_1^2v_2^4b_1+9i\sqrt{3}\,v_5v_1^3a_2\\
&-3i\sqrt{3}\,v_5v_1^3b_3-72i\sqrt{3}\,v_1^4v_2a_2+24i\sqrt{3}\,v_1^4v_2b_3\\
&-12i\sqrt{3}\,v_1^3v_2^2b_2+6i\sqrt{3}\,v_5v_1^2b_1-60i\sqrt{3}\,v_3v_1^3a_3\\
&-12i\sqrt{3}\,v_1^3v_2b_1-2i\sqrt{3}\,v_5v_3b_1+12i\sqrt{3}\,v_3v_1^2a_1-4v_5v_3v_1v_2^2a_3\\
&-48v_4v_3v_1^2v_2a_3-88i\sqrt{3}\,v_1^4v_2^3a_3+24i\sqrt{3}\,v_1^3v_2^4a_2+2v_5v_1v_2^3b_1\\
&+144v_4v_1^4v_2a_3+8v_4v_1^2v_2^3b_2+16v_3v_1^2v_2^3a_3-6v_5v_1^2v_2b_2\\
&+3v_5v_3v_1a_2-v_5v_3v_1b_3+2v_5v_3v_2b_2+24v_3v_1^2v_2a_2-8v_3v_1^2v_2b_3\\
&+4v_3v_1v_2^2b_2+6v_4v_3v_1b_2+4v_3v_1v_2b_1+180i\sqrt{3}\,v_1^5a_3\\
&-36i\sqrt{3}\,v_1^4a_1-88v_1^4v_2^3a_3+24v_1^3v_2^4a_2-8v_1^3v_2^4b_3+8v_1^2v_2^5b_2\\
&+8v_1^3v_2^3a_1+8v_1^2v_2^4b_1+24v_1^4v_2b_3-9v_5v_1^3a_2+3v_5v_1^3b_3-6v_5v_1^2b_1\\
&-18v_4v_1^3b_2-60v_3v_1^3a_3+2v_5v_3b_1+12v_3v_1^2a_1-72v_1^4v_2a_2\\
&-12v_1^3v_2^2b_2-12v_1^3v_2b_1+4i\sqrt{3}\,v_5v_3v_1v_2^2a_3-64v_4v_1^3v_2^4a_3\\
&+i\sqrt{3}\,v_5v_3v_1b_3-2i\sqrt{3}\,v_5v_1^3v_2^2a_3-6i\sqrt{3}\,v_5v_1^2v_2^3a_2\\
&+2i\sqrt{3}\,v_5v_1^2v_2^3b_3-2i\sqrt{3}\,v_5v_1v_2^4b_2-2i\sqrt{3}\,v_5v_1^2v_2^2a_1\\
&-2i\sqrt{3}\,v_5v_1v_2^3b_1+16i\sqrt{3}\,v_3v_1^2v_2^3a_3+6i\sqrt{3}\,v_5v_1^2v_2b_2\\
&-3i\sqrt{3}\,v_5v_3v_1a_2-2i\sqrt{3}\,v_5v_3v_2b_2+24i\sqrt{3}\,v_3v_1^2v_2a_2\\
&-8i\sqrt{3}\,v_3v_1^2v_2b_3+4i\sqrt{3}\,v_3v_1v_2^2b_2+4i\sqrt{3}\,v_3v_1v_2b_1\Big)=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1,v_2,v_3,v_4,v_5\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&-64b_2v_2^3v_4v_1^3 - 48b_2v_3v_4v_1^2 - 1152a_3v_2v_4v_1^5 \\
&+ \left(-128i\sqrt{3}\,a_3 - 128a_3\right) v_2^3v_3v_1^3 \\
&+ \left(48i\sqrt{3}\,a_2 - 16i\sqrt{3}\,b_3 - 48a_2 + 16b_3\right) v_2^3v_5v_1^3 \\
&+ \left(16i\sqrt{3}\,a_1 - 16a_1\right) v_2^2v_5v_1^3 \\
&+ \left(-192i\sqrt{3}\,a_2 + 64i\sqrt{3}\,b_3 - 192a_2 + 64b_3\right) v_2v_3v_1^3 \\
&+ \left(-48i\sqrt{3}\,b_2 + 48b_2\right) v_2v_5v_1^3 \\
&+ \left(16i\sqrt{3}\,b_2 - 16b_2\right) v_2^4v_5v_1^2 + \left(16i\sqrt{3}\,b_1 - 16b_1\right) v_2^3v_5v_1^2 \\
&+ \left(-32i\sqrt{3}\,b_2 - 32b_2\right) v_2^2v_3v_1^2 + \left(-32i\sqrt{3}\,b_1 - 32b_1\right) v_2v_3v_1^2 \\
&+ \left(24i\sqrt{3}\,a_2 - 8i\sqrt{3}\,b_3 - 24a_2 + 8b_3\right) v_3v_5v_1^2 \\
&+ \left(16i\sqrt{3}\,b_1 - 16b_1\right) v_3v_5v_1 \\
&+ \left(16i\sqrt{3}\,a_3 - 16a_3\right) v_2^2v_5v_1^4 + 384a_3v_2v_3v_4v_1^3 \\
&+ 512a_3v_2^4v_4v_1^4 + \left(-1440i\sqrt{3}\,a_3 - 1440a_3\right) v_1^6 \\
&+ \left(288i\sqrt{3}\,a_1 + 288a_1\right) v_1^5 + \left(-32i\sqrt{3}\,a_3 + 32a_3\right) v_2^2v_3v_5v_1^2 \\
&+ \left(16i\sqrt{3}\,b_2 - 16b_2\right) v_2v_3v_5v_1 + \left(704i\sqrt{3}\,a_3 + 704a_3\right) v_2^3v_1^5 \\
&+ \left(576i\sqrt{3}\,a_2 - 192i\sqrt{3}\,b_3 + 576a_2 - 192b_3\right) v_2v_1^5 \\
&+ \left(-192i\sqrt{3}\,a_2 + 64i\sqrt{3}\,b_3 - 192a_2 + 64b_3\right) v_2^4v_1^4 \\
&+ \left(-64i\sqrt{3}\,a_1 - 64a_1\right) v_2^3v_1^4 + \left(96i\sqrt{3}\,b_2 + 96b_2\right) v_2^2v_1^4 \\
&+ \left(96i\sqrt{3}\,b_1 + 96b_1\right) v_2v_1^4 + \left(480i\sqrt{3}\,a_3 + 480a_3\right) v_3v_1^4 \\
&+ \left(-72i\sqrt{3}\,a_2 + 24i\sqrt{3}\,b_3 + 72a_2 - 24b_3\right) v_5v_1^4 \\
&+ \left(-64i\sqrt{3}\,b_2 - 64b_2\right) v_2^5v_1^3 + \left(-64i\sqrt{3}\,b_1 - 64b_1\right) v_2^4v_1^3 \\
&+ \left(-96i\sqrt{3}\,a_1 - 96a_1\right) v_3v_1^3 \\
&+ \left(-48i\sqrt{3}\,b_1 + 48b_1\right) v_5v_1^3 + 144b_2v_4v_1^4 = 0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$-1152a_3 = 0$$
$$384a_3 = 0$$
$$512a_3 = 0$$
$$-64b_2 = 0$$
$$-48b_2 = 0$$
$$144b_2 = 0$$
$$-1440i\sqrt{3}\,a_3 - 1440a_3 = 0$$
$$-128i\sqrt{3}\,a_3 - 128a_3 = 0$$
$$-96i\sqrt{3}\,a_1 - 96a_1 = 0$$
$$-64i\sqrt{3}\,a_1 - 64a_1 = 0$$
$$-64i\sqrt{3}\,b_1 - 64b_1 = 0$$
$$-64i\sqrt{3}\,b_2 - 64b_2 = 0$$
$$-48i\sqrt{3}\,b_1 + 48b_1 = 0$$
$$-48i\sqrt{3}\,b_2 + 48b_2 = 0$$
$$-32i\sqrt{3}\,a_3 + 32a_3 = 0$$
$$-32i\sqrt{3}\,b_1 - 32b_1 = 0$$
$$-32i\sqrt{3}\,b_2 - 32b_2 = 0$$
$$16i\sqrt{3}\,a_1 - 16a_1 = 0$$
$$16i\sqrt{3}\,a_3 - 16a_3 = 0$$
$$16i\sqrt{3}\,b_1 - 16b_1 = 0$$
$$16i\sqrt{3}\,b_2 - 16b_2 = 0$$
$$96i\sqrt{3}\,b_1 + 96b_1 = 0$$
$$96i\sqrt{3}\,b_2 + 96b_2 = 0$$
$$288i\sqrt{3}\,a_1 + 288a_1 = 0$$
$$480i\sqrt{3}\,a_3 + 480a_3 = 0$$
$$704i\sqrt{3}\,a_3 + 704a_3 = 0$$
$$-192i\sqrt{3}\,a_2 + 64i\sqrt{3}\,b_3 - 192a_2 + 64b_3 = 0$$
$$-72i\sqrt{3}\,a_2 + 24i\sqrt{3}\,b_3 + 72a_2 - 24b_3 = 0$$
$$24i\sqrt{3}\,a_2 - 8i\sqrt{3}\,b_3 - 24a_2 + 8b_3 = 0$$
$$48i\sqrt{3}\,a_2 - 16i\sqrt{3}\,b_3 - 48a_2 + 16b_3 = 0$$
$$576i\sqrt{3}\,a_2 - 192i\sqrt{3}\,b_3 + 576a_2 - 192b_3 = 0$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= a_2 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= 3a_2 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= 3p \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, p) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dp}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial p}\right) S(x,p) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Solving equation (3)

Writing the ode as

$$p'(x) = -\frac{i\left(-12p^2 + 4\sqrt{-4p^3x^3 + 9p^4}\right)^{\frac{2}{3}}\sqrt{3} - 4i\sqrt{3}\,px + \left(-12p^2 + 4\sqrt{-4p^3x^3 + 9p^4}\right)^{\frac{2}{3}} + 4px}{2\left(-12p^2 + 4\sqrt{-4p^3x^3 + 9p^4}\right)^{\frac{1}{3}}}$$

$$p'(x) = \omega(x, p)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_p - \xi_x) - \omega^2\xi_p - \omega_x\xi - \omega_p\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = pa_3 + xa_2 + a_1 \tag{1E}$$

$$\eta = pb_3 + xb_2 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{\left(i\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}}\sqrt{3}-4i\sqrt{3}\,px+\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}}+4px\right)(b_3-a_2)}{2\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{1}{3}}} \tag{5E}$$

$$- \frac{\left(i\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}}\sqrt{3}-4i\sqrt{3}\,px+\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}}+4px\right)^2 a_3}{4\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}}}$$

$$- \left(-\frac{-\frac{16i\sqrt{3}\,p^3x^2}{\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{1}{3}}\sqrt{-4p^3x^3+9p^4}}-4i\sqrt{3}\,p-\frac{16p^3x^2}{\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{1}{3}}\sqrt{-4p^3x^3+9p^4}}+4p}{2\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{1}{3}}}\right.$$

$$\left.- \frac{4\left(i\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}}\sqrt{3}-4i\sqrt{3}\,px+\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}}+4px\right)p^3x^2}{\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{4}{3}}\sqrt{-4p^3x^3+9p^4}}\right)(pa_3$$

$$+ xa_2 + a_1) - \left(-\frac{\frac{2i\sqrt{3}\left(-24p+\frac{-24p^2x^3+72p^3}{\sqrt{-4p^3x^3+9p^4}}\right)}{3\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{1}{3}}}-4i\sqrt{3}\,x+\frac{-16p+\frac{2\left(-24p^2x^3+72p^3\right)}{3\sqrt{-4p^3x^3+9p^4}}}{\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{1}{3}}}+4x}{2\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{1}{3}}}\right.$$

$$+ \frac{\left(i\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}}\sqrt{3}-4i\sqrt{3}\,px+\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{2}{3}}+4px\right)\left(-24p+\frac{-24p^2x^3}{\sqrt{-4p^3x}}\right.}{6\left(-12p^2+4\sqrt{-4p^3x^3+9p^4}\right)^{\frac{4}{3}}}$$

$$+ xb_2 + b_1) = 0$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Simplifying the above gives

Expression too large to display (6E)

Since the PDE has radicals, simplifying gives

Expression too large to display

Looking at the above PDE shows the following are all the terms with $\{p, x\}$ in them.

$$\left\{p, x, \sqrt{p^3 (-4x^3 + 9p)}, \left(-12p^2+4\sqrt{p^3 (-4x^3 + 9p)}\right)^{\frac{1}{3}}, \left(-12p^2+4\sqrt{p^3 (-4x^3 + 9p)}\right)^{\frac{2}{3}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{p, x\}$ in them

$$\left\{p = v_1, x = v_2, \sqrt{p^3 (-4x^3 + 9p)} = v_3, \left(-12p^2 + 4\sqrt{p^3 (-4x^3 + 9p)}\right)^{\frac{1}{3}} = v_4, \left(-12p^2 + 4\sqrt{p^3 (-4x^3 + 9p)}\right)^{\frac{2}{3}} = v_5\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
8v_1\Big(&-8i\sqrt{3}\,v_3v_1^2v_2b_3+4i\sqrt{3}\,v_3v_1v_2^2b_2+4i\sqrt{3}\,v_3v_1v_2b_1\\
&+8i\sqrt{3}\,v_1^3v_2^3a_1+8i\sqrt{3}\,v_1^2v_2^4b_1+4v_5v_3v_1v_2^2a_3+48v_4v_3v_1^2v_2a_3\\
&+9i\sqrt{3}\,v_5v_1^3a_2-3i\sqrt{3}\,v_5v_1^3b_3-72i\sqrt{3}\,v_1^4v_2a_2+24i\sqrt{3}\,v_1^4v_2b_3\\
&-12i\sqrt{3}\,v_1^3v_2^2b_2+6i\sqrt{3}\,v_5v_1^2b_1-60i\sqrt{3}\,v_3v_1^3a_3\\
&-12i\sqrt{3}\,v_1^3v_2b_1-2i\sqrt{3}\,v_5v_3b_1+12i\sqrt{3}\,v_3v_1^2a_1\\
&-88i\sqrt{3}\,v_1^4v_2^3a_3+24i\sqrt{3}\,v_1^3v_2^4a_2-8i\sqrt{3}\,v_1^3v_2^4b_3+8i\sqrt{3}\,v_1^2v_2^5b_2\\
&-2v_5v_1^3v_2^2a_3-6v_5v_1^2v_2^3a_2+2v_5v_1^2v_2^3b_3-2v_5v_1v_2^4b_2\\
&-2v_5v_1^2v_2^2a_1-2v_5v_1v_2^3b_1-144v_4v_1^4v_2a_3-8v_4v_1^2v_2^3b_2\\
&-16v_3v_1^2v_2^3a_3+6v_5v_1^2v_2b_2-3v_5v_3v_1a_2+v_5v_3v_1b_3-2v_5v_3v_2b_2\\
&-24v_3v_1^2v_2a_2+8v_3v_1^2v_2b_3-4v_3v_1v_2^2b_2-6v_4v_3v_1b_2\\
&-4v_3v_1v_2b_1+180i\sqrt{3}\,v_1^5a_3-180v_1^5a_3+36v_1^4a_1+8v_1^3v_2^4b_3\\
&-24v_1^3v_2^4a_2+88v_1^4v_2^3a_3-8v_1^2v_2^5b_2-8v_1^3v_2^3a_1-8v_1^2v_2^4b_1\\
&-24v_1^4v_2b_3+9v_5v_1^3a_2-3v_5v_1^3b_3+6v_5v_1^2b_1+18v_4v_1^3b_2\\
&+60v_3v_1^3a_3-2v_5v_3b_1-12v_3v_1^2a_1+72v_1^4v_2a_2+12v_1^3v_2^2b_2\\
&+12v_1^3v_2b_1+i\sqrt{3}\,v_5v_3v_1b_3-2i\sqrt{3}\,v_5v_1^3v_2^2a_3-6i\sqrt{3}\,v_5v_1^2v_2^3a_2\\
&+2i\sqrt{3}\,v_5v_1^2v_2^3b_3-2i\sqrt{3}\,v_5v_1v_2^4b_2-2i\sqrt{3}\,v_5v_1^2v_2^2a_1\\
&-2i\sqrt{3}\,v_5v_1v_2^3b_1+16i\sqrt{3}\,v_3v_1^2v_2^3a_3+6i\sqrt{3}\,v_5v_1^2v_2b_2\\
&-3i\sqrt{3}\,v_5v_3v_1a_2-2i\sqrt{3}\,v_5v_3v_2b_2+24i\sqrt{3}\,v_3v_1^2v_2a_2\\
&+4i\sqrt{3}\,v_5v_3v_1v_2^2a_3+64v_4v_1^3v_2^4a_3-36i\sqrt{3}\,v_1^4a_1\Big)=0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&384a_3v_2v_3v_4v_1^3+\left(32i\sqrt{3}\,a_3+32a_3\right)v_2^2v_3v_5v_1^2\\
&+\left(-16i\sqrt{3}\,b_2-16b_2\right)v_2v_3v_5v_1\\
&+144b_2v_4v_1^4+\left(-704i\sqrt{3}\,a_3+704a_3\right)v_2^3v_1^5\\
&+\left(-576i\sqrt{3}\,a_2+192i\sqrt{3}\,b_3+576a_2-192b_3\right)v_2v_1^5\\
&+\left(192i\sqrt{3}\,a_2-64i\sqrt{3}\,b_3-192a_2+64b_3\right)v_2^4v_1^4\\
&+\left(64i\sqrt{3}\,a_1-64a_1\right)v_2^3v_1^4+\left(-96i\sqrt{3}\,b_2+96b_2\right)v_2^2v_1^4\\
&+\left(-96i\sqrt{3}\,b_1+96b_1\right)v_2v_1^4+\left(-480i\sqrt{3}\,a_3+480a_3\right)v_3v_1^4\\
&+\left(72i\sqrt{3}\,a_2-24i\sqrt{3}\,b_3+72a_2-24b_3\right)v_5v_1^4\\
&+\left(64i\sqrt{3}\,b_2-64b_2\right)v_2^5v_1^3+\left(64i\sqrt{3}\,b_1-64b_1\right)v_2^4v_1^3\\
&+\left(96i\sqrt{3}\,a_1-96a_1\right)v_3v_1^3+\left(48i\sqrt{3}\,b_1+48b_1\right)v_5v_1^3\\
&+\left(1440i\sqrt{3}\,a_3-1440a_3\right)v_1^6+\left(-288i\sqrt{3}\,a_1+288a_1\right)v_1^5\\
&+\left(128i\sqrt{3}\,a_3-128a_3\right)v_2^3v_3v_1^3\\
&+\left(-48i\sqrt{3}\,a_2+16i\sqrt{3}\,b_3-48a_2+16b_3\right)v_2^3v_5v_1^3\\
&+\left(-16i\sqrt{3}\,a_1-16a_1\right)v_2^2v_5v_1^3\\
&+\left(192i\sqrt{3}\,a_2-64i\sqrt{3}\,b_3-192a_2+64b_3\right)v_2v_3v_1^3\\
&+\left(48i\sqrt{3}\,b_2+48b_2\right)v_2v_5v_1^3+\left(-16i\sqrt{3}\,b_2-16b_2\right)v_2^4v_5v_1^2\\
&+\left(-16i\sqrt{3}\,b_1-16b_1\right)v_2^3v_5v_1^2\\
&+\left(32i\sqrt{3}\,b_2-32b_2\right)v_2^2v_3v_1^2+\left(32i\sqrt{3}\,b_1-32b_1\right)v_2v_3v_1^2\\
&+\left(-24i\sqrt{3}\,a_2+8i\sqrt{3}\,b_3-24a_2+8b_3\right)v_3v_5v_1^2\\
&+\left(-16i\sqrt{3}\,b_1-16b_1\right)v_3v_5v_1+\left(-16i\sqrt{3}\,a_3-16a_3\right)v_2^2v_5v_1^4\\
&-64b_2v_2^3v_4v_1^3-48b_2v_3v_4v_1^2-1152a_3v_2v_4v_1^5+512a_3v_2^4v_4v_1^4=0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-1152a_3 &= 0\\
384a_3 &= 0\\
512a_3 &= 0\\
-64b_2 &= 0\\
-48b_2 &= 0\\
144b_2 &= 0\\
-704i\sqrt{3}\,a_3 + 704a_3 &= 0\\
-480i\sqrt{3}\,a_3 + 480a_3 &= 0\\
-288i\sqrt{3}\,a_1 + 288a_1 &= 0\\
-96i\sqrt{3}\,b_1 + 96b_1 &= 0\\
-96i\sqrt{3}\,b_2 + 96b_2 &= 0\\
-16i\sqrt{3}\,a_1 - 16a_1 &= 0\\
-16i\sqrt{3}\,a_3 - 16a_3 &= 0\\
-16i\sqrt{3}\,b_1 - 16b_1 &= 0\\
-16i\sqrt{3}\,b_2 - 16b_2 &= 0\\
32i\sqrt{3}\,a_3 + 32a_3 &= 0\\
32i\sqrt{3}\,b_1 - 32b_1 &= 0\\
32i\sqrt{3}\,b_2 - 32b_2 &= 0\\
48i\sqrt{3}\,b_1 + 48b_1 &= 0\\
48i\sqrt{3}\,b_2 + 48b_2 &= 0\\
64i\sqrt{3}\,a_1 - 64a_1 &= 0\\
64i\sqrt{3}\,b_1 - 64b_1 &= 0\\
64i\sqrt{3}\,b_2 - 64b_2 &= 0\\
96i\sqrt{3}\,a_1 - 96a_1 &= 0\\
128i\sqrt{3}\,a_3 - 128a_3 &= 0\\
1440i\sqrt{3}\,a_3 - 1440a_3 &= 0\\
-576i\sqrt{3}\,a_2 + 192i\sqrt{3}\,b_3 + 576a_2 - 192b_3 &= 0\\
-48i\sqrt{3}\,a_2 + 16i\sqrt{3}\,b_3 - 48a_2 + 16b_3 &= 0\\
-24i\sqrt{3}\,a_2 + 8i\sqrt{3}\,b_3 - 24a_2 + 8b_3 &= 0\\
72i\sqrt{3}\,a_2 - 24i\sqrt{3}\,b_3 + 72a_2 - 24b_3 &= 0\\
192i\sqrt{3}\,a_2 - 64i\sqrt{3}\,b_3 - 192a_2 + 64b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 3a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= 3p
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, p) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dp}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial p}\right) S(x, p) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
2^{\frac{2}{3}}\left(1+i\sqrt{3}\right)\left(\int^{\frac{y'}{x^3}} -\frac{\left(-\sqrt{9\_a-4}+3\sqrt{\_a}\right)^{\frac{1}{3}}}{\sqrt{\_a}\left(\left(2i\sqrt{3}-2\right)2^{\frac{1}{3}}\left(\sqrt{9\_a-4}-3\sqrt{\_a}\right)^{\frac{2}{3}}+8-3\sqrt{\_a}\left(1+i\sqrt{3}\right)2^{\frac{2}{3}}\left(-\sqrt{9\_a-\right.\right.}}
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int \operatorname{RootOf}\left(i\sqrt{3}\,2^{\frac{2}{3}}\left(\int^{\frac{\_Z}{x^3}} \frac{\left(-\sqrt{9\_\right.}{\sqrt{\_a}\left(-2i\sqrt{3}\,2^{\frac{1}{3}}\left(\sqrt{9\_a-4}-3\sqrt{\_a}\right)^{\frac{2}{3}}+3i\sqrt{\_a}\sqrt{3}\,2^{\frac{2}{3}}\left(-\sqrt{9\_a-4}+\right.\right.}\right.\right. \\
&= \int \operatorname{RootOf}\left(i\sqrt{3}\,2^{\frac{2}{3}}\left(\int^{\frac{\_Z}{x^3}} \frac{\left(-\sqrt{9\_\right.}{\sqrt{\_a}\left(-2i\sqrt{3}\,2^{\frac{1}{3}}\left(\sqrt{9\_a-4}-3\sqrt{\_a}\right)^{\frac{2}{3}}+3i\sqrt{\_a}\sqrt{3}\,2^{\frac{2}{3}}\left(-\sqrt{9\_a-4}+\right.\right.}\right.\right.
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y &= \int \operatorname{RootOf}\Bigg(i\sqrt{3}\,2^{\frac{2}{3}}\left(\int^{\frac{\_Z}{x^3}} \frac{\left(-\sqrt{9\_a-\right.}{\sqrt{\_a}\left(-2i\sqrt{3}\,2^{\frac{1}{3}}\left(\sqrt{9\_a-4}-3\sqrt{\_a}\right)^{\frac{2}{3}}+3i\sqrt{\_a}\,\sqrt{3}\,2^{\frac{2}{3}}\left(-\sqrt{9\_a-4}+3\right.\right.} \right. \\
&\quad +2^{\frac{2}{3}}\left(\int^{\frac{\_Z}{x^3}} \frac{\left(-\sqrt{9\_a-4}+3\sqrt{\_a}\right)^{\frac{1}{3}}}{\sqrt{\_a}\left(-2i\sqrt{3}\,2^{\frac{1}{3}}\left(\sqrt{9\_a-4}-3\sqrt{\_a}\right)^{\frac{2}{3}}+3i\sqrt{\_a}\,\sqrt{3}\,2^{\frac{2}{3}}\left(-\sqrt{9\_a-4}+3\sqrt{\_a}\right)^{\frac{1}{3}}+2\,2^{\frac{1}{3}}\left(\right.\right.} \right. \\
&\quad + \ln(x) - c_1\Bigg)\,dx + c_4
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= \int \operatorname{RootOf}\Bigg(i\sqrt{3}\,2^{\frac{2}{3}}\left(\int^{\frac{\_Z}{x^3}} \frac{\left(-\sqrt{9\_a-\right.}{\sqrt{\_a}\left(-2i\sqrt{3}\,2^{\frac{1}{3}}\left(\sqrt{9\_a-4}-3\sqrt{\_a}\right)^{\frac{2}{3}}+3i\sqrt{\_a}\,\sqrt{3}\,2^{\frac{2}{3}}\left(-\sqrt{9\_a-4}+3\right.\right.} \right. \\
&\quad +2^{\frac{2}{3}}\left(\int^{\frac{\_Z}{x^3}} \frac{\left(-\sqrt{9\_a-4}+3\sqrt{\_a}\right)^{\frac{1}{3}}}{\sqrt{\_a}\left(-2i\sqrt{3}\,2^{\frac{1}{3}}\left(\sqrt{9\_a-4}-3\sqrt{\_a}\right)^{\frac{2}{3}}+3i\sqrt{\_a}\,\sqrt{3}\,2^{\frac{2}{3}}\left(-\sqrt{9\_a-4}+3\sqrt{\_a}\right)^{\frac{1}{3}}+2\,2^{\frac{1}{3}}\left(\right.\right.} \right. \\
&\quad + \ln(x) - c_1\Bigg)\,dx + c_4
\end{aligned}
$$

Warning, solution could not be verified

Maple trace

```
`Methods for second order ODEs:
   *** Sublevel 2 ***
   Methods for second order ODEs:
   Successful isolation of d^2y/dx^2: 3 solutions were found. Trying to solve each resulting
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      trying 2nd order, 2 integrating factors of the form mu(x,y)
      trying differential order: 2; missing variables
      `, `-> Computing symmetries using: way = 3
      -> Calling odsolve with the ODE`, diff(_b(_a), _a) = (4*_b(_a)*_a+(-12*_b(_a)^2+4*(_b(
         symmetry methods on request
      `, `1st order, trying reduction of order with given symmetries:`[_a, 3*_b]
```

✓ Solution by Maple

Time used: 0.5 (sec). Leaf size: 174

```
dsolve(diff(y(x),x$2)^3=12*diff(y(x),x)*(x*diff(y(x),x$2)-2*diff(y(x),x)),y(x), singsol=all)
```

$$y(x) = \frac{x^4}{9} + c_1$$
$$y(x) = c_1$$
$$y(x) = \int \operatorname{RootOf}\left(-6\ln(x) - \left(\int^{\_Z} \frac{3\_f\sqrt{\frac{1}{\_f(9\_f-4)}}\, 2^{\frac{1}{3}}\left(\left(3\sqrt{\frac{1}{\_f(9\_f-4)}}\,\_f+1\right)^2(9\_f-4)^4\right)^{\frac{1}{3}} - 2\,2^{\frac{2}{3}}\left(\left(3\sqrt{\frac{1}{\_f(9\_f-4)}}\,\_f+1\right.\right.}{\_f(9\_f-4)} \right) + 6c_1\right) x^3 dx + c_2$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(y''[x])^3==12*y'[x]*(x*y''[x]-2*y'[x]),y[x],x,IncludeSingularSolutions -> True]
```

Not solved

4.39 problem 42

Internal problem ID [6859]
Internal file name [`OUTPUT/6106_Friday_July_29_2022_03_09_42_AM_61425486/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 42.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{3yy'y'' - y'^3 = -1}$$

4.39.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$3yp(y)^2 \left(\frac{d}{dy}p(y)\right) - p(y)^3 = -1$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned} p' &= F(y,p) \\ &= f(y)g(p) \\ &= \frac{p^3-1}{3y\,p^2} \end{aligned}$$

Where $f(y) = \frac{1}{3y}$ and $g(p) = \frac{p^3-1}{p^2}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{p^3-1}{p^2}}\,dp &= \frac{1}{3y}\,dy \\ \int \frac{1}{\frac{p^3-1}{p^2}}\,dp &= \int \frac{1}{3y}\,dy \\ \frac{\ln\left(p^3-1\right)}{3} &= \frac{\ln\left(y\right)}{3} + c_1 \end{aligned}$$

Raising both side to exponential gives

$$\left(p^3-1\right)^{\frac{1}{3}} = \mathrm{e}^{\frac{\ln(y)}{3}+c_1}$$

Which simplifies to

$$\left(p^3-1\right)^{\frac{1}{3}} = c_2 y^{\frac{1}{3}}$$

Which simplifies to

$$\left(p(y)^3-1\right)^{\frac{1}{3}} = c_2 y^{\frac{1}{3}}\mathrm{e}^{c_1}$$

The solution is

$$\left(p(y)^3-1\right)^{\frac{1}{3}} = c_2 y^{\frac{1}{3}}\mathrm{e}^{c_1}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\left(y'^3-1\right)^{\frac{1}{3}} = c_2 y^{\frac{1}{3}}\mathrm{e}^{c_1}$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$
\begin{align}
y' &= \left(y\,\mathrm{e}^{3c_1} c_2^3 + 1\right)^{\frac{1}{3}} \tag{1}\\
y' &= -\frac{\left(y\,\mathrm{e}^{3c_1} c_2^3 + 1\right)^{\frac{1}{3}}}{2} + \frac{i\sqrt{3}\left(y\,\mathrm{e}^{3c_1} c_2^3 + 1\right)^{\frac{1}{3}}}{2} \tag{2}\\
y' &= -\frac{\left(y\,\mathrm{e}^{3c_1} c_2^3 + 1\right)^{\frac{1}{3}}}{2} - \frac{i\sqrt{3}\left(y\,\mathrm{e}^{3c_1} c_2^3 + 1\right)^{\frac{1}{3}}}{2} \tag{3}
\end{align}
$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$
\int \frac{1}{\left(y\,\mathrm{e}^{3c_1} c_2^3 + 1\right)^{\frac{1}{3}}} dy = \int dx
$$

$$
\frac{3\left(y\,\mathrm{e}^{3c_1} c_2^3 + 1\right)^{\frac{2}{3}} \mathrm{e}^{-3c_1}}{2c_2^3} = x + c_3
$$

Solving equation (2)

Integrating both sides gives

$$
\int \frac{1}{-\frac{\left(y\,\mathrm{e}^{3c_1} c_2^3+1\right)^{\frac{1}{3}}}{2} + \frac{i\sqrt{3}\left(y\,\mathrm{e}^{3c_1} c_2^3+1\right)^{\frac{1}{3}}}{2}} dy = \int dx
$$

$$
\frac{3\left(y\,\mathrm{e}^{3c_1} c_2^3 + 1\right)^{\frac{2}{3}} \mathrm{e}^{-3c_1}}{\left(i\sqrt{3} - 1\right) c_2^3} = x + c_4
$$

Solving equation (3)

Integrating both sides gives

$$
\int \frac{1}{-\frac{\left(y\,\mathrm{e}^{3c_1} c_2^3+1\right)^{\frac{1}{3}}}{2} - \frac{i\sqrt{3}\left(y\,\mathrm{e}^{3c_1} c_2^3+1\right)^{\frac{1}{3}}}{2}} dy = \int dx
$$

$$
-\frac{3\left(y\,\mathrm{e}^{3c_1} c_2^3 + 1\right)^{\frac{2}{3}} \mathrm{e}^{-3c_1}}{\left(1 + i\sqrt{3}\right) c_2^3} = x + c_5
$$

Summary
The solution(s) found are the following

$$y = -\frac{\left(-4\left(\mathrm{e}^{3c_1} c_2^3 (x+c_3)\right)^{\frac{3}{2}} + 3\sqrt{6}\right) \mathrm{e}^{-3c_1} \sqrt{6}}{18c_2^3} \tag{1}$$

$$y = \frac{\left(\left(3i\sqrt{3}\,\mathrm{e}^{3c_1} c_2^3 c_4 + 3i\sqrt{3}\,\mathrm{e}^{3c_1} c_2^3 x - 3c_4 \mathrm{e}^{3c_1} c_2^3 - 3\,\mathrm{e}^{3c_1} c_2^3 x\right)^{\frac{3}{2}} - 27\right) \mathrm{e}^{-3c_1}}{27c_2^3} \tag{2}$$

$$y = \frac{\left(\left(-3i\sqrt{3}\,\mathrm{e}^{3c_1} c_2^3 c_5 - 3i\sqrt{3}\,\mathrm{e}^{3c_1} c_2^3 x - 3c_5 \mathrm{e}^{3c_1} c_2^3 - 3\,\mathrm{e}^{3c_1} c_2^3 x\right)^{\frac{3}{2}} - 27\right) \mathrm{e}^{-3c_1}}{27c_2^3} \tag{3}$$

Verification of solutions

$$y = -\frac{\left(-4\left(\mathrm{e}^{3c_1} c_2^3 (x+c_3)\right)^{\frac{3}{2}} + 3\sqrt{6}\right) \mathrm{e}^{-3c_1} \sqrt{6}}{18c_2^3}$$

Verified OK.

$$y = \frac{\left(\left(3i\sqrt{3}\,\mathrm{e}^{3c_1} c_2^3 c_4 + 3i\sqrt{3}\,\mathrm{e}^{3c_1} c_2^3 x - 3c_4 \mathrm{e}^{3c_1} c_2^3 - 3\,\mathrm{e}^{3c_1} c_2^3 x\right)^{\frac{3}{2}} - 27\right) \mathrm{e}^{-3c_1}}{27c_2^3}$$

Verified OK.

$$y = \frac{\left(\left(-3i\sqrt{3}\,\mathrm{e}^{3c_1} c_2^3 c_5 - 3i\sqrt{3}\,\mathrm{e}^{3c_1} c_2^3 x - 3c_5 \mathrm{e}^{3c_1} c_2^3 - 3\,\mathrm{e}^{3c_1} c_2^3 x\right)^{\frac{3}{2}} - 27\right) \mathrm{e}^{-3c_1}}{27c_2^3}$$

Verified OK.

4.39.2 Maple step by step solution

Let's solve

$3yy'y'' - y'^3 = -1$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

$y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

$u(y)\left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

$3yu(y)^2\left(\frac{d}{dy}u(y)\right) - u(y)^3 = -1$

- Separate variables

$\frac{\left(\frac{d}{dy}u(y)\right)u(y)^2}{u(y)^3-1} = \frac{1}{3y}$

- Integrate both sides with respect to y

$\int \frac{\left(\frac{d}{dy}u(y)\right)u(y)^2}{u(y)^3-1}dy = \int \frac{1}{3y}dy + c_1$

- Evaluate integral

$\frac{\ln\left(u(y)^3-1\right)}{3} = \frac{\ln(y)}{3} + c_1$

- Solve for $u(y)$

$u(y) = \frac{\left((\mathrm{e}^{-3c_1}+y)\left(\mathrm{e}^{-3c_1}\right)^2\right)^{\frac{1}{3}}}{\mathrm{e}^{-3c_1}}$

- Solve 1st ODE for $u(y)$

$u(y) = \frac{\left((\mathrm{e}^{-3c_1}+y)\left(\mathrm{e}^{-3c_1}\right)^2\right)^{\frac{1}{3}}}{\mathrm{e}^{-3c_1}}$

- Revert to original variables with substitution $u(y) = y', y = y$

$y' = \frac{\left((\mathrm{e}^{-3c_1}+y)\left(\mathrm{e}^{-3c_1}\right)^2\right)^{\frac{1}{3}}}{\mathrm{e}^{-3c_1}}$

- Separate variables

$\frac{y'}{\left((\mathrm{e}^{-3c_1}+y)\left(\mathrm{e}^{-3c_1}\right)^2\right)^{\frac{1}{3}}} = \frac{1}{\mathrm{e}^{-3c_1}}$

- Integrate both sides with respect to x

$\int \frac{y'}{\left((\mathrm{e}^{-3c_1}+y)\left(\mathrm{e}^{-3c_1}\right)^2\right)^{\frac{1}{3}}}dx = \int \frac{1}{\mathrm{e}^{-3c_1}}dx + c_2$

- Evaluate integral

$\frac{3\left(\left(\mathrm{e}^{-3c_1}\right)^2 y+\left(\mathrm{e}^{-3c_1}\right)^3\right)^{\frac{2}{3}}}{2\left(\mathrm{e}^{-3c_1}\right)^2} = \frac{x}{\mathrm{e}^{-3c_1}} + c_2$

- Solve for y

$$\left\{RootOf\left(2c_2(\mathrm{e}^{-3c_1})^2+2x\,\mathrm{e}^{-3c_1}-3\left((\mathrm{e}^{-3c_1})^2\_Z+(\mathrm{e}^{-3c_1})^3\right)^{\frac{2}{3}}\right)\right\}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-(1/3)*(_b(_a)^3-1)/(_b(_a)*_a) =
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, 0]
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 119

```
dsolve(3*y(x)*diff(y(x),x)*diff(y(x),x$2)=diff(y(x),x)^3-1,y(x), singsol=all)
```

$$\frac{3(c_1y(x)+1)^{\frac{2}{3}}+(-2x-2c_2)\,c_1}{2c_1}=0$$

$$\frac{-i(x+c_2)\,c_1\sqrt{3}+(-x-c_2)\,c_1-3(c_1y(x)+1)^{\frac{2}{3}}}{c_1\left(1+i\sqrt{3}\right)}=0$$

$$\frac{-3i(c_1y(x)+1)^{\frac{2}{3}}+(-x-c_2)\,c_1\sqrt{3}-i(x+c_2)\,c_1}{c_1\left(\sqrt{3}+i\right)}=0$$

✓ Solution by Mathematica

Time used: 45.036 (sec). Leaf size: 126

```
DSolve[3*y[x]*y'[x]*y''[x]==(y'[x])^3-1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{9}e^{-3c_1}\left(-9+2\sqrt{6}\left(e^{3c_1}(x+c_2)\right)^{3/2}\right)$$
$$y(x) \to \frac{1}{9}e^{-3c_1}\left(-9+2\sqrt{6}\left(-\sqrt[3]{-1}e^{3c_1}(x+c_2)\right)^{3/2}\right)$$
$$y(x) \to \frac{1}{9}e^{-3c_1}\left(-9+2\sqrt{6}\left((-1)^{2/3}e^{3c_1}(x+c_2)\right)^{3/2}\right)$$

4.40 problem 43

Internal problem ID [6860]
Internal file name [`OUTPUT/6107_Friday_July_29_2022_03_09_43_AM_35066495/index.tex`]

Book: Elementary differential equations. By Earl D. Rainville, Phillip E. Bedient. Macmilliam Publishing Co. NY. 6th edition. 1981.
Section: CHAPTER 16. Nonlinear equations. Section 101. Independent variable missing. EXERCISES Page 324
Problem number: 43.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second__order__ode__missing__x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{4y{y'}^{2}y''-{y'}^{4}=3}$$

4.40.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$4yp(y)^{3}\left(\frac{d}{dy}p(y)\right)-p(y)^{4}=3$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$
\begin{aligned}
p' &= F(y, p) \\
&= f(y)g(p) \\
&= \frac{p^4+3}{4y\,p^3}
\end{aligned}
$$

Where $f(y) = \frac{1}{4y}$ and $g(p) = \frac{p^4+3}{p^3}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{p^4+3}{p^3}}\,dp &= \frac{1}{4y}\,dy \\
\int \frac{1}{\frac{p^4+3}{p^3}}\,dp &= \int \frac{1}{4y}\,dy \\
\frac{\ln(p^4+3)}{4} &= \frac{\ln(y)}{4} + c_1
\end{aligned}
$$

Raising both side to exponential gives

$$
\left(p^4+3\right)^{\frac{1}{4}} = e^{\frac{\ln(y)}{4}+c_1}
$$

Which simplifies to

$$
\left(p^4+3\right)^{\frac{1}{4}} = c_2 y^{\frac{1}{4}}
$$

Which simplifies to

$$
\left(p(y)^4+3\right)^{\frac{1}{4}} = c_2 y^{\frac{1}{4}} e^{c_1}
$$

The solution is

$$
\left(p(y)^4+3\right)^{\frac{1}{4}} = c_2 y^{\frac{1}{4}} e^{c_1}
$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
\left(y'^4+3\right)^{\frac{1}{4}} = c_2 y^{\frac{1}{4}} e^{c_1}
$$

Solving the given ode for y' results in 4 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \left(c_2^4 y\, \mathrm{e}^{4c_1} - 3\right)^{\frac{1}{4}} \tag{1}$$

$$y' = i\left(c_2^4 y\, \mathrm{e}^{4c_1} - 3\right)^{\frac{1}{4}} \tag{2}$$

$$y' = -\left(c_2^4 y\, \mathrm{e}^{4c_1} - 3\right)^{\frac{1}{4}} \tag{3}$$

$$y' = -i\left(c_2^4 y\, \mathrm{e}^{4c_1} - 3\right)^{\frac{1}{4}} \tag{4}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\left(c_2^4 y\, \mathrm{e}^{4c_1} - 3\right)^{\frac{1}{4}}} dy = \int dx$$

$$\frac{4\left(c_2^4 y\, \mathrm{e}^{4c_1} - 3\right)^{\frac{3}{4}} \mathrm{e}^{-4c_1}}{3c_2^4} = x + c_3$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{i}{\left(c_2^4 y\, \mathrm{e}^{4c_1} - 3\right)^{\frac{1}{4}}} dy = \int dx$$

$$-\frac{4i\left(c_2^4 y\, \mathrm{e}^{4c_1} - 3\right)^{\frac{3}{4}} \mathrm{e}^{-4c_1}}{3c_2^4} = x + c_4$$

Solving equation (3)

Integrating both sides gives

$$\int -\frac{1}{\left(c_2^4 y\, \mathrm{e}^{4c_1} - 3\right)^{\frac{1}{4}}} dy = \int dx$$

$$-\frac{4\left(c_2^4 y\, \mathrm{e}^{4c_1} - 3\right)^{\frac{3}{4}} \mathrm{e}^{-4c_1}}{3c_2^4} = x + c_5$$

Solving equation (4)

Integrating both sides gives

$$\int \frac{i}{\left(c_2^4 y\, \mathrm{e}^{4c_1}-3\right)^{\frac{1}{4}}} dy = \int dx$$

$$\frac{4i\left(c_2^4 y\, \mathrm{e}^{4c_1}-3\right)^{\frac{3}{4}} \mathrm{e}^{-4c_1}}{3c_2^4} = x + c_6$$

Summary
The solution(s) found are the following

$$y = \frac{\left(3\left(\mathrm{e}^{4c_1} c_2^4 (x+c_3)\right)^{\frac{4}{3}} + 4\, 6^{\frac{2}{3}}\right) 6^{\frac{1}{3}}\, \mathrm{e}^{-4c_1}}{8c_2^4} \tag{1}$$

$$y = \frac{\left(\frac{3\, 3^{\frac{1}{3}} 4^{\frac{2}{3}} \left(i\mathrm{e}^{4c_1} c_2^4 (x+c_4)\right)^{\frac{4}{3}}}{16} + 3\right) \mathrm{e}^{-4c_1}}{c_2^4} \tag{2}$$

$$y = \frac{\left(\left(-\frac{3c_5 \mathrm{e}^{4c_1} c_2^4}{4} - \frac{3\, \mathrm{e}^{4c_1} c_2^4 x}{4}\right)^{\frac{4}{3}} + 3\right) \mathrm{e}^{-4c_1}}{c_2^4} \tag{3}$$

$$y = \frac{\left(\left(-\frac{3i\mathrm{e}^{4c_1} c_2^4 (x+c_6)}{4}\right)^{\frac{4}{3}} + 3\right) \mathrm{e}^{-4c_1}}{c_2^4} \tag{4}$$

Verification of solutions

$$y = \frac{\left(3(e^{4c_1}c_2^4(x+c_3))^{\frac{4}{3}} + 4\,6^{\frac{2}{3}}\right)6^{\frac{1}{3}}e^{-4c_1}}{8c_2^4}$$

Verified OK.

$$y = \frac{\left(\frac{3\,3^{\frac{1}{3}}4^{\frac{2}{3}}\left(ie^{4c_1}c_2^4(x+c_4)\right)^{\frac{4}{3}}}{16} + 3\right)e^{-4c_1}}{c_2^4}$$

Verified OK.

$$y = \frac{\left(\left(-\frac{3c_5e^{4c_1}c_2^4}{4} - \frac{3\,e^{4c_1}c_2^4x}{4}\right)^{\frac{4}{3}} + 3\right)e^{-4c_1}}{c_2^4}$$

Verified OK.

$$y = \frac{\left(\left(-\frac{3ie^{4c_1}c_2^4(x+c_6)}{4}\right)^{\frac{4}{3}} + 3\right)e^{-4c_1}}{c_2^4}$$

Verified OK.

4.40.2 Maple step by step solution

Let's solve

$4yy''y'^2 - y'^4 = 3$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $4yu(y)^3\left(\frac{d}{dy}u(y)\right) - u(y)^4 = 3$
- Separate variables

  $\frac{\left(\frac{d}{dy}u(y)\right)u(y)^3}{u(y)^4+3} = \frac{1}{4y}$
- Integrate both sides with respect to y

  $\int \frac{\left(\frac{d}{dy}u(y)\right)u(y)^3}{u(y)^4+3}dy = \int \frac{1}{4y}dy + c_1$
- Evaluate integral

  $\frac{\ln\left(u(y)^4+3\right)}{4} = \frac{\ln(y)}{4} + c_1$
- Solve for $u(y)$

  $\left\{u(y) = \left(y(\mathrm{e}^{c_1})^4 - 3\right)^{\frac{1}{4}}, u(y) = -\left(y(\mathrm{e}^{c_1})^4 - 3\right)^{\frac{1}{4}}\right\}$
- Solve 1st ODE for $u(y)$

  $u(y) = \left(y(\mathrm{e}^{c_1})^4 - 3\right)^{\frac{1}{4}}$
- Revert to original variables with substitution $u(y) = y', y = y$

  $y' = \left((\mathrm{e}^{c_1})^4 y - 3\right)^{\frac{1}{4}}$
- Separate variables

  $\frac{y'}{\left((\mathrm{e}^{c_1})^4 y-3\right)^{\frac{1}{4}}} = 1$
- Integrate both sides with respect to x

  $\int \frac{y'}{\left((\mathrm{e}^{c_1})^4 y-3\right)^{\frac{1}{4}}}dx = \int 1dx + c_2$
- Evaluate integral

  $\frac{4\left((\mathrm{e}^{c_1})^4 y-3\right)^{\frac{3}{4}}}{3(\mathrm{e}^{c_1})^4} = x + c_2$
- Solve for y

  $y = \frac{\left(3\left((\mathrm{e}^{c_1})^4(x+c_2)\right)^{\frac{4}{3}}+4\,6^{\frac{2}{3}}\right)6^{\frac{1}{3}}}{8(\mathrm{e}^{c_1})^4}$
- Solve 2nd ODE for $u(y)$

  $u(y) = -\left(y(\mathrm{e}^{c_1})^4 - 3\right)^{\frac{1}{4}}$

- Revert to original variables with substitution $u(y) = y', y = y$

$$y' = -\left(\left(e^{c_1}\right)^4 y - 3\right)^{\frac{1}{4}}$$

- Separate variables

$$\frac{y'}{\left(\left(e^{c_1}\right)^4 y - 3\right)^{\frac{1}{4}}} = -1$$

- Integrate both sides with respect to x

$$\int \frac{y'}{\left(\left(e^{c_1}\right)^4 y - 3\right)^{\frac{1}{4}}} dx = \int (-1)\, dx + c_2$$

- Evaluate integral

$$\frac{4\left(\left(e^{c_1}\right)^4 y - 3\right)^{\frac{3}{4}}}{3\left(e^{c_1}\right)^4} = -x + c_2$$

- Solve for y

$$y = \frac{\left(3\left(\left(e^{c_1}\right)^4 (-x + c_2)\right)^{\frac{4}{3}} + 4\, 6^{\frac{2}{3}}\right) 6^{\frac{1}{3}}}{8\left(e^{c_1}\right)^4}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-(1/4)*(_b(_a)^4+3)/(_a*_b(_a)^2)
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, 0]
```

✓ Solution by Maple

Time used: 0.079 (sec). Leaf size: 111

```
dsolve(4*y(x)*diff(y(x),x)^2*diff(y(x),x$2)=diff(y(x),x)^4+3,y(x), singsol=all)
```

$$\frac{-4(c_1y(x)-3)^{\frac{3}{4}}+(-3x-3c_2)\,c_1}{3c_1}=0$$
$$\frac{4(c_1y(x)-3)^{\frac{3}{4}}+(-3x-3c_2)\,c_1}{3c_1}=0$$
$$\frac{-4i(c_1y(x)-3)^{\frac{3}{4}}+(-3x-3c_2)\,c_1}{3c_1}=0$$
$$\frac{4i(c_1y(x)-3)^{\frac{3}{4}}+(-3x-3c_2)\,c_1}{3c_1}=0$$

✓ Solution by Mathematica

Time used: 60.242 (sec). Leaf size: 156

```
DSolve[4*y[x]*(y'[x])^2*y''[x]==(y'[x])^4+3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to \frac{3}{8}e^{-4c_1}\left(8+\sqrt[3]{6}\left(-e^{4c_1}(x+c_2)\right)^{4/3}\right)$$
$$y(x)\to \frac{3}{8}e^{-4c_1}\left(8+\sqrt[3]{6}\left(-ie^{4c_1}(x+c_2)\right)^{4/3}\right)$$
$$y(x)\to \frac{3}{8}e^{-4c_1}\left(8+\sqrt[3]{6}\left(ie^{4c_1}(x+c_2)\right)^{4/3}\right)$$
$$y(x)\to \frac{3}{8}e^{-4c_1}\left(8+\sqrt[3]{6}\left(e^{4c_1}(x+c_2)\right)^{4/3}\right)$$