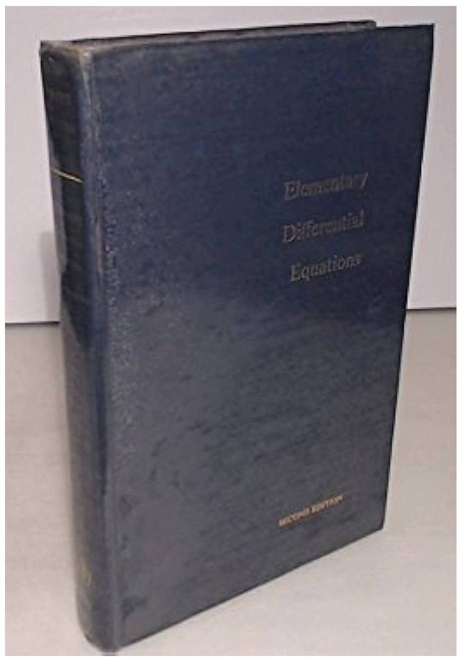
A Solution Manual For

Elementary Differential equations, Chaundy, 1969



Nasser M. Abbasi

May 15, 2024

Contents

1 Exercises 3, page 60

1.1 problem 1(a)

Internal problem ID [3029]
Internal file name [`OUTPUT/2521_Sunday_June_05_2022_03_18_01_AM_48492365/index.tex`]

Book: Elementary Differential equations, Chaundy, 1969
Section: Exercises 3, page 60
Problem number: 1(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differential-Type", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{yy' = x}$$

1.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x}{y} \end{aligned}$$

Where $f(x) = x$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\, dy = x\, dx$$

$$\int \frac{1}{\frac{1}{y}}\,dy = \int x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + c_1$$

Which results in

$$y = \sqrt{x^2 + 2c_1}$$

$$y = -\sqrt{x^2 + 2c_1}$$

Summary

The solution(s) found are the following

$$y = \sqrt{x^2 + 2c_1} \tag{1}$$

$$y = -\sqrt{x^2 + 2c_1} \tag{2}$$

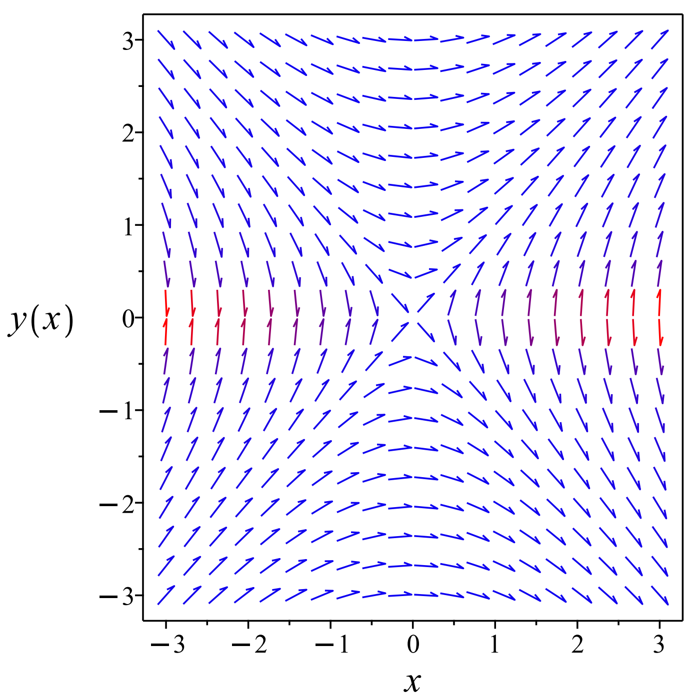


Figure 1: Slope field plot

Verification of solutions

$$y = \sqrt{x^2 + 2c_1}$$

Verified OK.

$$y = -\sqrt{x^2 + 2c_1}$$

Verified OK.

1.1.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x(u'(x)\,x + u(x)) = x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u^2 - 1}{ux} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2-1}{u}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{u^2-1}{u}}\,du &= -\frac{1}{x}\,dx \\ \int \frac{1}{\frac{u^2-1}{u}}\,du &= \int -\frac{1}{x}\,dx \\ \frac{\ln(u-1)}{2} + \frac{\ln(u+1)}{2} &= -\ln(x) + c_2 \end{aligned}$$

The above can be written as

$$\begin{aligned} \left(\frac{1}{2}\right)(\ln(u-1) + \ln(u+1)) &= -\ln(x) + 2c_2 \\ \ln(u-1) + \ln(u+1) &= (2)(-\ln(x) + 2c_2) \\ &= -2\ln(x) + 4c_2 \end{aligned}$$

Raising both side to exponential gives

$$e^{\ln(u-1)+\ln(u+1)} = e^{-2\ln(x)+2c_2}$$

Which simplifies to

$$u^2 - 1 = \frac{2c_2}{x^2}$$
$$= \frac{c_3}{x^2}$$

The solution is

$$u(x)^2 - 1 = \frac{c_3}{x^2}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y^2}{x^2} - 1 = \frac{c_3}{x^2}$$
$$\frac{y^2}{x^2} - 1 = \frac{c_3}{x^2}$$

Which simplifies to

$$-(-y + x)(y + x) = c_3$$

Summary

The solution(s) found are the following

$$-(-y + x)(y + x) = c_3 \tag{1}$$

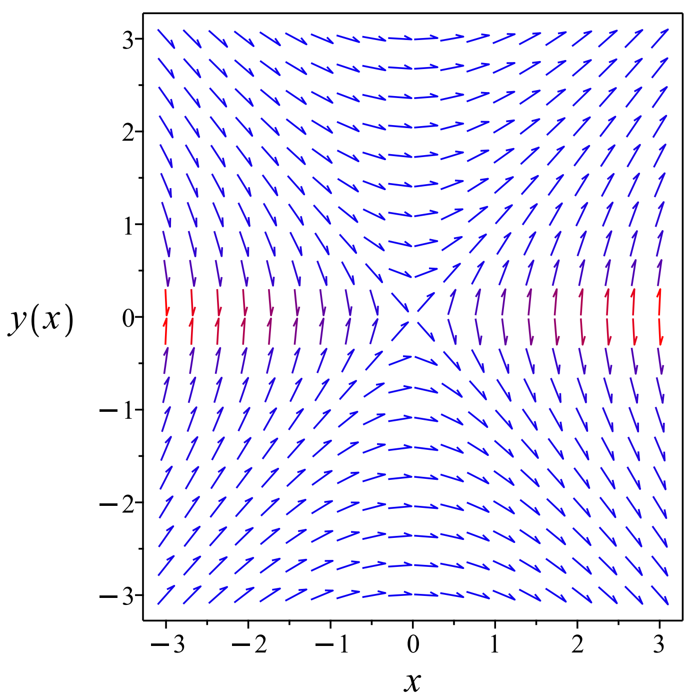


Figure 2: Slope field plot

Verification of solutions

$$-(-y+x)(y+x)=c_3$$

Verified OK.

1.1.3 Solving as differentialType ode

Writing the ode as

$$y'=\frac{x}{y} \tag{1}$$

Which becomes

$$(y)\,dy=(x)\,dx \tag{2}$$

But the RHS is complete differential because

$$(x)\,dx=d\left(\frac{x^2}{2}\right)$$

Hence (2) becomes

$$(y)\,dy=d\left(\frac{x^2}{2}\right)$$

Integrating both sides gives gives these solutions

$$y=\sqrt{x^2+2c_1}+c_1$$
$$y=-\sqrt{x^2+2c_1}+c_1$$

Summary

The solution(s) found are the following

$$y=\sqrt{x^2+2c_1}+c_1 \tag{1}$$
$$y=-\sqrt{x^2+2c_1}+c_1 \tag{2}$$

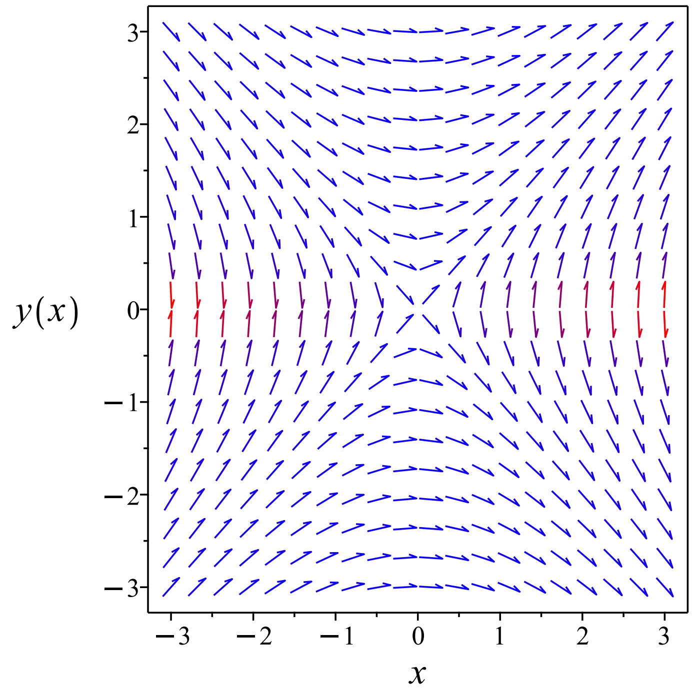


Figure 3: Slope field plot

<u>Verification of solutions</u>

$$y = \sqrt{x^2 + 2c_1} + c_1$$

Verified OK.

$$y = -\sqrt{x^2 + 2c_1} + c_1$$

Verified OK.

1.1.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x}{y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 1: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= \frac{1}{x} \\ \eta(x,y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{x}} dx \end{aligned}$$

Which results in

$$S = \frac{x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

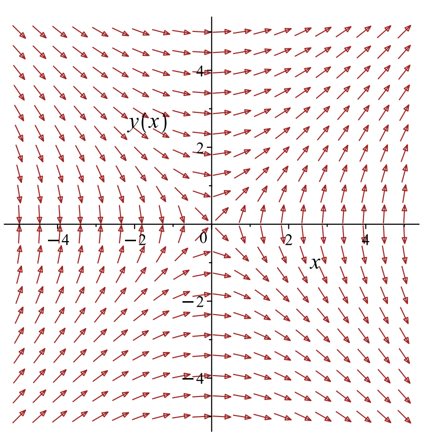
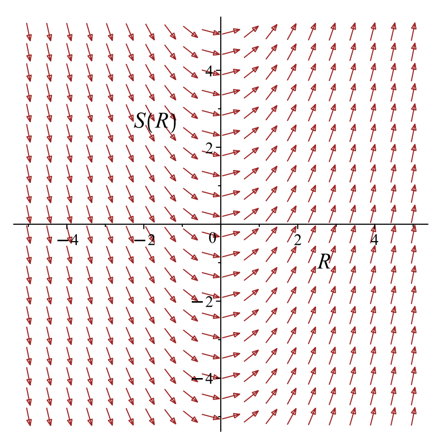
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^2}{2} = \frac{y^2}{2} + c_1$$

Which simplifies to

$$\frac{x^2}{2} = \frac{y^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x}{y}$ 	$R = y$ $S = \frac{x^2}{2}$	$\frac{dS}{dR} = R$ 

Summary
The solution(s) found are the following

$$\frac{x^2}{2} = \frac{y^2}{2} + c_1 \tag{1}$$

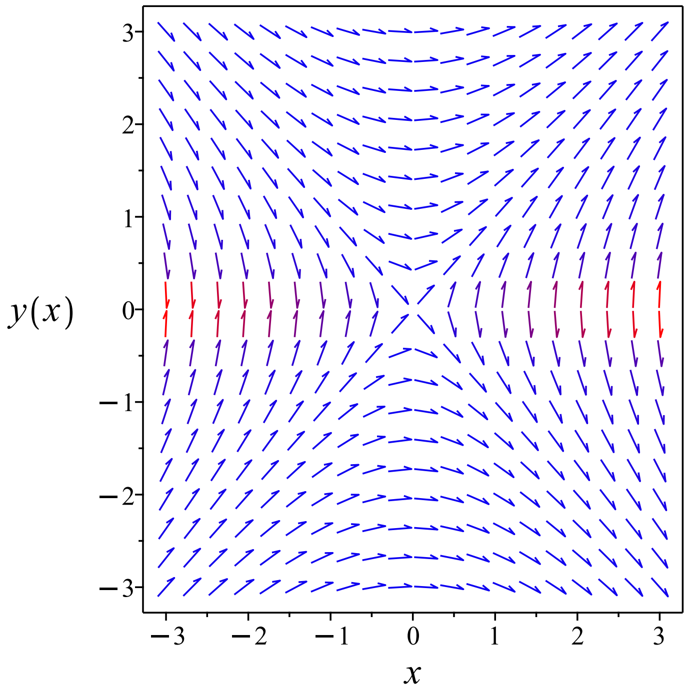


Figure 4: Slope field plot

Verification of solutions

$$\frac{x^2}{2} = \frac{y^2}{2} + c_1$$

Verified OK.

1.1.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(y)\,\mathrm{d}y &= (x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x + (y)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x\\ N(x, y) &= y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int -x \,\mathrm{d}x$$

$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y$. Therefore equation (4) becomes

$$y = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int (y) \,\mathrm{d}y$$

$$f(y) = \frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \frac{y^2}{2}$$

Summary

The solution(s) found are the following

$$-\frac{x^2}{2} + \frac{y^2}{2} = c_1 \tag{1}$$

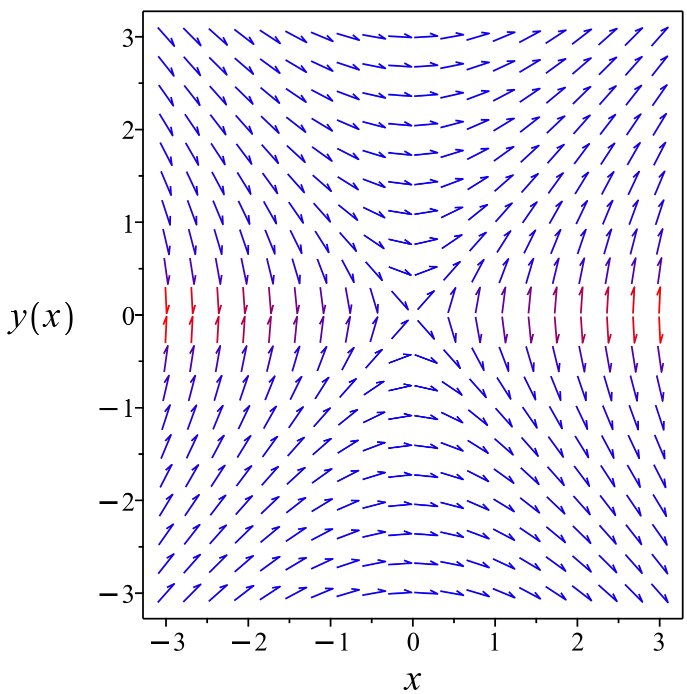


Figure 5: Slope field plot

Verification of solutions

$$-\frac{x^2}{2} + \frac{y^2}{2} = c_1$$

Verified OK.

1.1.6 Maple step by step solution

Let's solve

$yy' = x$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int yy' dx = \int x dx + c_1$

- Evaluate integral

  $\frac{y^2}{2} = \frac{x^2}{2} + c_1$

- Solve for y

  $\left\{y = \sqrt{x^2 + 2c_1}, y = -\sqrt{x^2 + 2c_1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve(diff(y(x),x)*y(x)=x,y(x), singsol=all)
```

$$y(x) = \sqrt{x^2 + c_1}$$
$$y(x) = -\sqrt{x^2 + c_1}$$

✓ Solution by Mathematica

Time used: 0.091 (sec). Leaf size: 35

```
DSolve[y'[x]*y[x]==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{x^2 + 2c_1}$$
$$y(x) \to \sqrt{x^2 + 2c_1}$$

1.2 problem 1(b)

Internal problem ID [3030]
Internal file name [`OUTPUT/2522_Sunday_June_05_2022_03_18_03_AM_29034540/index.tex`]

Book: Elementary Differential equations, Chaundy, 1969
Section: Exercises 3, page 60
Problem number: 1(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y = x^3}$$

1.2.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -1 \\ q(x) &= x^3 \end{aligned}$$

Hence the ode is

$$y' - y = x^3$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int (-1)dx} \\ &= e^{-x} \end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(x^3\right)\\ \frac{d}{dx}\left(e^{-x} y\right) &= \left(e^{-x}\right)\left(x^3\right)\\ d\left(e^{-x} y\right) &= \left(x^3 e^{-x}\right)\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{-x} y &= \int x^3 e^{-x}\, dx\\ e^{-x} y &= -\left(x^3+3x^2+6x+6\right) e^{-x} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-x}$ results in

$$y = -e^{x}\left(x^3+3x^2+6x+6\right) e^{-x} + c_1 e^{x}$$

which simplifies to

$$y = -x^3 - 3x^2 - 6x - 6 + c_1 e^{x}$$

Summary

The solution(s) found are the following

$$y = -x^3 - 3x^2 - 6x - 6 + c_1 e^{x} \tag{1}$$

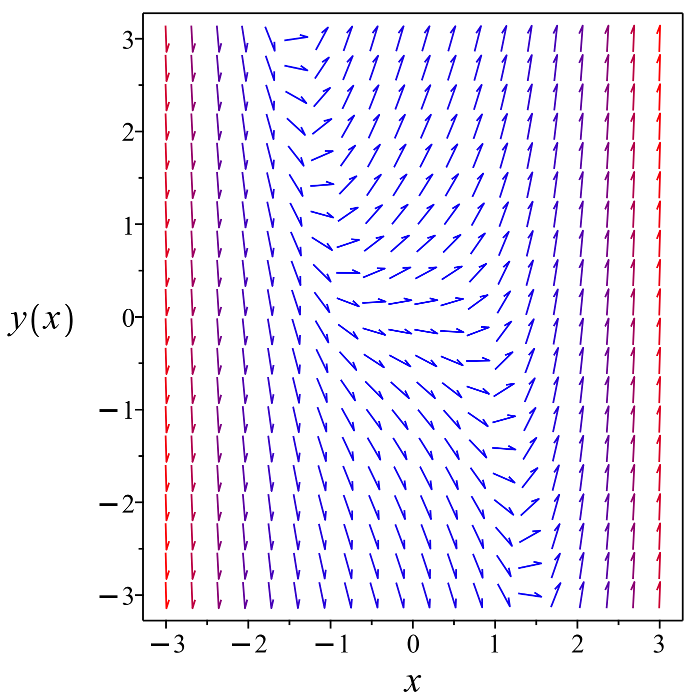


Figure 6: Slope field plot

Verification of solutions

$$y = -x^3 - 3x^2 - 6x - 6 + c_1 e^x$$

Verified OK.

1.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = x^3 + y$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 4: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{x}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^x} dy \end{aligned}$$

Which results in

$$S = e^{-x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = x^3 + y$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -e^{-x} y \\ S_y &= e^{-x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x^3 e^{-x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^3 e^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\left(R^3 + 3R^2 + 6R + 6\right) \mathrm{e}^{-R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{-x} y = -\left(x^3 + 3x^2 + 6x + 6\right) \mathrm{e}^{-x} + c_1$$

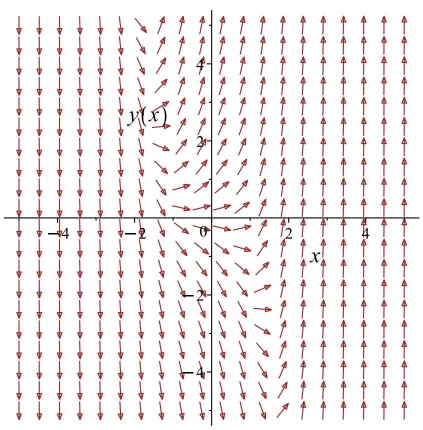
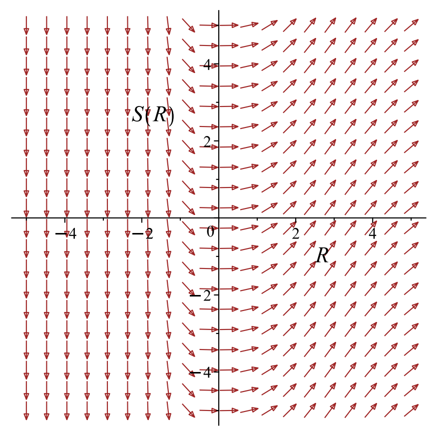
Which simplifies to

$$\left(x^3 + 3x^2 + 6x + y + 6\right) \mathrm{e}^{-x} - c_1 = 0$$

Which gives

$$y = -\left(x^3 \mathrm{e}^{-x} + 3\,\mathrm{e}^{-x} x^2 + 6x\,\mathrm{e}^{-x} + 6\,\mathrm{e}^{-x} - c_1\right) \mathrm{e}^{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x^3 + y$ 	$R = x$ $S = \mathrm{e}^{-x} y$	$\frac{dS}{dR} = R^3 \mathrm{e}^{-R}$ 

Summary

The solution(s) found are the following

$$y = -\left(x^3 \mathrm{e}^{-x} + 3\,\mathrm{e}^{-x} x^2 + 6x\,\mathrm{e}^{-x} + 6\,\mathrm{e}^{-x} - c_1\right) \mathrm{e}^{x} \tag{1}$$

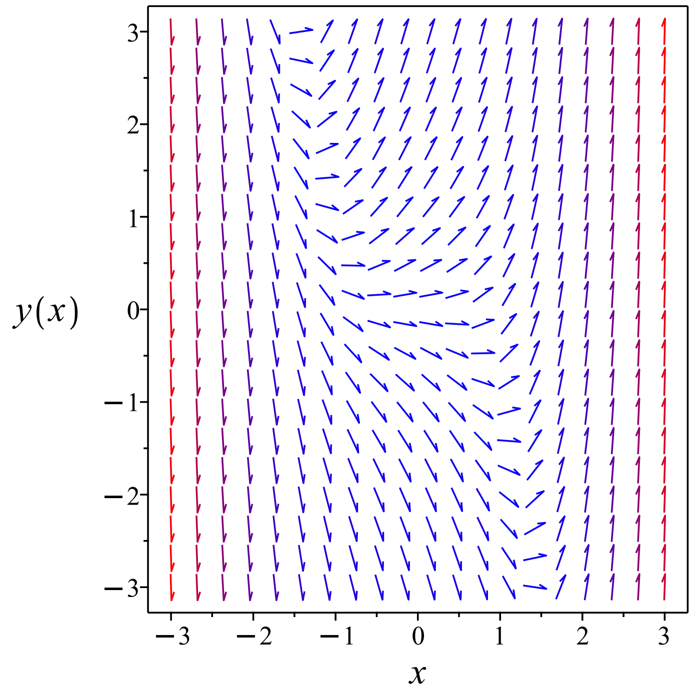


Figure 7: Slope field plot

Verification of solutions

$$y = -\left(x^3 \mathrm{e}^{-x} + 3\,\mathrm{e}^{-x} x^2 + 6x\,\mathrm{e}^{-x} + 6\,\mathrm{e}^{-x} - c_1\right) \mathrm{e}^{x}$$

Verified OK.

1.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(x^3 + y\right) \mathrm{d}x \\ \left(-x^3 - y\right) \mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x^3 - y \\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^3 - y\right) \\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((-1) - (0)) \\ &= -1 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -1 \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-x} \\ &= \mathrm{e}^{-x} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{-x}(-x^3 - y) \\ &= -\mathrm{e}^{-x}(x^3 + y) \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{-x}(1) \\ &= \mathrm{e}^{-x} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-\mathrm{e}^{-x}(x^3 + y)\right) + \left(\mathrm{e}^{-x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\mathrm{e}^{-x} (x^3 + y) \, \mathrm{d}x$$
$$\phi = \left(x^3 + 3x^2 + 6x + y + 6\right) \mathrm{e}^{-x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x}$. Therefore equation (4) becomes

$$\mathrm{e}^{-x} = \mathrm{e}^{-x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \left(x^3 + 3x^2 + 6x + y + 6\right) \mathrm{e}^{-x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \left(x^3 + 3x^2 + 6x + y + 6\right) \mathrm{e}^{-x}$$

The solution becomes

$$y = -\left(x^3 \mathrm{e}^{-x} + 3 \,\mathrm{e}^{-x} x^2 + 6x \,\mathrm{e}^{-x} + 6 \,\mathrm{e}^{-x} - c_1\right) \mathrm{e}^{x}$$

Summary
The solution(s) found are the following

$$y = -\left(x^3 e^{-x} + 3\,e^{-x} x^2 + 6x\,e^{-x} + 6\,e^{-x} - c_1\right) e^{x} \tag{1}$$

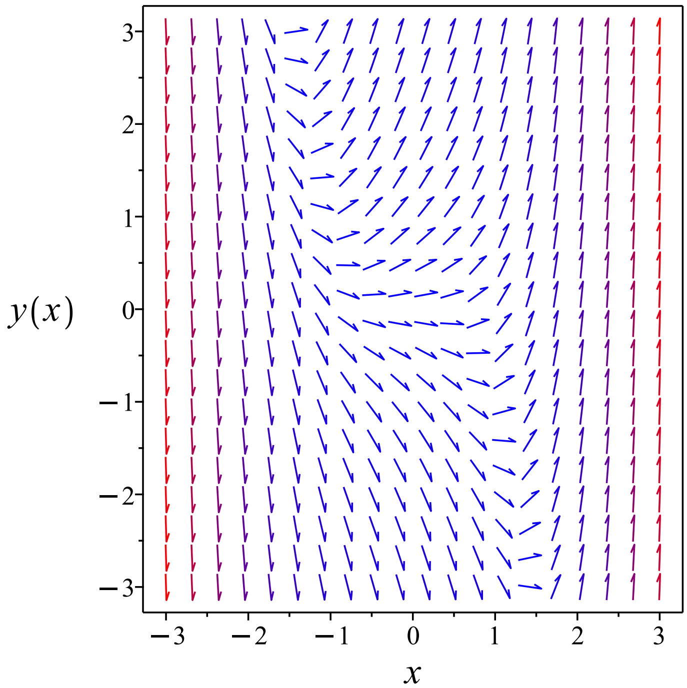


Figure 8: Slope field plot

Verification of solutions

$$y = -\left(x^3 e^{-x} + 3\,e^{-x} x^2 + 6x\,e^{-x} + 6\,e^{-x} - c_1\right) e^{x}$$

Verified OK.

1.2.4 Maple step by step solution

Let's solve

$y' - y = x^3$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = y + x^3$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y = x^3$
- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' - y) = \mu(x)\,x^3$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y' - y) = \mu'(x)\,y + \mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x) = -\mu(x)$
- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{-x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\,x^3 dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\,x^3 dx + c_1$
- Solve for y

  $y = \frac{\int \mu(x)x^3 dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{-x}$

  $y = \frac{\int x^3 \mathrm{e}^{-x} dx + c_1}{\mathrm{e}^{-x}}$
- Evaluate the integrals on the rhs

  $y = \frac{-(x^3+3x^2+6x+6)\mathrm{e}^{-x} + c_1}{\mathrm{e}^{-x}}$
- Simplify

  $y = -x^3 - 3x^2 - 6x - 6 + c_1 \mathrm{e}^{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 23

```
dsolve(diff(y(x),x)-y(x)=x^3,y(x), singsol=all)
```

$$y(x) = -x^3 - 3x^2 - 6x - 6 + e^x c_1$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 26

```
DSolve[y'[x]-y[x]==x^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x^3 - 3x^2 - 6x + c_1 e^x - 6$$

1.3 problem 1(c)

Internal problem ID [3031]
Internal file name [`OUTPUT/2523_Sunday_June_05_2022_03_18_05_AM_49581105/index.tex`]

Book: Elementary Differential equations, Chaundy, 1969
Section: Exercises 3, page 60
Problem number: 1(c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + y\cot(x) = x}$$

1.3.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \cot(x) \\ q(x) &= x \end{aligned}$$

Hence the ode is

$$y' + y\cot(x) = x$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \cot(x)dx} \\ &= \sin(x) \end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(x)$$
$$\frac{d}{dx}(\sin(x) y) = (\sin(x))(x)$$
$$d(\sin(x) y) = (x \sin(x)) \, dx$$

Integrating gives

$$\sin(x) y = \int x \sin(x) \, dx$$
$$\sin(x) y = \sin(x) - \cos(x) x + c_1$$

Dividing both sides by the integrating factor $\mu = \sin(x)$ results in

$$y = \csc(x)(\sin(x) - \cos(x) x) + c_1 \csc(x)$$

which simplifies to

$$y = -\cot(x) x + 1 + c_1 \csc(x)$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\cot(x) x + 1 + c_1 \csc(x) \tag{1}$$

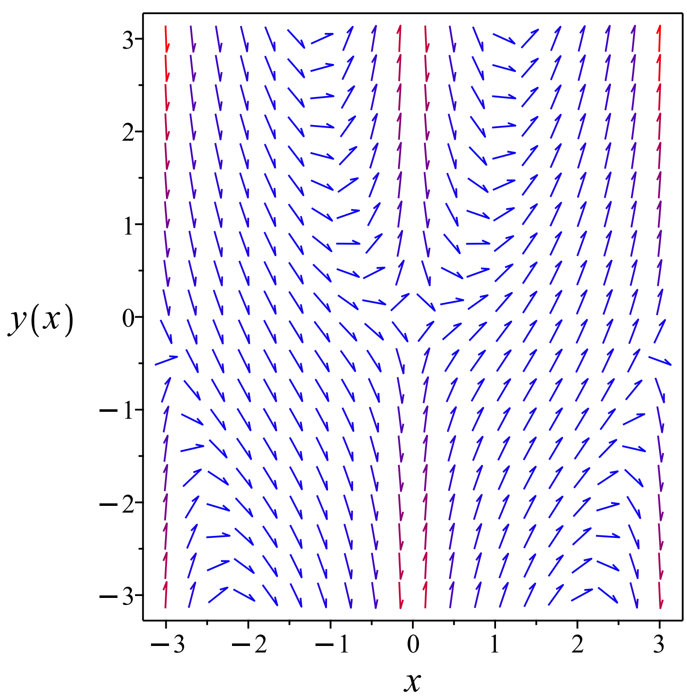


Figure 9: Slope field plot

Verification of solutions

$$y = -\cot(x)\, x + 1 + c_1 \csc(x)$$

Verified OK.

1.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -y\cot(x) + x \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 7: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{\sin(x)}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\sin(x)}} dy \end{aligned}$$

Which results in

$$S = \sin(x)\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -y\cot(x) + x$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \cos(x)\, y \\ S_y &= \sin(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x\sin(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R\sin(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \sin(R) - R\cos(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\sin(x) = \sin(x) - \cos(x)\, x + c_1$$

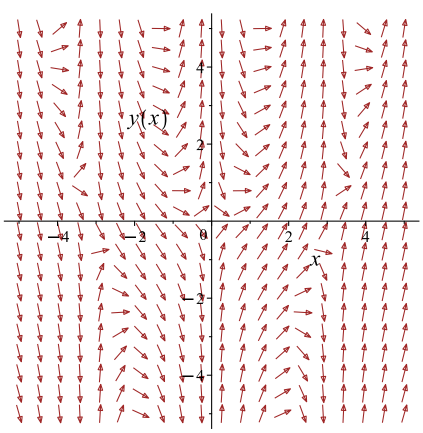
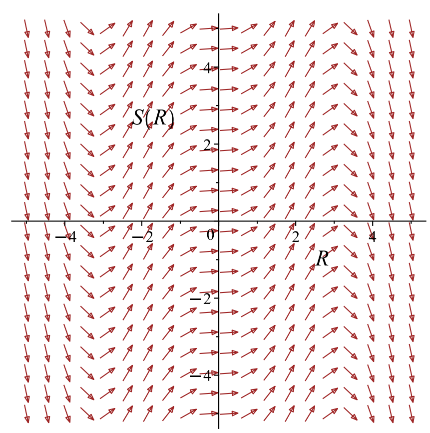
Which simplifies to

$$y\sin(x) = \sin(x) - \cos(x)\, x + c_1$$

Which gives

$$y = -\frac{\cos(x)\, x - \sin(x) - c_1}{\sin(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -y\cot(x) + x$  	$R = x$ $S = \sin(x)\, y$	$\frac{dS}{dR} = R\sin(R)$ 

Summary

The solution(s) found are the following

$$y = -\frac{\cos(x)\, x - \sin(x) - c_1}{\sin(x)} \tag{1}$$

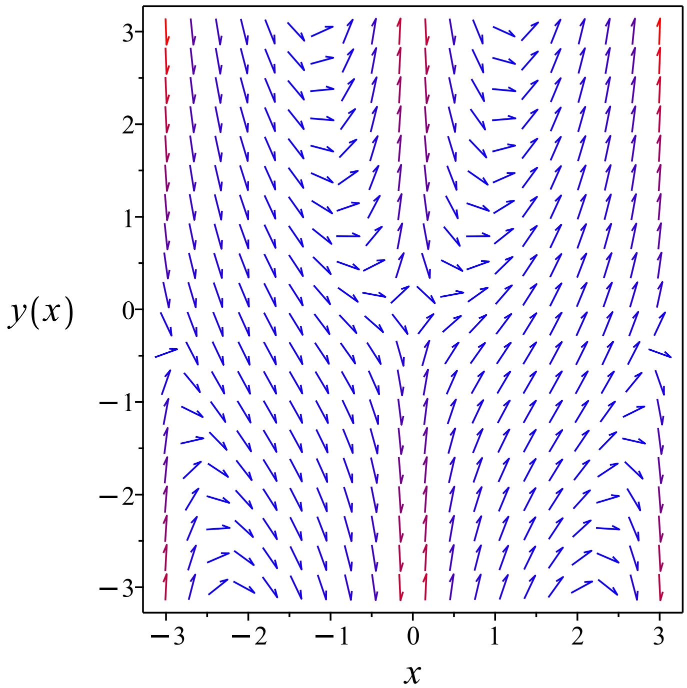


Figure 10: Slope field plot

Verification of solutions

$$y = -\frac{\cos(x)\,x - \sin(x) - c_1}{\sin(x)}$$

Verified OK.

1.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= (-y \cot (x) + x)\, \mathrm{d}x \\
(y \cot (x) - x)\, \mathrm{d}x + \mathrm{d}y &= 0
\end{aligned} \tag{2A}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= y \cot (x) - x \\
N(x, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y \cot (x) - x) \\
&= \cot (x)
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((\cot(x)) - (0)) \\ &= \cot(x) \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \cot(x) \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\ln(\sin(x))} \\ &= \sin(x) \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \sin(x)\,(y\cot(x) - x) \\ &= \cos(x)\, y - x\sin(x) \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \sin(x)\,(1) \\ &= \sin(x) \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (\cos(x)\, y - x\sin(x)) + (\sin(x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \cos(x) \, y - x \sin(x) \, \mathrm{d}x$$

$$\phi = (y-1)\sin(x) + \cos(x)\, x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \sin(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sin(x)$. Therefore equation (4) becomes

$$\sin(x) = \sin(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (y-1)\sin(x) + \cos(x)\, x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (y-1)\sin(x) + \cos(x)\, x$$

The solution becomes

$$y = -\frac{\cos(x)\, x - \sin(x) - c_1}{\sin(x)}$$

Summary
The solution(s) found are the following

$$y = -\frac{\cos(x)\, x - \sin(x) - c_1}{\sin(x)} \tag{1}$$

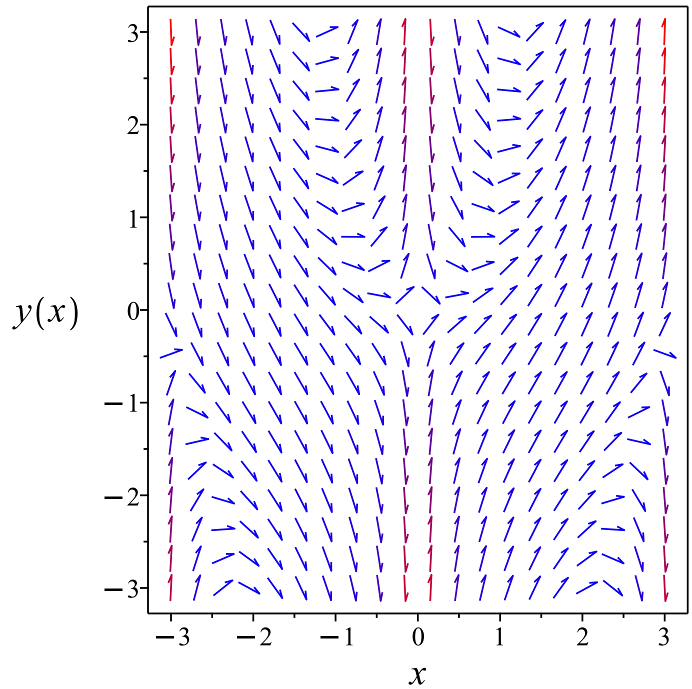


Figure 11: Slope field plot

Verification of solutions

$$y = -\frac{\cos(x)\, x - \sin(x) - c_1}{\sin(x)}$$

Verified OK.

1.3.4 Maple step by step solution

Let's solve

$y' + y \cot(x) = x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = -y \cot(x) + x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$y' + y \cot(x) = x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$\mu(x)(y' + y \cot(x)) = \mu(x) x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x) y)$

$\mu(x)(y' + y \cot(x)) = \mu'(x) y + \mu(x) y'$

- Isolate $\mu'(x)$

$\mu'(x) = \mu(x) \cot(x)$

- Solve to find the integrating factor

$\mu(x) = \sin(x)$

- Integrate both sides with respect to x

$\int \left(\frac{d}{dx}(\mu(x) y)\right) dx = \int \mu(x) x dx + c_1$

- Evaluate the integral on the lhs

$\mu(x) y = \int \mu(x) x dx + c_1$

- Solve for y

$y = \frac{\int \mu(x) x dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \sin(x)$

$y = \frac{\int x \sin(x) dx + c_1}{\sin(x)}$

- Evaluate the integrals on the rhs

$y = \frac{\sin(x) - \cos(x) x + c_1}{\sin(x)}$

- Simplify

$y = -\cot(x) x + 1 + c_1 \csc(x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x)+y(x)*cot(x)=x,y(x), singsol=all)
```

$$y(x) = -\cot(x)\,x + 1 + \csc(x)\,c_1$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 17

```
DSolve[y'[x]+y[x]*Cot[x]==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x\cot(x) + c_1\csc(x) + 1$$

1.4 problem 1(d)

Internal problem ID [3032]
Internal file name [`OUTPUT/2524_Sunday_June_05_2022_03_18_10_AM_35961804/index.tex`]

Book: Elementary Differential equations, Chaundy, 1969
Section: Exercises 3, page 60
Problem number: 1(d).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first__order__ode__lie__symmetry__lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + y\cot(x) = \tan(x)}$$

1.4.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \cot(x)$$
$$q(x) = \tan(x)$$

Hence the ode is

$$y' + y\cot(x) = \tan(x)$$

The integrating factor μ is

$$\mu = e^{\int \cot(x)dx}$$
$$= \sin(x)$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(\tan(x))$$
$$\frac{d}{dx}(\sin(x)\,y) = (\sin(x))(\tan(x))$$
$$d(\sin(x)\,y) = (\tan(x)\sin(x))\, dx$$

Integrating gives

$$\sin(x)\,y = \int \tan(x)\sin(x)\, dx$$
$$\sin(x)\,y = -\sin(x) + \ln(\sec(x) + \tan(x)) + c_1$$

Dividing both sides by the integrating factor $\mu = \sin(x)$ results in

$$y = \csc(x)(-\sin(x) + \ln(\sec(x) + \tan(x))) + c_1 \csc(x)$$

which simplifies to

$$y = \csc(x)(-\sin(x) + \ln(\sec(x) + \tan(x)) + c_1)$$

Summary
The solution(s) found are the following

$$y = \csc(x)(-\sin(x) + \ln(\sec(x) + \tan(x)) + c_1) \tag{1}$$

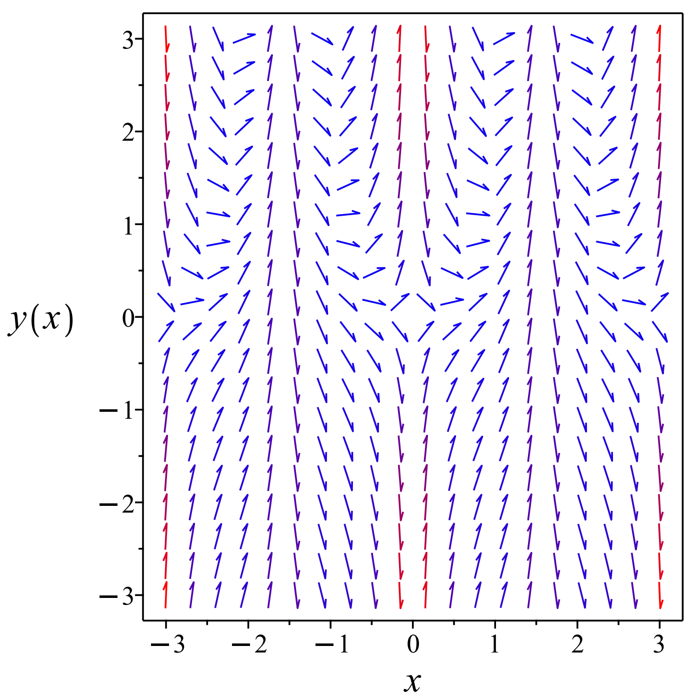


Figure 12: Slope field plot

Verification of solutions

$$y = \csc(x)(-\sin(x) + \ln(\sec(x) + \tan(x)) + c_1)$$

Verified OK.

1.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -y\cot(x) + \tan(x) \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 10: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{\sin(x)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\sin(x)}} dy \end{aligned}$$

Which results in

$$S = \sin(x) y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -y \cot(x) + \tan(x)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \cos(x) y \\ S_y &= \sin(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \tan(x) \sin(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \tan(R) \sin(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\sin(R) + \ln(\sec(R) + \tan(R)) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y \sin(x) = -\sin(x) + \ln(\sec(x) + \tan(x)) + c_1$$

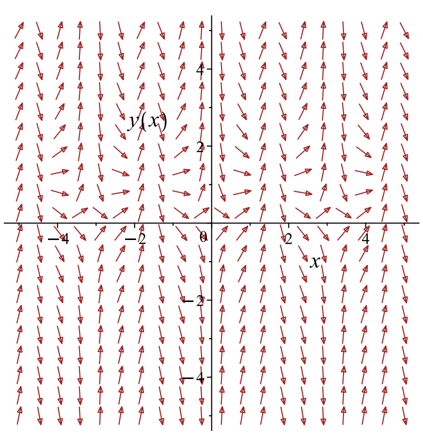
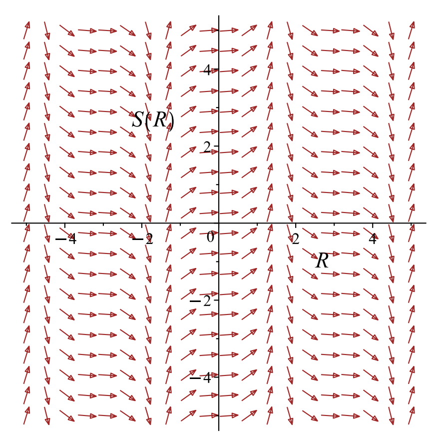
Which simplifies to

$$y \sin(x) = -\sin(x) + \ln(\sec(x) + \tan(x)) + c_1$$

Which gives

$$y = -\frac{\sin(x) - \ln(\sec(x) + \tan(x)) - c_1}{\sin(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -y\cot(x) + \tan(x)$

</td><td>$R = x$
$S = \sin(x)\, y$</td><td>$\frac{dS}{dR} = \tan(R)\sin(R)$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = -\frac{\sin(x) - \ln(\sec(x) + \tan(x)) - c_1}{\sin(x)} \tag{1}$$

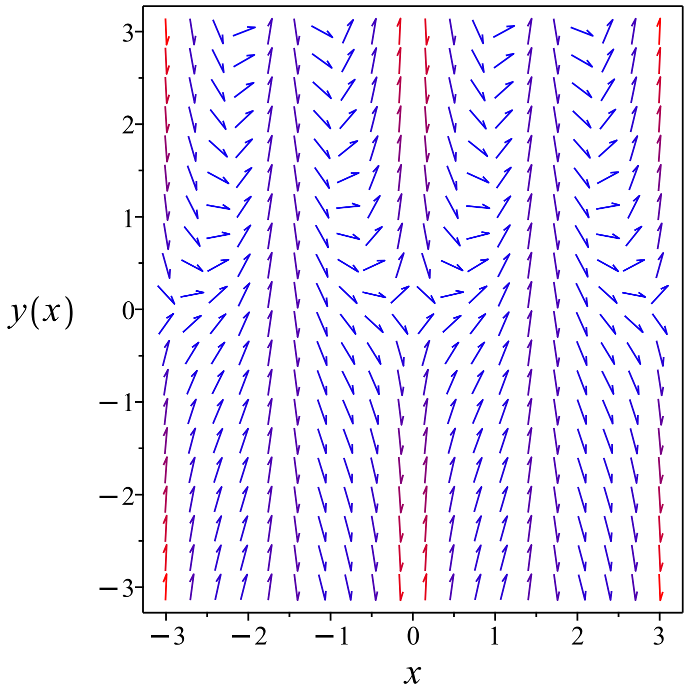


Figure 13: Slope field plot

Verification of solutions

$$y = -\frac{\sin(x) - \ln(\sec(x) + \tan(x)) - c_1}{\sin(x)}$$

Verified OK.

1.4.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= (-y\cot(x) + \tan(x))\,\mathrm{d}x\\ (y\cot(x) - \tan(x))\,\mathrm{d}x + \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y\cot(x) - \tan(x)\\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y\cot(x) - \tan(x))\\ &= \cot(x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((\cot(x)) - (0)) \\ &= \cot(x) \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \cot(x) \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\ln(\sin(x))} \\ &= \sin(x) \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \sin(x)\,(y\cot(x) - \tan(x)) \\ &= \cos(x)\,y - \tan(x)\sin(x) \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \sin(x)\,(1) \\ &= \sin(x) \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (\cos(x)\,y - \tan(x)\sin(x)) + (\sin(x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \cos(x)\, y - \tan(x)\sin(x) \,\mathrm{d}x$$
$$\phi = \sin(x)\, y + \sin(x) - \ln(\sec(x) + \tan(x)) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \sin(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sin(x)$. Therefore equation (4) becomes

$$\sin(x) = \sin(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \sin(x)\, y + \sin(x) - \ln(\sec(x) + \tan(x)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \sin(x)\, y + \sin(x) - \ln(\sec(x) + \tan(x))$$

The solution becomes

$$y = -\frac{\sin(x) - \ln(\sec(x) + \tan(x)) - c_1}{\sin(x)}$$

Summary
The solution(s) found are the following

$$y = -\frac{\sin(x) - \ln(\sec(x) + \tan(x)) - c_1}{\sin(x)} \tag{1}$$

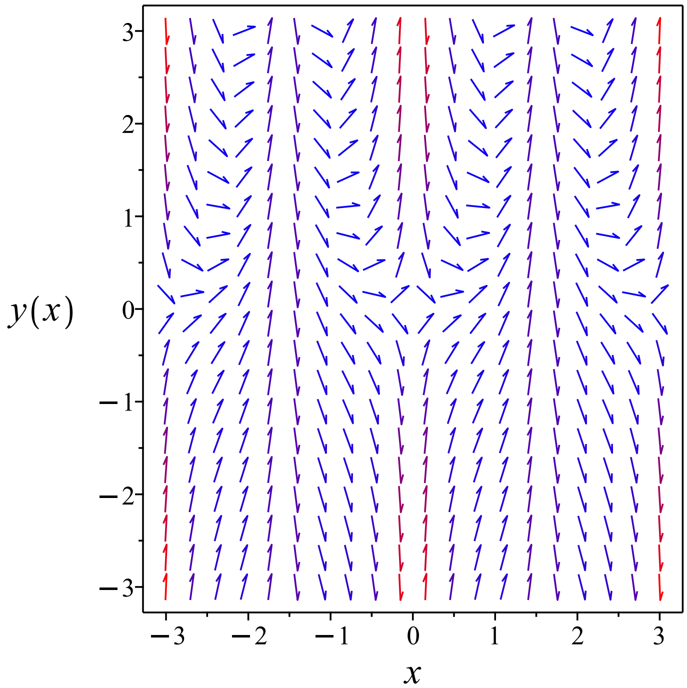


Figure 14: Slope field plot

Verification of solutions

$$y = -\frac{\sin(x) - \ln(\sec(x) + \tan(x)) - c_1}{\sin(x)}$$

Verified OK.

1.4.4 Maple step by step solution

Let's solve

$y' + y\cot(x) = \tan(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = -y\cot(x) + \tan(x)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y\cot(x) = \tan(x)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + y\cot(x)\right) = \mu(x)\tan(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + y\cot(x)\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)\cot(x)$

- Solve to find the integrating factor

  $\mu(x) = \sin(x)$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right)dx = \int \mu(x)\tan(x)\,dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\tan(x)\,dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)\tan(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \sin(x)$

  $y = \frac{\int \tan(x)\sin(x)dx + c_1}{\sin(x)}$

- Evaluate the integrals on the rhs

  $y = \frac{-\sin(x) + \ln(\sec(x) + \tan(x)) + c_1}{\sin(x)}$

- Simplify

  $y = \csc(x)\left(-\sin(x) + \ln(\sec(x) + \tan(x)) + c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x)+y(x)*cot(x)=tan(x),y(x), singsol=all)
```

$$y(x) = \csc(x)\left(-\sin(x) + \ln(\sec(x) + \tan(x)) + c_1\right)$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 18

```
DSolve[y'[x]+y[x]*Cot[x]==Tan[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \csc(x)\text{arctanh}(\sin(x)) + c_1 \csc(x) - 1$$

1.5 problem 1(e)

Internal problem ID [3033]
Internal file name [`OUTPUT/2525_Sunday_June_05_2022_03_18_12_AM_49144292/index.tex`]

Book: Elementary Differential equations, Chaundy, 1969
Section: Exercises 3, page 60
Problem number: 1(e).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + y\tan(x) = \cot(x)}$$

1.5.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \tan(x) \\ q(x) &= \cot(x) \end{aligned}$$

Hence the ode is

$$y' + y\tan(x) = \cot(x)$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \tan(x)dx} \\ &= \frac{1}{\cos(x)} \end{aligned}$$

Which simplifies to

$$\mu = \sec(x)$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(\cot(x)) \\ \frac{d}{dx}(\sec(x)\, y) &= (\sec(x))(\cot(x)) \\ d(\sec(x)\, y) &= \csc(x)\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\sec(x)\, y &= \int \csc(x)\, dx \\ \sec(x)\, y &= -\ln(\csc(x) + \cot(x)) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sec(x)$ results in

$$y = -\cos(x)\ln(\csc(x) + \cot(x)) + c_1 \cos(x)$$

which simplifies to

$$y = \cos(x)(-\ln(\csc(x) + \cot(x)) + c_1)$$

Summary

The solution(s) found are the following

$$y = \cos(x)(-\ln(\csc(x) + \cot(x)) + c_1) \tag{1}$$

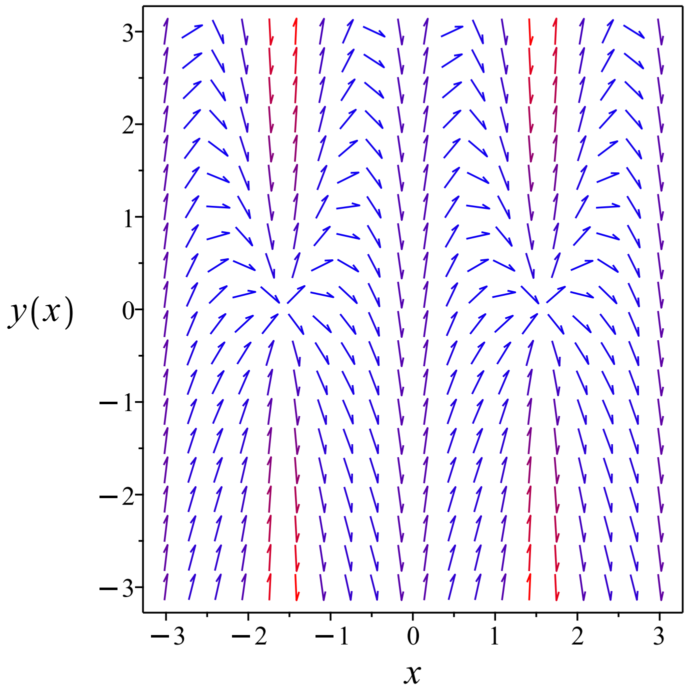


Figure 15: Slope field plot

Verification of solutions

$$y = \cos(x)\left(-\ln(\csc(x) + \cot(x)) + c_1\right)$$

Verified OK.

1.5.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\tan(x)\,y + \cot(x) \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 13: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= \cos(x) \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\cos(x)} dy \end{aligned}$$

Which results in

$$S = \frac{y}{\cos(x)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\tan(x)\, y + \cot(x)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \sec(x)\tan(x)\, y \\ S_y &= \sec(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \csc(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \csc(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(\csc(R) + \cot(R)) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y \sec(x) = -\ln(\csc(x) + \cot(x)) + c_1$$

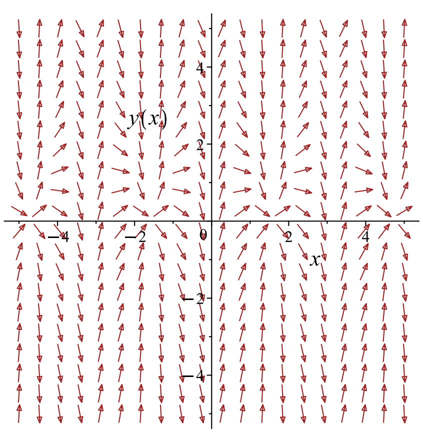
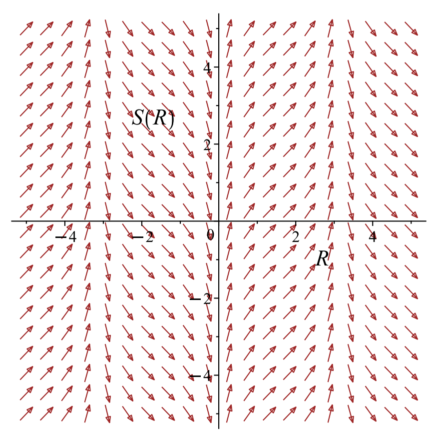
Which simplifies to

$$y \sec(x) = -\ln(\csc(x) + \cot(x)) + c_1$$

Which gives

$$y = -\frac{\ln(\csc(x) + \cot(x)) - c_1}{\sec(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\tan(x) y + \cot(x)$ 	$R = x$ $S = \sec(x) y$	$\frac{dS}{dR} = \csc(R)$ 

Summary
The solution(s) found are the following

$$y = -\frac{\ln(\csc(x) + \cot(x)) - c_1}{\sec(x)} \tag{1}$$

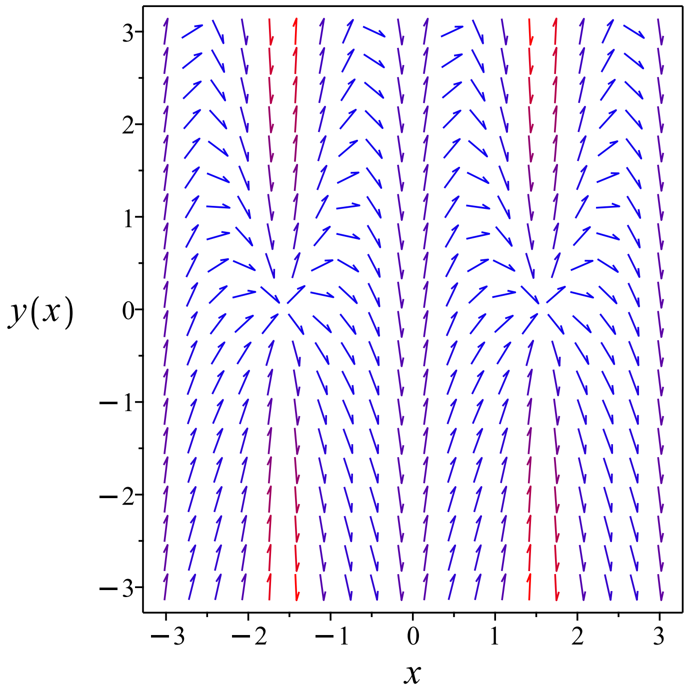


Figure 16: Slope field plot

Verification of solutions

$$y = -\frac{\ln(\csc(x) + \cot(x)) - c_1}{\sec(x)}$$

Verified OK.

1.5.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = (-\tan(x)\,y + \cot(x))\,\mathrm{d}x$$
$$(\tan(x)\,y - \cot(x))\,\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = \tan(x)\,y - \cot(x)$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(\tan(x)\,y - \cot(x))$$
$$= \tan(x)$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(1)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((\tan(x)) - (0)) \\ &= \tan(x) \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \tan(x) \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\ln(\cos(x))} \\ &= \sec(x) \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \sec(x)(\tan(x) y - \cot(x)) \\ &= \sec(x) \tan(x) y - \csc(x) \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \sec(x)(1) \\ &= \sec(x) \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (\sec(x)\tan(x) y - \csc(x)) + (\sec(x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \sec(x)\tan(x)\, y - \csc(x) \,\mathrm{d}x$$

$$\phi = \sec(x)\, y + \ln(\csc(x) + \cot(x)) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \sec(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sec(x)$. Therefore equation (4) becomes

$$\sec(x) = \sec(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \sec(x)\, y + \ln(\csc(x) + \cot(x)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \sec(x)\, y + \ln(\csc(x) + \cot(x))$$

The solution becomes

$$y = -\frac{\ln(\csc(x) + \cot(x)) - c_1}{\sec(x)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\ln(\csc(x) + \cot(x)) - c_1}{\sec(x)} \tag{1}$$

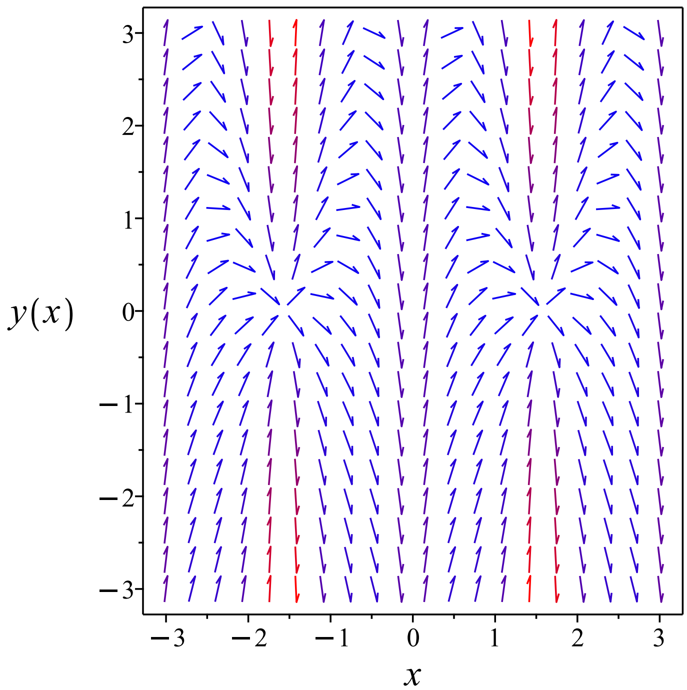


Figure 17: Slope field plot

Verification of solutions

$$y = -\frac{\ln(\csc(x) + \cot(x)) - c_1}{\sec(x)}$$

Verified OK.

1.5.4 Maple step by step solution

Let's solve

$y' + y\tan(x) = \cot(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$$y' = -y\tan(x) + \cot(x)$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' + y\tan(x) = \cot(x)$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(y' + y\tan(x)\right) = \mu(x)\cot(x)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$$\mu(x)\left(y' + y\tan(x)\right) = \mu'(x)\,y + \mu(x)\,y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = \mu(x)\tan(x)$$

- Solve to find the integrating factor

$$\mu(x) = \frac{1}{\cos(x)}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\cot(x)\,dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\,y = \int \mu(x)\cot(x)\,dx + c_1$$

- Solve for y

$$y = \frac{\int \mu(x)\cot(x)dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \frac{1}{\cos(x)}$

$$y = \cos(x)\left(\int \frac{\cot(x)}{\cos(x)}dx + c_1\right)$$

- Evaluate the integrals on the rhs

$$y = \cos(x)\left(\ln(\csc(x) - \cot(x)) + c_1\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x)+y(x)*tan(x)=cot(x),y(x), singsol=all)
```

$$y(x) = (-\ln(\csc(x) + \cot(x)) + c_1)\cos(x)$$

✓ Solution by Mathematica

Time used: 0.058 (sec). Leaf size: 16

```
DSolve[y'[x]+y[x]*Tan[x]==Cot[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \cos(x)(-\operatorname{arctanh}(\cos(x)) + c_1)$$

1.6 problem 1(f)

Internal problem ID [3034]
Internal file name [`OUTPUT/2526_Sunday_June_05_2022_03_18_14_AM_48549037/index.tex`]

Book: Elementary Differential equations, Chaundy, 1969
Section: Exercises 3, page 60
Problem number: 1(f).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + y\ln(x) = x^{-x}}$$

1.6.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \ln(x) \\ q(x) &= x^{-x} \end{aligned}$$

Hence the ode is

$$y' + y\ln(x) = x^{-x}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \ln(x)dx} \\ &= e^{\ln(x)x-x} \end{aligned}$$

Which simplifies to

$$\mu = x^x e^{-x}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(x^{-x}\right) \\
\frac{d}{dx}\left(x^x e^{-x} y\right) &= \left(x^x e^{-x}\right)\left(x^{-x}\right) \\
d\left(x^x e^{-x} y\right) &= e^{-x}\,dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
x^x e^{-x} y &= \int e^{-x}\,dx \\
x^x e^{-x} y &= -e^{-x} + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^x e^{-x}$ results in

$$y = -x^{-x} e^x e^{-x} + c_1 x^{-x} e^x$$

which simplifies to

$$y = (-1 + c_1 e^x)\, x^{-x}$$

Summary

The solution(s) found are the following

$$y = (-1 + c_1 e^x)\, x^{-x} \tag{1}$$

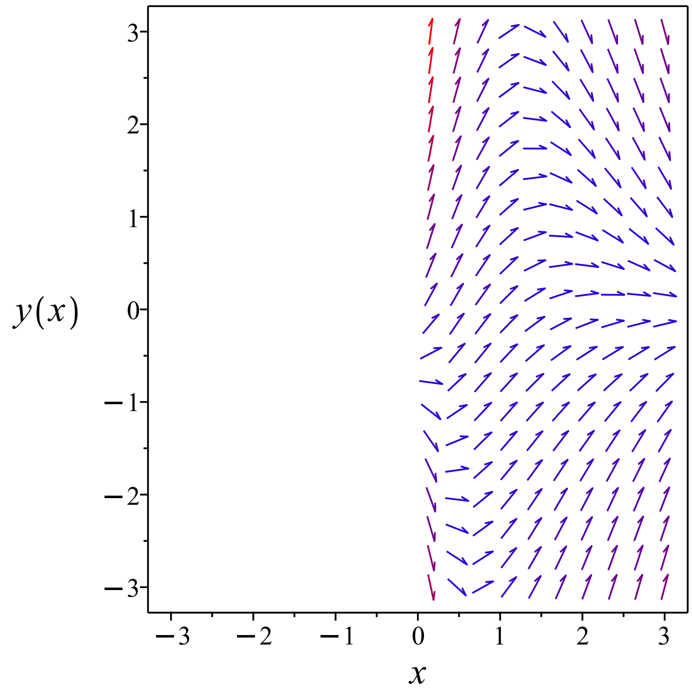


Figure 18: Slope field plot

Verification of solutions

$$y = (-1 + c_1 e^x) x^{-x}$$

Verified OK.

1.6.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\ln(x) y + x^{-x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 16: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \mathrm{e}^{-\ln(x)x+x}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-\ln(x)x+x}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{\ln(x)x-x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\ln(x) y + x^{-x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= x^x \mathrm{e}^{-x} y \ln(x) \\ S_y &= x^x \mathrm{e}^{-x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \mathrm{e}^{-x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \mathrm{e}^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\mathrm{e}^{-R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x^x \mathrm{e}^{-x} y = -\mathrm{e}^{-x} + c_1$$

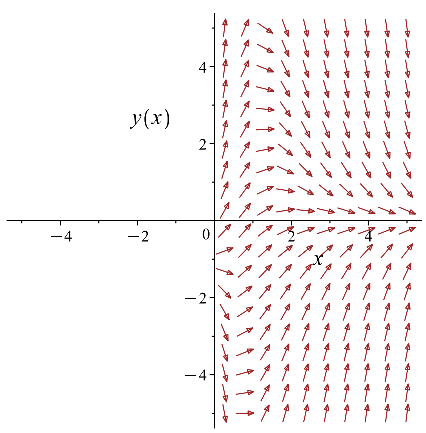
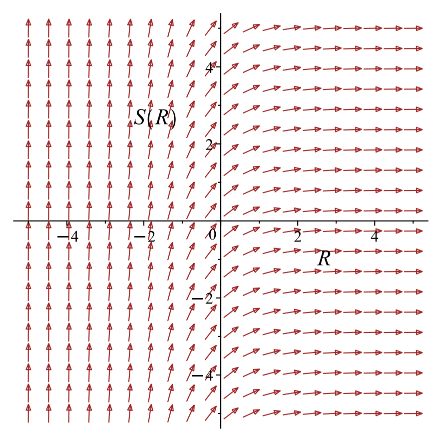
Which simplifies to

$$x^x \mathrm{e}^{-x} y = -\mathrm{e}^{-x} + c_1$$

Which gives

$$y = -\left(\mathrm{e}^{-x} - c_1\right) x^{-x} \mathrm{e}^{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\ln(x)\, y + x^{-x}$  	$R = x$ $S = x^x \mathrm{e}^{-x} y$	$\frac{dS}{dR} = \mathrm{e}^{-R}$ 

Summary

The solution(s) found are the following

$$y = -\left(\mathrm{e}^{-x} - c_1\right) x^{-x} \mathrm{e}^{x} \tag{1}$$

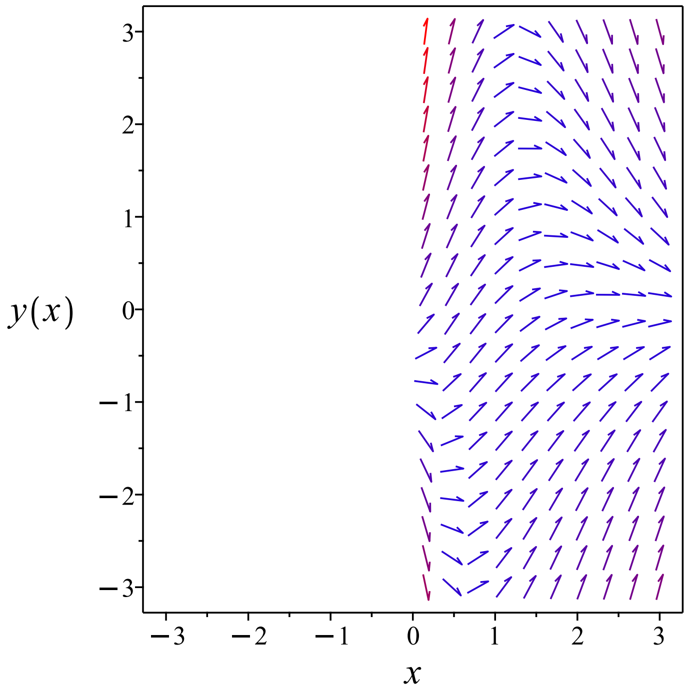


Figure 19: Slope field plot

Verification of solutions

$$y = -\left(\mathrm{e}^{-x} - c_1\right) x^{-x} \mathrm{e}^{x}$$

Verified OK.

1.6.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-\ln(x)\, y + x^{-x}\right) \mathrm{d}x$$
$$\left(\ln(x)\, y - x^{-x}\right) \mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = \ln(x)\, y - x^{-x}$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\ln(x)\, y - x^{-x}\right) \\ &= \ln(x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((\ln(x)) - (0)) \\ &= \ln(x) \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \ln(x) \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\ln(x)x - x} \\ &= x^x \mathrm{e}^{-x} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= x^x \mathrm{e}^{-x}\left(\ln(x)\, y - x^{-x}\right) \\ &= \mathrm{e}^{-x}(\ln(x)\, y\, x^x - 1) \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= x^x \mathrm{e}^{-x}(1) \\ &= x^x \mathrm{e}^{-x} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\mathrm{e}^{-x}(\ln(x)\, y\, x^x - 1)\right) + \left(x^x \mathrm{e}^{-x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \mathrm{e}^{-x}(\ln(x)\,y\,x^x - 1)\,\mathrm{d}x$$
$$\phi = \mathrm{e}^{-x}(x^x y + 1) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^x \mathrm{e}^{-x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^x \mathrm{e}^{-x}$. Therefore equation (4) becomes

$$x^x \mathrm{e}^{-x} = x^x \mathrm{e}^{-x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{-x}(x^x y + 1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \mathrm{e}^{-x}(x^x y + 1)$$

The solution becomes

$$y = -\left(\mathrm{e}^{-x} - c_1\right) x^{-x} \mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = -\left(\mathrm{e}^{-x} - c_1\right) x^{-x} \mathrm{e}^{x} \tag{1}$$

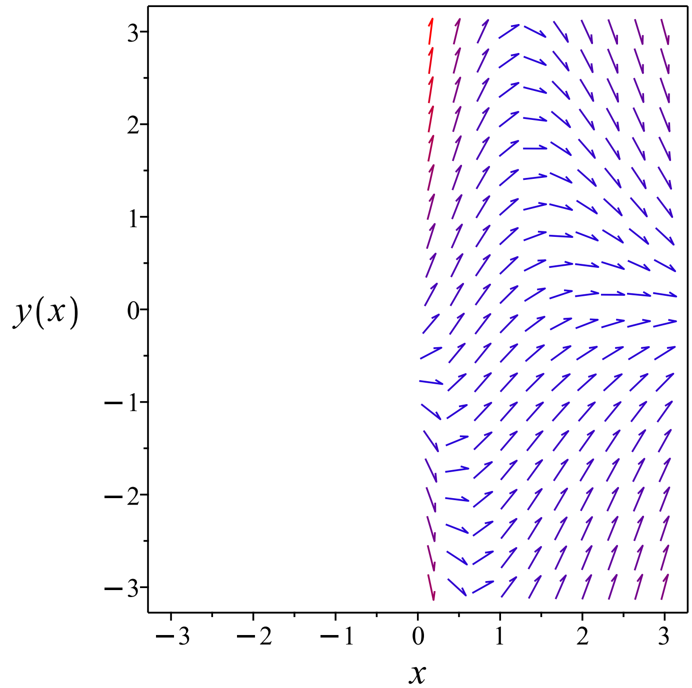


Figure 20: Slope field plot

Verification of solutions

$$y = -\left(e^{-x} - c_1\right) x^{-x} e^{x}$$

Verified OK.

1.6.4 Maple step by step solution

Let's solve

$y' + y \ln(x) = x^{-x}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -y \ln(x) + x^{-x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y \ln(x) = x^{-x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x) \left(y' + y \ln(x)\right) = \mu(x) x^{-x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y'+y\ln(x)) = \mu'(x)\,y+\mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)\ln(x)$

- Solve to find the integrating factor

  $\mu(x) = x^x \mathrm{e}^{-x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\,x^{-x} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\,x^{-x} dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) x^{-x} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x^x \mathrm{e}^{-x}$

  $y = \frac{\int x^{-x} x^x \mathrm{e}^{-x} dx + c_1}{x^x \mathrm{e}^{-x}}$

- Evaluate the integrals on the rhs

  $y = \frac{-\mathrm{e}^{-x} + c_1}{x^x \mathrm{e}^{-x}}$

- Simplify

  $y = (-1 + c_1 \mathrm{e}^x)\,x^{-x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(x),x)+y(x)*ln(x)=x^(-x),y(x), singsol=all)
```

$$y(x) = (e^{x}c_1 - 1)\,x^{-x}$$

✓ Solution by Mathematica

Time used: 0.08 (sec). Leaf size: 19

```
DSolve[y'[x]+y[x]*Log[x]==x^(-x),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^{-x}(-1 + c_1 e^{x})$$

1.7 problem 2(a)

Internal problem ID [3035]
Internal file name [`OUTPUT/2527_Sunday_June_05_2022_03_18_16_AM_35390428/index.tex`]

Book: Elementary Differential equations, Chaundy, 1969
Section: Exercises 3, page 60
Problem number: 2(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{xy' + y = x}$$

1.7.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{x}$$
$$q(x) = 1$$

Hence the ode is

$$y' + \frac{y}{x} = 1$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{1}{x}dx} \\
&= x
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu \\
\frac{\mathrm{d}}{\mathrm{d}x}(xy) &= x \\
\mathrm{d}(xy) &= x\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
xy &= \int x \,\mathrm{d}x \\
xy &= \frac{x^2}{2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$
y = \frac{x}{2} + \frac{c_1}{x}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \frac{x}{2} + \frac{c_1}{x} \tag{1}
$$

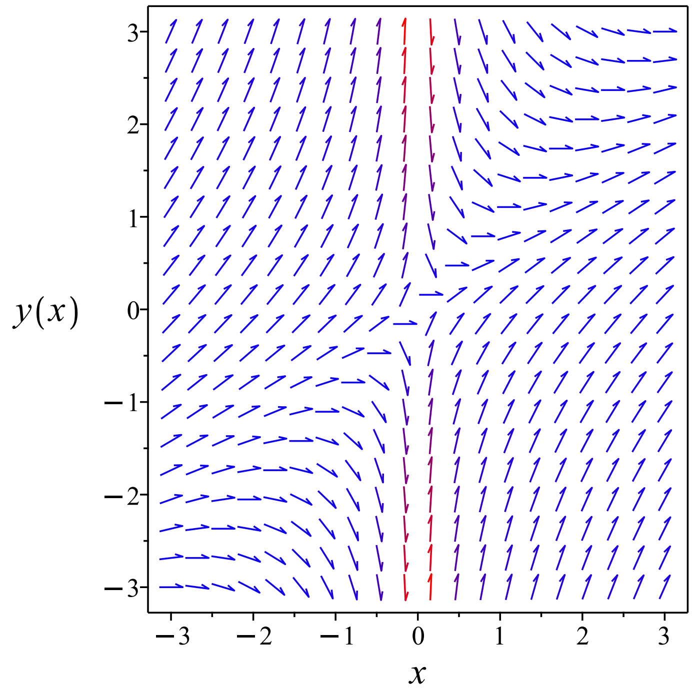


Figure 21: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{x}{2} + \frac{c_1}{x}$$

Verified OK.

1.7.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x(u'(x)\,x + u(x)) + u(x)\,x = x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{-2u+1}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = -2u + 1$. Integrating both sides gives

$$\frac{1}{-2u+1}\,du = \frac{1}{x}\,dx$$

$$\int \frac{1}{-2u+1}\,du = \int \frac{1}{x}\,dx$$

$$-\frac{\ln(-2u+1)}{2} = \ln(x) + c_2$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{-2u+1}} = e^{\ln(x)+c_2}$$

Which simplifies to

$$\frac{1}{\sqrt{-2u+1}} = c_3 x$$

Therefore the solution y is

$$\begin{aligned} y &= ux \\ &= \frac{(c_3^2 e^{2c_2} x^2 - 1)\, e^{-2c_2}}{2x\, c_3^2} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{(c_3^2 e^{2c_2} x^2 - 1)\, e^{-2c_2}}{2x\, c_3^2} \tag{1}$$

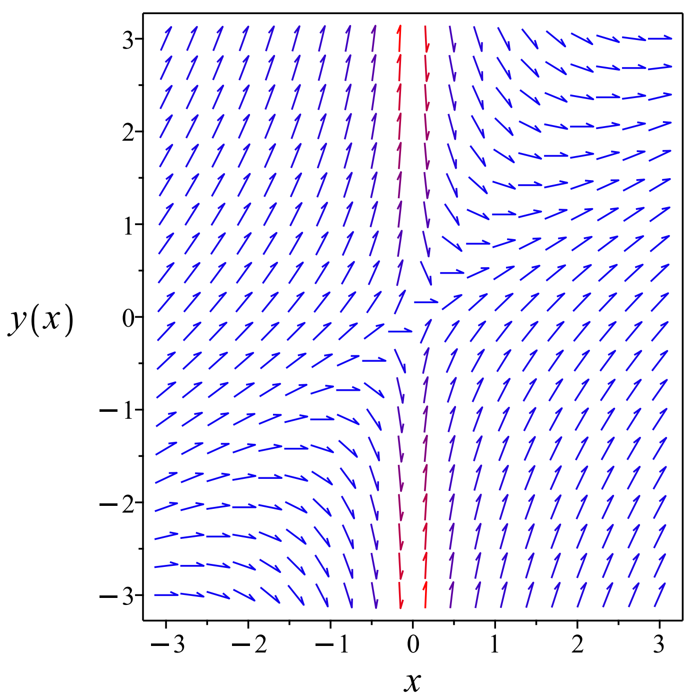


Figure 22: Slope field plot

Verification of solutions

$$y = \frac{(c_3^2 \mathrm{e}^{2c_2} x^2 - 1)\, \mathrm{e}^{-2c_2}}{2x\, c_3^2}$$

Verified OK.

1.7.3 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-y + x}{x} \tag{1}$$

Which becomes

$$0 = (-x)\, dy + (-y + x)\, dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\, dy + (-y + x)\, dx = d\left(\frac{1}{2}x^2 - xy\right)$$

Hence (2) becomes

$$0 = d\left(\frac{1}{2}x^2 - xy\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{x^2 + 2c_1}{2x} + c_1$$

Summary
The solution(s) found are the following

$$y = \frac{x^2 + 2c_1}{2x} + c_1 \tag{1}$$

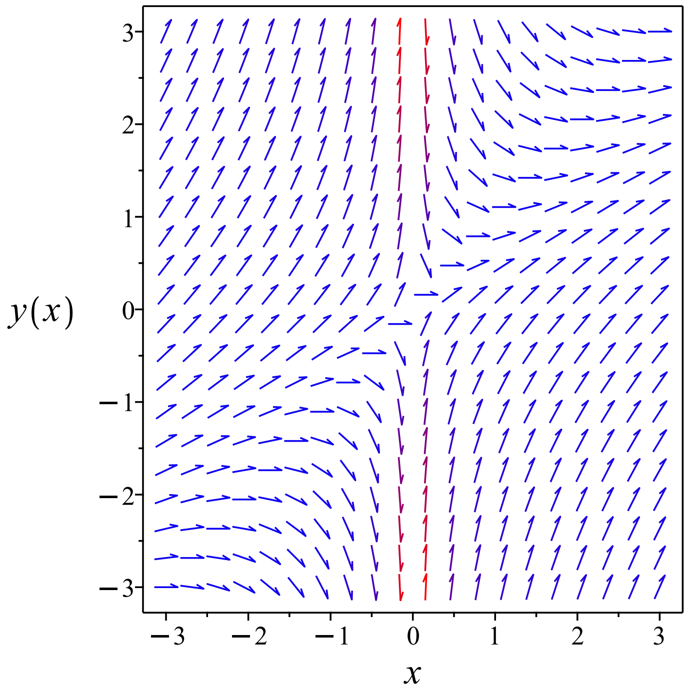


Figure 23: Slope field plot

Verification of solutions

$$y = \frac{x^2 + 2c_1}{2x} + c_1$$

Verified OK.

1.7.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y - x}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 19: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x}} dy \end{aligned}$$

Which results in

$$S = xy$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y - x}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y \\ S_y &= x \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$yx = \frac{x^2}{2} + c_1$$

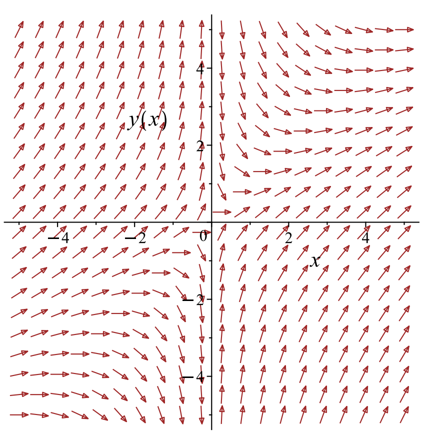
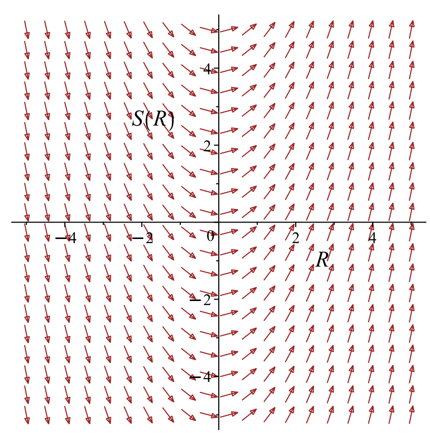
Which simplifies to

$$yx = \frac{x^2}{2} + c_1$$

Which gives

$$y = \frac{x^2 + 2c_1}{2x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y-x}{x}$ 	$R = x$ $S = xy$	$\frac{dS}{dR} = R$ 

Summary

The solution(s) found are the following

$$y = \frac{x^2 + 2c_1}{2x} \tag{1}$$

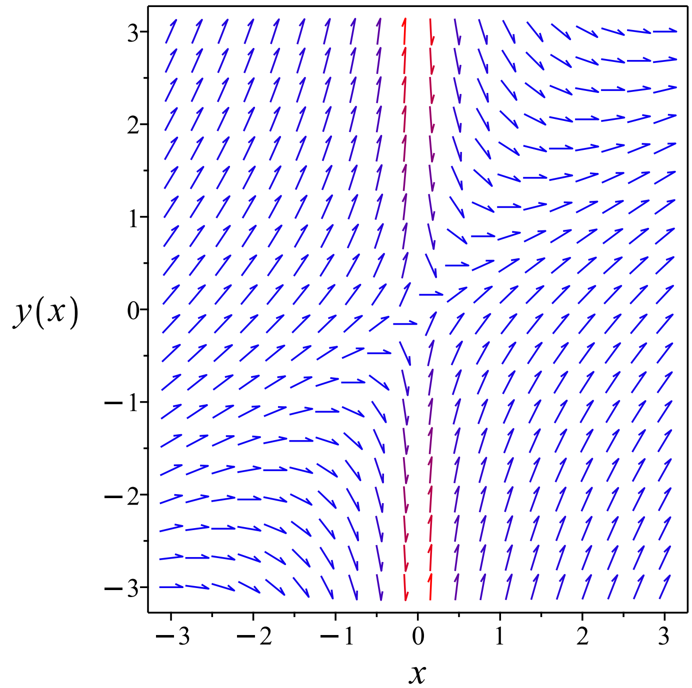


Figure 24: Slope field plot

Verification of solutions

$$y = \frac{x^2 + 2c_1}{2x}$$

Verified OK.

1.7.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(x)\,\mathrm{d}y = (-y + x)\,\mathrm{d}x$$
$$(y - x)\,\mathrm{d}x + (x)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = y - x$$
$$N(x, y) = x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y - x) \\ &= 1 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\ &= 1 \end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int y-x\,\mathrm{d}x$$
$$\phi=-\frac{x(x-2y)}{2}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=x+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=x$. Therefore equation (4) becomes

$$x=x+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=0$$

Therefore

$$f(y)=c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi=-\frac{x(x-2y)}{2}+c_1$$

But since ϕ itself is a constant function, then let $\phi=c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1=-\frac{x(x-2y)}{2}$$

The solution becomes

$$y = \frac{x^2 + 2c_1}{2x}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2 + 2c_1}{2x} \tag{1}$$

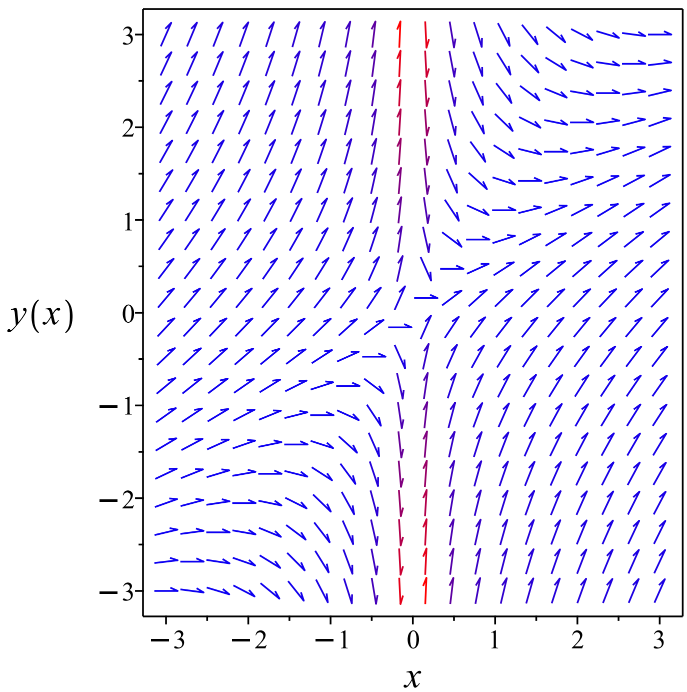


Figure 25: Slope field plot

Verification of solutions

$$y = \frac{x^2 + 2c_1}{2x}$$

Verified OK.

1.7.6 Maple step by step solution

Let's solve

$xy' + y = x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 1 - \frac{y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{x} = 1$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = \mu(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\,dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\,dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x$

  $y = \frac{\int x dx + c_1}{x}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{x^2}{2} + c_1}{x}$

- Simplify

$y = \frac{x^2+2c_1}{2x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(x*diff(y(x),x)+y(x)=x,y(x), singsol=all)
```

$$y(x) = \frac{x}{2} + \frac{c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 17

```
DSolve[x*y'[x]+y[x]==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x}{2} + \frac{c_1}{x}$$

1.8 problem 2(b)

Internal problem ID [3036]
Internal file name [`OUTPUT/2528_Sunday_June_05_2022_03_18_18_AM_29826810/index.tex`]

Book: Elementary Differential equations, Chaundy, 1969
Section: Exercises 3, page 60
Problem number: 2(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{xy' - y = x^3}$$

1.8.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = x^2$$

Hence the ode is

$$y' - \frac{y}{x} = x^2$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(x^2\right) \\ \frac{d}{dx}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)\left(x^2\right) \\ d\left(\frac{y}{x}\right) &= x\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x} &= \int x\, dx \\ \frac{y}{x} &= \frac{x^2}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = \frac{1}{2}x^3 + c_1 x$$

Summary
The solution(s) found are the following

$$y = \frac{1}{2}x^3 + c_1 x \tag{1}$$

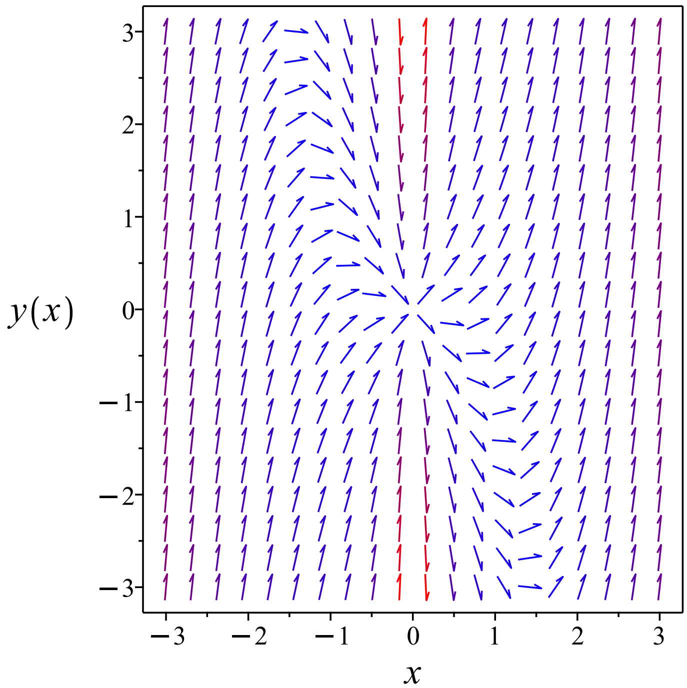


Figure 26: Slope field plot

Verification of solutions

$$y = \frac{1}{2}x^3 + c_1 x$$

Verified OK.

1.8.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x(u'(x)\,x + u(x)) - u(x)\,x = x^3$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int x \,\mathrm{d}x \\ &= \frac{x^2}{2} + c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= ux \\ &= x\left(\frac{x^2}{2} + c_2\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x\left(\frac{x^2}{2} + c_2\right) \tag{1}$$

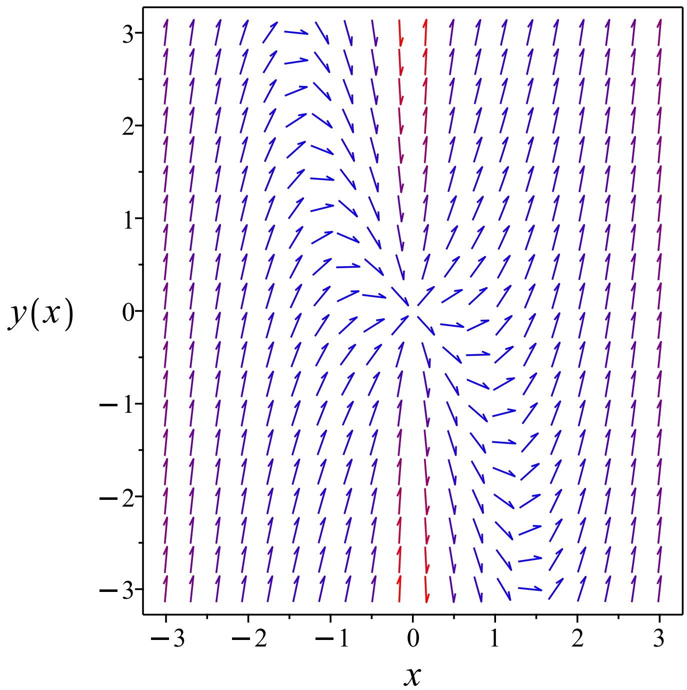


Figure 27: Slope field plot

Verification of solutions

$$y = x\left(\frac{x^2}{2} + c_2\right)$$

Verified OK.

1.8.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^3 + y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 22: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R=x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S=\frac{y}{x}$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR}=\frac{S_x+\omega(x,y)S_y}{R_x+\omega(x,y)R_y} \tag{2}$$

Where in the above R_x,R_y,S_x,S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y)=\frac{x^3+y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR}=x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x,y in terms of R,S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR}=R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = \frac{x^2}{2} + c_1$$

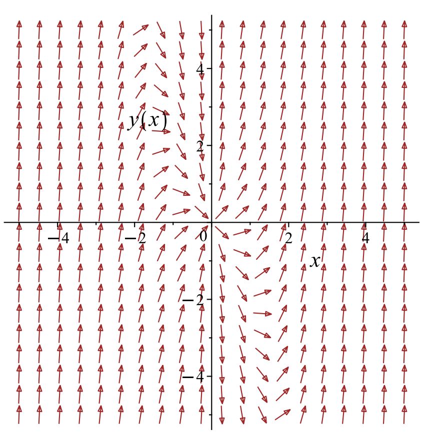
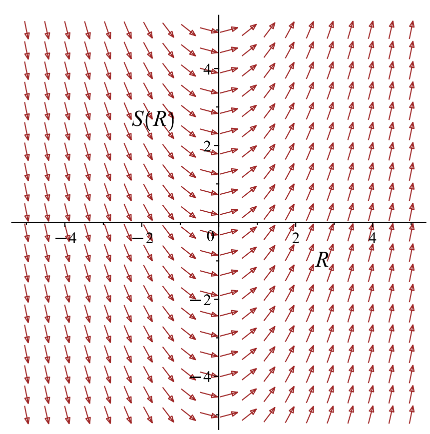
Which simplifies to

$$\frac{y}{x} = \frac{x^2}{2} + c_1$$

Which gives

$$y = \frac{x(x^2 + 2c_1)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^3+y}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = R$ 

Summary

The solution(s) found are the following

$$y = \frac{x(x^2 + 2c_1)}{2} \tag{1}$$

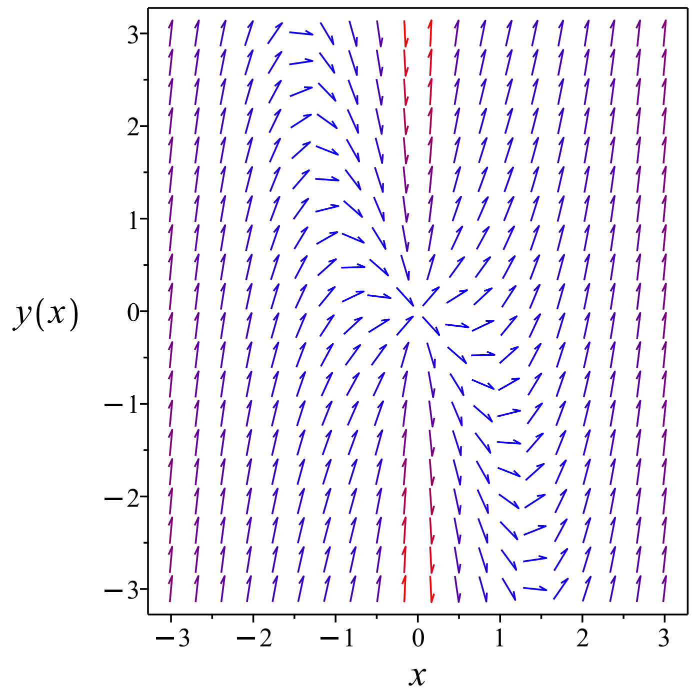


Figure 28: Slope field plot

Verification of solutions

$$y = \frac{x(x^2 + 2c_1)}{2}$$

Verified OK.

1.8.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y&=\left(x^3+y\right)\mathrm{d}x\\ \left(-x^3-y\right)\mathrm{d}x+(x)\,\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-x^3-y\\ N(x,y)&=x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-x^3-y\right)\\ &=-1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x}((-1) - (1))\\ &= -\frac{2}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\frac{2}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(x)}\\ &= \frac{1}{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^2}(-x^3 - y)\\ &= \frac{-x^3 - y}{x^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x^2}(x)\\ &= \frac{1}{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{-x^3 - y}{x^2}\right) + \left(\frac{1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{-x^3 - y}{x^2}\,\mathrm{d}x$$
$$\phi = \frac{-x^3 + 2y}{2x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x}$. Therefore equation (4) becomes

$$\frac{1}{x} = \frac{1}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-x^3 + 2y}{2x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-x^3 + 2y}{2x}$$

The solution becomes

$$y = \frac{x(x^2 + 2c_1)}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{x(x^2 + 2c_1)}{2} \tag{1}$$

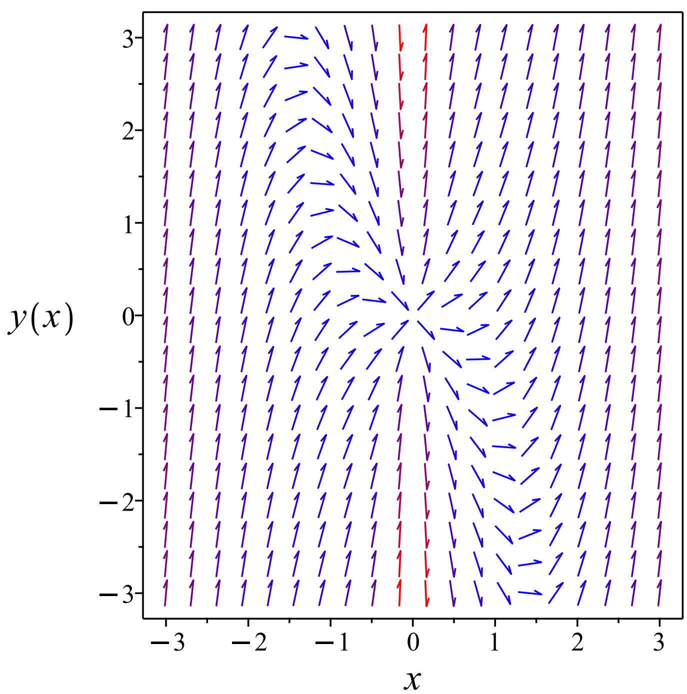


Figure 29: Slope field plot

Verification of solutions

$$y = \frac{x(x^2 + 2c_1)}{2}$$

Verified OK.

1.8.5 Maple step by step solution

Let's solve

$xy' - y = x^3$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{x} + x^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x} = x^2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu(x)\, x^2$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, x^2 dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, x^2 dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) x^2 dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x}$

$$y = x\left(\int x dx + c_1\right)$$

- Evaluate the integrals on the rhs

$$y = x\left(\frac{x^2}{2} + c_1\right)$$

- Simplify

$$y = \frac{x(x^2+2c_1)}{2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(x*diff(y(x),x)-y(x)=x^3,y(x), singsol=all)
```

$$y(x) = \frac{(x^2 + 2c_1)\, x}{2}$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 17

```
DSolve[x*y'[x]-y[x]==x^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^3}{2} + c_1 x$$

1.9 problem 2(c)

Internal problem ID [3037]
Internal file name [`OUTPUT/2529_Sunday_June_05_2022_03_18_20_AM_42066283/index.tex`]

Book: Elementary Differential equations, Chaundy, 1969
Section: Exercises 3, page 60
Problem number: 2(c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{xy' + ny = x^n}$$

1.9.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{n}{x}$$
$$q(x) = x^{n-1}$$

Hence the ode is

$$y' + \frac{ny}{x} = x^{n-1}$$

The integrating factor μ is

$$\mu = e^{\int \frac{n}{x} dx}$$
$$= e^{n \ln(x)}$$

Which simplifies to

$$\mu = x^n$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(x^{n-1}\right) \\
\frac{d}{dx}(x^n y) &= (x^n)\left(x^{n-1}\right) \\
d(x^n y) &= x^{2n-1}\,dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
x^n y &= \int x^{2n-1}\,dx \\
x^n y &= \frac{x^{2n}}{2n} + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^n$ results in

$$y = \frac{x^{-n}x^{2n}}{2n} + c_1 x^{-n}$$

which simplifies to

$$y = \frac{x^n}{2n} + c_1 x^{-n}$$

Summary

The solution(s) found are the following

$$y = \frac{x^n}{2n} + c_1 x^{-n} \tag{1}$$

Verification of solutions

$$y = \frac{x^n}{2n} + c_1 x^{-n}$$

Verified OK.

1.9.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{-ny + x^n}{x} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 25: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = 0$$
$$\eta(x,y) = \mathrm{e}^{-n\ln(x)} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-n\ln(x)}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{n\ln(x)} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-ny + x^n}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= ny\, x^{n-1} \\ S_y &= x^n \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x^{2n-1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^{2n-1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^{2n}}{2n} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x^n y = \frac{x^{2n}}{2n} + c_1$$

Which simplifies to

$$x^n y = \frac{x^{2n}}{2n} + c_1$$

Which gives

$$y = \frac{(2nc_1 + x^{2n})\, x^{-n}}{2n}$$

Summary

The solution(s) found are the following

$$y = \frac{(2nc_1 + x^{2n})\, x^{-n}}{2n} \tag{1}$$

Verification of solutions

$$y = \frac{(2nc_1 + x^{2n})\, x^{-n}}{2n}$$

Verified OK.

1.9.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y&=(-ny+x^n)\,\mathrm{d}x\\ (ny-x^n)\,\mathrm{d}x+(x)\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=ny-x^n\\ N(x,y)&=x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(ny - x^n) \\ &= n\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x}((n) - (1)) \\ &= \frac{n-1}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \,\mathrm{d}x} \\ &= e^{\int \frac{n-1}{x} \,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{(n-1)\ln(x)} \\ &= x^{n-1}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= x^{n-1}(ny - x^n) \\ &= (ny - x^n)\,x^{n-1}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= x^{n-1}(x) \\
&= x^n
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\left(ny - x^n\right)x^{n-1}\right) + \left(x^n\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \left(ny - x^n\right)x^{n-1}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = x^n y - \frac{x^{2n}}{2n} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = x^n + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^n$. Therefore equation (4) becomes

$$
x^n = x^n + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x^n y - \frac{x^{2n}}{2n} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^n y - \frac{x^{2n}}{2n}$$

The solution becomes

$$y = \frac{(2nc_1 + x^{2n})\, x^{-n}}{2n}$$

Summary

The solution(s) found are the following

$$y = \frac{(2nc_1 + x^{2n})\, x^{-n}}{2n} \tag{1}$$

Verification of solutions

$$y = \frac{(2nc_1 + x^{2n})\, x^{-n}}{2n}$$

Verified OK.

1.9.4 Maple step by step solution

Let's solve

$xy' + ny = x^n$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{ny}{x} + \frac{x^n}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' + \frac{ny}{x} = \frac{x^n}{x}$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(y' + \frac{ny}{x}\right) = \frac{\mu(x)\, x^n}{x}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

$$\mu(x)\left(y' + \frac{ny}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = \frac{\mu(x)\, n}{x}$$

- Solve to find the integrating factor

$$\mu(x) = x^n$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)\, x^n}{x} dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\, y = \int \frac{\mu(x)\, x^n}{x} dx + c_1$$

- Solve for y

$$y = \frac{\int \frac{\mu(x) x^n}{x} dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = x^n$

$$y = \frac{\int \frac{(x^n)^2}{x} dx + c_1}{x^n}$$

- Evaluate the integrals on the rhs

$$y = \frac{\frac{(x^n)^2}{2n} + c_1}{x^n}$$

- Simplify

$$y = \frac{x^n}{2n} + c_1 x^{-n}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(x*diff(y(x),x)+n*y(x)=x^n,y(x), singsol=all)
```

$$y(x) = \frac{x^n}{2n} + x^{-n}c_1$$

✓ Solution by Mathematica

Time used: 0.05 (sec). Leaf size: 24

```
DSolve[x*y'[x]+n*y[x]==x^n,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^n}{2n} + c_1x^{-n}$$

1.10 problem 2(d)

Internal problem ID [3038]
Internal file name [`OUTPUT/2530_Sunday_June_05_2022_03_18_22_AM_68192248/index.tex`]

Book: Elementary Differential equations, Chaundy, 1969
Section: Exercises 3, page 60
Problem number: 2(d).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{xy' - ny = x^n}$$

1.10.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{n}{x}$$
$$q(x) = x^{n-1}$$

Hence the ode is

$$y' - \frac{ny}{x} = x^{n-1}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{n}{x}dx}$$
$$= e^{-n\ln(x)}$$

Which simplifies to

$$\mu = x^{-n}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(x^{n-1}\right) \\
\frac{d}{dx}\left(x^{-n} y\right) &= \left(x^{-n}\right)\left(x^{n-1}\right) \\
d\left(x^{-n} y\right) &= \frac{1}{x}\, dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
x^{-n} y &= \int \frac{1}{x}\, dx \\
x^{-n} y &= \ln(x) + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^{-n}$ results in

$$y = x^n \ln(x) + c_1 x^n$$

which simplifies to

$$y = (\ln(x) + c_1)\, x^n$$

Summary
The solution(s) found are the following

$$y = (\ln(x) + c_1)\, x^n \tag{1}$$

Verification of solutions

$$y = (\ln(x) + c_1)\, x^n$$

Verified OK.

1.10.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}
y' &= \frac{ny + x^n}{x} \\
y' &= \omega(x, y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 28: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \mathrm{e}^{n \ln(x)}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{n \ln(x)}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{-n \ln(x)} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{ny + x^n}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -ny\, x^{-1-n} \\ S_y &= x^{-n} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x^{-n}y = \ln(x) + c_1$$

Which simplifies to

$$x^{-n}y = \ln(x) + c_1$$

Which gives

$$y = (\ln(x) + c_1) x^n$$

Summary
The solution(s) found are the following

$$y = (\ln(x) + c_1) x^n \tag{1}$$

Verification of solutions

$$y = (\ln(x) + c_1) x^n$$

Verified OK.

1.10.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x) \, \mathrm{d}y &= (ny + x^n) \, \mathrm{d}x \\ (-ny - x^n) \, \mathrm{d}x + (x) \, \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -ny - x^n \\ N(x, y) &= x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-ny - x^n) \\ &= -n\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x}((-n) - (1)) \\ &= \frac{-1-n}{x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \frac{-1-n}{x} \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{(-1-n)\ln(x)} \\ &= x^{-1-n} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= x^{-1-n}(-ny - x^n) \\ &= \frac{-1 - x^{-n}ny}{x} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= x^{-1-n}(x) \\ &= x^{-n} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{-1 - x^{-n}ny}{x}\right) + \left(x^{-n}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{-1 - x^{-n} n y}{x} \, \mathrm{d}x$$

$$\phi = x^{-n} y + \frac{\ln \left(x^{-n}\right)}{n} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^{-n} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^{-n}$. Therefore equation (4) becomes

$$x^{-n} = x^{-n} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x^{-n} y + \frac{\ln \left(x^{-n}\right)}{n} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^{-n} y + \frac{\ln \left(x^{-n}\right)}{n}$$

The solution becomes

$$y = -\frac{(-nc_1 + \ln(x^{-n}))\, x^n}{n}$$

Summary

The solution(s) found are the following

$$y = -\frac{(-nc_1 + \ln(x^{-n}))\, x^n}{n} \tag{1}$$

Verification of solutions

$$y = -\frac{(-nc_1 + \ln(x^{-n}))\, x^n}{n}$$

Verified OK.

1.10.4 Maple step by step solution

Let's solve

$xy' - ny = x^n$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{ny}{x} + \frac{x^n}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{ny}{x} = \frac{x^n}{x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{ny}{x}\right) = \frac{\mu(x)x^n}{x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{ny}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)n}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x^n}$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x) x^n}{x} dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\, y = \int \frac{\mu(x) x^n}{x} dx + c_1$$

- Solve for y

$$y = \frac{\int \frac{\mu(x) x^n}{x} dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \frac{1}{x^n}$

$$y = x^n \left(\int \frac{1}{x} dx + c_1\right)$$

- Evaluate the integrals on the rhs

$$y = (\ln (x) + c_1)\, x^n$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 12

```
dsolve(x*diff(y(x),x)-n*y(x)=x^n,y(x), singsol=all)
```

$$y(x) = (\ln (x) + c_1)\, x^n$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 14

```
DSolve[x*y'[x]-n*y[x]==x^n,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^n(\log(x) + c_1)$$

1.11 problem 2(e)

Internal problem ID [3039]
Internal file name [`OUTPUT/2531_Sunday_June_05_2022_03_18_24_AM_5097122/index.tex`]

Book: Elementary Differential equations, Chaundy, 1969
Section: Exercises 3, page 60
Problem number: 2(e).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{(x^3 + x)\, y' + y = x}$$

1.11.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{x\,(x^2 + 1)}$$
$$q(x) = \frac{1}{x^2 + 1}$$

Hence the ode is

$$y' + \frac{y}{x\,(x^2 + 1)} = \frac{1}{x^2 + 1}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{1}{x\left(x^2+1\right)}dx} \\
&= e^{-\frac{\ln\left(x^2+1\right)}{2}+\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
\mu = \frac{x}{\sqrt{x^2+1}}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{1}{x^2+1}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{xy}{\sqrt{x^2+1}}\right) &= \left(\frac{x}{\sqrt{x^2+1}}\right)\left(\frac{1}{x^2+1}\right) \\
\mathrm{d}\left(\frac{xy}{\sqrt{x^2+1}}\right) &= \left(\frac{x}{(x^2+1)^{\frac{3}{2}}}\right)\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{xy}{\sqrt{x^2+1}} &= \int \frac{x}{(x^2+1)^{\frac{3}{2}}}\mathrm{d}x \\
\frac{xy}{\sqrt{x^2+1}} &= -\frac{1}{\sqrt{x^2+1}} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{x}{\sqrt{x^2+1}}$ results in

$$
y = -\frac{1}{x} + \frac{c_1\sqrt{x^2+1}}{x}
$$

which simplifies to

$$
y = \frac{c_1\sqrt{x^2+1} - 1}{x}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\sqrt{x^2+1} - 1}{x} \tag{1}
$$

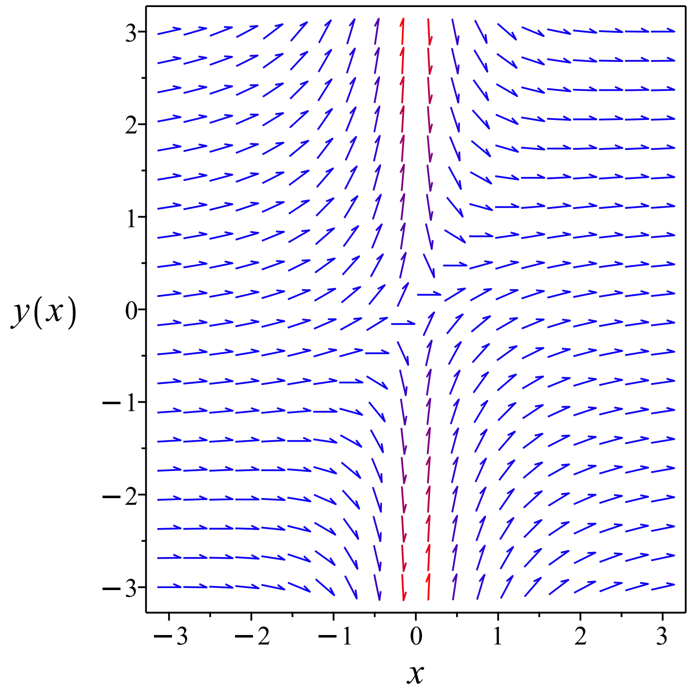


Figure 30: Slope field plot

Verification of solutions

$$y = \frac{c_1\sqrt{x^2+1}-1}{x}$$

Verified OK.

1.11.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y-x}{x\,(x^2+1)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 31: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \mathrm{e}^{\frac{\ln\left(x^2+1\right)}{2} - \ln(x)}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{\ln(x^2+1)}{2} - \ln(x)}} dy \end{aligned}$$

Which results in

$$S = \frac{xy}{\sqrt{x^2+1}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y-x}{x(x^2+1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y}{(x^2+1)^{\frac{3}{2}}} \\ S_y &= \frac{x}{\sqrt{x^2+1}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{x}{(x^2+1)^{\frac{3}{2}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R}{(R^2+1)^{\frac{3}{2}}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{1}{\sqrt{R^2+1}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{xy}{\sqrt{x^2+1}} = -\frac{1}{\sqrt{x^2+1}} + c_1$$

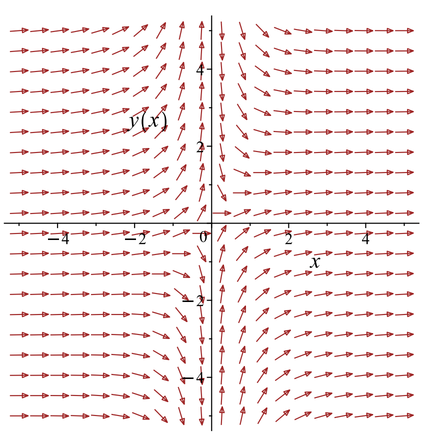
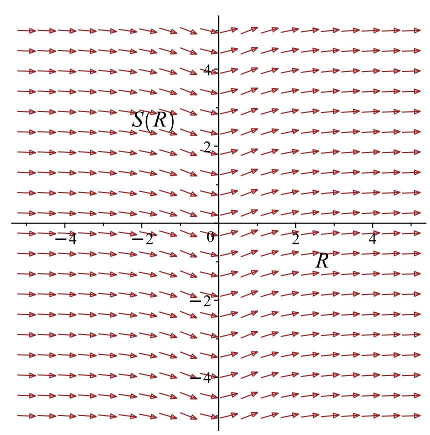
Which simplifies to

$$\frac{xy}{\sqrt{x^2+1}} = -\frac{1}{\sqrt{x^2+1}} + c_1$$

Which gives

$$y = \frac{c_1\sqrt{x^2+1} - 1}{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y-x}{x(x^2+1)}$ 	$R = x$ $S = \frac{xy}{\sqrt{x^2+1}}$	$\frac{dS}{dR} = \frac{R}{(R^2+1)^{\frac{3}{2}}}$ 

Summary
The solution(s) found are the following

$$y = \frac{c_1\sqrt{x^2+1}-1}{x} \tag{1}$$

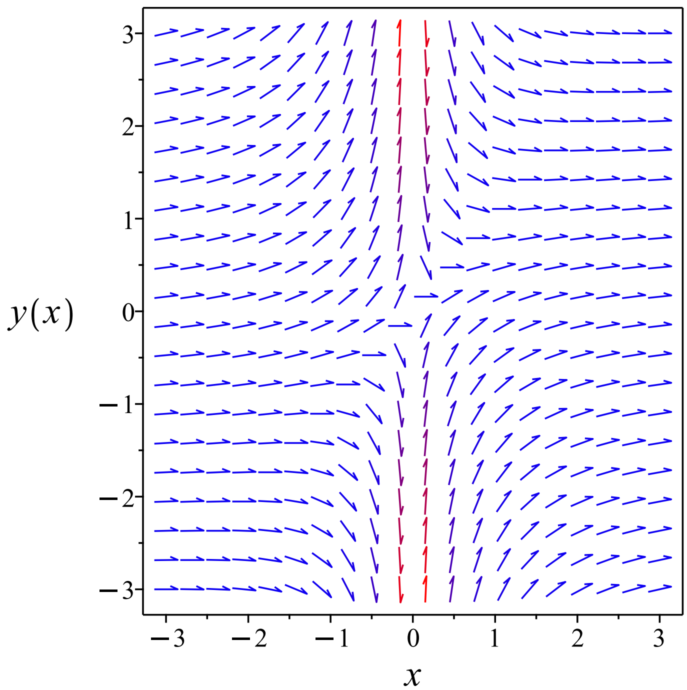


Figure 31: Slope field plot

Verification of solutions

$$y = \frac{c_1\sqrt{x^2+1}-1}{x}$$

Verified OK.

1.11.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x^3+x\right)\mathrm{d}y&=(-y+x)\,\mathrm{d}x\\ (y-x)\,\mathrm{d}x+\left(x^3+x\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=y-x\\ N(x,y)&=x^3+x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(y-x)\\ &=1\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^3 + x\right) \\
&= 3x^2 + 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x^3 + x}\left((1) - \left(3x^2 + 1\right)\right) \\
&= -\frac{3x}{x^2 + 1}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{3x}{x^2+1} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{3 \ln\left(x^2+1\right)}{2}} \\
&= \frac{1}{\left(x^2 + 1\right)^{\frac{3}{2}}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{\left(x^2 + 1\right)^{\frac{3}{2}}}(y - x) \\
&= \frac{y - x}{\left(x^2 + 1\right)^{\frac{3}{2}}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{\left(x^2 + 1\right)^{\frac{3}{2}}}\left(x^3 + x\right) \\
&= \frac{x}{\sqrt{x^2 + 1}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\left(\frac{y-x}{\left(x^2+1\right)^{\frac{3}{2}}}\right) + \left(\frac{x}{\sqrt{x^2+1}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{y-x}{\left(x^2+1\right)^{\frac{3}{2}}}\,\mathrm{d}x$$

$$\phi = \frac{xy+1}{\sqrt{x^2+1}} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x}{\sqrt{x^2+1}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x}{\sqrt{x^2+1}}$. Therefore equation (4) becomes

$$\frac{x}{\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{xy+1}{\sqrt{x^2+1}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{xy+1}{\sqrt{x^2+1}}$$

The solution becomes

$$y = \frac{c_1\sqrt{x^2+1}-1}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\sqrt{x^2+1}-1}{x} \tag{1}$$

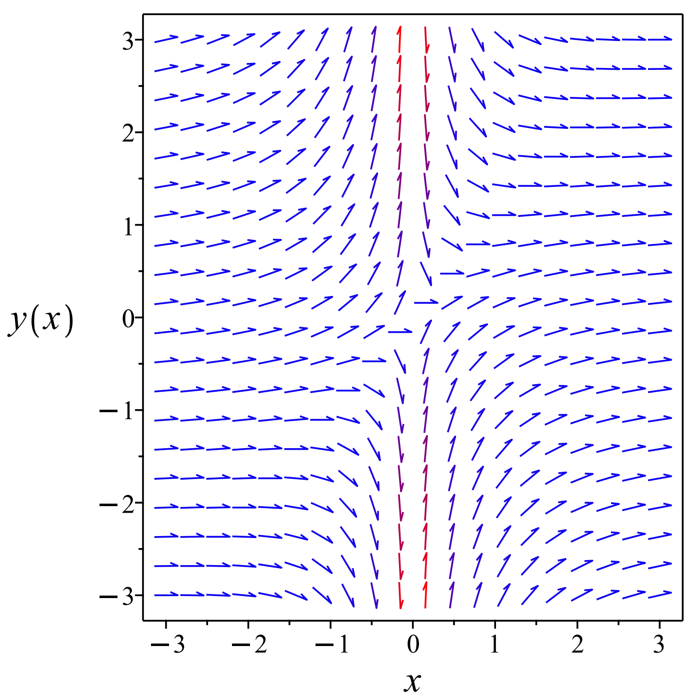


Figure 32: Slope field plot

Verification of solutions

$$y = \frac{c_1 \sqrt{x^2+1} - 1}{x}$$

Verified OK.

1.11.4 Maple step by step solution

Let's solve

$(x^3+x)\, y' + y = x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{x(x^2+1)} + \frac{1}{x^2+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{x(x^2+1)} = \frac{1}{x^2+1}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x) \left(y' + \frac{y}{x(x^2+1)}\right) = \frac{\mu(x)}{x^2+1}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x) \left(y' + \frac{y}{x(x^2+1)}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x(x^2+1)}$

- Solve to find the integrating factor

  $\mu(x) = \frac{x}{\sqrt{x^2+1}}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)}{x^2+1} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)}{x^2+1} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)}{x^2+1} dx + c_1}{\mu(x)}$

- Substitute $\mu(x)=\frac{x}{\sqrt{x^2+1}}$

  $$y=\frac{\sqrt{x^2+1}\left(\int \frac{x}{\left(x^2+1\right)^{\frac{3}{2}}}dx+c_1\right)}{x}$$

- Evaluate the integrals on the rhs

  $$y=\frac{\sqrt{x^2+1}\left(-\frac{1}{\sqrt{x^2+1}}+c_1\right)}{x}$$

- Simplify

  $$y=\frac{c_1\sqrt{x^2+1}-1}{x}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve((x^3+x)*diff(y(x),x)+y(x)=x,y(x), singsol=all)
```

$$y(x)=\frac{\sqrt{x^2+1}\,c_1-1}{x}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 23

```
DSolve[(x^3+x)*y'[x]+y[x]==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{-1+c_1\sqrt{x^2+1}}{x}$$

1.12 problem 3(a)

Internal problem ID [3040]
Internal file name [`OUTPUT/2532_Sunday_June_05_2022_03_18_27_AM_74647826/index.tex`]

Book: Elementary Differential equations, Chaundy, 1969
Section: Exercises 3, page 60
Problem number: 3(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{\cot(x)\, y' + y = x}$$

1.12.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \tan(x) \\ q(x) &= \tan(x)\, x \end{aligned}$$

Hence the ode is

$$y' + y\tan(x) = \tan(x)\, x$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \tan(x)dx} \\ &= \frac{1}{\cos(x)} \end{aligned}$$

Which simplifies to

$$\mu = \sec(x)$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(\tan(x)\,x)\\ \frac{d}{dx}(\sec(x)\,y) &= (\sec(x))(\tan(x)\,x)\\ d(\sec(x)\,y) &= (x\sec(x)\tan(x))\;dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\sec(x)\,y &= \int x\sec(x)\tan(x)\;dx\\ \sec(x)\,y &= \frac{x}{\cos(x)} - \ln(\sec(x)+\tan(x)) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sec(x)$ results in

$$y = \cos(x)\left(\frac{x}{\cos(x)} - \ln(\sec(x)+\tan(x))\right) + c_1\cos(x)$$

which simplifies to

$$y = -\ln(\sec(x)+\tan(x))\cos(x) + c_1\cos(x) + x$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\ln(\sec(x)+\tan(x))\cos(x) + c_1\cos(x) + x \tag{1}$$

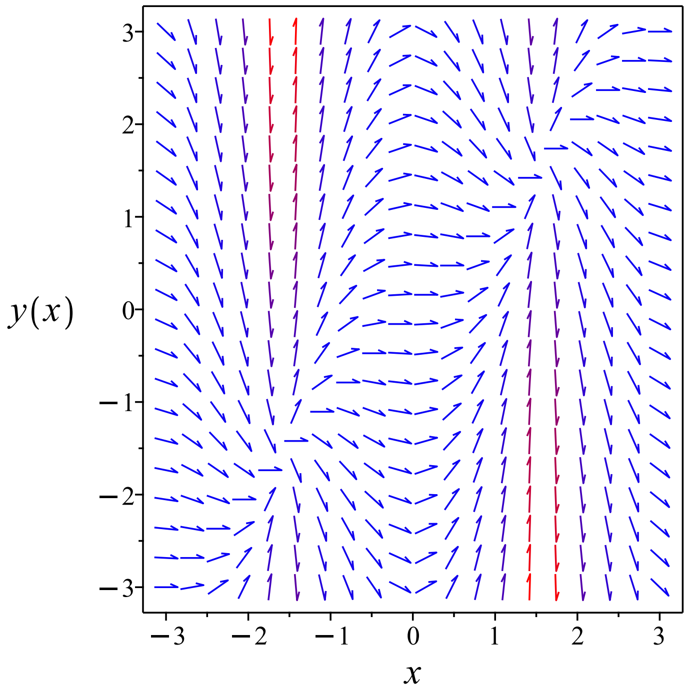


Figure 33: Slope field plot

Verification of solutions

$$y = -\ln(\sec(x) + \tan(x))\cos(x) + c_1\cos(x) + x$$

Verified OK.

1.12.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y - x}{\cot(x)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 34: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= \cos(x) \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\cos(x)} dy \end{aligned}$$

Which results in

$$S = \frac{y}{\cos(x)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y-x}{\cot(x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \sec(x)\tan(x)\,y \\ S_y &= \sec(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x\sec(x)\tan(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R\sec(R)\tan(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R}{\cos(R)} - \ln(\sec(R) + \tan(R)) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y \sec(x) = \frac{x}{\cos(x)} - \ln(\sec(x) + \tan(x)) + c_1$$

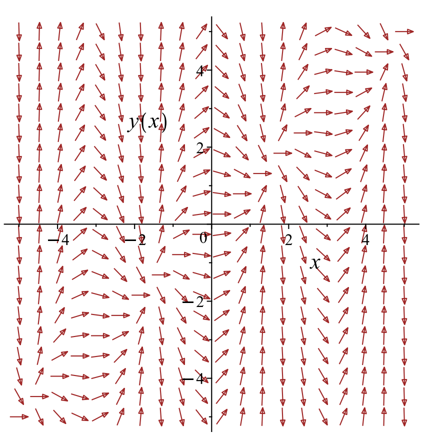
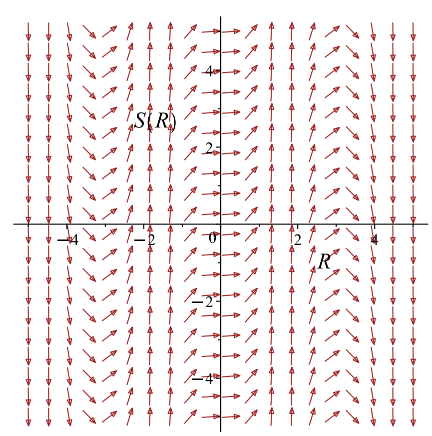
Which simplifies to

$$\ln(\sec(x) + \tan(x)) + \sec(x)(y - x) - c_1 = 0$$

Which gives

$$y = \frac{\sec(x) x - \ln(\sec(x) + \tan(x)) + c_1}{\sec(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y-x}{\cot(x)}$ 	$R = x$ $S = \sec(x) y$	$\frac{dS}{dR} = R \sec(R) \tan(R)$ 

Summary
The solution(s) found are the following

$$y = \frac{\sec(x)\,x - \ln(\sec(x) + \tan(x)) + c_1}{\sec(x)} \tag{1}$$

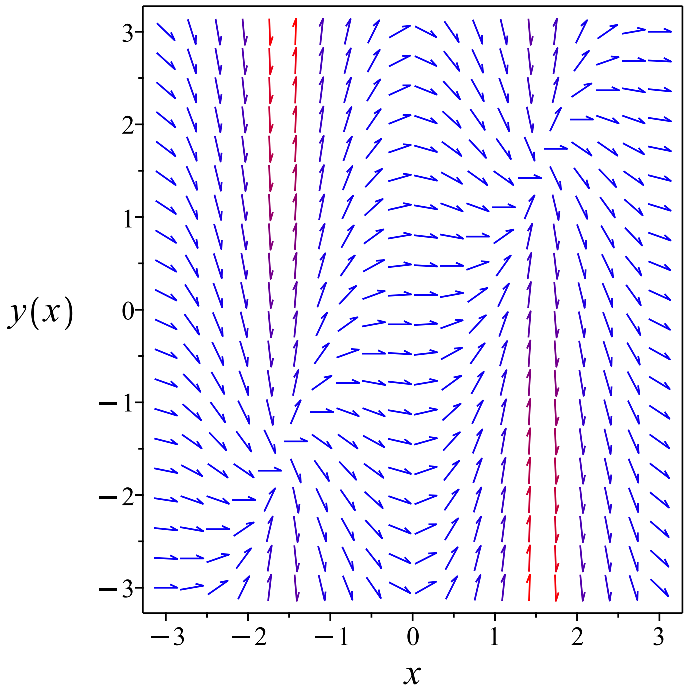


Figure 34: Slope field plot

Verification of solutions

$$y = \frac{\sec(x)\,x - \ln(\sec(x) + \tan(x)) + c_1}{\sec(x)}$$

Verified OK.

1.12.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\cot(x))\,\mathrm{d}y &= (-y+x)\,\mathrm{d}x\\ (y-x)\,\mathrm{d}x + (\cot(x))\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y-x\\ N(x,y) &= \cot(x)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y-x)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\cot(x))\\ &= -\csc(x)^2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \tan(x)\left((1) - \left(-1-\cot(x)^2\right)\right)\\ &= 2\tan(x) + \cot(x)\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int 2\tan(x)+\cot(x)\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(\cos(x))+\ln(\sin(x))}\\ &= \frac{\sin(x)}{\cos(x)^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{\sin(x)}{\cos(x)^2}(y-x)\\ &= \sec(x)\tan(x)(y-x)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{\sin(x)}{\cos(x)^2}(\cot(x))\\ &= \sec(x)\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
(\sec(x)\tan(x)(y-x)) + (\sec(x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \sec(x)\tan(x)(y-x)\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \ln(\sec(x)+\tan(x)) + \sec(x)(y-x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \sec(x) + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sec(x)$. Therefore equation (4) becomes

$$
\sec(x) = \sec(x) + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = \ln(\sec(x)+\tan(x)) + \sec(x)(y-x) + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(\sec(x) + \tan(x)) + \sec(x)(y - x)$$

The solution becomes

$$y = \frac{\sec(x) x - \ln(\sec(x) + \tan(x)) + c_1}{\sec(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{\sec(x) x - \ln(\sec(x) + \tan(x)) + c_1}{\sec(x)} \tag{1}$$

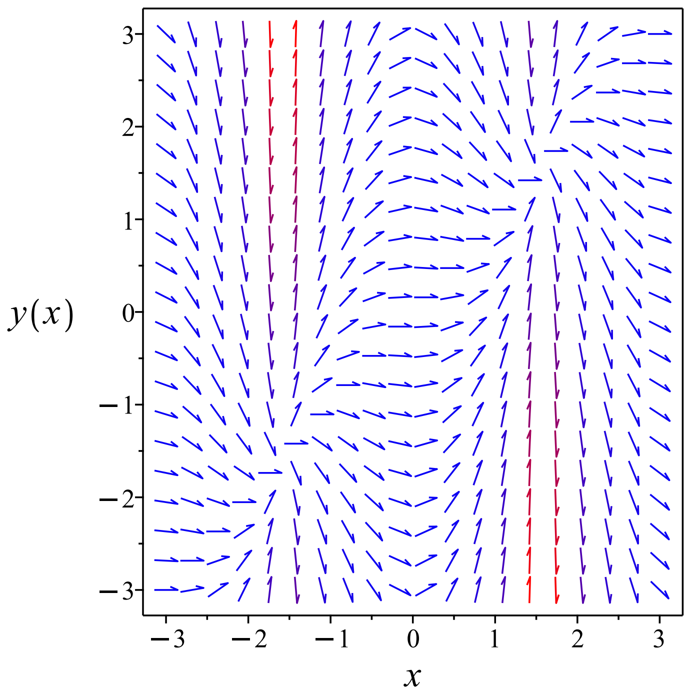


Figure 35: Slope field plot

Verification of solutions

$$y = \frac{\sec(x) x - \ln(\sec(x) + \tan(x)) + c_1}{\sec(x)}$$

Verified OK.

1.12.4 Maple step by step solution

Let's solve

$\cot(x) y' + y = x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{\cot(x)} + \frac{x}{\cot(x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{\cot(x)} = \frac{x}{\cot(x)}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{\cot(x)}\right) = \frac{\mu(x)x}{\cot(x)}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x) y)$

  $\mu(x)\left(y' + \frac{y}{\cot(x)}\right) = \mu'(x) y + \mu(x) y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{\cot(x)}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{\cos(x)}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x) y)\right) dx = \int \frac{\mu(x)x}{\cot(x)} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x) y = \int \frac{\mu(x)x}{\cot(x)} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)x}{\cot(x)} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{\cos(x)}$

  $y = \cos(x)\left(\int \frac{x}{\cos(x)\cot(x)} dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = \cos(x)\left(\frac{x}{\cos(x)} - \ln(\sec(x) + \tan(x)) + c_1\right)$

- Simplify

  $$y = -\ln(\sec(x) + \tan(x))\cos(x) + c_1\cos(x) + x$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(cot(x)*diff(y(x),x)+y(x)=x,y(x), singsol=all)
```

$$y(x) = x + \cos(x)\left(-\ln(\sec(x) + \tan(x)) + c_1\right)$$

✓ Solution by Mathematica

Time used: 0.081 (sec). Leaf size: 45

```
DSolve[Cot[x]*y'[x]+y[x]==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x + \cos(x)\left(\log\left(\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)\right) - \log\left(\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)\right) + c_1\right)$$

1.13 problem 3(b)

Internal problem ID [3041]
Internal file name [`OUTPUT/2533_Sunday_June_05_2022_03_18_29_AM_13567649/index.tex`]

Book: Elementary Differential equations, Chaundy, 1969
Section: Exercises 3, page 60
Problem number: 3(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{\cot(x)\,y' + y = \tan(x)}$$

1.13.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \tan(x) \\ q(x) &= \tan(x)^2 \end{aligned}$$

Hence the ode is

$$y' + y\tan(x) = \tan(x)^2$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \tan(x)dx} \\ &= \frac{1}{\cos(x)} \end{aligned}$$

Which simplifies to

$$\mu = \sec(x)$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\tan(x)^2\right)$$
$$\frac{d}{dx}(\sec(x)\, y) = (\sec(x))\left(\tan(x)^2\right)$$
$$d(\sec(x)\, y) = \left(\tan(x)^2 \sec(x)\right)\, dx$$

Integrating gives

$$\sec(x)\, y = \int \tan(x)^2 \sec(x)\, dx$$
$$\sec(x)\, y = \frac{\sin(x)^3}{2\cos(x)^2} + \frac{\sin(x)}{2} - \frac{\ln(\sec(x) + \tan(x))}{2} + c_1$$

Dividing both sides by the integrating factor $\mu = \sec(x)$ results in

$$y = \cos(x)\left(\frac{\sin(x)^3}{2\cos(x)^2} + \frac{\sin(x)}{2} - \frac{\ln(\sec(x) + \tan(x))}{2}\right) + c_1 \cos(x)$$

which simplifies to

$$y = \frac{\tan(x)}{2} - \frac{\ln(\sec(x) + \tan(x))\cos(x)}{2} + c_1 \cos(x)$$

Summary

The solution(s) found are the following

$$y = \frac{\tan(x)}{2} - \frac{\ln(\sec(x) + \tan(x))\cos(x)}{2} + c_1 \cos(x) \tag{1}$$

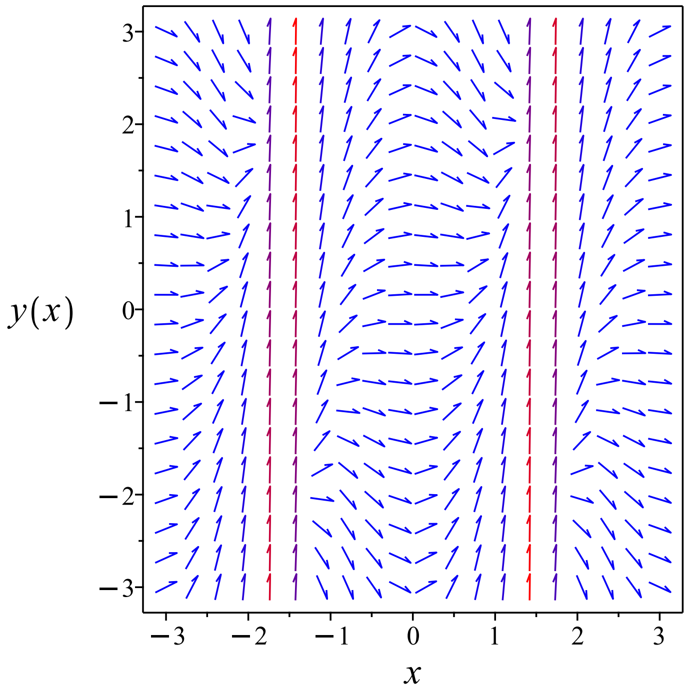


Figure 36: Slope field plot

Verification of solutions

$$y = \frac{\tan(x)}{2} - \frac{\ln(\sec(x) + \tan(x))\cos(x)}{2} + c_1 \cos(x)$$

Verified OK.

1.13.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y - \tan(x)}{\cot(x)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 37: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \cos(x) \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\cos(x)} dy \end{aligned}$$

Which results in

$$S = \frac{y}{\cos(x)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y - \tan(x)}{\cot(x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \sec(x) \tan(x) y \\ S_y &= \sec(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \tan(x)^2 \sec(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \tan(R)^2 \sec(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{\sin(R)^3}{2\cos(R)^2} + \frac{\sin(R)}{2} - \frac{\ln(\sec(R) + \tan(R))}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\sec(x) = \frac{\sin(x)^3}{2\cos(x)^2} + \frac{\sin(x)}{2} - \frac{\ln(\sec(x) + \tan(x))}{2} + c_1$$

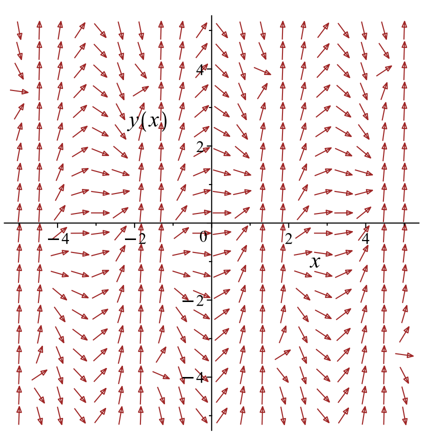
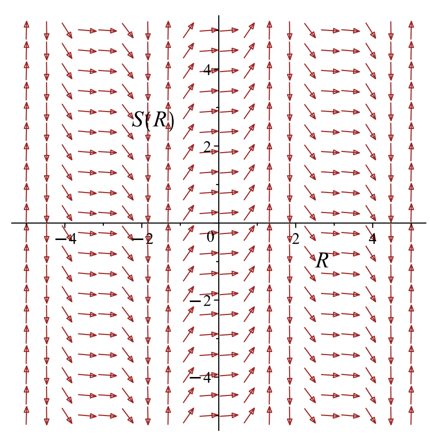
Which simplifies to

$$\frac{\ln(\sec(x) + \tan(x))}{2} + \frac{(2y - \tan(x))\sec(x)}{2} - c_1 = 0$$

Which gives

$$y = \frac{\sec(x)\tan(x) - \ln(\sec(x) + \tan(x)) + 2c_1}{2\sec(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{y - \tan(x)}{\cot(x)}$

</td><td>$R = x$
$S = \sec(x)\,y$</td><td>$\frac{dS}{dR} = \tan(R)^2\sec(R)$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \frac{\sec(x)\tan(x) - \ln(\sec(x) + \tan(x)) + 2c_1}{2\sec(x)} \tag{1}$$

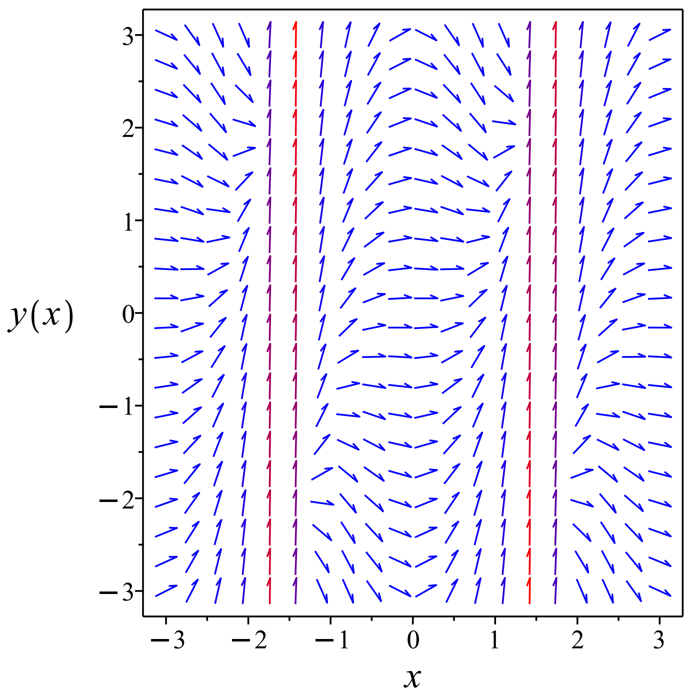


Figure 37: Slope field plot

Verification of solutions

$$y = \frac{\sec(x)\tan(x) - \ln(\sec(x) + \tan(x)) + 2c_1}{2\sec(x)}$$

Verified OK.

1.13.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(\cot(x))\,\mathrm{d}y&=(-y+\tan(x))\,\mathrm{d}x\\ (y-\tan(x))\,\mathrm{d}x+(\cot(x))\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=y-\tan(x)\\ N(x,y)&=\cot(x)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y - \tan(x)) \\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\cot(x)) \\ &= -\csc(x)^2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \tan(x)\left((1) - \left(-1 - \cot(x)^2\right)\right) \\ &= 2\tan(x) + \cot(x)\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int 2\tan(x)+\cot(x) \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(\cos(x))+\ln(\sin(x))} \\ &= \frac{\sin(x)}{\cos(x)^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{\sin(x)}{\cos(x)^2}(y - \tan(x)) \\ &= \sec(x)\tan(x)(y - \tan(x))\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{\sin(x)}{\cos(x)^2}(\cot(x)) \\ &= \sec(x)\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$(\sec(x)\tan(x)(y-\tan(x))) + (\sec(x))\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \sec(x)\tan(x)(y-\tan(x))\,\mathrm{d}x$$
$$\phi = \frac{\ln(\sec(x)+\tan(x))}{2} + \frac{(2y-\tan(x))\sec(x)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = \sec(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \sec(x)$. Therefore equation (4) becomes

$$\sec(x) = \sec(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\ln(\sec(x) + \tan(x))}{2} + \frac{(2y - \tan(x))\sec(x)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\ln(\sec(x) + \tan(x))}{2} + \frac{(2y - \tan(x))\sec(x)}{2}$$

The solution becomes

$$y = \frac{\sec(x)\tan(x) - \ln(\sec(x) + \tan(x)) + 2c_1}{2\sec(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{\sec(x)\tan(x) - \ln(\sec(x) + \tan(x)) + 2c_1}{2\sec(x)} \tag{1}$$

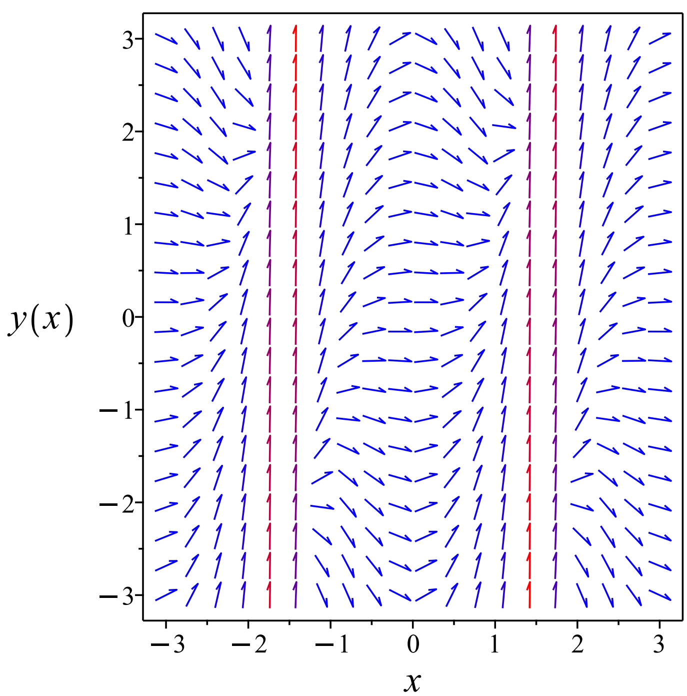


Figure 38: Slope field plot

Verification of solutions

$$y = \frac{\sec(x)\tan(x) - \ln(\sec(x) + \tan(x)) + 2c_1}{2\sec(x)}$$

Verified OK.

1.13.4 Maple step by step solution

Let's solve

$\cot(x)\, y' + y = \tan(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{\cot(x)} + \frac{\tan(x)}{\cot(x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{\cot(x)} = \frac{\tan(x)}{\cot(x)}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{\cot(x)}\right) = \frac{\mu(x)\tan(x)}{\cot(x)}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{y}{\cot(x)}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{\cot(x)}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{\cos(x)}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)\tan(x)}{\cot(x)} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)\tan(x)}{\cot(x)} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)\tan(x)}{\cot(x)} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{\cos(x)}$

  $$y = \cos(x) \left(\int \frac{\tan(x)}{\cos(x)\cot(x)} dx + c_1\right)$$

- Evaluate the integrals on the rhs

  $$y = \cos(x) \left(\frac{\sin(x)^3}{2\cos(x)^2} + \frac{\sin(x)}{2} - \frac{\ln(\sec(x)+\tan(x))}{2} + c_1\right)$$

- Simplify

  $$y = \frac{\tan(x)}{2} - \frac{\ln(\sec(x)+\tan(x))\cos(x)}{2} + c_1 \cos(x)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(cot(x)*diff(y(x),x)+y(x)=tan(x),y(x), singsol=all)
```

$$y(x) = \frac{\tan(x)}{2} - \frac{\cos(x)\ln(\sec(x)+\tan(x))}{2} + \cos(x) c_1$$

✓ Solution by Mathematica

Time used: 0.071 (sec). Leaf size: 25

```
DSolve[Cot[x]*y'[x]+y[x]==Tan[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}(\cos(x)(-\text{arctanh}(\sin(x))) + \tan(x) + 2c_1\cos(x))$$

1.14 problem 3(c)

Internal problem ID [3042]
Internal file name [`OUTPUT/2534_Sunday_June_05_2022_03_18_31_AM_93861156/index.tex`]

Book: Elementary Differential equations, Chaundy, 1969
Section: Exercises 3, page 60
Problem number: 3(c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y' \tan (x) + y = \cot (x)}$$

1.14.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \cot (x)$$
$$q(x) = \cot (x)^2$$

Hence the ode is

$$y' + y \cot (x) = \cot (x)^2$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int \cot(x)dx}$$
$$= \sin (x)$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\cot(x)^2\right)$$
$$\frac{d}{dx}(\sin(x)\, y) = (\sin(x))\left(\cot(x)^2\right)$$
$$d(\sin(x)\, y) = (\cos(x)\cot(x))\, dx$$

Integrating gives

$$\sin(x)\, y = \int \cos(x)\cot(x)\, dx$$
$$\sin(x)\, y = \cos(x) + \ln(\csc(x) - \cot(x)) + c_1$$

Dividing both sides by the integrating factor $\mu = \sin(x)$ results in

$$y = \csc(x)(\cos(x) + \ln(\csc(x) - \cot(x))) + c_1 \csc(x)$$

which simplifies to

$$y = \csc(x)(\cos(x) + \ln(\csc(x) - \cot(x)) + c_1)$$

Summary

The solution(s) found are the following

$$y = \csc(x)(\cos(x) + \ln(\csc(x) - \cot(x)) + c_1) \tag{1}$$

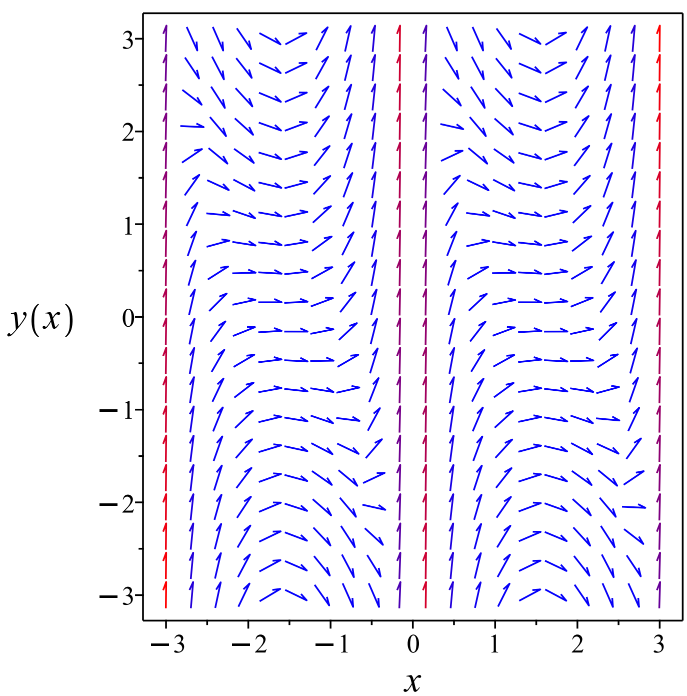


Figure 39: Slope field plot

Verification of solutions

$$y = \csc(x)(\cos(x) + \ln(\csc(x) - \cot(x)) + c_1)$$

Verified OK.

1.14.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y - \cot(x)}{\tan(x)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 40: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0\\ \eta(x,y) &= \frac{1}{\sin(x)}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\sin(x)}} dy \end{aligned}$$

Which results in

$$S = \sin(x)\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y - \cot(x)}{\tan(x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \cos(x)\, y \\ S_y &= \sin(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \cos(x)\cot(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \cos(R)\cot(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \cos(R) + \ln(\csc(R) - \cot(R)) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\sin(x) = \cos(x) + \ln(\csc(x) - \cot(x)) + c_1$$

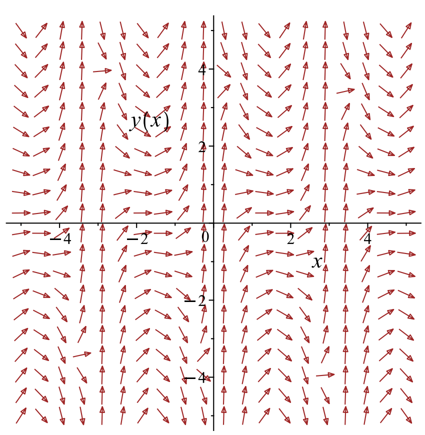
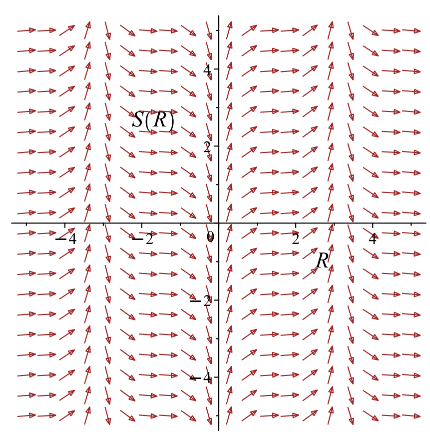
Which simplifies to

$$y\sin(x) = \cos(x) + \ln(\csc(x) - \cot(x)) + c_1$$

Which gives

$$y = \frac{\cos(x) + \ln(\csc(x) - \cot(x)) + c_1}{\sin(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y-\cot(x)}{\tan(x)}$ 	$R = x$ $S = \sin(x)\,y$	$\frac{dS}{dR} = \cos(R)\cot(R)$ 

Summary

The solution(s) found are the following

$$y = \frac{\cos(x) + \ln(\csc(x) - \cot(x)) + c_1}{\sin(x)} \tag{1}$$

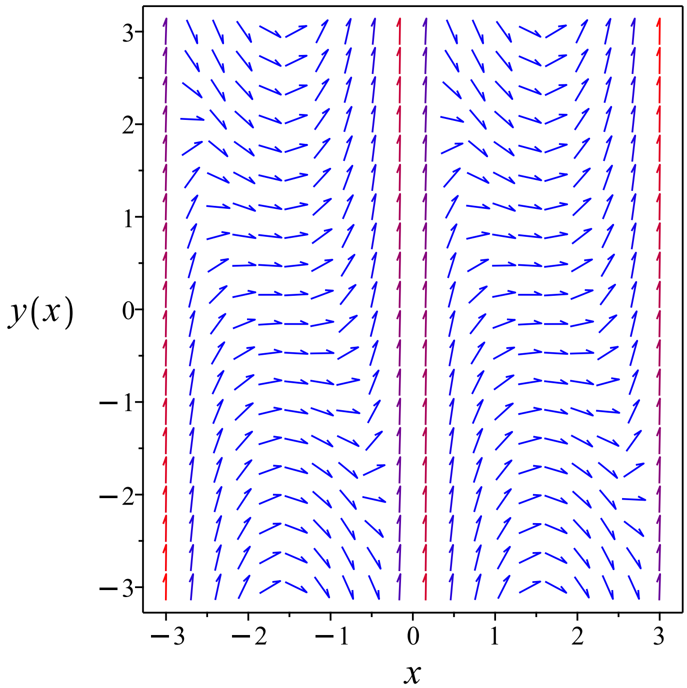


Figure 40: Slope field plot

Verification of solutions

$$y = \frac{\cos(x) + \ln(\csc(x) - \cot(x)) + c_1}{\sin(x)}$$

Verified OK.

1.14.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(\tan(x))\,\mathrm{d}y = (-y + \cot(x))\,\mathrm{d}x$$
$$(y - \cot(x))\,\mathrm{d}x + (\tan(x))\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = y - \cot(x)$$
$$N(x, y) = \tan(x)$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y - \cot(x)) \\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\tan(x)) \\ &= \sec(x)^2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \cot(x)\left((1) - \left(1 + \tan(x)^2\right)\right) \\ &= -\tan(x) \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -\tan(x) \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\ln(\cos(x))} \\ &= \cos(x) \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \cos(x)(y - \cot(x)) \\ &= (y - \cot(x))\cos(x) \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \cos(x)(\tan(x)) \\ &= \sin(x) \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ ((y - \cot(x))\cos(x)) + (\sin(x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int (y - \cot(x)) \cos(x) \, \mathrm{d}x \\
\phi &= \sin(x) \, y - \cos(x) - \ln(\csc(x) - \cot(x)) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \sin(x) + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sin(x)$. Therefore equation (4) becomes

$$
\sin(x) = \sin(x) + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = \sin(x) \, y - \cos(x) - \ln(\csc(x) - \cot(x)) + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = \sin(x) \, y - \cos(x) - \ln(\csc(x) - \cot(x))
$$

The solution becomes

$$
y = \frac{\cos(x) + \ln(\csc(x) - \cot(x)) + c_1}{\sin(x)}
$$

Summary

The solution(s) found are the following

$$y = \frac{\cos(x) + \ln(\csc(x) - \cot(x)) + c_1}{\sin(x)} \tag{1}$$

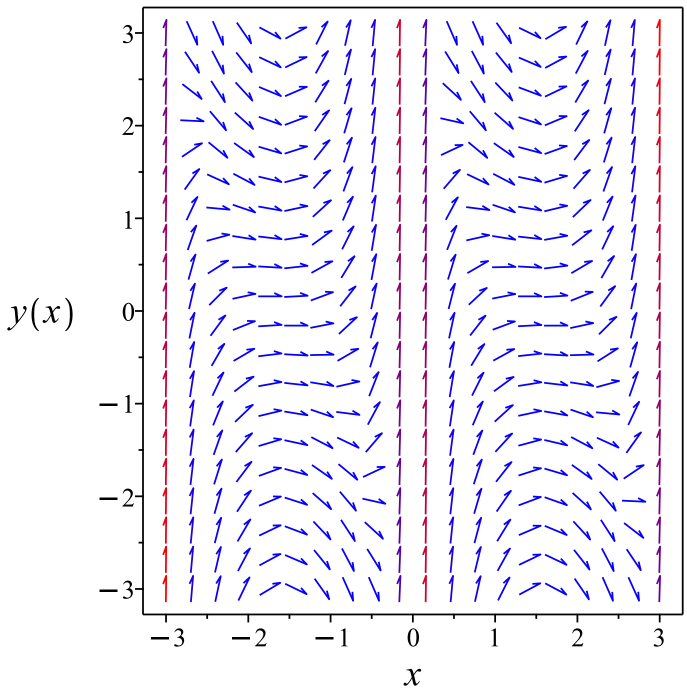


Figure 41: Slope field plot

Verification of solutions

$$y = \frac{\cos(x) + \ln(\csc(x) - \cot(x)) + c_1}{\sin(x)}$$

Verified OK.

1.14.4 Maple step by step solution

Let's solve

$y' \tan(x) + y = \cot(x)$

- Highest derivative means the order of the ODE is 1

y'

- Isolate the derivative

$$y' = -\frac{y}{\tan(x)} + \frac{\cot(x)}{\tan(x)}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' + \frac{y}{\tan(x)} = \frac{\cot(x)}{\tan(x)}$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(y' + \frac{y}{\tan(x)}\right) = \frac{\mu(x)\cot(x)}{\tan(x)}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$$\mu(x)\left(y' + \frac{y}{\tan(x)}\right) = \mu'(x)\,y + \mu(x)\,y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = \frac{\mu(x)}{\tan(x)}$$

- Solve to find the integrating factor

$$\mu(x) = \sin(x)$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \frac{\mu(x)\cot(x)}{\tan(x)} dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\,y = \int \frac{\mu(x)\cot(x)}{\tan(x)} dx + c_1$$

- Solve for y

$$y = \frac{\int \frac{\mu(x)\cot(x)}{\tan(x)} dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \sin(x)$

$$y = \frac{\int \frac{\sin(x)\cot(x)}{\tan(x)} dx + c_1}{\sin(x)}$$

- Evaluate the integrals on the rhs

$$y = \frac{\cos(x) + \ln(\csc(x) - \cot(x)) + c_1}{\sin(x)}$$

- Simplify

$$y = \csc(x)\left(\cos(x) + \ln(\csc(x) - \cot(x)) + c_1\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(tan(x)*diff(y(x),x)+y(x)=cot(x),y(x), singsol=all)
```

$$y(x) = \csc(x)(\cos(x) + \ln(\csc(x) - \cot(x)) + c_1)$$

✓ Solution by Mathematica

Time used: 0.085 (sec). Leaf size: 29

```
DSolve[Tan[x]*y'[x]+y[x]==Cot[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \csc(x)\left(\cos(x) + \log\left(\sin\left(\frac{x}{2}\right)\right) - \log\left(\cos\left(\frac{x}{2}\right)\right) + c_1\right)$$

1.15 problem 3(a)

Internal problem ID [3043]
Internal file name [`OUTPUT/2535_Sunday_June_05_2022_03_18_33_AM_46989943/index.tex`]

Book: Elementary Differential equations, Chaundy, 1969
Section: Exercises 3, page 60
Problem number: 3(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' \tan(x) - y = -\cos(x)}$$

1.15.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\cot(x) \\ q(x) &= -\cos(x)\cot(x) \end{aligned}$$

Hence the ode is

$$y' - y\cot(x) = -\cos(x)\cot(x)$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\cot(x)dx} \\ &= \frac{1}{\sin(x)} \end{aligned}$$

Which simplifies to

$$\mu = \csc(x)$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)(-\cos(x)\cot(x)) \\
\frac{d}{dx}(\csc(x)\, y) &= (\csc(x))(-\cos(x)\cot(x)) \\
d(\csc(x)\, y) &= \left(-\cot(x)^2\right)\, dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\csc(x)\, y &= \int -\cot(x)^2\, dx \\
\csc(x)\, y &= \cot(x) - \frac{\pi}{2} + x + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \csc(x)$ results in

$$y = \sin(x)\left(\cot(x) - \frac{\pi}{2} + x\right) + c_1 \sin(x)$$

which simplifies to

$$y = \sin(x)\left(\cot(x) - \frac{\pi}{2} + x + c_1\right)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \sin(x)\left(\cot(x) - \frac{\pi}{2} + x + c_1\right) \tag{1}$$

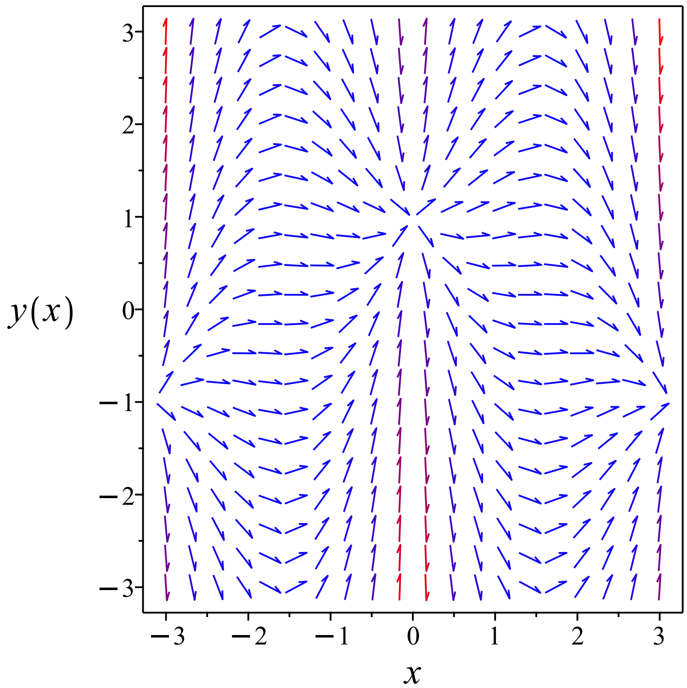


Figure 42: Slope field plot

Verification of solutions

$$y = \sin(x) \left(\cot(x) - \frac{\pi}{2} + x + c_1\right)$$

Verified OK.

1.15.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{-y + \cos(x)}{\tan(x)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 43: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \sin(x) \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\sin(x)} dy \end{aligned}$$

Which results in

$$S = \frac{y}{\sin(x)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-y + \cos(x)}{\tan(x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\csc(x)\cot(x)\, y \\ S_y &= \csc(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\cot(x)^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\cot(R)^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \cot(R) - \frac{\pi}{2} + R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\csc(x)\, y = \cot(x) - \frac{\pi}{2} + x + c_1$$

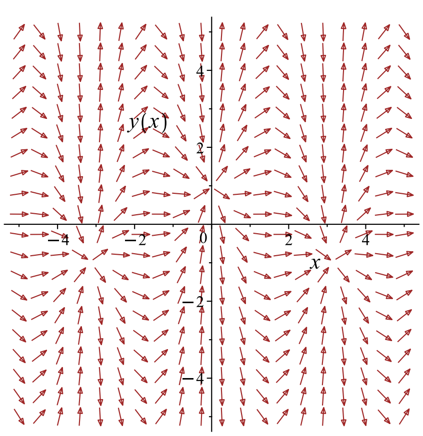
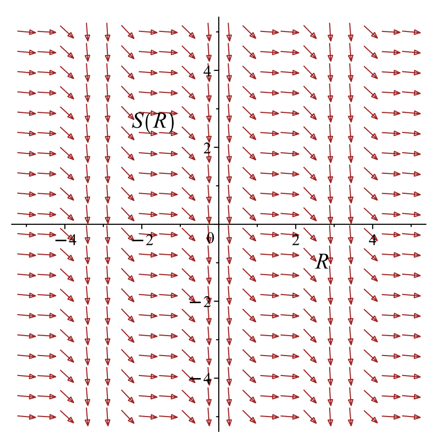
Which simplifies to

$$\csc(x)\, y = \cot(x) - \frac{\pi}{2} + x + c_1$$

Which gives

$$y = -\frac{-2\cot(x) + \pi - 2x - 2c_1}{2\csc(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{-y+\cos(x)}{\tan(x)}$
</td><td>$R = x$
$S = \csc(x)\, y$</td><td>$\frac{dS}{dR} = -\cot(R)^2$
</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = -\frac{-2\cot(x) + \pi - 2x - 2c_1}{2\csc(x)} \tag{1}$$

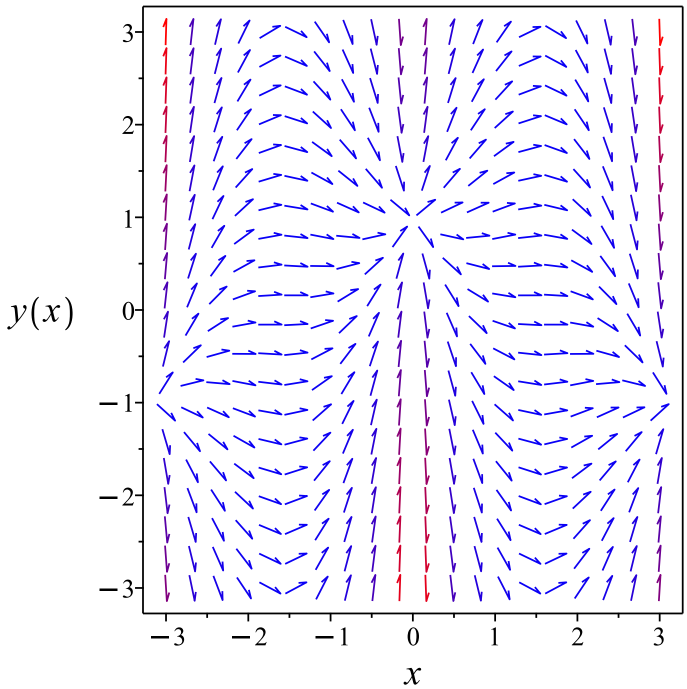


Figure 43: Slope field plot

Verification of solutions

$$y = -\frac{-2\cot(x) + \pi - 2x - 2c_1}{2\csc(x)}$$

Verified OK.

1.15.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\tan(x))\,\mathrm{d}y &= (y - \cos(x))\,\mathrm{d}x\\ (-y + \cos(x))\,\mathrm{d}x + (\tan(x))\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -y + \cos(x)\\ N(x,y) &= \tan(x)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y+\cos(x)) \\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\tan(x)) \\ &= \sec(x)^2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \cot(x)\left((-1) - \left(1+\tan(x)^2\right)\right) \\ &= -2\cot(x) - \tan(x)\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x} \\ &= e^{\int -2\cot(x)-\tan(x)\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(\sin(x))+\ln(\cos(x))} \\ &= \frac{\cos(x)}{\sin(x)^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{\cos(x)}{\sin(x)^2}(-y+\cos(x)) \\ &= -\cot(x)(\csc(x)\,y - \cot(x))\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{\cos(x)}{\sin(x)^2}(\tan(x)) \\ &= \csc(x)\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\cot\left(x\right)\left(\csc\left(x\right)y-\cot\left(x\right)\right)\right) + \left(\csc\left(x\right)\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\cot\left(x\right)\left(\csc\left(x\right)y-\cot\left(x\right)\right)\mathrm{d}x \\
\phi &= -x-\cot\left(x\right)+\csc\left(x\right)y+f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \csc\left(x\right) + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \csc\left(x\right)$. Therefore equation (4) becomes

$$
\csc\left(x\right) = \csc\left(x\right) + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = -x-\cot\left(x\right)+\csc\left(x\right)y+c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x - \cot(x) + \csc(x) y$$

The solution becomes

$$y = \frac{\cot(x) + x + c_1}{\csc(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{\cot(x) + x + c_1}{\csc(x)} \tag{1}$$

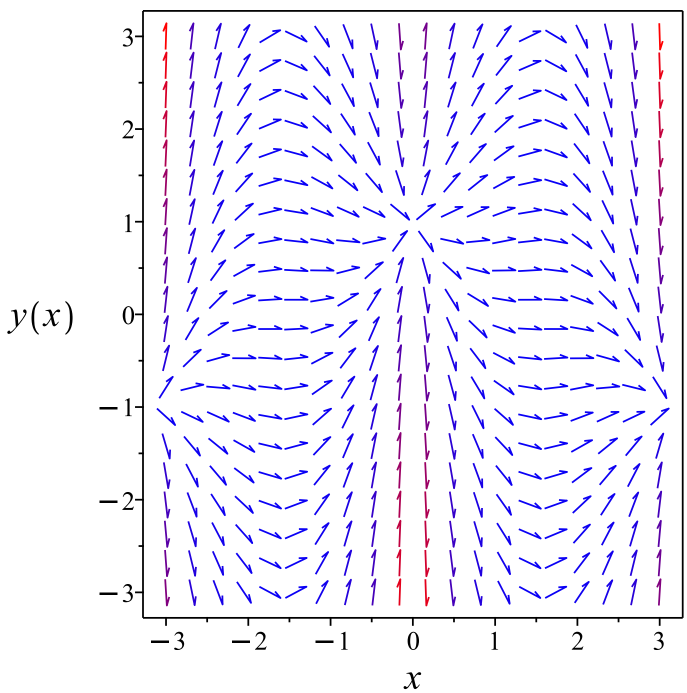


Figure 44: Slope field plot

Verification of solutions

$$y = \frac{\cot(x) + x + c_1}{\csc(x)}$$

Verified OK.

1.15.4 Maple step by step solution

Let's solve

$y' \tan(x) - y = -\cos(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{\tan(x)} - \frac{\cos(x)}{\tan(x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{\tan(x)} = -\frac{\cos(x)}{\tan(x)}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{\tan(x)}\right) = -\frac{\mu(x)\cos(x)}{\tan(x)}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{y}{\tan(x)}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{\tan(x)}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{\sin(x)}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int -\frac{\mu(x)\cos(x)}{\tan(x)} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int -\frac{\mu(x)\cos(x)}{\tan(x)} dx + c_1$

- Solve for y

  $y = \frac{\int -\frac{\mu(x)\cos(x)}{\tan(x)} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{\sin(x)}$

  $y = \sin(x)\left(\int -\frac{\cos(x)}{\sin(x)\tan(x)} dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = \sin(x)\left(\cot(x) + x + c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(tan(x)*diff(y(x),x)=y(x)-cos(x),y(x), singsol=all)
```

$$y(x) = \left(\cot\left(x\right) - \frac{\pi}{2} + x + c_1\right)\sin\left(x\right)$$

✓ Solution by Mathematica

Time used: 0.061 (sec). Leaf size: 28

```
DSolve[Tan[x]*y'[x]==y[x]-Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \cos(x)\,\text{Hypergeometric2F1}\left(-\frac{1}{2}, 1, \frac{1}{2}, -\tan^2(x)\right) + c_1 \sin(x)$$

1.16 problem 4(a)

Internal problem ID [3044]
Internal file name [`OUTPUT/2536_Sunday_June_05_2022_03_18_36_AM_93063267/index.tex`]

Book: Elementary Differential equations, Chaundy, 1969
Section: Exercises 3, page 60
Problem number: 4(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \cos(x)\, y = \sin(2x)}$$

1.16.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \cos(x) \\ q(x) &= \sin(2x) \end{aligned}$$

Hence the ode is

$$y' + \cos(x)\, y = \sin(2x)$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \cos(x) dx} \\ &= e^{\sin(x)} \end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(\sin(2x))$$
$$\frac{d}{dx}\left(e^{\sin(x)}y\right) = \left(e^{\sin(x)}\right)(\sin(2x))$$
$$d\left(e^{\sin(x)}y\right) = \left(\sin(2x)\,e^{\sin(x)}\right)\,dx$$

Integrating gives

$$e^{\sin(x)}y = \int \sin(2x)\,e^{\sin(x)}\,dx$$
$$e^{\sin(x)}y = 2\sin(x)\,e^{\sin(x)} - 2\,e^{\sin(x)} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{\sin(x)}$ results in

$$y = e^{-\sin(x)}\left(2\sin(x)\,e^{\sin(x)} - 2\,e^{\sin(x)}\right) + c_1 e^{-\sin(x)}$$

which simplifies to

$$y = 2\sin(x) - 2 + c_1 e^{-\sin(x)}$$

Summary

The solution(s) found are the following

$$y = 2\sin(x) - 2 + c_1 e^{-\sin(x)} \tag{1}$$

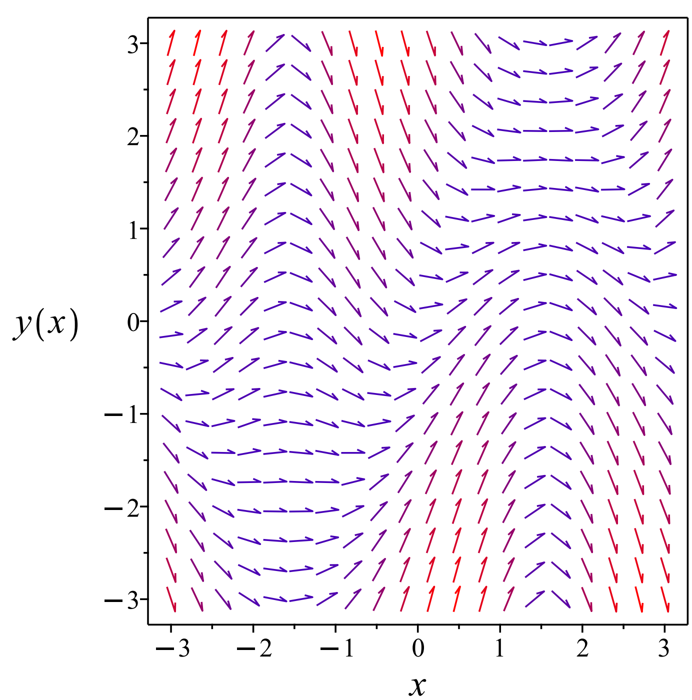


Figure 45: Slope field plot

Verification of solutions

$$y = 2\sin(x) - 2 + c_1 e^{-\sin(x)}$$

Verified OK.

1.16.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\cos(x)\, y + \sin(2x) \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 46: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x, y) = 0$$
$$\eta(x, y) = \mathrm{e}^{-\sin(x)} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-\sin(x)}} dy \end{aligned}$$

Which results in

$$S = e^{\sin(x)} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\cos(x) y + \sin(2x)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \cos(x) e^{\sin(x)} y \\ S_y &= e^{\sin(x)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \sin(2x) e^{\sin(x)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \sin(2R) e^{\sin(R)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 + 2\,\mathrm{e}^{\sin(R)}(-1 + \sin(R)) \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{\sin(x)} y = 2\,\mathrm{e}^{\sin(x)}(\sin(x) - 1) + c_1$$

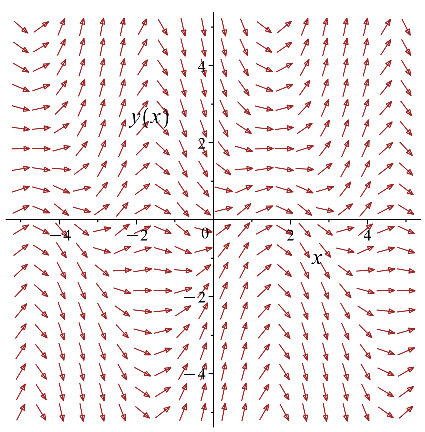
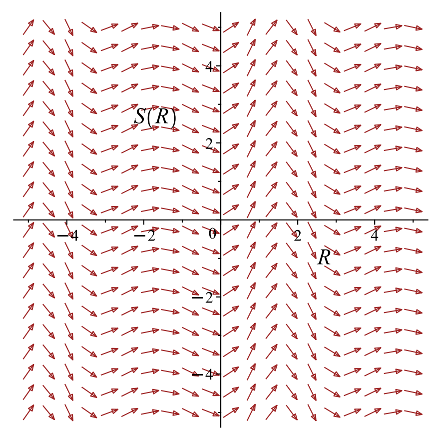
Which simplifies to

$$\mathrm{e}^{\sin(x)} y = 2\,\mathrm{e}^{\sin(x)}(\sin(x) - 1) + c_1$$

Which gives

$$y = \mathrm{e}^{-\sin(x)}\left(2\sin(x)\,\mathrm{e}^{\sin(x)} - 2\,\mathrm{e}^{\sin(x)} + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\cos(x)\,y + \sin(2x)$  	$R = x$ $S = \mathrm{e}^{\sin(x)} y$	$\frac{dS}{dR} = \sin(2R)\,\mathrm{e}^{\sin(R)}$ 

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\sin(x)}\left(2\sin(x)\,\mathrm{e}^{\sin(x)} - 2\,\mathrm{e}^{\sin(x)} + c_1\right) \tag{1}$$

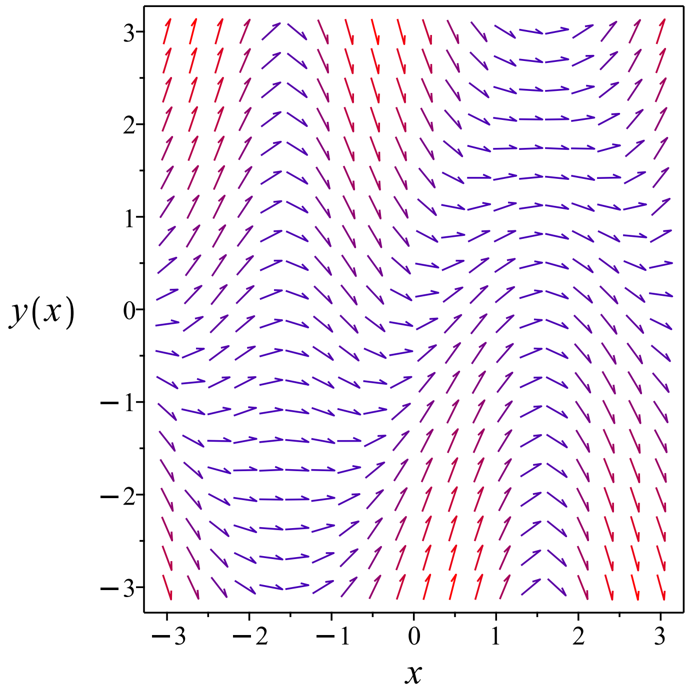


Figure 46: Slope field plot

Verification of solutions

$$y = e^{-\sin(x)} \left(2 \sin(x) e^{\sin(x)} - 2 e^{\sin(x)} + c_1\right)$$

Verified OK.

1.16.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= (-\cos(x)\,y + \sin(2x))\,\mathrm{d}x\\ (\cos(x)\,y - \sin(2x))\,\mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= \cos(x)\,y - \sin(2x)\\ N(x,y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\cos(x)\,y - \sin(2x))\\ &= \cos(x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((\cos(x)) - (0)) \\ &= \cos(x) \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \cos(x) \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\sin(x)} \\ &= \mathrm{e}^{\sin(x)} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{\sin(x)}(\cos(x)\, y - \sin(2x)) \\ &= \mathrm{e}^{\sin(x)} \cos(x)\,(-2\sin(x) + y) \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{\sin(x)}(1) \\ &= \mathrm{e}^{\sin(x)} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\mathrm{e}^{\sin(x)} \cos(x)\,(-2\sin(x) + y)\right) + \left(\mathrm{e}^{\sin(x)}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \mathrm{e}^{\sin(x)} \cos(x) \left(-2\sin(x) + y\right) \mathrm{d}x$$
$$\phi = \left(y - 2\sin(x) + 2\right) \mathrm{e}^{\sin(x)} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{\sin(x)} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{\sin(x)}$. Therefore equation (4) becomes

$$\mathrm{e}^{\sin(x)} = \mathrm{e}^{\sin(x)} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \left(y - 2\sin(x) + 2\right) \mathrm{e}^{\sin(x)} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \left(y - 2\sin(x) + 2\right) \mathrm{e}^{\sin(x)}$$

The solution becomes

$$y = \mathrm{e}^{-\sin(x)} \left(2\sin(x) \mathrm{e}^{\sin(x)} - 2\,\mathrm{e}^{\sin(x)} + c_1\right)$$

Summary

The solution(s) found are the following

$$y = e^{-\sin(x)} \left(2 \sin (x) e^{\sin(x)} - 2 e^{\sin(x)} + c_1\right) \tag{1}$$

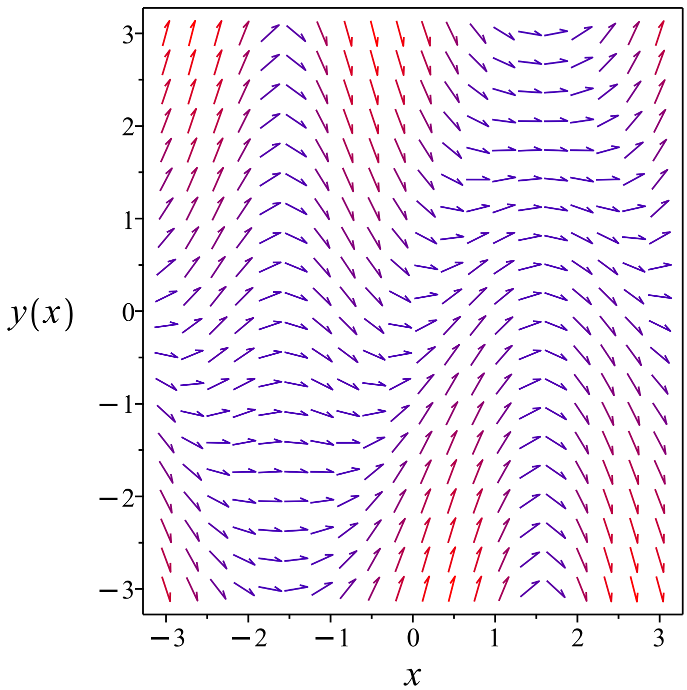


Figure 47: Slope field plot

Verification of solutions

$$y = e^{-\sin(x)} \left(2 \sin (x) e^{\sin(x)} - 2 e^{\sin(x)} + c_1\right)$$

Verified OK.

1.16.4 Maple step by step solution

Let's solve

$y' + \cos (x) y = \sin (2x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\cos (x) y + \sin (2x)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \cos(x) y = \sin(2x)$
- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x) (y' + \cos(x) y) = \mu(x) \sin(2x)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x) y)$

  $\mu(x) (y' + \cos(x) y) = \mu'(x) y + \mu(x) y'$
- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x) \cos(x)$
- Solve to find the integrating factor

  $\mu(x) = e^{\sin(x)}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x) y)\right) dx = \int \mu(x) \sin(2x) dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x) y = \int \mu(x) \sin(2x) dx + c_1$
- Solve for y

  $y = \frac{\int \mu(x) \sin(2x)dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = e^{\sin(x)}$

  $y = \frac{\int \sin(2x)e^{\sin(x)}dx + c_1}{e^{\sin(x)}}$
- Evaluate the integrals on the rhs

  $y = \frac{2\sin(x)e^{\sin(x)} - 2\,e^{\sin(x)} + c_1}{e^{\sin(x)}}$
- Simplify

  $y = 2\sin(x) - 2 + c_1 e^{-\sin(x)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x)+y(x)*cos(x)=sin(2*x),y(x), singsol=all)
```

$$y(x) = 2\sin(x) - 2 + e^{-\sin(x)}c_1$$

✓ Solution by Mathematica

Time used: 0.083 (sec). Leaf size: 20

```
DSolve[y'[x]+y[x]*Cos[x]==Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2\sin(x) + c_1 e^{-\sin(x)} - 2$$

1.17 problem 4(b)

Internal problem ID [3045]
Internal file name [`OUTPUT/2537_Sunday_June_05_2022_03_18_38_AM_92517129/index.tex`]

Book: Elementary Differential equations, Chaundy, 1969
Section: Exercises 3, page 60
Problem number: 4(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' \cos (x) + y = \sin (2x)}$$

1.17.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \sec (x) \\ q(x) &= 2 \sin (x) \end{aligned}$$

Hence the ode is

$$y' + y \sec (x) = 2 \sin (x)$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \sec(x)dx} \\ &= \sec (x) + \tan (x) \end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(2\sin(x))$$
$$\frac{d}{dx}((\sec(x)+\tan(x))\,y) = (\sec(x)+\tan(x))(2\sin(x))$$
$$d((\sec(x)+\tan(x))\,y) = ((2\sin(x)+2)\tan(x))\,dx$$

Integrating gives

$$(\sec(x)+\tan(x))\,y = \int (2\sin(x)+2)\tan(x)\,dx$$
$$(\sec(x)+\tan(x))\,y = -2\sin(x) - 2\ln(\sin(x)-1) + c_1$$

Dividing both sides by the integrating factor $\mu = \sec(x)+\tan(x)$ results in

$$y = \frac{-2\sin(x) - 2\ln(\sin(x)-1)}{\sec(x)+\tan(x)} + \frac{c_1}{\sec(x)+\tan(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{-2\sin(x) - 2\ln(\sin(x)-1)}{\sec(x)+\tan(x)} + \frac{c_1}{\sec(x)+\tan(x)} \tag{1}$$

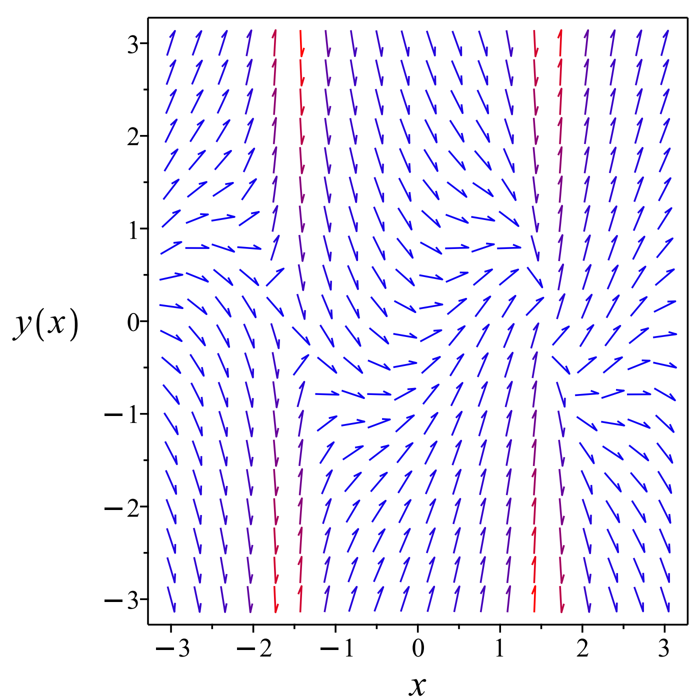


Figure 48: Slope field plot

Verification of solutions

$$y = \frac{-2\sin(x) - 2\ln(\sin(x) - 1)}{\sec(x) + \tan(x)} + \frac{c_1}{\sec(x) + \tan(x)}$$

Verified OK.

1.17.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-y + \sin(2x)}{\cos(x)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 49: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{\sec(x) + \tan(x)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\sec(x)+\tan(x)}} dy \end{aligned}$$

Which results in

$$S = (\sec(x) + \tan(x))\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-y + \sin(2x)}{\cos(x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{\sin(x) - 1} \\ S_y &= \sec(x) + \tan(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\tan(x)(\cos(x) + 1 + \sin(x))^2}{\cos(x) + 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\tan(R)(\cos(R) + 1 + \sin(R))^2}{\cos(R) + 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -2\sin(R) - 2\ln(\sin(R) - 1) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$(\sec(x) + \tan(x))\, y = -2\sin(x) - 2\ln(\sin(x) - 1) + c_1$$

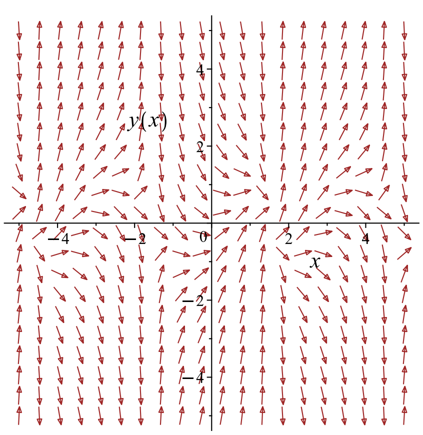
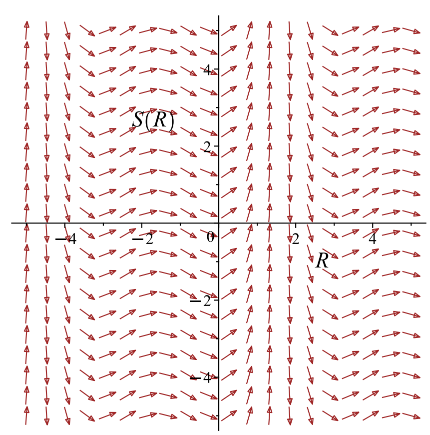
Which simplifies to

$$(\sec(x) + \tan(x))\, y = -2\sin(x) - 2\ln(\sin(x) - 1) + c_1$$

Which gives

$$y = -\frac{2\sin(x) + 2\ln(\sin(x) - 1) - c_1}{\sec(x) + \tan(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-y+\sin(2x)}{\cos(x)}$ 	$R = x$ $S = (\sec(x) + \tan(x))\, y$	$\frac{dS}{dR} = \frac{\tan(R)(\cos(R)+1+\sin(R))^2}{\cos(R)+1}$ 

Summary

The solution(s) found are the following

$$y = -\frac{2\sin(x) + 2\ln(\sin(x) - 1) - c_1}{\sec(x) + \tan(x)} \tag{1}$$

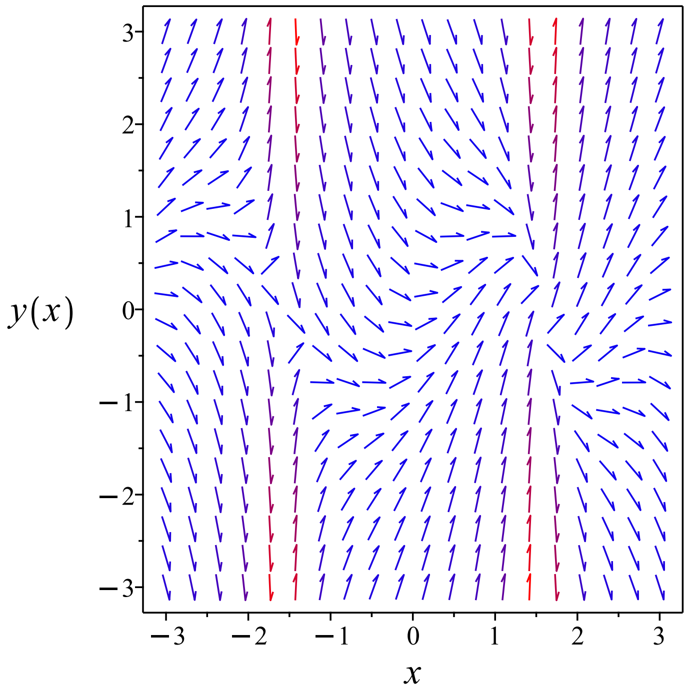


Figure 49: Slope field plot

Verification of solutions

$$y = -\frac{2\sin(x) + 2\ln(\sin(x) - 1) - c_1}{\sec(x) + \tan(x)}$$

Verified OK.

1.17.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(\cos(x))\,\mathrm{d}y = (-y + \sin(2x))\,\mathrm{d}x$$
$$(y - \sin(2x))\,\mathrm{d}x + (\cos(x))\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = y - \sin(2x)$$
$$N(x, y) = \cos(x)$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y - \sin(2x)) \\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\cos(x)) \\ &= -\sin(x)\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \sec(x)\left((1) - (-\sin(x))\right) \\
&= \sec(x) + \tan(x)
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \sec(x)+\tan(x) \, \mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{\ln(\sec(x)+\tan(x))-\ln(\cos(x))} \\
&= \frac{\sec(x) + \tan(x)}{\cos(x)}
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{\sec(x) + \tan(x)}{\cos(x)}(y - \sin(2x)) \\
&= \frac{-y + 2\sin(x)\cos(x)}{\sin(x) - 1}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{\sec(x) + \tan(x)}{\cos(x)}(\cos(x)) \\
&= \sec(x) + \tan(x)
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-y + 2\sin(x)\cos(x)}{\sin(x) - 1}\right) + (\sec(x) + \tan(x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{-y + 2\sin(x)\cos(x)}{\sin(x) - 1} \, \mathrm{d}x$$

$$\phi = \frac{4\tan\left(\frac{x}{2}\right)}{1 + \tan\left(\frac{x}{2}\right)^2} - 2\ln\left(\sec\left(\frac{x}{2}\right)^2\right) - \frac{2y}{-1 + \tan\left(\frac{x}{2}\right)} + 4\ln\left(-1 + \tan\left(\frac{x}{2}\right)\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{2}{-1 + \tan\left(\frac{x}{2}\right)} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sec(x) + \tan(x)$. Therefore equation (4) becomes

$$\sec(x) + \tan(x) = -\frac{2}{-1 + \tan\left(\frac{x}{2}\right)} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{\tan(x)\tan\left(\frac{x}{2}\right) + \sec(x)\tan\left(\frac{x}{2}\right) - \tan(x) - \sec(x) + 2}{-1 + \tan\left(\frac{x}{2}\right)}$$

$$= -1$$

Integrating the above w.r.t y results in

$$\int f'(y) \, \mathrm{d}y = \int (-1) \, \mathrm{d}y$$

$$f(y) = -y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{4\tan\left(\frac{x}{2}\right)}{1+\tan\left(\frac{x}{2}\right)^2} - 2\ln\left(\sec\left(\frac{x}{2}\right)^2\right) - \frac{2y}{-1+\tan\left(\frac{x}{2}\right)} + 4\ln\left(-1+\tan\left(\frac{x}{2}\right)\right) - y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{4\tan\left(\frac{x}{2}\right)}{1+\tan\left(\frac{x}{2}\right)^2} - 2\ln\left(\sec\left(\frac{x}{2}\right)^2\right) - \frac{2y}{-1+\tan\left(\frac{x}{2}\right)} + 4\ln\left(-1+\tan\left(\frac{x}{2}\right)\right) - y$$

The solution becomes

$$y = -\frac{2\tan\left(\frac{x}{2}\right)^3\ln\left(\sec\left(\frac{x}{2}\right)^2\right) - 4\tan\left(\frac{x}{2}\right)^3\ln\left(-1+\tan\left(\frac{x}{2}\right)\right) + \tan\left(\frac{x}{2}\right)^3 c_1 - 2\tan\left(\frac{x}{2}\right)^2\ln\left(\sec\left(\frac{x}{2}\right)^2\right) + 4\,\mathrm{t}}{}$$

Summary
The solution(s) found are the following

$$y = -\frac{2\tan\left(\frac{x}{2}\right)^3\ln\left(\sec\left(\frac{x}{2}\right)^2\right) - 4\tan\left(\frac{x}{2}\right)^3\ln\left(-1+\tan\left(\frac{x}{2}\right)\right) + \tan\left(\frac{x}{2}\right)^3 c_1 - 2\tan\left(\frac{x}{2}\right)^2\ln\left(\sec\left(\frac{x}{2}\right)^2\right) + 4\,\mathrm{t}}{} \tag{1}$$

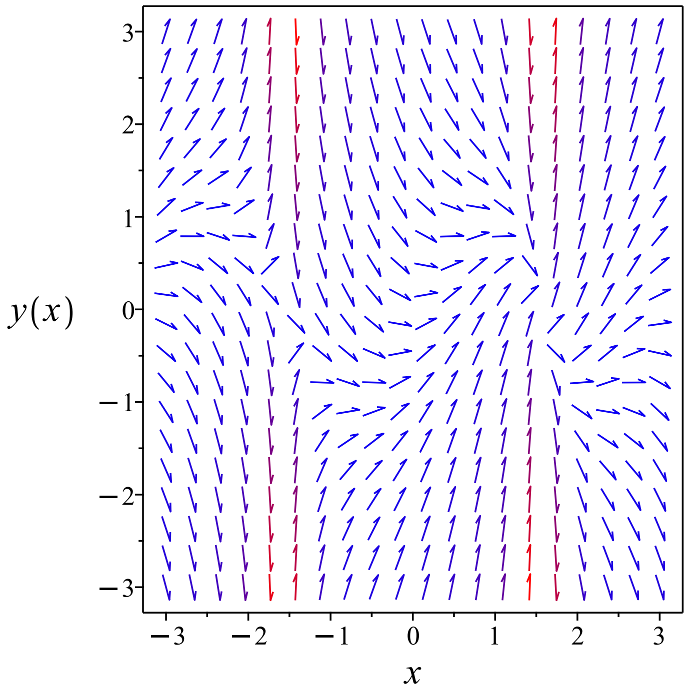


Figure 50: Slope field plot

Verification of solutions

$y =$

$$-\frac{2\tan\left(\frac{x}{2}\right)^{3}\ln\left(\sec\left(\frac{x}{2}\right)^{2}\right)-4\tan\left(\frac{x}{2}\right)^{3}\ln\left(-1+\tan\left(\frac{x}{2}\right)\right)+\tan\left(\frac{x}{2}\right)^{3}c_1-2\tan\left(\frac{x}{2}\right)^{2}\ln\left(\sec\left(\frac{x}{2}\right)^{2}\right)+4\,\mathrm{t}}{}$$

Verified OK.

1.17.4 Maple step by step solution

Let's solve

$y'\cos(x)+y=\sin(2x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y'=-\frac{y}{\cos(x)}+\frac{\sin(2x)}{\cos(x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y'+\frac{y}{\cos(x)}=\frac{\sin(2x)}{\cos(x)}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y'+\frac{y}{\cos(x)}\right)=\frac{\mu(x)\sin(2x)}{\cos(x)}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y'+\frac{y}{\cos(x)}\right)=\mu'(x)\,y+\mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x)=\frac{\mu(x)}{\cos(x)}$

- Solve to find the integrating factor

  $\mu(x)=\sec(x)+\tan(x)$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\frac{\mu(x)\sin(2x)}{\cos(x)}dx+c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y=\int\frac{\mu(x)\sin(2x)}{\cos(x)}dx+c_1$

- Solve for y

  $y=\frac{\int\frac{\mu(x)\sin(2x)}{\cos(x)}dx+c_1}{\mu(x)}$

- Substitute $\mu(x)=\sec(x)+\tan(x)$

  $y=\frac{\int\frac{(\sec(x)+\tan(x))\sin(2x)}{\cos(x)}dx+c_1}{\sec(x)+\tan(x)}$

- Evaluate the integrals on the rhs

  $y=\frac{-2\sin(x)-2\ln(\sin(x)-1)+c_1}{\sec(x)+\tan(x)}$

- Simplify

  $y=\frac{(-2\sin(x)-2\ln(\sin(x)-1)+c_1)(\cos(x)-\sin(x)+1)}{\cos(x)+1+\sin(x)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve(cos(x)*diff(y(x),x)+y(x)=sin(2*x),y(x), singsol=all)
```

$$y(x) = \frac{(\cos(x) - \sin(x) + 1)(-2\sin(x) - 2\ln(\sin(x) - 1) + c_1)}{\cos(x) + \sin(x) + 1}$$

✓ Solution by Mathematica

Time used: 0.091 (sec). Leaf size: 42

```
DSolve[Cos[x]*y'[x]+y[x]==Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2\operatorname{arctanh}\left(\tan\left(\frac{x}{2}\right)\right)}\left(-2\sin(x) - 4\log\left(\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)\right) + c_1\right)$$

1.18 problem 4(c)

Internal problem ID [3046]
Internal file name [`OUTPUT/2538_Sunday_June_05_2022_03_18_40_AM_54738457/index.tex`]

Book: Elementary Differential equations, Chaundy, 1969
Section: Exercises 3, page 60
Problem number: 4(c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y' + y\sin(x) = \sin(2x)}$$

1.18.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \sin(x)$$
$$q(x) = \sin(2x)$$

Hence the ode is

$$y' + y\sin(x) = \sin(2x)$$

The integrating factor μ is

$$\mu = e^{\int \sin(x)dx}$$
$$= e^{-\cos(x)}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(\sin(2x))$$

$$\frac{d}{dx}\left(e^{-\cos(x)}y\right) = \left(e^{-\cos(x)}\right)(\sin(2x))$$

$$d\left(e^{-\cos(x)}y\right) = \left(\sin(2x)\,e^{-\cos(x)}\right)\,dx$$

Integrating gives

$$e^{-\cos(x)}y = \int \sin(2x)\,e^{-\cos(x)}\,dx$$

$$e^{-\cos(x)}y = 2\cos(x)\,e^{-\cos(x)} + 2\,e^{-\cos(x)} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{-\cos(x)}$ results in

$$y = e^{\cos(x)}\left(2\cos(x)\,e^{-\cos(x)} + 2\,e^{-\cos(x)}\right) + c_1 e^{\cos(x)}$$

which simplifies to

$$y = c_1 e^{\cos(x)} + 2\cos(x) + 2$$

Summary

The solution(s) found are the following

$$y = c_1 e^{\cos(x)} + 2\cos(x) + 2 \tag{1}$$

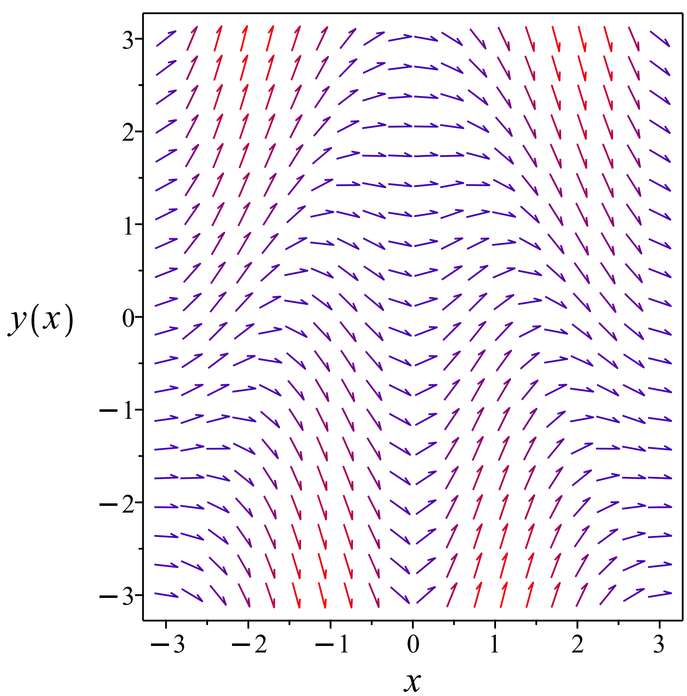


Figure 51: Slope field plot

Verification of solutions

$$y = c_1 e^{\cos(x)} + 2\cos(x) + 2$$

Verified OK.

1.18.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\sin(x)\, y + \sin(2x) \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 52: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= e^{\cos(x)}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\cos(x)}} dy \end{aligned}$$

Which results in

$$S = e^{-\cos(x)} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\sin(x)\, y + \sin(2x)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \sin(x)\, e^{-\cos(x)} y \\ S_y &= e^{-\cos(x)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \sin(2x)\, e^{-\cos(x)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \sin(2R)\, e^{-\cos(R)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 + 2\,\mathrm{e}^{-\cos(R)}(1 + \cos(R)) \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{-\cos(x)} y = 2\,\mathrm{e}^{-\cos(x)}(\cos(x) + 1) + c_1$$

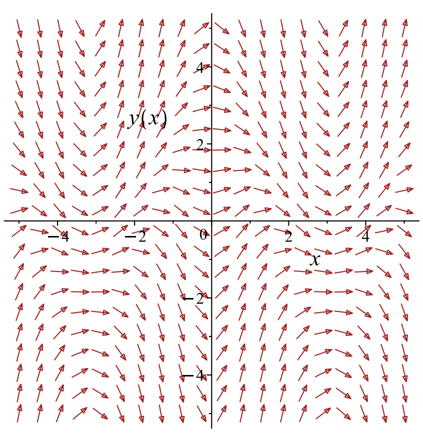
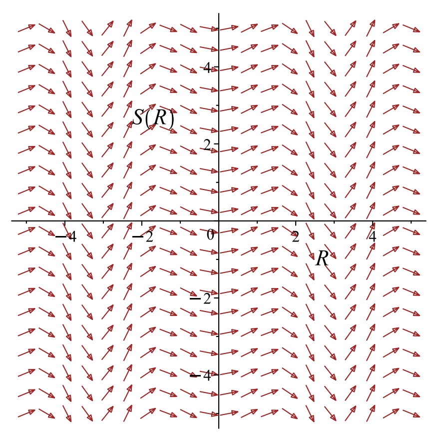
Which simplifies to

$$(y - 2\cos(x) - 2)\,\mathrm{e}^{-\cos(x)} - c_1 = 0$$

Which gives

$$y = \mathrm{e}^{\cos(x)}\left(2\cos(x)\,\mathrm{e}^{-\cos(x)} + 2\,\mathrm{e}^{-\cos(x)} + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\sin(x)\,y + \sin(2x)$

</td><td>$R = x$
$S = \mathrm{e}^{-\cos(x)} y$</td><td>$\frac{dS}{dR} = \sin(2R)\,\mathrm{e}^{-\cos(R)}$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\cos(x)}\left(2\cos(x)\,\mathrm{e}^{-\cos(x)} + 2\,\mathrm{e}^{-\cos(x)} + c_1\right) \tag{1}$$

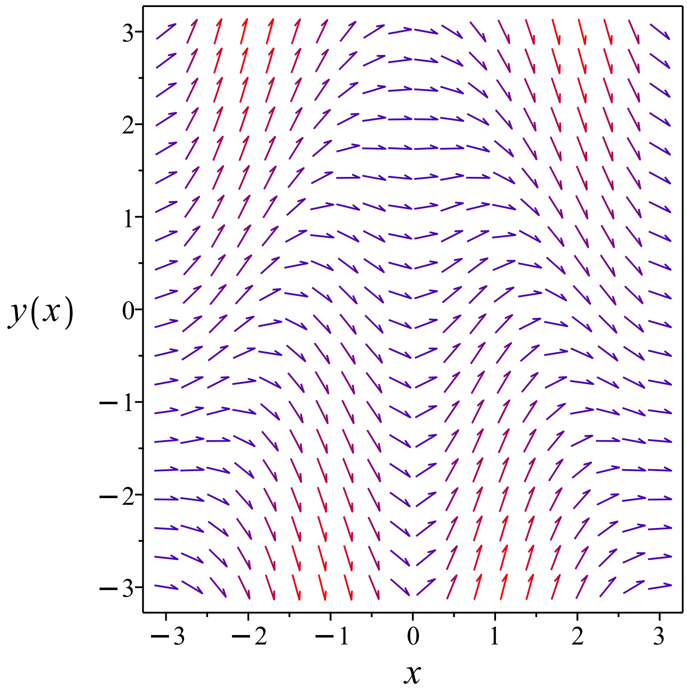


Figure 52: Slope field plot

Verification of solutions

$$y = e^{\cos(x)}\left(2\cos(x)\, e^{-\cos(x)} + 2\, e^{-\cos(x)} + c_1\right)$$

Verified OK.

1.18.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= (-\sin(x)\, y + \sin(2x))\, \mathrm{d}x \\
(\sin(x)\, y - \sin(2x))\, \mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= \sin(x)\, y - \sin(2x) \\
N(x, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\sin(x)\, y - \sin(2x)) \\
&= \sin(x)
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((\sin(x)) - (0)) \\ &= \sin(x) \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, dx} \\ &= e^{\int \sin(x) \, dx} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\cos(x)} \\ &= \mathrm{e}^{-\cos(x)} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{-\cos(x)}(\sin(x)\, y - \sin(2x)) \\ &= \mathrm{e}^{-\cos(x)} \sin(x)\,(-2\cos(x) + y) \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{-\cos(x)}(1) \\ &= \mathrm{e}^{-\cos(x)} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{dy}{dx} &= 0 \\ \left(\mathrm{e}^{-\cos(x)} \sin(x)\,(-2\cos(x) + y)\right) + \left(\mathrm{e}^{-\cos(x)}\right)\frac{dy}{dx} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \mathrm{e}^{-\cos(x)} \sin(x) \left(-2\cos(x) + y\right) \mathrm{d}x$$

$$\phi = (y - 2\cos(x) - 2)\,\mathrm{e}^{-\cos(x)} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-\cos(x)} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-\cos(x)}$. Therefore equation (4) becomes

$$\mathrm{e}^{-\cos(x)} = \mathrm{e}^{-\cos(x)} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (y - 2\cos(x) - 2)\,\mathrm{e}^{-\cos(x)} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (y - 2\cos(x) - 2)\,\mathrm{e}^{-\cos(x)}$$

The solution becomes

$$y = \mathrm{e}^{\cos(x)} \left(2\cos(x)\,\mathrm{e}^{-\cos(x)} + 2\,\mathrm{e}^{-\cos(x)} + c_1\right)$$

Summary

The solution(s) found are the following

$$y = e^{\cos(x)} \left(2\cos(x)\, e^{-\cos(x)} + 2\, e^{-\cos(x)} + c_1\right) \tag{1}$$

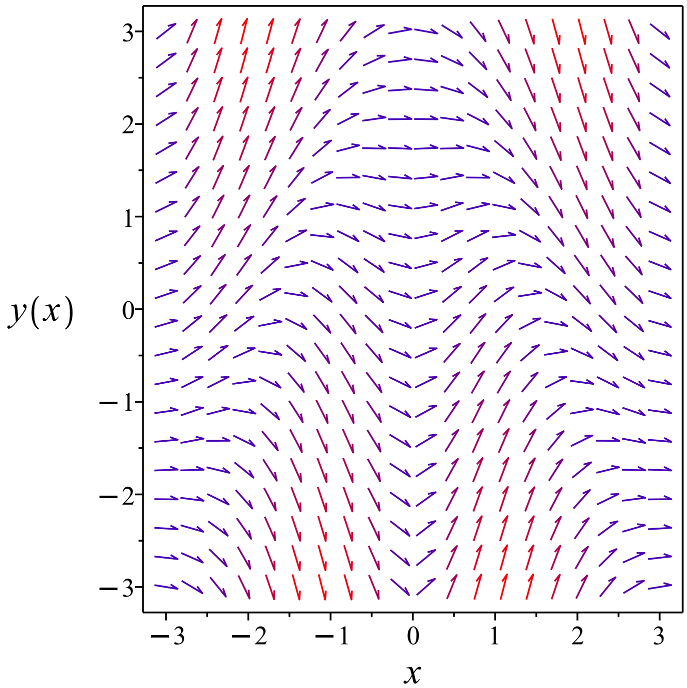


Figure 53: Slope field plot

Verification of solutions

$$y = e^{\cos(x)} \left(2\cos(x)\, e^{-\cos(x)} + 2\, e^{-\cos(x)} + c_1\right)$$

Verified OK.

1.18.4 Maple step by step solution

Let's solve

$y' + y\sin(x) = \sin(2x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -y\sin(x) + \sin(2x)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y\sin(x) = \sin(2x)$
- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)(y' + y\sin(x)) = \mu(x)\sin(2x)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)(y' + y\sin(x)) = \mu'(x)\,y + \mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)\sin(x)$
- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{-\cos(x)}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\sin(2x)\,dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\sin(2x)\,dx + c_1$
- Solve for y

  $y = \frac{\int \mu(x)\sin(2x)dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{-\cos(x)}$

  $y = \frac{\int \sin(2x)\mathrm{e}^{-\cos(x)}dx + c_1}{\mathrm{e}^{-\cos(x)}}$
- Evaluate the integrals on the rhs

  $y = \frac{2\cos(x)\mathrm{e}^{-\cos(x)} + 2\,\mathrm{e}^{-\cos(x)} + c_1}{\mathrm{e}^{-\cos(x)}}$
- Simplify

  $y = c_1\mathrm{e}^{\cos(x)} + 2\cos(x) + 2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x)+y(x)*sin(x)=sin(2*x),y(x), singsol=all)
```

$$y(x) = 2\cos(x) + 2 + e^{\cos(x)} c_1$$

✓ Solution by Mathematica

Time used: 0.075 (sec). Leaf size: 18

```
DSolve[y'[x]+y[x]*Sin[x]==Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2\cos(x) + c_1 e^{\cos(x)} + 2$$

1.19 problem 4(d)

Internal problem ID [3047]
Internal file name [`OUTPUT/2539_Sunday_June_05_2022_03_18_43_AM_65009450/index.tex`]

Book: Elementary Differential equations, Chaundy, 1969
Section: Exercises 3, page 60
Problem number: 4(d).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' \sin(x) + y = \sin(2x)}$$

1.19.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \csc(x) \\ q(x) &= 2\cos(x) \end{aligned}$$

Hence the ode is

$$y' + \csc(x)\, y = 2\cos(x)$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \csc(x)dx} \\ &= \csc(x) - \cot(x) \end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(2\cos(x))$$
$$\frac{d}{dx}((\csc(x) - \cot(x))\, y) = (\csc(x) - \cot(x))(2\cos(x))$$
$$d((\csc(x) - \cot(x))\, y) = ((-2\cos(x) + 2)\cot(x))\, dx$$

Integrating gives

$$(\csc(x) - \cot(x))\, y = \int (-2\cos(x) + 2)\cot(x)\, dx$$
$$(\csc(x) - \cot(x))\, y = -2\cos(x) + 2\ln(\cos(x) + 1) + c_1$$

Dividing both sides by the integrating factor $\mu = \csc(x) - \cot(x)$ results in

$$y = \frac{-2\cos(x) + 2\ln(\cos(x) + 1)}{\csc(x) - \cot(x)} + \frac{c_1}{\csc(x) - \cot(x)}$$

which simplifies to

$$y = \csc(x)(-2\cos(x) + 2\ln(\cos(x) + 1) + c_1)(\cos(x) + 1)$$

Summary

The solution(s) found are the following

$$y = \csc(x)(-2\cos(x) + 2\ln(\cos(x) + 1) + c_1)(\cos(x) + 1) \tag{1}$$

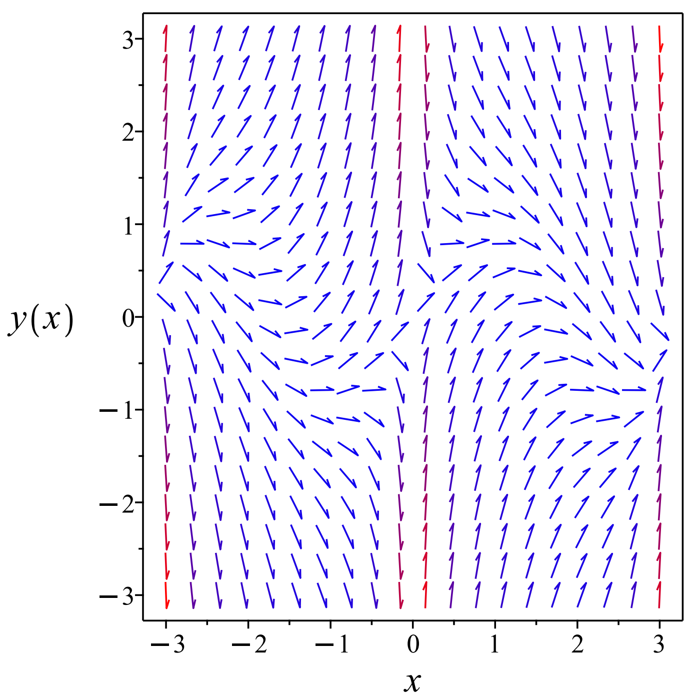


Figure 54: Slope field plot

Verification of solutions

$$
y = \csc(x)\left(-2\cos(x) + 2\ln(\cos(x) + 1) + c_1\right)(\cos(x) + 1)
$$

Verified OK.

1.19.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{-y + \sin(2x)}{\sin(x)} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 55: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \csc(x) + \cot(x)
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\csc(x) + \cot(x)} dy \end{aligned}$$

Which results in

$$S = \frac{y}{\csc(x) + \cot(x)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-y + \sin(2x)}{\sin(x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y}{\cos(x) + 1} \\ S_y &= \frac{1}{\csc(x) + \cot(x)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\sin(2x)}{\cos(x) + 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\sin(2R)}{\cos(R) + 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -2\cos(R) + 2\ln(\cos(R) + 1) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{\csc(x) + \cot(x)} = -2\cos(x) + 2\ln(\cos(x) + 1) + c_1$$

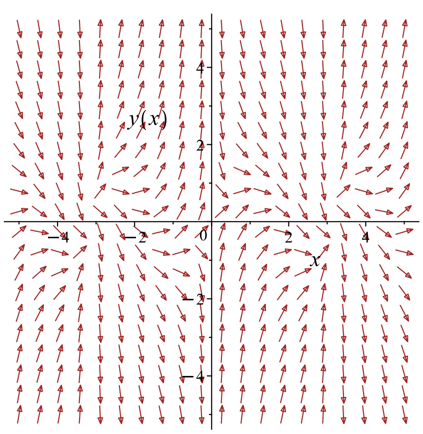
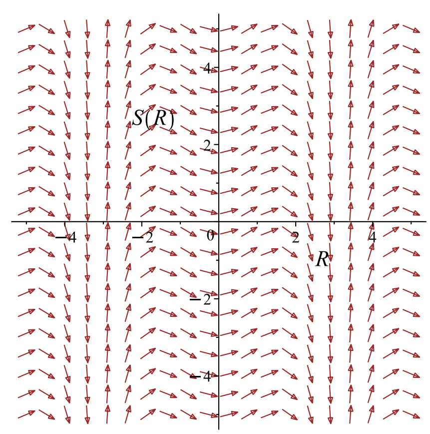
Which simplifies to

$$\frac{y}{\csc(x) + \cot(x)} = -2\cos(x) + 2\ln(\cos(x) + 1) + c_1$$

Which gives

$$y = -2\cos(x)\cot(x) - 2\cos(x)\csc(x) + 2\ln(\cos(x) + 1)\cot(x) + c_1\cot(x) + 2\ln(\cos(x) + 1)\csc(x) +$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-y+\sin(2x)}{\sin(x)}$ 	$R = x$ $S = \frac{y}{\csc(x) + \cot(x)}$	$\frac{dS}{dR} = \frac{\sin(2R)}{\cos(R)+1}$ 

Summary

The solution(s) found are the following

$$\begin{aligned} y = &-2\cos(x)\cot(x) - 2\cos(x)\csc(x) + 2\ln(\cos(x) + 1)\cot(x) \\ &+ c_1\cot(x) + 2\ln(\cos(x) + 1)\csc(x) + c_1\csc(x) \end{aligned} \tag{1}$$

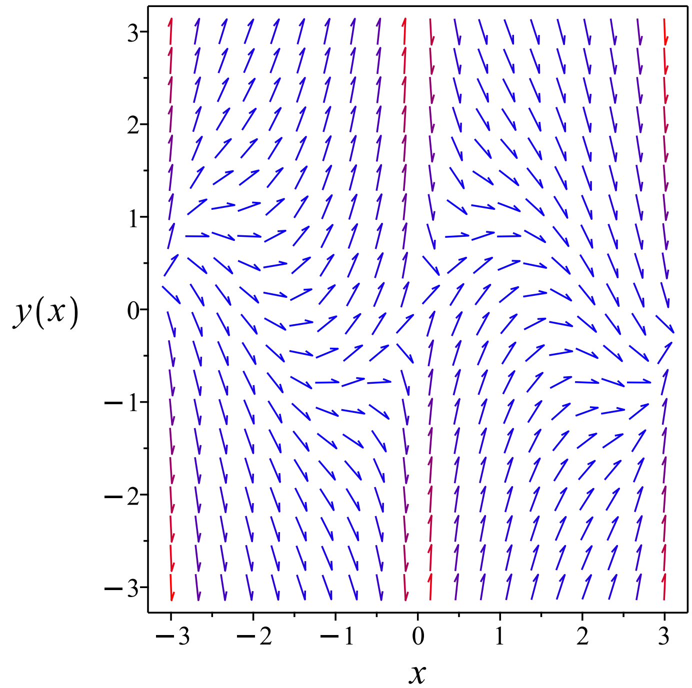


Figure 55: Slope field plot

<u>Verification of solutions</u>

$$
\begin{aligned}
y = &-2\cos(x)\cot(x) - 2\cos(x)\csc(x) + 2\ln(\cos(x)+1)\cot(x) \\
&+ c_1\cot(x) + 2\ln(\cos(x)+1)\csc(x) + c_1\csc(x)
\end{aligned}
$$

Verified OK.

1.19.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$
M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y) = 0
$$

Hence

$$
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(\sin(x))\,\mathrm{d}y &= (-y + \sin(2x))\,\mathrm{d}x \\
(y - \sin(2x))\,\mathrm{d}x + (\sin(x))\,\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= y - \sin(2x) \\
N(x,y) &= \sin(x)
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y - \sin(2x)) \\
&= 1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\sin(x)) \\
&= \cos(x)
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \csc(x)\left((1) - (\cos(x))\right) \\
&= \csc(x) - \cot(x)
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, dx} \\
&= e^{\int \csc(x) - \cot(x) \, dx}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\ln(\sin(x)) - \ln(\csc(x) + \cot(x))} \\
&= \frac{1}{(\csc(x) + \cot(x))\sin(x)}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{(\csc(x) + \cot(x))\sin(x)}(y - \sin(2x)) \\
&= \frac{y - 2\sin(x)\cos(x)}{\cos(x) + 1}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{(\csc(x) + \cot(x))\sin(x)}(\sin(x)) \\
&= \frac{1}{\csc(x) + \cot(x)}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{dy}{dx} &= 0 \\
\left(\frac{y - 2\sin(x)\cos(x)}{\cos(x) + 1}\right) + \left(\frac{1}{\csc(x) + \cot(x)}\right)\frac{dy}{dx} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{y - 2\sin(x)\cos(x)}{\cos(x) + 1} \, \mathrm{d}x$$
$$\phi = \tan\left(\frac{x}{2}\right) y + 4\cos\left(\frac{x}{2}\right)^2 + 2\ln\left(\sec\left(\frac{x}{2}\right)^2\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \tan\left(\frac{x}{2}\right) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{\csc(x)+\cot(x)}$. Therefore equation (4) becomes

$$\frac{1}{\csc(x) + \cot(x)} = \tan\left(\frac{x}{2}\right) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \tan\left(\frac{x}{2}\right) y + 4\cos\left(\frac{x}{2}\right)^2 + 2\ln\left(\sec\left(\frac{x}{2}\right)^2\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \tan\left(\frac{x}{2}\right) y + 4\cos\left(\frac{x}{2}\right)^2 + 2\ln\left(\sec\left(\frac{x}{2}\right)^2\right)$$

The solution becomes

$$y = -\frac{4\cos\left(\frac{x}{2}\right)^2 + 2\ln\left(\sec\left(\frac{x}{2}\right)^2\right) - c_1}{\tan\left(\frac{x}{2}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{4\cos\left(\frac{x}{2}\right)^2 + 2\ln\left(\sec\left(\frac{x}{2}\right)^2\right) - c_1}{\tan\left(\frac{x}{2}\right)} \tag{1}$$

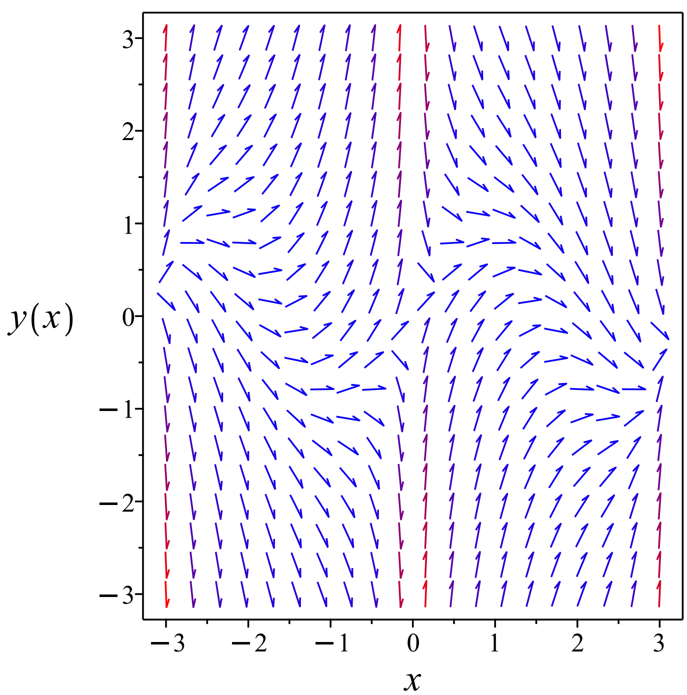


Figure 56: Slope field plot

Verification of solutions

$$y = -\frac{4\cos\left(\frac{x}{2}\right)^2 + 2\ln\left(\sec\left(\frac{x}{2}\right)^2\right) - c_1}{\tan\left(\frac{x}{2}\right)}$$

Verified OK.

1.19.4 Maple step by step solution

Let's solve

$y'\sin(x) + y = \sin(2x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{\sin(x)} + \frac{\sin(2x)}{\sin(x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{\sin(x)} = \frac{\sin(2x)}{\sin(x)}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{\sin(x)}\right) = \frac{\mu(x)\sin(2x)}{\sin(x)}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{y}{\sin(x)}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{\sin(x)}$

- Solve to find the integrating factor

  $\mu(x) = \cot(x) - \csc(x)$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx = \int \frac{\mu(x)\sin(2x)}{\sin(x)}dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \frac{\mu(x)\sin(2x)}{\sin(x)}dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)\sin(2x)}{\sin(x)}dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \cot(x) - \csc(x)$

$$y = \frac{\int \frac{(\cot(x)-\csc(x))\sin(2x)}{\sin(x)}dx + c_1}{\cot(x)-\csc(x)}$$

- Evaluate the integrals on the rhs

$$y = \frac{2\cos(x) - 2\ln(\cos(x)+1) + c_1}{\cot(x)-\csc(x)}$$

- Simplify

$$y = -\csc(x)\,(2\cos(x) - 2\ln(\cos(x)+1) + c_1)\,(\cos(x)+1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(sin(x)*diff(y(x),x)+y(x)=sin(2*x),y(x), singsol=all)
```

$$y(x) = \csc(x)\,(-2\cos(x) + 2\ln(\cos(x)+1) + c_1)\,(\cos(x)+1)$$

✓ Solution by Mathematica

Time used: 0.288 (sec). Leaf size: 38

```
DSolve[Sin[x]*y'[x]+y[x]==Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{\operatorname{arctanh}(\cos(x))}\left(-2\sqrt{\sin^2(x)}\csc(x)\left(\cos(x) + \log\left(\sec^2\left(\frac{x}{2}\right)\right)\right) + c_1\right)$$

1.20 problem 5(a)

Internal problem ID [3048]
Internal file name [`OUTPUT/2540_Sunday_June_05_2022_03_18_45_AM_70471759/index.tex`]

Book: Elementary Differential equations, Chaundy, 1969
Section: Exercises 3, page 60
Problem number: 5(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first__order__ode__lie__symmetry__lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{\sqrt{x^2+1}\, y' + y = 2x}$$

1.20.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{\sqrt{x^2+1}}$$
$$q(x) = \frac{2x}{\sqrt{x^2+1}}$$

Hence the ode is

$$y' + \frac{y}{\sqrt{x^2+1}} = \frac{2x}{\sqrt{x^2+1}}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{1}{\sqrt{x^2+1}}dx} \\
&= \sqrt{x^2+1}+x
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{2x}{\sqrt{x^2+1}}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(\sqrt{x^2+1}+x\right)y\right) &= \left(\sqrt{x^2+1}+x\right)\left(\frac{2x}{\sqrt{x^2+1}}\right) \\
\mathrm{d}\left(\left(\sqrt{x^2+1}+x\right)y\right) &= \left(\frac{2x\left(\sqrt{x^2+1}+x\right)}{\sqrt{x^2+1}}\right)\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\left(\sqrt{x^2+1}+x\right)y &= \int \frac{2x\left(\sqrt{x^2+1}+x\right)}{\sqrt{x^2+1}}\,\mathrm{d}x \\
\left(\sqrt{x^2+1}+x\right)y &= x^2+\sqrt{x^2+1}\,x-\operatorname{arcsinh}(x)+c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \sqrt{x^2+1}+x$ results in

$$
y = \frac{x^2+\sqrt{x^2+1}\,x-\operatorname{arcsinh}(x)}{\sqrt{x^2+1}+x}+\frac{c_1}{\sqrt{x^2+1}+x}
$$

which simplifies to

$$
y = \frac{\sqrt{x^2+1}\,x+x^2-\operatorname{arcsinh}(x)+c_1}{\sqrt{x^2+1}+x}
$$

Summary

The solution(s) found are the following

$$
y = \frac{\sqrt{x^2+1}\,x+x^2-\operatorname{arcsinh}(x)+c_1}{\sqrt{x^2+1}+x} \tag{1}
$$

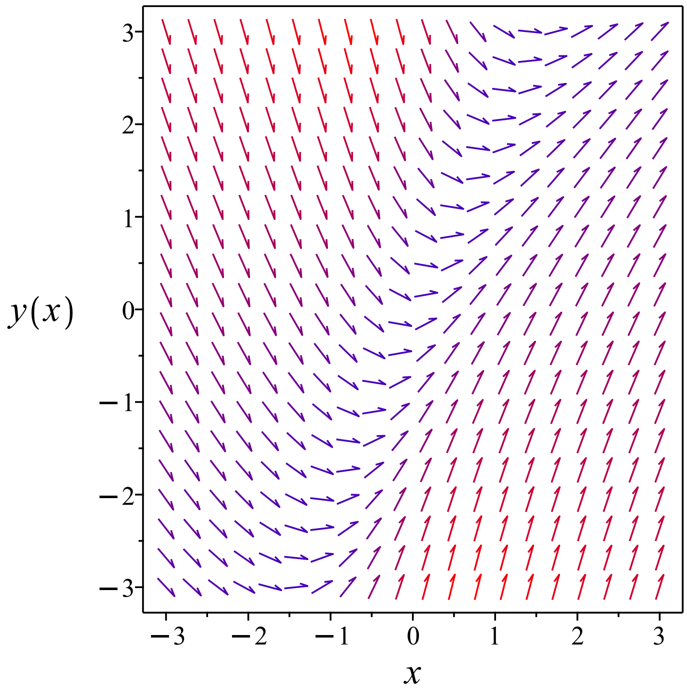


Figure 57: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{x^2+1}\,x + x^2 - \operatorname{arcsinh}(x) + c_1}{\sqrt{x^2+1}+x}$$

Verified OK.

1.20.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{-2x+y}{\sqrt{x^2+1}}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 58: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \frac{1}{\sqrt{x^2+1}+x} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\sqrt{x^2+1}+x}} dy \end{aligned}$$

Which results in

$$S = \left(\sqrt{x^2+1}+x\right) y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-2x+y}{\sqrt{x^2+1}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \left(\frac{x}{\sqrt{x^2+1}}+1\right) y \\ S_y &= \sqrt{x^2+1}+x \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{2x\left(\sqrt{x^2+1}+x\right)}{\sqrt{x^2+1}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{2R\left(\sqrt{R^2+1}+R\right)}{\sqrt{R^2+1}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R^2 + R\sqrt{R^2+1} - \operatorname{arcsinh}(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\left(\sqrt{x^2+1}+x\right) y = \sqrt{x^2+1}\, x + x^2 - \operatorname{arcsinh}(x) + c_1$$

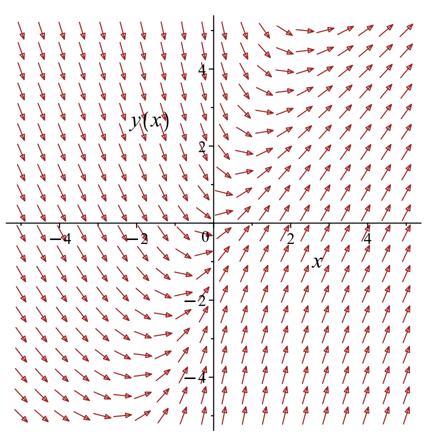
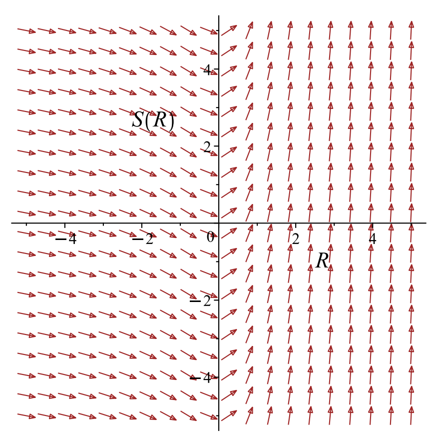
Which simplifies to

$$(y-x)\sqrt{x^2+1} - x^2 + yx - c_1 + \operatorname{arcsinh}(x) = 0$$

Which gives

$$y = \frac{\sqrt{x^2+1}\, x + x^2 - \operatorname{arcsinh}(x) + c_1}{\sqrt{x^2+1}+x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-2x+y}{\sqrt{x^2+1}}$ 	$R = x$ $S = \left(\sqrt{x^2+1}+x\right) y$	$\frac{dS}{dR} = \frac{2R\left(\sqrt{R^2+1}+R\right)}{\sqrt{R^2+1}}$ 

Summary
The solution(s) found are the following

$$y = \frac{\sqrt{x^2+1}\,x + x^2 - \operatorname{arcsinh}(x) + c_1}{\sqrt{x^2+1} + x} \tag{1}$$

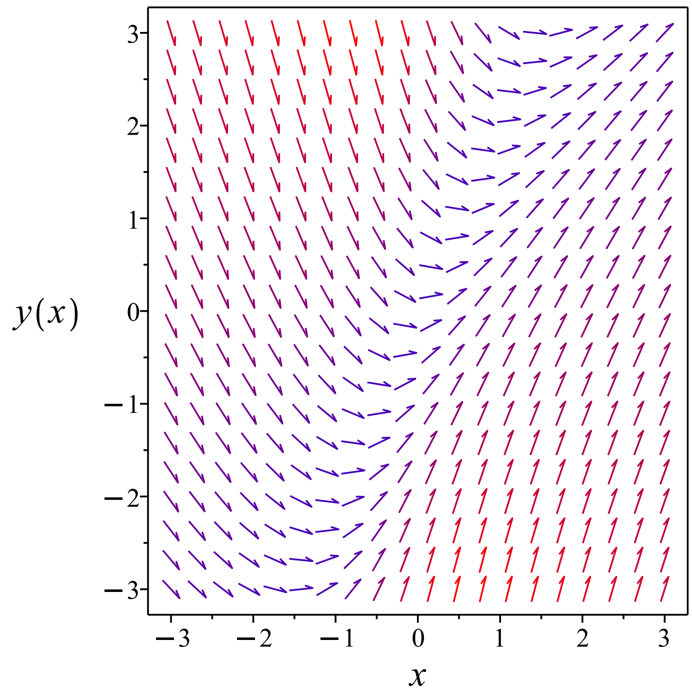


Figure 58: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{x^2+1}\,x + x^2 - \operatorname{arcsinh}(x) + c_1}{\sqrt{x^2+1} + x}$$

Verified OK.

1.20.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\sqrt{x^2+1}\right)\mathrm{d}y &= (2x-y)\,\mathrm{d}x\\ (-2x+y)\,\mathrm{d}x + \left(\sqrt{x^2+1}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -2x+y\\ N(x,y) &= \sqrt{x^2+1}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-2x+y) \\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\sqrt{x^2+1}\right) \\ &= \frac{x}{\sqrt{x^2+1}}\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{\sqrt{x^2+1}}\left((1) - \left(\frac{x}{\sqrt{x^2+1}}\right)\right) \\ &= \frac{\sqrt{x^2+1}-x}{x^2+1}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \frac{\sqrt{x^2+1}-x}{x^2+1} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\operatorname{arcsinh}(x) - \frac{\ln\left(x^2+1\right)}{2}} \\ &= \frac{x}{\sqrt{x^2+1}} + 1\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{x}{\sqrt{x^2+1}} + 1(-2x+y) \\ &= (-2x+y)\left(\frac{x}{\sqrt{x^2+1}} + 1\right)\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{x}{\sqrt{x^2+1}} + 1\left(\sqrt{x^2+1}\right) \\
&= \sqrt{x^2+1} + x
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left((-2x+y)\left(\frac{x}{\sqrt{x^2+1}} + 1\right)\right) + \left(\sqrt{x^2+1} + x\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int (-2x+y)\left(\frac{x}{\sqrt{x^2+1}} + 1\right)\mathrm{d}x
\end{aligned}
$$

$$
\phi = (y-x)\sqrt{x^2+1} - x^2 + xy + \operatorname{arcsinh}(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \sqrt{x^2+1} + x + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sqrt{x^2+1} + x$. Therefore equation (4) becomes

$$
\sqrt{x^2+1} + x = \sqrt{x^2+1} + x + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (y - x)\sqrt{x^2 + 1} - x^2 + xy + \operatorname{arcsinh}(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (y - x)\sqrt{x^2 + 1} - x^2 + xy + \operatorname{arcsinh}(x)$$

The solution becomes

$$y = \frac{\sqrt{x^2 + 1}\, x + x^2 - \operatorname{arcsinh}(x) + c_1}{\sqrt{x^2 + 1} + x}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{x^2 + 1}\, x + x^2 - \operatorname{arcsinh}(x) + c_1}{\sqrt{x^2 + 1} + x} \tag{1}$$

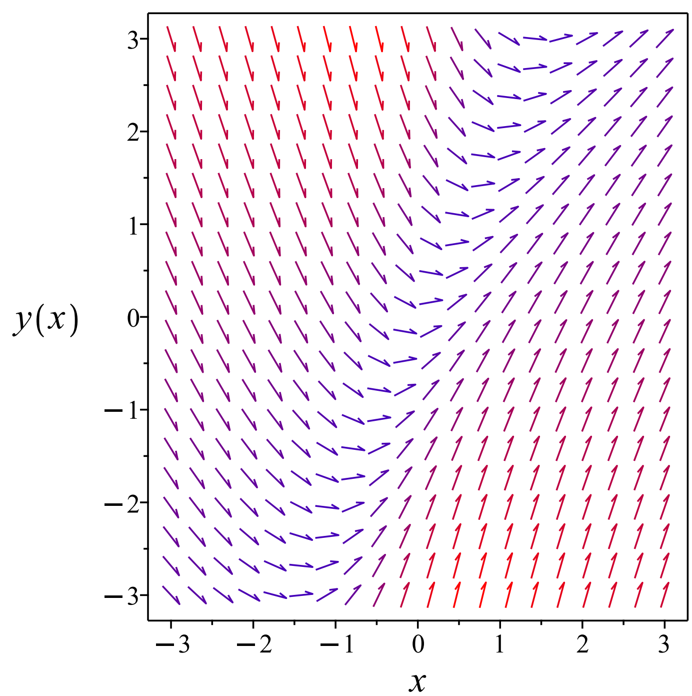


Figure 59: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{x^2+1}\,x + x^2 - \operatorname{arcsinh}(x) + c_1}{\sqrt{x^2+1}+x}$$

Verified OK.

1.20.4 Maple step by step solution

Let's solve

$\sqrt{x^2+1}\,y' + y = 2x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{\sqrt{x^2+1}} + \frac{2x}{\sqrt{x^2+1}}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{\sqrt{x^2+1}} = \frac{2x}{\sqrt{x^2+1}}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{\sqrt{x^2+1}}\right) = \frac{2\mu(x)x}{\sqrt{x^2+1}}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{y}{\sqrt{x^2+1}}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{\sqrt{x^2+1}}$

- Solve to find the integrating factor

  $\mu(x) = \sqrt{x^2+1} + x$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \frac{2\mu(x)x}{\sqrt{x^2+1}} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \frac{2\mu(x)x}{\sqrt{x^2+1}} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{2\mu(x)x}{\sqrt{x^2+1}} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \sqrt{x^2+1}+x$

$$y = \frac{\int \frac{2x\left(\sqrt{x^2+1}+x\right)}{\sqrt{x^2+1}}dx + c_1}{\sqrt{x^2+1}+x}$$

- Evaluate the integrals on the rhs

$$y = \frac{\sqrt{x^2+1}\,x + x^2 - \operatorname{arcsinh}(x) + c_1}{\sqrt{x^2+1}+x}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve(sqrt(1+x^2)*diff(y(x),x)+y(x)=2*x,y(x), singsol=all)
```

$$y(x) = \frac{x^2 + x\sqrt{x^2+1} - \operatorname{arcsinh}(x) + c_1}{x + \sqrt{x^2+1}}$$

✓ Solution by Mathematica

Time used: 0.136 (sec). Leaf size: 50

```
DSolve[Sqrt[1+x^2]*y'[x]+y[x]==2*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(\sqrt{x^2+1} - x\right)\left(x^2 + \sqrt{x^2+1}x + \log\left(\sqrt{x^2+1} - x\right) + c_1\right)$$

1.21 problem 5(b)

Internal problem ID [3049]
Internal file name [`OUTPUT/2541_Sunday_June_05_2022_03_18_47_AM_17618221/index.tex`]

Book: Elementary Differential equations, Chaundy, 1969
Section: Exercises 3, page 60
Problem number: 5(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{\sqrt{x^2+1}\, y'-y=2\sqrt{x^2+1}}$$

1.21.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$\begin{aligned} p(x)&=-\frac{1}{\sqrt{x^2+1}}\\ q(x)&=2 \end{aligned}$$

Hence the ode is

$$y'-\frac{y}{\sqrt{x^2+1}}=2$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{1}{\sqrt{x^2+1}}dx} \\ &= \frac{1}{\sqrt{x^2+1}+x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(2) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{\sqrt{x^2+1}+x}\right) &= \left(\frac{1}{\sqrt{x^2+1}+x}\right)(2) \\ \mathrm{d}\left(\frac{y}{\sqrt{x^2+1}+x}\right) &= \left(\frac{2}{\sqrt{x^2+1}+x}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{\sqrt{x^2+1}+x} &= \int \frac{2}{\sqrt{x^2+1}+x}\,\mathrm{d}x \\ \frac{y}{\sqrt{x^2+1}+x} &= \sqrt{x^2+1}\,x + \operatorname{arcsinh}(x) - x^2 + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{x^2+1}+x}$ results in

$$y = \left(\sqrt{x^2+1}+x\right)\left(\sqrt{x^2+1}\,x + \operatorname{arcsinh}(x) - x^2\right) + c_1\left(\sqrt{x^2+1}+x\right)$$

which simplifies to

$$y = (\operatorname{arcsinh}(x) + c_1)\sqrt{x^2+1} + x(c_1 + \operatorname{arcsinh}(x) + 1)$$

Summary

The solution(s) found are the following

$$y = (\operatorname{arcsinh}(x) + c_1)\sqrt{x^2+1} + x(c_1 + \operatorname{arcsinh}(x) + 1) \tag{1}$$

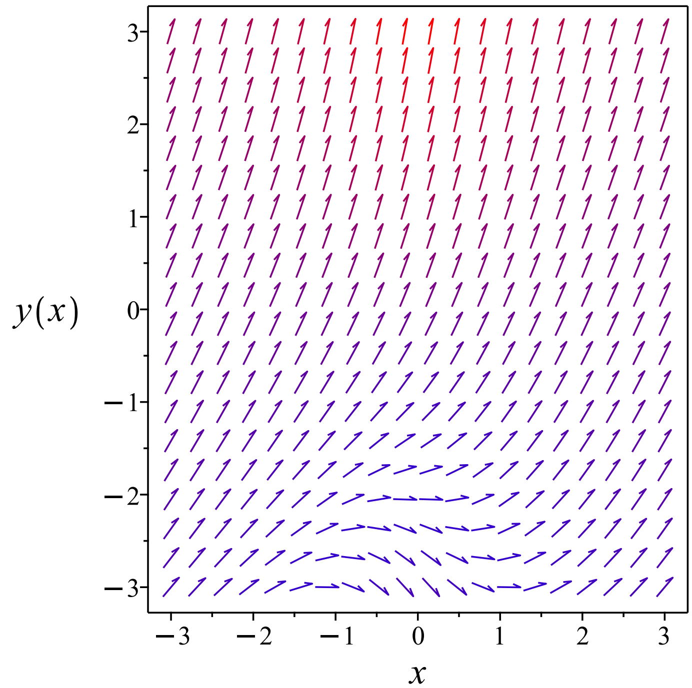


Figure 60: Slope field plot

Verification of solutions

$$y = (\operatorname{arcsinh}(x) + c_1)\sqrt{x^2+1} + x(c_1 + \operatorname{arcsinh}(x) + 1)$$

Verified OK.

1.21.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y + 2\sqrt{x^2+1}}{\sqrt{x^2+1}}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 61: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \sqrt{x^2 + 1} + x
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\sqrt{x^2+1}+x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{\sqrt{x^2+1}+x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y + 2\sqrt{x^2+1}}{\sqrt{x^2+1}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y}{\left(-\sqrt{x^2+1}-x\right)\sqrt{x^2+1}} \\ S_y &= \frac{1}{\sqrt{x^2+1}+x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{2}{\sqrt{x^2+1}+x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{2}{\sqrt{R^2+1}+R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R\sqrt{R^2+1} + \operatorname{arcsinh}(R) - R^2 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{\sqrt{x^2+1}+x} = \sqrt{x^2+1}\,x + \operatorname{arcsinh}(x) - x^2 + c_1$$

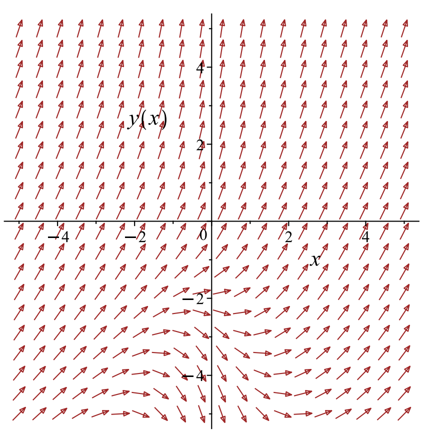
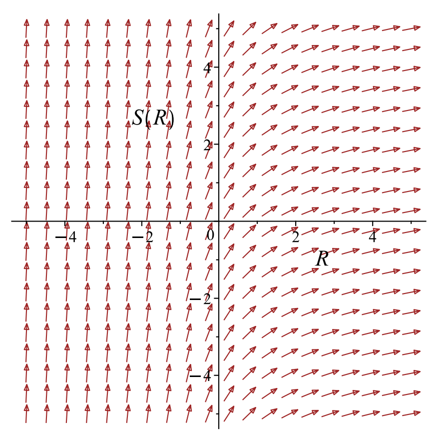
Which simplifies to

$$\frac{y}{\sqrt{x^2+1}+x} = \sqrt{x^2+1}\,x + \operatorname{arcsinh}(x) - x^2 + c_1$$

Which gives

$$y = \sqrt{x^2+1}\,\operatorname{arcsinh}(x) + c_1\sqrt{x^2+1} + \operatorname{arcsinh}(x)\,x + c_1 x + x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+2\sqrt{x^2+1}}{\sqrt{x^2+1}}$ 	$R = x$ $S = \frac{y}{\sqrt{x^2+1}+x}$	$\frac{dS}{dR} = \frac{2}{\sqrt{R^2+1}+R}$ 

Summary

The solution(s) found are the following

$$y = \sqrt{x^2+1}\,\operatorname{arcsinh}(x) + c_1\sqrt{x^2+1} + \operatorname{arcsinh}(x)\,x + c_1 x + x \tag{1}$$

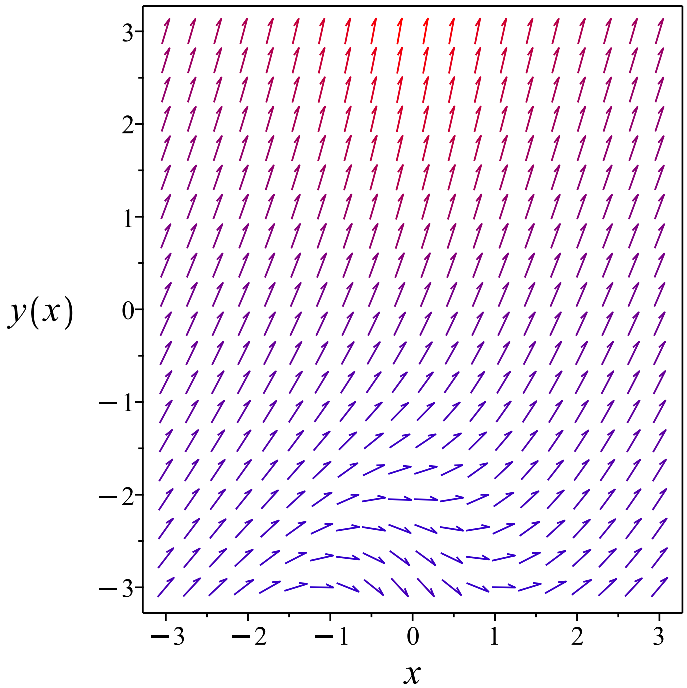


Figure 61: Slope field plot

Verification of solutions

$$y = \sqrt{x^2+1}\,\mathrm{arcsinh}(x) + c_1\sqrt{x^2+1} + \mathrm{arcsinh}(x)\,x + c_1 x + x$$

Verified OK.

1.21.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\sqrt{x^2+1}\right) \mathrm{d}y = \left(y + 2\sqrt{x^2+1}\right) \mathrm{d}x$$
$$\left(-y - 2\sqrt{x^2+1}\right) \mathrm{d}x + \left(\sqrt{x^2+1}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -y - 2\sqrt{x^2+1}$$
$$N(x, y) = \sqrt{x^2+1}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y - 2\sqrt{x^2+1}\right) \\ &= -1\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\sqrt{x^2+1}\right) \\
&= \frac{x}{\sqrt{x^2+1}}
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{\sqrt{x^2+1}}\left((-1) - \left(\frac{x}{\sqrt{x^2+1}}\right)\right) \\
&= \frac{-\sqrt{x^2+1} - x}{x^2+1}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{-\sqrt{x^2+1}-x}{x^2+1} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\operatorname{arcsinh}(x) - \frac{\ln\left(x^2+1\right)}{2}} \\
&= \frac{1}{\left(\sqrt{x^2+1}+x\right)\sqrt{x^2+1}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{\left(\sqrt{x^2+1}+x\right)\sqrt{x^2+1}}\left(-y - 2\sqrt{x^2+1}\right) \\
&= -\frac{y + 2\sqrt{x^2+1}}{\left(\sqrt{x^2+1}+x\right)\sqrt{x^2+1}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{\left(\sqrt{x^2+1}+x\right)\sqrt{x^2+1}}\left(\sqrt{x^2+1}\right) \\
&= \frac{1}{\sqrt{x^2+1}+x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\left(-\frac{y + 2\sqrt{x^2+1}}{\left(\sqrt{x^2+1}+x\right)\sqrt{x^2+1}}\right) + \left(\frac{1}{\sqrt{x^2+1}+x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\frac{y + 2\sqrt{x^2+1}}{\left(\sqrt{x^2+1}+x\right)\sqrt{x^2+1}}\,\mathrm{d}x$$

$$\phi = (y-x)\sqrt{x^2+1} + x^2 - xy - \operatorname{arcsinh}(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \sqrt{x^2+1} - x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{\sqrt{x^2+1}+x}$. Therefore equation (4) becomes

$$\frac{1}{\sqrt{x^2+1}+x} = \sqrt{x^2+1} - x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (y - x)\sqrt{x^2+1} + x^2 - xy - \operatorname{arcsinh}(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (y - x)\sqrt{x^2+1} + x^2 - xy - \operatorname{arcsinh}(x)$$

The solution becomes

$$y = \frac{\sqrt{x^2+1}\,x + \operatorname{arcsinh}(x) - x^2 + c_1}{\sqrt{x^2+1} - x}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{x^2+1}\,x + \operatorname{arcsinh}(x) - x^2 + c_1}{\sqrt{x^2+1} - x} \tag{1}$$

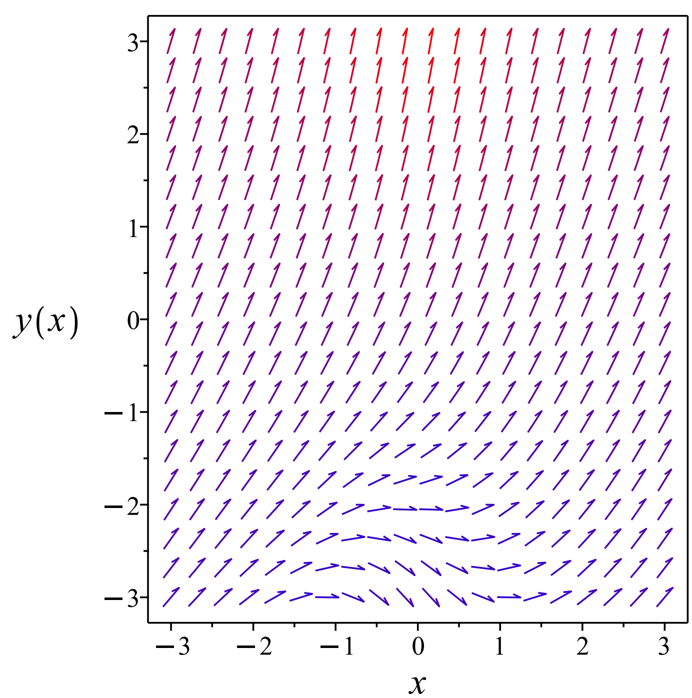


Figure 62: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{x^2+1}\,x + \operatorname{arcsinh}(x) - x^2 + c_1}{\sqrt{x^2+1} - x}$$

Verified OK.

1.21.4 Maple step by step solution

Let's solve

$\sqrt{x^2+1}\,y' - y = 2\sqrt{x^2+1}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 2 + \frac{y}{\sqrt{x^2+1}}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{\sqrt{x^2+1}} = 2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{\sqrt{x^2+1}}\right) = 2\mu(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' - \frac{y}{\sqrt{x^2+1}}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{\sqrt{x^2+1}}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{\sqrt{x^2+1}+x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int 2\mu(x)\,dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int 2\mu(x)\,dx + c_1$

- Solve for y

  $y = \frac{\int 2\mu(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{\sqrt{x^2+1}+x}$

  $$y = \left(\sqrt{x^2+1}+x\right)\left(\int \frac{2}{\sqrt{x^2+1}+x}dx + c_1\right)$$

- Evaluate the integrals on the rhs

  $$y = \left(\sqrt{x^2+1}+x\right)\left(\sqrt{x^2+1}\,x + \operatorname{arcsinh}(x) - x^2 + c_1\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve(sqrt(1+x^2)*diff(y(x),x)-y(x)=2*sqrt(1+x^2),y(x), singsol=all)
```

$$y(x) = \left(x\sqrt{x^2+1} + \operatorname{arcsinh}(x) - x^2 + c_1\right)\left(x + \sqrt{x^2+1}\right)$$

✓ Solution by Mathematica

Time used: 0.082 (sec). Leaf size: 55

```
DSolve[Sqrt[1+x^2]*y'[x]-y[x]==2*Sqrt[1+x^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2 - \sqrt{x^2+1}x + \log\left(\sqrt{x^2+1}-x\right) - c_1}{x - \sqrt{x^2+1}}$$

1.22 problem 5(c)

Internal problem ID [3050]
Internal file name [`OUTPUT/2542_Sunday_June_05_2022_03_18_49_AM_37191220/index.tex`]

Book: Elementary Differential equations, Chaundy, 1969
Section: Exercises 3, page 60
Problem number: 5(c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{\sqrt{(x+a)(x+b)}\,(2y'-3)+y=0}$$

1.22.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$p(x)=\frac{1}{2\sqrt{(x+a)(x+b)}}$$
$$q(x)=\frac{3}{2}$$

Hence the ode is

$$y'+\frac{y}{2\sqrt{(x+a)(x+b)}}=\frac{3}{2}$$

The integrating factor μ is

$$\mu = e^{\int \frac{1}{2\sqrt{(x+a)(x+b)}}dx}$$

$$= e^{\frac{\sqrt{(x+b)^2+(-b+a)(x+b)}+\frac{(-b+a)\ln\left(\frac{b}{2}+\frac{a}{2}+x+\sqrt{(x+b)^2+(-b+a)(x+b)}\right)}{2}}{2a-2b} - \frac{\sqrt{(x+a)^2+(b-a)(x+a)}+\frac{(b-a)\ln\left(\frac{b}{2}+\frac{a}{2}+x+\sqrt{(x+a)^2+(b-a)(x+a)}\right)}{2}}{2(-b+a)}}$$

Which simplifies to

$$\mu = \frac{\sqrt{2}\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}}{2}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{3}{2}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sqrt{2}\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}\,y}{2}\right) = \left(\frac{\sqrt{2}\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}}{2}\right)\left(\frac{3}{2}\right)$$

$$\mathrm{d}\left(\frac{\sqrt{2}\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}\,y}{2}\right) = \left(\frac{3\sqrt{2}\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}}{4}\right)\mathrm{d}x$$

Integrating gives

$$\frac{\sqrt{2}\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}\,y}{2} = \int \frac{3\sqrt{2}\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}}{4}\,\mathrm{d}x$$

$$\frac{\sqrt{2}\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}\,y}{2} = \int \frac{3\sqrt{2}\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}}{4}dx + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{\sqrt{2}\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}}{2}$ results in

$$y = \frac{\sqrt{2}\left(\int \frac{3\sqrt{2}\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}}{4}dx\right)}{\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}} + \frac{c_1\sqrt{2}}{\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}}$$

which simplifies to

$$y = \frac{2\sqrt{2}\,c_1 + 3\left(\int \sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}dx\right)}{2\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}}$$

Summary

The solution(s) found are the following

$$y = \frac{2\sqrt{2}\, c_1 + 3\left(\int \sqrt{a+b+2x+2\sqrt{(x+a)\,(x+b)}}dx\right)}{2\sqrt{a+b+2x+2\sqrt{(x+a)\,(x+b)}}} \tag{1}$$

Verification of solutions

$$y = \frac{2\sqrt{2}\, c_1 + 3\left(\int \sqrt{a+b+2x+2\sqrt{(x+a)\,(x+b)}}dx\right)}{2\sqrt{a+b+2x+2\sqrt{(x+a)\,(x+b)}}}$$

Verified OK.

1.22.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{3\sqrt{(x+a)\,(x+b)} - y}{2\sqrt{(x+a)\,(x+b)}}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 64: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{2}{\sqrt{2a + 2b + 4x + 4\sqrt{x^2 + (a+b)\, x + ab}}}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{2}{\sqrt{2a+2b+4x+4\sqrt{x^2+(a+b)x+ab}}}} dy \end{aligned}$$

Which results in

$$S = \frac{\sqrt{2a+2b+4x+4\sqrt{x^2+(a+b)\,x+ab}}\,y}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{3\sqrt{(x+a)\,(x+b)} - y}{2\sqrt{(x+a)\,(x+b)}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y\left(a+b+2x+2\sqrt{x+a}\,\sqrt{x+b}\right)}{2\sqrt{2a+2b+4x+4\sqrt{x+a}\,\sqrt{x+b}}\,\sqrt{x+a}\,\sqrt{x+b}} \\ S_y &= \frac{\sqrt{2a+2b+4x+4\sqrt{x+a}\,\sqrt{x+b}}}{2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{3\left(\frac{\sqrt{x+b}\left(a+b+2x+\frac{2y}{3}\right)\sqrt{x+a}}{2} + \left(\frac{a}{6}+\frac{b}{6}+\frac{x}{3}\right)y + (x+a)\,(x+b)\right)\sqrt{(x+a)\,(x+b)} - y\left(\frac{\sqrt{x+b}\,(2x+b+a)\sqrt{x-}}{2}\right.}{\sqrt{2a+2b+4x+4\sqrt{x+a}\,\sqrt{x+b}}\,\sqrt{x+a}\,\sqrt{x+b}\,\sqrt{(x+a)\,(x+b)}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{3\sqrt{2a+2b+4R+4\sqrt{R+a}\,\sqrt{R+b}}}{4}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int \frac{3\sqrt{2a+2b+4R+4\sqrt{R+a}\,\sqrt{R+b}}}{4}dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\sqrt{2a+2b+4x+4\sqrt{x+a}\,\sqrt{x+b}}\,y}{2} = \int \frac{3\sqrt{2a+2b+4x+4\sqrt{x+a}\,\sqrt{x+b}}}{4}dx + c_1$$

Which simplifies to

$$\frac{\sqrt{2a+2b+4x+4\sqrt{x+a}\,\sqrt{x+b}}\,y}{2} = \int \frac{3\sqrt{2a+2b+4x+4\sqrt{x+a}\,\sqrt{x+b}}}{4}dx + c_1$$

Which gives

$$y = \frac{2\left(\int \frac{3\sqrt{2a+2b+4x+4\sqrt{x+a}\,\sqrt{x+b}}}{4}dx\right) + 2c_1}{\sqrt{2a+2b+4x+4\sqrt{x+a}\,\sqrt{x+b}}}$$

Summary

The solution(s) found are the following

$$y = \frac{2\left(\int \frac{3\sqrt{2a+2b+4x+4\sqrt{x+a}\,\sqrt{x+b}}}{4}dx\right) + 2c_1}{\sqrt{2a+2b+4x+4\sqrt{x+a}\,\sqrt{x+b}}} \tag{1}$$

Verification of solutions

$$y = \frac{2\left(\int \frac{3\sqrt{2a+2b+4x+4\sqrt{x+a}\,\sqrt{x+b}}}{4}dx\right) + 2c_1}{\sqrt{2a+2b+4x+4\sqrt{x+a}\,\sqrt{x+b}}}$$

Verified OK.

1.22.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(2\sqrt{(x+a)(x+b)}\right)\mathrm{d}y&=\left(3\sqrt{(x+a)(x+b)}-y\right)\mathrm{d}x\\ \left(-3\sqrt{(x+a)(x+b)}+y\right)\mathrm{d}x+\left(2\sqrt{(x+a)(x+b)}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-3\sqrt{(x+a)(x+b)}+y\\ N(x,y)&=2\sqrt{(x+a)(x+b)}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-3\sqrt{(x+a)\,(x+b)}+y\right)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(2\sqrt{(x+a)\,(x+b)}\right)\\ &= \frac{2x+b+a}{\sqrt{(x+a)\,(x+b)}}\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{2\sqrt{(x+a)\,(x+b)}}\left((1) - \left(\frac{2x+b+a}{\sqrt{(x+a)\,(x+b)}}\right)\right)\\ &= \frac{\sqrt{(x+a)\,(x+b)}-2x-b-a}{2\,(x+a)\,(x+b)}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int \frac{\sqrt{(x+a)(x+b)}-2x-b-a}{2(x+a)(x+b)}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\frac{\sqrt{(x+b)^2+(-b+a)(x+b)}+\frac{(-b+a)\ln\left(\frac{b}{2}+\frac{a}{2}+x+\sqrt{(x+b)^2+(-b+a)(x+b)}\right)}{2}}{2a-2b} - \frac{\sqrt{(x+a)^2+(b-a)(x+a)}+\frac{(b-a)\ln\left(\frac{b}{2}+\frac{a}{2}+x+\sqrt{(x+a)^2+(b-a)(x+a)}\right)}{2}}{2(-b+a)} - \ln((}\\ &= \frac{\sqrt{a+b+2x+2\sqrt{(x+a)\,(x+b)}}\,\sqrt{2}}{2\sqrt{(x+a)\,(x+b)}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}\,\sqrt{2}}{2\sqrt{(x+a)(x+b)}}\left(-3\sqrt{(x+a)(x+b)}+y\right) \\
&= \frac{\left(-3\sqrt{(x+a)(x+b)}+y\right)\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}\,\sqrt{2}}{2\sqrt{(x+a)(x+b)}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}\,\sqrt{2}}{2\sqrt{(x+a)(x+b)}}\left(2\sqrt{(x+a)(x+b)}\right) \\
&= \sqrt{2}\,\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\left(\frac{\left(-3\sqrt{(x+a)(x+b)}+y\right)\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}\,\sqrt{2}}{2\sqrt{(x+a)(x+b)}}\right)+\left(\sqrt{2}\,\sqrt{a+b+2x+2\sqrt{(x+a)}\right.
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x
$$

$$
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{\left(-3\sqrt{(x+a)(x+b)}+y\right)\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}\,\sqrt{2}}{2\sqrt{(x+a)(x+b)}}\,\mathrm{d}x
$$

$$
\phi = \int^{x} \frac{\left(-3\sqrt{(\_a+a)(\_a+b)}+y\right)\sqrt{a+b+2\_a+2\sqrt{(\_a+a)(\_a+b)}}\,\sqrt{2}}{2\sqrt{(\_a+a)(\_a+b)}} d\_a + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \sqrt{2}\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sqrt{2}\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}}$. Therefore equation (4) becomes

$$
\sqrt{2}\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}} = \sqrt{2}\sqrt{a+b+2x+2\sqrt{(x+a)(x+b)}} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = \int^{x} \frac{\left(-3\sqrt{(\_a+a)(\_a+b)}+y\right)\sqrt{a+b+2\_a+2\sqrt{(\_a+a)(\_a+b)}}\,\sqrt{2}}{2\sqrt{(\_a+a)(\_a+b)}} d\_a + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = \int^{x} \frac{\left(-3\sqrt{(\_a+a)(\_a+b)}+y\right)\sqrt{a+b+2\_a+2\sqrt{(\_a+a)(\_a+b)}}\,\sqrt{2}}{2\sqrt{(\_a+a)(\_a+b)}} d\_a
$$

Summary
The solution(s) found are the following

$$\int^{x} \frac{\left(-3\sqrt{\left(\_a+a\right)\left(\_a+b\right)}+y\right)\sqrt{a+b+2\_a+2\sqrt{\left(\_a+a\right)\left(\_a+b\right)}}\,\sqrt{2}}{2\sqrt{\left(\_a+a\right)\left(\_a+b\right)}}d\_a = c_1 \tag{1}$$

Verification of solutions

$$\int^{x} \frac{\left(-3\sqrt{\left(\_a+a\right)\left(\_a+b\right)}+y\right)\sqrt{a+b+2\_a+2\sqrt{\left(\_a+a\right)\left(\_a+b\right)}}\,\sqrt{2}}{2\sqrt{\left(\_a+a\right)\left(\_a+b\right)}}d\_a = c_1$$

Verified OK.

1.22.4 Maple step by step solution

Let's solve

$\sqrt{\left(x+a\right)\left(x+b\right)}\,(2y'-3)+y=0$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{3}{2} - \frac{y}{2\sqrt{(x+a)(x+b)}}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{2\sqrt{(x+a)(x+b)}} = \frac{3}{2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{2\sqrt{(x+a)(x+b)}}\right) = \frac{3\mu(x)}{2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{y}{2\sqrt{(x+a)(x+b)}}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{2\sqrt{(x+a)(x+b)}}$

- Solve to find the integrating factor

  $\mu(x) = \sqrt{2a+2b+4x+4\sqrt{\left(x+a\right)\left(x+b\right)}}$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{3\mu(x)}{2} dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\, y = \int \frac{3\mu(x)}{2} dx + c_1$$

- Solve for y

$$y = \frac{\int \frac{3\mu(x)}{2} dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \sqrt{2a + 2b + 4x + 4\sqrt{(x+a)\,(x+b)}}$

$$y = \frac{\int \frac{3\sqrt{2a+2b+4x+4\sqrt{(x+a)(x+b)}}}{2} dx + c_1}{\sqrt{2a+2b+4x+4\sqrt{(x+a)(x+b)}}}$$

- Simplify

$$y = \frac{3\left(\int \sqrt{2a+2b+4x+4\sqrt{(x+a)(x+b)}}dx\right) + 2c_1}{2\sqrt{2a+2b+4x+4\sqrt{(x+a)(x+b)}}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 60

```
dsolve(sqrt((x+a)*(x+b))*(2*diff(y(x),x)-3)+y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{3\left(\int \sqrt{2a + 2b + 4x + 4\sqrt{(x+a)\,(x+b)}}dx\right) + 4c_1}{2\sqrt{2a + 2b + 4x + 4\sqrt{(x+a)\,(x+b)}}}$$

✓ Solution by Mathematica

Time used: 0.433 (sec). Leaf size: 115

```
DSolve[Sqrt[(x+a)*(x+b)]*(2*y'[x]-3)+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \exp\left(-\frac{\sqrt{a+x}\sqrt{b+x}\operatorname{arctanh}\left(\frac{\sqrt{b+x}}{\sqrt{a+x}}\right)}{\sqrt{(a+x)(b+x)}}\right)\left(\int_1^x \frac{3}{2}\exp\left(\frac{\operatorname{arctanh}\left(\frac{\sqrt{b+K[1]}}{\sqrt{a+K[1]}}\right)\sqrt{a+K[1]}\sqrt{b+K[1]}}{\sqrt{(a+K[1])(b+K[1])}}\right)dK[1] + c_1\right)$$

1.23 problem 5(d)

Internal problem ID [3051]
Internal file name [`OUTPUT/2543_Sunday_June_05_2022_03_18_52_AM_9840611/index.tex`]

Book: Elementary Differential equations, Chaundy, 1969
Section: Exercises 3, page 60
Problem number: 5(d).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{\sqrt{(x+a)\,(x+b)}\,y'+y=\sqrt{x+a}-\sqrt{x+b}}$$

1.23.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$p(x)=\frac{1}{\sqrt{(x+a)\,(x+b)}}$$
$$q(x)=\frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{(x+a)\,(x+b)}}$$

Hence the ode is

$$y'+\frac{y}{\sqrt{(x+a)\,(x+b)}}=\frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{(x+a)\,(x+b)}}$$

The integrating factor μ is

$$\mu = e^{\int \frac{1}{\sqrt{(x+a)(x+b)}}dx}$$

$$= e^{\frac{\sqrt{(x+b)^2+(-b+a)(x+b)}+\frac{(-b+a)\ln\left(\frac{b}{2}+\frac{a}{2}+x+\sqrt{(x+b)^2+(-b+a)(x+b)}\right)}{2}}{-b+a}-\frac{\sqrt{(x+a)^2+(b-a)(x+a)}+\frac{(b-a)\ln\left(\frac{b}{2}+\frac{a}{2}+x+\sqrt{(x+a)^2+(b-a)(x+a)}\right)}{2}}{-b+a}}$$

Which simplifies to

$$\mu = \frac{b}{2}+\frac{a}{2}+x+\sqrt{(x+a)(x+b)}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{(x+a)(x+b)}}\right)$$

$$\frac{d}{dx}\left(\left(\frac{b}{2}+\frac{a}{2}+x+\sqrt{(x+a)(x+b)}\right)y\right) = \left(\frac{b}{2}+\frac{a}{2}+x+\sqrt{(x+a)(x+b)}\right)\left(\frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{(x+a)(x+b)}}\right)$$

$$d\left(\left(\frac{b}{2}+\frac{a}{2}+x+\sqrt{(x+a)(x+b)}\right)y\right) = \left(\frac{(\sqrt{x+a}-\sqrt{x+b})\left(a+b+2x+2\sqrt{(x+a)(x+b)}\right)}{2\sqrt{(x+a)(x+b)}}\right)$$

Integrating gives

$$\left(\frac{b}{2}+\frac{a}{2}+x+\sqrt{(x+a)(x+b)}\right)y = \int \frac{(\sqrt{x+a}-\sqrt{x+b})\left(a+b+2x+2\sqrt{(x+a)(x+b)}\right)}{2\sqrt{(x+a)(x+b)}}\,dx$$

$$\left(\frac{b}{2}+\frac{a}{2}+x+\sqrt{(x+a)(x+b)}\right)y = \frac{2(x+a)^{\frac{3}{2}}}{3}-\frac{2(x+b)^{\frac{3}{2}}}{3}+\frac{\sqrt{x+a}\,(x+b)(2x-b+3a)}{3\sqrt{(x+a)(x+b)}}-\frac{\sqrt{x+b}\,(}{3\sqrt{}}$$

Dividing both sides by the integrating factor $\mu = \frac{b}{2}+\frac{a}{2}+x+\sqrt{(x+a)(x+b)}$ results in

$$y = \frac{\frac{4(x+a)^{\frac{3}{2}}}{3}-\frac{4(x+b)^{\frac{3}{2}}}{3}+\frac{2\sqrt{x+a}\,(x+b)(2x-b+3a)}{3\sqrt{(x+a)(x+b)}}-\frac{2\sqrt{x+b}\,(x+a)(2x-a+3b)}{3\sqrt{(x+a)(x+b)}}}{a+b+2x+2\sqrt{(x+a)(x+b)}}+\frac{2c_1}{a+b+2x+2\sqrt{(x+a)(x+b)}}$$

which simplifies to

$$y = \frac{2\left((2a+2x)\sqrt{x+a}+(-2b-2x)\sqrt{x+b}+3c_1\right)\sqrt{(x+a)(x+b)}+6(x+b)\left(-\frac{b}{3}+a+\frac{2x}{3}\right)\sqrt{x+a}}{\sqrt{(x+a)(x+b)}\left(3a+3b+6x+6\sqrt{(x+a)(x+b)}\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{2\left(\left(2a+2x\right)\sqrt{x+a}+\left(-2b-2x\right)\sqrt{x+b}+3c_1\right)\sqrt{\left(x+a\right)\left(x+b\right)}+6(x+b)\left(-\frac{b}{3}+a+\frac{2x}{3}\right)\sqrt{x+a}+}{\sqrt{\left(x+a\right)\left(x+b\right)}\left(3a+3b+6x+6\sqrt{\left(x+a\right)\left(x+b\right)}\right)} \tag{1}$$

Verification of solutions

$$y = \frac{2\left(\left(2a+2x\right)\sqrt{x+a}+\left(-2b-2x\right)\sqrt{x+b}+3c_1\right)\sqrt{\left(x+a\right)\left(x+b\right)}+6(x+b)\left(-\frac{b}{3}+a+\frac{2x}{3}\right)\sqrt{x+a}+}{\sqrt{\left(x+a\right)\left(x+b\right)}\left(3a+3b+6x+6\sqrt{\left(x+a\right)\left(x+b\right)}\right)}$$

Verified OK.

1.23.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-y+\sqrt{x+a}-\sqrt{x+b}}{\sqrt{\left(x+a\right)\left(x+b\right)}}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 67: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \frac{1}{\frac{a}{2} + \frac{b}{2} + x + \sqrt{x^2 + (a+b)\,x + ab}} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\frac{a}{2}+\frac{b}{2}+x+\sqrt{x^2+(a+b)x+ab}}} dy \end{aligned}$$

Which results in

$$S = \left(\frac{a}{2} + \frac{b}{2} + x + \sqrt{x^2 + (a+b)\,x + ab}\right) y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-y + \sqrt{x+a} - \sqrt{x+b}}{\sqrt{(x+a)\,(x+b)}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{\left(a + b + 2x + 2\sqrt{x+a}\,\sqrt{x+b}\right) y}{2\sqrt{x+a}\,\sqrt{x+b}} \\ S_y &= \frac{a}{2} + \frac{b}{2} + x + \sqrt{x+a}\,\sqrt{x+b} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\left(a + b + 2x + 2\sqrt{x+a}\,\sqrt{x+b}\right)\left(\frac{y}{\sqrt{x+a}\,\sqrt{x+b}} + \frac{-y+\sqrt{x+a}-\sqrt{x+b}}{\sqrt{(x+a)(x+b)}}\right)}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\left(2\sqrt{R+a}\,\sqrt{R+b} + a + b + 2R\right)\left(\sqrt{R+a} - \sqrt{R+b}\right)}{2\sqrt{R+a}\,\sqrt{R+b}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \sqrt{R+b}\,a - \sqrt{R+b}\,b + \sqrt{R+a}\,a - \sqrt{R+a}\,b + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\left(a+b+2x+2\sqrt{x+a}\,\sqrt{x+b}\right)y}{2} = \sqrt{x+b}\,a - \sqrt{x+b}\,b + \sqrt{x+a}\,a - \sqrt{x+a}\,b + c_1$$

Which simplifies to

$$\left(y\sqrt{x+b} - a + b\right)\sqrt{x+a} + (b-a)\sqrt{x+b} + \frac{y(2x+b+a)}{2} - c_1 = 0$$

Which gives

$$y = \frac{2\sqrt{x+b}\,a - 2\sqrt{x+b}\,b + 2\sqrt{x+a}\,a - 2\sqrt{x+a}\,b + 2c_1}{a+b+2x+2\sqrt{x+a}\,\sqrt{x+b}}$$

Summary

The solution(s) found are the following

$$y = \frac{2\sqrt{x+b}\,a - 2\sqrt{x+b}\,b + 2\sqrt{x+a}\,a - 2\sqrt{x+a}\,b + 2c_1}{a+b+2x+2\sqrt{x+a}\,\sqrt{x+b}} \tag{1}$$

Verification of solutions

$$y = \frac{2\sqrt{x+b}\,a - 2\sqrt{x+b}\,b + 2\sqrt{x+a}\,a - 2\sqrt{x+a}\,b + 2c_1}{a+b+2x+2\sqrt{x+a}\,\sqrt{x+b}}$$

Verified OK.

1.23.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\sqrt{(x+a)\,(x+b)}\right)\mathrm{d}y&=\left(-y+\sqrt{x+a}-\sqrt{x+b}\right)\mathrm{d}x\\ \left(y-\sqrt{x+a}+\sqrt{x+b}\right)\mathrm{d}x+\left(\sqrt{(x+a)\,(x+b)}\right)\mathrm{d}y&=0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=y-\sqrt{x+a}+\sqrt{x+b}\\ N(x,y)&=\sqrt{(x+a)\,(x+b)}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(y-\sqrt{x+a}+\sqrt{x+b}\right)\\ &=1\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\sqrt{(x+a)(x+b)}\right) \\
&= \frac{2x+b+a}{2\sqrt{(x+a)(x+b)}}
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{\sqrt{(x+a)(x+b)}}\left((1) - \left(\frac{2x+b+a}{2\sqrt{(x+a)(x+b)}}\right)\right) \\
&= \frac{2\sqrt{(x+a)(x+b)} - 2x - b - a}{2(x+a)(x+b)}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}x} \\
&= e^{\int \frac{2\sqrt{(x+a)(x+b)} - 2x - b - a}{2(x+a)(x+b)} \,\mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\frac{\sqrt{(x+b)^2+(-b+a)(x+b)} + \frac{(-b+a)\ln\left(\frac{b}{2}+\frac{a}{2}+x+\sqrt{(x+b)^2+(-b+a)(x+b)}\right)}{2}}{-b+a} - \frac{\sqrt{(x+a)^2+(b-a)(x+a)} + \frac{(b-a)\ln\left(\frac{b}{2}+\frac{a}{2}+x+\sqrt{(x+a)^2+(b-a)(x+a)}\right)}{2}}{-b+a} - \ln((} \\
&= \frac{a+b+2x+2\sqrt{(x+a)(x+b)}}{2\sqrt{(x+a)(x+b)}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{a+b+2x+2\sqrt{(x+a)(x+b)}}{2\sqrt{(x+a)(x+b)}}\left(y - \sqrt{x+a} + \sqrt{x+b}\right) \\
&= \frac{\left(y - \sqrt{x+a} + \sqrt{x+b}\right)\left(a+b+2x+2\sqrt{(x+a)(x+b)}\right)}{2\sqrt{(x+a)(x+b)}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{a+b+2x+2\sqrt{(x+a)(x+b)}}{2\sqrt{(x+a)(x+b)}}\left(\sqrt{(x+a)(x+b)}\right) \\
&= \frac{b}{2}+\frac{a}{2}+x+\sqrt{(x+a)(x+b)}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} =
$$

$$
\left(\frac{\left(y-\sqrt{x+a}+\sqrt{x+b}\right)\left(a+b+2x+2\sqrt{(x+a)(x+b)}\right)}{2\sqrt{(x+a)(x+b)}}\right)+\left(\frac{b}{2}+\frac{a}{2}+x+\sqrt{(x+a)(x+b)}\right)\frac{\mathrm{d}y}{\mathrm{d}x} =
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int\overline{M}\,\mathrm{d}x
$$

$$
\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int\frac{\left(y-\sqrt{x+a}+\sqrt{x+b}\right)\left(a+b+2x+2\sqrt{(x+a)(x+b)}\right)}{2\sqrt{(x+a)(x+b)}}\,\mathrm{d}x
$$

$$
\begin{aligned}
\phi = &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3)\\
&-\frac{\left((2a+2x)\sqrt{x+a}+(-2b-2x)\sqrt{x+b}-3xy\right)\sqrt{(x+a)(x+b)}+3(x+b)\left(-\frac{b}{3}+a+\frac{2x}{3}\right)+}{3\sqrt{(x+a)(x+b)}} \\
&+f(y)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial\phi}{\partial y} &= -\frac{-3\sqrt{(x+a)(x+b)}\,x+(-3x-3b)(x+a)}{3\sqrt{(x+a)(x+b)}}+f'(y) \qquad (4)\\
&= \frac{\sqrt{(x+a)(x+b)}\,x+(x+a)(x+b)}{\sqrt{(x+a)(x+b)}}+f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{b}{2} + \frac{a}{2} + x + \sqrt{(x+a)(x+b)}$. Therefore equation (4) becomes

$$\frac{b}{2} + \frac{a}{2} + x + \sqrt{(x+a)(x+b)} = \frac{\sqrt{(x+a)(x+b)}\, x + (x+a)(x+b)}{\sqrt{(x+a)(x+b)}} + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{a}{2} + \frac{b}{2}$$

Integrating the above w.r.t y gives

$$\int f'(y)\, \mathrm{d}y = \int \left(\frac{a}{2} + \frac{b}{2}\right) \mathrm{d}y$$

$$f(y) = \left(\frac{a}{2} + \frac{b}{2}\right) y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\left((2a+2x)\sqrt{x+a} + (-2b-2x)\sqrt{x+b} - 3xy\right)\sqrt{(x+a)(x+b)} + 3(x+b)\left(-\frac{b}{3} + a + \frac{2x}{3}\right)\sqrt{x+a}\ \cdots}{3\sqrt{(x+a)(x+b)}} + \left(\frac{a}{2} + \frac{b}{2}\right) y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\left((2a+2x)\sqrt{x+a} + (-2b-2x)\sqrt{x+b} - 3xy\right)\sqrt{(x+a)(x+b)} + 3(x+b)\left(-\frac{b}{3} + a + \frac{2x}{3}\right)\sqrt{x+a}\ \cdots}{3\sqrt{(x+a)(x+b)}} + \left(\frac{a}{2} + \frac{b}{2}\right) y$$

The solution becomes

$$y = \frac{\frac{2\sqrt{x+b}\, a^2}{3} - 2\sqrt{x+b}\, ba + 2\sqrt{x+a}\, ab - \frac{2\sqrt{x+b}\, ax}{3} + 2\sqrt{x+a}\, ax + \frac{4\sqrt{(x+a)(x+b)}\,\sqrt{x+a}\, a}{3} - \frac{2\sqrt{x+a}\, b^2}{3} - 2\sqrt{x+\cdots}}{\sqrt{(x+a)(x+b)}\, a + \sqrt{(x+a)(x\cdots}}$$

Summary

The solution(s) found are the following

$$y = \frac{\frac{2\sqrt{x+b}\,a^2}{3} - 2\sqrt{x+b}\,ba + 2\sqrt{x+a}\,ab - \frac{2\sqrt{x+b}\,ax}{3} + 2\sqrt{x+a}\,ax + \frac{4\sqrt{(x+a)(x+b)}\,\sqrt{x+a}\,a}{3} - \frac{2\sqrt{x+a}\,b^2}{3} - 2\sqrt{x+ \cdots}{\sqrt{(x+a)\,(x+b)}\,a + \sqrt{(x+a)\,(x \cdots}} \tag{1}$$

Verification of solutions

$$y = \frac{\frac{2\sqrt{x+b}\,a^2}{3} - 2\sqrt{x+b}\,ba + 2\sqrt{x+a}\,ab - \frac{2\sqrt{x+b}\,ax}{3} + 2\sqrt{x+a}\,ax + \frac{4\sqrt{(x+a)(x+b)}\,\sqrt{x+a}\,a}{3} - \frac{2\sqrt{x+a}\,b^2}{3} - 2\sqrt{x+ \cdots}{\sqrt{(x+a)\,(x+b)}\,a + \sqrt{(x+a)\,(x \cdots}}$$

Verified OK.

1.23.4 Maple step by step solution

Let's solve

$\sqrt{(x+a)\,(x+b)}\,y' + y = \sqrt{x+a} - \sqrt{x+b}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{\sqrt{(x+a)(x+b)}} + \frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{(x+a)(x+b)}}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{\sqrt{(x+a)(x+b)}} = \frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{(x+a)(x+b)}}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{\sqrt{(x+a)(x+b)}}\right) = \frac{\mu(x)\left(\sqrt{x+a}-\sqrt{x+b}\right)}{\sqrt{(x+a)(x+b)}}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{y}{\sqrt{(x+a)(x+b)}}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{\sqrt{(x+a)(x+b)}}$

- Solve to find the integrating factor

  $\mu(x) = a + b + 2x + 2\sqrt{(x+a)\,(x+b)}$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \frac{\mu(x)\left(\sqrt{x+a}-\sqrt{x+b}\right)}{\sqrt{(x+a)(x+b)}} dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\,y = \int \frac{\mu(x)\left(\sqrt{x+a}-\sqrt{x+b}\right)}{\sqrt{(x+a)(x+b)}} dx + c_1$$

- Solve for y

$$y = \frac{\int \frac{\mu(x)\left(\sqrt{x+a}-\sqrt{x+b}\right)}{\sqrt{(x+a)(x+b)}} dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = a + b + 2x + 2\sqrt{(x+a)\,(x+b)}$

$$y = \frac{\int \frac{\left(\sqrt{x+a}-\sqrt{x+b}\right)\left(a+b+2x+2\sqrt{(x+a)(x+b)}\right)}{\sqrt{(x+a)(x+b)}} dx + c_1}{a+b+2x+2\sqrt{(x+a)(x+b)}}$$

- Evaluate the integrals on the rhs

$$y = \frac{\frac{4(x+a)^{\frac{3}{2}}}{3} - \frac{4(x+b)^{\frac{3}{2}}}{3} + \frac{2\sqrt{x+a}\,(x+b)(2x-b+3a)}{3\sqrt{(x+a)(x+b)}} - \frac{2\sqrt{x+b}\,(x+a)(2x-a+3b)}{3\sqrt{(x+a)(x+b)}} + c_1}{a+b+2x+2\sqrt{(x+a)(x+b)}}$$

- Simplify

$$y = \frac{2\left(\left((2a+2x)\sqrt{x+a}+(-2b-2x)\sqrt{x+b}+\frac{3c_1}{2}\right)\sqrt{(x+a)(x+b)}+3(x+b)\left(-\frac{b}{3}+a+\frac{2x}{3}\right)\sqrt{x+a}+\sqrt{x+b}\,(x+a)(-2x+a-3b)\right)}{\sqrt{(x+a)(x+b)}\left(3a+3b+6x+6\sqrt{(x+a)(x+b)}\right)}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 114

```
dsolve(sqrt((x+a)*(x+b))*diff(y(x),x)+y(x)=sqrt(x+a)-sqrt(x+b),y(x), singsol=all)
```

$$y(x) = \frac{2\left((2a+2x)\sqrt{x+a}+(-2b-2x)\sqrt{x+b}+3c_1\right)\sqrt{(x+a)\,(x+b)}+6\left(-\frac{b}{3}+a+\frac{2x}{3}\right)(x+b)\sqrt{x+a}+}{\sqrt{(x+a)\,(x+b)}\,\left(3a+3b+6x+6\sqrt{(x+a)\,(x+b)}\right)}$$

✓ Solution by Mathematica

Time used: 2.411 (sec). Leaf size: 145

```
DSolve[Sqrt[(x+a)*(x+b)]*y'[x]+y[x]==Sqrt[x+a]-Sqrt[x+b],y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \exp\left(-\frac{2\sqrt{a+x}\sqrt{b+x}\text{arctanh}\left(\frac{\sqrt{b+x}}{\sqrt{a+x}}\right)}{\sqrt{(a+x)(b+x)}}\right)\left(\int_1^x \frac{\exp\left(\frac{2\text{arctanh}\left(\frac{\sqrt{b+K[1]}}{\sqrt{a+K[1]}}\right)\sqrt{a+K[1]}\sqrt{b+K[1]}}{\sqrt{(a+K[1])(b+K[1])}}\right)\left(\sqrt{a+K[1]}\right.}{\sqrt{(a+K[1])(b+K[1])}} + c_1\right)$$