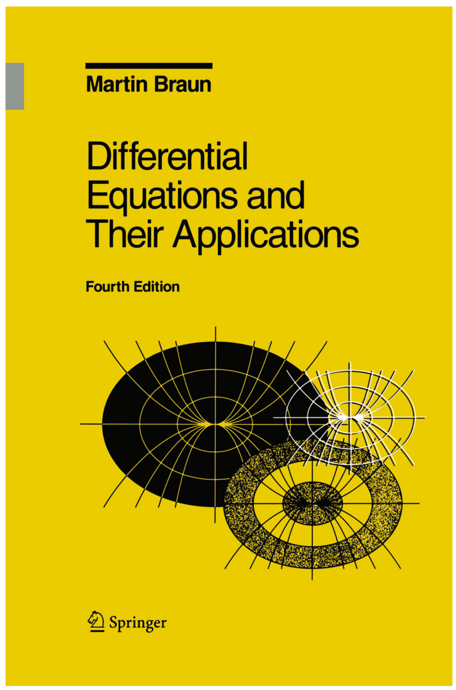
A Solution Manual For

Differential equations and their applications, 4th ed., M. Braun



Nasser M. Abbasi

May 15, 2024

Contents

1 Section 3.8, Systems of differential equations. The eigenva1ue-eigenvector method. Page 339

1.1 problem 1

Internal problem ID [1824]
Internal file name [`OUTPUT/1825_Sunday_June_05_2022_02_34_15_AM_73582960/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.8, Systems of differential equations. The eigenva1ue-eigenvector method. Page 339
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 6x_1(t) - 3x_2(t) \\
x_2'(t) &= 2x_1(t) + x_2(t)
\end{aligned}
$$

1.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} -2\,e^{3t} + 3\,e^{4t} & -3\,e^{4t} + 3\,e^{3t} \\ 2\,e^{4t} - 2\,e^{3t} & 3\,e^{3t} - 2\,e^{4t} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} -2\,\mathrm{e}^{3t}+3\,\mathrm{e}^{4t} & -3\,\mathrm{e}^{4t}+3\,\mathrm{e}^{3t} \\ 2\,\mathrm{e}^{4t}-2\,\mathrm{e}^{3t} & 3\,\mathrm{e}^{3t}-2\,\mathrm{e}^{4t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} (-2\,\mathrm{e}^{3t}+3\,\mathrm{e}^{4t})\,c_1 + (-3\,\mathrm{e}^{4t}+3\,\mathrm{e}^{3t})\,c_2 \\ (2\,\mathrm{e}^{4t}-2\,\mathrm{e}^{3t})\,c_1 + (3\,\mathrm{e}^{3t}-2\,\mathrm{e}^{4t})\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} (-2c_1+3c_2)\,\mathrm{e}^{3t} + 3\,\mathrm{e}^{4t}(c_1-c_2) \\ (-2c_1+3c_2)\,\mathrm{e}^{3t} + 2\,\mathrm{e}^{4t}(c_1-c_2) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

1.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 6-\lambda & -3 \\ 2 & 1-\lambda \end{bmatrix} \right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 7\lambda + 12 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3$$
$$\lambda_2 = 4$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
3	1	real eigenvalue
4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & -3 & 0 \\ 2 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} 3 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & -3 & 0 \\ 2 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{3t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
3	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
4	1	1	No	$\begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{3t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}\end{aligned}$$

Since eigenvalue 4 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{4t} \\ &= \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} e^{4t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{3\,e^{4t}}{2} \\ e^{4t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{3t} + \frac{3c_2 e^{4t}}{2} \\ c_1 e^{3t} + c_2 e^{4t} \end{bmatrix}$$

The following is the phase plot of the system.

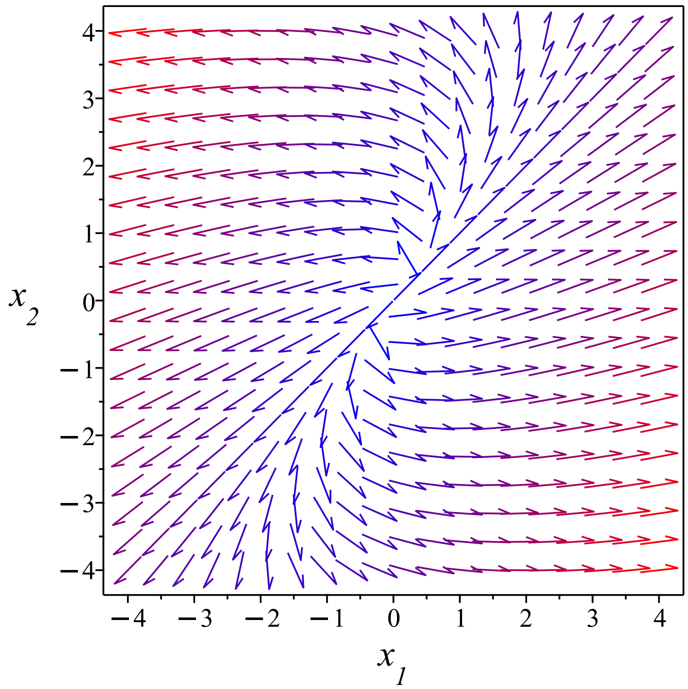


Figure 1: Phase plot

1.1.3 Maple step by step solution

Let's solve

$[x_1'(t) = 6x_1(t) - 3x_2(t), x_2'(t) = 2x_1(t) + x_2(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix}$$

- Rewrite the system as

  $$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right], \left[4, \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

  $$\left[3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\underline{\vec{x}}_1 = \mathrm{e}^{3t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

  $$\left[4, \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\underline{\vec{x}}_2 = \mathrm{e}^{4t} \cdot \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

  $$\underline{\vec{x}} = c_1 \underline{\vec{x}}_1 + c_2 \underline{\vec{x}}_2$$

- Substitute solutions into the general solution

  $$\underline{\vec{x}} = c_1 \mathrm{e}^{3t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{4t} \cdot \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{3t} + \frac{3c_2 e^{4t}}{2} \\ c_1 e^{3t} + c_2 e^{4t} \end{bmatrix}$$

- Solution to the system of ODEs

  $$\left\{x_1(t) = c_1 e^{3t} + \frac{3c_2 e^{4t}}{2}, x_2(t) = c_1 e^{3t} + c_2 e^{4t}\right\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 35

```
dsolve([diff(x__1(t),t)=6*x__1(t)-3*x__2(t),diff(x__2(t),t)=2*x__1(t)+1*x__2(t)],singsol=all
```

$$x_1(t) = c_1 e^{3t} + c_2 e^{4t}$$
$$x_2(t) = c_1 e^{3t} + \frac{2c_2 e^{4t}}{3}$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 60

```
DSolve[{x1'[t]==6*x1[t]-3*x2[t],x2'[t]==2*x1[t]+1*x2[t]},{x1[t],x2[t]},t,IncludeSingularSolu
```

$$\text{x1}(t) \to e^{3t}(c_1(3e^t - 2) - 3c_2(e^t - 1))$$
$$\text{x2}(t) \to e^{3t}(2c_1(e^t - 1) + c_2(3 - 2e^t))$$

1.2 problem 2

Internal problem ID [1825]
Internal file name [`OUTPUT/1826_Sunday_June_05_2022_02_34_18_AM_98845134/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.8, Systems of differential equations. The eigenva1ue-eigenvector method. Page 339
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= -2x_1(t) + x_2(t) \\
x_2'(t) &= -4x_1(t) + 3x_2(t)
\end{aligned}
$$

1.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \frac{4\,\mathrm{e}^{-t}}{3} - \frac{\mathrm{e}^{2t}}{3} & \frac{\mathrm{e}^{2t}}{3} - \frac{\mathrm{e}^{-t}}{3} \\ -\frac{4\,\mathrm{e}^{2t}}{3} + \frac{4\,\mathrm{e}^{-t}}{3} & -\frac{\mathrm{e}^{-t}}{3} + \frac{4\,\mathrm{e}^{2t}}{3} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{4\,\mathrm{e}^{-t}}{3} - \frac{\mathrm{e}^{2t}}{3} & \frac{\mathrm{e}^{2t}}{3} - \frac{\mathrm{e}^{-t}}{3} \\ -\frac{4\,\mathrm{e}^{2t}}{3} + \frac{4\,\mathrm{e}^{-t}}{3} & -\frac{\mathrm{e}^{-t}}{3} + \frac{4\,\mathrm{e}^{2t}}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{4\,\mathrm{e}^{-t}}{3} - \frac{\mathrm{e}^{2t}}{3}\right) c_1 + \left(\frac{\mathrm{e}^{2t}}{3} - \frac{\mathrm{e}^{-t}}{3}\right) c_2 \\ \left(-\frac{4\,\mathrm{e}^{2t}}{3} + \frac{4\,\mathrm{e}^{-t}}{3}\right) c_1 + \left(-\frac{\mathrm{e}^{-t}}{3} + \frac{4\,\mathrm{e}^{2t}}{3}\right) c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(4c_1 - c_2)\mathrm{e}^{-t}}{3} - \frac{\mathrm{e}^{2t}(c_1 - c_2)}{3} \\ \frac{(4c_1 - c_2)\mathrm{e}^{-t}}{3} - \frac{4\,\mathrm{e}^{2t}(c_1 - c_2)}{3} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

1.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -2-\lambda & 1 \\ -4 & 3-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 - \lambda - 2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 1 \\ -4 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ -4 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 - 4R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -4 & 1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -4 & 1 & 0 \\ -4 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} -4 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{4}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{4} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{4} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	1	1	No	$\begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{2t}\\ &= \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix} e^{2t}\end{aligned}$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-t}\\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{\mathrm{e}^{2t}}{4} \\ \mathrm{e}^{2t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1 \mathrm{e}^{2t}}{4} + c_2 \mathrm{e}^{-t} \\ c_1 \mathrm{e}^{2t} + c_2 \mathrm{e}^{-t} \end{bmatrix}$$

The following is the phase plot of the system.

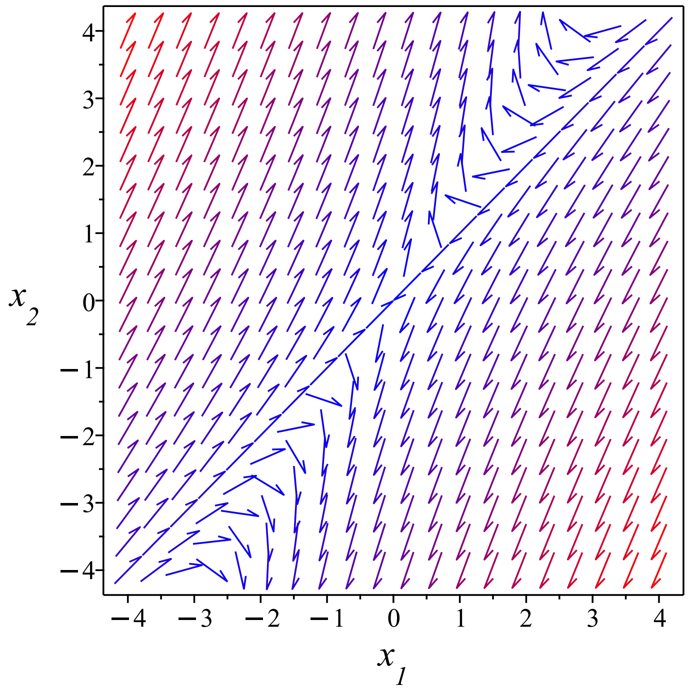


Figure 2: Phase plot

1.2.3 Maple step by step solution

Let's solve

$$[x_1'(t) = -2x_1(t) + x_2(t), x_2'(t) = -4x_1(t) + 3x_2(t)]$$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_1 = e^{-t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_2 = e^{2t} \cdot \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{\vec{x}} = c_1 \underline{\vec{x}}_1 + c_2 \underline{\vec{x}}_2$$

- Substitute solutions into the general solution

$$\underline{\vec{x}} = c_1 e^{-t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \cdot \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} + \frac{c_2 e^{2t}}{4} \\ c_1 e^{-t} + c_2 e^{2t} \end{bmatrix}$$

- Solution to the system of ODEs

  $$\left\{x_1(t) = c_1 e^{-t} + \frac{c_2 e^{2t}}{4}, x_2(t) = c_1 e^{-t} + c_2 e^{2t}\right\}$$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve([diff(x__1(t),t)=-2*x__1(t)+1*x__2(t),diff(x__2(t),t)=-4*x__1(t)+3*x__2(t)],singsol=a
```

$$x_1(t) = e^{-t} c_1 + c_2 e^{2t}$$
$$x_2(t) = e^{-t} c_1 + 4 c_2 e^{2t}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 72

```
DSolve[{x1'[t]==-2*x1[t]+1*x2[t],x2'[t]==-4*x1[t]+3*x2[t]},{x1[t],x2[t]},t,IncludeSingularSo
```

$$\text{x1}(t) \to \frac{1}{3} e^{-t} \left(c_2 \left(e^{3t} - 1\right) - c_1 \left(e^{3t} - 4\right)\right)$$
$$\text{x2}(t) \to \frac{1}{3} e^{-t} \left(c_2 \left(4 e^{3t} - 1\right) - 4 c_1 \left(e^{3t} - 1\right)\right)$$

1.3 problem 3

Internal problem ID [1826]
Internal file name [`OUTPUT/1827_Sunday_June_05_2022_02_34_19_AM_54133011/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.8, Systems of differential equations. The eigenva1ue-eigenvector method. Page 339
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 3x_1(t) + 2x_2(t) + 4x_3(t)\\
x_2'(t) &= 2x_1(t) + 2x_3(t)\\
x_3'(t) &= 4x_1(t) + 2x_2(t) + 3x_3(t)
\end{aligned}
$$

1.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{5\,e^{-t}}{9} + \frac{4\,e^{8t}}{9} & -\frac{2\,e^{-t}}{9} + \frac{2\,e^{8t}}{9} & -\frac{4\,e^{-t}}{9} + \frac{4\,e^{8t}}{9} \\ -\frac{2\,e^{-t}}{9} + \frac{2\,e^{8t}}{9} & \frac{8\,e^{-t}}{9} + \frac{e^{8t}}{9} & -\frac{2\,e^{-t}}{9} + \frac{2\,e^{8t}}{9} \\ -\frac{4\,e^{-t}}{9} + \frac{4\,e^{8t}}{9} & -\frac{2\,e^{-t}}{9} + \frac{2\,e^{8t}}{9} & \frac{5\,e^{-t}}{9} + \frac{4\,e^{8t}}{9} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{5\,\mathrm{e}^{-t}}{9}+\frac{4\,\mathrm{e}^{8t}}{9} & -\frac{2\,\mathrm{e}^{-t}}{9}+\frac{2\,\mathrm{e}^{8t}}{9} & -\frac{4\,\mathrm{e}^{-t}}{9}+\frac{4\,\mathrm{e}^{8t}}{9} \\ -\frac{2\,\mathrm{e}^{-t}}{9}+\frac{2\,\mathrm{e}^{8t}}{9} & \frac{8\,\mathrm{e}^{-t}}{9}+\frac{\mathrm{e}^{8t}}{9} & -\frac{2\,\mathrm{e}^{-t}}{9}+\frac{2\,\mathrm{e}^{8t}}{9} \\ -\frac{4\,\mathrm{e}^{-t}}{9}+\frac{4\,\mathrm{e}^{8t}}{9} & -\frac{2\,\mathrm{e}^{-t}}{9}+\frac{2\,\mathrm{e}^{8t}}{9} & \frac{5\,\mathrm{e}^{-t}}{9}+\frac{4\,\mathrm{e}^{8t}}{9} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{5\,\mathrm{e}^{-t}}{9}+\frac{4\,\mathrm{e}^{8t}}{9}\right)c_1+\left(-\frac{2\,\mathrm{e}^{-t}}{9}+\frac{2\,\mathrm{e}^{8t}}{9}\right)c_2+\left(-\frac{4\,\mathrm{e}^{-t}}{9}+\frac{4\,\mathrm{e}^{8t}}{9}\right)c_3 \\ \left(-\frac{2\,\mathrm{e}^{-t}}{9}+\frac{2\,\mathrm{e}^{8t}}{9}\right)c_1+\left(\frac{8\,\mathrm{e}^{-t}}{9}+\frac{\mathrm{e}^{8t}}{9}\right)c_2+\left(-\frac{2\,\mathrm{e}^{-t}}{9}+\frac{2\,\mathrm{e}^{8t}}{9}\right)c_3 \\ \left(-\frac{4\,\mathrm{e}^{-t}}{9}+\frac{4\,\mathrm{e}^{8t}}{9}\right)c_1+\left(-\frac{2\,\mathrm{e}^{-t}}{9}+\frac{2\,\mathrm{e}^{8t}}{9}\right)c_2+\left(\frac{5\,\mathrm{e}^{-t}}{9}+\frac{4\,\mathrm{e}^{8t}}{9}\right)c_3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(5c_1-2c_2-4c_3)\mathrm{e}^{-t}}{9}+\frac{4\left(c_1+\frac{c_2}{2}+c_3\right)\mathrm{e}^{8t}}{9} \\ \frac{(-2c_1+8c_2-2c_3)\mathrm{e}^{-t}}{9}+\frac{2\left(c_1+\frac{c_2}{2}+c_3\right)\mathrm{e}^{8t}}{9} \\ \frac{(-4c_1-2c_2+5c_3)\mathrm{e}^{-t}}{9}+\frac{4\left(c_1+\frac{c_2}{2}+c_3\right)\mathrm{e}^{8t}}{9} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

1.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 6\lambda^2 - 15\lambda - 8 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 8$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
8	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 4 & 2 & 4 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 2 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 4 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 2 & 4 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 4 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2} - s\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} - s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} - s \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} -\frac{t}{2} - s \\ t \\ s \end{bmatrix} &= \begin{bmatrix} -\frac{t}{2} \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} -\frac{t}{2} - s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)$$

Which are normalized to

$$\left(\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)$$

Considering the eigenvalue $\lambda_2 = 8$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} - (8)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 2 & -8 & 2 & 0 \\ 4 & 2 & -5 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{2R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 0 & -\frac{36}{5} & \frac{18}{5} & 0 \\ 4 & 2 & -5 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{4R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 0 & -\frac{36}{5} & \frac{18}{5} & 0 \\ 0 & \frac{18}{5} & -\frac{9}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_2}{2} \Longrightarrow \left[\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 0 & -\frac{36}{5} & \frac{18}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -5 & 2 & 4 \\ 0 & -\frac{36}{5} & \frac{18}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = t, v_2 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} t \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} t \\ \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} t \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated

with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
8	1	1	No	$\begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$
-1	2	2	No	$\begin{bmatrix} -\frac{1}{2} & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 8 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{8t} \\ &= \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix} e^{8t} \end{aligned}$$

eigenvalue -1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

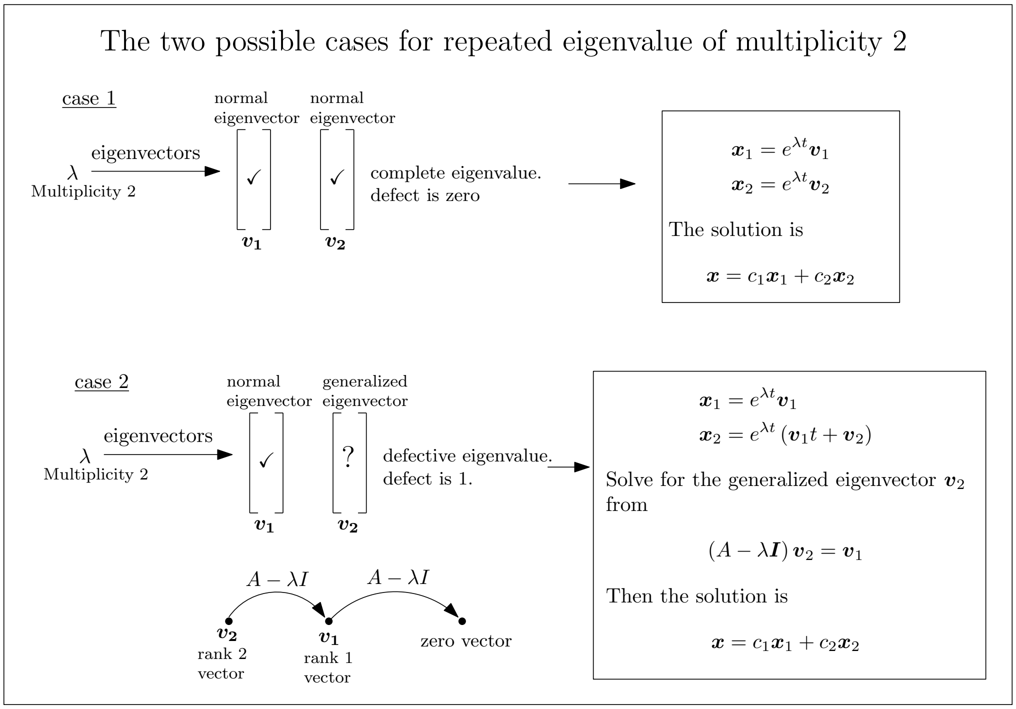


Figure 3: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-t}\\ &= \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} e^{-t}\end{aligned}$$

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{-t}\\ &= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{8t} \\ \frac{e^{8t}}{2} \\ e^{8t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{e^{-t}}{2} \\ e^{-t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{(-c_2-2c_3)e^{-t}}{2} + c_1 e^{8t} \\ \frac{c_1 e^{8t}}{2} + c_2 e^{-t} \\ c_1 e^{8t} + c_3 e^{-t} \end{bmatrix}$$

1.3.3 Maple step by step solution

Let's solve

$[x_1'(t) = 3x_1(t) + 2x_2(t) + 4x_3(t), x_2'(t) = 2x_1(t) + 2x_3(t), x_3'(t) = 4x_1(t) + 2x_2(t) + 3x_3(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

- Rewrite the system as

  $\underline{x}^{\rightarrow}{}'(t) = A \cdot \underline{x}^{\rightarrow}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-1, \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}\right], \left[-1, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right], \left[8, \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[-1, \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}\right]$$

- First solution from eigenvalue -1

  $$\underline{x}^{\rightarrow}{}_1(t) = \mathrm{e}^{-t} \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = -1$ is the eigenvalue, an

  $\underline{x}^{\rightarrow}{}_2(t) = \mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right)$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\underline{x}^{\rightarrow}{}_2(t)$ into the homogeneous system

  $\lambda\, \mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \left(\mathrm{e}^{\lambda t} A\right) \cdot \left(t\vec{v} + \vec{p}\right)$

- Use the fact that $\vec{v}$ is an eigenvector of A

  $\lambda\, \mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \mathrm{e}^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$

- Simplify equation

  $\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$

- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{x}_2(t)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue -1

$$\left(\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} - (-1) \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \overrightarrow{p} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p} = \begin{bmatrix} -\frac{1}{8} \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue -1

$$\overrightarrow{x}_2(t) = \mathrm{e}^{-t} \cdot \left(t \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{8} \\ 0 \\ 0 \end{bmatrix}\right)$$

- Consider eigenpair

$$\left[8, \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_3 = \mathrm{e}^{8t} \cdot \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

  $\overrightarrow{x} = c_1 \overrightarrow{x}_1(t) + c_2 \overrightarrow{x}_2(t) + c_3 \overrightarrow{x}_3$

- Substitute solutions into the general solution

$$\underline{\vec{x}} = c_1 \mathrm{e}^{-t} \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + c_2 \mathrm{e}^{-t} \cdot \left(t \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{8} \\ 0 \\ 0 \end{bmatrix} \right) + c_3 \mathrm{e}^{8t} \cdot \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{((-4t-1)c_2-4c_1)\mathrm{e}^{-t}}{8} + c_3 \mathrm{e}^{8t} \\ (c_2 t + c_1)\,\mathrm{e}^{-t} + \frac{c_3 \mathrm{e}^{8t}}{2} \\ c_3 \mathrm{e}^{8t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \frac{((-4t-1)c_2-4c_1)\mathrm{e}^{-t}}{8} + c_3 \mathrm{e}^{8t}, x_2(t) = (c_2 t + c_1)\,\mathrm{e}^{-t} + \frac{c_3 \mathrm{e}^{8t}}{2}, x_3(t) = c_3 \mathrm{e}^{8t}\right\}$$

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 67

```
dsolve([diff(x__1(t),t)=3*x__1(t)+2*x__2(t)+4*x__3(t),diff(x__2(t),t)=2*x__1(t)+0*x__2(t)+2*
```

$$x_1(t) = 2c_2 \mathrm{e}^{8t} + 2c_3 \mathrm{e}^{-t} + \mathrm{e}^{-t} c_1$$
$$x_2(t) = c_2 \mathrm{e}^{8t} + c_3 \mathrm{e}^{-t}$$
$$x_3(t) = 2c_2 \mathrm{e}^{8t} - \frac{5c_3 \mathrm{e}^{-t}}{2} - \mathrm{e}^{-t} c_1$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 135

```
DSolve[{x1'[t]==3*x1[t]+2*x2[t]+4*x3[t],x2'[t]==2*x1[t]+0*x2[t]+2*x3[t],x3'[t]==4*x1[t]+2*x2
```

$$\text{x1}(t) \to \frac{1}{9} e^{-t} \left(c_1 \left(4e^{9t} + 5\right) + 2(c_2 + 2c_3)\left(e^{9t} - 1\right)\right)$$
$$\text{x2}(t) \to \frac{1}{9} e^{-t} \left(2c_1 \left(e^{9t} - 1\right) + c_2 \left(e^{9t} + 8\right) + 2c_3 \left(e^{9t} - 1\right)\right)$$
$$\text{x3}(t) \to \frac{1}{9} e^{-t} \left(4c_1 \left(e^{9t} - 1\right) + 2c_2 \left(e^{9t} - 1\right) + c_3 \left(4e^{9t} + 5\right)\right)$$

1.4 problem 4

Internal problem ID [1827]
Internal file name [`OUTPUT/1828_Sunday_June_05_2022_02_34_22_AM_35330707/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.8, Systems of differential equations. The eigenva1ue-eigenvector method. Page 339
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned}
x_1'(t) &= 7x_1(t) - x_2(t) + 6x_3(t) \\
x_2'(t) &= -10x_1(t) + 4x_2(t) - 12x_3(t) \\
x_3'(t) &= -2x_1(t) + x_2(t) - x_3(t)
\end{aligned}$$

1.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} 2\,e^{2t} - 4\,e^{3t} + 3\,e^{5t} & -e^{3t} + e^{2t} & 3\,e^{5t} - 3\,e^{3t} \\ 8\,e^{3t} - 2\,e^{2t} - 6\,e^{5t} & -e^{2t} + 2\,e^{3t} & -6\,e^{5t} + 6\,e^{3t} \\ 4\,e^{3t} - 2\,e^{2t} - 2\,e^{5t} & e^{3t} - e^{2t} & 3\,e^{3t} - 2\,e^{5t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} 2\,e^{2t}-4\,e^{3t}+3\,e^{5t} & -e^{3t}+e^{2t} & 3\,e^{5t}-3\,e^{3t} \\ 8\,e^{3t}-2\,e^{2t}-6\,e^{5t} & -e^{2t}+2\,e^{3t} & -6\,e^{5t}+6\,e^{3t} \\ 4\,e^{3t}-2\,e^{2t}-2\,e^{5t} & e^{3t}-e^{2t} & 3\,e^{3t}-2\,e^{5t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} (2\,e^{2t}-4\,e^{3t}+3\,e^{5t})\,c_1+(-e^{3t}+e^{2t})\,c_2+(3\,e^{5t}-3\,e^{3t})\,c_3 \\ (8\,e^{3t}-2\,e^{2t}-6\,e^{5t})\,c_1+(-e^{2t}+2\,e^{3t})\,c_2+(-6\,e^{5t}+6\,e^{3t})\,c_3 \\ (4\,e^{3t}-2\,e^{2t}-2\,e^{5t})\,c_1+(e^{3t}-e^{2t})\,c_2+(3\,e^{3t}-2\,e^{5t})\,c_3 \end{bmatrix} \\
&= \begin{bmatrix} (-4c_1-c_2-3c_3)\,e^{3t}+(2c_1+c_2)\,e^{2t}+3\,e^{5t}(c_1+c_3) \\ (8c_1+2c_2+6c_3)\,e^{3t}+(-2c_1-c_2)\,e^{2t}-6\,e^{5t}(c_1+c_3) \\ (4c_1+c_2+3c_3)\,e^{3t}+(-2c_1-c_2)\,e^{2t}-2\,e^{5t}(c_1+c_3) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

1.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I)=0$$

Expanding gives

$$\det\left(\begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 7-\lambda & -1 & 6 \\ -10 & 4-\lambda & -12 \\ -2 & 1 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 10\lambda^2 + 31\lambda - 30 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3$$
$$\lambda_2 = 5$$
$$\lambda_3 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
3	1	real eigenvalue
5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -1 & 6 \\ -10 & 2 & -12 \\ -2 & 1 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 5 & -1 & 6 & 0 \\ -10 & 2 & -12 & 0 \\ -2 & 1 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} 5 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 1 & -3 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{2R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} 5 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{3}{5} & -\frac{3}{5} & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 5 & -1 & 6 & 0 \\ 0 & \frac{3}{5} & -\frac{3}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 5 & -1 & 6 \\ 0 & \frac{3}{5} & -\frac{3}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -1 & 6 \\ -10 & 1 & -12 \\ -2 & 1 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 4 & -1 & 6 & 0 \\ -10 & 1 & -12 & 0 \\ -2 & 1 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{5R_1}{2} \implies \left[\begin{array}{ccc|c} 4 & -1 & 6 & 0 \\ 0 & -\frac{3}{2} & 3 & 0 \\ -2 & 1 & -4 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{2} \implies \left[\begin{array}{ccc|c} 4 & -1 & 6 & 0 \\ 0 & -\frac{3}{2} & 3 & 0 \\ 0 & \frac{1}{2} & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_2}{3} \implies \left[\begin{array}{ccc|c} 4 & -1 & 6 & 0 \\ 0 & -\frac{3}{2} & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & -1 & 6 \\ 0 & -\frac{3}{2} & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = 2t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} - (5) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 6 \\ -10 & -1 & -12 \\ -2 & 1 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ -10 & -1 & -12 & 0 \\ -2 & 1 & -6 & 0 \end{array}\right]$$

$$R_2 = R_2 + 5R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 0 & -6 & 18 & 0 \\ -2 & 1 & -6 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 0 & -6 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -1 & 6 \\ 0 & -6 & 18 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{3t}{2}, v_2 = 3t\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{3t}{2} \\ 3t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{3t}{2} \\ 3t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{3t}{2} \\ 3t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{3}{2} \\ 3 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{3t}{2} \\ 3t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} \\ 3 \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{3t}{2} \\ 3t \\ t \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
3	1	1	No	$\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$
5	1	1	No	$\begin{bmatrix} -\frac{3}{2} \\ 3 \\ 1 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{3t}$$
$$= \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} e^{3t}$$

Since eigenvalue 5 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{5t}$$
$$= \begin{bmatrix} -\frac{3}{2} \\ 3 \\ 1 \end{bmatrix} e^{5t}$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_3(t) = \vec{v}_3 e^{2t}$$
$$= \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} e^{2t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{3t} \\ 2\,\mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{3\,\mathrm{e}^{5t}}{2} \\ 3\,\mathrm{e}^{5t} \\ \mathrm{e}^{5t} \end{bmatrix} + c_3 \begin{bmatrix} -\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -c_1 \mathrm{e}^{3t} - \frac{3c_2 \mathrm{e}^{5t}}{2} - c_3 \mathrm{e}^{2t} \\ 2c_1 \mathrm{e}^{3t} + 3c_2 \mathrm{e}^{5t} + c_3 \mathrm{e}^{2t} \\ c_1 \mathrm{e}^{3t} + c_2 \mathrm{e}^{5t} + c_3 \mathrm{e}^{2t} \end{bmatrix}$$

1.4.3 Maple step by step solution

Let's solve

$[x_1'(t) = 7x_1(t) - x_2(t) + 6x_3(t), x_2'(t) = -10x_1(t) + 4x_2(t) - 12x_3(t), x_3'(t) = -2x_1(t) + x_2(t) -$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix}$$

- Rewrite the system as

$\underline{x}^{\rightarrow}{}'(t) = A \cdot \underline{x}^{\rightarrow}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}\right], \left[5, \begin{bmatrix} -\frac{3}{2} \\ 3 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_1 = e^{2t} \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_2 = e^{3t} \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[5, \begin{bmatrix} -\frac{3}{2} \\ 3 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_3 = e^{5t} \cdot \begin{bmatrix} -\frac{3}{2} \\ 3 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{\vec{x}} = c_1\underline{\vec{x}}_1 + c_2\underline{\vec{x}}_2 + c_3\underline{\vec{x}}_3$$

- Substitute solutions into the general solution

$$\underline{\vec{x}} = c_1 e^{2t} \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{3t} \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + c_3 e^{5t} \cdot \begin{bmatrix} -\frac{3}{2} \\ 3 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -c_1 e^{2t} - c_2 e^{3t} - \frac{3c_3 e^{5t}}{2} \\ c_1 e^{2t} + 2c_2 e^{3t} + 3c_3 e^{5t} \\ c_1 e^{2t} + c_2 e^{3t} + c_3 e^{5t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = -c_1 e^{2t} - c_2 e^{3t} - \frac{3c_3 e^{5t}}{2}, x_2(t) = c_1 e^{2t} + 2c_2 e^{3t} + 3c_3 e^{5t}, x_3(t) = c_1 e^{2t} + c_2 e^{3t} + c_3 e^{5t}\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 74

```
dsolve([diff(x__1(t),t)=7*x__1(t)-1*x__2(t)+6*x__3(t),diff(x__2(t),t)=-10*x__1(t)+4*x__2(t)-
```

$$x_1(t) = c_1 e^{3t} + c_2 e^{2t} + c_3 e^{5t}$$
$$x_2(t) = -2c_1 e^{3t} - c_2 e^{2t} - 2c_3 e^{5t}$$
$$x_3(t) = -c_1 e^{3t} - c_2 e^{2t} - \frac{2c_3 e^{5t}}{3}$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 153

```
DSolve[{x1'[t]==7*x1[t]-1*x2[t]+6*x3[t],x2'[t]==-10*x1[t]+4*x2[t]-12*x3[t],x3'[t]==-2*x1[t]+
```

$$\text{x1}(t) \to e^{2t}\left(c_1\left(-4e^t + 3e^{3t} + 2\right) - c_2\left(e^t - 1\right) + 3c_3 e^t\left(e^{2t} - 1\right)\right)$$
$$\text{x2}(t) \to -e^{2t}\left(c_1\left(-8e^t + 6e^{3t} + 2\right) + c_2\left(1 - 2e^t\right) + 6c_3 e^t\left(e^{2t} - 1\right)\right)$$
$$\text{x3}(t) \to e^{2t}\left(-2c_1\left(-2e^t + e^{3t} + 1\right) + c_2\left(e^t - 1\right) + c_3 e^t\left(3 - 2e^{2t}\right)\right)$$

1.5 problem 5

Internal problem ID [1828]
Internal file name [`OUTPUT/1829_Sunday_June_05_2022_02_34_24_AM_41905812/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.8, Systems of differential equations. The eigenva1ue-eigenvector method. Page 339
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= -7x_1(t) + 6x_3(t) \\
x_2'(t) &= 5x_2(t) \\
x_3'(t) &= 6x_1(t) + 2x_3(t)
\end{aligned}
$$

1.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -7 & 0 & 6 \\ 0 & 5 & 0 \\ 6 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(e^{15t}+4)e^{-10t}}{5} & 0 & \frac{2(e^{15t}-1)e^{-10t}}{5} \\ 0 & e^{5t} & 0 \\ \frac{2(e^{15t}-1)e^{-10t}}{5} & 0 & \frac{(4e^{15t}+1)e^{-10t}}{5} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{(e^{15t}+4)e^{-10t}}{5} & 0 & \frac{2(e^{15t}-1)e^{-10t}}{5} \\ 0 & e^{5t} & 0 \\ \frac{2(e^{15t}-1)e^{-10t}}{5} & 0 & \frac{(4e^{15t}+1)e^{-10t}}{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(e^{15t}+4)e^{-10t}c_1}{5} + \frac{2(e^{15t}-1)e^{-10t}c_3}{5} \\ e^{5t}c_2 \\ \frac{2(e^{15t}-1)e^{-10t}c_1}{5} + \frac{(4e^{15t}+1)e^{-10t}c_3}{5} \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{-10t}((c_1+2c_3)e^{15t}+4c_1-2c_3)}{5} \\ e^{5t}c_2 \\ \frac{2((c_1+2c_3)e^{15t}-c_1+\frac{c_3}{2})e^{-10t}}{5} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

1.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -7 & 0 & 6 \\ 0 & 5 & 0 \\ 6 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -7 & 0 & 6 \\ 0 & 5 & 0 \\ 6 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -7-\lambda & 0 & 6 \\ 0 & 5-\lambda & 0 \\ 6 & 0 & 2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 75\lambda + 250 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 5$$
$$\lambda_2 = -10$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
5	1	real eigenvalue
-10	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -10$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -7 & 0 & 6 \\ 0 & 5 & 0 \\ 6 & 0 & 2 \end{bmatrix} - (-10)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & 0 & 6 \\ 0 & 15 & 0 \\ 6 & 0 & 12 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 3 & 0 & 6 & 0 \\ 0 & 15 & 0 & 0 \\ 6 & 0 & 12 & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} 3 & 0 & 6 & 0 \\ 0 & 15 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & 0 & 6 \\ 0 & 15 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -2t, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} -2t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -2t \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -2t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -2t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -7 & 0 & 6 \\ 0 & 5 & 0 \\ 6 & 0 & 2 \end{bmatrix} - (5)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -12 & 0 & 6 \\ 0 & 0 & 0 \\ 6 & 0 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -12 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 6 & 0 & -3 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{2} \implies \left[\begin{array}{ccc|c} -12 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -12 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = \frac{s}{2}\}$

Hence the solution is

$$\begin{bmatrix} \frac{s}{2} \\ t \\ s \end{bmatrix} = \begin{bmatrix} \frac{s}{2} \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{s}{2} \\ t \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{s}{2} \\ 0 \\ s \end{bmatrix}$$
$$= t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} \frac{s}{2} \\ t \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right)$$

Which are normalized to

$$\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
5	2	2	No	$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$
-10	1	1	No	$\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 5 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

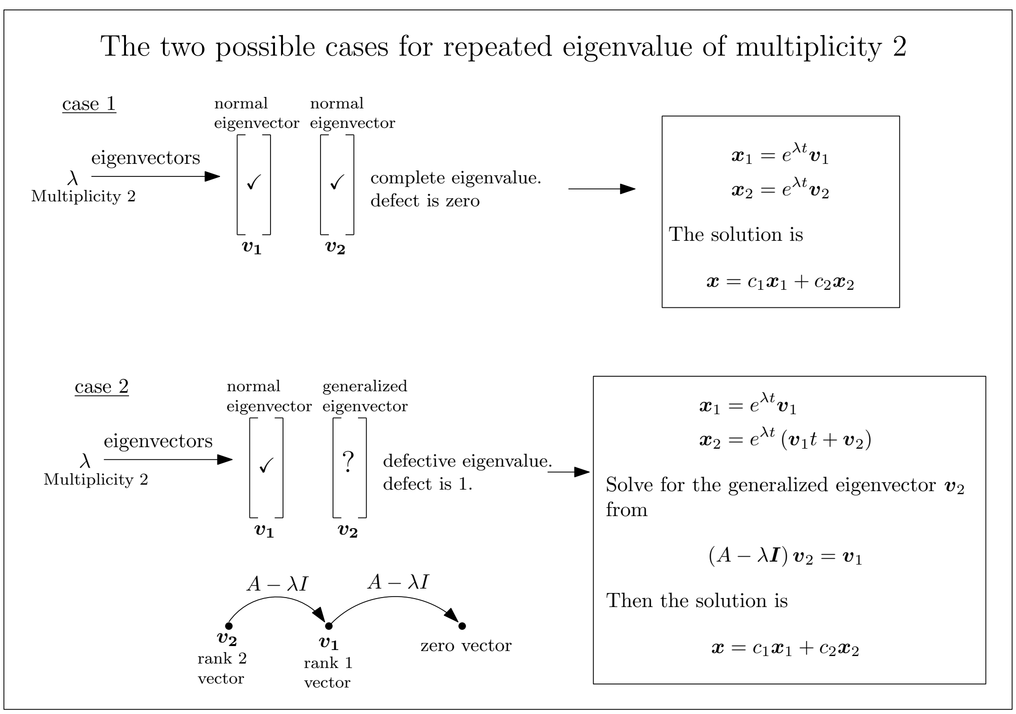


Figure 4: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{5t} \\
&= \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} e^{5t}
\end{aligned}
$$

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{5t} \\
&= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{5t}
\end{aligned}
$$

Since eigenvalue -10 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_3(t) &= \vec{v}_3 e^{-10t} \\
&= \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} e^{-10t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{e^{5t}}{2} \\ 0 \\ e^{5t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{5t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -2\, e^{-10t} \\ 0 \\ e^{-10t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{(c_1 e^{15t} - 4c_3) e^{-10t}}{2} \\ c_2 e^{5t} \\ (c_1 e^{15t} + c_3)\, e^{-10t} \end{bmatrix}
$$

1.5.3 Maple step by step solution

Let's solve

$[x_1'(t) = -7x_1(t) + 6x_3(t), x_2'(t) = 5x_2(t), x_3'(t) = 6x_1(t) + 2x_3(t)]$

- Define vector

$$\underline{\vec{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{\vec{x}}'(t) = \begin{bmatrix} -7 & 0 & 6 \\ 0 & 5 & 0 \\ 6 & 0 & 2 \end{bmatrix} \cdot \underline{\vec{x}}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{\vec{x}}'(t) = \begin{bmatrix} -7 & 0 & 6 \\ 0 & 5 & 0 \\ 6 & 0 & 2 \end{bmatrix} \cdot \underline{\vec{x}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -7 & 0 & 6 \\ 0 & 5 & 0 \\ 6 & 0 & 2 \end{bmatrix}$$

- Rewrite the system as

$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-10, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right], \left[5, \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}\right], \left[5, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\begin{bmatrix} -10, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}\_\_1 = \mathrm{e}^{-10t} \cdot \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\begin{bmatrix} 5, \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \end{bmatrix}$$

- First solution from eigenvalue 5

$$\overrightarrow{x}\_\_2(t) = \mathrm{e}^{5t} \cdot \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 5$ is the eigenvalue, and

$$\overrightarrow{x}\_\_3(t) = \mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the t multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\overrightarrow{x}\_\_3(t)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \left(\mathrm{e}^{\lambda t}A\right) \cdot \left(t\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \mathrm{e}^{\lambda t}\left(\lambda t\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

$$\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{x}\_\_3(t)$ to be a solution to the homogeneous system

$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 5

$$\left(\begin{bmatrix} -7 & 0 & 6 \\ 0 & 5 & 0 \\ 6 & 0 & 2 \end{bmatrix} - 5 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} -\frac{1}{24} \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 5

$$\underline{\vec{x}}_3(t) = \mathrm{e}^{5t} \cdot \left(t \cdot \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{24} \\ 0 \\ 0 \end{bmatrix}\right)$$

- General solution to the system of ODEs

$\underline{\vec{x}} = c_1 \underline{\vec{x}}_1 + c_2 \underline{\vec{x}}_2(t) + c_3 \underline{\vec{x}}_3(t)$

- Substitute solutions into the general solution

$$\underline{\vec{x}} = c_1 \mathrm{e}^{-10t} \cdot \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{5t} \cdot \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} + c_3 \mathrm{e}^{5t} \cdot \left(t \cdot \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{24} \\ 0 \\ 0 \end{bmatrix}\right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{\left(\left(\left(t - \frac{1}{12}\right) c_3 + c_2\right) \mathrm{e}^{15t} - 4c_1\right) \mathrm{e}^{-10t}}{2} \\ 0 \\ \left(\left(c_3 t + c_2\right) \mathrm{e}^{15t} + c_1\right) \mathrm{e}^{-10t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \frac{\left(\left(\left(t - \frac{1}{12}\right) c_3 + c_2\right) \mathrm{e}^{15t} - 4c_1\right) \mathrm{e}^{-10t}}{2}, x_2(t) = 0, x_3(t) = \left(\left(c_3 t + c_2\right) \mathrm{e}^{15t} + c_1\right) \mathrm{e}^{-10t}\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
dsolve([diff(x__1(t),t)=-7*x__1(t)+0*x__2(t)+6*x__3(t),diff(x__2(t),t)=0*x__1(t)+5*x__2(t)+0
```

$$x_1(t) = c_1 e^{-10t} + c_2 e^{5t}$$
$$x_2(t) = c_3 e^{5t}$$
$$x_3(t) = -\frac{c_1 e^{-10t}}{2} + 2c_2 e^{5t}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 158

```
DSolve[{x1'[t]==-7*x1[t]+0*x2[t]+6*x3[t],x2'[t]==0*x1[t]+5*x2[t]+0*x3[t],x3'[t]==6*x1[t]+0*x
```

$$\text{x1}(t) \to \frac{1}{5}e^{-10t}\left(c_1\left(e^{15t}+4\right)+2c_2\left(e^{15t}-1\right)\right)$$
$$\text{x3}(t) \to \frac{1}{5}e^{-10t}\left(2c_1\left(e^{15t}-1\right)+c_2\left(4e^{15t}+1\right)\right)$$
$$\text{x2}(t) \to c_3 e^{5t}$$
$$\text{x1}(t) \to \frac{1}{5}e^{-10t}\left(c_1\left(e^{15t}+4\right)+2c_2\left(e^{15t}-1\right)\right)$$
$$\text{x3}(t) \to \frac{1}{5}e^{-10t}\left(2c_1\left(e^{15t}-1\right)+c_2\left(4e^{15t}+1\right)\right)$$
$$\text{x2}(t) \to 0$$

1.6 problem 6

Internal problem ID [1829]
Internal file name [`OUTPUT/1830_Sunday_June_05_2022_02_34_26_AM_42392674/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.8, Systems of differential equations. The eigenva1ue-eigenvector method. Page 339
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_1(t) + 2x_2(t) + 3x_3(t) + 6x_4(t) \\
x_2'(t) &= 3x_1(t) + 6x_2(t) + 9x_3(t) + 18x_4(t) \\
x_3'(t) &= 5x_1(t) + 10x_2(t) + 15x_3(t) + 30x_4(t) \\
x_4'(t) &= 7x_1(t) + 14x_2(t) + 21x_3(t) + 42x_4(t)
\end{aligned}
$$

1.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 3 & 6 & 9 & 18 \\ 5 & 10 & 15 & 30 \\ 7 & 14 & 21 & 42 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{63}{64}+\frac{e^{64t}}{64} & \frac{e^{64t}}{32}-\frac{1}{32} & \frac{3e^{64t}}{64}-\frac{3}{64} & \frac{3e^{64t}}{32}-\frac{3}{32} \\ \frac{3e^{64t}}{64}-\frac{3}{64} & \frac{29}{32}+\frac{3e^{64t}}{32} & \frac{9e^{64t}}{64}-\frac{9}{64} & \frac{9e^{64t}}{32}-\frac{9}{32} \\ \frac{5e^{64t}}{64}-\frac{5}{64} & \frac{5e^{64t}}{32}-\frac{5}{32} & \frac{49}{64}+\frac{15e^{64t}}{64} & \frac{15e^{64t}}{32}-\frac{15}{32} \\ \frac{7e^{64t}}{64}-\frac{7}{64} & \frac{7e^{64t}}{32}-\frac{7}{32} & \frac{21e^{64t}}{64}-\frac{21}{64} & \frac{11}{32}+\frac{21e^{64t}}{32} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \frac{63}{64}+\frac{e^{64t}}{64} & \frac{e^{64t}}{32}-\frac{1}{32} & \frac{3e^{64t}}{64}-\frac{3}{64} & \frac{3e^{64t}}{32}-\frac{3}{32} \\ \frac{3e^{64t}}{64}-\frac{3}{64} & \frac{29}{32}+\frac{3e^{64t}}{32} & \frac{9e^{64t}}{64}-\frac{9}{64} & \frac{9e^{64t}}{32}-\frac{9}{32} \\ \frac{5e^{64t}}{64}-\frac{5}{64} & \frac{5e^{64t}}{32}-\frac{5}{32} & \frac{49}{64}+\frac{15e^{64t}}{64} & \frac{15e^{64t}}{32}-\frac{15}{32} \\ \frac{7e^{64t}}{64}-\frac{7}{64} & \frac{7e^{64t}}{32}-\frac{7}{32} & \frac{21e^{64t}}{64}-\frac{21}{64} & \frac{11}{32}+\frac{21e^{64t}}{32} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{63}{64}+\frac{e^{64t}}{64}\right)c_1+\left(\frac{e^{64t}}{32}-\frac{1}{32}\right)c_2+\left(\frac{3e^{64t}}{64}-\frac{3}{64}\right)c_3+\left(\frac{3e^{64t}}{32}-\frac{3}{32}\right)c_4 \\ \left(\frac{3e^{64t}}{64}-\frac{3}{64}\right)c_1+\left(\frac{29}{32}+\frac{3e^{64t}}{32}\right)c_2+\left(\frac{9e^{64t}}{64}-\frac{9}{64}\right)c_3+\left(\frac{9e^{64t}}{32}-\frac{9}{32}\right)c_4 \\ \left(\frac{5e^{64t}}{64}-\frac{5}{64}\right)c_1+\left(\frac{5e^{64t}}{32}-\frac{5}{32}\right)c_2+\left(\frac{49}{64}+\frac{15e^{64t}}{64}\right)c_3+\left(\frac{15e^{64t}}{32}-\frac{15}{32}\right)c_4 \\ \left(\frac{7e^{64t}}{64}-\frac{7}{64}\right)c_1+\left(\frac{7e^{64t}}{32}-\frac{7}{32}\right)c_2+\left(\frac{21e^{64t}}{64}-\frac{21}{64}\right)c_3+\left(\frac{11}{32}+\frac{21e^{64t}}{32}\right)c_4 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(c_1+2c_2+3c_3+6c_4)e^{64t}}{64}+\frac{63c_1}{64}-\frac{c_2}{32}-\frac{3c_3}{64}-\frac{3c_4}{32} \\ \frac{3(c_1+2c_2+3c_3+6c_4)e^{64t}}{64}-\frac{3c_1}{64}+\frac{29c_2}{32}-\frac{9c_3}{64}-\frac{9c_4}{32} \\ \frac{5(c_1+2c_2+3c_3+6c_4)e^{64t}}{64}-\frac{5c_1}{64}-\frac{5c_2}{32}+\frac{49c_3}{64}-\frac{15c_4}{32} \\ \frac{7(c_1+2c_2+3c_3+6c_4)e^{64t}}{64}-\frac{7c_1}{64}-\frac{7c_2}{32}-\frac{21c_3}{64}+\frac{11c_4}{32} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

1.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 3 & 6 & 9 & 18 \\ 5 & 10 & 15 & 30 \\ 7 & 14 & 21 & 42 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 2 & 3 & 6 \\ 3 & 6 & 9 & 18 \\ 5 & 10 & 15 & 30 \\ 7 & 14 & 21 & 42 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 2 & 3 & 6 \\ 3 & 6-\lambda & 9 & 18 \\ 5 & 10 & 15-\lambda & 30 \\ 7 & 14 & 21 & 42-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^4 - 64\lambda^3 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 64$$
$$\lambda_2 = 0$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
64	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 & 3 & 6 \\ 3 & 6 & 9 & 18 \\ 5 & 10 & 15 & 30 \\ 7 & 14 & 21 & 42 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 3 & 6 & 9 & 18 \\ 5 & 10 & 15 & 30 \\ 7 & 14 & 21 & 42 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 6 & 0 \\ 3 & 6 & 9 & 18 & 0 \\ 5 & 10 & 15 & 30 & 0 \\ 7 & 14 & 21 & 42 & 0 \end{array}\right]$$

$$R_2 = R_2 - 3R_1 \implies \left[\begin{array}{cccc|c} 1 & 2 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 5 & 10 & 15 & 30 & 0 \\ 7 & 14 & 21 & 42 & 0 \end{array}\right]$$

$$R_3 = R_3 - 5R_1 \implies \left[\begin{array}{cccc|c} 1 & 2 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 7 & 14 & 21 & 42 & 0 \end{array}\right]$$

$$R_4 = R_4 - 7R_1 \Longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3, v_4\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Let $v_4 = r$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -2t - 3s - 6r\}$

Hence the solution is

$$\begin{bmatrix} -2t - 3s - 6r \\ t \\ s \\ r \end{bmatrix} = \begin{bmatrix} -2t - 3s - 6r \\ t \\ s \\ r \end{bmatrix}$$

Since there are three free Variable, we have found three eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} -2t - 3s - 6r \\ t \\ s \\ r \end{bmatrix} &= \begin{bmatrix} -2t \\ t \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3s \\ 0 \\ s \\ 0 \end{bmatrix} \\ &= t \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} -6 \\ 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t=1$ and $s=1$ and $r=1$ then the above becomes

$$\begin{bmatrix} -2t-3s-6r \\ t \\ s \\ r \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -6 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence the three eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

Considering the eigenvalue $\lambda_2 = 64$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 & 3 & 6 \\ 3 & 6 & 9 & 18 \\ 5 & 10 & 15 & 30 \\ 7 & 14 & 21 & 42 \end{bmatrix} - (64) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -63 & 2 & 3 & 6 \\ 3 & -58 & 9 & 18 \\ 5 & 10 & -49 & 30 \\ 7 & 14 & 21 & -22 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} -63 & 2 & 3 & 6 & 0 \\ 3 & -58 & 9 & 18 & 0 \\ 5 & 10 & -49 & 30 & 0 \\ 7 & 14 & 21 & -22 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{21} \Longrightarrow \left[\begin{array}{cccc|c} -63 & 2 & 3 & 6 & 0 \\ 0 & -\frac{1216}{21} & \frac{64}{7} & \frac{128}{7} & 0 \\ 5 & 10 & -49 & 30 & 0 \\ 7 & 14 & 21 & -22 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{5R_1}{63} \Longrightarrow \left[\begin{array}{cccc|c} -63 & 2 & 3 & 6 & 0 \\ 0 & -\frac{1216}{21} & \frac{64}{7} & \frac{128}{7} & 0 \\ 0 & \frac{640}{63} & -\frac{1024}{21} & \frac{640}{21} & 0 \\ 7 & 14 & 21 & -22 & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{R_1}{9} \Longrightarrow \left[\begin{array}{cccc|c} -63 & 2 & 3 & 6 & 0 \\ 0 & -\frac{1216}{21} & \frac{64}{7} & \frac{128}{7} & 0 \\ 0 & \frac{640}{63} & -\frac{1024}{21} & \frac{640}{21} & 0 \\ 0 & \frac{128}{9} & \frac{64}{3} & -\frac{64}{3} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{10R_2}{57} \Longrightarrow \left[\begin{array}{cccc|c} -63 & 2 & 3 & 6 & 0 \\ 0 & -\frac{1216}{21} & \frac{64}{7} & \frac{128}{7} & 0 \\ 0 & 0 & -\frac{896}{19} & \frac{640}{19} & 0 \\ 0 & \frac{128}{9} & \frac{64}{3} & -\frac{64}{3} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{14R_2}{57} \Longrightarrow \left[\begin{array}{cccc|c} -63 & 2 & 3 & 6 & 0 \\ 0 & -\frac{1216}{21} & \frac{64}{7} & \frac{128}{7} & 0 \\ 0 & 0 & -\frac{896}{19} & \frac{640}{19} & 0 \\ 0 & 0 & \frac{448}{19} & -\frac{320}{19} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{R_3}{2} \Longrightarrow \left[\begin{array}{cccc|c} -63 & 2 & 3 & 6 & 0 \\ 0 & -\frac{1216}{21} & \frac{64}{7} & \frac{128}{7} & 0 \\ 0 & 0 & -\frac{896}{19} & \frac{640}{19} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -63 & 2 & 3 & 6 \\ 0 & -\frac{1216}{21} & \frac{64}{7} & \frac{128}{7} \\ 0 & 0 & -\frac{896}{19} & \frac{640}{19} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{7}, v_2 = \frac{3t}{7}, v_3 = \frac{5t}{7}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{7} \\ \frac{3t}{7} \\ \frac{5t}{7} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{7} \\ \frac{3t}{7} \\ \frac{5t}{7} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{7} \\ \frac{3t}{7} \\ \frac{5t}{7} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{7} \\ \frac{3}{7} \\ \frac{5}{7} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{7} \\ \frac{3t}{7} \\ \frac{5t}{7} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{7} \\ \frac{3}{7} \\ \frac{5}{7} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{7} \\ \frac{3t}{7} \\ \frac{5t}{7} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
64	1	1	No	$\begin{bmatrix} \frac{1}{7} \\ \frac{3}{7} \\ \frac{5}{7} \\ 1 \end{bmatrix}$
0	3	3	No	$\begin{bmatrix} -6 & -3 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 64 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{64t} \\
&= \begin{bmatrix} \frac{1}{7} \\ \frac{3}{7} \\ \frac{5}{7} \\ 1 \end{bmatrix} e^{64t}
\end{aligned}
$$

eigenvalue 0 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

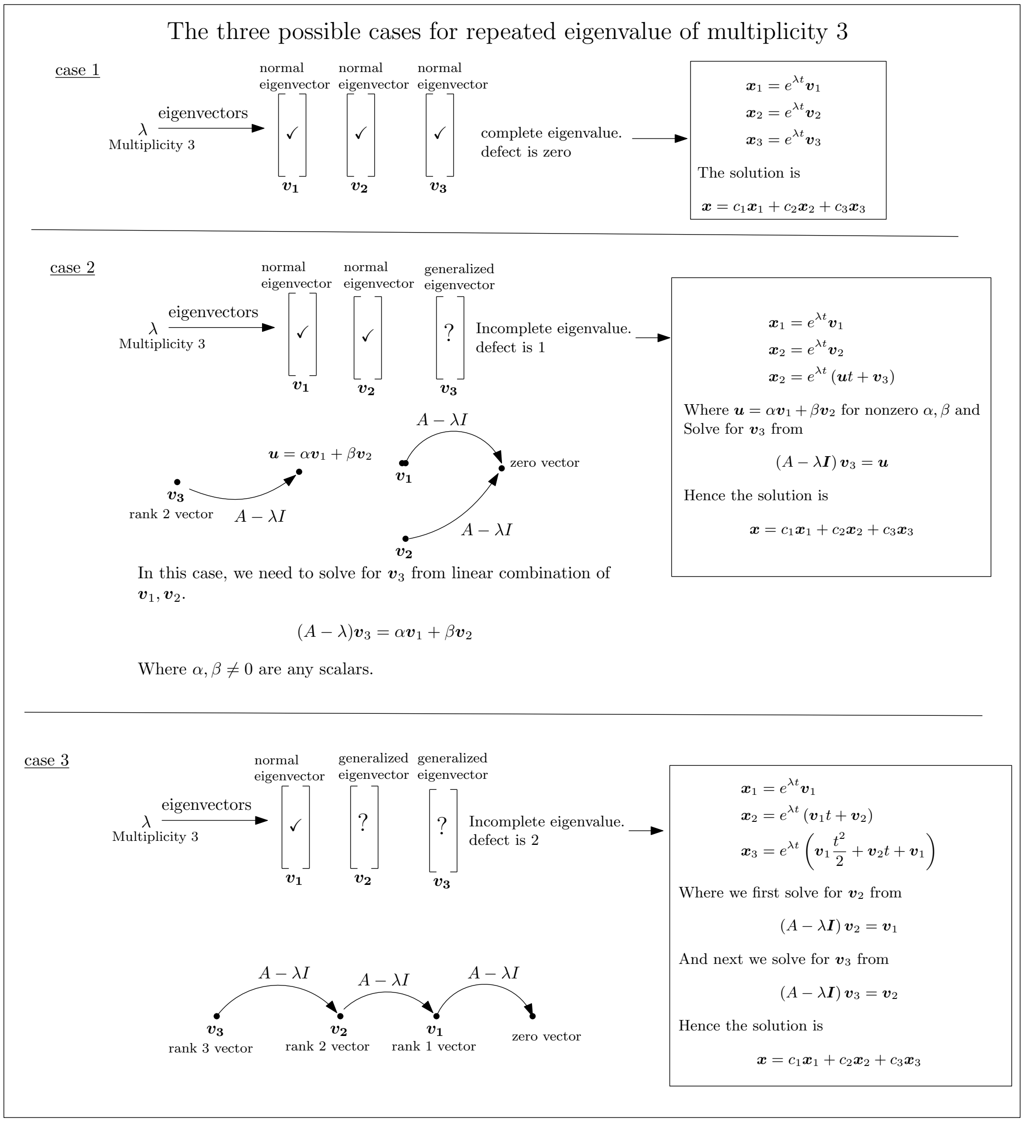


Figure 5: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3 which is the same as its geometric multiplicity 3, then it is complete eigenvalue and this falls into case 1 shown above.

Hence the corresponding eigenvector basis are

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^0 \\
&= \begin{bmatrix} -6 \\ 0 \\ 0 \\ 1 \end{bmatrix} e^0
\end{aligned}
$$

$$
\begin{aligned}
\vec{x}_3(t) &= \vec{v}_3 e^0 \\
&= \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} e^0
\end{aligned}
$$

$$
\begin{aligned}
\vec{x}_4(t) &= \vec{v}_4 e^0 \\
&= \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} e^0
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t) + c_4 \vec{x}_4(t)
$$

Which is written as

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{e^{64t}}{7} \\ \frac{3\,e^{64t}}{7} \\ \frac{5\,e^{64t}}{7} \\ e^{64t} \end{bmatrix} + c_2 \begin{bmatrix} -6 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1 e^{64t}}{7} - 6c_2 - 3c_3 - 2c_4 \\ \frac{3c_1 e^{64t}}{7} + c_4 \\ \frac{5c_1 e^{64t}}{7} + c_3 \\ c_1 e^{64t} + c_2 \end{bmatrix}
$$

1.6.3 Maple step by step solution

Let's solve

$[x_1'(t) = x_1(t) + 2x_2(t) + 3x_3(t) + 6x_4(t), x_2'(t) = 3x_1(t) + 6x_2(t) + 9x_3(t) + 18x_4(t), x_3'(t) = 5x_1($

- Define vector

$$\underline{x^{\rightarrow}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 3 & 6 & 9 & 18 \\ 5 & 10 & 15 & 30 \\ 7 & 14 & 21 & 42 \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 3 & 6 & 9 & 18 \\ 5 & 10 & 15 & 30 \\ 7 & 14 & 21 & 42 \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 3 & 6 & 9 & 18 \\ 5 & 10 & 15 & 30 \\ 7 & 14 & 21 & 42 \end{bmatrix}$$

- Rewrite the system as

$\underline{x^{\rightarrow}}'(t) = A \cdot \underline{x^{\rightarrow}}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} -6 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right], \left[0, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[64, \begin{bmatrix} \frac{1}{7} \\ \frac{3}{7} \\ \frac{5}{7} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} -6 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_1 = \begin{bmatrix} -6 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[64, \begin{bmatrix} \frac{1}{7} \\ \frac{3}{7} \\ \frac{5}{7} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_4 = \mathrm{e}^{64t} \cdot \begin{bmatrix} \frac{1}{7} \\ \frac{3}{7} \\ \frac{5}{7} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{\vec{x}} = c_1 \underline{\vec{x}}_1 + c_2 \underline{\vec{x}}_2 + c_3 \underline{\vec{x}}_3 + c_4 \underline{\vec{x}}_4$$

- Substitute solutions into the general solution

$$\underline{\vec{x}} = c_4 \mathrm{e}^{64t} \cdot \begin{bmatrix} \frac{1}{7} \\ \frac{3}{7} \\ \frac{5}{7} \\ 1 \end{bmatrix} + \begin{bmatrix} -6c_1 - 3c_2 - 2c_3 \\ c_3 \\ c_2 \\ c_1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} \frac{c_4 e^{64t}}{7} - 2c_3 - 3c_2 - 6c_1 \\ \frac{3c_4 e^{64t}}{7} + c_3 \\ \frac{5c_4 e^{64t}}{7} + c_2 \\ c_4 e^{64t} + c_1 \end{bmatrix}
$$

- Solution to the system of ODEs

$$
\left\{x_1(t) = \frac{c_4 e^{64t}}{7} - 2c_3 - 3c_2 - 6c_1, x_2(t) = \frac{3c_4 e^{64t}}{7} + c_3, x_3(t) = \frac{5c_4 e^{64t}}{7} + c_2, x_4(t) = c_4 e^{64t} + c_1\right\}
$$

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 63

```
dsolve([diff(x__1(t),t)=1*x__1(t)+2*x__2(t)+3*x__3(t)+6*x__4(t),diff(x__2(t),t)=3*x__1(t)+6*
```

$$
\begin{aligned}
x_1(t) &= c_3 + c_4 e^{64t} \\
x_2(t) &= 3c_3 + 3c_4 e^{64t} + c_2 \\
x_3(t) &= 5c_3 + 5c_4 e^{64t} + c_1 \\
x_4(t) &= 7c_4 e^{64t} - \frac{11c_3}{3} - \frac{c_2}{3} - \frac{c_1}{2}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.089 (sec). Leaf size: 554

```
DSolve[{x1'[t]==1*x1[t]+2*x2[t]+3*x3[t]+6*x4[t],x2'[t]==3*x1[t]+6*x2[t]+9*x3[t]+19*x4[t],x3'
```

x1(t)

$$\rightarrow \frac{e^{-\sqrt{1038}t}\left(2076(7c_1-c_4)e^{\sqrt{1038}t}-\left(7\sqrt{1038}c_1+14\sqrt{1038}c_2+21\sqrt{1038}c_3+10\sqrt{1038}c_4-1038c_4\right)e^{32t}+\right.}{14532}$$

x2(t)

$$\rightarrow \frac{\left(7\left(519+13\sqrt{1038}\right)c_1+14\left(519+13\sqrt{1038}\right)c_2+273\sqrt{1038}c_3+10899c_3-389\sqrt{1038}c_4-8304c_4\right)e^{-(}}{}$$

x3(t)

$$\rightarrow \frac{e^{-\sqrt{1038}t}\left(2076(7c_3-5c_4)e^{\sqrt{1038}t}-5\left(7\sqrt{1038}c_1+14\sqrt{1038}c_2+21\sqrt{1038}c_3+10\sqrt{1038}c_4-1038c_4\right)e^{32}\right.}{14532}$$

x4(t)

$$\rightarrow \frac{e^{-\left(\left(\sqrt{1038}-32\right)t\right)}\left(7c_1\left(e^{2\sqrt{1038}t}-1\right)+14c_2\left(e^{2\sqrt{1038}t}-1\right)+21c_3e^{2\sqrt{1038}t}+\sqrt{1038}c_4e^{2\sqrt{1038}t}+10c_4e^{2\sqrt{103}}\right.}{2\sqrt{1038}}$$

1.7 problem 7

Internal problem ID [1830]
Internal file name [OUTPUT/1831_Sunday_June_05_2022_02_34_29_AM_45158599/index.tex]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.8, Systems of differential equations. The eigenva1ue-eigenvector method. Page 339
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = x_1(t) + x_2(t)$$
$$x_2'(t) = 4x_1(t) + x_2(t)$$

With initial conditions

$$[x_1(0) = 2, x_2(0) = 3]$$

1.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{-t}}{2} + \frac{e^{3t}}{2} & \frac{e^{3t}}{4} - \frac{e^{-t}}{4} \\ e^{3t} - e^{-t} & \frac{e^{-t}}{2} + \frac{e^{3t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \frac{e^{-t}}{2}+\frac{e^{3t}}{2} & \frac{e^{3t}}{4}-\frac{e^{-t}}{4} \\ e^{3t}-e^{-t} & \frac{e^{-t}}{2}+\frac{e^{3t}}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{-t}}{4}+\frac{7e^{3t}}{4} \\ \frac{7e^{3t}}{2}-\frac{e^{-t}}{2} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

1.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 2\lambda - 3 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 4 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & 1 & 0 \\ 4 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$
3	1	1	No	$\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{-t} \\
&= \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} e^{-t}
\end{aligned}
$$

Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{3t} \\
&= \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} e^{3t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{e^{-t}}{2} \\ e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{e^{3t}}{2} \\ e^{3t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_1 e^{-t}}{2} + \frac{c_2 e^{3t}}{2} \\ c_1 e^{-t} + c_2 e^{3t} \end{bmatrix}
$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$
\begin{bmatrix} x_1(0) = 2 \\ x_2(0) = 3 \end{bmatrix} \tag{1}
$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$
\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{c_1}{2} + \frac{c_2}{2} \\ c_1 + c_2 \end{bmatrix}
$$

Solving for the constants of integrations gives

$$\left[\begin{array}{c} c_1 = -\frac{1}{2} \\ c_2 = \frac{7}{2} \end{array}\right]$$

Substituting these constants back in original solution in Eq. (1) gives

$$\left[\begin{array}{c} x_1(t) \\ x_2(t) \end{array}\right] = \left[\begin{array}{c} \frac{e^{-t}}{4} + \frac{7 e^{3t}}{4} \\ \frac{7 e^{3t}}{2} - \frac{e^{-t}}{2} \end{array}\right]$$

The following is the phase plot of the system.

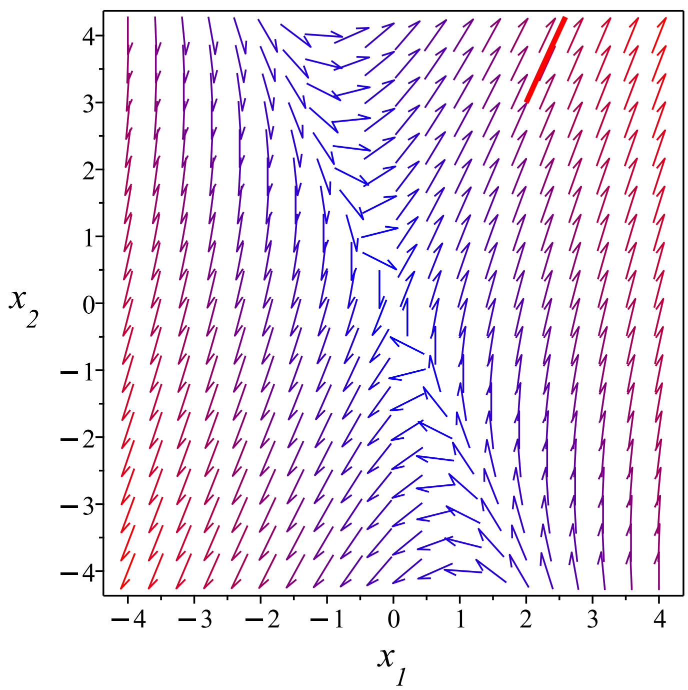
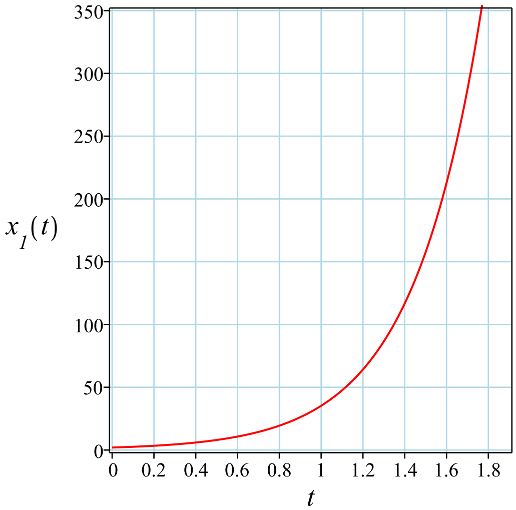
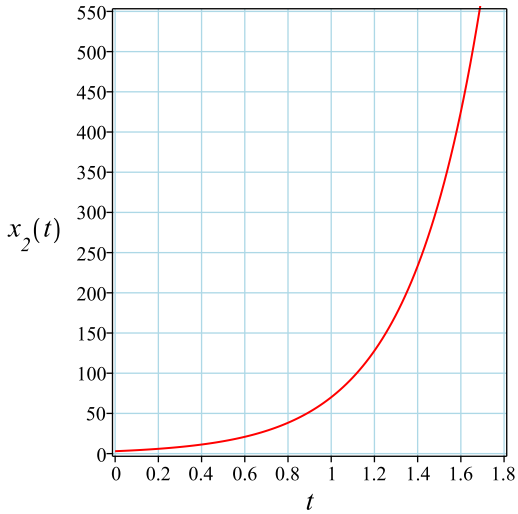


Figure 6: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 34

```
dsolve([diff(x__1(t),t) = x__1(t)+x__2(t), diff(x__2(t),t) = 4*x__1(t)+x__2(t), x__1(0) = 2,
```

$$x_1(t) = \frac{7\,e^{3t}}{4} + \frac{e^{-t}}{4}$$
$$x_2(t) = \frac{7\,e^{3t}}{2} - \frac{e^{-t}}{2}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 44

```
DSolve[{x1'[t]==1*x1[t]+1*x2[t],x2'[t]==4*x1[t]+1*x2[t]},{x1[0]==2,x2[0]==3},{x1[t],x2[t]},t
```

$$\text{x1}(t) \to \frac{1}{4}e^{-t}\left(7e^{4t}+1\right)$$
$$\text{x2}(t) \to \frac{1}{2}e^{-t}\left(7e^{4t}-1\right)$$

1.8 problem 8

Internal problem ID [1831]
Internal file name [`OUTPUT/1832_Sunday_June_05_2022_02_34_31_AM_49847952/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.8, Systems of differential equations. The eigenva1ue-eigenvector method. Page 339
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x_1'(t) &= x_1(t) - 3x_2(t) \\ x_2'(t) &= -2x_1(t) + 2x_2(t) \end{aligned}$$

With initial conditions

$$[x_1(0) = 0, x_2(0) = 5]$$

1.8.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{3\,e^{-t}}{5} + \frac{2\,e^{4t}}{5} & -\frac{3\,e^{4t}}{5} + \frac{3\,e^{-t}}{5} \\ -\frac{2\,e^{4t}}{5} + \frac{2\,e^{-t}}{5} & \frac{2\,e^{-t}}{5} + \frac{3\,e^{4t}}{5} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \frac{3\,\mathrm{e}^{-t}}{5} + \frac{2\,\mathrm{e}^{4t}}{5} & -\frac{3\,\mathrm{e}^{4t}}{5} + \frac{3\,\mathrm{e}^{-t}}{5} \\ -\frac{2\,\mathrm{e}^{4t}}{5} + \frac{2\,\mathrm{e}^{-t}}{5} & \frac{2\,\mathrm{e}^{-t}}{5} + \frac{3\,\mathrm{e}^{4t}}{5} \end{bmatrix} \begin{bmatrix} 0 \\ 5 \end{bmatrix} \\
&= \begin{bmatrix} -3\,\mathrm{e}^{4t} + 3\,\mathrm{e}^{-t} \\ 2\,\mathrm{e}^{-t} + 3\,\mathrm{e}^{4t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

1.8.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -3 \\ -2 & 2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 3\lambda - 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 4$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -3 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & -3 & 0 \\ -2 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{3t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -3 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3 & -3 & 0 \\ -2 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{3} \implies \left[\begin{array}{cc|c} -3 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
4	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 4 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{4t}$$
$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4t}$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-t}$$
$$= \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} e^{-t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{4t} \\ \mathrm{e}^{4t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{3\,\mathrm{e}^{-t}}{2} \\ \mathrm{e}^{-t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -c_1 \mathrm{e}^{4t} + \frac{3c_2 \mathrm{e}^{-t}}{2} \\ c_1 \mathrm{e}^{4t} + c_2 \mathrm{e}^{-t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = 0 \\ x_2(0) = 5 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} -c_1 + \frac{3c_2}{2} \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 3 \\ c_2 = 2 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -3\,e^{4t} + 3\,e^{-t} \\ 2\,e^{-t} + 3\,e^{4t} \end{bmatrix}$$

The following is the phase plot of the system.

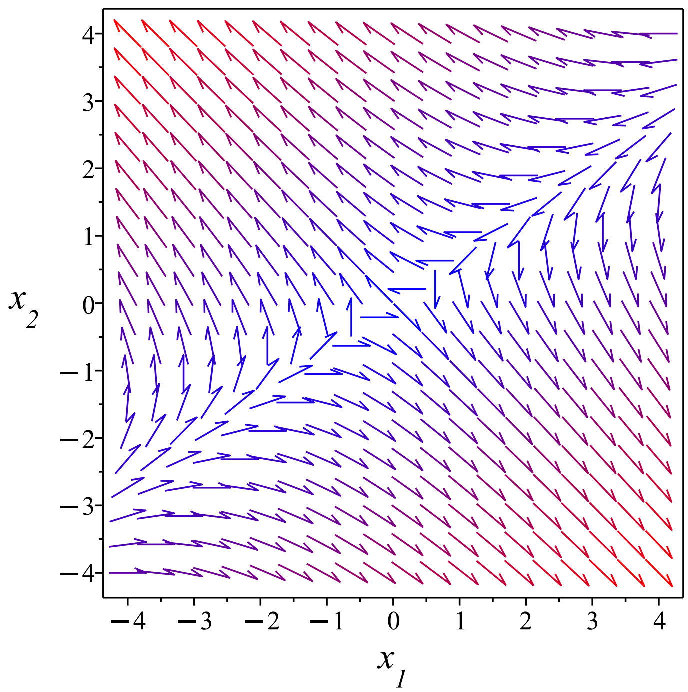
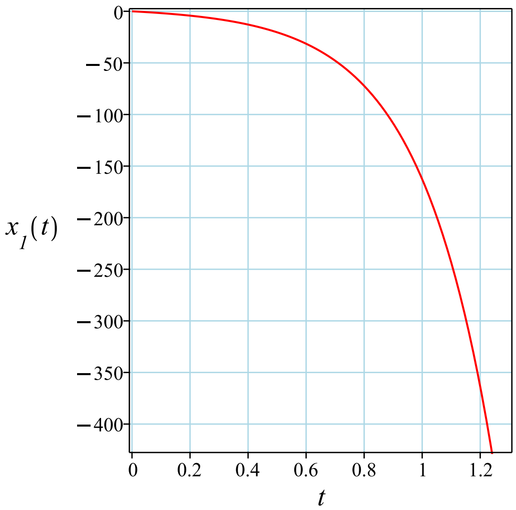
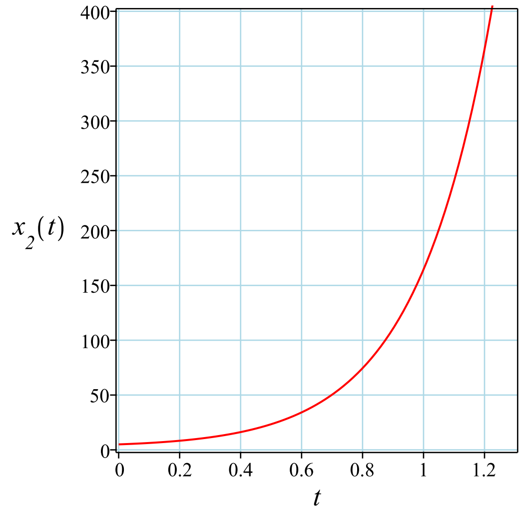


Figure 7: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 34

```
dsolve([diff(x__1(t),t) = x__1(t)-3*x__2(t), diff(x__2(t),t) = -2*x__1(t)+2*x__2(t), x__1(0)
```

$$x_1(t) = 3\,\mathrm{e}^{-t} - 3\,\mathrm{e}^{4t}$$
$$x_2(t) = 2\,\mathrm{e}^{-t} + 3\,\mathrm{e}^{4t}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 37

```
DSolve[{x1'[t]==1*x1[t]-3*x2[t],x2'[t]==-2*x1[t]+2*x2[t]},{x1[0]==0,x2[0]==5},{x1[t],x2[t]},
```

$$\mathrm{x1}(t) \to -3e^{-t}\left(e^{5t} - 1\right)$$
$$\mathrm{x2}(t) \to e^{-t}\left(3e^{5t} + 2\right)$$

1.9 problem 9

Internal problem ID [1832]
Internal file name [`OUTPUT/1833_Sunday_June_05_2022_02_34_33_AM_6328522/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.8, Systems of differential equations. The eigenva1ue-eigenvector method. Page 339
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 3x_1(t) + x_2(t) - x_3(t) \\
x_2'(t) &= x_1(t) + 3x_2(t) - x_3(t) \\
x_3'(t) &= 3x_1(t) + 3x_2(t) - x_3(t)
\end{aligned}
$$

With initial conditions

$$[x_1(0) = 1, x_2(0) = -2, x_3(0) = -1]$$

1.9.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} 2\,e^{2t} - e^{t} & e^{2t} - e^{t} & -e^{2t} + e^{t} \\ e^{2t} - e^{t} & 2\,e^{2t} - e^{t} & -e^{2t} + e^{t} \\ 3\,e^{2t} - 3\,e^{t} & 3\,e^{2t} - 3\,e^{t} & 3\,e^{t} - 2\,e^{2t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} 2\,e^{2t} - e^{t} & e^{2t} - e^{t} & -e^{2t} + e^{t} \\ e^{2t} - e^{t} & 2\,e^{2t} - e^{t} & -e^{2t} + e^{t} \\ 3\,e^{2t} - 3\,e^{t} & 3\,e^{2t} - 3\,e^{t} & 3\,e^{t} - 2\,e^{2t} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} e^{2t} \\ -2\,e^{2t} \\ -e^{2t} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

1.9.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & 1 & -1 \\ 1 & 3-\lambda & -1 \\ 3 & 3 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ 3 & 3 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 1 & 2 & -1 & 0 \\ 3 & 3 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 3 & 3 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{3R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{1}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{3}, v_2 = \frac{t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{3} \\ \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{3} \\ \frac{t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{3} \\ \frac{t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{3} \\ \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{3} \\ \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 3 & 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 1 & -1 & 0 \\ 3 & 3 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 3 & -3 & 0 \end{array}\right]$$

$$R_3 = R_3 - 3R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t + s\}$

Hence the solution is

$$\begin{bmatrix} -t+s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -t+s \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} -t+s \\ t \\ s \end{bmatrix} &= \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} s \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} -t+s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated

with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	2	2	No	$\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 2 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

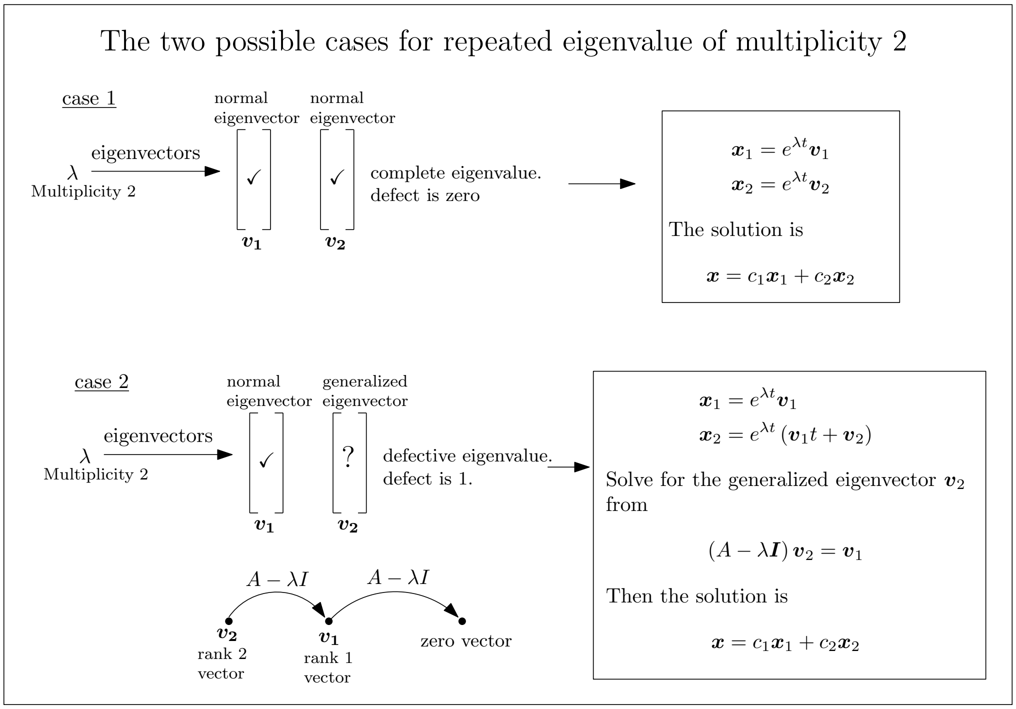


Figure 8: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{2t} \\
&= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t}
\end{aligned}
$$

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{2t} \\
&= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{2t}
\end{aligned}
$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^t \\ &= \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix} e^t\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{2t} \\ 0 \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} -e^{2t} \\ e^{2t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} \frac{e^t}{3} \\ \frac{e^t}{3} \\ e^t \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} (c_1 - c_2)\, e^{2t} + \frac{c_3 e^t}{3} \\ c_2 e^{2t} + \frac{c_3 e^t}{3} \\ c_1 e^{2t} + c_3 e^t \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = 1 \\ x_2(0) = -2 \\ x_3(0) = -1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 - c_2 + \frac{c_3}{3} \\ c_2 + \frac{c_3}{3} \\ c_1 + c_3 \end{bmatrix}$$

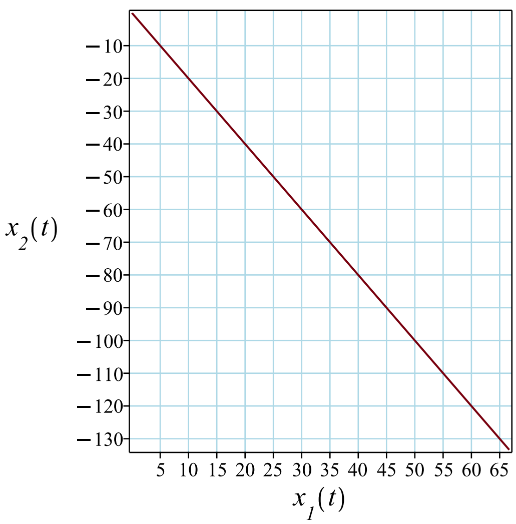
Solving for the constants of integrations gives

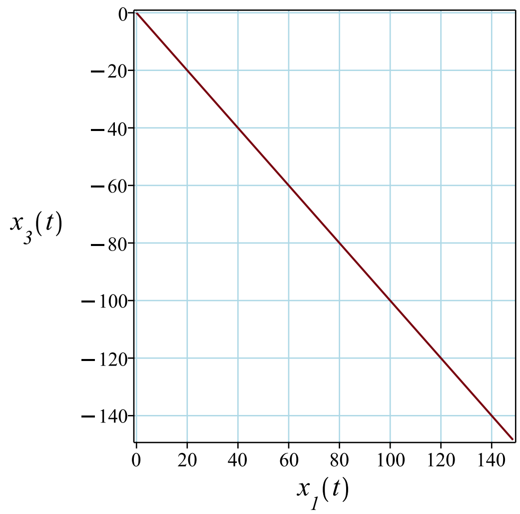
$$\begin{bmatrix} c_1 = -1 \\ c_2 = -2 \\ c_3 = 0 \end{bmatrix}$$

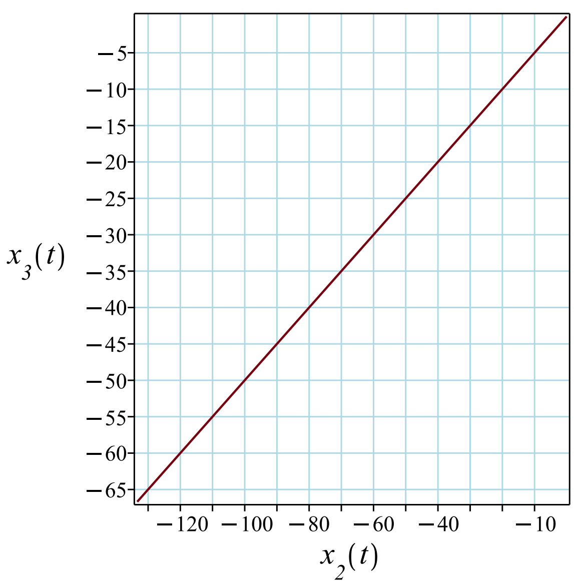
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} e^{2t} \\ -2\,e^{2t} \\ -e^{2t} \end{bmatrix}$$

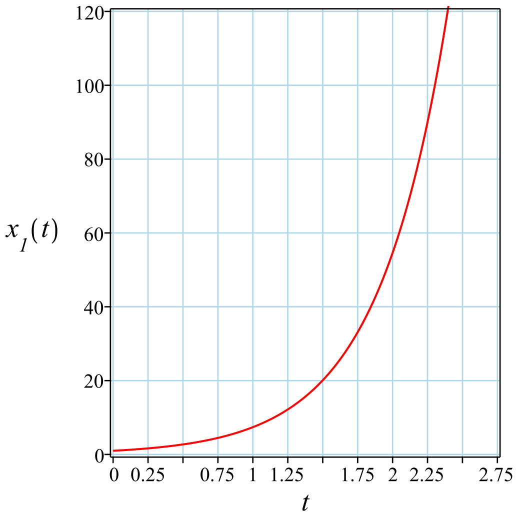
The following are plots of each solution against another.

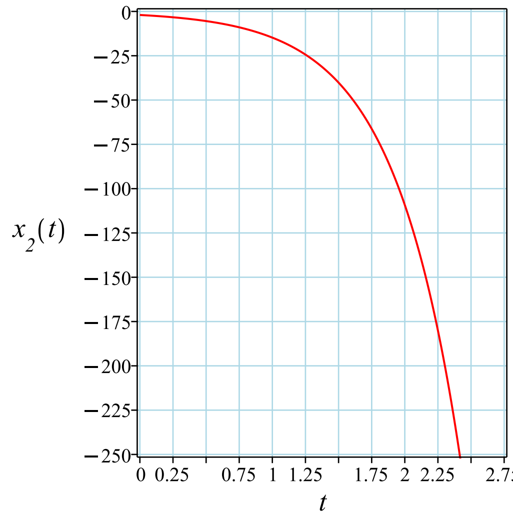


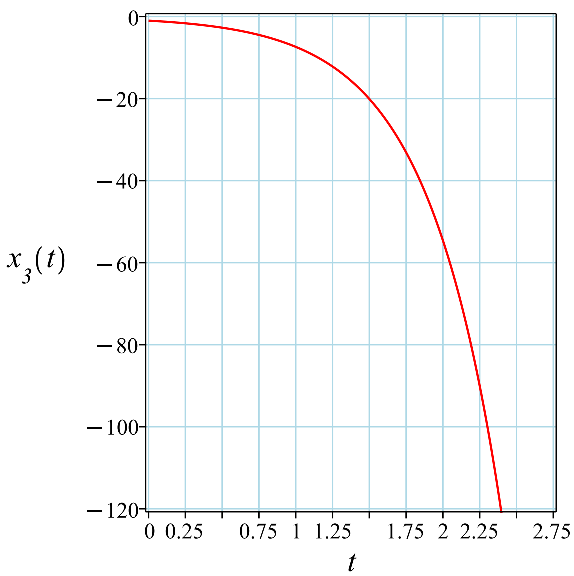




The following are plots of each solution.







✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 27

```
dsolve([diff(x__1(t),t) = 3*x__1(t)+x__2(t)-x__3(t), diff(x__2(t),t) = x__1(t)+3*x__2(t)-x__
```

$$x_1(t) = e^{2t}$$
$$x_2(t) = -2\,e^{2t}$$
$$x_3(t) = -e^{2t}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 30

```
DSolve[{x1'[t]==3*x1[t]+1*x2[t]-1*x3[t],x2'[t]==1*x1[t]+3*x2[t]-1*x3[t],x3'[t]==3*x1[t]+3*x2
```

$$
\begin{aligned}
\text{x1}(t) &\to e^{2t} \\
\text{x2}(t) &\to -2e^{2t} \\
\text{x3}(t) &\to -e^{2t}
\end{aligned}
$$

1.10 problem 10

Internal problem ID [1833]
Internal file name [`OUTPUT/1834_Sunday_June_05_2022_02_34_35_AM_9724454/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.8, Systems of differential equations. The eigenva1ue-eigenvector method. Page 339
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_1(t) - x_2(t) \\
x_2'(t) &= x_1(t) + 2x_2(t) + x_3(t) \\
x_3'(t) &= x_1(t) + 10x_2(t) + 2x_3(t)
\end{aligned}
$$

With initial conditions

$$[x_1(0) = -1, x_2(0) = -4, x_3(0) = 13]$$

1.10.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ 1 & 10 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{9\,e^{t}}{8}-\frac{e^{5t}}{24}-\frac{e^{-t}}{12} & -\frac{e^{5t}}{8}-\frac{e^{t}}{8}+\frac{e^{-t}}{4} & -\frac{e^{5t}}{24}+\frac{e^{t}}{8}-\frac{e^{-t}}{12} \\ \frac{e^{5t}}{6}-\frac{e^{-t}}{6} & \frac{e^{-t}}{2}+\frac{e^{5t}}{2} & \frac{e^{5t}}{6}-\frac{e^{-t}}{6} \\ -\frac{9\,e^{t}}{8}+\frac{7\,e^{-t}}{12}+\frac{13\,e^{5t}}{24} & \frac{13\,e^{5t}}{8}+\frac{e^{t}}{8}-\frac{7\,e^{-t}}{4} & \frac{7\,e^{-t}}{12}+\frac{13\,e^{5t}}{24}-\frac{e^{t}}{8} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} \frac{9\,e^{t}}{8}-\frac{e^{5t}}{24}-\frac{e^{-t}}{12} & -\frac{e^{5t}}{8}-\frac{e^{t}}{8}+\frac{e^{-t}}{4} & -\frac{e^{5t}}{24}+\frac{e^{t}}{8}-\frac{e^{-t}}{12} \\ \frac{e^{5t}}{6}-\frac{e^{-t}}{6} & \frac{e^{-t}}{2}+\frac{e^{5t}}{2} & \frac{e^{5t}}{6}-\frac{e^{-t}}{6} \\ -\frac{9\,e^{t}}{8}+\frac{7\,e^{-t}}{12}+\frac{13\,e^{5t}}{24} & \frac{13\,e^{5t}}{8}+\frac{e^{t}}{8}-\frac{7\,e^{-t}}{4} & \frac{7\,e^{-t}}{12}+\frac{13\,e^{5t}}{24}-\frac{e^{t}}{8} \end{bmatrix} \begin{bmatrix} -1 \\ -4 \\ 13 \end{bmatrix} \\ &= \begin{bmatrix} e^{t}-2\,e^{-t} \\ -4\,e^{-t} \\ -e^{t}+14\,e^{-t} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

1.10.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ 1 & 10 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ 1 & 10 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -1 & 0 \\ 1 & 2-\lambda & 1 \\ 1 & 10 & 2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 5\lambda^2 - \lambda + 5 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned} \lambda_1 &= -1 \\ \lambda_2 &= 1 \\ \lambda_3 &= 5 \end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
1	1	real eigenvalue
5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ 1 & 10 & 2 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & 1 \\ 1 & 10 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 10 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 0 & \frac{7}{2} & 1 & 0 \\ 1 & 10 & 3 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 0 & \frac{7}{2} & 1 & 0 \\ 0 & \frac{21}{2} & 3 & 0 \end{array}\right]$$

$$R_3 = R_3 - 3R_2 \Longrightarrow \left[\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 0 & \frac{7}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{7}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{7}, v_2 = -\frac{2t}{7}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{7} \\ -\frac{2t}{7} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{7} \\ -\frac{2t}{7} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{7} \\ -\frac{2t}{7} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{7} \\ -\frac{2}{7} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{7} \\ -\frac{2t}{7} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{7} \\ -\frac{2}{7} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{7} \\ -\frac{2t}{7} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 7 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ 1 & 10 & 2 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & 10 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 10 & 1 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 10 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 9 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 + 9R_2 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ 1 & 10 & 2 \end{bmatrix} - (5)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & -1 & 0 \\ 1 & -3 & 1 \\ 1 & 10 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -4 & -1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ 1 & 10 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} -4 & -1 & 0 & 0 \\ 0 & -\frac{13}{4} & 1 & 0 \\ 1 & 10 & -3 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} -4 & -1 & 0 & 0 \\ 0 & -\frac{13}{4} & 1 & 0 \\ 0 & \frac{39}{4} & -3 & 0 \end{array}\right]$$

$$R_3 = R_3 + 3R_2 \Longrightarrow \left[\begin{array}{ccc|c} -4 & -1 & 0 & 0 \\ 0 & -\frac{13}{4} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & -1 & 0 \\ 0 & -\frac{13}{4} & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{13}, v_2 = \frac{4t}{13}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{13} \\ \frac{4t}{13} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{13} \\ \frac{4t}{13} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{13} \\ \frac{4t}{13} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{13} \\ \frac{4}{13} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{13} \\ \frac{4t}{13} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{13} \\ \frac{4}{13} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{13} \\ \frac{4t}{13} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 13 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} -\frac{1}{7} \\ -\frac{2}{7} \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$
5	1	1	No	$\begin{bmatrix} -\frac{1}{13} \\ \frac{4}{13} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} -\frac{1}{7} \\ -\frac{2}{7} \\ 1 \end{bmatrix} e^{-t}\end{aligned}$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{t} \\ &= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{t}\end{aligned}$$

Since eigenvalue 5 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_3(t) = \vec{v}_3 e^{5t}$$
$$= \begin{bmatrix} -\frac{1}{13} \\ \frac{4}{13} \\ 1 \end{bmatrix} e^{5t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{e^{-t}}{7} \\ -\frac{2\,e^{-t}}{7} \\ e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} -e^{t} \\ 0 \\ e^{t} \end{bmatrix} + c_3 \begin{bmatrix} -\frac{e^{5t}}{13} \\ \frac{4\,e^{5t}}{13} \\ e^{5t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_1 e^{-t}}{7} - c_2 e^{t} - \frac{c_3 e^{5t}}{13} \\ -\frac{2c_1 e^{-t}}{7} + \frac{4c_3 e^{5t}}{13} \\ c_1 e^{-t} + c_2 e^{t} + c_3 e^{5t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = -1 \\ x_2(0) = -4 \\ x_3(0) = 13 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} -1 \\ -4 \\ 13 \end{bmatrix} = \begin{bmatrix} -\frac{c_1}{7} - c_2 - \frac{c_3}{13} \\ -\frac{2c_1}{7} + \frac{4c_3}{13} \\ c_1 + c_2 + c_3 \end{bmatrix}$$

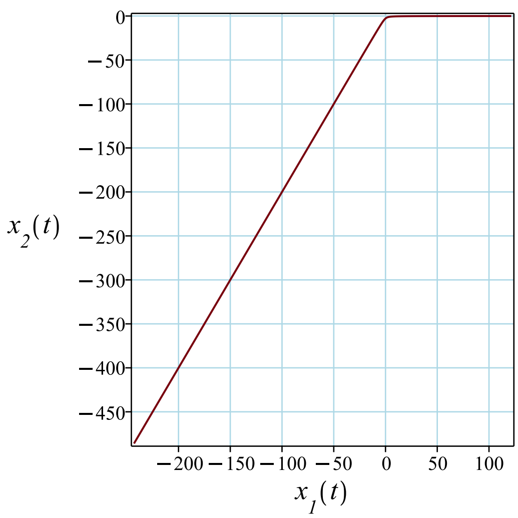
Solving for the constants of integrations gives

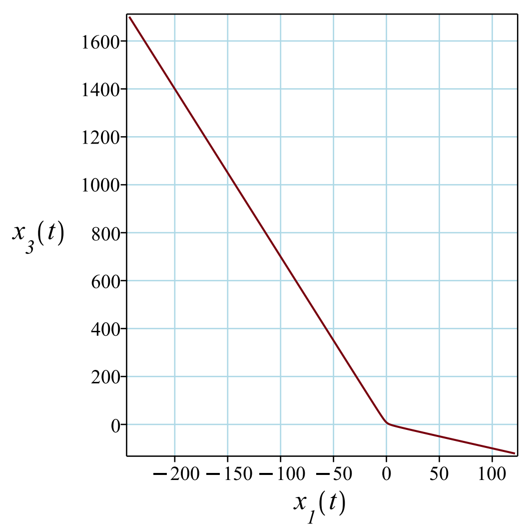
$$\begin{bmatrix} c_1 = 14 \\ c_2 = -1 \\ c_3 = 0 \end{bmatrix}$$

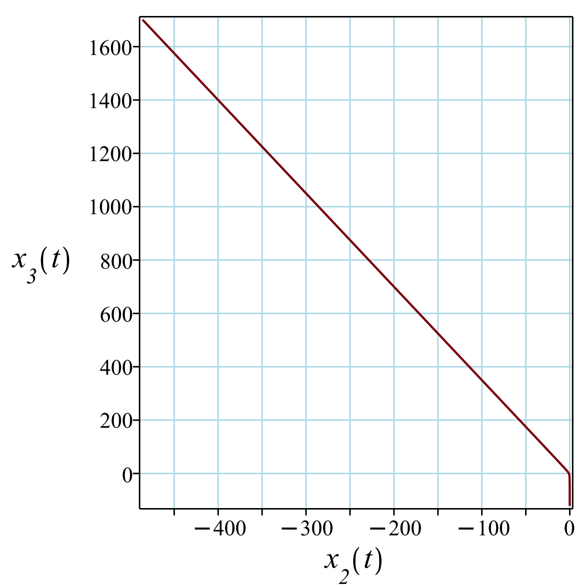
Substituting these constants back in original solution in Eq. (1) gives

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} e^{t} - 2\,e^{-t} \\ -4\,e^{-t} \\ -e^{t} + 14\,e^{-t} \end{bmatrix}
$$

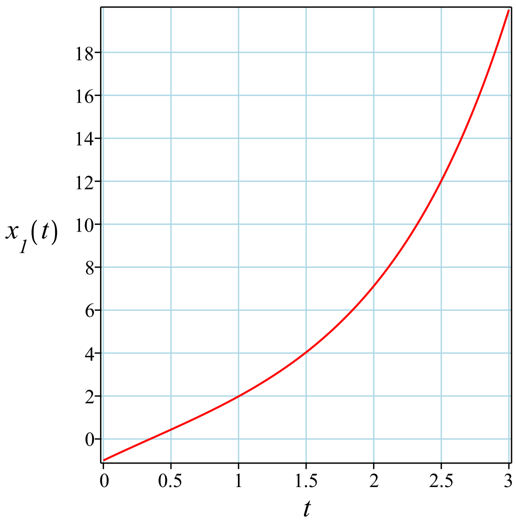
The following are plots of each solution against another.

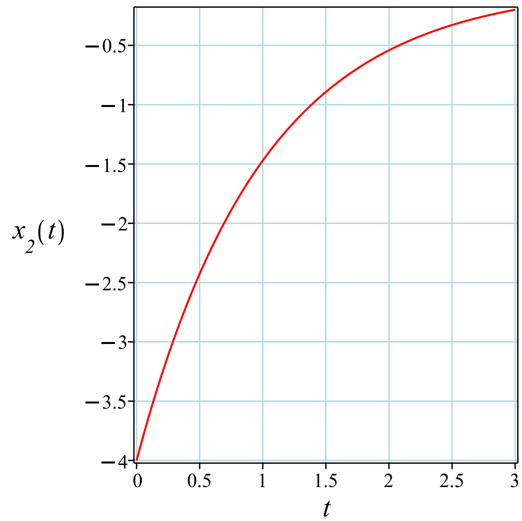


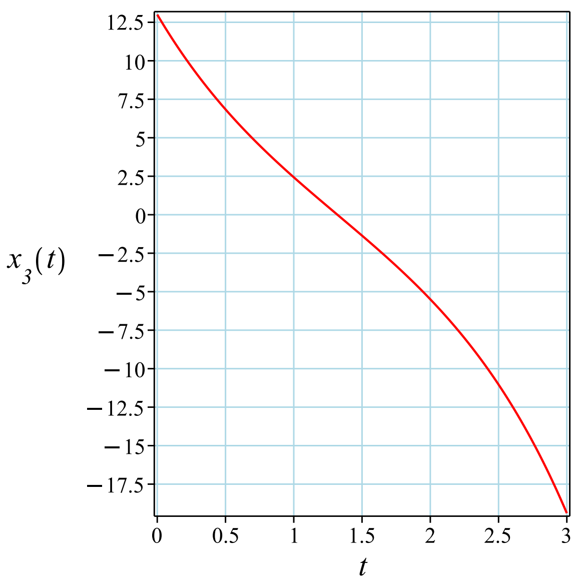




The following are plots of each solution.







✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 37

```
dsolve([diff(x__1(t),t) = x__1(t)-x__2(t), diff(x__2(t),t) = x__1(t)+2*x__2(t)+x__3(t), diff
```

$$x_1(t) = e^t - 2\,e^{-t}$$
$$x_2(t) = -4\,e^{-t}$$
$$x_3(t) = -e^t + 14\,e^{-t}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 42

```
DSolve[{x1'[t]==1*x1[t]-1*x2[t]-0*x3[t],x2'[t]==1*x1[t]+2*x2[t]+1*x3[t],x3'[t]==1*x1[t]+10*x
```

$$x1(t) \to e^t - 2e^{-t}$$
$$x2(t) \to -4e^{-t}$$
$$x3(t) \to 14e^{-t} - e^t$$

1.11 problem 11

Internal problem ID [1834]
Internal file name [`OUTPUT/1835_Sunday_June_05_2022_02_34_38_AM_33026560/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.8, Systems of differential equations. The eigenva1ue-eigenvector method. Page 339
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_1(t) - 3x_2(t) + 2x_3(t) \\
x_2'(t) &= -x_2(t) \\
x_3'(t) &= -x_2(t) - 2x_3(t)
\end{aligned}
$$

With initial conditions

$$[x_1(0) = -2, x_2(0) = 0, x_3(0) = 3]$$

1.11.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ 0 & -1 & 0 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{t} & -\frac{(11\,e^{3t}-15\,e^{t}+4)\,e^{-2t}}{6} & \frac{2(e^{3t}-1)\,e^{-2t}}{3} \\ 0 & e^{-t} & 0 \\ 0 & -e^{-t}+e^{-2t} & e^{-2t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} e^{t} & -\frac{(11\,e^{3t}-15\,e^{t}+4)\,e^{-2t}}{6} & \frac{2(e^{3t}-1)\,e^{-2t}}{3} \\ 0 & e^{-t} & 0 \\ 0 & -e^{-t}+e^{-2t} & e^{-2t} \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} -2\,e^{t}+2(e^{3t}-1)\,e^{-2t} \\ 0 \\ 3\,e^{-2t} \end{bmatrix} \\ &= \begin{bmatrix} -2\,e^{-2t} \\ 0 \\ 3\,e^{-2t} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

1.11.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ 0 & -1 & 0 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -3 & 2 \\ 0 & -1 & 0 \\ 0 & -1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -3 & 2 \\ 0 & -1-\lambda & 0 \\ 0 & -1 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + 2\lambda^2 - \lambda - 2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 1$$
$$\lambda_3 = -2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
-2	1	real eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -3 & 2 \\ 0 & -1 & 0 \\ 0 & -1 & -2 \end{bmatrix} - (-2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -3 & 2 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 3 & -3 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} 3 & -3 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -3 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{3}, v_2 = 0\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2t}{3} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2t}{3} \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2t}{3} \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{2}{3} \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2t}{3} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ 0 \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{2t}{3} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -3 & 2 \\ 0 & -1 & 0 \\ 0 & -1 & -2 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 & 2 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 2 & -3 & 2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -3 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{5t}{2}, v_2 = -t\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{5t}{2} \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{5t}{2} \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{5t}{2} \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{5}{2} \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{5t}{2} \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} \\ -1 \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{5t}{2} \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -5 \\ -2 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -3 & 2 \\ 0 & -1 & 0 \\ 0 & -1 & -2 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -3 & 2 \\ 0 & -2 & 0 \\ 0 & -1 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & -3 & 2 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & -1 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} 0 & -3 & 2 & 0 \\ 0 & 0 & -\frac{4}{3} & 0 \\ 0 & -1 & -3 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} 0 & -3 & 2 & 0 \\ 0 & 0 & -\frac{4}{3} & 0 \\ 0 & 0 & -\frac{11}{3} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{11R_2}{4} \Longrightarrow \left[\begin{array}{ccc|c} 0 & -3 & 2 & 0 \\ 0 & 0 & -\frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & -3 & 2 \\ 0 & 0 & -\frac{4}{3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2, v_3\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} -\frac{5}{2} \\ -1 \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
-2	1	1	No	$\begin{bmatrix} -\frac{2}{3} \\ 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} -\frac{5}{2} \\ -1 \\ 1 \end{bmatrix} e^{-t} \end{aligned}$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^t \\ &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t\end{aligned}$$

Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{-2t} \\ &= \begin{bmatrix} -\frac{2}{3} \\ 0 \\ 1 \end{bmatrix} e^{-2t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{5\,\mathrm{e}^{-t}}{2} \\ -\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{t} \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -\frac{2\,\mathrm{e}^{-2t}}{3} \\ 0 \\ \mathrm{e}^{-2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{(6c_2\mathrm{e}^{3t} - 15c_1\mathrm{e}^{t} - 4c_3)\mathrm{e}^{-2t}}{6} \\ -c_1\mathrm{e}^{-t} \\ c_1\mathrm{e}^{-t} + c_3\mathrm{e}^{-2t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = -2 \\ x_2(0) = 0 \\ x_3(0) = 3 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} c_2 - \frac{5c_1}{2} - \frac{2c_3}{3} \\ -c_1 \\ c_1 + c_3 \end{bmatrix}$$

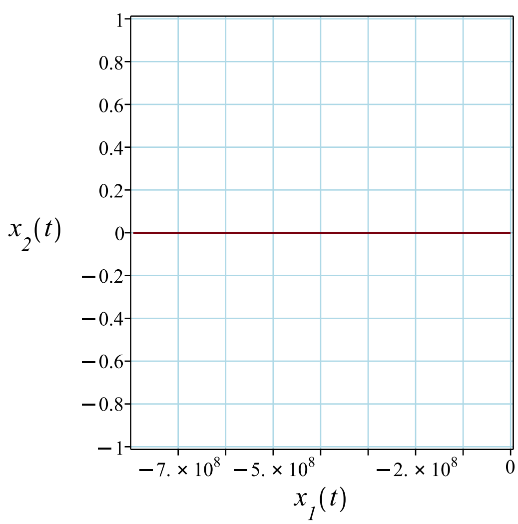
Solving for the constants of integrations gives

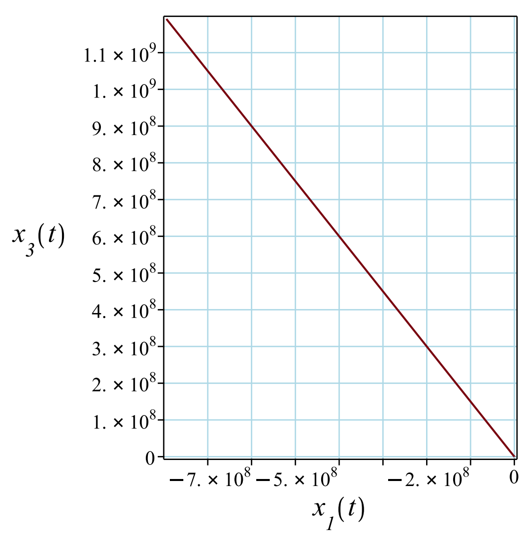
$$\begin{bmatrix} c_1 = 0 \\ c_2 = 0 \\ c_3 = 3 \end{bmatrix}$$

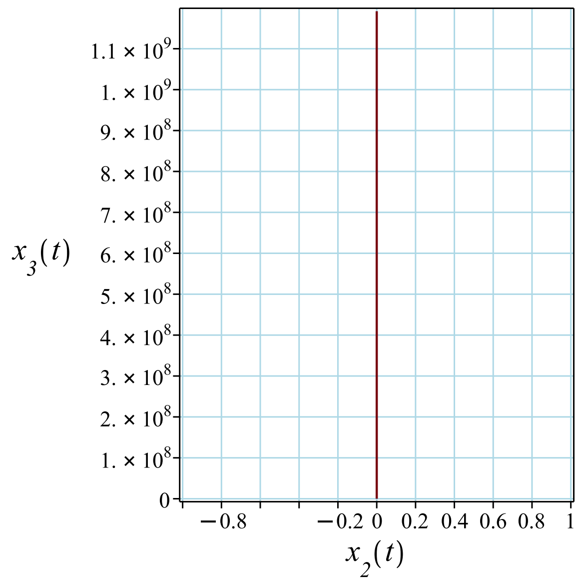
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -2\,\mathrm{e}^{-2t} \\ 0 \\ 3\,\mathrm{e}^{-2t} \end{bmatrix}$$

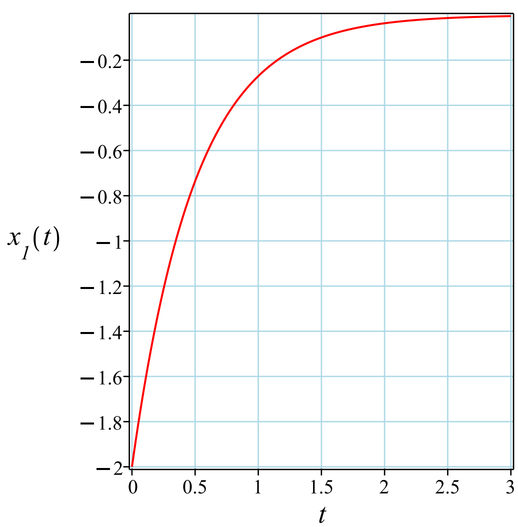
The following are plots of each solution against another.

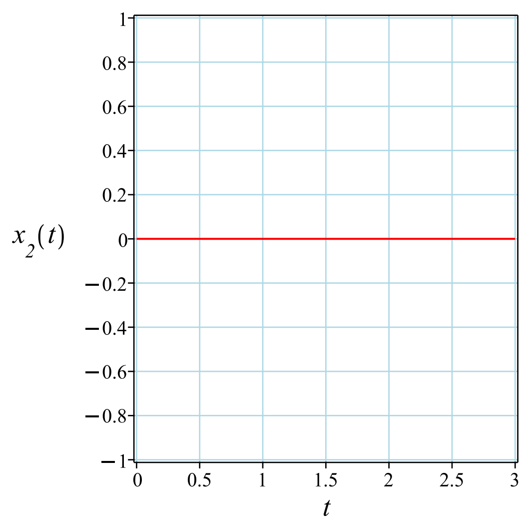


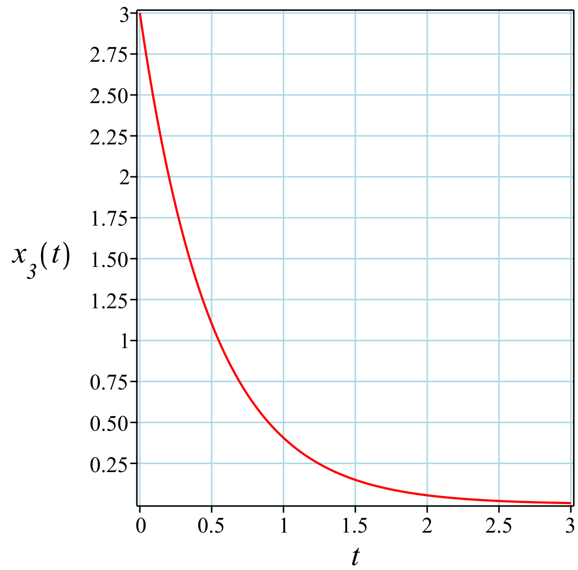




The following are plots of each solution.







✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 24

```
dsolve([diff(x__1(t),t) = x__1(t)-3*x__2(t)+2*x__3(t), diff(x__2(t),t) = -x__2(t), diff(x__3
```

$$x_1(t) = -2\,\mathrm{e}^{-2t}$$
$$x_2(t) = 0$$
$$x_3(t) = 3\,\mathrm{e}^{-2t}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 26

```
DSolve[{x1'[t]==1*x1[t]-3*x2[t]+2*x3[t],x2'[t]==0*x1[t]-1*x2[t]+0*x3[t],x3'[t]==0*x1[t]-1*x2
```

$$\text{x1}(t) \to -2e^{-2t}$$
$$\text{x2}(t) \to 0$$
$$\text{x3}(t) \to 3e^{-2t}$$

1.12 problem 12

Internal problem ID [1835]
Internal file name [OUTPUT/1836_Sunday_June_05_2022_02_34_40_AM_5214114/index.tex]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.8, Systems of differential equations. The eigenva1ue-eigenvector method. Page 339
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 3x_1(t) + x_2(t) - 2x_3(t) \\
x_2'(t) &= -x_1(t) + 2x_2(t) + x_3(t) \\
x_3'(t) &= 4x_1(t) + x_2(t) - 3x_3(t)
\end{aligned}
$$

With initial conditions

$$[x_1(0) = 1, x_2(0) = 4, x_3(0) = -7]$$

1.12.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 3 & 1 & -2 \\ -1 & 2 & 1 \\ 4 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -\frac{7\,e^{-t}}{6} + \frac{5\,e^{t}}{2} - \frac{e^{2t}}{3} & e^{2t} - e^{t} & \frac{e^{2t}}{3} - \frac{3\,e^{t}}{2} + \frac{7\,e^{-t}}{6} \\ -\frac{e^{2t}}{3} + \frac{e^{-t}}{3} & e^{2t} & \frac{e^{2t}}{3} - \frac{e^{-t}}{3} \\ \frac{5\,e^{t}}{2} - \frac{13\,e^{-t}}{6} - \frac{e^{2t}}{3} & e^{2t} - e^{t} & \frac{13\,e^{-t}}{6} + \frac{e^{2t}}{3} - \frac{3\,e^{t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} -\frac{7\,e^{-t}}{6} + \frac{5\,e^{t}}{2} - \frac{e^{2t}}{3} & e^{2t} - e^{t} & \frac{e^{2t}}{3} - \frac{3\,e^{t}}{2} + \frac{7\,e^{-t}}{6} \\ -\frac{e^{2t}}{3} + \frac{e^{-t}}{3} & e^{2t} & \frac{e^{2t}}{3} - \frac{e^{-t}}{3} \\ \frac{5\,e^{t}}{2} - \frac{13\,e^{-t}}{6} - \frac{e^{2t}}{3} & e^{2t} - e^{t} & \frac{13\,e^{-t}}{6} + \frac{e^{2t}}{3} - \frac{3\,e^{t}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{28\,e^{-t}}{3} + 9\,e^{t} + \frac{4\,e^{2t}}{3} \\ \frac{4\,e^{2t}}{3} + \frac{8\,e^{-t}}{3} \\ 9\,e^{t} - \frac{52\,e^{-t}}{3} + \frac{4\,e^{2t}}{3} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

1.12.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 3 & 1 & -2 \\ -1 & 2 & 1 \\ 4 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & 1 & -2 \\ -1 & 2 & 1 \\ 4 & 1 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 4 & 1 & -3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 1$$
$$\lambda_3 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 1 & -2 \\ -1 & 2 & 1 \\ 4 & 1 & -3 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 & -2 \\ -1 & 3 & 1 \\ 4 & 1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 4 & 1 & -2 & 0 \\ -1 & 3 & 1 & 0 \\ 4 & 1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} 4 & 1 & -2 & 0 \\ 0 & \frac{13}{4} & \frac{1}{2} & 0 \\ 4 & 1 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 4 & 1 & -2 & 0 \\ 0 & \frac{13}{4} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & 1 & -2 \\ 0 & \frac{13}{4} & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{7t}{13}, v_2 = -\frac{2t}{13}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{7t}{13} \\ -\frac{2t}{13} \\ t \end{bmatrix} = \begin{bmatrix} \frac{7t}{13} \\ -\frac{2t}{13} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{7t}{13} \\ -\frac{2t}{13} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{7}{13} \\ -\frac{2}{13} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{7t}{13} \\ -\frac{2t}{13} \\ t \end{bmatrix} = \begin{bmatrix} \frac{7}{13} \\ -\frac{2}{13} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{7t}{13} \\ -\frac{2t}{13} \\ t \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 13 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 1 & -2 \\ -1 & 2 & 1 \\ 4 & 1 & -3 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & -2 \\ -1 & 1 & 1 \\ 4 & 1 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 1 & -2 & 0 \\ -1 & 1 & 1 & 0 \\ 4 & 1 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{2} \implies \left[\begin{array}{ccc|c} 2 & 1 & -2 & 0 \\ 0 & \frac{3}{2} & 0 & 0 \\ 4 & 1 & -4 & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_1 \implies \left[\begin{array}{ccc|c} 2 & 1 & -2 & 0 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{2R_2}{3} \implies \left[\begin{array}{ccc|c} 2 & 1 & -2 & 0 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 1 & -2 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 1 & -2 \\ -1 & 2 & 1 \\ 4 & 1 & -3 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -2 \\ -1 & 0 & 1 \\ 4 & 1 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ -1 & 0 & 1 & 0 \\ 4 & 1 & -5 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 4 & 1 & -5 & 0 \end{array}\right]$$

$$R_3 = R_3 - 4R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right]$$

$$R_3 = R_3 + 3R_2 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} \frac{7}{13} \\ -\frac{2}{13} \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} \frac{7}{13} \\ -\frac{2}{13} \\ 1 \end{bmatrix} e^{-t} \end{aligned}$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{t} \\ &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{t}\end{aligned}$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{2t} \\ &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{7\,\mathrm{e}^{-t}}{13} \\ -\frac{2\,\mathrm{e}^{-t}}{13} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{t} \\ 0 \\ \mathrm{e}^{t} \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{7c_1\mathrm{e}^{-t}}{13} + c_2\mathrm{e}^{t} + c_3\mathrm{e}^{2t} \\ -\frac{2c_1\mathrm{e}^{-t}}{13} + c_3\mathrm{e}^{2t} \\ c_1\mathrm{e}^{-t} + c_2\mathrm{e}^{t} + c_3\mathrm{e}^{2t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = 1 \\ x_2(0) = 4 \\ x_3(0) = -7 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix} = \begin{bmatrix} \frac{7c_1}{13} + c_2 + c_3 \\ -\frac{2c_1}{13} + c_3 \\ c_1 + c_2 + c_3 \end{bmatrix}$$

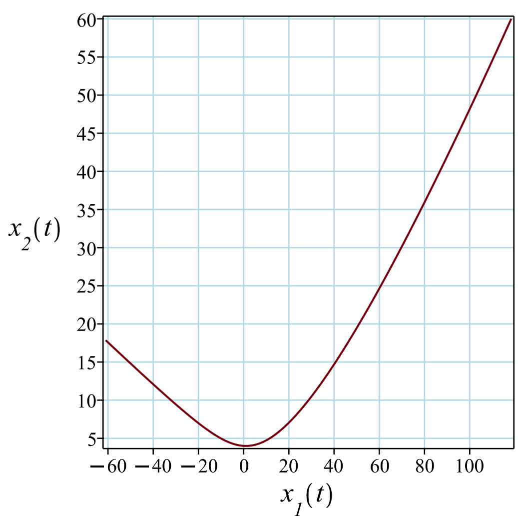
Solving for the constants of integrations gives

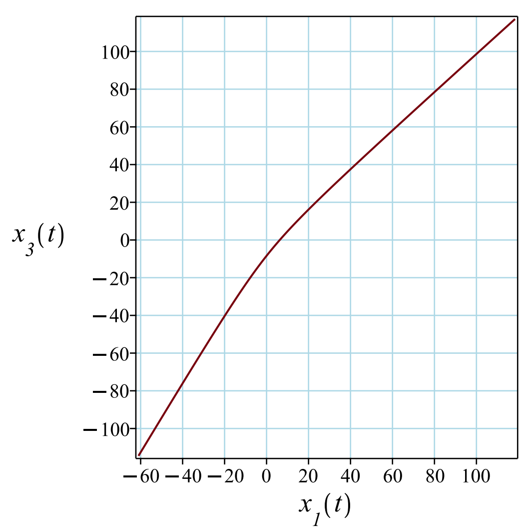
$$\begin{bmatrix} c_1 = -\frac{52}{3} \\ c_2 = 9 \\ c_3 = \frac{4}{3} \end{bmatrix}$$

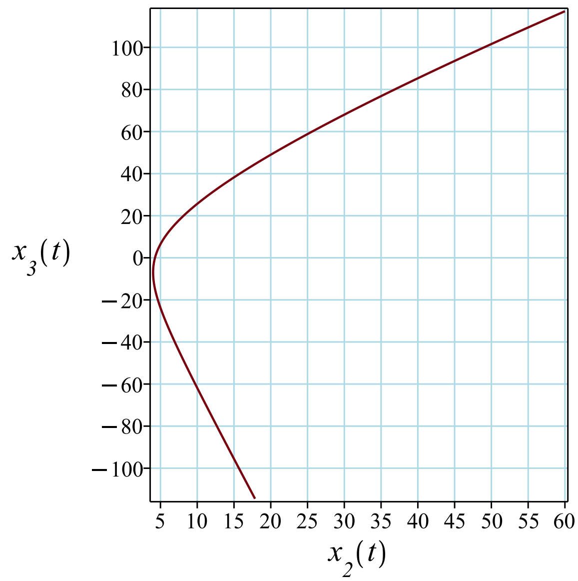
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{28\,\mathrm{e}^{-t}}{3} + 9\,\mathrm{e}^{t} + \frac{4\,\mathrm{e}^{2t}}{3} \\ \frac{4\,\mathrm{e}^{2t}}{3} + \frac{8\,\mathrm{e}^{-t}}{3} \\ 9\,\mathrm{e}^{t} - \frac{52\,\mathrm{e}^{-t}}{3} + \frac{4\,\mathrm{e}^{2t}}{3} \end{bmatrix}$$

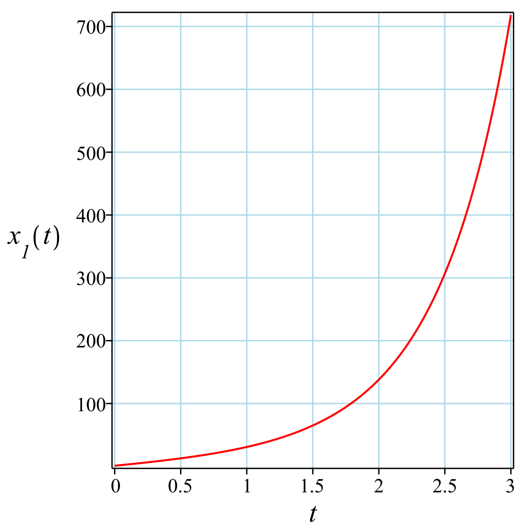
The following are plots of each solution against another.

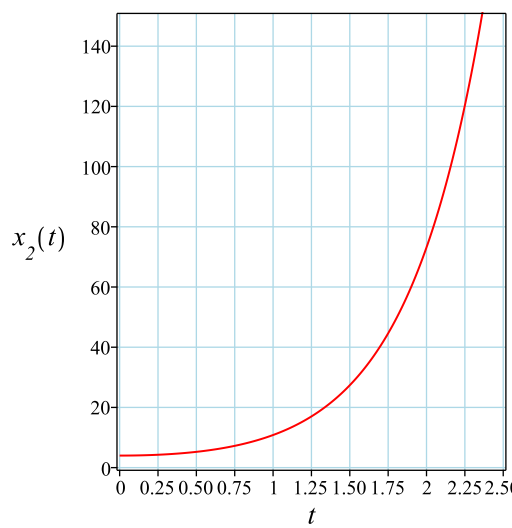


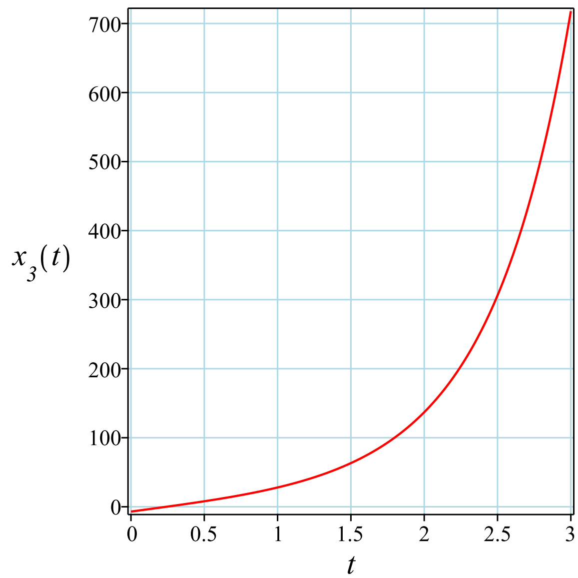




The following are plots of each solution.







✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 58

```
dsolve([diff(x__1(t),t) = 3*x__1(t)+x__2(t)-2*x__3(t), diff(x__2(t),t) = -x__1(t)+2*x__2(t)+
```

$$x_1(t) = 9\,e^t - \frac{28\,e^{-t}}{3} + \frac{4\,e^{2t}}{3}$$
$$x_2(t) = \frac{8\,e^{-t}}{3} + \frac{4\,e^{2t}}{3}$$
$$x_3(t) = 9\,e^t - \frac{52\,e^{-t}}{3} + \frac{4\,e^{2t}}{3}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 75

```
DSolve[{x1'[t]==3*x1[t]+1*x2[t]-2*x3[t],x2'[t]==-1*x1[t]+2*x2[t]+1*x3[t],x3'[t]==4*x1[t]+1*x
```

$$\text{x1}(t) \to -\frac{28e^{-t}}{3} + 9e^t + \frac{4e^{2t}}{3}$$
$$\text{x2}(t) \to \frac{4}{3}e^{-t}\left(e^{3t}+2\right)$$
$$\text{x3}(t) \to -\frac{52e^{-t}}{3} + 9e^t + \frac{4e^{2t}}{3}$$

2 Section 3.9, Systems of differential equations. Complex roots. Page 344

2.1 problem 1

Internal problem ID [1836]
Internal file name [`OUTPUT/1837_Sunday_June_05_2022_02_34_42_AM_76804885/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.9, Systems of differential equations. Complex roots. Page 344
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = -3x_1(t) + 2x_2(t)$$
$$x_2'(t) = -x_1(t) - x_2(t)$$

2.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$\begin{aligned} e^{At} &= \begin{bmatrix} e^{-2t}\cos(t) - e^{-2t}\sin(t) & 2\,e^{-2t}\sin(t) \\ -e^{-2t}\sin(t) & e^{-2t}\cos(t) + e^{-2t}\sin(t) \end{bmatrix} \\ &= \begin{bmatrix} e^{-2t}(-\sin(t) + \cos(t)) & 2\,e^{-2t}\sin(t) \\ -e^{-2t}\sin(t) & e^{-2t}(\cos(t) + \sin(t)) \end{bmatrix} \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-2t}(-\sin(t)+\cos(t)) & 2\,e^{-2t}\sin(t) \\ -e^{-2t}\sin(t) & e^{-2t}(\cos(t)+\sin(t)) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-2t}(-\sin(t)+\cos(t))\,c_1 + 2\,e^{-2t}\sin(t)\,c_2 \\ -e^{-2t}\sin(t)\,c_1 + e^{-2t}(\cos(t)+\sin(t))\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} ((-c_1+2c_2)\sin(t)+c_1\cos(t))\,e^{-2t} \\ -(\sin(t)\,(c_1-c_2)-c_2\cos(t))\,e^{-2t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

2.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -3-\lambda & 2 \\ -1 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 4\lambda + 5 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2 + i$$
$$\lambda_2 = -2 - i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-2+i$	1	complex eigenvalue
$-2-i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2 - i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} - (-2-i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1+i & 2 \\ -1 & 1+i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1+i & 2 & 0 \\ -1 & 1+i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{1}{2} - \frac{i}{2}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} -1+i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1+i & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (1+i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (1+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (1+i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (1+\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (1+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -2+i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} - (-2+i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1-i & 2 \\ -1 & 1-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1-i & 2 & 0 \\ -1 & 1-i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{1}{2} + \frac{i}{2}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} -1-i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1-i & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (1 - i)\, t\}$

Hence the solution is

$$\begin{bmatrix} (1 - \mathrm{I})\, t \\ t \end{bmatrix} = \begin{bmatrix} (1 - i)\, t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (1 - \mathrm{I})\, t \\ t \end{bmatrix} = t \begin{bmatrix} 1 - i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (1 - \mathrm{I})\, t \\ t \end{bmatrix} = \begin{bmatrix} 1 - i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$-2 + i$	1	1	No	$\begin{bmatrix} 1 - i \\ 1 \end{bmatrix}$
$-2 - i$	1	1	No	$\begin{bmatrix} 1 + i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} (1-i)\,\mathrm{e}^{(-2+i)t} \\ \mathrm{e}^{(-2+i)t} \end{bmatrix} + c_2 \begin{bmatrix} (1+i)\,\mathrm{e}^{(-2-i)t} \\ \mathrm{e}^{(-2-i)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} (1-i)\,c_1\mathrm{e}^{(-2+i)t} + (1+i)\,c_2\mathrm{e}^{(-2-i)t} \\ c_1\mathrm{e}^{(-2+i)t} + c_2\mathrm{e}^{(-2-i)t} \end{bmatrix}$$

The following is the phase plot of the system.

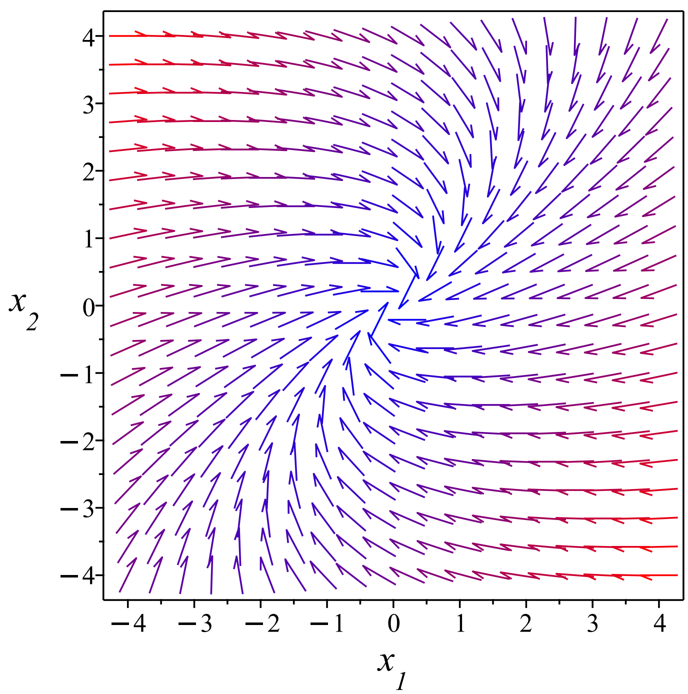


Figure 9: Phase plot

2.1.3 Maple step by step solution

Let's solve

$[x_1'(t) = -3x_1(t) + 2x_2(t)\,, x_2'(t) = -x_1(t) - x_2(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\underline{x}^{\rightarrow}{}'(t) = A \cdot \underline{x}^{\rightarrow}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2 - \mathrm{I}, \begin{bmatrix} 1 + \mathrm{I} \\ 1 \end{bmatrix}\right], \left[-2 + \mathrm{I}, \begin{bmatrix} 1 - \mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2 - \mathrm{I}, \begin{bmatrix} 1 + \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(-2-\mathrm{I})t} \cdot \begin{bmatrix} 1 + \mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{-2t} \cdot (\cos(t) - \mathrm{I}\sin(t)) \cdot \begin{bmatrix} 1 + \mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{-2t} \cdot \begin{bmatrix} (1+I)(\cos(t) - I\sin(t)) \\ \cos(t) - I\sin(t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{\vec{x}}_{1}(t) = e^{-2t} \cdot \begin{bmatrix} \cos(t) + \sin(t) \\ \cos(t) \end{bmatrix}, \underline{\vec{x}}_{2}(t) = e^{-2t} \cdot \begin{bmatrix} -\sin(t) + \cos(t) \\ -\sin(t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\underline{\vec{x}} = c_1 \underline{\vec{x}}_{1}(t) + c_2 \underline{\vec{x}}_{2}(t)$$

- Substitute solutions into the general solution

$$\underline{\vec{x}} = c_1 e^{-2t} \cdot \begin{bmatrix} \cos(t) + \sin(t) \\ \cos(t) \end{bmatrix} + e^{-2t} c_2 \cdot \begin{bmatrix} -\sin(t) + \cos(t) \\ -\sin(t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} (\cos(t)(c_1 + c_2) + (c_1 - c_2)\sin(t))\, e^{-2t} \\ e^{-2t}(-c_2 \sin(t) + c_1 \cos(t)) \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x_1(t) = (\cos(t)(c_1 + c_2) + (c_1 - c_2)\sin(t))\, e^{-2t}, x_2(t) = e^{-2t}(-c_2 \sin(t) + c_1 \cos(t))\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 46

```
dsolve([diff(x__1(t),t)=-3*x__1(t)+2*x__2(t),diff(x__2(t),t)=-1*x__1(t)-1*x__2(t)],singsol=a
```

$$x_1(t) = e^{-2t}(c_1 \sin(t) + c_2 \cos(t))$$
$$x_2(t) = \frac{e^{-2t}(c_1 \sin(t) - c_2 \sin(t) + c_1 \cos(t) + c_2 \cos(t))}{2}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 53

```
DSolve[{x1'[t]==-3*x1[t]+2*x2[t],x2'[t]==-1*x1[t]-1*x2[t]},{x1[t],x2[t]},t,IncludeSingularSo
```

$$\text{x1}(t) \to e^{-2t}(c_1 \cos(t) - (c_1 - 2c_2)\sin(t))$$
$$\text{x2}(t) \to e^{-2t}(c_2 \cos(t) + (c_2 - c_1)\sin(t))$$

2.2 problem 2

Internal problem ID [1837]
Internal file name [`OUTPUT/1838_Sunday_June_05_2022_02_34_45_AM_81520183/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.9, Systems of differential equations. Complex roots. Page 344
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = x_1(t) - 5x_2(t)$$
$$x_2'(t) = x_1(t) - 3x_2(t)$$
$$x_3'(t) = x_3(t)$$

2.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & -5 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-t}\cos(t) + 2e^{-t}\sin(t) & -5e^{-t}\sin(t) & 0 \\ e^{-t}\sin(t) & e^{-t}\cos(t) - 2e^{-t}\sin(t) & 0 \\ 0 & 0 & e^{t} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t}(\cos(t) + 2\sin(t)) & -5e^{-t}\sin(t) & 0 \\ e^{-t}\sin(t) & e^{-t}(\cos(t) - 2\sin(t)) & 0 \\ 0 & 0 & e^{t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-t}(\cos(t) + 2\sin(t)) & -5e^{-t}\sin(t) & 0 \\ e^{-t}\sin(t) & e^{-t}(\cos(t) - 2\sin(t)) & 0 \\ 0 & 0 & e^{t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}(\cos(t) + 2\sin(t))\, c_1 - 5e^{-t}\sin(t)\, c_2 \\ e^{-t}\sin(t)\, c_1 + e^{-t}(\cos(t) - 2\sin(t))\, c_2 \\ e^{t}c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}((2c_1 - 5c_2)\sin(t) + c_1\cos(t)) \\ ((c_1 - 2c_2)\sin(t) + c_2\cos(t))\, e^{-t} \\ e^{t}c_3 \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

2.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & -5 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -5 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -5 & 0 \\ 1 & -3-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + \lambda^2 - 2 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned} \lambda_1 &= -1 + i \\ \lambda_2 &= -1 - i \\ \lambda_3 &= 1 \end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-1 - i$	1	complex eigenvalue
1	1	real eigenvalue
$-1 + i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -5 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -5 & 0 \\ 1 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & -5 & 0 & 0 \\ 1 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} 1 & -4 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -4 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1 - i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -5 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} - (-1 - i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2+i & -5 & 0 \\ 1 & -2+i & 0 \\ 0 & 0 & 2+i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2+i & -5 & 0 & 0 \\ 1 & -2+i & 0 & 0 \\ 0 & 0 & 2+i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{2}{5} + \frac{i}{5}\right) R_1 \implies \left[\begin{array}{ccc|c} 2+i & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2+i & 0 \end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 2+i & -5 & 0 & 0 \\ 0 & 0 & 2+i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2+i & -5 & 0 \\ 0 & 0 & 2+i \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (2-i)\,t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} (2-\mathrm{I})\,t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} (2-i)\,t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (2-\mathrm{I})\,t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 2-i \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (2-\mathrm{I})\,t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 2-i \\ 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = -1 + i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -5 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} - (-1+i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2-i & -5 & 0 \\ 1 & -2-i & 0 \\ 0 & 0 & 2-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2-i & -5 & 0 & 0 \\ 1 & -2-i & 0 & 0 \\ 0 & 0 & 2-i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{2}{5} - \frac{i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2-i & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2-i & 0 \end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 2-i & -5 & 0 & 0 \\ 0 & 0 & 2-i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2-i & -5 & 0 \\ 0 & 0 & 2-i \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (2+i)\,t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} (2+\mathrm{I})\,t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} (2+i)\,t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (2+\mathrm{I})\,t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 2+i \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (2+\mathrm{I})\,t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 2+i \\ 1 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$-1+i$	1	1	No	$\begin{bmatrix} 2+i \\ 1 \\ 0 \end{bmatrix}$
$-1-i$	1	1	No	$\begin{bmatrix} 2-i \\ 1 \\ 0 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^t \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^t \end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} (2+i)\,\mathrm{e}^{(-1+i)t} \\ \mathrm{e}^{(-1+i)t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} (2-i)\,\mathrm{e}^{(-1-i)t} \\ \mathrm{e}^{(-1-i)t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ \mathrm{e}^{t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} (2+i)\,c_1\mathrm{e}^{(-1+i)t} + (2-i)\,c_2\mathrm{e}^{(-1-i)t} \\ c_1\mathrm{e}^{(-1+i)t} + c_2\mathrm{e}^{(-1-i)t} \\ c_3\mathrm{e}^{t} \end{bmatrix}$$

2.2.3 Maple step by step solution

Let's solve

$[x_1'(t) = x_1(t) - 5x_2(t)\,, x_2'(t) = x_1(t) - 3x_2(t)\,, x_3'(t) = x_3(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 1 & -5 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 1 & -5 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & -5 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Rewrite the system as

  $$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[1, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right], \left[-1-\mathrm{I}, \begin{bmatrix} 2-\mathrm{I} \\ 1 \\ 0 \end{bmatrix}\right], \left[-1+\mathrm{I}, \begin{bmatrix} 2+\mathrm{I} \\ 1 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

  $$\left[1, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\underline{\vec{x}}_{1} = \mathrm{e}^{t} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

  $$\left[-1-\mathrm{I}, \begin{bmatrix} 2-\mathrm{I} \\ 1 \\ 0 \end{bmatrix}\right]$$

- Solution from eigenpair

  $$\mathrm{e}^{(-1-\mathrm{I})t} \cdot \begin{bmatrix} 2-\mathrm{I} \\ 1 \\ 0 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{-t}\cdot\left(\cos(t)-\mathrm{I}\sin(t)\right)\cdot\begin{bmatrix}2-\mathrm{I}\\1\\0\end{bmatrix}$$

- Simplify expression

$$e^{-t}\cdot\begin{bmatrix}(2-\mathrm{I})\left(\cos(t)-\mathrm{I}\sin(t)\right)\\\cos(t)-\mathrm{I}\sin(t)\\0\end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{\vec{x}}_2(t)=e^{-t}\cdot\begin{bmatrix}2\cos(t)-\sin(t)\\\cos(t)\\0\end{bmatrix},\underline{\vec{x}}_3(t)=e^{-t}\cdot\begin{bmatrix}-\cos(t)-2\sin(t)\\-\sin(t)\\0\end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\underline{\vec{x}}=c_1\underline{\vec{x}}_1+c_2\underline{\vec{x}}_2(t)+c_3\underline{\vec{x}}_3(t)$$

- Substitute solutions into the general solution

$$\underline{\vec{x}}=c_1e^{t}\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}+c_2e^{-t}\cdot\begin{bmatrix}2\cos(t)-\sin(t)\\\cos(t)\\0\end{bmatrix}+c_3e^{-t}\cdot\begin{bmatrix}-\cos(t)-2\sin(t)\\-\sin(t)\\0\end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{bmatrix}=\begin{bmatrix}2\left(\left(c_2-\frac{c_3}{2}\right)\cos(t)-\frac{\sin(t)(c_2+2c_3)}{2}\right)e^{-t}\\e^{-t}\left(c_2\cos(t)-c_3\sin(t)\right)\\c_1e^{t}\end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t)=2\left(\left(c_2-\frac{c_3}{2}\right)\cos(t)-\frac{\sin(t)(c_2+2c_3)}{2}\right)e^{-t},x_2(t)=e^{-t}\left(c_2\cos(t)-c_3\sin(t)\right),x_3(t)=c_1e^{t}\right\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 55

```
dsolve([diff(x__1(t),t)=1*x__1(t)-5*x__2(t)+0*x__3(t),diff(x__2(t),t)=1*x__1(t)-3*x__2(t)+0*
```

$$x_1(t) = e^{-t}(c_1 \sin(t) + c_2 \cos(t))$$
$$x_2(t) = \frac{e^{-t}(-c_1 \cos(t) + c_2 \sin(t) + 2c_1 \sin(t) + 2c_2 \cos(t))}{5}$$
$$x_3(t) = c_3 e^t$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 120

```
DSolve[{x1'[t]==1*x1[t]-5*x2[t]+0*x3[t],x2'[t]==1*x1[t]-3*x2[t]+0*x3[t],x3'[t]==0*x1[t]-0*x2
```

$$\text{x1}(t) \rightarrow e^{-t}(c_1 \cos(t) + (2c_1 - 5c_2) \sin(t))$$
$$\text{x2}(t) \rightarrow e^{-t}(c_2 \cos(t) + (c_1 - 2c_2) \sin(t))$$
$$\text{x3}(t) \rightarrow c_3 e^t$$
$$\text{x1}(t) \rightarrow e^{-t}(c_1 \cos(t) + (2c_1 - 5c_2) \sin(t))$$
$$\text{x2}(t) \rightarrow e^{-t}(c_2 \cos(t) + (c_1 - 2c_2) \sin(t))$$
$$\text{x3}(t) \rightarrow 0$$

2.3 problem 3

Internal problem ID [1838]
Internal file name [`OUTPUT/1839_Sunday_June_05_2022_02_34_47_AM_73909231/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.9, Systems of differential equations. Complex roots. Page 344
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x_1'(t) &= x_1(t) \\ x_2'(t) &= 3x_1(t) + x_2(t) - 2x_3(t) \\ x_3'(t) &= 2x_1(t) + 2x_2(t) + x_3(t) \end{aligned}$$

2.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & -2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} e^t & 0 & 0 \\ e^t \cos(2t) + \frac{3\,e^t \sin(2t)}{2} - e^t & e^t \cos(2t) & -e^t \sin(2t) \\ -\frac{3\,e^t \cos(2t)}{2} + e^t \sin(2t) + \frac{3\,e^t}{2} & e^t \sin(2t) & e^t \cos(2t) \end{bmatrix} \\
&= \begin{bmatrix} e^t & 0 & 0 \\ \frac{e^t(-2+2\cos(2t)+3\sin(2t))}{2} & e^t \cos(2t) & -e^t \sin(2t) \\ -\frac{e^t(-3+3\cos(2t)-2\sin(2t))}{2} & e^t \sin(2t) & e^t \cos(2t) \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^t & 0 & 0 \\ \frac{e^t(-2+2\cos(2t)+3\sin(2t))}{2} & e^t \cos(2t) & -e^t \sin(2t) \\ -\frac{e^t(-3+3\cos(2t)-2\sin(2t))}{2} & e^t \sin(2t) & e^t \cos(2t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^t c_1 \\ \frac{e^t(-2+2\cos(2t)+3\sin(2t))c_1}{2} + e^t \cos(2t)\, c_2 - e^t \sin(2t)\, c_3 \\ -\frac{e^t(-3+3\cos(2t)-2\sin(2t))c_1}{2} + e^t \sin(2t)\, c_2 + e^t \cos(2t)\, c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^t c_1 \\ \left((c_1 + c_2)\cos(2t) + \left(\frac{3c_1}{2} - c_3\right)\sin(2t) - c_1\right) e^t \\ -\frac{3\left(\left(c_1 - \frac{2c_3}{3}\right)\cos(2t) + \frac{2(-c_1-c_2)\sin(2t)}{3} - c_1\right)e^t}{2} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

2.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & -2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & -2 \\ 2 & 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 0 & 0 \\ 3 & 1-\lambda & -2 \\ 2 & 2 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 3\lambda^2 + 7\lambda - 5 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= 1 + 2i \\ \lambda_3 &= 1 - 2i \end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
$1 + 2i$	1	complex eigenvalue
$1 - 2i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & -2 \\ 2 & 2 & 1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & -2 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 3 & 0 & -2 & 0 \\ 2 & 2 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} 3 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{2R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} 3 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & \frac{4}{3} & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 3 & 0 & -2 & 0 \\ 0 & 2 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & 0 & -2 \\ 0 & 2 & \frac{4}{3} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{3}, v_2 = -\frac{2t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2t}{3} \\ -\frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{3} \\ -\frac{2t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2t}{3} \\ -\frac{2t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2t}{3} \\ -\frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{2t}{3} \\ -\frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1 - 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & -2 \\ 2 & 2 & 1 \end{bmatrix} - (1 - 2i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2i & 0 & 0 \\ 3 & 2i & -2 \\ 2 & 2 & 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2i & 0 & 0 & 0 \\ 3 & 2i & -2 & 0 \\ 2 & 2 & 2i & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{3iR_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 2i & 0 & 0 & 0 \\ 0 & 2i & -2 & 0 \\ 2 & 2 & 2i & 0 \end{array}\right]$$

$$R_3 = iR_1 + R_3 \Longrightarrow \left[\begin{array}{ccc|c} 2i & 0 & 0 & 0 \\ 0 & 2i & -2 & 0 \\ 0 & 2 & 2i & 0 \end{array}\right]$$

$$R_3 = iR_2 + R_3 \Longrightarrow \left[\begin{array}{ccc|c} 2i & 0 & 0 & 0 \\ 0 & 2i & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2i & 0 & 0 \\ 0 & 2i & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = -it\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ \text{-I}\,t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ \text{-I}\,t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ \text{-I}t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 1 + 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & -2 \\ 2 & 2 & 1 \end{bmatrix} - (1+2i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2i & 0 & 0 \\ 3 & -2i & -2 \\ 2 & 2 & -2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2i & 0 & 0 & 0 \\ 3 & -2i & -2 & 0 \\ 2 & 2 & -2i & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3iR_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2i & 0 & 0 & 0 \\ 0 & -2i & -2 & 0 \\ 2 & 2 & -2i & 0 \end{array}\right]$$

$$R_3 = -iR_1 + R_3 \Longrightarrow \left[\begin{array}{ccc|c} -2i & 0 & 0 & 0 \\ 0 & -2i & -2 & 0 \\ 0 & 2 & -2i & 0 \end{array}\right]$$

$$R_3 = -iR_2 + R_3 \Longrightarrow \left[\begin{array}{ccc|c} -2i & 0 & 0 & 0 \\ 0 & -2i & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2i & 0 & 0 \\ 0 & -2i & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = it\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ \mathrm{I}t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
1	1	1	No	$\begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix}$
$1+2i$	1	1	No	$\begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$
$1-2i$	1	1	No	$\begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{t} \\ &= \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix} e^{t} \end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{2\,\mathrm{e}^{t}}{3} \\ -\frac{2\,\mathrm{e}^{t}}{3} \\ \mathrm{e}^{t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ i\mathrm{e}^{(1+2i)t} \\ \mathrm{e}^{(1+2i)t} \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -i\mathrm{e}^{(1-2i)t} \\ \mathrm{e}^{(1-2i)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{2c_1 e^t}{3} \\ -\frac{2c_1 e^t}{3} + i c_2 e^{(1+2i)t} - i c_3 e^{(1-2i)t} \\ c_1 e^t + c_2 e^{(1+2i)t} + c_3 e^{(1-2i)t} \end{bmatrix}$$

2.3.3 Maple step by step solution

Let's solve

$[x_1'(t) = x_1(t), x_2'(t) = 3x_1(t) + x_2(t) - 2x_3(t), x_3'(t) = 2x_1(t) + 2x_2(t) + x_3(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & -2 \\ 2 & 2 & 1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & -2 \\ 2 & 2 & 1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$$

- Rewrite the system as

$\underline{x}^{\rightarrow\prime}(t) = A \cdot \underline{x}^{\rightarrow}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix}\right], \left[1-2\,\mathrm{I}, \begin{bmatrix} 0 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right], \left[1+2\,\mathrm{I}, \begin{bmatrix} 0 \\ \mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}\underline{\quad}_1 = \mathrm{e}^t \cdot \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1-2\,\mathrm{I}, \begin{bmatrix} 0 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(1-2\,\mathrm{I})t} \cdot \begin{bmatrix} 0 \\ -\mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^t \cdot (\cos(2t) - \mathrm{I}\sin(2t)) \cdot \begin{bmatrix} 0 \\ -\mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\mathrm{e}^t \cdot \begin{bmatrix} 0 \\ -\mathrm{I}(\cos(2t) - \mathrm{I}\sin(2t)) \\ \cos(2t) - \mathrm{I}\sin(2t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{x}^{\rightarrow}_{2}(t) = e^{t} \cdot \begin{bmatrix} 0 \\ -\sin(2t) \\ \cos(2t) \end{bmatrix}, \underline{x}^{\rightarrow}_{3}(t) = e^{t} \cdot \begin{bmatrix} 0 \\ -\cos(2t) \\ -\sin(2t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\underline{x}^{\rightarrow} = c_1 \underline{x}^{\rightarrow}_{1} + c_2 \underline{x}^{\rightarrow}_{2}(t) + c_3 \underline{x}^{\rightarrow}_{3}(t)$$

- Substitute solutions into the general solution

$$\underline{x}^{\rightarrow} = c_1 e^{t} \cdot \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix} + c_2 e^{t} \cdot \begin{bmatrix} 0 \\ -\sin(2t) \\ \cos(2t) \end{bmatrix} + c_3 e^{t} \cdot \begin{bmatrix} 0 \\ -\cos(2t) \\ -\sin(2t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{2c_1 e^{t}}{3} \\ -\frac{e^{t}(3c_3\cos(2t)+3c_2\sin(2t)+2c_1)}{3} \\ e^{t}(c_1 + c_2\cos(2t) - c_3\sin(2t)) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \frac{2c_1 e^{t}}{3}, x_2(t) = -\frac{e^{t}(3c_3\cos(2t)+3c_2\sin(2t)+2c_1)}{3}, x_3(t) = e^{t}(c_1 + c_2\cos(2t) - c_3\sin(2t))\right\}$$

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 70

```
dsolve([diff(x__1(t),t)=1*x__1(t)-0*x__2(t)+0*x__3(t),diff(x__2(t),t)=3*x__1(t)+1*x__2(t)-2*
```

$$x_1(t) = c_3 e^{t}$$
$$x_2(t) = e^{t}(c_2\sin(2t) + c_1\cos(2t) - c_3\cos(2t) - c_3)$$
$$x_3(t) = \frac{e^{t}(2c_1\sin(2t) - 2c_3\sin(2t) - 2c_2\cos(2t) + 3c_3)}{2}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 91

```
DSolve[{x1'[t]==1*x1[t]-0*x2[t]+0*x3[t],x2'[t]==3*x1[t]+1*x2[t]-2*x3[t],x3'[t]==2*x1[t]+2*x2
```

$$\begin{aligned}
\text{x1}(t) &\to c_1 e^t \\
\text{x2}(t) &\to \frac{1}{2}e^t(2(c_1+c_2)\cos(2t)+(3c_1-2c_3)\sin(2t)-2c_1) \\
\text{x3}(t) &\to \frac{1}{2}e^t((2c_3-3c_1)\cos(2t)+2(c_1+c_2)\sin(2t)+3c_1)
\end{aligned}$$

2.4 problem 4

Internal problem ID [1839]
Internal file name [`OUTPUT/1840_Sunday_June_05_2022_02_34_50_AM_46350731/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.9, Systems of differential equations. Complex roots. Page 344
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = x_1(t) + x_3(t)$$
$$x_2'(t) = x_2(t) - x_3(t)$$
$$x_3'(t) = -2x_1(t) - x_3(t)$$

2.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(t) + \sin(t) & 0 & \sin(t) \\ e^t - \cos(t) - \sin(t) & e^t & -\sin(t) \\ -2\sin(t) & 0 & -\sin(t) + \cos(t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \cos(t)+\sin(t) & 0 & \sin(t) \\ \mathrm{e}^t-\cos(t)-\sin(t) & \mathrm{e}^t & -\sin(t) \\ -2\sin(t) & 0 & -\sin(t)+\cos(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} (\cos(t)+\sin(t))\,c_1+\sin(t)\,c_3 \\ (\mathrm{e}^t-\cos(t)-\sin(t))\,c_1+\mathrm{e}^t c_2-\sin(t)\,c_3 \\ -2\sin(t)\,c_1+(-\sin(t)+\cos(t))\,c_3 \end{bmatrix} \\
&= \begin{bmatrix} (c_1+c_3)\sin(t)+c_1\cos(t) \\ (c_1+c_2)\,\mathrm{e}^t+(-c_1-c_3)\sin(t)-c_1\cos(t) \\ (-2c_1-c_3)\sin(t)+c_3\cos(t) \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

2.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & -1 \\ -2 & 0 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - \lambda^2 + \lambda - 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = i$$
$$\lambda_2 = -i$$
$$\lambda_3 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
$-i$	1	complex eigenvalue
i	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & -1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -2 & 0 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ -2 & 0 & -2 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 3 gives

$$\left[\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 0 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & -1 \end{bmatrix} - (-i)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1+i & 0 & 1 \\ 0 & 1+i & -1 \\ -2 & 0 & -1+i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1+i & 0 & 1 & 0 \\ 0 & 1+i & -1 & 0 \\ -2 & 0 & -1+i & 0 \end{array}\right]$$

$$R_3 = R_3 + (1-i)\,R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1+i & 0 & 1 & 0 \\ 0 & 1+i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1+i & 0 & 1 \\ 0 & 1+i & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{1}{2} + \frac{i}{2}\right)t, v_2 = \left(\frac{1}{2} - \frac{i}{2}\right)t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{1}{2} + \frac{I}{2}\right)t \\ \left(\frac{1}{2} - \frac{I}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2} + \frac{i}{2}\right)t \\ \left(\frac{1}{2} - \frac{i}{2}\right)t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{1}{2}+\frac{I}{2}\right)t \\ \left(\frac{1}{2}-\frac{I}{2}\right)t \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{1}{2}+\frac{i}{2} \\ \frac{1}{2}-\frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{1}{2}+\frac{I}{2}\right)t \\ \left(\frac{1}{2}-\frac{I}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}+\frac{i}{2} \\ \frac{1}{2}-\frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{1}{2}+\frac{I}{2}\right)t \\ \left(\frac{1}{2}-\frac{I}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} -1+i \\ 1-i \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & -1 \end{bmatrix} - (i)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-i & 0 & 1 \\ 0 & 1-i & -1 \\ -2 & 0 & -1-i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1-i & 0 & 1 & 0 \\ 0 & 1-i & -1 & 0 \\ -2 & 0 & -1-i & 0 \end{array}\right]$$

$$R_3 = R_3 + (1+i)\,R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1-i & 0 & 1 & 0 \\ 0 & 1-i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1-i & 0 & 1 \\ 0 & 1-i & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{1}{2} - \frac{i}{2}\right) t, v_2 = \left(\frac{1}{2} + \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{I}{2}\right) t \\ \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2} - \frac{i}{2}\right) t \\ \left(\frac{1}{2} + \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{I}{2}\right) t \\ \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} - \frac{i}{2} \\ \frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{I}{2}\right) t \\ \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} - \frac{i}{2} \\ \frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{I}{2}\right) t \\ \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -1-i \\ 1+i \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
i	1	1	No	$\begin{bmatrix} -\frac{1}{2}-\frac{i}{2} \\ \frac{1}{2}+\frac{i}{2} \\ 1 \end{bmatrix}$
$-i$	1	1	No	$\begin{bmatrix} -\frac{1}{2}+\frac{i}{2} \\ \frac{1}{2}-\frac{i}{2} \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^t \\ &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^t\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} \left(-\frac{1}{2}-\frac{i}{2}\right) \mathrm{e}^{it} \\ \left(\frac{1}{2}+\frac{i}{2}\right) \mathrm{e}^{it} \\ \mathrm{e}^{it} \end{bmatrix} + c_2 \begin{bmatrix} \left(-\frac{1}{2}+\frac{i}{2}\right) \mathrm{e}^{-it} \\ \left(\frac{1}{2}-\frac{i}{2}\right) \mathrm{e}^{-it} \\ \mathrm{e}^{-it} \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ \mathrm{e}^{t} \\ 0 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2}-\frac{i}{2}\right) c_1 \mathrm{e}^{it} + \left(-\frac{1}{2}+\frac{i}{2}\right) c_2 \mathrm{e}^{-it} \\ \left(\frac{1}{2}+\frac{i}{2}\right) c_1 \mathrm{e}^{it} + \left(\frac{1}{2}-\frac{i}{2}\right) c_2 \mathrm{e}^{-it} + c_3 \mathrm{e}^{t} \\ c_1 \mathrm{e}^{it} + c_2 \mathrm{e}^{-it} \end{bmatrix}$$

2.4.3 Maple step by step solution

Let's solve

$[x_1'(t) = x_1(t) + x_3(t), x_2'(t) = x_2(t) - x_3(t), x_3'(t) = -2x_1(t) - x_3(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & -1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & -1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & -1 \end{bmatrix}$$

- Rewrite the system as

$\underline{x}^{\rightarrow\prime}(t) = A \cdot \underline{x}^{\rightarrow}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ \frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} -\frac{1}{2}-\frac{\mathrm{I}}{2} \\ \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_{1} = \mathrm{e}^{t} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ \frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I}t} \cdot \begin{bmatrix} -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ \frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(t) - \mathrm{I}\sin(t)) \cdot \begin{bmatrix} -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ \frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} \left(-\frac{1}{2}+\frac{\mathrm{I}}{2}\right)(\cos(t) - \mathrm{I}\sin(t)) \\ \left(\frac{1}{2}-\frac{\mathrm{I}}{2}\right)(\cos(t) - \mathrm{I}\sin(t)) \\ \cos(t) - \mathrm{I}\sin(t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{x}^{\rightarrow}_{2}(t)=\left[\begin{array}{c}-\frac{\cos(t)}{2}+\frac{\sin(t)}{2}\\ \frac{\cos(t)}{2}-\frac{\sin(t)}{2}\\ \cos\left(t\right)\end{array}\right],\underline{x}^{\rightarrow}_{3}(t)=\left[\begin{array}{c}\frac{\cos(t)}{2}+\frac{\sin(t)}{2}\\ -\frac{\cos(t)}{2}-\frac{\sin(t)}{2}\\ -\sin\left(t\right)\end{array}\right]\right]$$

- General solution to the system of ODEs

$$\underline{x}^{\rightarrow}=c_1\underline{x}^{\rightarrow}_{1}+c_2\underline{x}^{\rightarrow}_{2}(t)+c_3\underline{x}^{\rightarrow}_{3}(t)$$

- Substitute solutions into the general solution

$$\underline{x}^{\rightarrow}=c_1\mathrm{e}^{t}\cdot\left[\begin{array}{c}0\\ 1\\ 0\end{array}\right]+\left[\begin{array}{c}c_3\left(\frac{\cos(t)}{2}+\frac{\sin(t)}{2}\right)+c_2\left(-\frac{\cos(t)}{2}+\frac{\sin(t)}{2}\right)\\ c_2\left(\frac{\cos(t)}{2}-\frac{\sin(t)}{2}\right)+c_3\left(-\frac{\cos(t)}{2}-\frac{\sin(t)}{2}\right)\\ c_2\cos\left(t\right)-c_3\sin\left(t\right)\end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c}x_1(t)\\ x_2(t)\\ x_3(t)\end{array}\right]=\left[\begin{array}{c}\frac{(-c_2+c_3)\cos(t)}{2}+\frac{(c_2+c_3)\sin(t)}{2}\\ \frac{(c_2-c_3)\cos(t)}{2}+\frac{(-c_2-c_3)\sin(t)}{2}+c_1\mathrm{e}^{t}\\ c_2\cos\left(t\right)-c_3\sin\left(t\right)\end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x_1(t)=\frac{(-c_2+c_3)\cos(t)}{2}+\frac{(c_2+c_3)\sin(t)}{2},x_2(t)=\frac{(c_2-c_3)\cos(t)}{2}+\frac{(-c_2-c_3)\sin(t)}{2}+c_1\mathrm{e}^{t},x_3(t)=c_2\cos\left(t\right)-c\right.$$

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 66

```
dsolve([diff(x__1(t),t)=1*x__1(t)-0*x__2(t)+1*x__3(t),diff(x__2(t),t)=0*x__1(t)+1*x__2(t)-1*
```

$$x_1(t)=\frac{c_3\sin\left(t\right)}{2}-\frac{c_2\cos\left(t\right)}{2}-\frac{c_2\sin\left(t\right)}{2}-\frac{c_3\cos\left(t\right)}{2}$$
$$x_2(t)=\frac{c_2\sin\left(t\right)}{2}-\frac{c_3\sin\left(t\right)}{2}+\frac{c_2\cos\left(t\right)}{2}+\frac{c_3\cos\left(t\right)}{2}+c_1\mathrm{e}^{t}$$
$$x_3(t)=c_2\sin\left(t\right)+c_3\cos\left(t\right)$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 67

```
DSolve[{x1'[t]==1*x1[t]-0*x2[t]+1*x3[t],x2'[t]==0*x1[t]+1*x2[t]-1*x3[t],x3'[t]==-2*x1[t]-0*x
```

$$
\begin{aligned}
\mathrm{x1}(t) &\to c_1\cos(t) + (c_1+c_3)\sin(t) \\
\mathrm{x2}(t) &\to (c_1+c_2)e^t - c_1\cos(t) - (c_1+c_3)\sin(t) \\
\mathrm{x3}(t) &\to c_3\cos(t) - (2c_1+c_3)\sin(t)
\end{aligned}
$$

2.5 problem 5

Internal problem ID [1840]
Internal file name [`OUTPUT/1841_Sunday_June_05_2022_02_34_53_AM_79439982/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.9, Systems of differential equations. Complex roots. Page 344
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = x_1(t) - x_2(t)$$
$$x_2'(t) = 5x_1(t) - 3x_2(t)$$

With initial conditions

$$[x_1(0) = 1, x_2(0) = 2]$$

2.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-t}\cos(t) + 2\,e^{-t}\sin(t) & -e^{-t}\sin(t) \\ 5\,e^{-t}\sin(t) & e^{-t}\cos(t) - 2\,e^{-t}\sin(t) \end{bmatrix}$$
$$= \begin{bmatrix} e^{-t}(\cos(t) + 2\sin(t)) & -e^{-t}\sin(t) \\ 5\,e^{-t}\sin(t) & e^{-t}(\cos(t) - 2\sin(t)) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^{-t}(\cos(t) + 2\sin(t)) & -e^{-t}\sin(t) \\ 5\,e^{-t}\sin(t) & e^{-t}(\cos(t) - 2\sin(t)) \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}(\cos(t) + 2\sin(t)) - 2\,e^{-t}\sin(t) \\ 5\,e^{-t}\sin(t) + 2\,e^{-t}(\cos(t) - 2\sin(t)) \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}\cos(t) \\ e^{-t}(\sin(t) + 2\cos(t)) \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

2.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -1 \\ 5 & -3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 2\lambda + 2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1 + i$$
$$\lambda_2 = -1 - i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-1 - i$	1	complex eigenvalue
$-1 + i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1 - i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} - (-1 - i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2+i & -1 \\ 5 & -2+i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2+i & -1 & 0 \\ 5 & -2+i & 0 \end{array}\right]$$

$$R_2 = R_2 + (-2 + i)\, R_1 \Longrightarrow \left[\begin{array}{cc|c} 2+i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2+i & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{2}{5} - \frac{i}{5}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{2}{5} - \frac{I}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{2}{5} - \frac{i}{5}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{2}{5} - \frac{I}{5}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{5} - \frac{i}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{2}{5} - \frac{I}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{5} - \frac{i}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{2}{5} - \frac{I}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 2 - i \\ 5 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1 + i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} - (-1 + i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 - i & -1 \\ 5 & -2 - i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 - i & -1 & 0 \\ 5 & -2 - i & 0 \end{array}\right]$$

$$R_2 = R_2 + (-2 - i) R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 - i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2-i & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{2}{5} + \frac{i}{5}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{2}{5} + \frac{I}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{2}{5} + \frac{i}{5}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{2}{5} + \frac{I}{5}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{5} + \frac{i}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{2}{5} + \frac{I}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{5} + \frac{i}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{2}{5} + \frac{I}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 2+i \\ 5 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity algebraic m	multiplicity geometric k	defective?	eigenvectors
$-1+i$	1	1	No	$\begin{bmatrix} \frac{2}{5}+\frac{i}{5} \\ 1 \end{bmatrix}$
$-1-i$	1	1	No	$\begin{bmatrix} \frac{2}{5}-\frac{i}{5} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \left(\frac{2}{5}+\frac{i}{5}\right) \mathrm{e}^{(-1+i)t} \\ \mathrm{e}^{(-1+i)t} \end{bmatrix} + c_2 \begin{bmatrix} \left(\frac{2}{5}-\frac{i}{5}\right) \mathrm{e}^{(-1-i)t} \\ \mathrm{e}^{(-1-i)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{2}{5}+\frac{i}{5}\right) c_1 \mathrm{e}^{(-1+i)t} + \left(\frac{2}{5}-\frac{i}{5}\right) c_2 \mathrm{e}^{(-1-i)t} \\ c_1 \mathrm{e}^{(-1+i)t} + c_2 \mathrm{e}^{(-1-i)t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = 1 \\ x_2(0) = 2 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \left(\frac{2}{5}+\frac{i}{5}\right) c_1 + \left(\frac{2}{5}-\frac{i}{5}\right) c_2 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 1 - \frac{i}{2} \\ c_2 = 1 + \frac{i}{2} \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{e^{(-1-i)t}}{2} + \frac{e^{(-1+i)t}}{2} \\ \left(1-\frac{i}{2}\right) e^{(-1+i)t} + \left(1+\frac{i}{2}\right) e^{(-1-i)t} \end{bmatrix}$$

The following is the phase plot of the system.

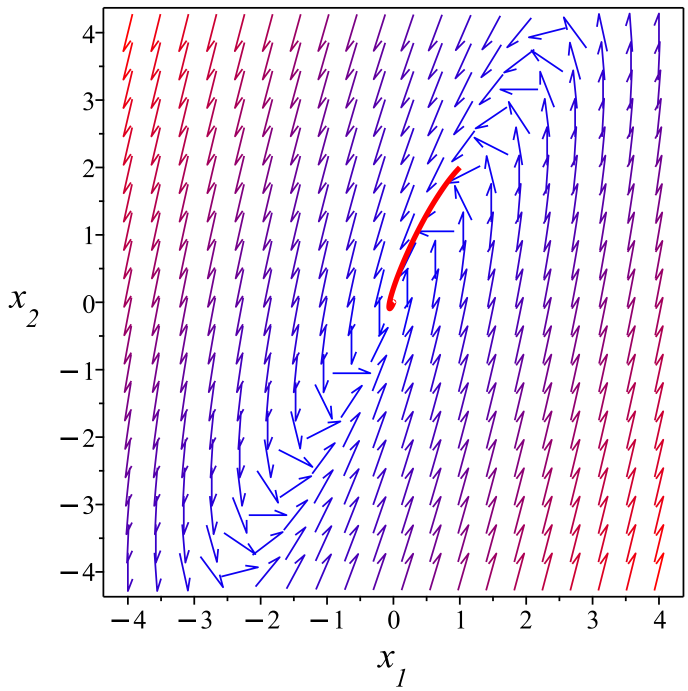


Figure 10: Phase plot

The following are plots of each solution.

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 30

```
dsolve([diff(x__1(t),t) = x__1(t)-x__2(t), diff(x__2(t),t) = 5*x__1(t)-3*x__2(t), x__1(0) =
```

$$x_1(t) = e^{-t} \cos(t)$$
$$x_2(t) = -e^{-t}(-2\cos(t) - \sin(t))$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 29

```
DSolve[{x1'[t]==1*x1[t]-1*x2[t],x2'[t]==5*x1[t]-3*x2[t]},{x1[0]==1,x2[0]==2},{x1[t],x2[t]},t
```

$$\text{x1}(t) \to e^{-t}\cos(t)$$
$$\text{x2}(t) \to e^{-t}(\sin(t) + 2\cos(t))$$

2.6 problem 6

Internal problem ID [1841]
Internal file name [`OUTPUT/1842_Sunday_June_05_2022_02_34_55_AM_35798951/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.9, Systems of differential equations. Complex roots. Page 344
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 3x_1(t) - 2x_2(t) \\
x_2'(t) &= 4x_1(t) - x_2(t)
\end{aligned}
$$

With initial conditions

$$
[x_1(0) = 1, x_2(0) = 5]
$$

2.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} \mathrm{e}^t \cos(2t) + \mathrm{e}^t \sin(2t) & -\mathrm{e}^t \sin(2t) \\ 2\,\mathrm{e}^t \sin(2t) & \mathrm{e}^t \cos(2t) - \mathrm{e}^t \sin(2t) \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^t(\cos(2t) + \sin(2t)) & -\mathrm{e}^t \sin(2t) \\ 2\,\mathrm{e}^t \sin(2t) & \mathrm{e}^t(\cos(2t) - \sin(2t)) \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \mathrm{e}^t(\cos(2t)+\sin(2t)) & -\mathrm{e}^t\sin(2t) \\ 2\,\mathrm{e}^t\sin(2t) & \mathrm{e}^t(\cos(2t)-\sin(2t)) \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^t(\cos(2t)+\sin(2t)) - 5\,\mathrm{e}^t\sin(2t) \\ 2\,\mathrm{e}^t\sin(2t) + 5\,\mathrm{e}^t(\cos(2t)-\sin(2t)) \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^t(\cos(2t)-4\sin(2t)) \\ \mathrm{e}^t(-3\sin(2t)+5\cos(2t)) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

2.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 2\lambda + 5 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1 + 2i$$
$$\lambda_2 = 1 - 2i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$1 + 2i$	1	complex eigenvalue
$1 - 2i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1 - 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} - (1 - 2i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 + 2i & -2 \\ 4 & -2 + 2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 + 2i & -2 & 0 \\ 4 & -2 + 2i & 0 \end{array}\right]$$

$$R_2 = R_2 + (-1 + i)\, R_1 \implies \left[\begin{array}{cc|c} 2 + 2i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 + 2i & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} - \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} - \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 1 - i \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1 + 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} - (1 + 2i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 - 2i & -2 \\ 4 & -2 - 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 - 2i & -2 & 0 \\ 4 & -2 - 2i & 0 \end{array}\right]$$

$$R_2 = R_2 + (-1 - i) R_1 \implies \left[\begin{array}{cc|c} 2 - 2i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2-2i & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} + \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} + \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$1+2i$	1	1	No	$\begin{bmatrix} \frac{1}{2}+\frac{i}{2} \\ 1 \end{bmatrix}$
$1-2i$	1	1	No	$\begin{bmatrix} \frac{1}{2}-\frac{i}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \left(\frac{1}{2}+\frac{i}{2}\right) e^{(1+2i)t} \\ e^{(1+2i)t} \end{bmatrix} + c_2 \begin{bmatrix} \left(\frac{1}{2}-\frac{i}{2}\right) e^{(1-2i)t} \\ e^{(1-2i)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}+\frac{i}{2}\right) c_1 e^{(1+2i)t} + \left(\frac{1}{2}-\frac{i}{2}\right) c_2 e^{(1-2i)t} \\ c_1 e^{(1+2i)t} + c_2 e^{(1-2i)t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = 1 \\ x_2(0) = 5 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}+\frac{i}{2}\right) c_1 + \left(\frac{1}{2}-\frac{i}{2}\right) c_2 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = \frac{5}{2}+\frac{3i}{2} \\ c_2 = \frac{5}{2}-\frac{3i}{2} \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}+2i\right) e^{(1+2i)t} + \left(\frac{1}{2}-2i\right) e^{(1-2i)t} \\ \left(\frac{5}{2}+\frac{3i}{2}\right) e^{(1+2i)t} + \left(\frac{5}{2}-\frac{3i}{2}\right) e^{(1-2i)t} \end{bmatrix}$$

The following is the phase plot of the system.

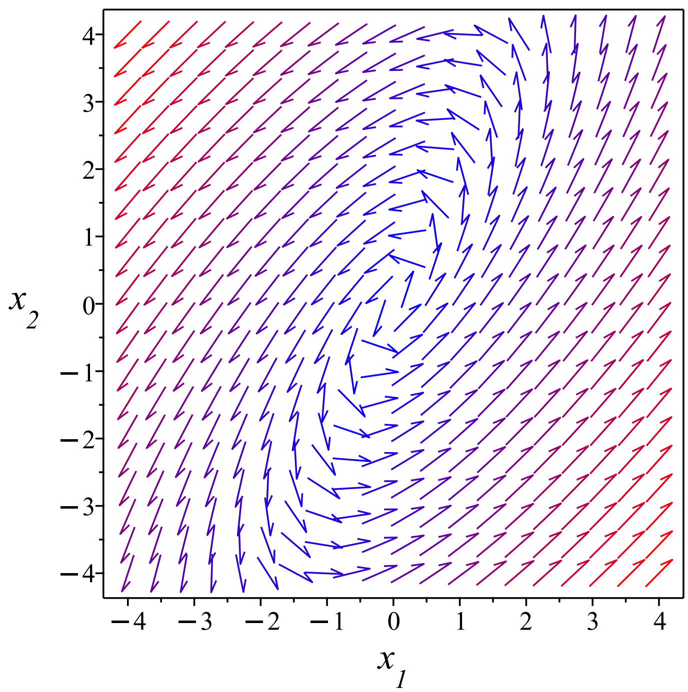


Figure 11: Phase plot

The following are plots of each solution.

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
dsolve([diff(x__1(t),t) = 3*x__1(t)-2*x__2(t), diff(x__2(t),t) = 4*x__1(t)-x__2(t), x__1(0)
```

$$x_1(t) = e^t(-4\sin(2t) + \cos(2t))$$
$$x_2(t) = -e^t(-5\cos(2t) + 3\sin(2t))$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 40

```
DSolve[{x1'[t]==3*x1[t]-2*x2[t],x2'[t]==4*x1[t]-1*x2[t]},{x1[0]==1,x2[0]==5},{x1[t],x2[t]},t
```

$$
\begin{aligned}
\mathrm{x1}(t) &\to e^t(\cos(2t) - 4\sin(2t)) \\
\mathrm{x2}(t) &\to e^t(5\cos(2t) - 3\sin(2t))
\end{aligned}
$$

2.7 problem 7

Internal problem ID [1842]
Internal file name [`OUTPUT/1843_Sunday_June_05_2022_02_34_57_AM_9880818/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.9, Systems of differential equations. Complex roots. Page 344
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = -3x_1(t) + 2x_3(t)$$
$$x_2'(t) = x_1(t) - x_2(t)$$
$$x_3'(t) = -2x_1(t) - x_2(t)$$

With initial conditions

$$[x_1(0) = 0, x_2(0) = -1, x_3(0) = -2]$$

2.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{2\,\mathrm{e}^{-2t}}{3} + \frac{\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} - \frac{2\sqrt{2}\,\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right)}{3} & -\frac{\sqrt{2}\,\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right)}{3} + \frac{2\,\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} - \frac{2\,\mathrm{e}^{-2t}}{3} & \frac{2\,\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} + 2\sqrt{2} \\ \frac{2\,\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} + \frac{\sqrt{2}\,\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right)}{6} - \frac{2\,\mathrm{e}^{-2t}}{3} & \frac{2\,\mathrm{e}^{-2t}}{3} + \frac{\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} + \frac{\sqrt{2}\,\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right)}{3} & -\frac{2\,\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} + \sqrt{} \\ -\frac{\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} + \frac{\mathrm{e}^{-2t}}{3} - \frac{5\sqrt{2}\,\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right)}{6} & -\frac{2\sqrt{2}\,\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right)}{3} + \frac{\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} - \frac{\mathrm{e}^{-2t}}{3} & \frac{4\,\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} + \sqrt{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} \frac{2\,\mathrm{e}^{-2t}}{3} + \frac{\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} - \frac{2\sqrt{2}\,\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right)}{3} & -\frac{\sqrt{2}\,\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right)}{3} + \frac{2\,\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} - \frac{2\,\mathrm{e}^{-2t}}{3} & \frac{2\,\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} + 2 \\ \frac{2\,\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} + \frac{\sqrt{2}\,\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right)}{6} - \frac{2\,\mathrm{e}^{-2t}}{3} & \frac{2\,\mathrm{e}^{-2t}}{3} + \frac{\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} + \frac{\sqrt{2}\,\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right)}{3} & -\frac{2\,\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} + \\ -\frac{\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} + \frac{\mathrm{e}^{-2t}}{3} - \frac{5\sqrt{2}\,\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right)}{6} & -\frac{2\sqrt{2}\,\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right)}{3} + \frac{\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} - \frac{\mathrm{e}^{-2t}}{3} & \frac{4\,\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} + \end{bmatrix} \\ &= \begin{bmatrix} -2\,\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right) - \sqrt{2}\,\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right) + 2\,\mathrm{e}^{-2t} \\ -2\,\mathrm{e}^{-2t} + \mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right) - \sqrt{2}\,\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right) \\ -3\,\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right) + \mathrm{e}^{-2t} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

2.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -3-\lambda & 0 & 2 \\ 1 & -1-\lambda & 0 \\ -2 & -1 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + 4\lambda^2 + 7\lambda + 6 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned} \lambda_1 &= -2 \\ \lambda_2 &= -1 + i\sqrt{2} \\ \lambda_3 &= -1 - i\sqrt{2} \end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-1 + i\sqrt{2}$	1	complex eigenvalue
-2	1	real eigenvalue
$-1 - i\sqrt{2}$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 2 \\ 1 & 1 & 0 \\ -2 & -1 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & -1 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ -2 & -1 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t, v_2 = -2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ -2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Let $t=1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1 - i\sqrt{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix} - \left(-1 - i\sqrt{2}\right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 + i\sqrt{2} & 0 & 2 \\ 1 & i\sqrt{2} & 0 \\ -2 & -1 & 1 + i\sqrt{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 + i\sqrt{2} & 0 & 2 & 0 \\ 1 & i\sqrt{2} & 0 & 0 \\ -2 & -1 & 1 + i\sqrt{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{-2 + i\sqrt{2}} \Longrightarrow \left[\begin{array}{ccc|c} -2 + i\sqrt{2} & 0 & 2 & 0 \\ 0 & i\sqrt{2} & -\frac{2}{-2+i\sqrt{2}} & 0 \\ -2 & -1 & 1 + i\sqrt{2} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{2R_1}{-2 + i\sqrt{2}} \Longrightarrow \left[\begin{array}{ccc|c} -2 + i\sqrt{2} & 0 & 2 & 0 \\ 0 & i\sqrt{2} & -\frac{2}{-2+i\sqrt{2}} & 0 \\ 0 & -1 & -\frac{\sqrt{2}}{2i+\sqrt{2}} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{i\sqrt{2}\,R_2}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 + i\sqrt{2} & 0 & 2 & 0 \\ 0 & i\sqrt{2} & -\frac{2}{-2+i\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2+i\sqrt{2} & 0 & 2 \\ 0 & i\sqrt{2} & -\frac{2}{-2+i\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{-2+i\sqrt{2}}, v_2 = -\frac{t}{1+i\sqrt{2}}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2t}{-2+I\sqrt{2}} \\ -\frac{t}{1+I\sqrt{2}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2t}{-2+i\sqrt{2}} \\ -\frac{t}{1+i\sqrt{2}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2t}{-2+I\sqrt{2}} \\ -\frac{t}{1+I\sqrt{2}} \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{2}{-2+i\sqrt{2}} \\ -\frac{1}{1+i\sqrt{2}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2t}{-2+I\sqrt{2}} \\ -\frac{t}{1+I\sqrt{2}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{-2+i\sqrt{2}} \\ -\frac{1}{1+i\sqrt{2}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{2t}{-2+I\sqrt{2}} \\ -\frac{t}{1+I\sqrt{2}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{-2+i\sqrt{2}} \\ -\frac{1}{1+i\sqrt{2}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = -1 + i\sqrt{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix} - \left(-1 + i\sqrt{2}\right)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 - i\sqrt{2} & 0 & 2 \\ 1 & -i\sqrt{2} & 0 \\ -2 & -1 & 1 - i\sqrt{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 - i\sqrt{2} & 0 & 2 & 0 \\ 1 & -i\sqrt{2} & 0 & 0 \\ -2 & -1 & 1 - i\sqrt{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{-2 - i\sqrt{2}} \Longrightarrow \left[\begin{array}{ccc|c} -2 - i\sqrt{2} & 0 & 2 & 0 \\ 0 & -i\sqrt{2} & \frac{2}{2+i\sqrt{2}} & 0 \\ -2 & -1 & 1 - i\sqrt{2} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{2R_1}{-2 - i\sqrt{2}} \Longrightarrow \left[\begin{array}{ccc|c} -2 - i\sqrt{2} & 0 & 2 & 0 \\ 0 & -i\sqrt{2} & \frac{2}{2+i\sqrt{2}} & 0 \\ 0 & -1 & \frac{\sqrt{2}}{2i-\sqrt{2}} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{i\sqrt{2}\,R_2}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 - i\sqrt{2} & 0 & 2 & 0 \\ 0 & -i\sqrt{2} & \frac{2}{2+i\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 - i\sqrt{2} & 0 & 2 \\ 0 & -i\sqrt{2} & \frac{2}{2+i\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{2+i\sqrt{2}}, v_2 = \frac{t}{-1+i\sqrt{2}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2t}{2+\mathrm{I}\sqrt{2}} \\ \frac{t}{-1+\mathrm{I}\sqrt{2}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{2+i\sqrt{2}} \\ \frac{t}{-1+i\sqrt{2}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2t}{2+\mathrm{I}\sqrt{2}} \\ \frac{t}{-1+\mathrm{I}\sqrt{2}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{2+i\sqrt{2}} \\ \frac{1}{-1+i\sqrt{2}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2t}{2+\mathrm{I}\sqrt{2}} \\ \frac{t}{-1+\mathrm{I}\sqrt{2}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{2+i\sqrt{2}} \\ \frac{1}{-1+i\sqrt{2}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{2t}{2+\mathrm{I}\sqrt{2}} \\ \frac{t}{-1+\mathrm{I}\sqrt{2}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{2+i\sqrt{2}} \\ \frac{1}{-1+i\sqrt{2}} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity algebraic m	multiplicity geometric k	defective?	eigenvectors
-2	1	1	No	$\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$
$-1+i\sqrt{2}$	1	1	No	$\begin{bmatrix} \frac{2}{2+i\sqrt{2}} \\ \frac{i(-2+i\sqrt{2})\sqrt{2}}{6} \\ 1 \end{bmatrix}$
$-1-i\sqrt{2}$	1	1	No	$\begin{bmatrix} \frac{2}{2-i\sqrt{2}} \\ -\frac{i(-2-i\sqrt{2})\sqrt{2}}{6} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-2t} \\ &= \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} e^{-2t} \end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} 2\,\mathrm{e}^{-2t} \\ -2\,\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{2\,\mathrm{e}^{(-1+i\sqrt{2})t}}{2+i\sqrt{2}} \\ \frac{i\mathrm{e}^{(-1+i\sqrt{2})t}(-2+i\sqrt{2})\sqrt{2}}{6} \\ \mathrm{e}^{(-1+i\sqrt{2})t} \end{bmatrix} + c_3 \begin{bmatrix} \frac{2\,\mathrm{e}^{(-1-i\sqrt{2})t}}{2-i\sqrt{2}} \\ -\frac{i\mathrm{e}^{(-1-i\sqrt{2})t}(-2-i\sqrt{2})\sqrt{2}}{6} \\ \mathrm{e}^{(-1-i\sqrt{2})t} \end{bmatrix}$$

Which becomes

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{\left(2+i\sqrt{2}\right)c_3 \mathrm{e}^{-\left(1+i\sqrt{2}\right)t}}{3} + \frac{c_2\left(2-i\sqrt{2}\right)\mathrm{e}^{\left(-1+i\sqrt{2}\right)t}}{3} + 2c_1 \mathrm{e}^{-2t} \\ \frac{\left(-1+i\sqrt{2}\right)c_3 \mathrm{e}^{-\left(1+i\sqrt{2}\right)t}}{3} + \frac{c_2\left(-1-i\sqrt{2}\right)\mathrm{e}^{\left(-1+i\sqrt{2}\right)t}}{3} - 2c_1 \mathrm{e}^{-2t} \\ c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{\left(-1+i\sqrt{2}\right)t} + c_3 \mathrm{e}^{-\left(1+i\sqrt{2}\right)t} \end{bmatrix}
$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$
\begin{bmatrix} x_1(0) = 0 \\ x_2(0) = -1 \\ x_3(0) = -2 \end{bmatrix} \tag{1}
$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$
\begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{i(c_3-c_2)\sqrt{2}}{3} + 2c_1 + \frac{2c_2}{3} + \frac{2c_3}{3} \\ \frac{i(c_3-c_2)\sqrt{2}}{3} - 2c_1 - \frac{c_2}{3} - \frac{c_3}{3} \\ c_1 + c_2 + c_3 \end{bmatrix}
$$

Solving for the constants of integrations gives

$$
\begin{bmatrix} c_1 = 1 \\ c_2 = -\frac{3}{2} \\ c_3 = -\frac{3}{2} \end{bmatrix}
$$

Substituting these constants back in original solution in Eq. (1) gives

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{\left(2+i\sqrt{2}\right)\mathrm{e}^{-\left(1+i\sqrt{2}\right)t}}{2} - \frac{\left(2-i\sqrt{2}\right)\mathrm{e}^{\left(-1+i\sqrt{2}\right)t}}{2} + 2\,\mathrm{e}^{-2t} \\ -\frac{\left(-1+i\sqrt{2}\right)\mathrm{e}^{-\left(1+i\sqrt{2}\right)t}}{2} - \frac{\left(-1-i\sqrt{2}\right)\mathrm{e}^{\left(-1+i\sqrt{2}\right)t}}{2} - 2\,\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} - \frac{3\,\mathrm{e}^{\left(-1+i\sqrt{2}\right)t}}{2} - \frac{3\,\mathrm{e}^{-\left(1+i\sqrt{2}\right)t}}{2} \end{bmatrix}
$$

The following are plots of each solution against another.

The following are plots of each solution.

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 95

```
dsolve([diff(x__1(t),t) = -3*x__1(t)+2*x__3(t), diff(x__2(t),t) = x__1(t)-x__2(t), diff(x__3
```

$$x_1(t) = 2\,e^{-2t} - \sqrt{2}\,e^{-t}\sin\left(\sqrt{2}\,t\right) - 2\,e^{-t}\cos\left(\sqrt{2}\,t\right)$$
$$x_2(t) = -2\,e^{-2t} + e^{-t}\cos\left(\sqrt{2}\,t\right) - \sqrt{2}\,e^{-t}\sin\left(\sqrt{2}\,t\right)$$
$$x_3(t) = e^{-2t} - 3\,e^{-t}\cos\left(\sqrt{2}\,t\right)$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 109

```
DSolve[{x1'[t]==-3*x1[t]-0*x2[t]+2*x3[t],x2'[t]==1*x1[t]-1*x2[t]-0*x3[t],x3'[t]==-2*x1[t]-1*
```

$$\text{x1}(t) \to -e^{-2t}\left(\sqrt{2}e^t\sin\left(\sqrt{2}t\right) + 2e^t\cos\left(\sqrt{2}t\right) - 2\right)$$
$$\text{x2}(t) \to e^{-2t}\left(-\sqrt{2}e^t\sin\left(\sqrt{2}t\right) + e^t\cos\left(\sqrt{2}t\right) - 2\right)$$
$$\text{x3}(t) \to e^{-2t}\left(1 - 3e^t\cos\left(\sqrt{2}t\right)\right)$$

2.8 problem 8

Internal problem ID [1843]
Internal file name [`OUTPUT/1844_Sunday_June_05_2022_02_35_01_AM_71795827/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.9, Systems of differential equations. Complex roots. Page 344
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 2x_2(t) \\
x_2'(t) &= -2x_1(t) \\
x_3'(t) &= -3x_4(t) \\
x_4'(t) &= 3x_3(t)
\end{aligned}
$$

With initial conditions

$$[x_1(0) = 1, x_2(0) = 1, x_3(0) = 1, x_4(0) = 0]$$

2.8.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(2t) & \sin(2t) & 0 & 0 \\ -\sin(2t) & \cos(2t) & 0 & 0 \\ 0 & 0 & \cos(3t) & -\sin(3t) \\ 0 & 0 & \sin(3t) & \cos(3t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} \cos(2t) & \sin(2t) & 0 & 0 \\ -\sin(2t) & \cos(2t) & 0 & 0 \\ 0 & 0 & \cos(3t) & -\sin(3t) \\ 0 & 0 & \sin(3t) & \cos(3t) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \cos(2t) + \sin(2t) \\ \cos(2t) - \sin(2t) \\ \cos(3t) \\ \sin(3t) \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

2.8.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 3 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 2 & 0 & 0 \\ -2 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & -3 \\ 0 & 0 & 3 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^4 + 13\lambda^2 + 36 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned} \lambda_1 &= 2i \\ \lambda_2 &= -2i \\ \lambda_3 &= 3i \\ \lambda_4 &= -3i \end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$2i$	1	complex eigenvalue
$-2i$	1	complex eigenvalue
$-3i$	1	complex eigenvalue
$3i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 3 & 0 \end{bmatrix} - (-3i)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3i & 2 & 0 & 0 \\ -2 & 3i & 0 & 0 \\ 0 & 0 & 3i & -3 \\ 0 & 0 & 3 & 3i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 3i & 2 & 0 & 0 & 0 \\ -2 & 3i & 0 & 0 & 0 \\ 0 & 0 & 3i & -3 & 0 \\ 0 & 0 & 3 & 3i & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2iR_1}{3} \Longrightarrow \left[\begin{array}{cccc|c} 3i & 2 & 0 & 0 & 0 \\ 0 & \frac{5i}{3} & 0 & 0 & 0 \\ 0 & 0 & 3i & -3 & 0 \\ 0 & 0 & 3 & 3i & 0 \end{array}\right]$$

$$R_4 = iR_3 + R_4 \Longrightarrow \left[\begin{array}{cccc|c} 3i & 2 & 0 & 0 & 0 \\ 0 & \frac{5i}{3} & 0 & 0 & 0 \\ 0 & 0 & 3i & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3i & 2 & 0 & 0 \\ 0 & \frac{5i}{3} & 0 & 0 \\ 0 & 0 & 3i & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = 0, v_3 = -it\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ 0 \\ \text{-I}t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ 0 \\ \text{-I}t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ -i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ 0 \\ \text{-I}t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 3 & 0 \end{bmatrix} - (-2i) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2i & 2 & 0 & 0 \\ -2 & 2i & 0 & 0 \\ 0 & 0 & 2i & -3 \\ 0 & 0 & 3 & 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 2i & 2 & 0 & 0 & 0 \\ -2 & 2i & 0 & 0 & 0 \\ 0 & 0 & 2i & -3 & 0 \\ 0 & 0 & 3 & 2i & 0 \end{array}\right]$$

$$R_2 = -iR_1 + R_2 \implies \left[\begin{array}{cccc|c} 2i & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2i & -3 & 0 \\ 0 & 0 & 3 & 2i & 0 \end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{cccc|c} 2i & 2 & 0 & 0 & 0 \\ 0 & 0 & 2i & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2i & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{3iR_2}{2} \implies \left[\begin{array}{cccc|c} 2i & 2 & 0 & 0 & 0 \\ 0 & 0 & 2i & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{5i}{2} & 0 \end{array}\right]$$

Since the current pivot $A(3,4)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 3 and row 4 gives

$$\left[\begin{array}{cccc|c} 2i & 2 & 0 & 0 & 0 \\ 0 & 0 & 2i & -3 & 0 \\ 0 & 0 & 0 & -\frac{5i}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2i & 2 & 0 & 0 \\ 0 & 0 & 2i & -3 \\ 0 & 0 & 0 & -\frac{5i}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3, v_4\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it, v_3 = 0, v_4 = 0\}$

Hence the solution is

$$\begin{bmatrix} \mathrm{I}t \\ t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} it \\ t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \mathrm{I}t \\ t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} i \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \mathrm{I}t \\ t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} i \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 3 & 0 \end{bmatrix} - (2i)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2i & 2 & 0 & 0 \\ -2 & -2i & 0 & 0 \\ 0 & 0 & -2i & -3 \\ 0 & 0 & 3 & -2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} -2i & 2 & 0 & 0 & 0 \\ -2 & -2i & 0 & 0 & 0 \\ 0 & 0 & -2i & -3 & 0 \\ 0 & 0 & 3 & -2i & 0 \end{array}\right]$$

$$R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{cccc|c} -2i & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2i & -3 & 0 \\ 0 & 0 & 3 & -2i & 0 \end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{cccc|c} -2i & 2 & 0 & 0 & 0 \\ 0 & 0 & -2i & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & -2i & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{3iR_2}{2} \Longrightarrow \left[\begin{array}{cccc|c} -2i & 2 & 0 & 0 & 0 \\ 0 & 0 & -2i & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{5i}{2} & 0 \end{array}\right]$$

Since the current pivot $A(3,4)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 3 and row 4 gives

$$\left[\begin{array}{cccc|c} -2i & 2 & 0 & 0 & 0 \\ 0 & 0 & -2i & -3 & 0 \\ 0 & 0 & 0 & \frac{5i}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2i & 2 & 0 & 0 \\ 0 & 0 & -2i & -3 \\ 0 & 0 & 0 & \frac{5i}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3, v_4\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it, v_3 = 0, v_4 = 0\}$

Hence the solution is

$$\begin{bmatrix} \text{-I}t \\ t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -it \\ t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \text{-I}t \\ t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \text{-I}t \\ t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -i \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_4 = 3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 3 & 0 \end{bmatrix} - (3i)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3i & 2 & 0 & 0 \\ -2 & -3i & 0 & 0 \\ 0 & 0 & -3i & -3 \\ 0 & 0 & 3 & -3i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} -3i & 2 & 0 & 0 & 0 \\ -2 & -3i & 0 & 0 & 0 \\ 0 & 0 & -3i & -3 & 0 \\ 0 & 0 & 3 & -3i & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{2iR_1}{3} \Longrightarrow \left[\begin{array}{cccc|c} -3i & 2 & 0 & 0 & 0 \\ 0 & -\frac{5i}{3} & 0 & 0 & 0 \\ 0 & 0 & -3i & -3 & 0 \\ 0 & 0 & 3 & -3i & 0 \end{array}\right]$$

$$R_4 = -iR_3 + R_4 \Longrightarrow \left[\begin{array}{cccc|c} -3i & 2 & 0 & 0 & 0 \\ 0 & -\frac{5i}{3} & 0 & 0 & 0 \\ 0 & 0 & -3i & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3i & 2 & 0 & 0 \\ 0 & -\frac{5i}{3} & 0 & 0 \\ 0 & 0 & -3i & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = 0, v_3 = it\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ 0 \\ \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ 0 \\ \mathrm{I}t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ 0 \\ \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$2i$	1	1	No	$\begin{bmatrix} -i \\ 1 \\ 0 \\ 0 \end{bmatrix}$
$-2i$	1	1	No	$\begin{bmatrix} i \\ 1 \\ 0 \\ 0 \end{bmatrix}$
$3i$	1	1	No	$\begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix}$
$-3i$	1	1	No	$\begin{bmatrix} 0 \\ 0 \\ -i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t) + c_4\vec{x}_4(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = c_1 \begin{bmatrix} -i\mathrm{e}^{2it} \\ \mathrm{e}^{2it} \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} i\mathrm{e}^{-2it} \\ \mathrm{e}^{-2it} \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ i\mathrm{e}^{3it} \\ \mathrm{e}^{3it} \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 0 \\ -i\mathrm{e}^{-3it} \\ \mathrm{e}^{-3it} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} -i\left(c_1 e^{2it} - c_2 e^{-2it}\right) \\ c_1 e^{2it} + c_2 e^{-2it} \\ -i\left(c_4 e^{-3it} - c_3 e^{3it}\right) \\ c_3 e^{3it} + c_4 e^{-3it} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = 1 \\ x_2(0) = 1 \\ x_3(0) = 1 \\ x_4(0) = 0 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -i(c_1 - c_2) \\ c_1 + c_2 \\ i(c_3 - c_4) \\ c_3 + c_4 \end{bmatrix}$$

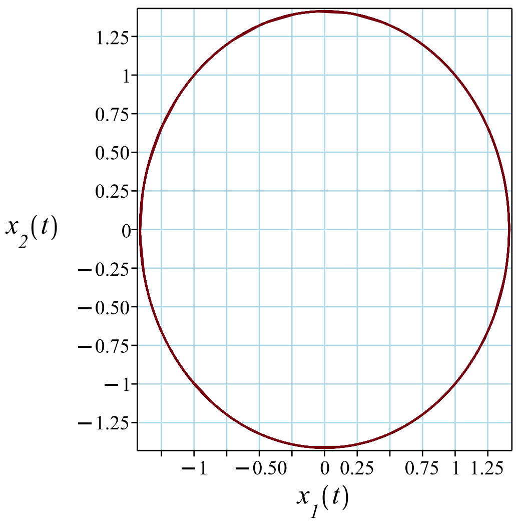
Solving for the constants of integrations gives

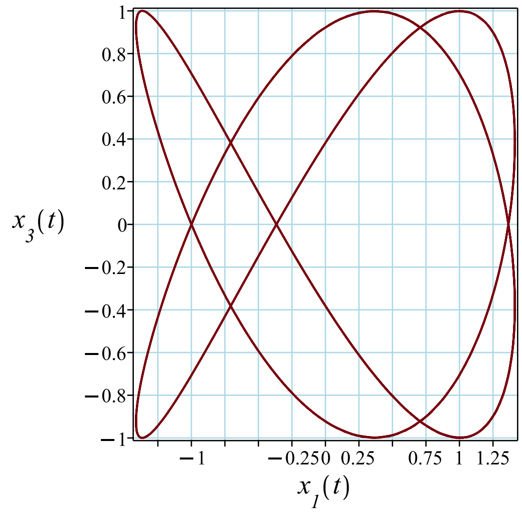
$$\begin{bmatrix} c_1 = \frac{1}{2} + \frac{i}{2} \\ c_2 = \frac{1}{2} - \frac{i}{2} \\ c_3 = -\frac{i}{2} \\ c_4 = \frac{i}{2} \end{bmatrix}$$

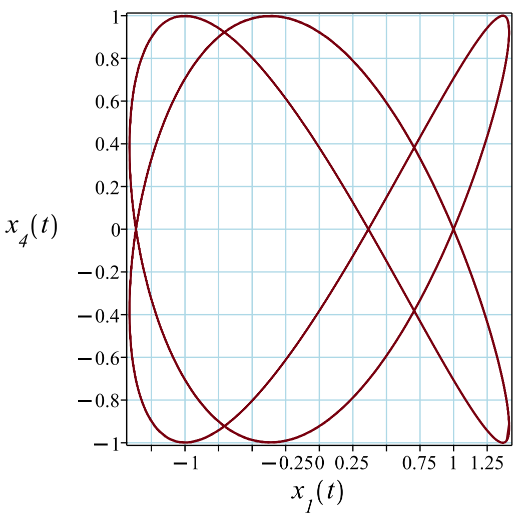
Substituting these constants back in original solution in Eq. (1) gives

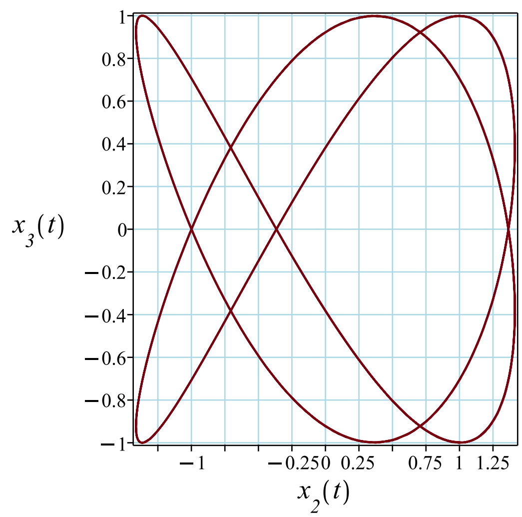
$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} -i\left(\left(\frac{1}{2} + \frac{i}{2}\right) e^{2it} + \left(-\frac{1}{2} + \frac{i}{2}\right) e^{-2it}\right) \\ \left(\frac{1}{2} + \frac{i}{2}\right) e^{2it} + \left(\frac{1}{2} - \frac{i}{2}\right) e^{-2it} \\ -i\left(\frac{i e^{-3it}}{2} + \frac{i e^{3it}}{2}\right) \\ -\frac{i e^{3it}}{2} + \frac{i e^{-3it}}{2} \end{bmatrix}$$

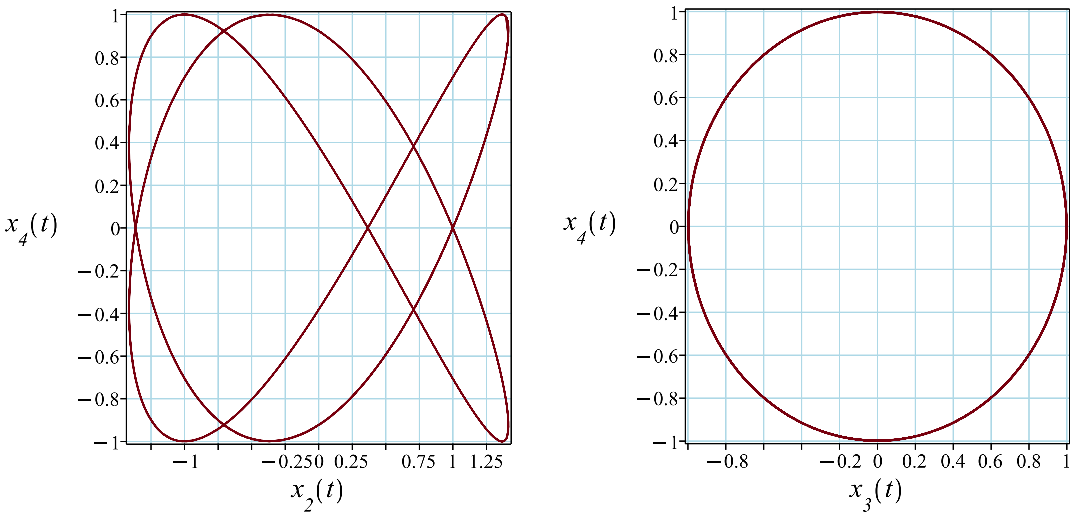
The following are plots of each solution against another.











The following are plots of each solution.

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 42

```
dsolve([diff(x__1(t),t) = 2*x__2(t), diff(x__2(t),t) = -2*x__1(t), diff(x__3(t),t) = -3*x__4
```

$$x_1(t) = \sin(2t) + \cos(2t)$$
$$x_2(t) = \cos(2t) - \sin(2t)$$
$$x_3(t) = \cos(3t)$$
$$x_4(t) = \sin(3t)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 42

```
DSolve[{x1'[t]==-0*x1[t]+2*x2[t]+0*x3[t]+0*x4[t],x2'[t]==-2*x1[t]-0*x2[t]-0*x3[t]+0*x4[t],x3
```

$$\text{x1}(t) \to \sin(2t) + \cos(2t)$$
$$\text{x2}(t) \to \cos(2t) - \sin(2t)$$
$$\text{x3}(t) \to \cos(3t)$$
$$\text{x4}(t) \to \sin(3t)$$

3 Section 3.10, Systems of differential equations. Equal roots. Page 352

3.1 problem Example 1, page 348

Internal problem ID [1844]
Internal file name [`OUTPUT/1845_Sunday_June_05_2022_02_35_05_AM_69074826/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.10, Systems of differential equations. Equal roots. Page 352
Problem number: Example 1, page 348.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_1(t) + x_2(t) \\
x_2'(t) &= x_2(t) \\
x_3'(t) &= 2x_3(t)
\end{aligned}
$$

3.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{t} & t\,\mathrm{e}^{t} & 0 \\ 0 & \mathrm{e}^{t} & 0 \\ 0 & 0 & \mathrm{e}^{2t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^t & t\,e^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^t c_1 + t\,e^t c_2 \\ e^t c_2 \\ e^{2t} c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^t (c_2 t + c_1) \\ e^t c_2 \\ e^{2t} c_3 \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

3.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(1-\lambda)(1-\lambda)(2-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2, v_3\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	2	1	Yes	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

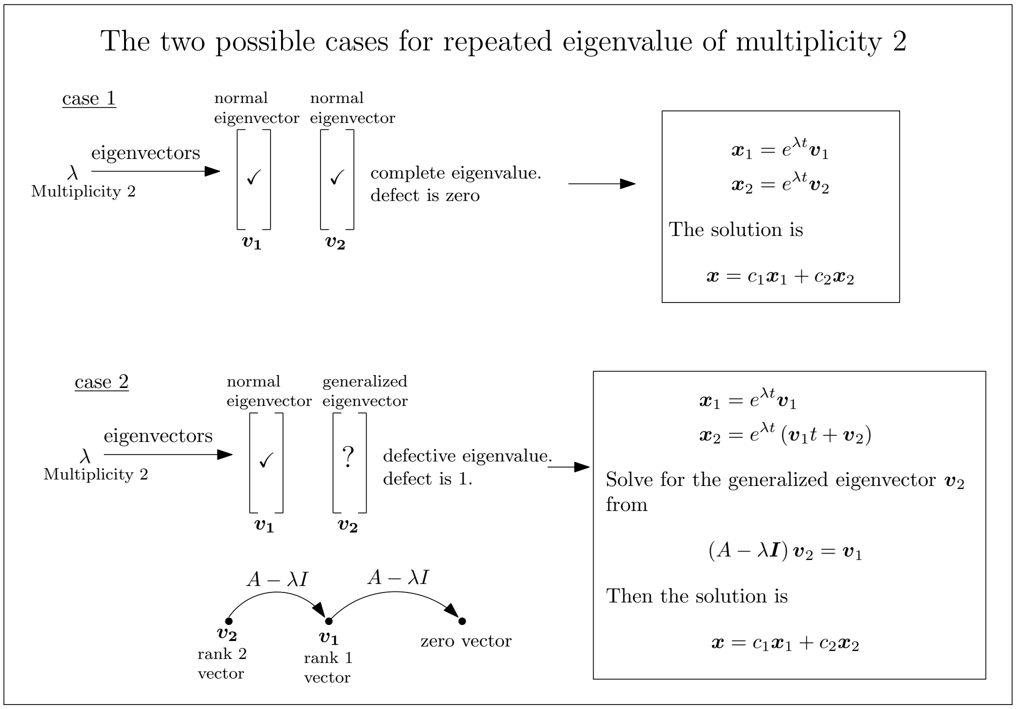


Figure 12: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 1. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mathrm{e}^{t} \\ &= \begin{bmatrix} \mathrm{e}^{t} \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

And

$$\begin{aligned} \vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) \mathrm{e}^{t} \\ &= \begin{bmatrix} \mathrm{e}^{t}(t+1) \\ \mathrm{e}^{t} \\ 0 \end{bmatrix} \end{aligned}$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_3(t) &= \vec{v}_3 e^{2t} \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t} \end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} e^t \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} e^t(t+1) \\ e^t \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ e^{2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} e^t(c_2 t + c_1 + c_2) \\ c_2 e^t \\ c_3 e^{2t} \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 29

```
dsolve([diff(x__1(t),t)=1*x__1(t)+1*x__2(t)+0*x__3(t),diff(x__2(t),t)=0*x__1(t)+1*x__2(t)-0*
```

$$x_1(t) = e^t(c_2 t + c_1)$$
$$x_2(t) = c_2 e^t$$
$$x_3(t) = c_3 e^{2t}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 64

```
DSolve[{x1'[t]==1*x1[t]+1*x2[t]+0*x3[t],x2'[t]==0*x1[t]+1*x2[t]-0*x3[t],x3'[t]==0*x1[t]-0*x2
```

$$\text{x1}(t) \to e^t(c_2 t + c_1)$$
$$\text{x2}(t) \to c_2 e^t$$
$$\text{x3}(t) \to c_3 e^{2t}$$
$$\text{x1}(t) \to e^t(c_2 t + c_1)$$
$$\text{x2}(t) \to c_2 e^t$$
$$\text{x3}(t) \to 0$$

3.2 problem Example 2, page 349

Internal problem ID [1845]
Internal file name [OUTPUT/1846_Sunday_June_05_2022_02_35_07_AM_53505999/index.tex]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.10, Systems of differential equations. Equal roots. Page 352
Problem number: Example 2, page 349.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 2x_1(t) + x_2(t) + 3x_3(t) \\
x_2'(t) &= 2x_2(t) - x_3(t) \\
x_3'(t) &= 2x_3(t)
\end{aligned}
$$

With initial conditions

$$[x_1(0) = 1, x_2(0) = 2, x_3(0) = 1]$$

3.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{2t} & e^{2t}t & -\frac{e^{2t}t(t-6)}{2} \\ 0 & e^{2t} & -e^{2t}t \\ 0 & 0 & e^{2t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^{2t} & e^{2t}t & -\frac{e^{2t}t(t-6)}{2} \\ 0 & e^{2t} & -e^{2t}t \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t} + 2\,e^{2t}t - \frac{e^{2t}t(t-6)}{2} \\ 2\,e^{2t} - e^{2t}t \\ e^{2t} \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}\left(1 + 5t - \frac{1}{2}t^2\right) \\ e^{2t}(-t+2) \\ e^{2t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

3.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & 1 & 3 \\ 0 & 2-\lambda & -1 \\ 0 & 0 & 2-\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(2-\lambda)(2-\lambda)(2-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 1 & 3 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2, v_3\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	3	1	Yes	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 2 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

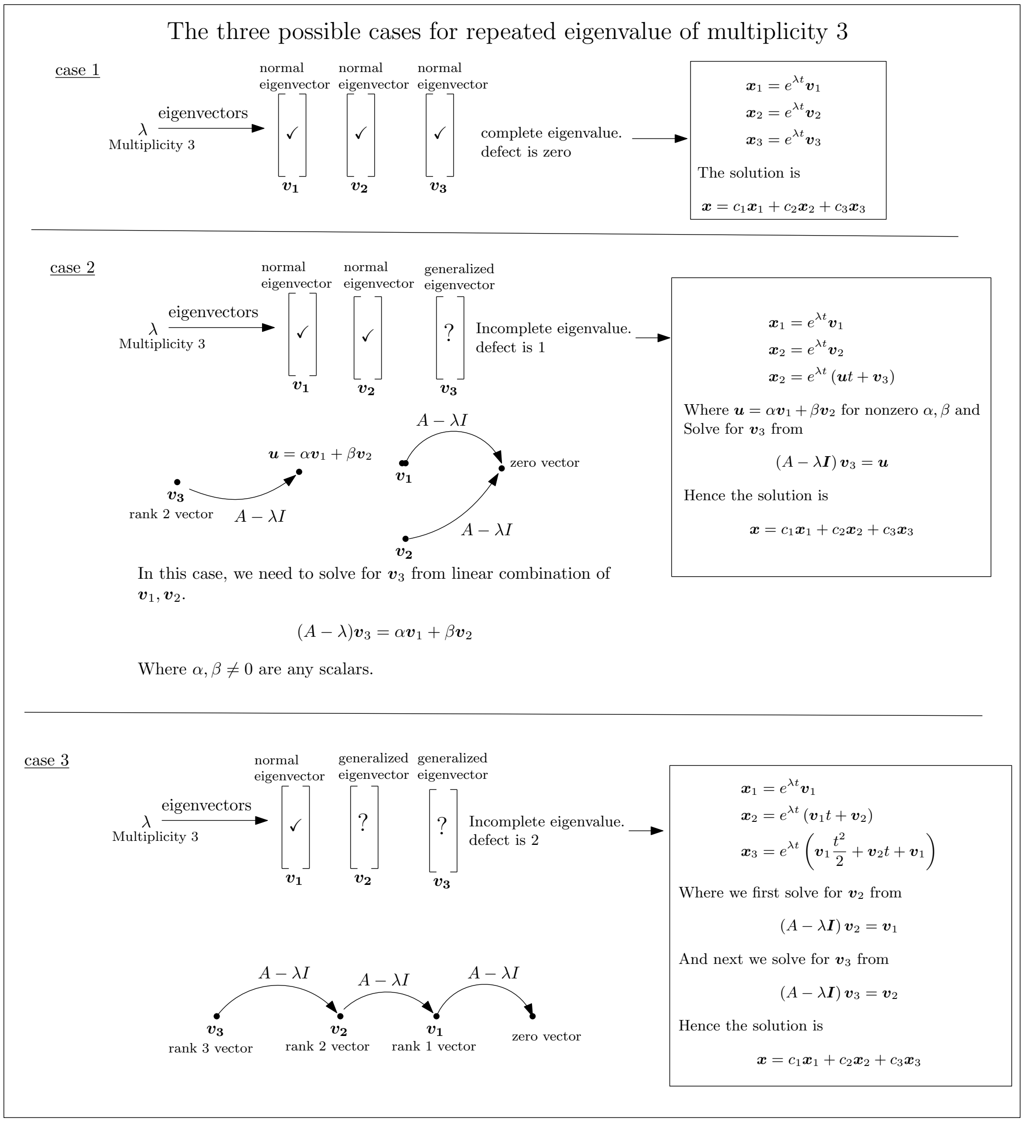


Figure 13: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 2. This falls into case 3 shown above. First we find generalized eigenvector $\vec{v}_2$ of rank 2 and then use this to find generalized eigenvector

$\vec{v}_3$ of rank 3.$\vec{v}_2$ is found by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence

$$\left(\begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Now $\vec{v}_3$ is found by solving

$$(A - \lambda I)\,\vec{v}_3 = \vec{v}_2$$

Where $\vec{v}_2$ is the (rank 2) generalized eigenvector found above. Hence

$$\left(\begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Solving for $\vec{v}_3$ gives

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$$

We have found three generalized eigenvectors for eigenvalue 2. Therefore the three basis solutions associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mathrm{e}^{2t} \\
&= \begin{bmatrix} \mathrm{e}^{2t} \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= e^{\lambda t}(\vec{v}_1 t + \vec{v}_2) \\
&= \mathrm{e}^{2t}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) \\
&= \begin{bmatrix} \mathrm{e}^{2t}(t+1) \\ \mathrm{e}^{2t} \\ 0 \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_3(t) &= \left(\vec{v}_1 \frac{t^2}{2} + \vec{v}_2 t + \vec{v}_3\right) e^{\lambda t} \\
&= \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{t^2}{2} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}\right) \mathrm{e}^{2t} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{2t}(t^2+2t+2)}{2} \\ \mathrm{e}^{2t}(t+4) \\ -\mathrm{e}^{2t} \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{2t} \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} e^{2t}(t+1) \\ e^{2t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} e^{2t}\left(t+\frac{1}{2}t^2+1\right) \\ e^{2t}(t+4) \\ -e^{2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{((t^2+2t+2)c_3+2c_2t+2c_1+2c_2)e^{2t}}{2} \\ ((t+4)\,c_3+c_2)\,e^{2t} \\ -c_3e^{2t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = 1 \\ x_2(0) = 2 \\ x_3(0) = 1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} c_3+c_1+c_2 \\ 4c_3+c_2 \\ -c_3 \end{bmatrix}$$

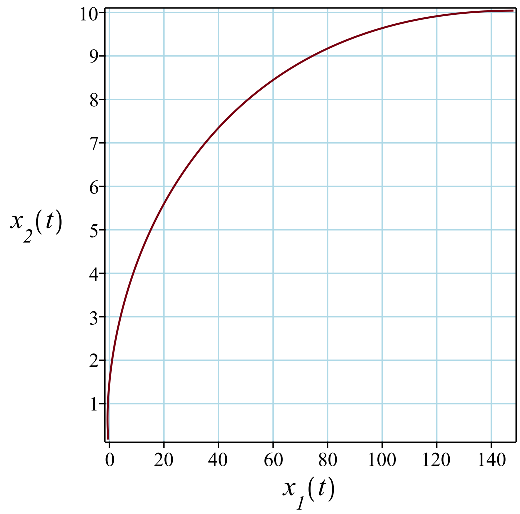
Solving for the constants of integrations gives

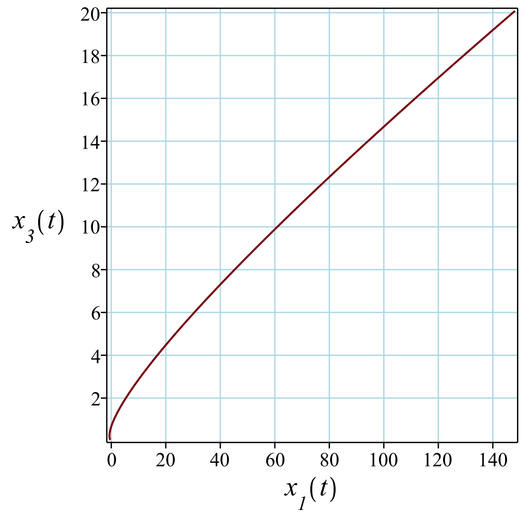
$$\begin{bmatrix} c_1 = -4 \\ c_2 = 6 \\ c_3 = -1 \end{bmatrix}$$

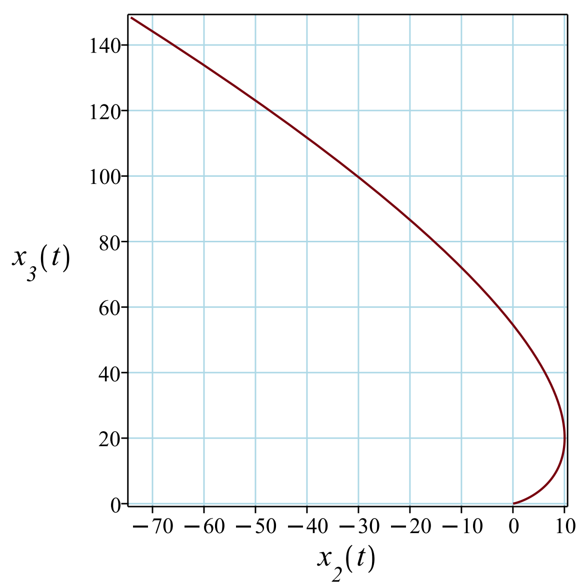
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{e^{2t}(-t^2+10t+2)}{2} \\ e^{2t}(-t+2) \\ e^{2t} \end{bmatrix}$$

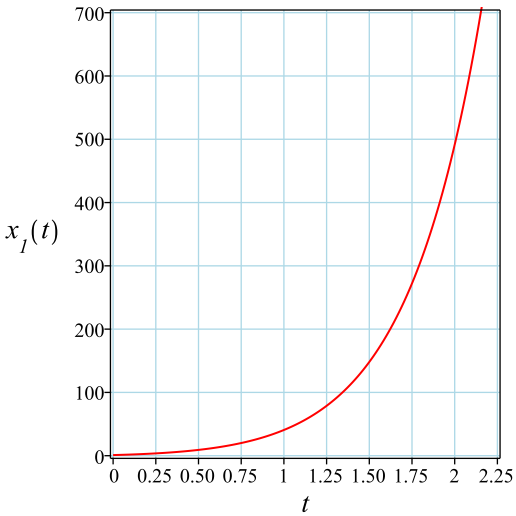
The following are plots of each solution against another.

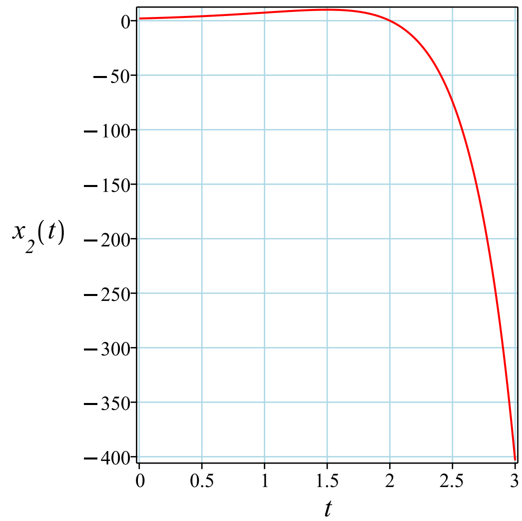


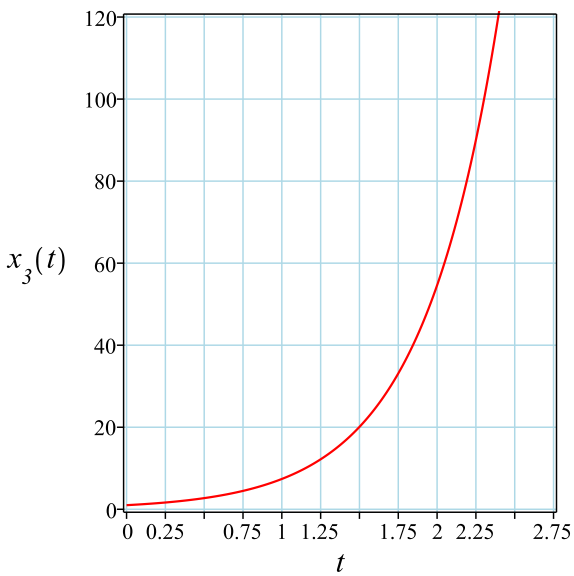




The following are plots of each solution.







✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 41

```
dsolve([diff(x__1(t),t) = 2*x__1(t)+x__2(t)+3*x__3(t), diff(x__2(t),t) = 2*x__2(t)-x__3(t),
```

$$x_1(t) = \frac{(-t^2 + 10t + 2)\,e^{2t}}{2}$$
$$x_2(t) = (-t + 2)\,e^{2t}$$
$$x_3(t) = e^{2t}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 43

```
DSolve[{x1'[t]==2*x1[t]+1*x2[t]+3*x3[t],x2'[t]==0*x1[t]+2*x2[t]-1*x3[t],x3'[t]==0*x1[t]-0*x2
```

$$
\begin{aligned}
\text{x1}(t) &\to -\frac{1}{2}e^{2t}\left(t^2-10t-2\right)\\
\text{x2}(t) &\to -e^{2t}(t-2)\\
\text{x3}(t) &\to e^{2t}
\end{aligned}
$$

3.3 problem 1

Internal problem ID [1846]
Internal file name [`OUTPUT/1847_Sunday_June_05_2022_02_35_09_AM_62664101/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.10, Systems of differential equations. Equal roots. Page 352
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= -x_2(t) + x_3(t) \\
x_2'(t) &= 2x_1(t) - 3x_2(t) + x_3(t) \\
x_3'(t) &= x_1(t) - x_2(t) - x_3(t)
\end{aligned}
$$

3.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 2 & -3 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{-t}(t+1) & -t\,\mathrm{e}^{-t} & t\,\mathrm{e}^{-t} \\ t\,\mathrm{e}^{-t} + \mathrm{e}^{-t} - \mathrm{e}^{-2t} & -t\,\mathrm{e}^{-t} + \mathrm{e}^{-2t} & t\,\mathrm{e}^{-t} \\ \mathrm{e}^{-t} - \mathrm{e}^{-2t} & -\mathrm{e}^{-t} + \mathrm{e}^{-2t} & \mathrm{e}^{-t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-t}(t+1) & -t\,e^{-t} & t\,e^{-t} \\ t\,e^{-t}+e^{-t}-e^{-2t} & -t\,e^{-t}+e^{-2t} & t\,e^{-t} \\ e^{-t}-e^{-2t} & -e^{-t}+e^{-2t} & e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}(t+1)\,c_1 - t\,e^{-t}c_2 + t\,e^{-t}c_3 \\ (t\,e^{-t}+e^{-t}-e^{-2t})\,c_1 + (-t\,e^{-t}+e^{-2t})\,c_2 + t\,e^{-t}c_3 \\ (e^{-t}-e^{-2t})\,c_1 + (-e^{-t}+e^{-2t})\,c_2 + e^{-t}c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}((c_1-c_2+c_3)\,t + c_1) \\ (c_1(t+1) - t(c_2-c_3))\,e^{-t} - e^{-2t}(c_1-c_2) \\ (c_1-c_2+c_3)\,e^{-t} - e^{-2t}(c_1-c_2) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

3.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 2 & -3 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & -1 & 1 \\ 2 & -3 & 1 \\ 1 & -1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & -1 & 1 \\ 2 & -3-\lambda & 1 \\ 1 & -1 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + 4\lambda^2 + 5\lambda + 2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
-2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -1 & 1 \\ 2 & -3 & 1 \\ 1 & -1 & -1 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -1 & 1 \\ 2 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -1 & 1 \\ 2 & -3 & 1 \\ 1 & -1 & -1 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \implies \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \implies \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \implies \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity algebraic m	multiplicity geometric k	defective?	eigenvectors
-2	1	1	No	$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$
-1	2	1	Yes	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-2t} \\ &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{-2t}\end{aligned}$$

eigenvalue -1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

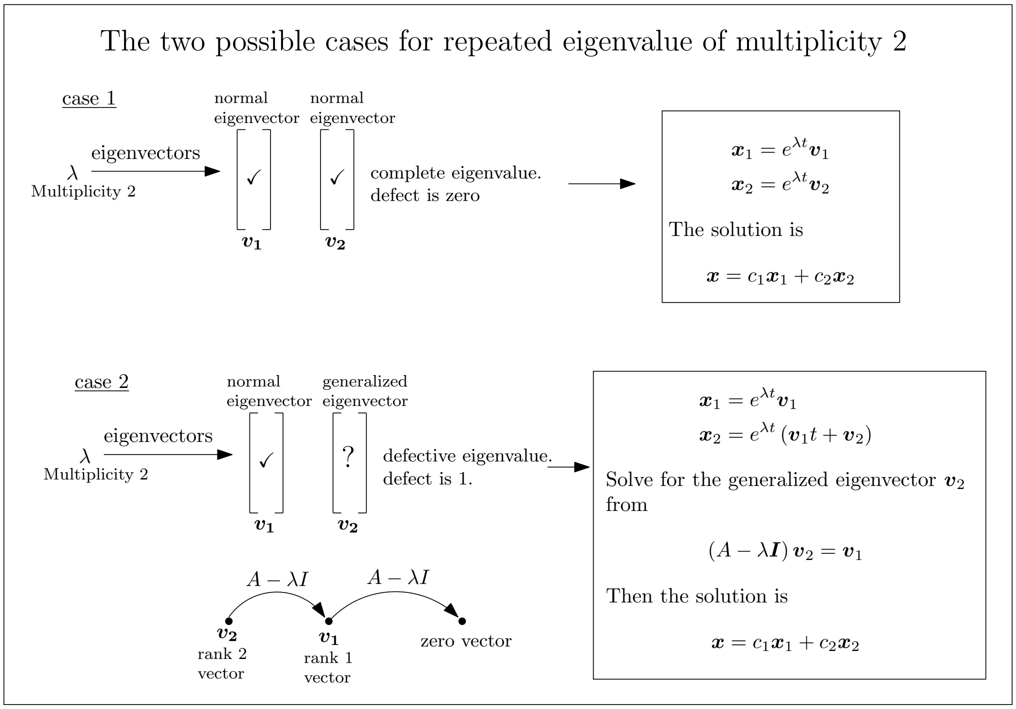


Figure 14: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 0 & -1 & 1 \\ 2 & -3 & 1 \\ 1 & -1 & -1 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -1. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned} \vec{x}_2(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \mathrm{e}^{-t} \\ &= \begin{bmatrix} \mathrm{e}^{-t} \\ \mathrm{e}^{-t} \\ 0 \end{bmatrix} \end{aligned}$$

And

$$\begin{aligned} \vec{x}_3(t) &= (\vec{v}_1 t + \vec{v}_2) e^{\lambda t} \\ &= \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \mathrm{e}^{-t} \\ &= \begin{bmatrix} \mathrm{e}^{-t}(t+1) \\ \mathrm{e}^{-t}(t+1) \\ \mathrm{e}^{-t} \end{bmatrix} \end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ \mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-t} \\ \mathrm{e}^{-t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{-t}(t+1) \\ \mathrm{e}^{-t}(t+1) \\ \mathrm{e}^{-t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-t}(c_3 t + c_2 + c_3) \\ ((t+1)\, c_3 + c_2)\, \mathrm{e}^{-t} + c_1 \mathrm{e}^{-2t} \\ c_1 \mathrm{e}^{-2t} + c_3 \mathrm{e}^{-t} \end{bmatrix}$$

3.3.3 Maple step by step solution

Let's solve

$[x_1'(t) = -x_2(t) + x_3(t)\,, x_2'(t) = 2x_1(t) - 3x_2(t) + x_3(t)\,, x_3'(t) = x_1(t) - x_2(t) - x_3(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 0 & -1 & 1 \\ 2 & -3 & 1 \\ 1 & -1 & -1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 0 & -1 & 1 \\ 2 & -3 & 1 \\ 1 & -1 & -1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 2 & -3 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

- Rewrite the system as

$\underline{x}^{\rightarrow}{}'(t) = A \cdot \underline{x}^{\rightarrow}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{1} = e^{-2t} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-1, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right]$$

- First solution from eigenvalue -1

$$\underline{\vec{x}}_{2}(t) = e^{-t} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = -1$ is the eigenvalue, an

$$\underline{\vec{x}}_{3}(t) = e^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\underline{\vec{x}}_{3}(t)$ into the homogeneous system

$$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = \left(e^{\lambda t}A\right) \cdot \left(t\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = e^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p}+\vec{v}=A\cdot\vec{p}$$

- Make use of the identity matrix I

$$(\lambda\cdot I)\cdot\vec{p}+\vec{v}=A\cdot\vec{p}$$

- Condition $\vec{p}$ must meet for $\underline{\vec{x}}_3(t)$ to be a solution to the homogeneous system

$$(A-\lambda\cdot I)\cdot\vec{p}=\vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue -1

$$\left(\begin{bmatrix}0 & -1 & 1\\ 2 & -3 & 1\\ 1 & -1 & -1\end{bmatrix}-(-1)\cdot\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\cdot\vec{p}=\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p}=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$$

- Second solution from eigenvalue -1

$$\underline{\vec{x}}_3(t)=\mathrm{e}^{-t}\cdot\left(t\cdot\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}+\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}\right)$$

- General solution to the system of ODEs

$$\underline{\vec{x}}=c_1\underline{\vec{x}}_1+c_2\underline{\vec{x}}_2(t)+c_3\underline{\vec{x}}_3(t)$$

- Substitute solutions into the general solution

$$\underline{\vec{x}}=c_1\mathrm{e}^{-2t}\cdot\begin{bmatrix}0\\ 1\\ 1\end{bmatrix}+c_2\mathrm{e}^{-t}\cdot\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}+c_3\mathrm{e}^{-t}\cdot\left(t\cdot\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}+\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}\right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}=\begin{bmatrix}\mathrm{e}^{-t}(c_3t+c_2+c_3)\\ (c_3t+c_2)\,\mathrm{e}^{-t}+c_1\mathrm{e}^{-2t}\\ c_1\mathrm{e}^{-2t}\end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \mathrm{e}^{-t}(c_3t + c_2 + c_3)\,, x_2(t) = (c_3t + c_2)\,\mathrm{e}^{-t} + c_1\mathrm{e}^{-2t}, x_3(t) = c_1\mathrm{e}^{-2t}\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 54

```
dsolve([diff(x__1(t),t)=0*x__1(t)-1*x__2(t)+1*x__3(t),diff(x__2(t),t)=2*x__1(t)-3*x__2(t)+1*
```

$$x_1(t) = \mathrm{e}^{-t}(c_3t + c_2)$$
$$x_2(t) = c_2\mathrm{e}^{-t} + c_3\mathrm{e}^{-t}t + c_1\mathrm{e}^{-2t}$$
$$x_3(t) = c_3\mathrm{e}^{-t} + c_1\mathrm{e}^{-2t}$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 99

```
DSolve[{x1'[t]==0*x1[t]-1*x2[t]+1*x3[t],x2'[t]==2*x1[t]-3*x2[t]+1*x3[t],x3'[t]==1*x1[t]-1*x2
```

$$\mathrm{x1}(t) \to e^{-t}(c_1(t+1) + (c_3 - c_2)t)$$
$$\mathrm{x2}(t) \to e^{-2t}\left(c_1\left(e^t(t+1) - 1\right) - c_2e^tt + c_3e^tt + c_2\right)$$
$$\mathrm{x3}(t) \to e^{-2t}\left(c_1\left(e^t - 1\right) - c_2e^t + c_3e^t + c_2\right)$$

3.4 problem 2

Internal problem ID [1847]
Internal file name [`OUTPUT/1848_Sunday_June_05_2022_02_35_11_AM_19255775/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.10, Systems of differential equations. Equal roots. Page 352
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_1(t) + x_2(t) + x_3(t)\\
x_2'(t) &= 2x_1(t) + x_2(t) - x_3(t)\\
x_3'(t) &= -3x_1(t) + 2x_2(t) + 4x_3(t)
\end{aligned}
$$

3.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{2t}(1-t) & e^{2t}t & e^{2t}t \\ -\frac{e^{2t}(t-4)t}{2} & e^{2t}\left(1-t+\frac{1}{2}t^2\right) & \frac{e^{2t}t(t-2)}{2} \\ \frac{e^{2t}t(t-6)}{2} & -\frac{e^{2t}(t-4)t}{2} & e^{2t}\left(1-\frac{1}{2}t^2+2t\right) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{2t}(1-t) & e^{2t}t & e^{2t}t \\ -\frac{e^{2t}(t-4)t}{2} & e^{2t}\left(1-t+\frac{1}{2}t^2\right) & \frac{e^{2t}t(t-2)}{2} \\ \frac{e^{2t}t(t-6)}{2} & -\frac{e^{2t}(t-4)t}{2} & e^{2t}\left(1-\frac{1}{2}t^2+2t\right) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}(1-t)\,c_1 + e^{2t}tc_2 + e^{2t}tc_3 \\ -\frac{e^{2t}(t-4)tc_1}{2} + e^{2t}\left(1-t+\frac{1}{2}t^2\right)c_2 + \frac{e^{2t}t(t-2)c_3}{2} \\ \frac{e^{2t}t(t-6)c_1}{2} - \frac{e^{2t}(t-4)tc_2}{2} + e^{2t}\left(1-\frac{1}{2}t^2+2t\right)c_3 \end{bmatrix} \\
&= \begin{bmatrix} -((c_1-c_2-c_3)\,t - c_1)\,e^{2t} \\ -\frac{((c_1-c_2-c_3)t^2+(-4c_1+2c_2+2c_3)t-2c_2)e^{2t}}{2} \\ \frac{((c_1-c_2-c_3)t^2+(-6c_1+4c_2+4c_3)t+2c_3)e^{2t}}{2} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

3.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 1 & 1 \\ 2 & 1-\lambda & -1 \\ -3 & 2 & 4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 2 & -1 & -1 & 0 \\ -3 & 2 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -3 & 2 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 - 3R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = -t\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated

with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	3	1	Yes	$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 2 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

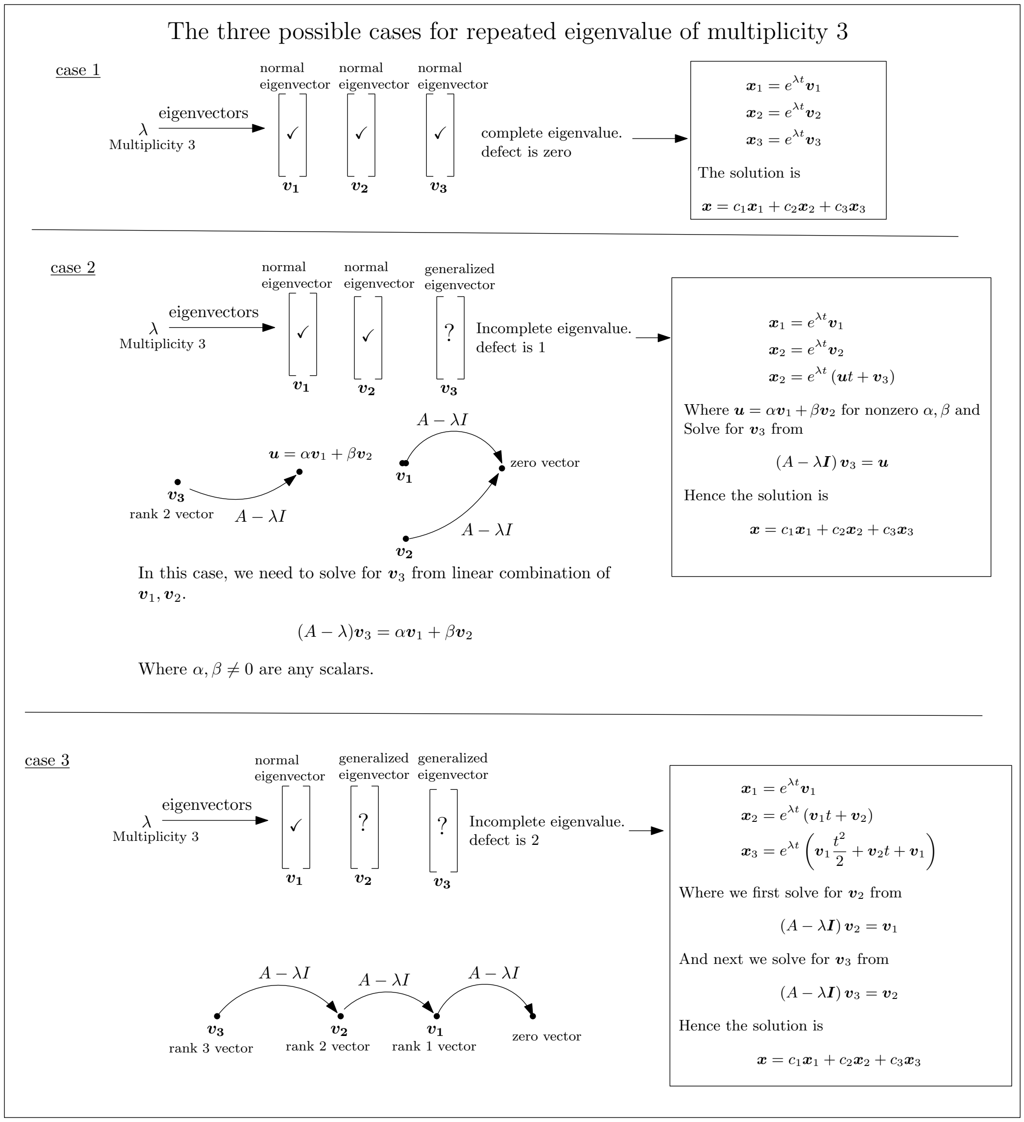


Figure 15: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 2. This falls into case 3 shown above. First we find generalized eigenvector $\vec{v}_2$ of rank 2 and then use this to find generalized eigenvector $\vec{v}_3$ of rank 3.$\vec{v}_2$ is found by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence

$$\left(\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

Now $\vec{v}_3$ is found by solving

$$(A - \lambda I)\,\vec{v}_3 = \vec{v}_2$$

Where $\vec{v}_2$ is the (rank 2) generalized eigenvector found above. Hence

$$\left(\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_3$ gives

$$\vec{v}_3 = \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix}$$

We have found three generalized eigenvectors for eigenvalue 2. Therefore the three basis

solutions associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \mathrm{e}^{2t} \\
&= \begin{bmatrix} 0 \\ -\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= e^{\lambda t}(\vec{v}_1 t + \vec{v}_2) \\
&= \mathrm{e}^{2t} \left(\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right) \\
&= \begin{bmatrix} -\mathrm{e}^{2t} \\ -\mathrm{e}^{2t}(2+t) \\ \mathrm{e}^{2t}(t+1) \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_3(t) &= \left(\vec{v}_1 \frac{t^2}{2} + \vec{v}_2 t + \vec{v}_3 \right) e^{\lambda t} \\
&= \left(\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \frac{t^2}{2} + \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix} \right) \mathrm{e}^{2t} \\
&= \begin{bmatrix} -\mathrm{e}^{2t}(t+3) \\ -\frac{\mathrm{e}^{2t}(t^2+4t+10)}{2} \\ \frac{\mathrm{e}^{2t}(t^2+2t+2)}{2} \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ -e^{2t} \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} -e^{2t} \\ e^{2t}(-t-2) \\ e^{2t}(t+1) \end{bmatrix} + c_3 \begin{bmatrix} e^{2t}(-t-3) \\ e^{2t}\left(-\frac{1}{2}t^2-2t-5\right) \\ e^{2t}\left(t+\frac{1}{2}t^2+1\right) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -((t+3)\,c_3+c_2)\,e^{2t} \\ -\frac{((t^2+4t+10)c_3+2c_2t+2c_1+4c_2)e^{2t}}{2} \\ \frac{((t^2+2t+2)c_3+2c_2t+2c_1+2c_2)e^{2t}}{2} \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 73

```
dsolve([diff(x__1(t),t)=1*x__1(t)+1*x__2(t)+1*x__3(t),diff(x__2(t),t)=2*x__1(t)+1*x__2(t)-1*
```

$$x_1(t) = e^{2t}(c_3t+c_2)$$
$$x_2(t) = \frac{(c_3t^2+2c_2t-2c_3t+2c_1)\,e^{2t}}{2}$$
$$x_3(t) = -\frac{e^{2t}(c_3t^2+2c_2t-4c_3t+2c_1-2c_2-2c_3)}{2}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 110

```
DSolve[{x1'[t]==1*x1[t]+1*x2[t]+1*x3[t],x2'[t]==2*x1[t]+1*x2[t]-1*x3[t],x3'[t]==-3*x1[t]+2*x
```

$$\text{x1}(t) \to e^{2t}((c_2+c_3)t-c_1(t-1))$$
$$\text{x2}(t) \to \frac{1}{2}e^{2t}\left(c_2\left(t^2-2t+2\right)-(c_1(t-4)t)+c_3(t-2)t\right)$$
$$\text{x3}(t) \to \frac{1}{2}e^{2t}\left((c_1-c_2-c_3)t^2-6c_1t+4(c_2+c_3)t+2c_3\right)$$

3.5 problem 3

Internal problem ID [1848]
Internal file name [`OUTPUT/1849_Sunday_June_05_2022_02_35_14_AM_70188850/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.10, Systems of differential equations. Equal roots. Page 352
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= -x_1(t) - x_2(t) \\
x_2'(t) &= -x_2(t) \\
x_3'(t) &= -2x_3(t)
\end{aligned}
$$

3.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{-t} & -t\,\mathrm{e}^{-t} & 0 \\ 0 & \mathrm{e}^{-t} & 0 \\ 0 & 0 & \mathrm{e}^{-2t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-t} & -t\,e^{-t} & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}c_1 - t\,e^{-t}c_2 \\ e^{-t}c_2 \\ e^{-2t}c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}(-c_2 t + c_1) \\ e^{-t}c_2 \\ e^{-2t}c_3 \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

3.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & -1 & 0 \\ 0 & -1-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(-1-\lambda)(-1-\lambda)(-2-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
-2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2, v_3\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
-2	1	1	No	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
-1	2	1	Yes	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-2t} \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-2t} \end{aligned}$$

eigenvalue -1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

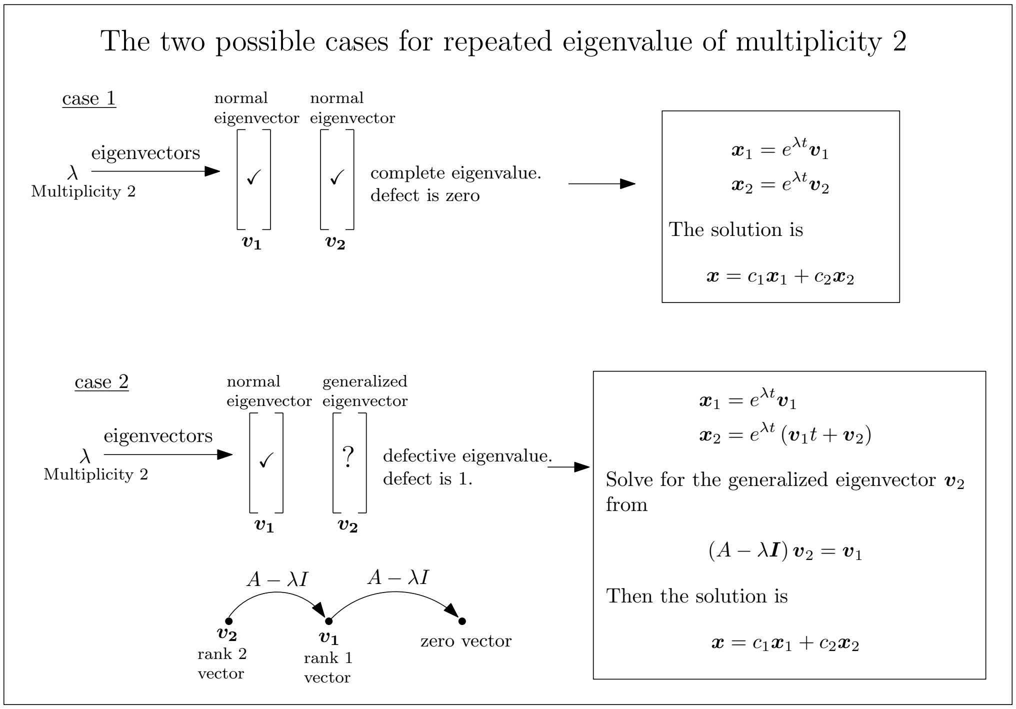


Figure 16: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -1. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mathrm{e}^{-t} \\ &= \begin{bmatrix} \mathrm{e}^{-t} \\ 0 \\ 0 \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_3(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right) \mathrm{e}^{-t} \\ &= \begin{bmatrix} \mathrm{e}^{-t}(t+1) \\ -\mathrm{e}^{-t} \\ 0 \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 0 \\ \mathrm{e}^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-t} \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{-t}(t+1) \\ -\mathrm{e}^{-t} \\ 0 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} e^{-t}(c_3 t + c_2 + c_3) \\ -c_3 e^{-t} \\ c_1 e^{-2t} \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 34

```
dsolve([diff(x__1(t),t)=-1*x__1(t)-1*x__2(t)+0*x__3(t),diff(x__2(t),t)=0*x__1(t)-1*x__2(t)+0
```

$$x_1(t) = (-c_2 t + c_1)\, e^{-t}$$
$$x_2(t) = c_2 e^{-t}$$
$$x_3(t) = c_3 e^{-2t}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 74

```
DSolve[{x1'[t]==-1*x1[t]-1*x2[t]+0*x3[t],x2'[t]==0*x1[t]-1*x2[t]+0*x3[t],x3'[t]==0*x1[t]-0*x
```

$$\text{x1}(t) \to e^{-t}(c_1 - c_2 t)$$
$$\text{x2}(t) \to c_2 e^{-t}$$
$$\text{x3}(t) \to c_3 e^{-2t}$$
$$\text{x1}(t) \to e^{-t}(c_1 - c_2 t)$$
$$\text{x2}(t) \to c_2 e^{-t}$$
$$\text{x3}(t) \to 0$$

3.6 problem 4

Internal problem ID [1849]
Internal file name [`OUTPUT/1850_Sunday_June_05_2022_02_35_15_AM_735843/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.10, Systems of differential equations. Equal roots. Page 352
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 2x_1(t) - x_3(t) \\
x_2'(t) &= 2x_2(t) + x_3(t) \\
x_3'(t) &= 2x_3(t) \\
x_4'(t) &= -x_3(t) + 2x_4(t)
\end{aligned}
$$

3.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{2t} & 0 & -e^{2t}t & 0 \\ 0 & e^{2t} & e^{2t}t & 0 \\ 0 & 0 & e^{2t} & 0 \\ 0 & 0 & -e^{2t}t & e^{2t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{2t} & 0 & -e^{2t}t & 0 \\ 0 & e^{2t} & e^{2t}t & 0 \\ 0 & 0 & e^{2t} & 0 \\ 0 & 0 & -e^{2t}t & e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}c_1 - e^{2t}tc_3 \\ e^{2t}c_2 + e^{2t}tc_3 \\ e^{2t}c_3 \\ -e^{2t}tc_3 + e^{2t}c_4 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}(-c_3t + c_1) \\ e^{2t}(c_3t + c_2) \\ e^{2t}c_3 \\ e^{2t}(-c_3t + c_4) \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

3.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & 0 & -1 & 0 \\ 0 & 2-\lambda & 1 & 0 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & -1 & 2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^4 - 8\lambda^3 + 24\lambda^2 - 32\lambda + 16 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cccc|c} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{array}\right]$$

$$R_4 = R_4 - R_1 \Longrightarrow \left[\begin{array}{cccc|c} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1, v_2, v_4\}$ and the leading variables are $\{v_3\}$. Let $v_1 = t$. Let $v_2 = s$. Let $v_4 = r$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ s \\ 0 \\ r \end{bmatrix} = \begin{bmatrix} t \\ s \\ 0 \\ r \end{bmatrix}$$

Since there are three free Variable, we have found three eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} t \\ s \\ 0 \\ r \end{bmatrix} &= \begin{bmatrix} t \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ s \\ 0 \\ 0 \end{bmatrix} \\ &= t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ and $r = 1$ then the above becomes

$$\begin{bmatrix} t \\ s \\ 0 \\ r \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence the three eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated

with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
2	4	3	Yes	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 2 is real and repated eigenvalue of multiplicity 4.There are four possible cases that can happen. This is illustrated in this diagram

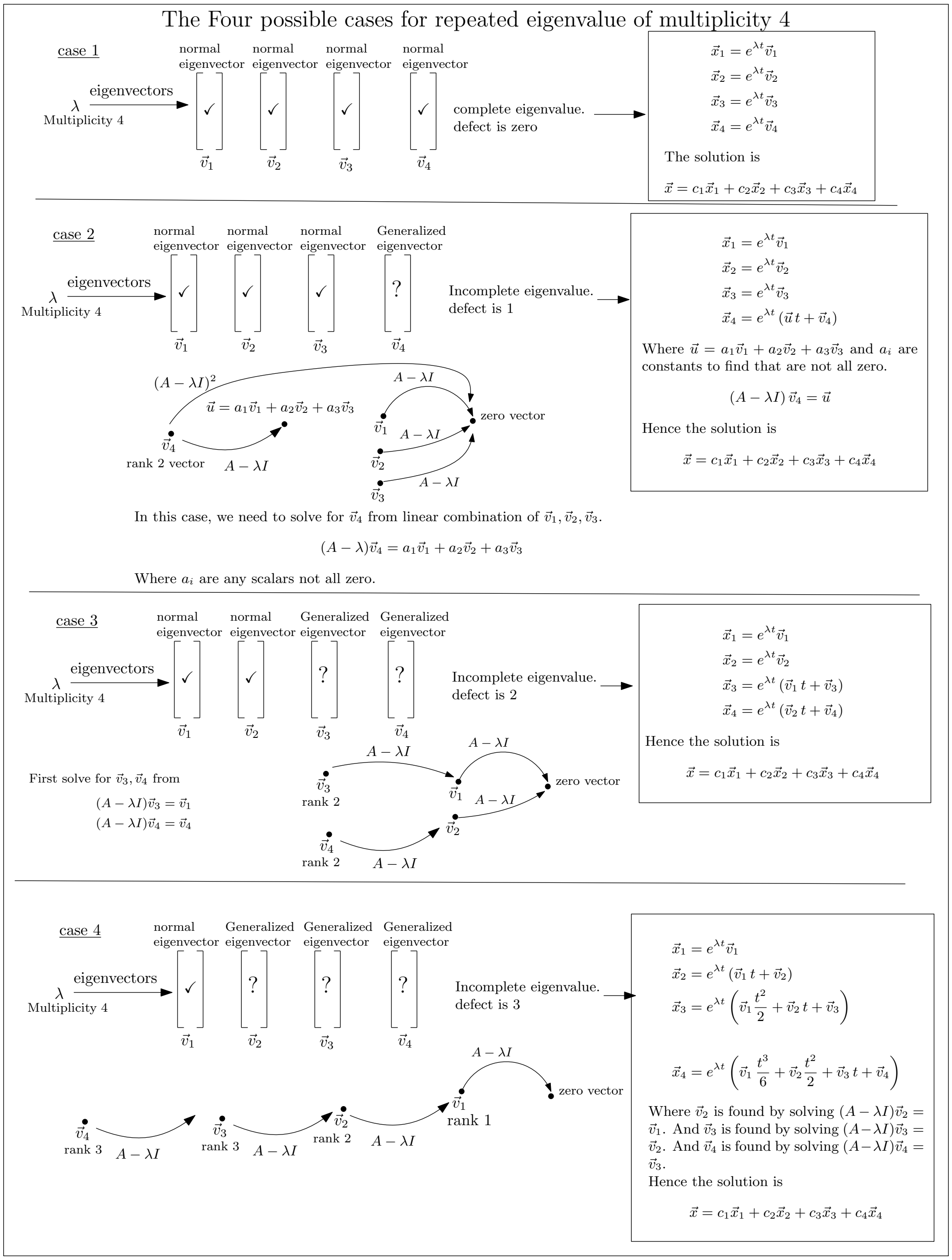


Figure 17: Possible case for repeated λ of multiplicity 4

This eigenvalue has algebraic multiplicity of 4, and geometric multiplicity 3, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to find rank-2 eigenvector $\vec{v}_4$. This eigenvector must therefore satisfy $(A - \lambda I)^2\,\vec{v}_4 = \vec{0}$.

But

$$(A-\lambda I)^2 = \left(\begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)^2$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore $\vec{v}_4$ could be any eigenvector vector we want (but not the zero vector). Let

$$\vec{v}_4 = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix}$$

To determine the actual $\vec{v}_4$ we need now to enforce the condition that $\vec{v}_4$ satisfies

$$(A-\lambda I)\,\vec{v}_4 = \vec{u} \tag{1}$$

Where $\vec{u}$ is linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$. Hence

$$\vec{u} = a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3$$

Where a_1, a_2, a_3 are arbitrary constants (not all zero). Eq. (1) becomes

$$(A - \lambda I) \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix} = a_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix} = a_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\eta_3 \\ \eta_3 \\ 0 \\ -\eta_3 \end{bmatrix} = \begin{bmatrix} a_3 \\ a_2 \\ 0 \\ a_1 \end{bmatrix}$$

Expanding the above gives the following equations equations

$$\begin{aligned} -\eta_3 &= a_3 \\ \eta_3 &= a_2 \\ -\eta_3 &= a_1 \end{aligned}$$

solving for a_1, a_2, a_3 from the above gives

$$\begin{aligned} -\eta_3 &= a_3 \\ \eta_3 &= a_2 \\ -\eta_3 &= a_1 \end{aligned}$$

Since a_1, a_2, a_3 are not all zero, then we just need to determine η_i values, not all zero, which satisfy the above equations for a_1, a_2, a_3 not all zero. By inspection we see that the following values satisfy this condition

$$[\eta_3 = 1]$$

Hence we found the missing generalized eigenvector

$$\vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Which implies that

$$\begin{aligned}
a_1 &= -1 \\
a_2 &= 1 \\
a_3 &= -1
\end{aligned}$$

Therefore

$$\begin{aligned}
\vec{u} &= a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 \\
&= -1\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + (1)\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \end{bmatrix}
\end{aligned}$$

Therefore the missing generalized eigenvector is now found. We have found four generalized eigenvectors for eigenvalue 2. Therefore the four basis solutions associated with this eigenvalue are

$$\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \mathrm{e}^{2t} \\
&= \begin{bmatrix} 0 \\ 0 \\ 0 \\ \mathrm{e}^{2t} \end{bmatrix}
\end{aligned}$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{\lambda t} \\
&= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \mathrm{e}^{2t} \\
&= \begin{bmatrix} 0 \\ \mathrm{e}^{2t} \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_3(t) &= \vec{v}_3 e^{\lambda t} \\
&= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mathrm{e}^{2t} \\
&= \begin{bmatrix} \mathrm{e}^{2t} \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_4(t) &= \left(\vec{u}t + \vec{v}_4\right) e^{\lambda t} \\
&= \left(\begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right) \mathrm{e}^{2t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t) + c_4 \vec{x}_4(t)
$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{2t} \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} e^{2t} \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} -e^{2t}t \\ e^{2t}t \\ e^{2t} \\ -e^{2t}t \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} e^{2t}(-tc_4 + c_3) \\ e^{2t}(tc_4 + c_2) \\ c_4 e^{2t} \\ e^{2t}(-tc_4 + c_1) \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 52

```
dsolve([diff(x__1(t),t)=2*x__1(t)+0*x__2(t)-1*x__3(t)+0*x__4(t),diff(x__2(t),t)=0*x__1(t)+2*
```

$$\begin{aligned} x_1(t) &= (-c_4 t + c_3)\, e^{2t} \\ x_2(t) &= (c_4 t + c_2)\, e^{2t} \\ x_3(t) &= c_4 e^{2t} \\ x_4(t) &= (-c_4 t + c_1)\, e^{2t} \end{aligned}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 63

```
DSolve[{x1'[t]==2*x1[t]+0*x2[t]-1*x3[t]+0*x4[t],x2'[t]==0*x1[t]+2*x2[t]+1*x3[t]+0*x4[t],x3'[
```

$$\begin{aligned} \text{x1}(t) &\to e^{2t}(c_1 - c_3 t) \\ \text{x2}(t) &\to e^{2t}(c_3 t + c_2) \\ \text{x3}(t) &\to c_3 e^{2t} \\ \text{x4}(t) &\to e^{2t}(c_4 - c_3 t) \end{aligned}$$

3.7 problem 5

Internal problem ID [1850]
Internal file name [OUTPUT/1851_Sunday_June_05_2022_02_35_18_AM_77697202/index.tex]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.10, Systems of differential equations. Equal roots. Page 352
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= -x_1(t) + x_2(t) + 2x_3(t) \\
x_2'(t) &= -x_1(t) + x_2(t) + x_3(t) \\
x_3'(t) &= -2x_1(t) + x_2(t) + 3x_3(t)
\end{aligned}
$$

With initial conditions

$$
[x_1(0) = 1, x_2(0) = 0, x_3(0) = 1]
$$

3.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ -1 & 1 & 1 \\ -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \mathrm{e}^t\left(1 - \frac{1}{2}t^2 - 2t\right) & t\,\mathrm{e}^t & \frac{\mathrm{e}^t t(t+4)}{2} \\ -t\,\mathrm{e}^t & \mathrm{e}^t & t\,\mathrm{e}^t \\ -\frac{\mathrm{e}^t t(t+4)}{2} & t\,\mathrm{e}^t & \mathrm{e}^t\left(1 + \frac{1}{2}t^2 + 2t\right) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \mathrm{e}^t\left(1-\frac{1}{2}t^2-2t\right) & t\,\mathrm{e}^t & \frac{\mathrm{e}^t t(t+4)}{2} \\ -t\,\mathrm{e}^t & \mathrm{e}^t & t\,\mathrm{e}^t \\ -\frac{\mathrm{e}^t t(t+4)}{2} & t\,\mathrm{e}^t & \mathrm{e}^t\left(1+\frac{1}{2}t^2+2t\right) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^t\left(1-\frac{1}{2}t^2-2t\right)+\frac{\mathrm{e}^t t(t+4)}{2} \\ 0 \\ -\frac{\mathrm{e}^t t(t+4)}{2}+\mathrm{e}^t\left(1+\frac{1}{2}t^2+2t\right) \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^t \\ 0 \\ \mathrm{e}^t \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

3.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ -1 & 1 & 1 \\ -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & 1 & 2 \\ -1 & 1 & 1 \\ -2 & 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & 1 & 2 \\ -1 & 1-\lambda & 1 \\ -2 & 1 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 1 & 2 \\ -1 & 1 & 1 \\ -2 & 1 & 3 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & 1 & 2 & 0 \\ -1 & 0 & 1 & 0 \\ -2 & 1 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 & 1 & 2 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ -2 & 1 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 1 & 2 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 1 & 2 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
1	3	1	Yes	$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 1 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

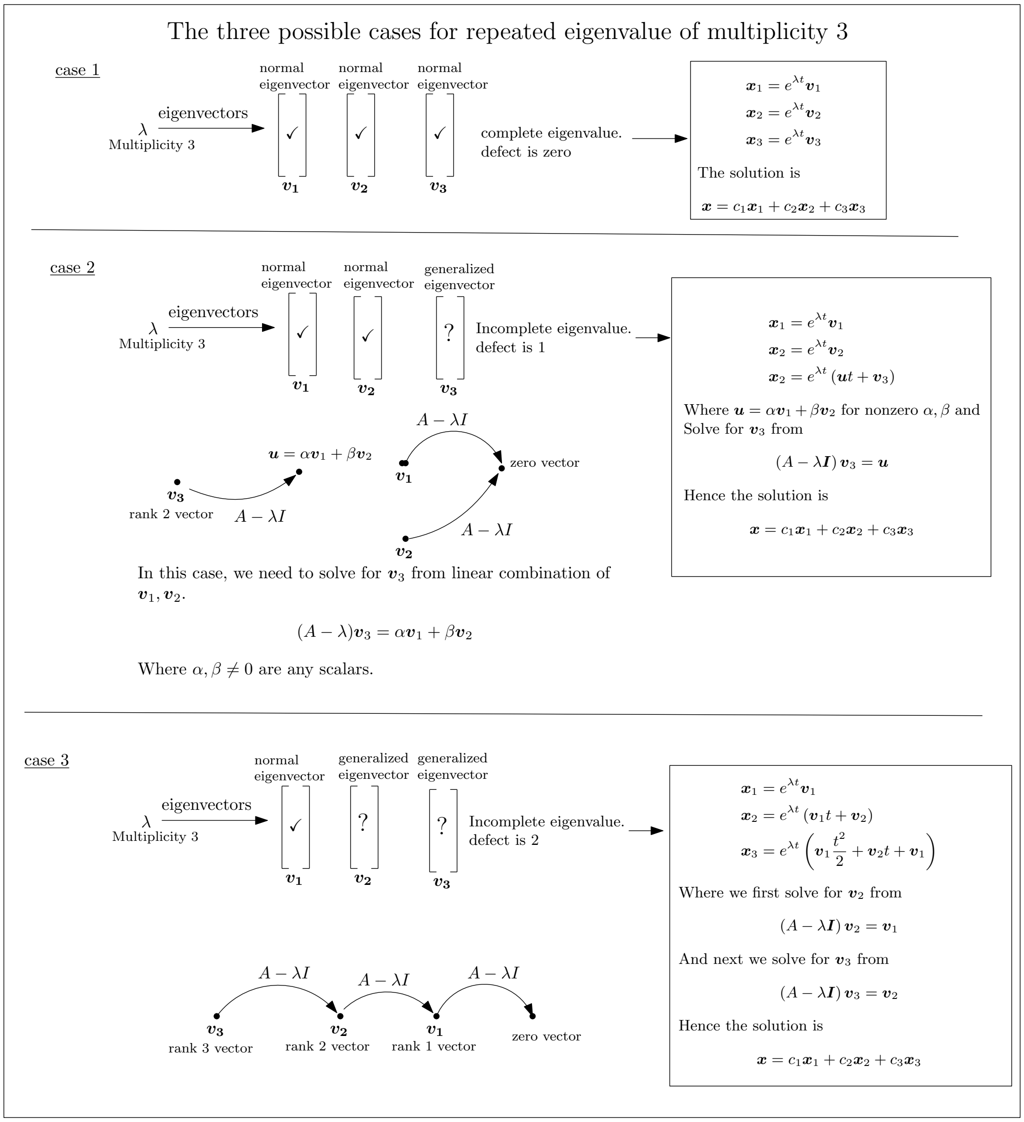


Figure 18: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 2. This falls into case 3 shown above. First we find generalized eigenvector $\vec{v}_2$ of rank 2 and then use this to find generalized eigenvector $\vec{v}_3$ of rank 3.$\vec{v}_2$ is found by solving

$$(A-\lambda I)\,\vec{v}_2=\vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence

$$\left(\begin{bmatrix} -1 & 1 & 2 \\ -1 & 1 & 1 \\ -2 & 1 & 3 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Now $\vec{v}_3$ is found by solving

$$(A - \lambda I)\,\vec{v}_3 = \vec{v}_2$$

Where $\vec{v}_2$ is the (rank 2) generalized eigenvector found above. Hence

$$\left(\begin{bmatrix} -1 & 1 & 2 \\ -1 & 1 & 1 \\ -2 & 1 & 3 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_3$ gives

$$\vec{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

We have found three generalized eigenvectors for eigenvalue 1. Therefore the three basis

solutions associated with this eigenvalue are

$$\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t}\\
&= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \mathrm{e}^{t}\\
&= \begin{bmatrix} \mathrm{e}^{t} \\ 0 \\ \mathrm{e}^{t} \end{bmatrix}
\end{aligned}$$

And

$$\begin{aligned}
\vec{x}_2(t) &= e^{\lambda t}(\vec{v}_1 t + \vec{v}_2)\\
&= \mathrm{e}^{t}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)\\
&= \begin{bmatrix} \mathrm{e}^{t}(t+1) \\ \mathrm{e}^{t} \\ \mathrm{e}^{t}(t+1) \end{bmatrix}
\end{aligned}$$

And

$$\begin{aligned}
\vec{x}_3(t) &= \left(\vec{v}_1 \frac{t^2}{2} + \vec{v}_2 t + \vec{v}_3\right) e^{\lambda t}\\
&= \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \frac{t^2}{2} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right) \mathrm{e}^{t}\\
&= \begin{bmatrix} \frac{\mathrm{e}^{t} t(2+t)}{2} \\ \mathrm{e}^{t}(t-1) \\ \frac{\mathrm{e}^{t}(t^2+2t+2)}{2} \end{bmatrix}
\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} e^t \\ 0 \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} e^t(t+1) \\ e^t \\ e^t(t+1) \end{bmatrix} + c_3 \begin{bmatrix} e^t\left(\frac{1}{2}t^2+t\right) \\ e^t(t-1) \\ e^t\left(t+\frac{1}{2}t^2+1\right) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{(c_3t^2+(2c_2+2c_3)t+2c_1+2c_2)e^t}{2} \\ ((t-1)c_3+c_2)e^t \\ \frac{((t^2+2t+2)c_3+2c_2t+2c_1+2c_2)e^t}{2} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = 1 \\ x_2(0) = 0 \\ x_3(0) = 1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1+c_2 \\ -c_3+c_2 \\ c_3+c_1+c_2 \end{bmatrix}$$

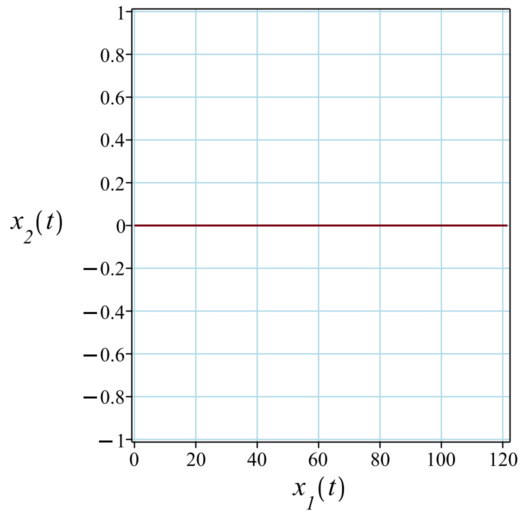
Solving for the constants of integrations gives

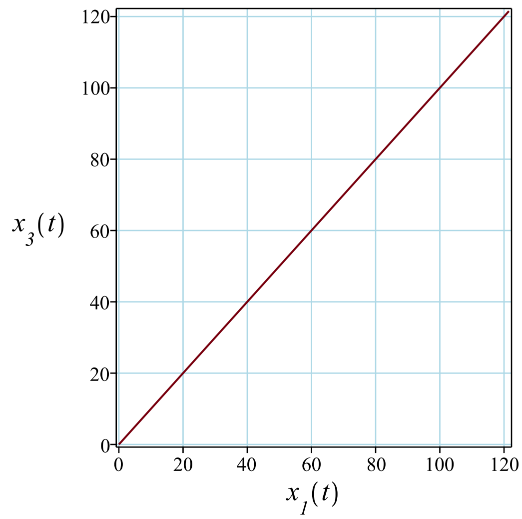
$$\begin{bmatrix} c_1 = 1 \\ c_2 = 0 \\ c_3 = 0 \end{bmatrix}$$

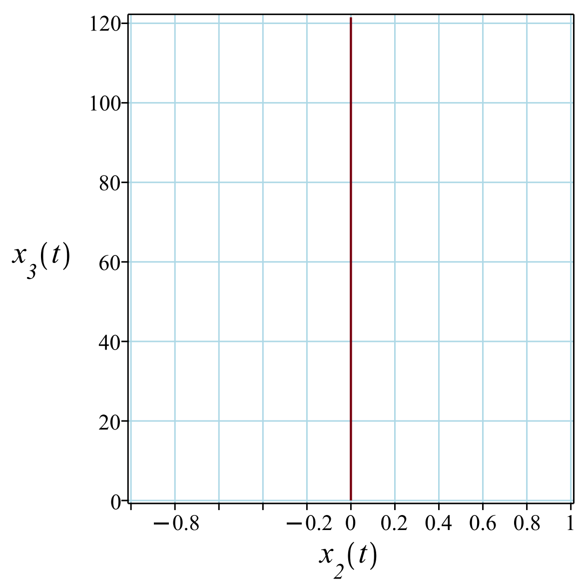
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} e^t \\ 0 \\ e^t \end{bmatrix}$$

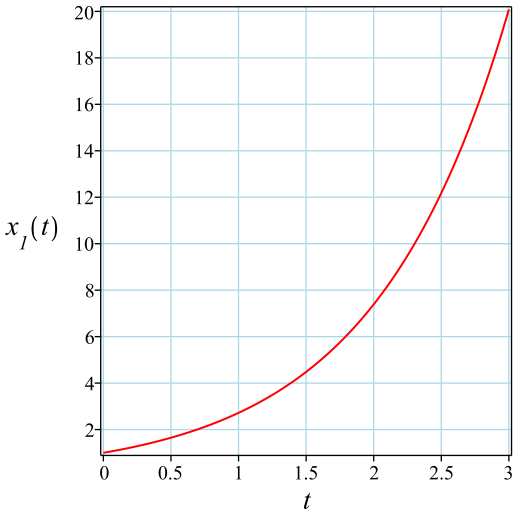
The following are plots of each solution against another.

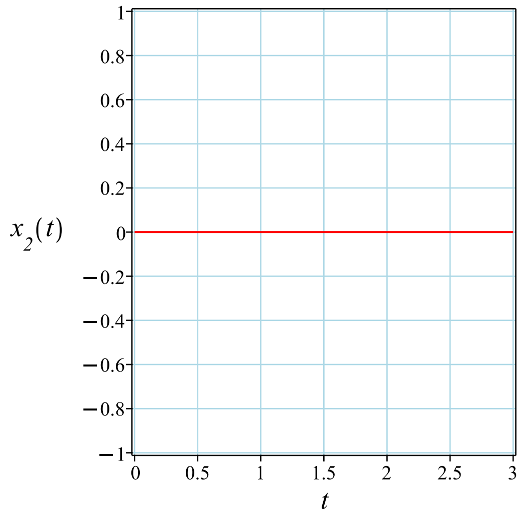


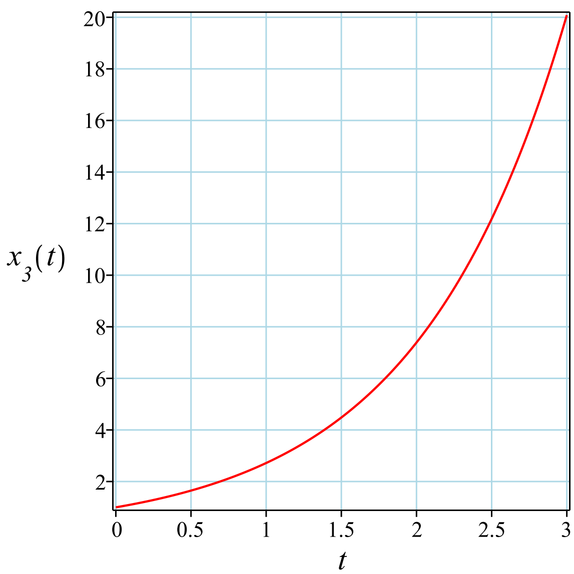




The following are plots of each solution.







✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 16

```
dsolve([diff(x__1(t),t) = -x__1(t)+x__2(t)+2*x__3(t), diff(x__2(t),t) = -x__1(t)+x__2(t)+x__
```

$$x_1(t) = e^t$$
$$x_2(t) = 0$$
$$x_3(t) = e^t$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 18

```
DSolve[{x1'[t]==-1*x1[t]+1*x2[t]+2*x3[t],x2'[t]==-1*x1[t]+1*x2[t]+1*x3[t],x3'[t]==-2*x1[t]+1
```

$$x1(t) \to e^t$$
$$x2(t) \to 0$$
$$x3(t) \to e^t$$

3.8 problem 6

Internal problem ID [1851]
Internal file name [`OUTPUT/1852_Sunday_June_05_2022_02_35_20_AM_64036813/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.10, Systems of differential equations. Equal roots. Page 352
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x_1'(t) &= -4x_1(t) - 4x_2(t) \\ x_2'(t) &= 10x_1(t) + 9x_2(t) + x_3(t) \\ x_3'(t) &= -4x_1(t) - 3x_2(t) + x_3(t) \end{aligned}$$

With initial conditions

$$[x_1(0) = 2, x_2(0) = 1, x_3(0) = -1]$$

3.8.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -4 & -4 & 0 \\ 10 & 9 & 1 \\ -4 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{2t}(-2t^2 - 6t + 1) & -2\,e^{2t}t(2+t) & -2\,e^{2t}t^2 \\ e^{2t}t(3t+10) & e^{2t}(3t^2 + 7t + 1) & e^{2t}(3t^2 + t) \\ -e^{2t}t(t+4) & -e^{2t}t(t+3) & e^{2t}(-t^2 - t + 1) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^{2t}(-2t^2-6t+1) & -2\,e^{2t}t(2+t) & -2\,e^{2t}t^2 \\ e^{2t}t(3t+10) & e^{2t}(3t^2+7t+1) & e^{2t}(3t^2+t) \\ -e^{2t}t(t+4) & -e^{2t}t(t+3) & e^{2t}(-t^2-t+1) \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 2\,e^{2t}(-2t^2-6t+1) - 2\,e^{2t}t(2+t) + 2\,e^{2t}t^2 \\ 2\,e^{2t}t(3t+10) + e^{2t}(3t^2+7t+1) - e^{2t}(3t^2+t) \\ -2\,e^{2t}t(t+4) - e^{2t}t(t+3) - e^{2t}(-t^2-t+1) \end{bmatrix} \\
&= \begin{bmatrix} (-4t^2-16t+2)\,e^{2t} \\ e^{2t}(6t^2+26t+1) \\ -2\left(t^2+5t+\frac{1}{2}\right)e^{2t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

3.8.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -4 & -4 & 0 \\ 10 & 9 & 1 \\ -4 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -4 & -4 & 0 \\ 10 & 9 & 1 \\ -4 & -3 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -4-\lambda & -4 & 0 \\ 10 & 9-\lambda & 1 \\ -4 & -3 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -4 & -4 & 0 \\ 10 & 9 & 1 \\ -4 & -3 & 1 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -6 & -4 & 0 \\ 10 & 7 & 1 \\ -4 & -3 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -6 & -4 & 0 & 0 \\ 10 & 7 & 1 & 0 \\ -4 & -3 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{5R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} -6 & -4 & 0 & 0 \\ 0 & \frac{1}{3} & 1 & 0 \\ -4 & -3 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{2R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} -6 & -4 & 0 & 0 \\ 0 & \frac{1}{3} & 1 & 0 \\ 0 & -\frac{1}{3} & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -6 & -4 & 0 & 0 \\ 0 & \frac{1}{3} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -6 & -4 & 0 \\ 0 & \frac{1}{3} & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t, v_2 = -3t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ -3t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ -3t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ -3t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ -3t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated

with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
2	3	1	Yes	$\left[\begin{array}{c} 2 \\ -3 \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 2 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

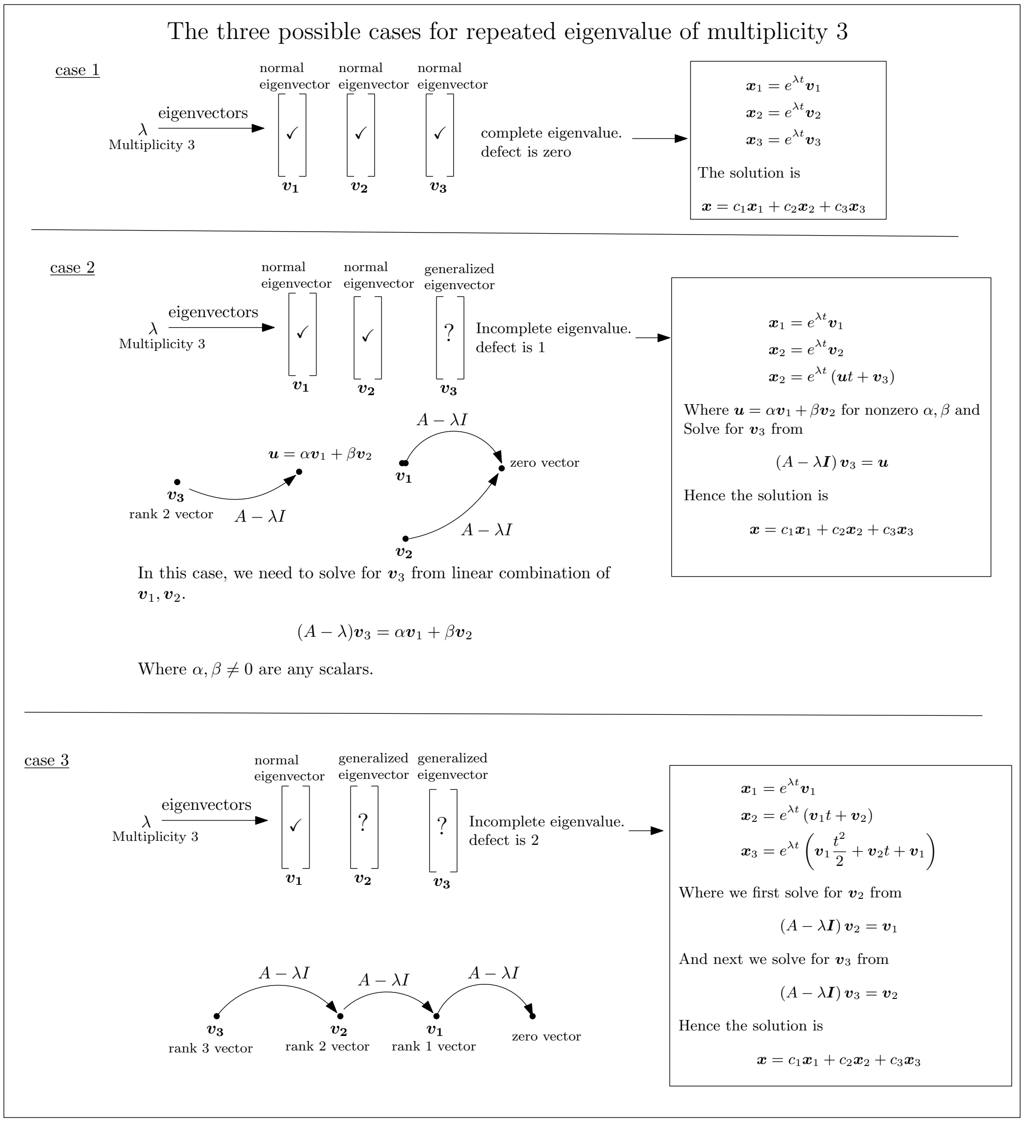


Figure 19: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 2. This falls into case 3 shown above. First we find generalized eigenvector $\vec{v}_2$ of rank 2 and then use this to find generalized eigenvector $\vec{v}_3$ of rank 3.$\vec{v}_2$ is found by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence

$$\left(\begin{bmatrix}-4 & -4 & 0\\ 10 & 9 & 1\\ -4 & -3 & 1\end{bmatrix} - (2)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}2\\ -3\\ 1\end{bmatrix}$$
$$\begin{bmatrix}-6 & -4 & 0\\ 10 & 7 & 1\\ -4 & -3 & -1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}2\\ -3\\ 1\end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix}1\\ -2\\ 1\end{bmatrix}$$

Now $\vec{v}_3$ is found by solving

$$(A - \lambda I)\,\vec{v}_3 = \vec{v}_2$$

Where $\vec{v}_2$ is the (rank 2) generalized eigenvector found above. Hence

$$\left(\begin{bmatrix}-4 & -4 & 0\\ 10 & 9 & 1\\ -4 & -3 & 1\end{bmatrix} - (2)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}1\\ -2\\ 1\end{bmatrix}$$
$$\begin{bmatrix}-6 & -4 & 0\\ 10 & 7 & 1\\ -4 & -3 & -1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}1\\ -2\\ 1\end{bmatrix}$$

Solving for $\vec{v}_3$ gives

$$\vec{v}_3 = \begin{bmatrix}\frac{5}{2}\\ -4\\ 1\end{bmatrix}$$

We have found three generalized eigenvectors for eigenvalue 2. Therefore the three basis

solutions associated with this eigenvalue are

$$\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \mathrm{e}^{2t} \\
&= \begin{bmatrix} 2\,\mathrm{e}^{2t} \\ -3\,\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix}
\end{aligned}$$

And

$$\begin{aligned}
\vec{x}_2(t) &= e^{\lambda t}(\vec{v}_1 t + \vec{v}_2) \\
&= \mathrm{e}^{2t} \left(\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right) \\
&= \begin{bmatrix} \mathrm{e}^{2t}(1+2t) \\ (-3t-2)\,\mathrm{e}^{2t} \\ \mathrm{e}^{2t}(t+1) \end{bmatrix}
\end{aligned}$$

And

$$\begin{aligned}
\vec{x}_3(t) &= \left(\vec{v}_1 \frac{t^2}{2} + \vec{v}_2 t + \vec{v}_3 \right) e^{\lambda t} \\
&= \left(\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \frac{t^2}{2} + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} t + \begin{bmatrix} \frac{5}{2} \\ -4 \\ 1 \end{bmatrix} \right) \mathrm{e}^{2t} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{2t}(2t^2+2t+5)}{2} \\ -\frac{\mathrm{e}^{2t}(3t^2+4t+8)}{2} \\ \frac{\mathrm{e}^{2t}(t^2+2t+2)}{2} \end{bmatrix}
\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} 2\,\mathrm{e}^{2t} \\ -3\,\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{2t}(1+2t) \\ (-3t-2)\,\mathrm{e}^{2t} \\ \mathrm{e}^{2t}(t+1) \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{2t}\left(t^2+t+\frac{5}{2}\right) \\ \mathrm{e}^{2t}\left(-\frac{3}{2}t^2-2t-4\right) \\ \mathrm{e}^{2t}\left(t+\frac{1}{2}t^2+1\right) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \left(\left(t^2+t+\frac{5}{2}\right)c_3+2c_2t+2c_1+c_2\right)\mathrm{e}^{2t} \\ \mathrm{e}^{2t}\left(-3c_1-3c_2t-2c_2-\frac{3}{2}c_3t^2-2c_3t-4c_3\right) \\ \frac{\left((t^2+2t+2)c_3+2c_2t+2c_1+2c_2\right)\mathrm{e}^{2t}}{2} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0)=2 \\ x_2(0)=1 \\ x_3(0)=-1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t=0$ gives

$$\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{5c_3}{2}+2c_1+c_2 \\ -3c_1-2c_2-4c_3 \\ c_3+c_1+c_2 \end{bmatrix}$$

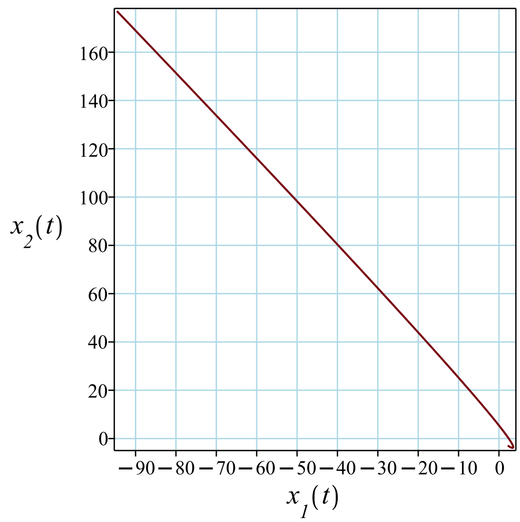
Solving for the constants of integrations gives

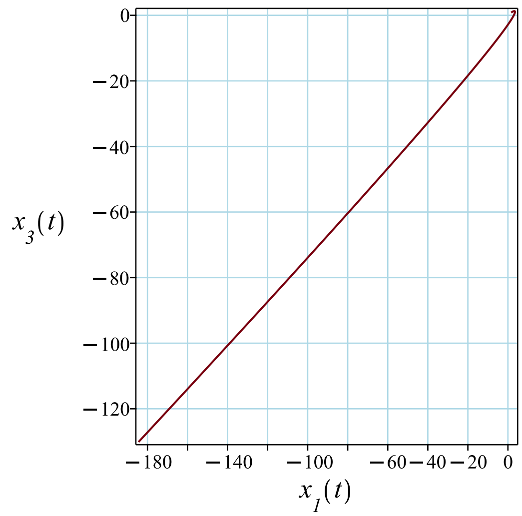
$$\begin{bmatrix} c_1=9 \\ c_2=-6 \\ c_3=-4 \end{bmatrix}$$

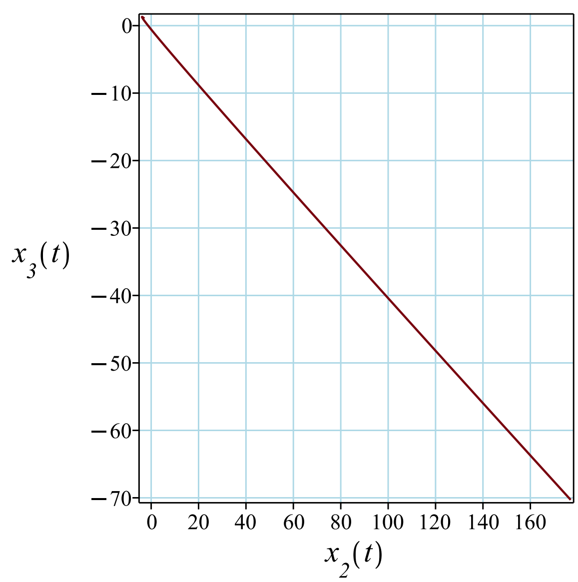
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} (-4t^2-16t+2)\,\mathrm{e}^{2t} \\ \mathrm{e}^{2t}(6t^2+26t+1) \\ \frac{(-4t^2-20t-2)\,\mathrm{e}^{2t}}{2} \end{bmatrix}$$

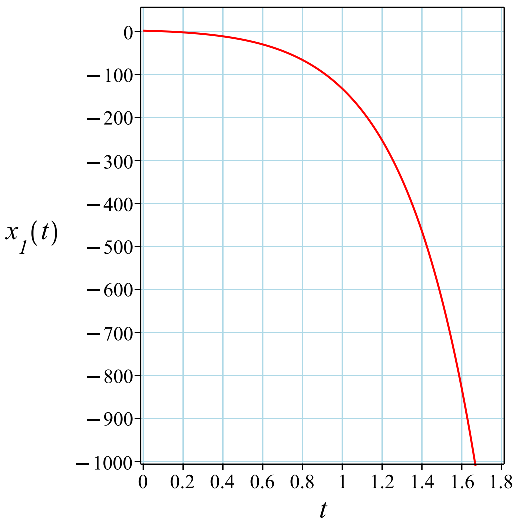
The following are plots of each solution against another.

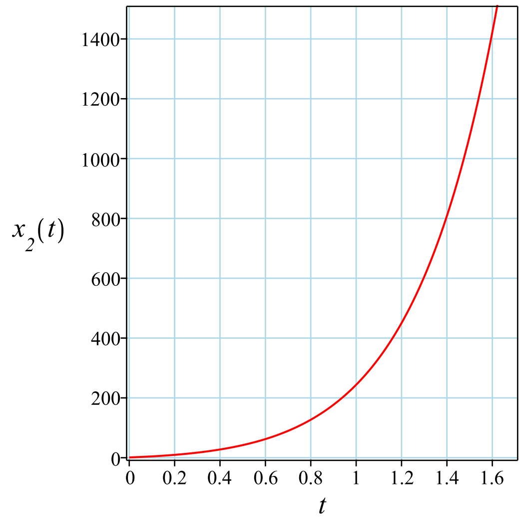


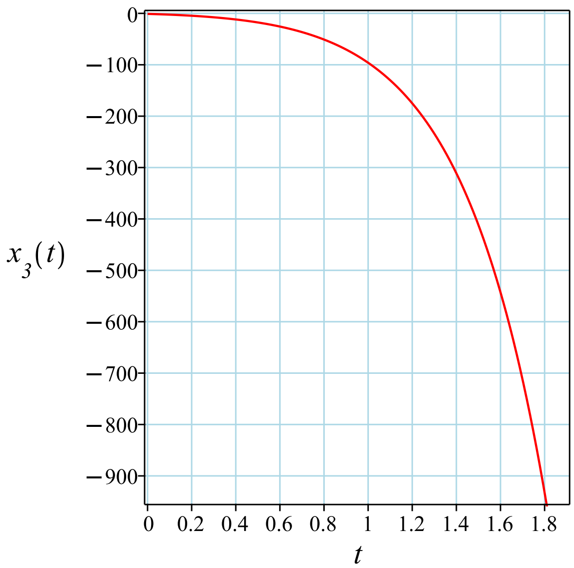




The following are plots of each solution.







✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 58

```
dsolve([diff(x__1(t),t) = -4*x__1(t)-4*x__2(t), diff(x__2(t),t) = 10*x__1(t)+9*x__2(t)+x__3(
```

$$x_1(t) = e^{2t}(-4t^2 - 16t + 2)$$
$$x_2(t) = -\frac{e^{2t}(-24t^2 - 104t - 4)}{4}$$
$$x_3(t) = \frac{e^{2t}(-8t^2 - 40t - 4)}{4}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 61

```
DSolve[{x1'[t]==-4*x1[t]-4*x2[t]+0*x3[t],x2'[t]==10*x1[t]+9*x2[t]+1*x3[t],x3'[t]==-4*x1[t]-3
```

$$\begin{aligned}
\text{x1}(t) &\to -2e^{2t}\left(2t^2+8t-1\right)\\
\text{x2}(t) &\to e^{2t}\left(6t^2+26t+1\right)\\
\text{x3}(t) &\to -e^{2t}\left(2t^2+10t+1\right)
\end{aligned}$$

3.9 problem 7

Internal problem ID [1852]
Internal file name [OUTPUT/1853_Sunday_June_05_2022_02_35_22_AM_67541300/index.tex]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.10, Systems of differential equations. Equal roots. Page 352
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_1(t) + 2x_2(t) - 3x_3(t) \\
x_2'(t) &= x_1(t) + x_2(t) + 2x_3(t) \\
x_3'(t) &= x_1(t) - x_2(t) + 4x_3(t)
\end{aligned}
$$

With initial conditions

$$
[x_1(0) = 1, x_2(0) = 0, x_3(0) = 0]
$$

3.9.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \mathrm{e}^{2t}(1-t) & -\frac{\mathrm{e}^{2t}(t-4)t}{2} & \frac{\mathrm{e}^{2t}t(t-6)}{2} \\ \mathrm{e}^{2t}t & \mathrm{e}^{2t}\left(1 - t + \frac{1}{2}t^2\right) & -\frac{\mathrm{e}^{2t}(t-4)t}{2} \\ \mathrm{e}^{2t}t & \frac{\mathrm{e}^{2t}t(t-2)}{2} & \mathrm{e}^{2t}\left(1 - \frac{1}{2}t^2 + 2t\right) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^{2t}(1-t) & -\frac{e^{2t}(t-4)t}{2} & \frac{e^{2t}t(t-6)}{2} \\ e^{2t}t & e^{2t}\left(1-t+\frac{1}{2}t^2\right) & -\frac{e^{2t}(t-4)t}{2} \\ e^{2t}t & \frac{e^{2t}t(t-2)}{2} & e^{2t}\left(1-\frac{1}{2}t^2+2t\right) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}(1-t) \\ e^{2t}t \\ e^{2t}t \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

3.9.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 2 & -3 \\ 1 & 1-\lambda & 2 \\ 1 & -1 & 4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 & -3 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 2 & -3 & 0 \\ 1 & -1 & 2 & 0 \\ 1 & -1 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 2 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 2 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$
R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 2 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} -1 & 2 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = t\}$

Hence the solution is

$$
\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$
\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}
$$

Let $t = 1$ the eigenvector becomes

$$
\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}
$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
2	3	1	Yes	$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 2 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

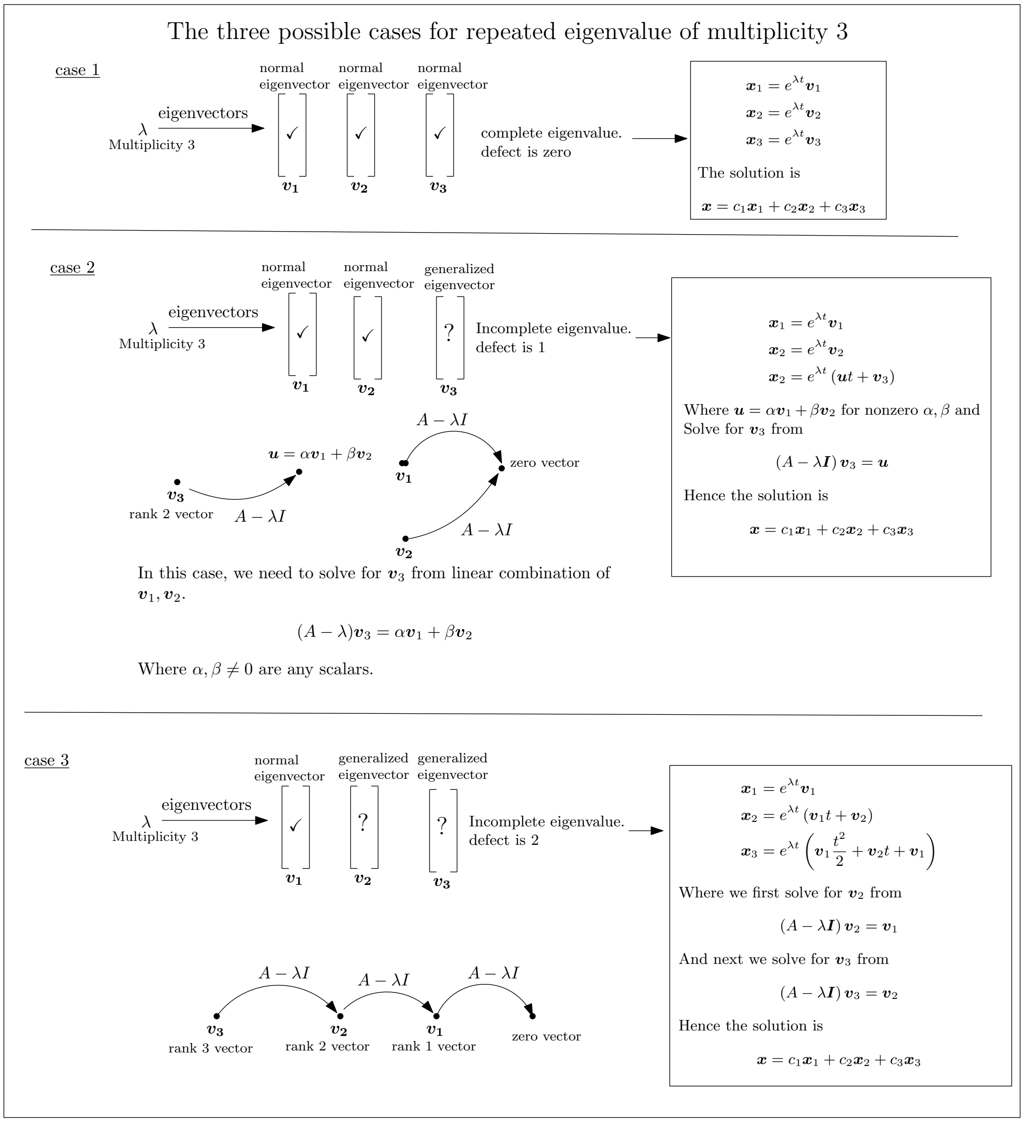


Figure 20: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 2. This falls into case 3 shown above. First we find generalized eigenvector $\vec{v}_2$ of rank 2 and then use this to find generalized eigenvector $\vec{v}_3$ of rank 3.$\vec{v}_2$ is found by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence

$$\left(\begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 & -3 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Now $\vec{v}_3$ is found by solving

$$(A - \lambda I)\,\vec{v}_3 = \vec{v}_2$$

Where $\vec{v}_2$ is the (rank 2) generalized eigenvector found above. Hence

$$\left(\begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 & -3 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_3$ gives

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

We have found three generalized eigenvectors for eigenvalue 2. Therefore the three basis

solutions associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \mathrm{e}^{2t} \\
&= \begin{bmatrix} -\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= e^{\lambda t}(\vec{v}_1 t + \vec{v}_2) \\
&= \mathrm{e}^{2t}\left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) \\
&= \begin{bmatrix} -\mathrm{e}^{2t} t \\ \mathrm{e}^{2t}(t+1) \\ \mathrm{e}^{2t}(t+1) \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_3(t) &= \left(\vec{v}_1 \frac{t^2}{2} + \vec{v}_2 t + \vec{v}_3\right) e^{\lambda t} \\
&= \left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \frac{t^2}{2} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right) \mathrm{e}^{2t} \\
&= \begin{bmatrix} -\frac{\mathrm{e}^{2t}(t^2-2)}{2} \\ \frac{\mathrm{e}^{2t}(t^2+2t+4)}{2} \\ \frac{\mathrm{e}^{2t}(t^2+2t+2)}{2} \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} -e^{2t}t \\ e^{2t}(t+1) \\ e^{2t}(t+1) \end{bmatrix} + c_3 \begin{bmatrix} e^{2t}\left(1-\frac{t^2}{2}\right) \\ e^{2t}\left(\frac{1}{2}t^2+t+2\right) \\ e^{2t}\left(t+\frac{1}{2}t^2+1\right) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} e^{2t}\left(-c_1 - tc_2 + c_3 - \frac{1}{2}c_3 t^2\right) \\ \frac{\left((t^2+2t+4)c_3+2tc_2+2c_1+2c_2\right)e^{2t}}{2} \\ \frac{\left((t^2+2t+2)c_3+2tc_2+2c_1+2c_2\right)e^{2t}}{2} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = 1 \\ x_2(0) = 0 \\ x_3(0) = 0 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -c_1 + c_3 \\ 2c_3 + c_1 + c_2 \\ c_3 + c_1 + c_2 \end{bmatrix}$$

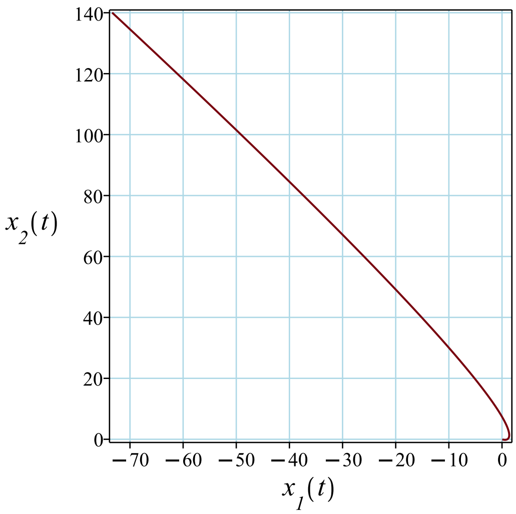
Solving for the constants of integrations gives

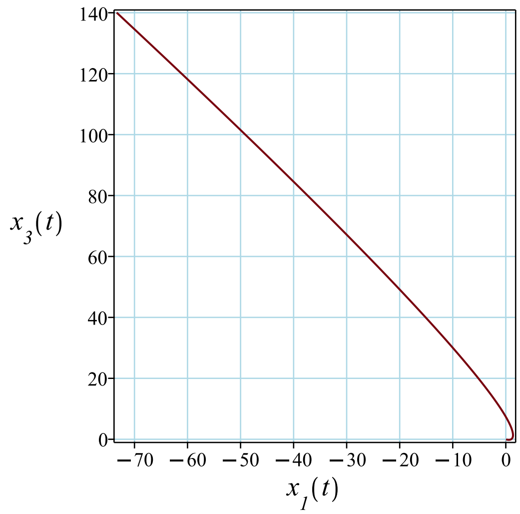
$$\begin{bmatrix} c_1 = -1 \\ c_2 = 1 \\ c_3 = 0 \end{bmatrix}$$

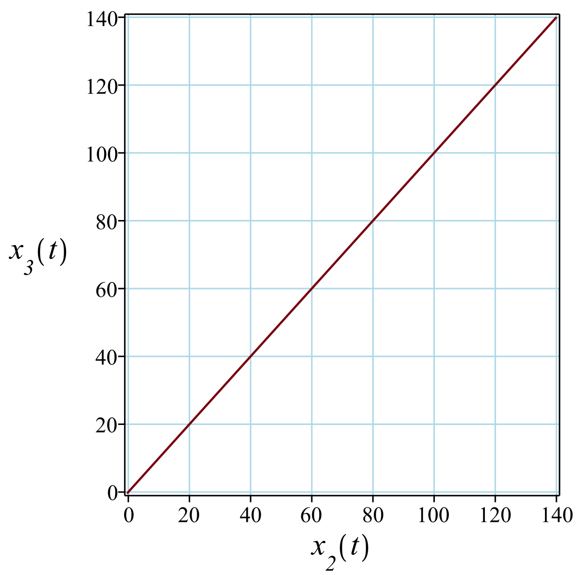
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} e^{2t}(1-t) \\ e^{2t}t \\ e^{2t}t \end{bmatrix}$$

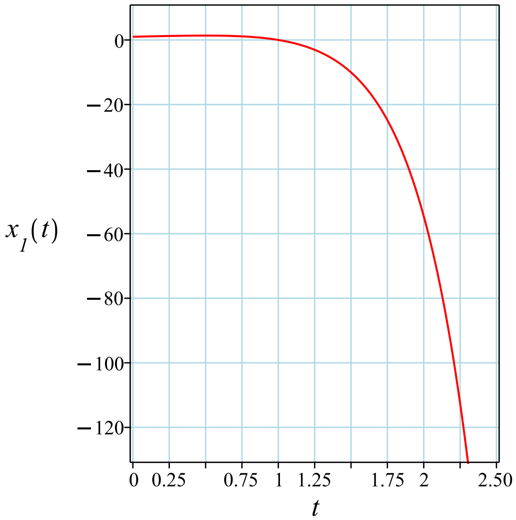
The following are plots of each solution against another.

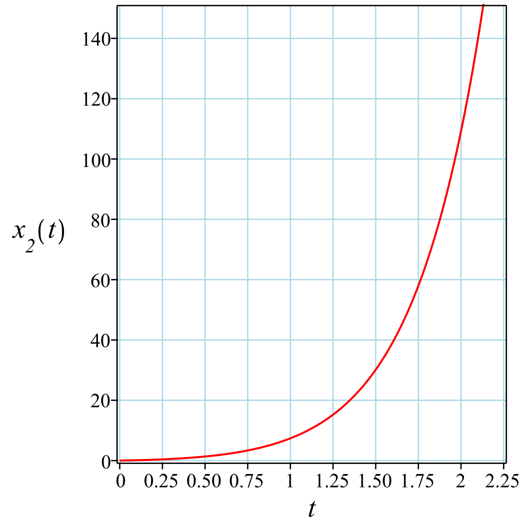


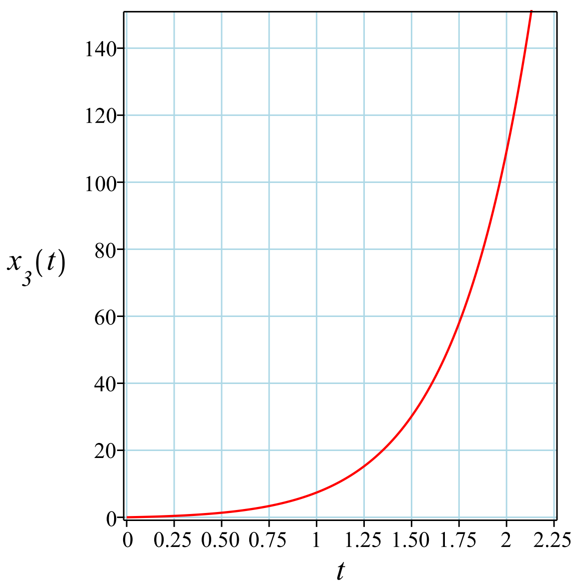




The following are plots of each solution.







✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
dsolve([diff(x__1(t),t) = x__1(t)+2*x__2(t)-3*x__3(t), diff(x__2(t),t) = x__1(t)+x__2(t)+2*x
```

$$x_1(t) = e^{2t}(-t+1)$$
$$x_2(t) = e^{2t}t$$
$$x_3(t) = e^{2t}t$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 35

```
DSolve[{x1'[t]==1*x1[t]+2*x2[t]-3*x3[t],x2'[t]==1*x1[t]+1*x2[t]+2*x3[t],x3'[t]==1*x1[t]-1*x2
```

$$x1(t) \to -e^{2t}(t-1)$$
$$x2(t) \to e^{2t}t$$
$$x3(t) \to e^{2t}t$$

3.10 problem 8

Internal problem ID [1853]
Internal file name [`OUTPUT/1854_Sunday_June_05_2022_02_35_25_AM_60018694/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.10, Systems of differential equations. Equal roots. Page 352
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 3x_1(t) \\
x_2'(t) &= x_1(t) + 3x_2(t) \\
x_3'(t) &= 3x_3(t) \\
x_4'(t) &= 2x_3(t) + 3x_4(t)
\end{aligned}
$$

With initial conditions

$$[x_1(0) = 1, x_2(0) = 1, x_3(0) = 1, x_4(0) = 1]$$

3.10.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{3t} & 0 & 0 & 0 \\ e^{3t}t & e^{3t} & 0 & 0 \\ 0 & 0 & e^{3t} & 0 \\ 0 & 0 & 2\,e^{3t}t & e^{3t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} e^{3t} & 0 & 0 & 0 \\ e^{3t}t & e^{3t} & 0 & 0 \\ 0 & 0 & e^{3t} & 0 \\ 0 & 0 & 2\,e^{3t}t & e^{3t} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{3t} \\ e^{3t}t + e^{3t} \\ e^{3t} \\ e^{3t} + 2\,e^{3t}t \end{bmatrix} \\ &= \begin{bmatrix} e^{3t} \\ e^{3t}(t+1) \\ e^{3t} \\ e^{3t}(1+2t) \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

3.10.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & 0 & 0 & 0 \\ 1 & 3-\lambda & 0 & 0 \\ 0 & 0 & 3-\lambda & 0 \\ 0 & 0 & 2 & 3-\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(3-\lambda)(3-\lambda)(3-\lambda)(3-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 2 & 3 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 4 gives

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_4\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Let $v_4 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \\ 0 \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 0 \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} 0 \\ t \\ 0 \\ s \end{bmatrix} &= \begin{bmatrix} 0 \\ t \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} 0 \\ t \\ 0 \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
3	4	2	Yes	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$

This case will be solved using the Jordan form of the matrix A. The Jordan form diagonalization is

$$A = PJP^{-1}$$

Which can be found to be

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 2 & 0 \end{bmatrix}^{-1}$$

Looking at the P matrix above, we see there are 2 chains. Therefore, we now construct the basis solution by following these chains as follows.

$$\vec{x}_1 = \begin{bmatrix} 0 \\ e^{3t} \\ 0 \\ 2\,e^{3t} \end{bmatrix}$$

$$\vec{x}_2 = \begin{bmatrix} e^{3t} \\ e^{3t} t \\ e^{3t} \\ 2\,e^{3t} t \end{bmatrix}$$

$$\vec{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2\,e^{3t} \end{bmatrix}$$

$$\vec{x}_4 = \begin{bmatrix} 0 \\ 0 \\ e^{3t} \\ 2\,e^{3t} t \end{bmatrix}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t) + c_4 \vec{x}_4(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ e^{3t} \\ 0 \\ 2\,e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} e^{3t} \\ e^{3t} t \\ e^{3t} \\ 2\,e^{3t} t \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2\,e^{3t} \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 0 \\ e^{3t} \\ 2\,e^{3t} t \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} c_2 e^{3t} \\ e^{3t}(tc_2 + c_1) \\ e^{3t}(c_2 + c_4) \\ 2((c_2 + c_4)\, t + c_1 + c_3)\, e^{3t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = 1 \\ x_2(0) = 1 \\ x_3(0) = 1 \\ x_4(0) = 1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_2 \\ c_1 \\ c_2 + c_4 \\ 2c_1 + 2c_3 \end{bmatrix}$$

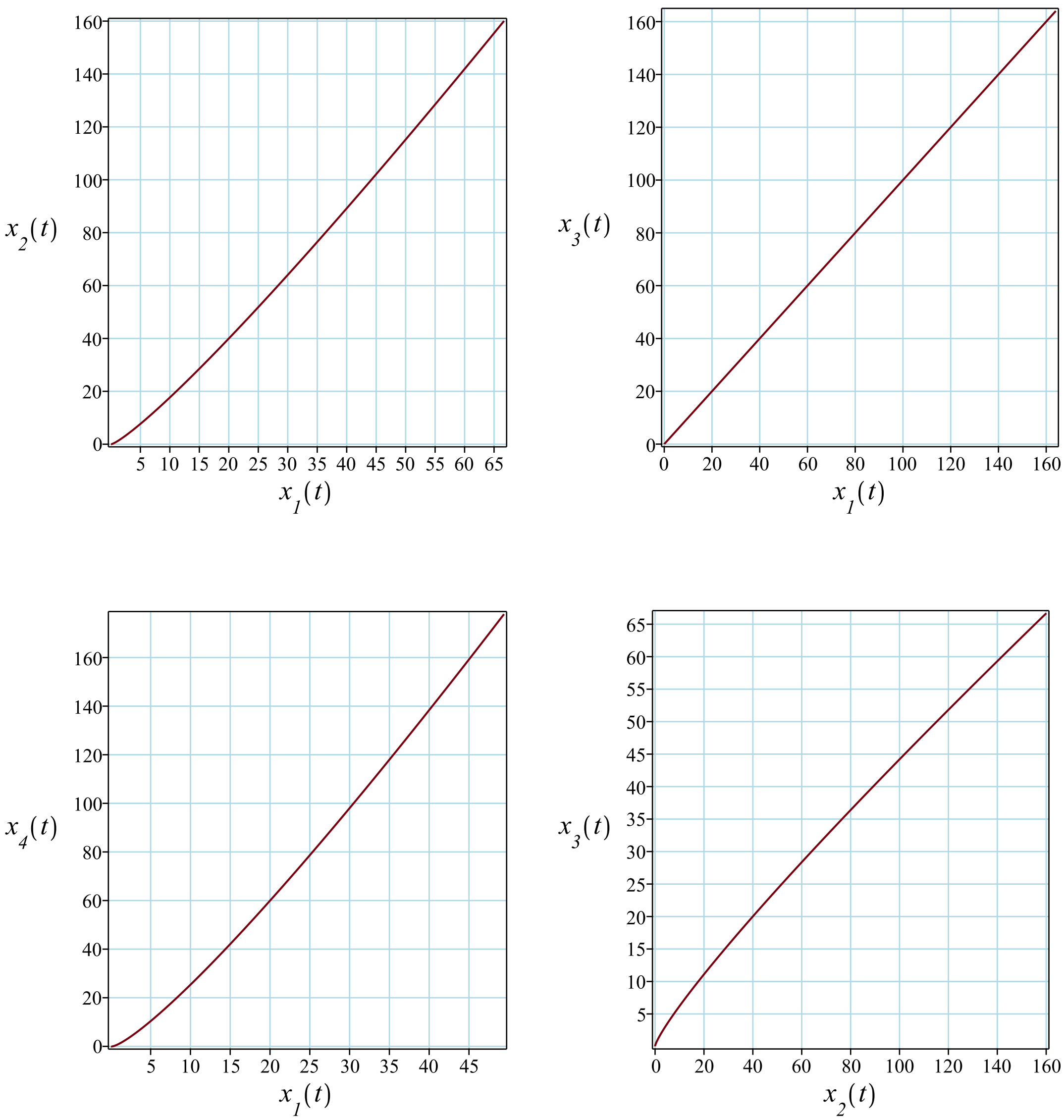
Solving for the constants of integrations gives

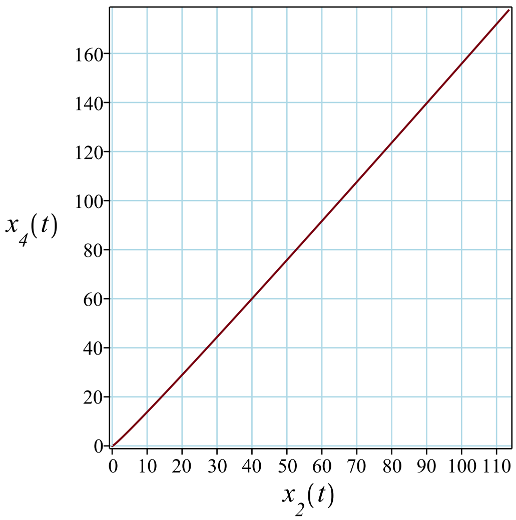
$$\begin{bmatrix} c_1 = 1 \\ c_2 = 1 \\ c_3 = -\frac{1}{2} \\ c_4 = 0 \end{bmatrix}$$

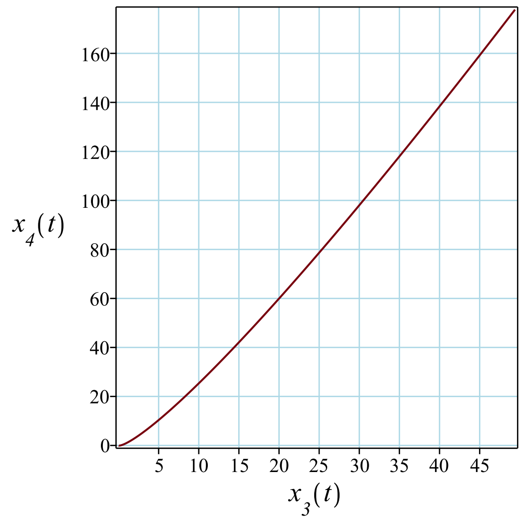
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} e^{3t} \\ e^{3t}(t + 1) \\ e^{3t} \\ 2\left(t + \frac{1}{2}\right) e^{3t} \end{bmatrix}$$

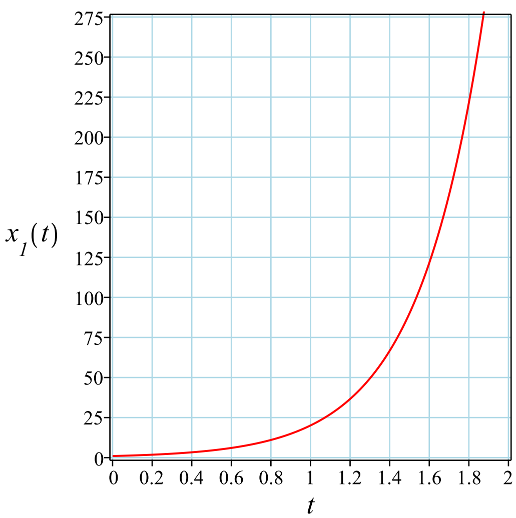
The following are plots of each solution against another.

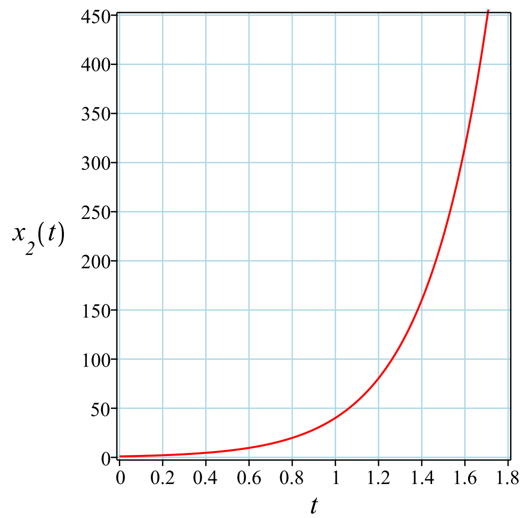


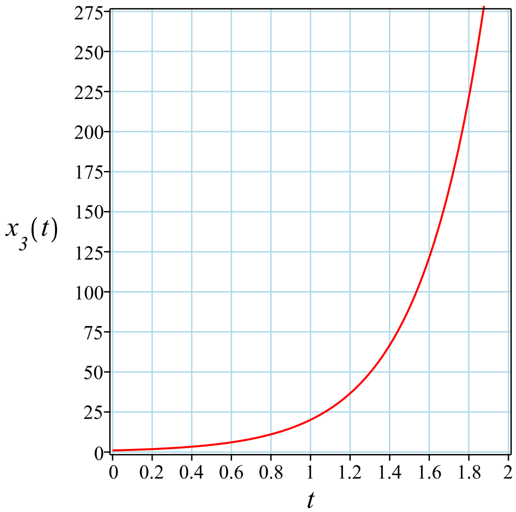


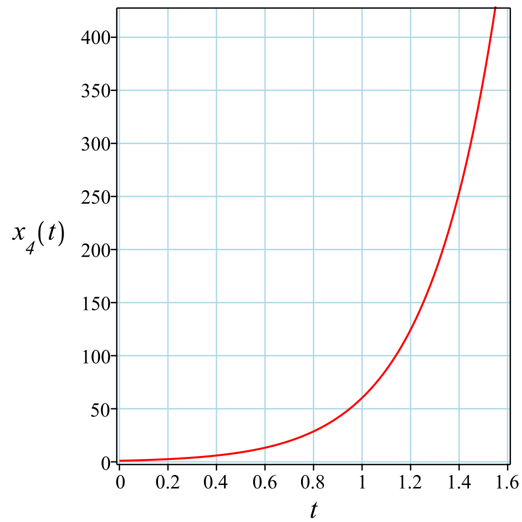


The following are plots of each solution.









✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 40

```
dsolve([diff(x__1(t),t) = 3*x__1(t), diff(x__2(t),t) = x__1(t)+3*x__2(t), diff(x__3(t),t) =
```

$$x_1(t) = e^{3t}$$
$$x_2(t) = (t+1)\,e^{3t}$$
$$x_3(t) = e^{3t}$$
$$x_4(t) = (2t+1)\,e^{3t}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 44

```
DSolve[{x1'[t]==3*x1[t]+0*x2[t]+0*x3[t]+0*x4[t],x2'[t]==1*x1[t]+3*x2[t]-0*x3[t]+0*x4[t],x3'[
```

$$\text{x1}(t) \to e^{3t}$$
$$\text{x2}(t) \to e^{3t}(t+1)$$
$$\text{x3}(t) \to e^{3t}$$
$$\text{x4}(t) \to e^{3t}(2t+1)$$

4 Section 3.12, Systems of differential equations. The nonhomogeneous equation. variation of parameters. Page 366

4.1 problem Example 1, page 361

Internal problem ID [1854]
Internal file name [`OUTPUT/1855_Sunday_June_05_2022_02_35_26_AM_67103142/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.12, Systems of differential equations. The nonhomogeneous equation. variation of parameters. Page 366
Problem number: Example 1, page 361.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_1(t) \\
x_2'(t) &= 2x_1(t) + x_2(t) - 2x_3(t) \\
x_3'(t) &= 3x_1(t) + 2x_2(t) + x_3(t) + 2\cos(t)^2 \mathrm{e}^t - \mathrm{e}^t
\end{aligned}
$$

4.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2\cos(t)^2 \mathrm{e}^t - \mathrm{e}^t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} \mathrm{e}^{t} & 0 & 0 \\ \frac{3\,\mathrm{e}^{t}\cos(2t)}{2} + \mathrm{e}^{t}\sin(2t) - \frac{3\,\mathrm{e}^{t}}{2} & \mathrm{e}^{t}\cos(2t) & -\mathrm{e}^{t}\sin(2t) \\ -\mathrm{e}^{t}\cos(2t) + \frac{3\,\mathrm{e}^{t}\sin(2t)}{2} + \mathrm{e}^{t} & \mathrm{e}^{t}\sin(2t) & \mathrm{e}^{t}\cos(2t) \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{t} & 0 & 0 \\ \frac{\mathrm{e}^{t}(-3+3\cos(2t)+2\sin(2t))}{2} & \mathrm{e}^{t}\cos(2t) & -\mathrm{e}^{t}\sin(2t) \\ -\frac{\mathrm{e}^{t}(-2+2\cos(2t)-3\sin(2t))}{2} & \mathrm{e}^{t}\sin(2t) & \mathrm{e}^{t}\cos(2t) \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \mathrm{e}^{t} & 0 & 0 \\ \frac{\mathrm{e}^{t}(-3+3\cos(2t)+2\sin(2t))}{2} & \mathrm{e}^{t}\cos(2t) & -\mathrm{e}^{t}\sin(2t) \\ -\frac{\mathrm{e}^{t}(-2+2\cos(2t)-3\sin(2t))}{2} & \mathrm{e}^{t}\sin(2t) & \mathrm{e}^{t}\cos(2t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{t}c_1 \\ \frac{\mathrm{e}^{t}(-3+3\cos(2t)+2\sin(2t))c_1}{2} + \mathrm{e}^{t}\cos(2t)\,c_2 - \mathrm{e}^{t}\sin(2t)\,c_3 \\ -\frac{\mathrm{e}^{t}(-2+2\cos(2t)-3\sin(2t))c_1}{2} + \mathrm{e}^{t}\sin(2t)\,c_2 + \mathrm{e}^{t}\cos(2t)\,c_3 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{t}c_1 \\ \frac{3\left(\left(c_1+\frac{2c_2}{3}\right)\cos(2t)+\frac{2(c_1-c_3)\sin(2t)}{3}-c_1\right)\mathrm{e}^{t}}{2} \\ -\left((c_1-c_3)\cos(2t)+\left(-\frac{3c_1}{2}-c_2\right)\sin(2t)-c_1\right)\mathrm{e}^{t} \end{bmatrix}
\end{aligned}
$$

The particular solution given by

$$
\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt
$$

But

$$
\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} \mathrm{e}^{-t} & 0 & 0 \\ \frac{(-3+3\cos(2t)-2\sin(2t))\mathrm{e}^{-t}}{2} & \mathrm{e}^{-t}\cos(2t) & \mathrm{e}^{-t}\sin(2t) \\ -\frac{(-2+2\cos(2t)+3\sin(2t))\mathrm{e}^{-t}}{2} & -\mathrm{e}^{-t}\sin(2t) & \mathrm{e}^{-t}\cos(2t) \end{bmatrix}
\end{aligned}
$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} \mathrm{e}^{t} & 0 & 0 \\ \frac{\mathrm{e}^{t}(-3+3\cos(2t)+2\sin(2t))}{2} & \mathrm{e}^{t}\cos(2t) & -\mathrm{e}^{t}\sin(2t) \\ -\frac{\mathrm{e}^{t}(-2+2\cos(2t)-3\sin(2t))}{2} & \mathrm{e}^{t}\sin(2t) & \mathrm{e}^{t}\cos(2t) \end{bmatrix} \int \begin{bmatrix} \mathrm{e}^{-t} & 0 \\ \frac{(-3+3\cos(2t)-2\sin(2t))\mathrm{e}^{-t}}{2} & \mathrm{e}^{-t}\cos(2t) \\ -\frac{(-2+2\cos(2t)+3\sin(2t))\mathrm{e}^{-t}}{2} & -\mathrm{e}^{-t}\sin(2t) \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{e}^{t} & 0 & 0 \\ \frac{\mathrm{e}^{t}(-3+3\cos(2t)+2\sin(2t))}{2} & \mathrm{e}^{t}\cos(2t) & -\mathrm{e}^{t}\sin(2t) \\ -\frac{\mathrm{e}^{t}(-2+2\cos(2t)-3\sin(2t))}{2} & \mathrm{e}^{t}\sin(2t) & \mathrm{e}^{t}\cos(2t) \end{bmatrix} \begin{bmatrix} 0 \\ \cos(t)^{2}\sin(t)^{2} \\ \frac{t}{2}+\frac{\sin(4t)}{8} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -\frac{\mathrm{e}^{t} t\sin(2t)}{2} \\ \frac{\left(2\cos(t)^{2} t+\sin(t)\cos(t)-t\right)\mathrm{e}^{t}}{2} \end{bmatrix}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$= \begin{bmatrix} \mathrm{e}^{t} c_1 \\ -\frac{((t-2c_1+2c_3)\sin(2t)+(-3c_1-2c_2)\cos(2t)+3c_1)\mathrm{e}^{t}}{2} \\ \left((t+2c_3)\cos(t)^{2}+3\left(c_1+\frac{2c_2}{3}+\frac{1}{6}\right)\sin(t)\cos(t)+2\sin(t)^{2} c_1-\frac{t}{2}-c_3\right)\mathrm{e}^{t} \end{bmatrix}$$

4.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2\cos(t)^{2}\mathrm{e}^{t}-\mathrm{e}^{t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 3\lambda^2 + 7\lambda - 5 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1 + 2i$$
$$\lambda_2 = 1 - 2i$$
$$\lambda_3 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
$1 + 2i$	1	complex eigenvalue
$1 - 2i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 \\ 3 & 2 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{3R_1}{2} \implies \left[\begin{array}{ccc|c} 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 2 & 0 & -2 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = t, v_2 = -\frac{3t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} t \\ -\frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} t \\ -\frac{3t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ -\frac{3t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ -\frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} t \\ -\frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1 - 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} - (1 - 2i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2i & 0 & 0 \\ 2 & 2i & -2 \\ 3 & 2 & 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2i & 0 & 0 & 0 \\ 2 & 2i & -2 & 0 \\ 3 & 2 & 2i & 0 \end{array}\right]$$

$$R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} 2i & 0 & 0 & 0 \\ 0 & 2i & -2 & 0 \\ 3 & 2 & 2i & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{3iR_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 2i & 0 & 0 & 0 \\ 0 & 2i & -2 & 0 \\ 0 & 2 & 2i & 0 \end{array}\right]$$

$$R_3 = iR_2 + R_3 \Longrightarrow \left[\begin{array}{ccc|c} 2i & 0 & 0 & 0 \\ 0 & 2i & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2i & 0 & 0 \\ 0 & 2i & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = -it\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ \text{-I}t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ \text{-I}t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ \text{-I}t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 1 + 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} - (1+2i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2i & 0 & 0 \\ 2 & -2i & -2 \\ 3 & 2 & -2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2i & 0 & 0 & 0 \\ 2 & -2i & -2 & 0 \\ 3 & 2 & -2i & 0 \end{array}\right]$$

$$R_2 = -iR_1 + R_2 \implies \left[\begin{array}{ccc|c} -2i & 0 & 0 & 0 \\ 0 & -2i & -2 & 0 \\ 3 & 2 & -2i & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{3iR_1}{2} \implies \left[\begin{array}{ccc|c} -2i & 0 & 0 & 0 \\ 0 & -2i & -2 & 0 \\ 0 & 2 & -2i & 0 \end{array}\right]$$

$$R_3 = -iR_2 + R_3 \implies \left[\begin{array}{ccc|c} -2i & 0 & 0 & 0 \\ 0 & -2i & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2i & 0 & 0 \\ 0 & -2i & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = it\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ \mathrm{I}t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$1+2i$	1	1	No	$\begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$
$1-2i$	1	1	No	$\begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^t \\
&= \begin{bmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{bmatrix} e^t
\end{aligned}
$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ i \mathrm{e}^{(1+2i)t} \\ \mathrm{e}^{(1+2i)t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -i \mathrm{e}^{(1-2i)t} \\ \mathrm{e}^{(1-2i)t} \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^t \\ -\frac{3\,\mathrm{e}^t}{2} \\ \mathrm{e}^t \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} 0 & 0 & e^t \\ i e^{(1+2i)t} & -i e^{(1-2i)t} & -\frac{3e^t}{2} \\ e^{(1+2i)t} & e^{(1-2i)t} & e^t \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \left(-\frac{1}{2} - \frac{3i}{4}\right) e^{(-1-2i)t} & -\frac{i e^{(-1-2i)t}}{2} & \frac{e^{(-1-2i)t}}{2} \\ \left(-\frac{1}{2} + \frac{3i}{4}\right) e^{(-1+2i)t} & \frac{i e^{(-1+2i)t}}{2} & \frac{e^{(-1+2i)t}}{2} \\ e^{-t} & 0 & 0 \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} 0 & 0 & e^t \\ i e^{(1+2i)t} & -i e^{(1-2i)t} & -\frac{3e^t}{2} \\ e^{(1+2i)t} & e^{(1-2i)t} & e^t \end{bmatrix} \int \begin{bmatrix} \left(-\frac{1}{2} - \frac{3i}{4}\right) e^{(-1-2i)t} & -\frac{i e^{(-1-2i)t}}{2} & \frac{e^{(-1-2i)t}}{2} \\ \left(-\frac{1}{2} + \frac{3i}{4}\right) e^{(-1+2i)t} & \frac{i e^{(-1+2i)t}}{2} & \frac{e^{(-1+2i)t}}{2} \\ e^{-t} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2\cos(t)^2 e^t - e^t \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & e^t \\ i e^{(1+2i)t} & -i e^{(1-2i)t} & -\frac{3e^t}{2} \\ e^{(1+2i)t} & e^{(1-2i)t} & e^t \end{bmatrix} \int \begin{bmatrix} \frac{e^{-2it}\cos(2t)}{2} \\ \frac{e^{2it}\cos(2t)}{2} \\ 0 \end{bmatrix} dt \\
&= \begin{bmatrix} 0 & 0 & e^t \\ i e^{(1+2i)t} & -i e^{(1-2i)t} & -\frac{3e^t}{2} \\ e^{(1+2i)t} & e^{(1-2i)t} & e^t \end{bmatrix} \begin{bmatrix} \frac{t}{4} + \frac{i e^{-4it}}{16} \\ -\frac{i e^{4it}}{16} + \frac{t}{4} \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ \frac{i e^{(1+2i)t} t}{4} - \frac{e^{(1-2i)t}}{16} - \frac{e^{(1+2i)t}}{16} - \frac{i e^{(1-2i)t} t}{4} \\ \frac{(i+4t) e^{(1-2i)t}}{16} - \frac{(i-4t) e^{(1+2i)t}}{16} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 0 \\ i c_1 \mathrm{e}^{(1+2i)t} \\ c_1 \mathrm{e}^{(1+2i)t} \end{bmatrix} + \begin{bmatrix} 0 \\ -i c_2 \mathrm{e}^{(1-2i)t} \\ c_2 \mathrm{e}^{(1-2i)t} \end{bmatrix} + \begin{bmatrix} c_3 \mathrm{e}^t \\ -\frac{3 c_3 \mathrm{e}^t}{2} \\ c_3 \mathrm{e}^t \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{i \mathrm{e}^{(1+2i)t} t}{4} - \frac{\mathrm{e}^{(1-2i)t}}{16} - \frac{\mathrm{e}^{(1+2i)t}}{16} - \frac{i \mathrm{e}^{(1-2i)t} t}{4} \\ \frac{(i+4t)\mathrm{e}^{(1-2i)t}}{16} - \frac{(i-4t)\mathrm{e}^{(1+2i)t}}{16} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} c_3 \mathrm{e}^t \\ \frac{(-4it-16ic_2-1)\mathrm{e}^{(1-2i)t}}{16} + \frac{(4it+16ic_1-1)\mathrm{e}^{(1+2i)t}}{16} - \frac{3c_3\mathrm{e}^t}{2} \\ \frac{(i+4t+16c_2)\mathrm{e}^{(1-2i)t}}{16} + \frac{(-i+4t+16c_1)\mathrm{e}^{(1+2i)t}}{16} + c_3 \mathrm{e}^t \end{bmatrix}$$

4.1.3 Maple step by step solution

Let's solve

$\left[x_1'(t) = x_1(t), x_2'(t) = 2x_1(t) + x_2(t) - 2x_3(t), x_3'(t) = 3x_1(t) + 2x_2(t) + x_3(t) + 2\cos(t)^2 \mathrm{e}^t - \mathrm{e}\right.$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 2\cos(t)^2 \mathrm{e}^t - \mathrm{e}^t \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 2\cos(t)^2 \mathrm{e}^t - \mathrm{e}^t \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \left[\begin{array}{c} 0 \\ 0 \\ 2\cos(t)^2 \mathrm{e}^t - \mathrm{e}^t \end{array}\right]$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{array}\right]$$

- Rewrite the system as

$$\underline{x}^{\rightarrow}{}'(t) = A \cdot \underline{x}^{\rightarrow}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \left[\begin{array}{c} 1 \\ -\frac{3}{2} \\ 1 \end{array}\right]\right], \left[1-2\,\mathrm{I}, \left[\begin{array}{c} 0 \\ -\mathrm{I} \\ 1 \end{array}\right]\right], \left[1+2\,\mathrm{I}, \left[\begin{array}{c} 0 \\ \mathrm{I} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c} 1 \\ -\frac{3}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{x}^{\rightarrow}{}_{1} = \mathrm{e}^t \cdot \left[\begin{array}{c} 1 \\ -\frac{3}{2} \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1-2\,\mathrm{I}, \left[\begin{array}{c} 0 \\ -\mathrm{I} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{(1-2I)t} \cdot \begin{bmatrix} 0 \\ -I \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$e^t \cdot (\cos(2t) - I\sin(2t)) \cdot \begin{bmatrix} 0 \\ -I \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^t \cdot \begin{bmatrix} 0 \\ -I(\cos(2t) - I\sin(2t)) \\ \cos(2t) - I\sin(2t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{\vec{x}}_2(t) = e^t \cdot \begin{bmatrix} 0 \\ -\sin(2t) \\ \cos(2t) \end{bmatrix}, \underline{\vec{x}}_3(t) = e^t \cdot \begin{bmatrix} 0 \\ -\cos(2t) \\ -\sin(2t) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\underline{\vec{x}}$

$$\underline{\vec{x}}(t) = c_1\underline{\vec{x}}_1 + c_2\underline{\vec{x}}_2(t) + c_3\underline{\vec{x}}_3(t) + \underline{\vec{x}}_p(t)$$

□ Fundamental matrix

  - Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} e^t & 0 & 0 \\ -\frac{3e^t}{2} & -e^t\sin(2t) & -e^t\cos(2t) \\ e^t & e^t\cos(2t) & -e^t\sin(2t) \end{bmatrix}$$

  - The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$$

  - Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} e^t & 0 & 0 \\ -\frac{3 e^t}{2} & -e^t \sin(2t) & -e^t \cos(2t) \\ e^t & e^t \cos(2t) & -e^t \sin(2t) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} e^t & 0 & 0 \\ \frac{e^t(-3+3\cos(2t)+2\sin(2t))}{2} & e^t \cos(2t) & -e^t \sin(2t) \\ -\frac{e^t(-2+2\cos(2t)-3\sin(2t))}{2} & e^t \sin(2t) & e^t \cos(2t) \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$

- Take the derivative of the particular solution

$\vec{x}_p{}'(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

$\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

$\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

$\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

$\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds \right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t) = \begin{bmatrix} 0 \\ -\frac{e^t t \sin(2t)}{2} \\ \frac{e^t(2\cos(2t)t+\sin(2t))}{4} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2(t) + c_3\vec{x}_3(t) + \begin{bmatrix} 0 \\ -\frac{e^t t \sin(2t)}{2} \\ \frac{e^t(2\cos(2t)t+\sin(2t))}{4} \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} c_1 e^t \\ -\frac{e^t((t+2c_2)\sin(2t)+2c_3\cos(2t)+3c_1)}{2} \\ \frac{\left((t+2c_2)\cos(2t)+\left(-2c_3+\frac{1}{2}\right)\sin(2t)+2c_1\right)e^t}{2} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = c_1 e^t, x_2(t) = -\frac{e^t((t+2c_2)\sin(2t)+2c_3\cos(2t)+3c_1)}{2}, x_3(t) = \frac{\left((t+2c_2)\cos(2t)+\left(-2c_3+\frac{1}{2}\right)\sin(2t)+2c_1\right)e^t}{2}\right\}$$

✓ Solution by Maple

Time used: 0.485 (sec). Leaf size: 93

```
dsolve([diff(x__1(t),t)=1*x__1(t)+0*x__2(t)+0*x__3(t),diff(x__2(t),t)=2*x__1(t)+1*x__2(t)-2*
```

$$x_1(t) = c_3 e^t$$

$$x_2(t) = \frac{e^t(-3c_3 - 3c_3\cos(2t) + 2c_1\cos(2t) + 2c_2\sin(2t) - \sin(2t)\,t)}{2}$$

$$x_3(t) = -\frac{e^t(4c_2\cos(2t) - 2t\cos(2t) - 4c_1\sin(2t) + 6c_3\sin(2t) - \sin(2t) - 4c_3)}{4}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 103

```
DSolve[{x1'[t]==1*x1[t]+0*x2[t]+0*x3[t],x2'[t]==2*x1[t]+1*x2[t]-2*x3[t],x3'[t]==3*x1[t]+2*x2
```

$$x1(t) \to c_1 e^t$$
$$x2(t) \to -\frac{1}{8}e^t((1-12c_1-8c_2)\cos(2t)+4(t-2c_1+2c_3)\sin(2t)+12c_1)$$
$$x3(t) \to \frac{1}{8}e^t(4(t-2c_1+2c_3)\cos(2t)+(1+12c_1+8c_2)\sin(2t)+8c_1)$$

4.2 problem Example 2, page 364

Internal problem ID [1855]
Internal file name [`OUTPUT/1856_Sunday_June_05_2022_02_35_31_AM_54055234/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.12, Systems of differential equations. The nonhomogeneous equation. variation of parameters. Page 366
Problem number: Example 2, page 364.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_1(t) + \mathrm{e}^{ct} \\
x_2'(t) &= 2x_1(t) + x_2(t) - 2x_3(t) \\
x_3'(t) &= 3x_1(t) + 2x_2(t) + x_3(t)
\end{aligned}
$$

4.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} \mathrm{e}^{ct} \\ 0 \\ 0 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \left[\begin{array}{ccc} \mathrm{e}^{t} & 0 & 0 \\ \frac{3\,\mathrm{e}^{t}\cos(2t)}{2}+\mathrm{e}^{t}\sin(2t)-\frac{3\,\mathrm{e}^{t}}{2} & \mathrm{e}^{t}\cos(2t) & -\mathrm{e}^{t}\sin(2t) \\ -\mathrm{e}^{t}\cos(2t)+\frac{3\,\mathrm{e}^{t}\sin(2t)}{2}+\mathrm{e}^{t} & \mathrm{e}^{t}\sin(2t) & \mathrm{e}^{t}\cos(2t) \end{array}\right] \\
&= \left[\begin{array}{ccc} \mathrm{e}^{t} & 0 & 0 \\ \frac{\mathrm{e}^{t}(-3+3\cos(2t)+2\sin(2t))}{2} & \mathrm{e}^{t}\cos(2t) & -\mathrm{e}^{t}\sin(2t) \\ -\frac{\mathrm{e}^{t}(-2+2\cos(2t)-3\sin(2t))}{2} & \mathrm{e}^{t}\sin(2t) & \mathrm{e}^{t}\cos(2t) \end{array}\right]
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \left[\begin{array}{ccc} \mathrm{e}^{t} & 0 & 0 \\ \frac{\mathrm{e}^{t}(-3+3\cos(2t)+2\sin(2t))}{2} & \mathrm{e}^{t}\cos(2t) & -\mathrm{e}^{t}\sin(2t) \\ -\frac{\mathrm{e}^{t}(-2+2\cos(2t)-3\sin(2t))}{2} & \mathrm{e}^{t}\sin(2t) & \mathrm{e}^{t}\cos(2t) \end{array}\right] \left[\begin{array}{c} c_1 \\ c_2 \\ c_3 \end{array}\right] \\
&= \left[\begin{array}{c} \mathrm{e}^{t}c_1 \\ \frac{\mathrm{e}^{t}(-3+3\cos(2t)+2\sin(2t))c_1}{2}+\mathrm{e}^{t}\cos(2t)\,c_2-\mathrm{e}^{t}\sin(2t)\,c_3 \\ -\frac{\mathrm{e}^{t}(-2+2\cos(2t)-3\sin(2t))c_1}{2}+\mathrm{e}^{t}\sin(2t)\,c_2+\mathrm{e}^{t}\cos(2t)\,c_3 \end{array}\right] \\
&= \left[\begin{array}{c} \mathrm{e}^{t}c_1 \\ \frac{3\left(\left(c_1+\frac{2c_2}{3}\right)\cos(2t)+\frac{2(c_1-c_3)\sin(2t)}{3}-c_1\right)\mathrm{e}^{t}}{2} \\ -\left((c_1-c_3)\cos(2t)+\left(-\frac{3c_1}{2}-c_2\right)\sin(2t)-c_1\right)\mathrm{e}^{t} \end{array}\right]
\end{aligned}
$$

The particular solution given by

$$
\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt
$$

But

$$
\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \left[\begin{array}{ccc} \mathrm{e}^{-t} & 0 & 0 \\ \frac{(-3+3\cos(2t)-2\sin(2t))\mathrm{e}^{-t}}{2} & \mathrm{e}^{-t}\cos(2t) & \mathrm{e}^{-t}\sin(2t) \\ -\frac{(-2+2\cos(2t)+3\sin(2t))\mathrm{e}^{-t}}{2} & -\mathrm{e}^{-t}\sin(2t) & \mathrm{e}^{-t}\cos(2t) \end{array}\right]
\end{aligned}
$$

Hence

$$
\vec{x}_p(t) = \left[\begin{array}{ccc} \mathrm{e}^{t} & 0 & 0 \\ \frac{\mathrm{e}^{t}(-3+3\cos(2t)+2\sin(2t))}{2} & \mathrm{e}^{t}\cos(2t) & -\mathrm{e}^{t}\sin(2t) \\ -\frac{\mathrm{e}^{t}(-2+2\cos(2t)-3\sin(2t))}{2} & \mathrm{e}^{t}\sin(2t) & \mathrm{e}^{t}\cos(2t) \end{array}\right] \int \left[\begin{array}{cc} \mathrm{e}^{-t} & 0 \;\cdots \\ \frac{(-3+3\cos(2t)-2\sin(2t))\mathrm{e}^{-t}}{2} & \mathrm{e}^{-t}\cos(2t) \;\cdots \\ -\frac{(-2+2\cos(2t)+3\sin(2t))\mathrm{e}^{-t}}{2} & -\mathrm{e}^{-t}\sin(2t) \;\cdots \end{array}\right.
$$

$$
= \left[\begin{array}{ccc} \mathrm{e}^{t} & 0 & 0 \\ \frac{\mathrm{e}^{t}(-3+3\cos(2t)+2\sin(2t))}{2} & \mathrm{e}^{t}\cos(2t) & -\mathrm{e}^{t}\sin(2t) \\ -\frac{\mathrm{e}^{t}(-2+2\cos(2t)-3\sin(2t))}{2} & \mathrm{e}^{t}\sin(2t) & \mathrm{e}^{t}\cos(2t) \end{array}\right] \left[\begin{array}{c} \frac{\mathrm{e}^{t(c-1)}}{c-1} \\ \frac{3\left(\left(c^{2}-\frac{2}{3}c-\frac{1}{3}\right)\cos(2t)-\frac{2(c-1)(c-4)\sin(2t)}{3}-c^{2}+2c-5\right)\mathrm{e}^{t(\cdots}}{2(c-1)(c^{2}-2c+5)} \\ -\frac{\left(\left(c^{2}-5c+4\right)\cos(2t)+\left(\frac{3}{2}c^{2}-c-\frac{1}{2}\right)\sin(2t)-c^{2}+2c-5\right)\mathrm{e}^{t(\cdots}}{(c-1)(c^{2}-2c+5)} \end{array}\right.
$$

$$
= \left[\begin{array}{c} \frac{\mathrm{e}^{ct}}{c-1} \\ \frac{2(c-4)\mathrm{e}^{ct}}{(c-1)(c^{2}-2c+5)} \\ \frac{(3c+1)\mathrm{e}^{ct}}{(c-1)(c^{2}-2c+5)} \end{array}\right]
$$

Hence the complete solution is

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

$$
= \left[\begin{array}{c} \frac{\mathrm{e}^{ct}+(c-1)c_{1}\mathrm{e}^{t}}{c-1} \\ \frac{3\left(c_{1}+\frac{2c_{2}}{3}\right)(c^{2}-2c+5)(c-1)\mathrm{e}^{t}\cos(2t)+2\,\mathrm{e}^{t}(c-1)(c^{2}-2c+5)(c_{1}-c_{3})\sin(2t)+4(c-4)\mathrm{e}^{ct}-3\,\mathrm{e}^{t}c_{1}(c-1)(c^{2}-2c+5)}{2(c-1)(c^{2}-2c+5)} \\ \frac{-2\,\mathrm{e}^{t}(c-1)(c^{2}-2c+5)(c_{1}-c_{3})\cos(2t)+3\left(c_{1}+\frac{2c_{2}}{3}\right)(c^{2}-2c+5)(c-1)\mathrm{e}^{t}\sin(2t)+(6c+2)\mathrm{e}^{ct}+2\,\mathrm{e}^{t}c_{1}(c-1)(c^{2}-2c+5)}{2(c-1)(c^{2}-2c+5)} \end{array}\right]
$$

4.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\left[\begin{array}{c} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{array}\right] = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{array}\right] \left[\begin{array}{c} x_1(t) \\ x_2(t) \\ x_3(t) \end{array}\right] + \left[\begin{array}{c} \mathrm{e}^{ct} \\ 0 \\ 0 \end{array}\right]
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 3\lambda^2 + 7\lambda - 5 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1 + 2i$$
$$\lambda_2 = 1 - 2i$$
$$\lambda_3 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
$1+2i$	1	complex eigenvalue
$1-2i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 \\ 3 & 2 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{3R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 2 & 0 & -2 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = t, v_2 = -\frac{3t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} t \\ -\frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} t \\ -\frac{3t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ -\frac{3t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ -\frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} t \\ -\frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1 - 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} - (1 - 2i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2i & 0 & 0 \\ 2 & 2i & -2 \\ 3 & 2 & 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2i & 0 & 0 & 0 \\ 2 & 2i & -2 & 0 \\ 3 & 2 & 2i & 0 \end{array}\right]$$

$$R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} 2i & 0 & 0 & 0 \\ 0 & 2i & -2 & 0 \\ 3 & 2 & 2i & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{3iR_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 2i & 0 & 0 & 0 \\ 0 & 2i & -2 & 0 \\ 0 & 2 & 2i & 0 \end{array}\right]$$

$$R_3 = iR_2 + R_3 \Longrightarrow \left[\begin{array}{ccc|c} 2i & 0 & 0 & 0 \\ 0 & 2i & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2i & 0 & 0 \\ 0 & 2i & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = -it\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ \text{-I}\,t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ \text{-I}\,t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ \text{-I}t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 1 + 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} - (1+2i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2i & 0 & 0 \\ 2 & -2i & -2 \\ 3 & 2 & -2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2i & 0 & 0 & 0 \\ 2 & -2i & -2 & 0 \\ 3 & 2 & -2i & 0 \end{array}\right]$$

$$R_2 = -iR_1 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -2i & 0 & 0 & 0 \\ 0 & -2i & -2 & 0 \\ 3 & 2 & -2i & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{3iR_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2i & 0 & 0 & 0 \\ 0 & -2i & -2 & 0 \\ 0 & 2 & -2i & 0 \end{array}\right]$$

$$R_3 = -iR_2 + R_3 \Longrightarrow \left[\begin{array}{ccc|c} -2i & 0 & 0 & 0 \\ 0 & -2i & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2i & 0 & 0 \\ 0 & -2i & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = it\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ \mathrm{I}t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$1+2i$	1	1	No	$\begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$
$1-2i$	1	1	No	$\begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^t \\ &= \begin{bmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{bmatrix} e^t \end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ i\mathrm{e}^{(1+2i)t} \\ \mathrm{e}^{(1+2i)t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -i\mathrm{e}^{(1-2i)t} \\ \mathrm{e}^{(1-2i)t} \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^t \\ -\frac{3\,\mathrm{e}^t}{2} \\ \mathrm{e}^t \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} 0 & 0 & e^{t} \\ i e^{(1+2i)t} & -i e^{(1-2i)t} & -\frac{3\,e^{t}}{2} \\ e^{(1+2i)t} & e^{(1-2i)t} & e^{t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\,dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \left(-\frac{1}{2}-\frac{3i}{4}\right) e^{(-1-2i)t} & -\frac{i e^{(-1-2i)t}}{2} & \frac{e^{(-1-2i)t}}{2} \\ \left(-\frac{1}{2}+\frac{3i}{4}\right) e^{(-1+2i)t} & \frac{i e^{(-1+2i)t}}{2} & \frac{e^{(-1+2i)t}}{2} \\ e^{-t} & 0 & 0 \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} 0 & 0 & e^{t} \\ i e^{(1+2i)t} & -i e^{(1-2i)t} & -\frac{3\,e^{t}}{2} \\ e^{(1+2i)t} & e^{(1-2i)t} & e^{t} \end{bmatrix} \int \begin{bmatrix} \left(-\frac{1}{2}-\frac{3i}{4}\right) e^{(-1-2i)t} & -\frac{i e^{(-1-2i)t}}{2} & \frac{e^{(-1-2i)t}}{2} \\ \left(-\frac{1}{2}+\frac{3i}{4}\right) e^{(-1+2i)t} & \frac{i e^{(-1+2i)t}}{2} & \frac{e^{(-1+2i)t}}{2} \\ e^{-t} & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{ct} \\ 0 \\ 0 \end{bmatrix} dt \\
&= \begin{bmatrix} 0 & 0 & e^{t} \\ i e^{(1+2i)t} & -i e^{(1-2i)t} & -\frac{3\,e^{t}}{2} \\ e^{(1+2i)t} & e^{(1-2i)t} & e^{t} \end{bmatrix} \int \begin{bmatrix} \left(-\frac{1}{2}-\frac{3i}{4}\right) e^{t(c-1-2i)} \\ \left(-\frac{1}{2}+\frac{3i}{4}\right) e^{t(c-1+2i)} \\ e^{t(c-1)} \end{bmatrix} dt \\
&= \begin{bmatrix} 0 & 0 & e^{t} \\ i e^{(1+2i)t} & -i e^{(1-2i)t} & -\frac{3\,e^{t}}{2} \\ e^{(1+2i)t} & e^{(1-2i)t} & e^{t} \end{bmatrix} \begin{bmatrix} \frac{(3-2i)e^{t(c-1-2i)}}{4ic-4i+8} \\ \frac{(-2+3i)e^{t(c-1+2i)}}{4c-4+8i} \\ \frac{e^{t(c-1)}}{c-1} \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{ct}}{c-1} \\ -\frac{2(c-4)e^{ct}}{-c^3+3c^2-7c+5} \\ \frac{(-3c-1)e^{ct}}{-c^3+3c^2-7c+5} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 0 \\ i c_1 \mathrm{e}^{(1+2i)t} \\ c_1 \mathrm{e}^{(1+2i)t} \end{bmatrix} + \begin{bmatrix} 0 \\ -i c_2 \mathrm{e}^{(1-2i)t} \\ c_2 \mathrm{e}^{(1-2i)t} \end{bmatrix} + \begin{bmatrix} c_3 \mathrm{e}^t \\ -\frac{3 c_3 \mathrm{e}^t}{2} \\ c_3 \mathrm{e}^t \end{bmatrix} + \begin{bmatrix} \frac{\mathrm{e}^{ct}}{c-1} \\ -\frac{2(c-4)\mathrm{e}^{ct}}{-c^3+3c^2-7c+5} \\ \frac{(-3c-1)\mathrm{e}^{ct}}{-c^3+3c^2-7c+5} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{e}^{ct}+(c-1)c_3\mathrm{e}^t}{c-1} \\ \frac{-2i(c^2-2c+5)(c-1)c_2\mathrm{e}^{(1-2i)t}+2i(c^2-2c+5)(c-1)c_1\mathrm{e}^{(1+2i)t}+(4c-16)\mathrm{e}^{ct}-3c_3\mathrm{e}^t(c-1)(c^2-2c+5)}{2c^3-6c^2+14c-10} \\ \frac{(c^2-2c+5)(c-1)c_2\mathrm{e}^{(1-2i)t}+(c^2-2c+5)(c-1)c_1\mathrm{e}^{(1+2i)t}+(3c+1)\mathrm{e}^{ct}+c_3\mathrm{e}^t(c-1)(c^2-2c+5)}{c^3-3c^2+7c-5} \end{bmatrix}$$

4.2.3 Maple step by step solution

Let's solve

$[x_1'(t) = x_1(t) + \mathrm{e}^{ct}, x_2'(t) = 2x_1(t) + x_2(t) - 2x_3(t), x_3'(t) = 3x_1(t) + 2x_2(t) + x_3(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} \mathrm{e}^{ct} \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} \mathrm{e}^{ct} \\ 0 \\ 0 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} e^{ct} \\ 0 \\ 0 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{bmatrix}\right], \left[1 - 2\,\mathrm{I}, \begin{bmatrix} 0 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right], \left[1 + 2\,\mathrm{I}, \begin{bmatrix} 0 \\ \mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_1 = e^t \cdot \begin{bmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1 - 2\,\mathrm{I}, \begin{bmatrix} 0 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{(1-2I)t} \cdot \begin{bmatrix} 0 \\ -I \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{t} \cdot (\cos(2t) - I\sin(2t)) \cdot \begin{bmatrix} 0 \\ -I \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{t} \cdot \begin{bmatrix} 0 \\ -I(\cos(2t) - I\sin(2t)) \\ \cos(2t) - I\sin(2t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{\vec{x}}_{2}(t) = e^{t} \cdot \begin{bmatrix} 0 \\ -\sin(2t) \\ \cos(2t) \end{bmatrix}, \underline{\vec{x}}_{3}(t) = e^{t} \cdot \begin{bmatrix} 0 \\ -\cos(2t) \\ -\sin(2t) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\underline{\vec{x}}$

$$\underline{\vec{x}}(t) = c_1\underline{\vec{x}}_{1} + c_2\underline{\vec{x}}_{2}(t) + c_3\underline{\vec{x}}_{3}(t) + \underline{\vec{x}}_{p}(t)$$

☐ Fundamental matrix

- ○ Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} e^{t} & 0 & 0 \\ -\frac{3e^{t}}{2} & -e^{t}\sin(2t) & -e^{t}\cos(2t) \\ e^{t} & e^{t}\cos(2t) & -e^{t}\sin(2t) \end{bmatrix}$$

- ○ The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$$

- ○ Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} e^t & 0 & 0 \\ -\frac{3e^t}{2} & -e^t \sin(2t) & -e^t \cos(2t) \\ e^t & e^t \cos(2t) & -e^t \sin(2t) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} e^t & 0 & 0 \\ \frac{e^t(-3+3\cos(2t)+2\sin(2t))}{2} & e^t \cos(2t) & -e^t \sin(2t) \\ -\frac{e^t(-2+2\cos(2t)-3\sin(2t))}{2} & e^t \sin(2t) & e^t \cos(2t) \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

  $\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$

- Take the derivative of the particular solution

  $\vec{x}_p'(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

  $\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t) = \begin{bmatrix} \frac{-e^t+e^{ct}}{c-1} \\ \frac{e^t\left(4(c-4)e^{t(c-1)}-3\left(c^2-\frac{2}{3}c-\frac{1}{3}\right)\cos(2t)-2(c-1)(c-4)\sin(2t)+3c^2-6c+15\right)}{2(c-1)(c^2-2c+5)} \\ \frac{\left((3c+1)e^{t(c-1)}+(c^2-5c+4)\cos(2t)+\left(-\frac{3}{2}c^2+c+\frac{1}{2}\right)\sin(2t)-c^2+2c-5\right)e^t}{(c-1)(c^2-2c+5)} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2(t) + c_3\vec{x}_3(t) + \begin{bmatrix} \frac{-e^t+e^{ct}}{c-1} \\ \frac{e^t\left(4(c-4)e^{t(c-1)}-3\left(c^2-\frac{2}{3}c-\frac{1}{3}\right)\cos(2t)-2(c-1)(c-4)\sin(2t)+3\right.}{2(c-1)(c^2-2c+5)} \\ \frac{\left((3c+1)e^{t(c-1)}+(c^2-5c+4)\cos(2t)+\left(-\frac{3}{2}c^2+c+\frac{1}{2}\right)\sin(2t)-c^2\right.}{(c-1)(c^2-2c+5)} \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{e^{ct}+(-1+(c-1)c_1)e^t}{c-1} \\ -\frac{3\left(\frac{4(-c+4)e^{t(c-1)}}{3}+\frac{2\left(c_3c^2+\left(-2c_3+\frac{3}{2}\right)c+5c_3+\frac{1}{2}\right)(c-1)\cos(2t)}{3}+\frac{2\left(c_2c^2+(1-2c_2)c+5c_2-4\right)(c-1)\sin(2t)}{3}+(c^2-2c\right.}{2(c-1)(c^2-2c+5)} \\ \frac{\left((3c+1)e^{t(c-1)}+\left(c_2c^2+(1-2c_2)c+5c_2-4\right)(c-1)\cos(2t)-\left(c_3c^2+\left(-2c_3+\frac{3}{2}\right)c+5c_3+\frac{1}{2}\right)(c-1)\sin(2t)+(c^2-2c\right.}{(c-1)(c^2-2c+5)} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \frac{e^{ct}+(-1+(c-1)c_1)e^t}{c-1}, x_2(t) = -\frac{3\left(\frac{4(-c+4)e^{t(c-1)}}{3}+\frac{2\left(c_3c^2+\left(-2c_3+\frac{3}{2}\right)c+5c_3+\frac{1}{2}\right)(c-1)\cos(2t)}{3}+\frac{2\left(c_2c^2+(1-2c_2)c+5\right.}{3}\right.}{2(c-1)(c^2-2c+5)}\right.$$

✓ Solution by Maple

Time used: 0.375 (sec). Leaf size: 408

```
dsolve([diff(x__1(t),t)=1*x__1(t)+0*x__2(t)+0*x__3(t)+exp(c*t),diff(x__2(t),t)=2*x__1(t)+1*x
```

$$x_1(t) = c_3e^t + \frac{e^{ct}}{c-1}$$

$$x_2(t) = \frac{2\,e^t\sin(2t)\,c_2c^3 + 2\,e^t\cos(2t)\,c_1c^3 - 3c^3e^tc_3\cos(2t) - 6\,e^t\sin(2t)\,c_2c^2 - 6\,e^t\cos(2t)\,c_1c^2 + 9c^2e^tc_3\cos(2}{}$$

$$x_3(t) = \frac{2\,e^t\sin(2t)\,c_1c^3 - 3c^3e^tc_3\sin(2t) - 2\,e^t\cos(2t)\,c_2c^3 - 6\,e^t\sin(2t)\,c_1c^2 + 9c^2e^tc_3\sin(2t) + 6\,e^t\cos(2t)\,c_2c}{}$$

✓ Solution by Mathematica

Time used: 0.496 (sec). Leaf size: 256

```
DSolve[{x1'[t]==1*x1[t]+0*x2[t]+0*x3[t]+Exp[c*t],x2'[t]==2*x1[t]+1*x2[t]-2*x3[t],x3'[t]==3*x
```

$$x1(t) \to e^t \left(\frac{e^{(c-1)t}}{c-1} + c_1\right)$$

x2(t)

$$\to \frac{e^t \left(-3c^3c_1 + 9c^2c_1 + (c^3 - 3c^2 + 7c - 5)(3c_1 + 2c_2)\cos(2t) + 2(c^3 - 3c^2 + 7c - 5)(c_1 - c_3)\sin(2t) + 4c\right.}{2(c-1)(c^2 - 2c + 5)}$$

x3(t)

$$\to \frac{e^t \left(-2(c^3 - 3c^2 + 7c - 5)(c_1 - c_3)\cos(2t) + (c^3 - 3c^2 + 7c - 5)(3c_1 + 2c_2)\sin(2t) + 2(c^3 - 3c^2 + 7c -\right.}{2(c-1)(c^2 - 2c + 5)}$$

4.3 problem 1

Internal problem ID [1856]
Internal file name [OUTPUT/1857_Sunday_June_05_2022_02_35_35_AM_69161224/index.tex]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.12, Systems of differential equations. The nonhomogeneous equation. variation of parameters. Page 366
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = 4x_1(t) + 5x_2(t) + 4\,e^t \cos(t)$$
$$x_2'(t) = -2x_1(t) - 2x_2(t)$$

With initial conditions

$$[x_1(0) = 0, x_2(0) = 0]$$

4.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 4\,e^t \cos(t) \\ 0 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} e^t \cos(t) + 3\sin(t)\, e^t & 5\sin(t)\, e^t \\ -2\sin(t)\, e^t & e^t \cos(t) - 3\sin(t)\, e^t \end{bmatrix} \\
&= \begin{bmatrix} e^t(\cos(t) + 3\sin(t)) & 5\sin(t)\, e^t \\ -2\sin(t)\, e^t & e^t(-3\sin(t) + \cos(t)) \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^t(\cos(t) + 3\sin(t)) & 5\sin(t)\, e^t \\ -2\sin(t)\, e^t & e^t(-3\sin(t) + \cos(t)) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

The particular solution given by

$$
\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t)\, dt
$$

But

$$
\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} (-3\sin(t) + \cos(t))\, e^{-t} & -5\, e^{-t}\sin(t) \\ 2\, e^{-t}\sin(t) & (\cos(t) + 3\sin(t))\, e^{-t} \end{bmatrix}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} e^t(\cos(t) + 3\sin(t)) & 5\sin(t)\, e^t \\ -2\sin(t)\, e^t & e^t(-3\sin(t) + \cos(t)) \end{bmatrix} \int \begin{bmatrix} (-3\sin(t) + \cos(t))\, e^{-t} & -5\, e^{-t}\sin(t \\ 2\, e^{-t}\sin(t) & (\cos(t) + 3\sin(t \end{bmatrix} \\
&= \begin{bmatrix} e^t(\cos(t) + 3\sin(t)) & 5\sin(t)\, e^t \\ -2\sin(t)\, e^t & e^t(-3\sin(t) + \cos(t)) \end{bmatrix} \begin{bmatrix} 6\cos(t)^2 + 2\sin(t)\cos(t) + 2t \\ -4\cos(t)^2 \end{bmatrix} \\
&= \begin{bmatrix} 2((t+3)\cos(t) + 3t\sin(t))\, e^t \\ -4\, e^t(\cos(t) + t\sin(t)) \end{bmatrix}
\end{aligned}
$$

Hence the complete solution is

$$\begin{aligned}\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \begin{bmatrix} 2((t+3)\cos(t) + 3t\sin(t))\,\mathrm{e}^t \\ -4\,\mathrm{e}^t(\cos(t) + t\sin(t)) \end{bmatrix}\end{aligned}$$

4.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 4\,\mathrm{e}^t\cos(t) \\ 0 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 4 & 5 \\ -2 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 4-\lambda & 5 \\ -2 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 2\lambda + 2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1 + i$$
$$\lambda_2 = 1 - i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$1 - i$	1	complex eigenvalue
$1 + i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1 - i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & 5 \\ -2 & -2 \end{bmatrix} - (1 - i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 + i & 5 \\ -2 & -3 + i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 + i & 5 & 0 \\ -2 & -3 + i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(\frac{3}{5} - \frac{i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 3 + i & 5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 + i & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{3}{2} + \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{3}{2}+\frac{\mathrm{I}}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{3}{2}+\frac{i}{2}\right)t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{3}{2}+\frac{\mathrm{I}}{2}\right)t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{3}{2}+\frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{3}{2}+\frac{\mathrm{I}}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}+\frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{3}{2}+\frac{\mathrm{I}}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} -3+i \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1 + i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & 5 \\ -2 & -2 \end{bmatrix} - (1+i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3-i & 5 \\ -2 & -3-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3-i & 5 & 0 \\ -2 & -3-i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(\frac{3}{5} + \frac{i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 3-i & 5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3-i & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{3}{2} - \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{3}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{3}{2} - \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{3}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{3}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{3}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{3}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -3 - i \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$1+i$	1	1	No	$\begin{bmatrix} -\frac{3}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$
$1-i$	1	1	No	$\begin{bmatrix} -\frac{3}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \left(-\frac{3}{2} - \frac{i}{2}\right) e^{(1+i)t} \\ e^{(1+i)t} \end{bmatrix} + c_2 \begin{bmatrix} \left(-\frac{3}{2} + \frac{i}{2}\right) e^{(1-i)t} \\ e^{(1-i)t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \left(-\frac{3}{2} - \frac{i}{2}\right) e^{(1+i)t} & \left(-\frac{3}{2} + \frac{i}{2}\right) e^{(1-i)t} \\ e^{(1+i)t} & e^{(1-i)t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} i e^{(-1-i)t} & \left(\frac{1}{2} + \frac{3i}{2}\right) e^{(-1-i)t} \\ -i e^{(-1+i)t} & \left(\frac{1}{2} - \frac{3i}{2}\right) e^{(-1+i)t} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \left(-\frac{3}{2} - \frac{i}{2}\right) e^{(1+i)t} & \left(-\frac{3}{2} + \frac{i}{2}\right) e^{(1-i)t} \\ e^{(1+i)t} & e^{(1-i)t} \end{bmatrix} \int \begin{bmatrix} i e^{(-1-i)t} & \left(\frac{1}{2} + \frac{3i}{2}\right) e^{(-1-i)t} \\ -i e^{(-1+i)t} & \left(\frac{1}{2} - \frac{3i}{2}\right) e^{(-1+i)t} \end{bmatrix} \begin{bmatrix} 4\, e^{t} \cos(t) \\ 0 \end{bmatrix} dt \\
&= \begin{bmatrix} \left(-\frac{3}{2} - \frac{i}{2}\right) e^{(1+i)t} & \left(-\frac{3}{2} + \frac{i}{2}\right) e^{(1-i)t} \\ e^{(1+i)t} & e^{(1-i)t} \end{bmatrix} \int \begin{bmatrix} 4i \cos(t)\, e^{-it} \\ -4i \cos(t)\, e^{it} \end{bmatrix} dt \\
&= \begin{bmatrix} \left(-\frac{3}{2} - \frac{i}{2}\right) e^{(1+i)t} & \left(-\frac{3}{2} + \frac{i}{2}\right) e^{(1-i)t} \\ e^{(1+i)t} & e^{(1-i)t} \end{bmatrix} \begin{bmatrix} 2((i-t)\sin(t) + it\cos(t))\, e^{-it} \\ -2((t+i)\sin(t) + it\cos(t))\, e^{it} \end{bmatrix} \\
&= \begin{bmatrix} 2\, e^{t}(t\cos(t) + \sin(t) + 3t\sin(t)) \\ -4t\, e^{t} \sin(t) \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \left(-\frac{3}{2}-\frac{i}{2}\right) c_1 \mathrm{e}^{(1+i)t} \\ c_1 \mathrm{e}^{(1+i)t} \end{bmatrix} + \begin{bmatrix} \left(-\frac{3}{2}+\frac{i}{2}\right) c_2 \mathrm{e}^{(1-i)t} \\ c_2 \mathrm{e}^{(1-i)t} \end{bmatrix} + \begin{bmatrix} 2\,\mathrm{e}^t (t\cos(t) + \sin(t) + 3t\sin(t)) \\ -4t\,\mathrm{e}^t \sin(t) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \left(-\frac{3}{2}+\frac{i}{2}\right) c_2 \mathrm{e}^{(1-i)t} + \left(-\frac{3}{2}-\frac{i}{2}\right) c_1 \mathrm{e}^{(1+i)t} + 2((1+3t)\sin(t) + t\cos(t))\,\mathrm{e}^t \\ c_1 \mathrm{e}^{(1+i)t} + c_2 \mathrm{e}^{(1-i)t} - 4t\,\mathrm{e}^t \sin(t) \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = 0 \\ x_2(0) = 0 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

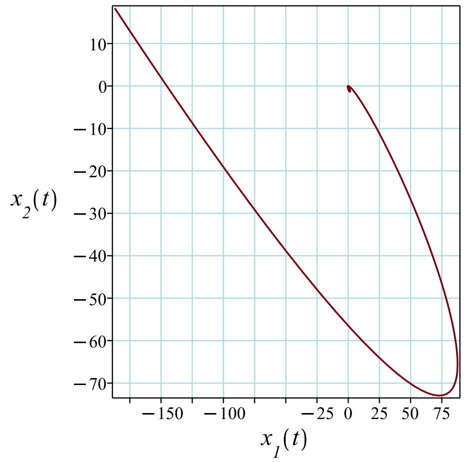
$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \left(-\frac{3}{2}-\frac{i}{2}\right) c_1 + \left(-\frac{3}{2}+\frac{i}{2}\right) c_2 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

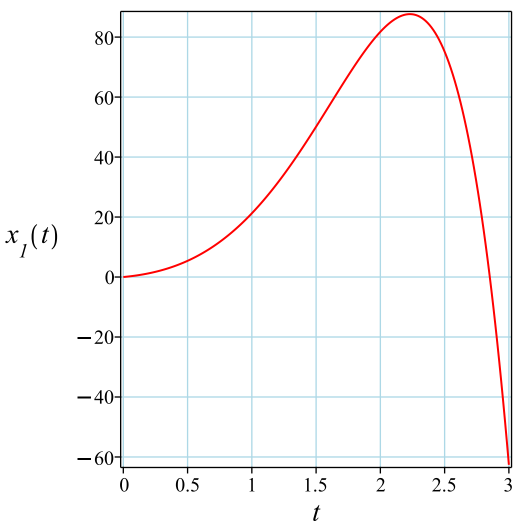
$$\begin{bmatrix} c_1 = 0 \\ c_2 = 0 \end{bmatrix}$$

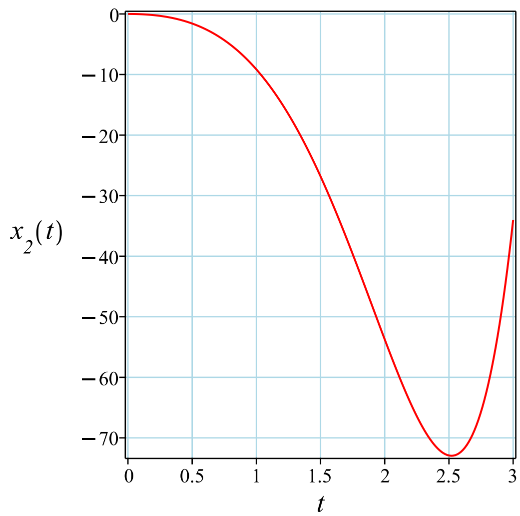
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2((1+3t)\sin(t) + t\cos(t))\,\mathrm{e}^t \\ -4t\,\mathrm{e}^t \sin(t) \end{bmatrix}$$



The following are plots of each solution.





✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 34

```
dsolve([diff(x__1(t),t) = 4*x__1(t)+5*x__2(t)+4*exp(t)*cos(t), diff(x__2(t),t) = -2*x__1(t)-
```

$$x_1(t) = \frac{e^t (12 \sin(t) t + 4 \cos(t) t + 4 \sin(t))}{2}$$
$$x_2(t) = -4 e^t \sin(t) t$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 33

```
DSolve[{x1'[t]==4*x1[t]+5*x2[t]+4*Exp[t]*Cos[t],x2'[t]==-2*x1[t]-2*x2[t]},{x1[0]==0,x2[0]==0
```

$$x1(t) \to 2e^t(3t\sin(t) + \sin(t) + t\cos(t))$$
$$x2(t) \to -4e^t t\sin(t)$$

4.4 problem 2

Internal problem ID [1857]
Internal file name [OUTPUT/1858_Sunday_June_05_2022_02_35_38_AM_51191583/index.tex]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.12, Systems of differential equations. The nonhomogeneous equation. variation of parameters. Page 366
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = 3x_1(t) - 4x_2(t) + e^t$$
$$x_2'(t) = x_1(t) - x_2(t) + e^t$$

With initial conditions

$$[x_1(0) = 1, x_2(0) = 1]$$

4.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} e^t \\ e^t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^t(1+2t) & -4t\,\mathrm{e}^t \\ t\,\mathrm{e}^t & \mathrm{e}^t(1-2t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} \mathrm{e}^t(1+2t) & -4t\,\mathrm{e}^t \\ t\,\mathrm{e}^t & \mathrm{e}^t(1-2t) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{e}^t(1+2t) - 4t\,\mathrm{e}^t \\ t\,\mathrm{e}^t + \mathrm{e}^t(1-2t) \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{e}^t(1-2t) \\ \mathrm{e}^t(1-t) \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} \mathrm{e}^{-t}(1-2t) & 4t\,\mathrm{e}^{-t} \\ -t\,\mathrm{e}^{-t} & \mathrm{e}^{-t}(1+2t) \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} \mathrm{e}^t(1+2t) & -4t\,\mathrm{e}^t \\ t\,\mathrm{e}^t & \mathrm{e}^t(1-2t) \end{bmatrix} \int \begin{bmatrix} \mathrm{e}^{-t}(1-2t) & 4t\,\mathrm{e}^{-t} \\ -t\,\mathrm{e}^{-t} & \mathrm{e}^{-t}(1+2t) \end{bmatrix} \begin{bmatrix} \mathrm{e}^t \\ \mathrm{e}^t \end{bmatrix} dt \\ &= \begin{bmatrix} \mathrm{e}^t(1+2t) & -4t\,\mathrm{e}^t \\ t\,\mathrm{e}^t & \mathrm{e}^t(1-2t) \end{bmatrix} \begin{bmatrix} t(t+1) \\ \frac{t(2+t)}{2} \end{bmatrix} \\ &= \begin{bmatrix} -\mathrm{e}^t t(t-1) \\ -\frac{\mathrm{e}^t t(t-2)}{2} \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
&= \begin{bmatrix} e^t(-t^2 - t + 1) \\ e^t\left(1 - \frac{t^2}{2}\right) \end{bmatrix}
\end{aligned}
$$

4.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} e^t \\ e^t \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 3 - \lambda & -4 \\ 1 & -1 - \lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 - 2\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & -4 & 0 \\ 1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 2 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
1	2	1	Yes	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

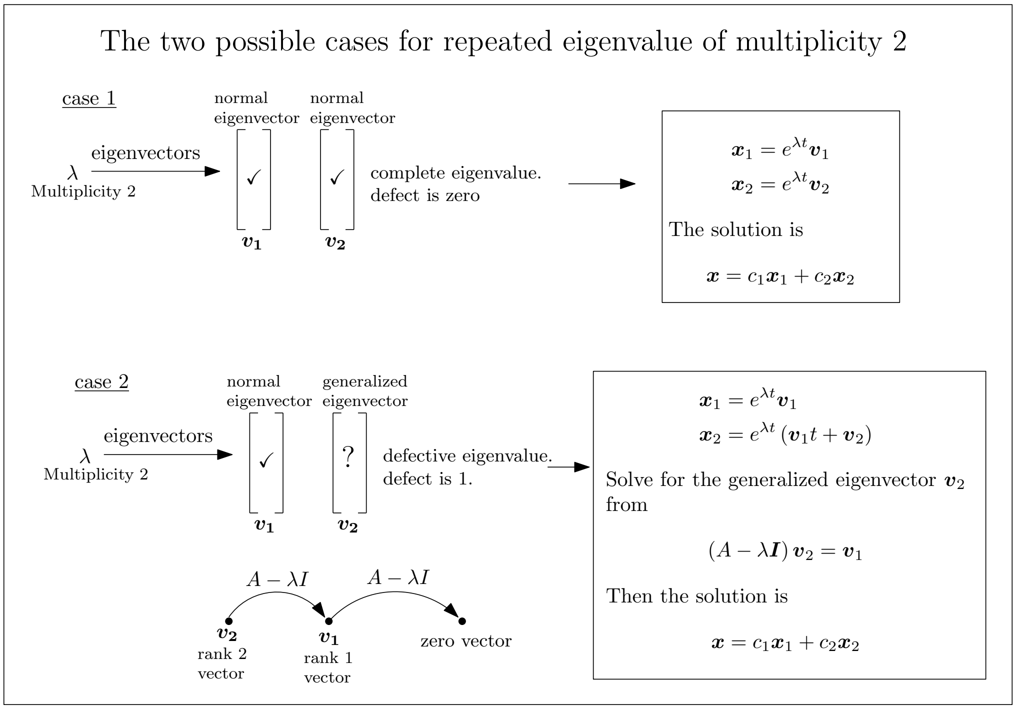


Figure 21: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 1. Therefore the two basis solution associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 2 \\ 1 \end{bmatrix} \mathrm{e}^{t} \\
&= \begin{bmatrix} 2\,\mathrm{e}^{t} \\ \mathrm{e}^{t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right) \mathrm{e}^{t} \\
&= \begin{bmatrix} \mathrm{e}^{t}(2t+3) \\ \mathrm{e}^{t}(t+1) \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 2\,\mathrm{e}^{t} \\ \mathrm{e}^{t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{t}(2t+3) \\ \mathrm{e}^{t}(t+1) \end{bmatrix}
$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$
\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}
$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$
\Phi(t) = \begin{bmatrix} 2\,\mathrm{e}^{t} & \mathrm{e}^{t}(2t+3) \\ \mathrm{e}^{t} & \mathrm{e}^{t}(t+1) \end{bmatrix}
$$

The particular solution is then given by

$$
\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt
$$

But

$$\Phi^{-1} = \begin{bmatrix} -e^{-t}(t+1) & e^{-t}(2t+3) \\ e^{-t} & -2e^{-t} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} 2e^t & e^t(2t+3) \\ e^t & e^t(t+1) \end{bmatrix} \int \begin{bmatrix} -e^{-t}(t+1) & e^{-t}(2t+3) \\ e^{-t} & -2e^{-t} \end{bmatrix} \begin{bmatrix} e^t \\ e^t \end{bmatrix} dt \\
&= \begin{bmatrix} 2e^t & e^t(2t+3) \\ e^t & e^t(t+1) \end{bmatrix} \int \begin{bmatrix} 2+t \\ -1 \end{bmatrix} dt \\
&= \begin{bmatrix} 2e^t & e^t(2t+3) \\ e^t & e^t(t+1) \end{bmatrix} \begin{bmatrix} \frac{t(t+4)}{2} \\ -t \end{bmatrix} \\
&= \begin{bmatrix} -e^t t(t-1) \\ -\frac{e^t t(t-2)}{2} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} 2c_1e^t \\ c_1e^t \end{bmatrix} + \begin{bmatrix} c_2e^t(2t+3) \\ c_2e^t(t+1) \end{bmatrix} + \begin{bmatrix} -e^t t(t-1) \\ -\frac{e^t t(t-2)}{2} \end{bmatrix}
\end{aligned}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -(t^2 + (-2c_2 - 1)t - 2c_1 - 3c_2)e^t \\ e^t\left(c_1 + c_2t + c_2 - \frac{1}{2}t^2 + t\right) \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = 1 \\ x_2(0) = 1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

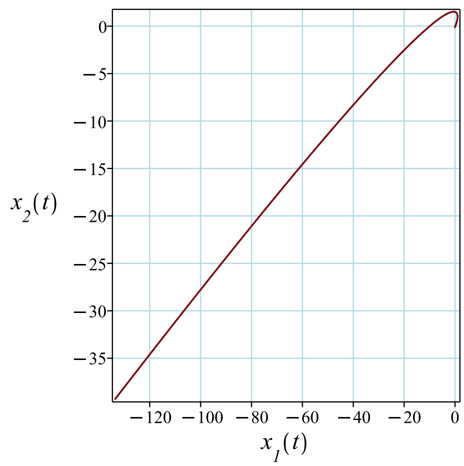
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2c_1 + 3c_2 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

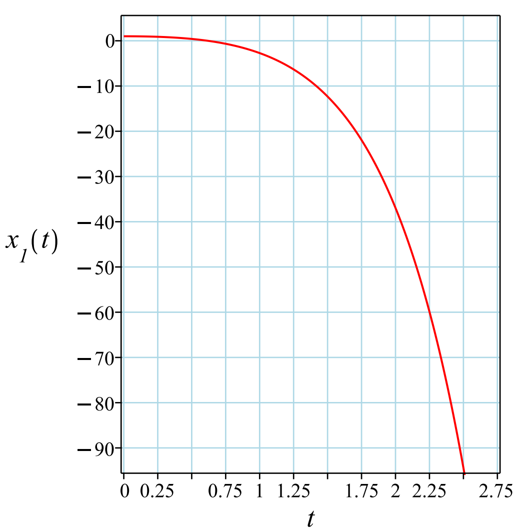
$$\begin{bmatrix} c_1 = 2 \\ c_2 = -1 \end{bmatrix}$$

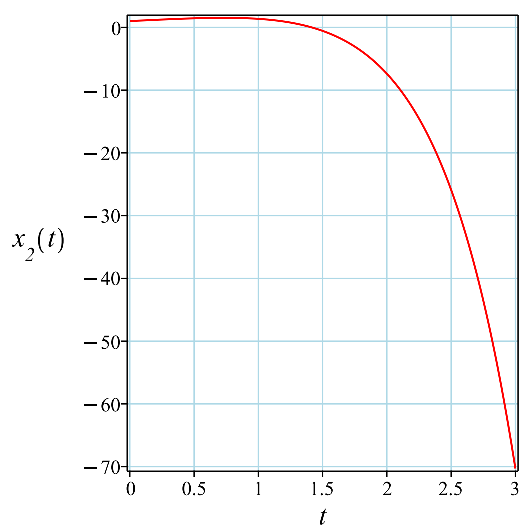
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -\mathrm{e}^t(t^2 + t - 1) \\ \mathrm{e}^t\left(1 - \frac{t^2}{2}\right) \end{bmatrix}$$



The following are plots of each solution.





✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 32

```
dsolve([diff(x__1(t),t) = 3*x__1(t)-4*x__2(t)+exp(t), diff(x__2(t),t) = x__1(t)-x__2(t)+exp(
```

$$x_1(t) = e^t(-t^2 - t + 1)$$
$$x_2(t) = \frac{e^t(-2t^2 + 4)}{4}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 31

```
DSolve[{x1'[t]==3*x1[t]-4*x2[t]+Exp[t],x2'[t]==1*x1[t]-1*x2[t]+Exp[t]},{x1[0]==1,x2[0]==1},{
```

$$\text{x1}(t) \to -e^t(t^2 + t - 1)$$
$$\text{x2}(t) \to -\frac{1}{2}e^t(t^2 - 2)$$

4.5 problem 3

Internal problem ID [1858]
Internal file name [`OUTPUT/1859_Sunday_June_05_2022_02_35_40_AM_24763410/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.12, Systems of differential equations. The nonhomogeneous equation. variation of parameters. Page 366
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 2x_1(t) - 5x_2(t) + \sin(t) \\
x_2'(t) &= x_1(t) - 2x_2(t) + \tan(t)
\end{aligned}
$$

With initial conditions

$$[x_1(0) = 0, x_2(0) = 0]$$

4.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \sin(t) \\ \tan(t) \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(t) + 2\sin(t) & -5\sin(t) \\ \sin(t) & \cos(t) - 2\sin(t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} \cos(t) + 2\sin(t) & -5\sin(t) \\ \sin(t) & \cos(t) - 2\sin(t) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\, dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} \cos(t) - 2\sin(t) & 5\sin(t) \\ -\sin(t) & \cos(t) + 2\sin(t) \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} \cos(t) + 2\sin(t) & -5\sin(t) \\ \sin(t) & \cos(t) - 2\sin(t) \end{bmatrix} \int \begin{bmatrix} \cos(t) - 2\sin(t) & 5\sin(t) \\ -\sin(t) & \cos(t) + 2\sin(t) \end{bmatrix} \begin{bmatrix} \sin(t) \\ \tan(t) \end{bmatrix} \\ &= \begin{bmatrix} \cos(t) + 2\sin(t) & -5\sin(t) \\ \sin(t) & \cos(t) - 2\sin(t) \end{bmatrix} \begin{bmatrix} -\frac{\cos(t)^2}{2} + \sin(t)\cos(t) - t - 5\sin(t) + 5\ln(\sec(t) + \\ 2\ln(\sec(t) + \tan(t)) + \frac{(\sin(t)-2)\cos(t)}{2} - \frac{t}{2} - 2\sin \end{bmatrix} \\ &= \begin{bmatrix} -\frac{\cos(t)}{2} + \frac{t\sin(t)}{2} + 5\cos(t)\ln(\sec(t) + \tan(t)) - t\cos(t) \\ -1 + (\sin(t) + 2\cos(t))\ln(\sec(t) + \tan(t)) - \frac{t\cos(t)}{2} \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \begin{bmatrix} -\frac{\cos(t)}{2} + \frac{t\sin(t)}{2} + 5\cos(t)\ln(\sec(t) + \tan(t)) - t\cos(t) \\ -1 + (\sin(t) + 2\cos(t))\ln(\sec(t) + \tan(t)) - \frac{t\cos(t)}{2} \end{bmatrix} \end{aligned}$$

4.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \sin(t) \\ \tan(t) \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-i$	1	complex eigenvalue
i	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} - (-i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2+i & -5 \\ 1 & -2+i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2+i & -5 & 0 \\ 1 & -2+i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{2}{5} + \frac{i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 2+i & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2+i & -5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (2-i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (2-\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (2-i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (2-\mathrm{I})\,t \\ t \end{bmatrix} = t\begin{bmatrix} 2-i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (2-\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} - (i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2-i & -5 & 0 \\ 1 & -2-i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{2}{5} - \frac{i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 2-i & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2-i & -5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (2+i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (2+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (2+i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (2+\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (2+\mathrm{I})\, t \\ t \end{bmatrix} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
i	1	1	No	$\begin{bmatrix} 2+i \\ 1 \end{bmatrix}$
$-i$	1	1	No	$\begin{bmatrix} 2-i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} (2+i)\, \mathrm{e}^{it} \\ \mathrm{e}^{it} \end{bmatrix} + c_2 \begin{bmatrix} (2-i)\, \mathrm{e}^{-it} \\ \mathrm{e}^{-it} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} (2+i)\, \mathrm{e}^{it} & (2-i)\, \mathrm{e}^{-it} \\ \mathrm{e}^{it} & \mathrm{e}^{-it} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -\frac{i e^{-it}}{2} & \left(\frac{1}{2}+i\right) e^{-it} \\ \frac{i e^{it}}{2} & \left(\frac{1}{2}-i\right) e^{it} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} (2+i) e^{it} & (2-i) e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \int \begin{bmatrix} -\frac{i e^{-it}}{2} & \left(\frac{1}{2}+i\right) e^{-it} \\ \frac{i e^{it}}{2} & \left(\frac{1}{2}-i\right) e^{it} \end{bmatrix} \begin{bmatrix} \sin(t) \\ \tan(t) \end{bmatrix} dt \\
&= \begin{bmatrix} (2+i) e^{it} & (2-i) e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \int \begin{bmatrix} -\frac{e^{-it}(i\sin(t)+(-1-2i)\tan(t))}{2} \\ \frac{e^{it}(i\sin(t)+(1-2i)\tan(t))}{2} \end{bmatrix} dt \\
&= \begin{bmatrix} (2+i) e^{it} & (2-i) e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \begin{bmatrix} \int_0^t \left(-\frac{i e^{-i\tau}\sin(\tau)}{2} + \left(\frac{1}{2}+i\right) e^{-i\tau}\tan(\tau)\right) d\tau \\ -\frac{t}{4} - \frac{i e^{2it}}{8} + \left(-\frac{1}{2}+i\right) e^{it} + \left(-1-\frac{i}{2}\right)\ln\left(e^{it}-i\right) + \left(1+\frac{i}{2}\right)\ln\left(e^{it} + \right. \end{bmatrix} \\
&= \begin{bmatrix} \left(1+\frac{i}{2}\right) e^{it}\left(\int_0^t e^{-i\tau}(2i\tan(\tau) - i\sin(\tau) + \tan(\tau))\, d\tau\right) - \frac{5\, e^{-it}\ln(e^{it}-i)}{2} + \frac{5\, e^{-it}\ln(e^{it}+i)}{2} + \left(-\frac{1}{2} + \frac{i}{4}\right. \\ \frac{e^{it}\left(\int_0^t e^{-i\tau}(2i\tan(\tau)-i\sin(\tau)+\tan(\tau))d\tau\right)}{2} + \left(-1-\frac{i}{2}\right) e^{-it}\ln\left(e^{it}-i\right) + \left(1+\frac{i}{2}\right) e^{-it}\ln\left(e^{it}+i\right) - \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} (2+i)\, c_1 e^{it} \\ c_1 e^{it} \end{bmatrix} + \begin{bmatrix} (2-i)\, c_2 e^{-it} \\ c_2 e^{-it} \end{bmatrix} + \begin{bmatrix} \left(1+\frac{i}{2}\right) e^{it}\left(\int_0^t e^{-i\tau}(2i\tan(\tau) - i\sin(\tau) + \tan(\tau)\right. \\ \frac{e^{it}\left(\int_0^t e^{-i\tau}(2i\tan(\tau)-i\sin(\tau)+\tan(\tau))d\tau\right)}{2} + (-1- \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{5i}{2} + \left(1+\frac{i}{2}\right) e^{it}\left(\int_0^t e^{-i\tau}(2i\tan(\tau) - i\sin(\tau) + \tan(\tau))\, d\tau\right) - \frac{5\, e^{-it}\ln(e^{it}-i)}{2} + \frac{5\, e^{-it}\ln(e^{it}+i)}{2} \\ \frac{e^{it}\left(\int_0^t e^{-i\tau}(2i\tan(\tau)-i\sin(\tau)+\tan(\tau))d\tau\right)}{2} + \left(-1-\frac{i}{2}\right) e^{-it}\ln\left(e^{it}-i\right) + \left(1+\frac{i}{2}\right) e^{-it}\ln\left(e^{it} + \right. \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = 0 \\ x_2(0) = 0 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (2+i)\, c_1 + (2-i)\, c_2 + \frac{5i\pi}{4} - \frac{1}{8} + \frac{9i}{4} \\ \left(-\frac{1}{4} + \frac{i}{2}\right)\pi - \frac{1}{2} + \frac{7i}{8} + c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 0 \\ c_2 = -\frac{i(2i\pi+4\pi+4i+7)}{8} \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{5i}{2} + \left(1+\frac{i}{2}\right) e^{it}\left(\int_0^t e^{-i\tau}(2i\tan(\tau) - i\sin(\tau) + \tan(\tau))\, d\tau\right) - \frac{5\, e^{-it}\ln(e^{it}-i)}{2} + \frac{5\, e^{-it}\ln(e^{it}+i)}{2} \\ \frac{e^{it}\left(\int_0^t e^{-i\tau}(2i\tan(\tau)-i\sin(\tau)+\tan(\tau))d\tau\right)}{2} + \left(-1-\frac{i}{2}\right) e^{-it}\ln(e^{it}-i) + \left(1+\frac{i}{2}\right) e^{-it}\ln(e^{it} \end{bmatrix}$$

The following are plots of each solution.

✓ Solution by Maple

Time used: 0.187 (sec). Leaf size: 149

```
dsolve([diff(x__1(t),t) = 2*x__1(t)-5*x__2(t)+sin(t), diff(x__2(t),t) = x__1(t)-2*x__2(t)+ta
```

$$x_1(t) = -4\sin(t) + 5\cos(t)\ln(\sec(t) + \tan(t)) - \cos(t)\, t + \frac{\sin(t)\, t}{2}$$

$$x_2(t) = \frac{10\sin(t)\ln(\sec(t)+\tan(t))\sec(t) + 20\cos(t)\ln(\sec(t)+\tan(t))\sec(t) - 15\sin(t)\sec(t) - 10\sec(t}{}$$

✓ Solution by Mathematica

Time used: 0.069 (sec). Leaf size: 58

```
DSolve[{x1'[t]==2*x1[t]-5*x2[t]+Sin[t],x2'[t]==1*x1[t]-2*x2[t]+Tan[t]},{x1[0]==0,x2[0]==0},{
```

$$x1(t) \to 5\cos(t)\text{arctanh}(\sin(t)) + \frac{1}{2}(t-8)\sin(t) - t\cos(t)$$

$$x2(t) \to \text{arctanh}(\sin(t))(\sin(t) + 2\cos(t)) - \frac{3\sin(t)}{2} - \frac{1}{2}t\cos(t) + \cos(t) - 1$$

4.6 problem 4

Internal problem ID [1859]
Internal file name [OUTPUT/1860_Sunday_June_05_2022_02_35_44_AM_29637151/index.tex]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.12, Systems of differential equations. The nonhomogeneous equation. variation of parameters. Page 366
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_2(t) + f_1(t) \\
x_2'(t) &= -x_1(t) + f_2(t)
\end{aligned}
$$

With initial conditions

$$
[x_1(0) = 0, x_2(0) = 0]
$$

4.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}\vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned}e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}\end{aligned}$$

Hence

$$\begin{aligned}\vec{x}_p(t) &= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \int \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} dt \\ &= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} \int (\cos(t) f_1(t) - \sin(t) f_2(t))\,dt \\ \int (\sin(t) f_1(t) + \cos(t) f_2(t))\,dt \end{bmatrix} \\ &= \begin{bmatrix} \cos(t) \left(\int (\cos(t) f_1(t) - \sin(t) f_2(t))\,dt\right) + \sin(t) \left(\int (\sin(t) f_1(t) + \cos(t) f_2(t))\,dt\right) \\ -\sin(t) \left(\int (\cos(t) f_1(t) - \sin(t) f_2(t))\,dt\right) + \cos(t) \left(\int (\sin(t) f_1(t) + \cos(t) f_2(t))\,dt\right) \end{bmatrix}\end{aligned}$$

Hence the complete solution is

$$\begin{aligned}\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \begin{bmatrix} \cos(t) \left(\int (\cos(t) f_1(t) - \sin(t) f_2(t))\,dt\right) + \sin(t) \left(\int (\sin(t) f_1(t) + \cos(t) f_2(t))\,dt\right) \\ -\sin(t) \left(\int (\cos(t) f_1(t) - \sin(t) f_2(t))\,dt\right) + \cos(t) \left(\int (\sin(t) f_1(t) + \cos(t) f_2(t))\,dt\right) \end{bmatrix}\end{aligned}$$

4.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-i$	1	complex eigenvalue
i	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - (-i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right]$$

$$R_2 = -iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it\}$

Hence the solution is

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = t\begin{bmatrix} i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \text{I}t \\ t \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - (i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right]$$

$$R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} -i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -i & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it\}$

Hence the solution is

$$\begin{bmatrix} \text{-I}t \\ t \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \text{-I}t \\ t \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \text{-I}t \\ t \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
i	1	1	No	$\begin{bmatrix} -i \\ 1 \end{bmatrix}$
$-i$	1	1	No	$\begin{bmatrix} i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} -i\mathrm{e}^{it} \\ \mathrm{e}^{it} \end{bmatrix} + c_2 \begin{bmatrix} i\mathrm{e}^{-it} \\ \mathrm{e}^{-it} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -i\mathrm{e}^{it} & i\mathrm{e}^{-it} \\ \mathrm{e}^{it} & \mathrm{e}^{-it} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{i e^{-it}}{2} & \frac{e^{-it}}{2} \\ -\frac{i e^{it}}{2} & \frac{e^{it}}{2} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} -i e^{it} & i e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \int \begin{bmatrix} \frac{i e^{-it}}{2} & \frac{e^{-it}}{2} \\ -\frac{i e^{it}}{2} & \frac{e^{it}}{2} \end{bmatrix} \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} dt \\
&= \begin{bmatrix} -i e^{it} & i e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \int \begin{bmatrix} \frac{e^{-it}(i f_1(t)+f_2(t))}{2} \\ \frac{e^{it}(-i f_1(t)+f_2(t))}{2} \end{bmatrix} dt \\
&= \begin{bmatrix} -i e^{it} & i e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \begin{bmatrix} \int_0^t \left(\frac{i e^{-i\tau} f_1(\tau)}{2} + \frac{e^{-i\tau} f_2(\tau)}{2}\right) d\tau \\ \int_0^t \left(-\frac{i e^{i\tau} f_1(\tau)}{2} + \frac{e^{i\tau} f_2(\tau)}{2}\right) d\tau \end{bmatrix} \\
&= \begin{bmatrix} -\frac{i\left(e^{it}\left(\int_0^t e^{-i\tau}(i f_1(\tau)+f_2(\tau))d\tau\right)+e^{-it}\left(\int_0^t e^{i\tau}(i f_1(\tau)-f_2(\tau))d\tau\right)\right)}{2} \\ \frac{e^{it}\left(\int_0^t e^{-i\tau}(i f_1(\tau)+f_2(\tau))d\tau\right)}{2} - \frac{e^{-it}\left(\int_0^t e^{i\tau}(i f_1(\tau)-f_2(\tau))d\tau\right)}{2} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -i c_1 e^{it} \\ c_1 e^{it} \end{bmatrix} + \begin{bmatrix} i c_2 e^{-it} \\ c_2 e^{-it} \end{bmatrix} + \begin{bmatrix} -\frac{i\left(e^{it}\left(\int_0^t e^{-i\tau}(i f_1(\tau)+f_2(\tau))d\tau\right)+e^{-it}\left(\int_0^t e^{i\tau}(i f_1(\tau)-f_2(\tau))d\tau\right)\right)}{2} \\ \frac{e^{it}\left(\int_0^t e^{-i\tau}(i f_1(\tau)+f_2(\tau))d\tau\right)}{2} - \frac{e^{-it}\left(\int_0^t e^{i\tau}(i f_1(\tau)-f_2(\tau))d\tau\right)}{2} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{i\left(e^{it}\left(\int_0^t e^{-i\tau}(i f_1(\tau)+f_2(\tau))d\tau\right)+2c_1 e^{it}+e^{-it}\left(\int_0^t e^{i\tau}(i f_1(\tau)-f_2(\tau))d\tau\right)-2c_2 e^{-it}\right)}{2} \\ c_1 e^{it} + c_2 e^{-it} + \frac{e^{it}\left(\int_0^t e^{-i\tau}(i f_1(\tau)+f_2(\tau))d\tau\right)}{2} - \frac{e^{-it}\left(\int_0^t e^{i\tau}(i f_1(\tau)-f_2(\tau))d\tau\right)}{2} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = 0 \\ x_2(0) = 0 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -i(c_1 - c_2) \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 0 \\ c_2 = 0 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{i\left(e^{it}\left(\int_0^t e^{-i\tau}(if_1(\tau)+f_2(\tau))d\tau\right)+e^{-it}\left(\int_0^t e^{i\tau}(if_1(\tau)-f_2(\tau))d\tau\right)\right)}{2} \\ \frac{e^{it}\left(\int_0^t e^{-i\tau}(if_1(\tau)+f_2(\tau))d\tau\right)}{2} - \frac{e^{-it}\left(\int_0^t e^{i\tau}(if_1(\tau)-f_2(\tau))d\tau\right)}{2} \end{bmatrix}$$

The following are plots of each solution.

✓ Solution by Maple

Time used: 0.187 (sec). Leaf size: 101

```
dsolve([diff(x__1(t),t) = x__2(t)+f__1(t), diff(x__2(t),t) = -x__1(t)+f__2(t), x__1(0) = 0,
```

$$x_1(t) = f_1(0)\sin(t) + \left(\int_0^t \cos(\_z1)\left(\frac{d}{d\_z1}f_1(\_z1) + f_2(\_z1)\right)d\_z1\right)\sin(t) - \left(\int_0^t \sin(\_z1)\left(\frac{d}{d\_z1}f_1(\_z1) + f_2(\_z1)\right)d\_z1\right)\cos(t)$$

$$x_2(t) = f_1(0)\cos(t) + \left(\int_0^t \cos(\_z1)\left(\frac{d}{d\_z1}f_1(\_z1) + f_2(\_z1)\right)d\_z1\right)\cos(t) + \left(\int_0^t \sin(\_z1)\left(\frac{d}{d\_z1}f_1(\_z1) + f_2(\_z1)\right)d\_z1\right)\sin(t) - f_1(t)$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 216

```
DSolve[{x1'[t]==0*x1[t]+1*x2[t]+f1[t],x2'[t]==-1*x1[t]-0*x2[t]+f2[t]},{x1[0]==0,x2[0]==0},{x
```

$$
\begin{aligned}
\mathrm{x1}(t) \to & -\cos(t)\int_1^0 (\cos(K[1])\mathrm{f1}(K[1]) - \mathrm{f2}(K[1])\sin(K[1]))dK[1] \\
& + \cos(t)\int_1^t (\cos(K[1])\mathrm{f1}(K[1]) - \mathrm{f2}(K[1])\sin(K[1]))dK[1] \\
& + \sin(t)\left(\int_1^t (\cos(K[2])\mathrm{f2}(K[2]) + \mathrm{f1}(K[2])\sin(K[2]))dK[2] \right. \\
& \left. - \int_1^0 (\cos(K[2])\mathrm{f2}(K[2]) + \mathrm{f1}(K[2])\sin(K[2]))dK[2]\right) \\
\mathrm{x2}(t) \to & \sin(t)\int_1^0 (\cos(K[1])\mathrm{f1}(K[1]) - \mathrm{f2}(K[1])\sin(K[1]))dK[1] \\
& - \sin(t)\int_1^t (\cos(K[1])\mathrm{f1}(K[1]) - \mathrm{f2}(K[1])\sin(K[1]))dK[1] \\
& + \cos(t)\left(\int_1^t (\cos(K[2])\mathrm{f2}(K[2]) + \mathrm{f1}(K[2])\sin(K[2]))dK[2] \right. \\
& \left. - \int_1^0 (\cos(K[2])\mathrm{f2}(K[2]) + \mathrm{f1}(K[2])\sin(K[2]))dK[2]\right)
\end{aligned}
$$

4.7 problem 5

Internal problem ID [1860]
Internal file name [`OUTPUT/1861_Sunday_June_05_2022_02_35_47_AM_16078708/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.12, Systems of differential equations. The nonhomogeneous equation. variation of parameters. Page 366
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x_1'(t) &= 2x_1(t) + x_3(t) + e^{2t} \\ x_2'(t) &= 2x_2(t) \\ x_3'(t) &= x_2(t) + 3x_3(t) + e^{2t} \end{aligned}$$

With initial conditions

$$[x_1(0) = 1, x_2(0) = 1, x_3(0) = 1]$$

4.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} e^{2t} \\ 0 \\ e^{2t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{2t} & (-t-1)\,e^{2t} + e^{3t} & e^{3t} - e^{2t} \\ 0 & e^{2t} & 0 \\ 0 & e^{3t} - e^{2t} & e^{3t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} e^{2t} & (-t-1)\,e^{2t} + e^{3t} & e^{3t} - e^{2t} \\ 0 & e^{2t} & 0 \\ 0 & e^{3t} - e^{2t} & e^{3t} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} (-t-1)\,e^{2t} + 2\,e^{3t} \\ e^{2t} \\ -e^{2t} + 2\,e^{3t} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} e^{-2t} & (1 + e^{t}(t-1))\,e^{-3t} & -e^{-3t}(e^{t} - 1) \\ 0 & e^{-2t} & 0 \\ 0 & -e^{-3t}(e^{t} - 1) & e^{-3t} \end{bmatrix} \end{aligned}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} e^{2t} & (-t-1)\,e^{2t} + e^{3t} & e^{3t} - e^{2t} \\ 0 & e^{2t} & 0 \\ 0 & e^{3t} - e^{2t} & e^{3t} \end{bmatrix} \int \begin{bmatrix} e^{-2t} & (1 + e^{t}(t-1))\,e^{-3t} & -e^{-3t}(e^{t} - 1) \\ 0 & e^{-2t} & 0 \\ 0 & -e^{-3t}(e^{t} - 1) & e^{-3t} \end{bmatrix} \begin{bmatrix} e^{2t} \\ 0 \\ e^{2t} \end{bmatrix} dt$$

$$= \begin{bmatrix} e^{2t} & (-t-1)\,e^{2t} + e^{3t} & e^{3t} - e^{2t} \\ 0 & e^{2t} & 0 \\ 0 & e^{3t} - e^{2t} & e^{3t} \end{bmatrix} \begin{bmatrix} -e^{-t} \\ 0 \\ -e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} -e^{2t} \\ 0 \\ -e^{2t} \end{bmatrix}$$

Hence the complete solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \begin{bmatrix} e^{2t}(-t-2) + 2\,e^{3t} \\ e^{2t} \\ -2\,e^{2t} + 2\,e^{3t} \end{bmatrix} \end{aligned}$$

4.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} e^{2t} \\ 0 \\ e^{2t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

Since the current pivot $A(1,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 3 gives

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2, v_3\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	2	1	Yes	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
3	1	1	No	$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 2 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

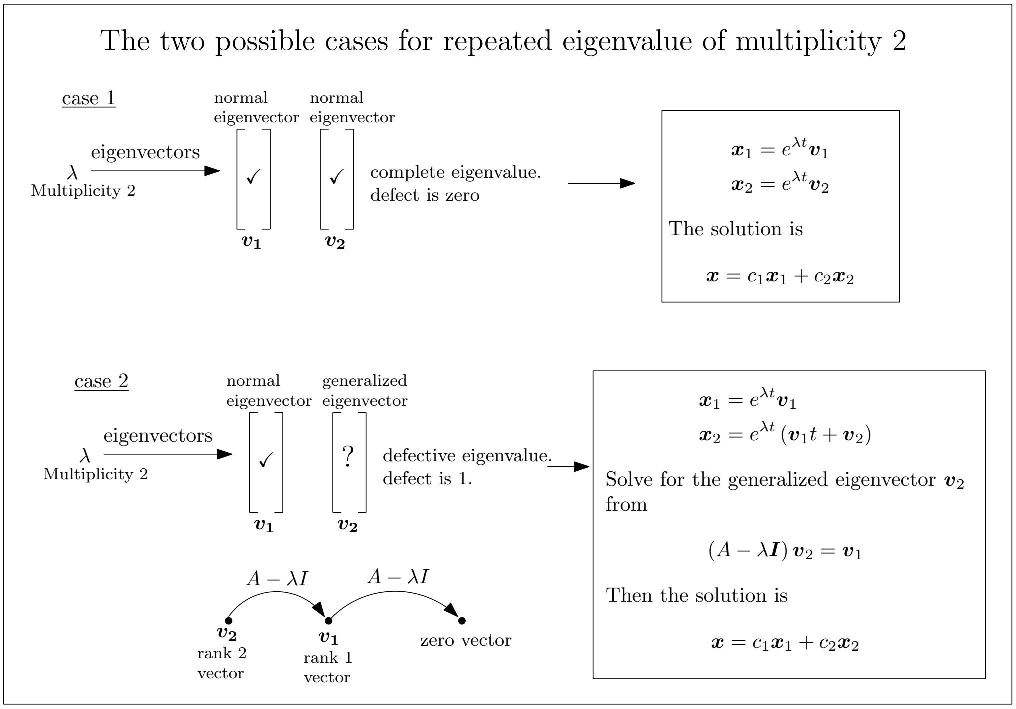


Figure 22: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I) \vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 2. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mathrm{e}^{2t} \\ &= \begin{bmatrix} \mathrm{e}^{2t} \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

And

$$\begin{aligned} \vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right) \mathrm{e}^{2t} \\ &= \begin{bmatrix} \mathrm{e}^{2t}(t+1) \\ -\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix} \end{aligned}$$

Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_3(t) &= \vec{v}_3 e^{3t} \\ &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{3t} \end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{2t} \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} e^{2t}(t+1) \\ -e^{2t} \\ e^{2t} \end{bmatrix} + c_3 \begin{bmatrix} e^{3t} \\ 0 \\ e^{3t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} e^{2t} & e^{2t}(t+1) & e^{3t} \\ 0 & -e^{2t} & 0 \\ 0 & e^{2t} & e^{3t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t) \, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} e^{-2t} & t\, e^{-2t} & -e^{-2t} \\ 0 & -e^{-2t} & 0 \\ 0 & e^{-3t} & e^{-3t} \end{bmatrix}$$

Hence

$$
\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} e^{2t} & e^{2t}(t+1) & e^{3t} \\ 0 & -e^{2t} & 0 \\ 0 & e^{2t} & e^{3t} \end{bmatrix} \int \begin{bmatrix} e^{-2t} & t\,e^{-2t} & -e^{-2t} \\ 0 & -e^{-2t} & 0 \\ 0 & e^{-3t} & e^{-3t} \end{bmatrix} \begin{bmatrix} e^{2t} \\ 0 \\ e^{2t} \end{bmatrix} dt \\
&= \begin{bmatrix} e^{2t} & e^{2t}(t+1) & e^{3t} \\ 0 & -e^{2t} & 0 \\ 0 & e^{2t} & e^{3t} \end{bmatrix} \int \begin{bmatrix} 0 \\ 0 \\ e^{-t} \end{bmatrix} dt \\
&= \begin{bmatrix} e^{2t} & e^{2t}(t+1) & e^{3t} \\ 0 & -e^{2t} & 0 \\ 0 & e^{2t} & e^{3t} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -e^{-t} \end{bmatrix} \\
&= \begin{bmatrix} -e^{2t} \\ 0 \\ -e^{2t} \end{bmatrix}
\end{aligned}
$$

Now that we found particular solution, the final solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} &= \begin{bmatrix} c_1 e^{2t} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} c_2 e^{2t}(t+1) \\ -c_2 e^{2t} \\ c_2 e^{2t} \end{bmatrix} + \begin{bmatrix} c_3 e^{3t} \\ 0 \\ c_3 e^{3t} \end{bmatrix} + \begin{bmatrix} -e^{2t} \\ 0 \\ -e^{2t} \end{bmatrix}
\end{aligned}
$$

Which becomes

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} ((t+1)\,c_2 + c_1 - 1)\,e^{2t} + c_3 e^{3t} \\ -c_2 e^{2t} \\ (c_2 - 1)\,e^{2t} + c_3 e^{3t} \end{bmatrix}
$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$
\begin{bmatrix} x_1(0) = 1 \\ x_2(0) = 1 \\ x_3(0) = 1 \end{bmatrix} \tag{1}
$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_2 + c_1 - 1 + c_3 \\ -c_2 \\ c_2 - 1 + c_3 \end{bmatrix}$$

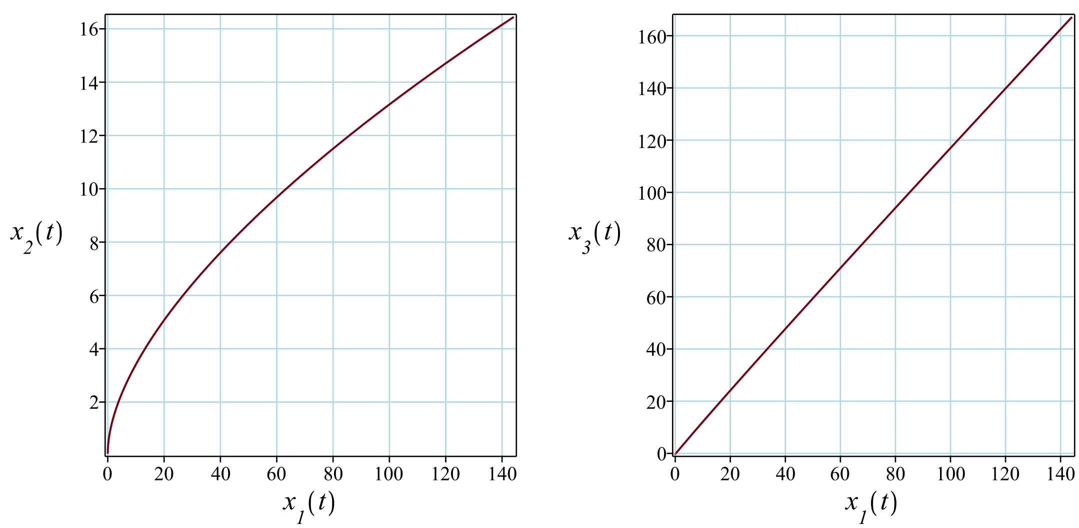
Solving for the constants of integrations gives

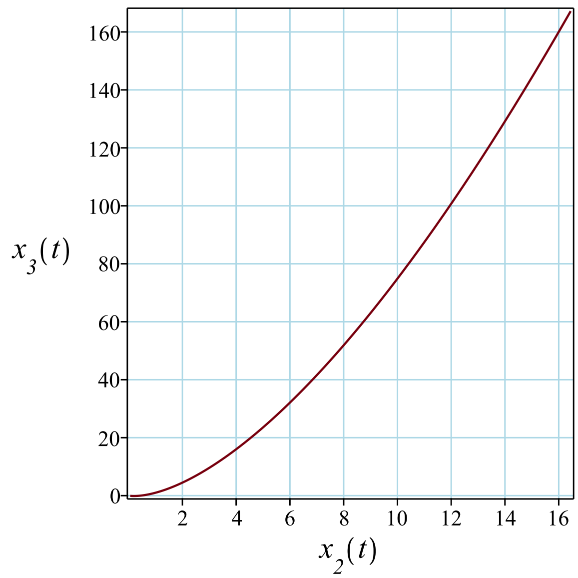
$$\begin{bmatrix} c_1 = 0 \\ c_2 = -1 \\ c_3 = 3 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

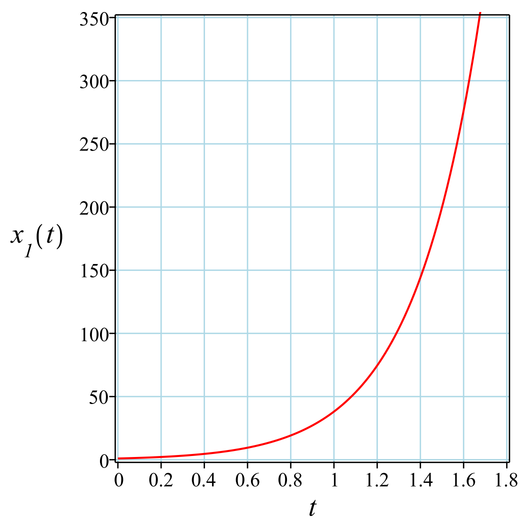
$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} e^{2t}(-t-2) + 3\,e^{3t} \\ e^{2t} \\ 3\,e^{3t} - 2\,e^{2t} \end{bmatrix}$$

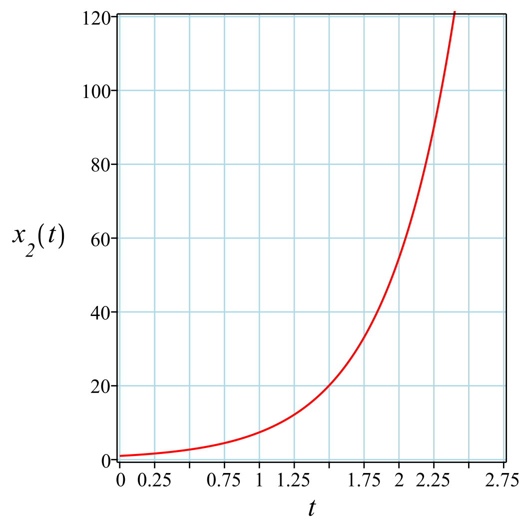
The following are plots of each solution against another.

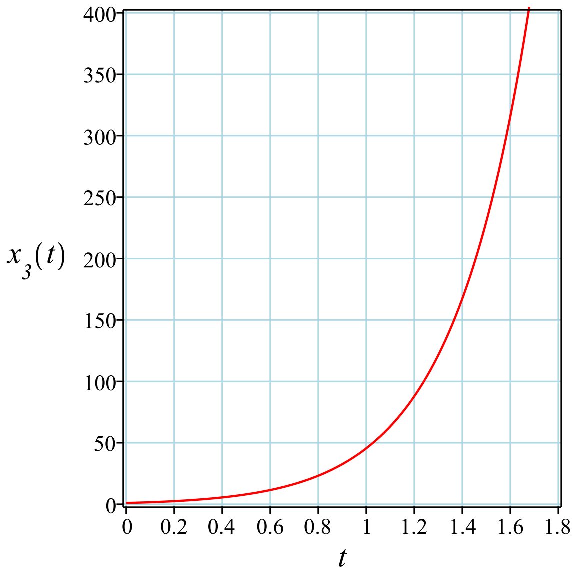




The following are plots of each solution.







✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 45

```
dsolve([diff(x__1(t),t) = 2*x__1(t)+x__3(t)+exp(2*t), diff(x__2(t),t) = 2*x__2(t), diff(x__3
```

$$x_1(t) = (-t-2)\,\mathrm{e}^{2t} + 3\,\mathrm{e}^{3t}$$
$$x_2(t) = \mathrm{e}^{2t}$$
$$x_3(t) = -2\,\mathrm{e}^{2t} + 3\,\mathrm{e}^{3t}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 45

```
DSolve[{x1'[t]==2*x1[t]+0*x2[t]+1*x3[t]+Exp[2*t],x2'[t]==0*x1[t]+2*x2[t]+0*x3[t],x3'[t]==0*x
```

$$\mathrm{x1}(t) \to e^{2t}\left(-t+3e^t-2\right)$$
$$\mathrm{x2}(t) \to e^{2t}$$
$$\mathrm{x3}(t) \to e^{2t}\left(3e^t-2\right)$$

4.8 problem 6

Internal problem ID [1861]
Internal file name [`OUTPUT/1862_Sunday_June_05_2022_02_35_50_AM_26128245/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.12, Systems of differential equations. The nonhomogeneous equation. variation of parameters. Page 366
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= -x_1(t) - x_2(t) - 2x_3(t) + \mathrm{e}^t \\
x_2'(t) &= x_1(t) + x_2(t) + x_3(t) \\
x_3'(t) &= 2x_1(t) + x_2(t) + 3x_3(t)
\end{aligned}
$$

With initial conditions

$$[x_1(0) = 0, x_2(0) = 0, x_3(0) = 0]$$

4.8.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -1 & -1 & -2 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} \mathrm{e}^t \\ 0 \\ 0 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^t\left(1 - \frac{1}{2}t^2 - 2t\right) & -t\,\mathrm{e}^t & -\frac{\mathrm{e}^t t(t+4)}{2} \\ t\,\mathrm{e}^t & \mathrm{e}^t & t\,\mathrm{e}^t \\ \frac{\mathrm{e}^t t(t+4)}{2} & t\,\mathrm{e}^t & \mathrm{e}^t\left(1 + \frac{1}{2}t^2 + 2t\right) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} \mathrm{e}^t\left(1 - \frac{1}{2}t^2 - 2t\right) & -t\,\mathrm{e}^t & -\frac{\mathrm{e}^t t(t+4)}{2} \\ t\,\mathrm{e}^t & \mathrm{e}^t & t\,\mathrm{e}^t \\ \frac{\mathrm{e}^t t(t+4)}{2} & t\,\mathrm{e}^t & \mathrm{e}^t\left(1 + \frac{1}{2}t^2 + 2t\right) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= \left(e^{At}\right)^{-1} \\ &= \begin{bmatrix} -\frac{(t^2-4t-2)\mathrm{e}^{-t}}{2} & t\,\mathrm{e}^{-t} & -\frac{\mathrm{e}^{-t}t(t-4)}{2} \\ -t\,\mathrm{e}^{-t} & \mathrm{e}^{-t} & -t\,\mathrm{e}^{-t} \\ \frac{\mathrm{e}^{-t}t(t-4)}{2} & -t\,\mathrm{e}^{-t} & \frac{(t^2-4t+2)\mathrm{e}^{-t}}{2} \end{bmatrix} \end{aligned}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} e^t\left(1-\frac{1}{2}t^2-2t\right) & -t\,e^t & -\frac{e^t t(t+4)}{2} \\ t\,e^t & e^t & t\,e^t \\ \frac{e^t t(t+4)}{2} & t\,e^t & e^t\left(1+\frac{1}{2}t^2+2t\right) \end{bmatrix} \int \begin{bmatrix} -\frac{(t^2-4t-2)e^{-t}}{2} & t\,e^{-t} & -\frac{e^{-t}t(t-4)}{2} \\ -t\,e^{-t} & e^{-t} & -t\,e^{-t} \\ \frac{e^{-t}t(t-4)}{2} & -t\,e^{-t} & \frac{(t^2-4t+2)e^{-t}}{2} \end{bmatrix} \begin{bmatrix} e^t \\ 0 \\ 0 \end{bmatrix} d$$

$$= \begin{bmatrix} e^t\left(1-\frac{1}{2}t^2-2t\right) & -t\,e^t & -\frac{e^t t(t+4)}{2} \\ t\,e^t & e^t & t\,e^t \\ \frac{e^t t(t+4)}{2} & t\,e^t & e^t\left(1+\frac{1}{2}t^2+2t\right) \end{bmatrix} \begin{bmatrix} -\frac{t(t^2-6t-6)}{6} \\ -\frac{t^2}{2} \\ \frac{t^2(t-6)}{6} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{e^t t(t^2+6t-6)}{6} \\ \frac{t^2 e^t}{2} \\ \frac{e^t t^2(t+6)}{6} \end{bmatrix}$$

Hence the complete solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \begin{bmatrix} -\frac{e^t t(t^2+6t-6)}{6} \\ \frac{t^2 e^t}{2} \\ \frac{e^t t^2(t+6)}{6} \end{bmatrix} \end{aligned}$$

4.8.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -1 & -1 & -2 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} e^t \\ 0 \\ 0 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I)=0$$

Expanding gives

$$\det\left(\begin{bmatrix}-1&-1&-2\\1&1&1\\2&1&3\end{bmatrix}-\lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)=0$$

Therefore

$$\det\left(\begin{bmatrix}-1-\lambda&-1&-2\\1&1-\lambda&1\\2&1&3-\lambda\end{bmatrix}\right)=0$$

Which gives the characteristic equation

$$\lambda^3-3\lambda^2+3\lambda-1=0$$

The roots of the above are the eigenvalues.

$$\lambda_1=1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1=1$

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\left(\begin{bmatrix}-1&-1&-2\\1&1&1\\2&1&3\end{bmatrix}-(1)\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}-2&-1&-2\\1&0&1\\2&1&2\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & -1 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 & -1 & -2 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 2 & 1 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -2 & -1 & -2 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & -1 & -2 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	3	1	Yes	$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 1 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

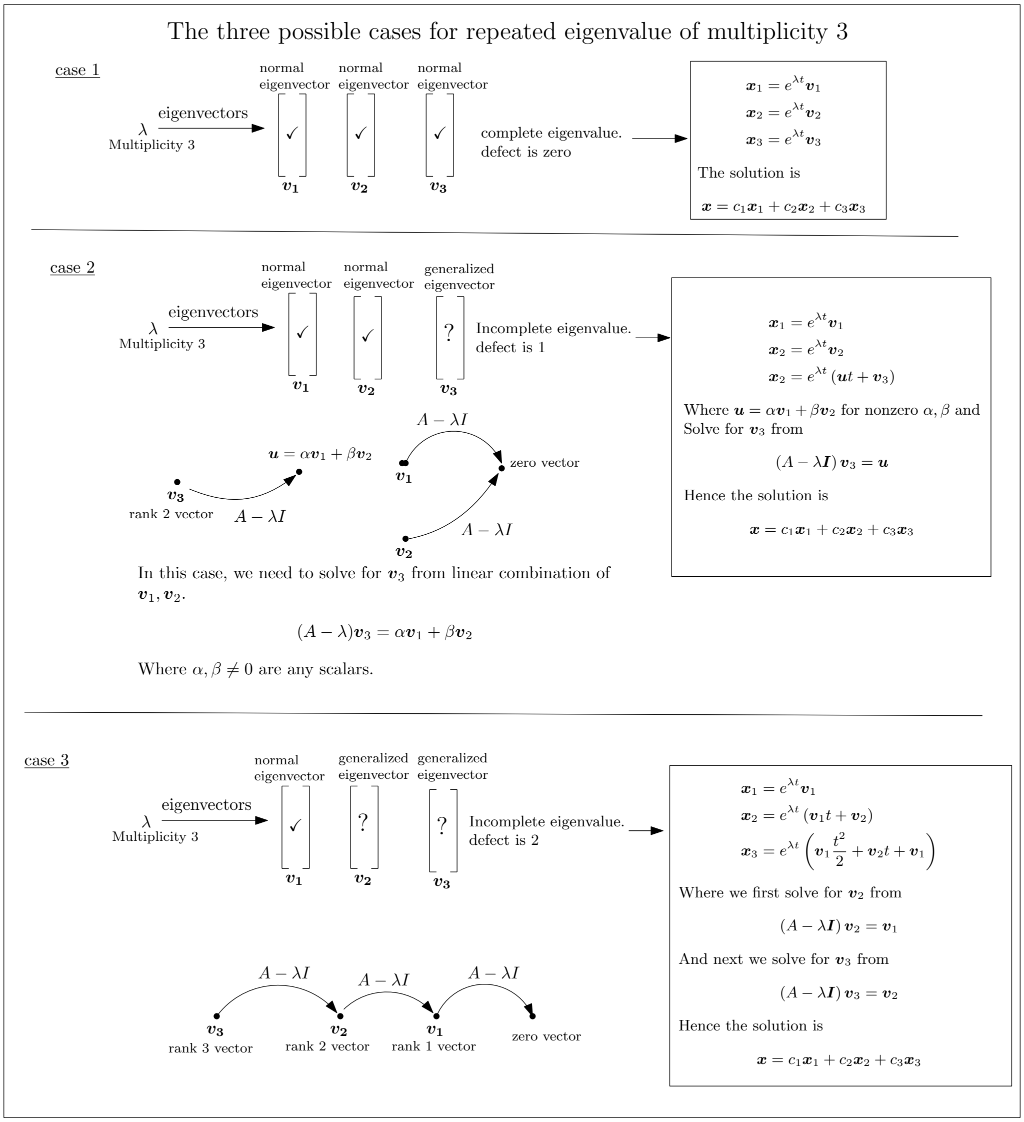


Figure 23: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 2. This falls into case 3 shown above. First we find generalized eigenvector $\vec{v}_2$ of rank 2 and then use this to find generalized eigenvector $\vec{v}_3$ of rank 3.$\vec{v}_2$ is found by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence

$$\left(\begin{bmatrix}-1 & -1 & -2\\ 1 & 1 & 1\\ 2 & 1 & 3\end{bmatrix} - (1)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}$$

$$\begin{bmatrix}-2 & -1 & -2\\ 1 & 0 & 1\\ 2 & 1 & 2\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix}-1\\ 1\\ 1\end{bmatrix}$$

Now $\vec{v}_3$ is found by solving

$$(A - \lambda I)\,\vec{v}_3 = \vec{v}_2$$

Where $\vec{v}_2$ is the (rank 2) generalized eigenvector found above. Hence

$$\left(\begin{bmatrix}-1 & -1 & -2\\ 1 & 1 & 1\\ 2 & 1 & 3\end{bmatrix} - (1)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}-1\\ 1\\ 1\end{bmatrix}$$

$$\begin{bmatrix}-2 & -1 & -2\\ 1 & 0 & 1\\ 2 & 1 & 2\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}-1\\ 1\\ 1\end{bmatrix}$$

Solving for $\vec{v}_3$ gives

$$\vec{v}_3 = \begin{bmatrix}0\\ -1\\ 1\end{bmatrix}$$

We have found three generalized eigenvectors for eigenvalue 1. Therefore the three basis

solutions associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \mathrm{e}^{t} \\
&= \begin{bmatrix} -\mathrm{e}^{t} \\ 0 \\ \mathrm{e}^{t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= e^{\lambda t}(\vec{v}_1 t + \vec{v}_2) \\
&= \mathrm{e}^{t}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right) \\
&= \begin{bmatrix} -\mathrm{e}^{t}(t+1) \\ \mathrm{e}^{t} \\ \mathrm{e}^{t}(t+1) \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_3(t) &= \left(\vec{v}_1 \frac{t^2}{2} + \vec{v}_2 t + \vec{v}_3\right) e^{\lambda t} \\
&= \left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \frac{t^2}{2} + \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right) \mathrm{e}^{t} \\
&= \begin{bmatrix} -\frac{\mathrm{e}^{t} t(2+t)}{2} \\ \mathrm{e}^{t}(t-1) \\ \frac{\mathrm{e}^{t}(t^2+2t+2)}{2} \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^t \\ 0 \\ \mathrm{e}^t \end{bmatrix} + c_2 \begin{bmatrix} (-t-1)\,\mathrm{e}^t \\ \mathrm{e}^t \\ \mathrm{e}^t(t+1) \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^t\left(-\frac{1}{2}t^2 - t\right) \\ \mathrm{e}^t(t-1) \\ \mathrm{e}^t\left(t + \frac{1}{2}t^2 + 1\right) \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -\mathrm{e}^t & (-t-1)\,\mathrm{e}^t & \mathrm{e}^t\left(-\frac{1}{2}t^2 - t\right) \\ 0 & \mathrm{e}^t & \mathrm{e}^t(t-1) \\ \mathrm{e}^t & \mathrm{e}^t(t+1) & \mathrm{e}^t\left(t + \frac{1}{2}t^2 + 1\right) \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{(t^2-2t-4)\mathrm{e}^{-t}}{2} & -\mathrm{e}^{-t}(t+1) & \frac{(t^2-2t-2)\mathrm{e}^{-t}}{2} \\ -(t-1)\,\mathrm{e}^{-t} & \mathrm{e}^{-t} & -(t-1)\,\mathrm{e}^{-t} \\ \mathrm{e}^{-t} & 0 & \mathrm{e}^{-t} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} -e^t & (-t-1)e^t & e^t\left(-\frac{1}{2}t^2 - t\right) \\ 0 & e^t & e^t(t-1) \\ e^t & e^t(t+1) & e^t\left(t + \frac{1}{2}t^2 + 1\right) \end{bmatrix} \int \begin{bmatrix} \frac{(t^2-2t-4)e^{-t}}{2} & -e^{-t}(t+1) & \frac{(t^2-2t-2)e^{-t}}{2} \\ -(t-1)e^{-t} & e^{-t} & -(t-1)e^{-t} \\ e^{-t} & 0 & e^{-t} \end{bmatrix} \begin{bmatrix} e^t \\ 0 \\ 0 \end{bmatrix} dt$$

$$= \begin{bmatrix} -e^t & (-t-1)e^t & e^t\left(-\frac{1}{2}t^2 - t\right) \\ 0 & e^t & e^t(t-1) \\ e^t & e^t(t+1) & e^t\left(t + \frac{1}{2}t^2 + 1\right) \end{bmatrix} \int \begin{bmatrix} \frac{1}{2}t^2 - t - 2 \\ 1 - t \\ 1 \end{bmatrix} dt$$

$$= \begin{bmatrix} -e^t & (-t-1)e^t & e^t\left(-\frac{1}{2}t^2 - t\right) \\ 0 & e^t & e^t(t-1) \\ e^t & e^t(t+1) & e^t\left(t + \frac{1}{2}t^2 + 1\right) \end{bmatrix} \begin{bmatrix} \frac{t(t^2-3t-12)}{6} \\ -\frac{t(t-2)}{2} \\ t \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{e^t t(t^2+6t-6)}{6} \\ \frac{t^2 e^t}{2} \\ \frac{e^t t^2(t+6)}{6} \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -c_1 e^t \\ 0 \\ c_1 e^t \end{bmatrix} + \begin{bmatrix} c_2(-t-1)e^t \\ c_2 e^t \\ c_2 e^t(t+1) \end{bmatrix} + \begin{bmatrix} c_3 e^t\left(-\frac{1}{2}t^2 - t\right) \\ c_3 e^t(t-1) \\ c_3 e^t\left(t + \frac{1}{2}t^2 + 1\right) \end{bmatrix} + \begin{bmatrix} -\frac{e^t t(t^2+6t-6)}{6} \\ \frac{t^2 e^t}{2} \\ \frac{e^t t^2(t+6)}{6} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{(t^3+(3c_3+6)t^2+(6c_2+6c_3-6)t+6c_1+6c_2)e^t}{6} \\ e^t\left(c_2 + c_3 t - c_3 + \frac{1}{2}t^2\right) \\ \frac{e^t\left(t^3+(3c_3+6)t^2+(6c_2+6c_3)t+6c_1+6c_2+6c_3\right)}{6} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = 0 \\ x_2(0) = 0 \\ x_3(0) = 0 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -c_1 - c_2 \\ c_2 - c_3 \\ c_1 + c_2 + c_3 \end{bmatrix}$$

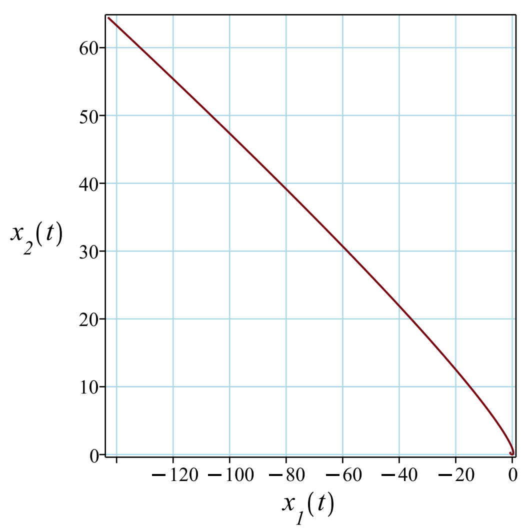
Solving for the constants of integrations gives

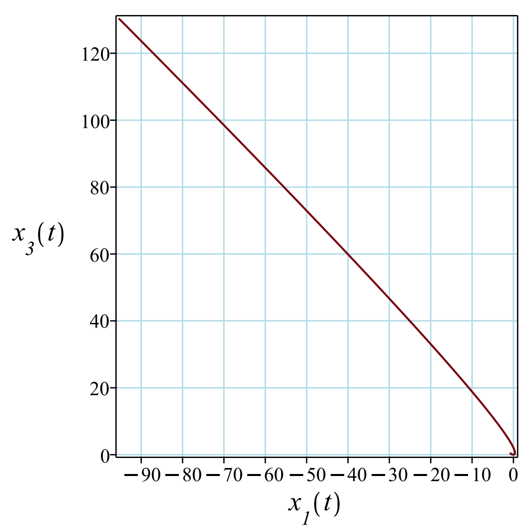
$$\begin{bmatrix} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{bmatrix}$$

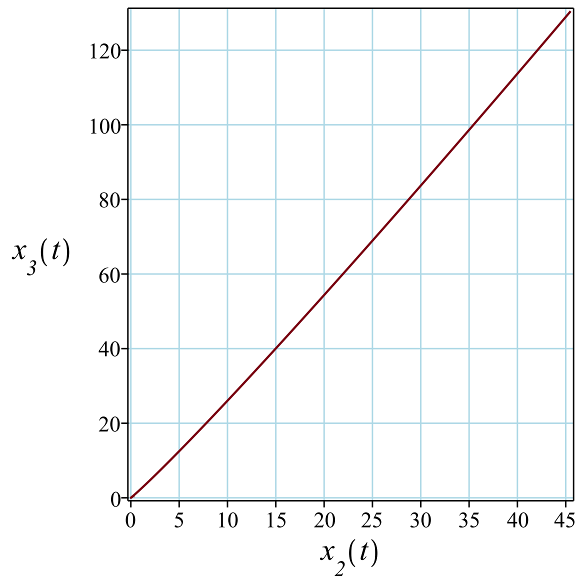
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{(t^3+6t^2-6t)\,\mathrm{e}^t}{6} \\ \frac{t^2\mathrm{e}^t}{2} \\ \frac{\mathrm{e}^t\,(t^3+6t^2)}{6} \end{bmatrix}$$

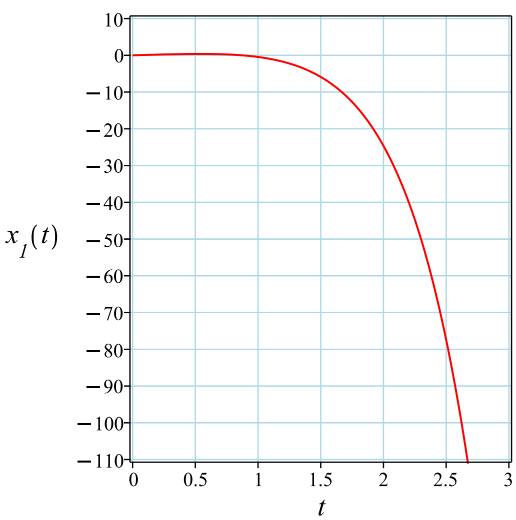
The following are plots of each solution against another.

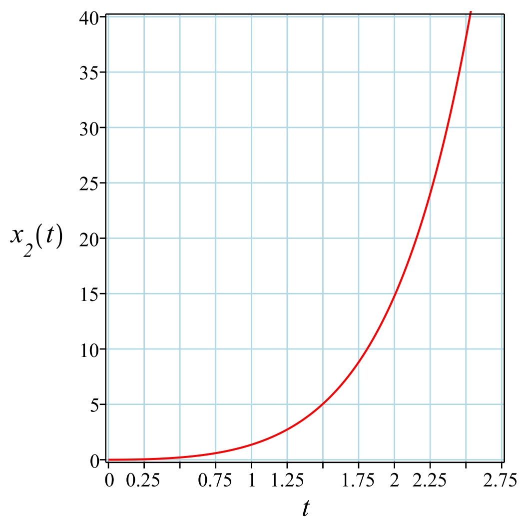


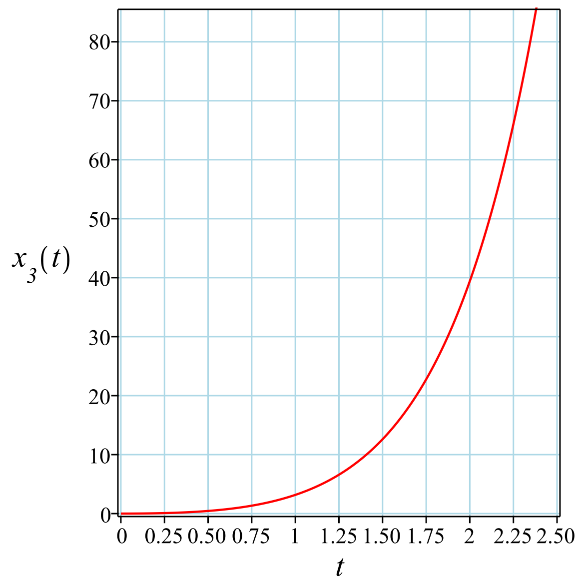




The following are plots of each solution.







✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 47

```
dsolve([diff(x__1(t),t) = -x__1(t)-x__2(t)-2*x__3(t)+exp(t), diff(x__2(t),t) = x__1(t)+x__2(
```

$$x_1(t) = -\frac{e^t(t^3 + 6t^2 - 6t)}{6}$$
$$x_2(t) = \frac{t^2 e^t}{2}$$
$$x_3(t) = \frac{(t^3 + 6t^2)\, e^t}{6}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 50

```
DSolve[{x1'[t]==-1*x1[t]-1*x2[t]-2*x3[t]+Exp[t],x2'[t]==1*x1[t]+1*x2[t]+1*x3[t],x3'[t]==2*x1
```

$$\text{x1}(t) \to -\frac{1}{6}e^t t(t^2 + 6t - 6)$$
$$\text{x2}(t) \to \frac{e^t t^2}{2}$$
$$\text{x3}(t) \to \frac{1}{6}e^t t^2(t + 6)$$

4.9 problem 10

Internal problem ID [1862]
Internal file name [`OUTPUT/1863_Sunday_June_05_2022_02_35_53_AM_74684757/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.12, Systems of differential equations. The nonhomogeneous equation. variation of parameters. Page 366
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 2x_1(t) + x_2(t) + \mathrm{e}^{3t} \\
x_2'(t) &= 3x_1(t) - 2x_2(t) + \mathrm{e}^{3t}
\end{aligned}
$$

4.9.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{\left(-2\sqrt{7}+7\right)\mathrm{e}^{-\sqrt{7}\,t}}{14} + \frac{\mathrm{e}^{\sqrt{7}\,t}\left(2\sqrt{7}+7\right)}{14} & \frac{\left(-\mathrm{e}^{-\sqrt{7}\,t}+\mathrm{e}^{\sqrt{7}\,t}\right)\sqrt{7}}{14} \\ \frac{3\left(-\mathrm{e}^{-\sqrt{7}\,t}+\mathrm{e}^{\sqrt{7}\,t}\right)\sqrt{7}}{14} & \frac{\left(2\sqrt{7}+7\right)\mathrm{e}^{-\sqrt{7}\,t}}{14} + \frac{\left(-2\sqrt{7}+7\right)\mathrm{e}^{\sqrt{7}\,t}}{14} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{\left(-2\sqrt{7}+7\right)\mathrm{e}^{-\sqrt{7}\,t}}{14} + \frac{\mathrm{e}^{\sqrt{7}\,t}\left(2\sqrt{7}+7\right)}{14} & \frac{\left(-\mathrm{e}^{-\sqrt{7}\,t}+\mathrm{e}^{\sqrt{7}\,t}\right)\sqrt{7}}{14} \\ \frac{3\left(-\mathrm{e}^{-\sqrt{7}\,t}+\mathrm{e}^{\sqrt{7}\,t}\right)\sqrt{7}}{14} & \frac{\left(2\sqrt{7}+7\right)\mathrm{e}^{-\sqrt{7}\,t}}{14} + \frac{\left(-2\sqrt{7}+7\right)\mathrm{e}^{\sqrt{7}\,t}}{14} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{\left(-2\sqrt{7}+7\right)\mathrm{e}^{-\sqrt{7}\,t}}{14} + \frac{\mathrm{e}^{\sqrt{7}\,t}\left(2\sqrt{7}+7\right)}{14}\right)c_1 + \frac{\left(-\mathrm{e}^{-\sqrt{7}\,t}+\mathrm{e}^{\sqrt{7}\,t}\right)\sqrt{7}\,c_2}{14} \\ \frac{3\left(-\mathrm{e}^{-\sqrt{7}\,t}+\mathrm{e}^{\sqrt{7}\,t}\right)\sqrt{7}\,c_1}{14} + \left(\frac{\left(2\sqrt{7}+7\right)\mathrm{e}^{-\sqrt{7}\,t}}{14} + \frac{\left(-2\sqrt{7}+7\right)\mathrm{e}^{\sqrt{7}\,t}}{14}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{\left((-2c_1-c_2)\sqrt{7}+7c_1\right)\mathrm{e}^{-\sqrt{7}\,t}}{14} + \frac{\left(\left(c_1+\frac{c_2}{2}\right)\sqrt{7}+\frac{7c_1}{2}\right)\mathrm{e}^{\sqrt{7}\,t}}{7} \\ \frac{\left((-3c_1+2c_2)\sqrt{7}+7c_2\right)\mathrm{e}^{-\sqrt{7}\,t}}{14} + \frac{3\left(\left(c_1-\frac{2c_2}{3}\right)\sqrt{7}+\frac{7c_2}{3}\right)\mathrm{e}^{\sqrt{7}\,t}}{14} \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} \frac{\left(2\sqrt{7}+7\right)\mathrm{e}^{-\sqrt{7}\,t}}{14} + \frac{\left(-2\sqrt{7}+7\right)\mathrm{e}^{\sqrt{7}\,t}}{14} & -\frac{\left(-\mathrm{e}^{-\sqrt{7}\,t}+\mathrm{e}^{\sqrt{7}\,t}\right)\sqrt{7}}{14} \\ -\frac{3\left(-\mathrm{e}^{-\sqrt{7}\,t}+\mathrm{e}^{\sqrt{7}\,t}\right)\sqrt{7}}{14} & \frac{\left(-2\sqrt{7}+7\right)\mathrm{e}^{-\sqrt{7}\,t}}{14} + \frac{\mathrm{e}^{\sqrt{7}\,t}\left(2\sqrt{7}+7\right)}{14} \end{bmatrix}
\end{aligned}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} \frac{\left(-2\sqrt{7}+7\right)e^{-\sqrt{7}\,t}}{14} + \frac{e^{\sqrt{7}\,t}\left(2\sqrt{7}+7\right)}{14} & \frac{\left(-e^{-\sqrt{7}\,t}+e^{\sqrt{7}\,t}\right)\sqrt{7}}{14} \\ \frac{3\left(-e^{-\sqrt{7}\,t}+e^{\sqrt{7}\,t}\right)\sqrt{7}}{14} & \frac{\left(2\sqrt{7}+7\right)e^{-\sqrt{7}\,t}}{14} + \frac{\left(-2\sqrt{7}+7\right)e^{\sqrt{7}\,t}}{14} \end{bmatrix} \int \begin{bmatrix} \frac{\left(2\sqrt{7}+7\right)e^{-\sqrt{7}\,t}}{14} + \frac{\left(-2\sqrt{7}+7\right)e^{\sqrt{7}\,t}}{14} \\ -\frac{3\left(-e^{-\sqrt{7}\,t}+e^{\sqrt{7}\,t}\right)\sqrt{7}}{14} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\left(-2\sqrt{7}+7\right)e^{-\sqrt{7}\,t}}{14} + \frac{e^{\sqrt{7}\,t}\left(2\sqrt{7}+7\right)}{14} & \frac{\left(-e^{-\sqrt{7}\,t}+e^{\sqrt{7}\,t}\right)\sqrt{7}}{14} \\ \frac{3\left(-e^{-\sqrt{7}\,t}+e^{\sqrt{7}\,t}\right)\sqrt{7}}{14} & \frac{\left(2\sqrt{7}+7\right)e^{-\sqrt{7}\,t}}{14} + \frac{\left(-2\sqrt{7}+7\right)e^{\sqrt{7}\,t}}{14} \end{bmatrix} \begin{bmatrix} \frac{4\sqrt{7}\,e^{-t\left(-3+\sqrt{7}\right)}}{7} - \frac{4\sqrt{7}\,e^{t\left(3+\sqrt{7}\right)}}{7} + 3 \\ \frac{5\sqrt{7}\,e^{-t\left(-3+\sqrt{7}\right)}}{14} - \frac{5\sqrt{7}\,e^{t\left(3+\sqrt{7}\right)}}{14} + e \end{bmatrix}$$

$$= \begin{bmatrix} 3\,e^{3t} \\ 2\,e^{3t} \end{bmatrix}$$

Hence the complete solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \begin{bmatrix} \frac{\left((-2c_1-c_2)\sqrt{7}+7c_1\right)e^{-\sqrt{7}\,t}}{14} + \frac{\left((2c_1+c_2)\sqrt{7}+7c_1\right)e^{\sqrt{7}\,t}}{14} + 3\,e^{3t} \\ \frac{\left((-3c_1+2c_2)\sqrt{7}+7c_2\right)e^{-\sqrt{7}\,t}}{14} + \frac{\left((3c_1-2c_2)\sqrt{7}+7c_2\right)e^{\sqrt{7}\,t}}{14} + 2\,e^{3t} \end{bmatrix} \end{aligned}$$

4.9.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & 1 \\ 3 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 7 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \sqrt{7}$$
$$\lambda_2 = -\sqrt{7}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$\sqrt{7}$	1	real eigenvalue
$-\sqrt{7}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \sqrt{7}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix} - \left(\sqrt{7}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2-\sqrt{7} & 1 \\ 3 & -2-\sqrt{7} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2-\sqrt{7} & 1 & 0 \\ 3 & -2-\sqrt{7} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{2-\sqrt{7}} \Longrightarrow \left[\begin{array}{cc|c} 2-\sqrt{7} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2-\sqrt{7} & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{\sqrt{7}-2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{\sqrt{7}-2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{\sqrt{7}-2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{\sqrt{7}-2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{\sqrt{7}-2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{\sqrt{7}-2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{7}-2} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -\sqrt{7}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix} - \left(-\sqrt{7}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{7}+2 & 1 \\ 3 & \sqrt{7}-2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \sqrt{7}+2 & 1 & 0 \\ 3 & \sqrt{7}-2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{\sqrt{7}+2} \implies \left[\begin{array}{cc|c} \sqrt{7}+2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \sqrt{7}+2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{\sqrt{7}+2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{\sqrt{7}+2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{\sqrt{7}+2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{\sqrt{7}+2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{\sqrt{7}+2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{\sqrt{7}+2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{7}+2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{\sqrt{7}+2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{7}+2} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$\sqrt{7}$	1	1	No	$\left[\begin{array}{c} \frac{1}{\sqrt{7}-2} \\ 1 \end{array}\right]$
$-\sqrt{7}$	1	1	No	$\left[\begin{array}{c} \frac{1}{-2-\sqrt{7}} \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $\sqrt{7}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\sqrt{7}\, t} \\
&= \left[\begin{array}{c} \frac{1}{\sqrt{7}-2} \\ 1 \end{array}\right] e^{\sqrt{7}\, t}
\end{aligned}
$$

Since eigenvalue $-\sqrt{7}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{-\sqrt{7}\, t} \\
&= \left[\begin{array}{c} \frac{1}{-2-\sqrt{7}} \\ 1 \end{array}\right] e^{-\sqrt{7}\, t}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\left[\begin{array}{c} x_1(t) \\ x_2(t) \end{array}\right] = c_1 \left[\begin{array}{c} \frac{e^{\sqrt{7}\, t}}{\sqrt{7}-2} \\ e^{\sqrt{7}\, t} \end{array}\right] + c_2 \left[\begin{array}{c} \frac{e^{-\sqrt{7}\, t}}{-2-\sqrt{7}} \\ e^{-\sqrt{7}\, t} \end{array}\right]
$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$
\Phi = \left[\vec{x}_1 \quad \vec{x}_2 \quad \cdots\right]
$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \frac{e^{\sqrt{7}\, t}}{\sqrt{7}-2} & \frac{e^{-\sqrt{7}\, t}}{-2-\sqrt{7}} \\ e^{\sqrt{7}\, t} & e^{-\sqrt{7}\, t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{3\sqrt{7}\, e^{-\sqrt{7}\, t}}{14} & \frac{\sqrt{7}\left(\sqrt{7}-2\right) e^{-\sqrt{7}\, t}}{14} \\ -\frac{3\sqrt{7}\, e^{\sqrt{7}\, t}}{14} & \frac{\sqrt{7}\, e^{\sqrt{7}\, t}\left(\sqrt{7}+2\right)}{14} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \frac{e^{\sqrt{7}\, t}}{\sqrt{7}-2} & \frac{e^{-\sqrt{7}\, t}}{-2-\sqrt{7}} \\ e^{\sqrt{7}\, t} & e^{-\sqrt{7}\, t} \end{bmatrix} \int \begin{bmatrix} \frac{3\sqrt{7}\, e^{-\sqrt{7}\, t}}{14} & \frac{\sqrt{7}\left(\sqrt{7}-2\right) e^{-\sqrt{7}\, t}}{14} \\ -\frac{3\sqrt{7}\, e^{\sqrt{7}\, t}}{14} & \frac{\sqrt{7}\, e^{\sqrt{7}\, t}\left(\sqrt{7}+2\right)}{14} \end{bmatrix} \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} dt \\
&= \begin{bmatrix} \frac{e^{\sqrt{7}\, t}}{\sqrt{7}-2} & \frac{e^{-\sqrt{7}\, t}}{-2-\sqrt{7}} \\ e^{\sqrt{7}\, t} & e^{-\sqrt{7}\, t} \end{bmatrix} \int \begin{bmatrix} \frac{e^{-t\left(-3+\sqrt{7}\right)}\left(\sqrt{7}+7\right)}{14} \\ -\frac{e^{t\left(3+\sqrt{7}\right)}\left(\sqrt{7}-7\right)}{14} \end{bmatrix} dt \\
&= \begin{bmatrix} \frac{e^{\sqrt{7}\, t}}{\sqrt{7}-2} & \frac{e^{-\sqrt{7}\, t}}{-2-\sqrt{7}} \\ e^{\sqrt{7}\, t} & e^{-\sqrt{7}\, t} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{7}\, e^{-t\left(-3+\sqrt{7}\right)}\left(1+\sqrt{7}\right)\left(3+\sqrt{7}\right)}{28} \\ -\frac{\sqrt{7}\, e^{t\left(3+\sqrt{7}\right)}\left(-3+\sqrt{7}\right)\left(-1+\sqrt{7}\right)}{28} \end{bmatrix} \\
&= \begin{bmatrix} 3\, e^{3t} \\ 2\, e^{3t} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1 e^{\sqrt{7}\, t}}{\sqrt{7}-2} \\ c_1 e^{\sqrt{7}\, t} \end{bmatrix} + \begin{bmatrix} \frac{c_2 e^{-\sqrt{7}\, t}}{-2-\sqrt{7}} \\ c_2 e^{-\sqrt{7}\, t} \end{bmatrix} + \begin{bmatrix} 3\, e^{3t} \\ 2\, e^{3t} \end{bmatrix}$$

Which becomes

$$\left[\begin{array}{c} x_1(t) \\ x_2(t) \end{array}\right] = \left[\begin{array}{c} -\frac{c_2\left(\sqrt{7}-2\right)\mathrm{e}^{-\sqrt{7}\,t}}{3} + \frac{c_1\left(\sqrt{7}+2\right)\mathrm{e}^{\sqrt{7}\,t}}{3} + 3\,\mathrm{e}^{3t} \\ c_1\mathrm{e}^{\sqrt{7}\,t} + c_2\mathrm{e}^{-\sqrt{7}\,t} + 2\,\mathrm{e}^{3t} \end{array}\right]$$

4.9.3 Maple step by step solution

Let's solve

$\left[x_1'(t) = 2x_1(t) + x_2(t) + \left(\mathrm{e}^t\right)^3, x_2'(t) = 3x_1(t) - 2x_2(t) + \left(\mathrm{e}^t\right)^3\right]$

- Define vector

$$\underline{\vec{x}}(t) = \left[\begin{array}{c} x_1(t) \\ x_2(t) \end{array}\right]$$

- Convert system into a vector equation

$$\underline{\vec{x}}'(t) = \left[\begin{array}{cc} 2 & 1 \\ 3 & -2 \end{array}\right] \cdot \underline{\vec{x}}(t) + \left[\begin{array}{c} \left(\mathrm{e}^t\right)^3 \\ \left(\mathrm{e}^t\right)^3 \end{array}\right]$$

- System to solve

$$\underline{\vec{x}}'(t) = \left[\begin{array}{cc} 2 & 1 \\ 3 & -2 \end{array}\right] \cdot \underline{\vec{x}}(t) + \left[\begin{array}{c} \left(\mathrm{e}^t\right)^3 \\ \left(\mathrm{e}^t\right)^3 \end{array}\right]$$

- Define the forcing function

$$\vec{f}(t) = \left[\begin{array}{c} \left(\mathrm{e}^t\right)^3 \\ \left(\mathrm{e}^t\right)^3 \end{array}\right]$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cc} 2 & 1 \\ 3 & -2 \end{array}\right]$$

- Rewrite the system as

$$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[\sqrt{7}, \begin{bmatrix} \frac{1}{\sqrt{7}-2} \\ 1 \end{bmatrix}\right], \left[-\sqrt{7}, \begin{bmatrix} \frac{1}{-2-\sqrt{7}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[\sqrt{7}, \begin{bmatrix} \frac{1}{\sqrt{7}-2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_1 = e^{\sqrt{7}\,t} \cdot \begin{bmatrix} \frac{1}{\sqrt{7}-2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-\sqrt{7}, \begin{bmatrix} \frac{1}{-2-\sqrt{7}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_2 = e^{-\sqrt{7}\,t} \cdot \begin{bmatrix} \frac{1}{-2-\sqrt{7}} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\underline{\vec{x}}$

$$\underline{\vec{x}}(t) = c_1\underline{\vec{x}}_1 + c_2\underline{\vec{x}}_2 + \underline{\vec{x}}_p(t)$$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} \frac{e^{\sqrt{7}\,t}}{\sqrt{7}-2} & \frac{e^{-\sqrt{7}\,t}}{-2-\sqrt{7}} \\ e^{\sqrt{7}\,t} & e^{-\sqrt{7}\,t} \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} \frac{e^{\sqrt{7}\,t}}{\sqrt{7}-2} & \frac{e^{-\sqrt{7}\,t}}{-2-\sqrt{7}} \\ e^{\sqrt{7}\,t} & e^{-\sqrt{7}\,t} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{\sqrt{7}-2} & \frac{1}{-2-\sqrt{7}} \\ 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} \frac{\left(\left(\sqrt{7}-2\right)e^{-\sqrt{7}\,t}+e^{\sqrt{7}\,t}\left(\sqrt{7}+2\right)\right)\sqrt{7}}{14} & \frac{\left(-e^{-\sqrt{7}\,t}+e^{\sqrt{7}\,t}\right)\sqrt{7}}{14} \\ \frac{3\left(-e^{-\sqrt{7}\,t}+e^{\sqrt{7}\,t}\right)\sqrt{7}}{14} & \frac{\left(2\sqrt{7}+7\right)e^{-\sqrt{7}\,t}}{14}+\frac{\left(-2\sqrt{7}+7\right)e^{\sqrt{7}\,t}}{14} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$$\underline{\vec{x}}_p(t) = \Phi(t) \cdot \vec{v}(t)$$

- Take the derivative of the particular solution

$$\underline{\vec{x}}_p'(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$$

- Substitute particular solution and its derivative into the system of ODEs

$$\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$$A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$$

- Cancel like terms

$$\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$$

- Integrate to solve for $\vec{v}(t)$

$$\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\,ds$$

- Plug $\vec{v}(t)$ into the equation for the particular solution

$$\underline{\vec{x}}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\,ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\underline{\vec{x}}_p(t) = \begin{bmatrix} \frac{\left(-21+8\sqrt{7}\right)e^{-\sqrt{7}\,t}}{14}+\frac{\left(-21-8\sqrt{7}\right)e^{\sqrt{7}\,t}}{14}+3\,e^{3t} \\ 2\,e^{3t}-\frac{5\sqrt{7}\,e^{\sqrt{7}\,t}}{14}+\frac{5\sqrt{7}\,e^{-\sqrt{7}\,t}}{14}-e^{\sqrt{7}\,t}-e^{-\sqrt{7}\,t} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2 + \begin{bmatrix} \frac{(-21+8\sqrt{7})e^{-\sqrt{7}\,t}}{14} + \frac{(-21-8\sqrt{7})e^{\sqrt{7}\,t}}{14} + 3\,e^{3t} \\ 2\,e^{3t} - \frac{5\sqrt{7}\,e^{\sqrt{7}\,t}}{14} + \frac{5\sqrt{7}\,e^{-\sqrt{7}\,t}}{14} - e^{\sqrt{7}\,t} - e^{-\sqrt{7}\,t} \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{((-14c_2+24)\sqrt{7}+28c_2-63)e^{-\sqrt{7}\,t}}{42} + \frac{((14c_1-24)\sqrt{7}+28c_1-63)e^{\sqrt{7}\,t}}{42} + 3\,e^{3t} \\ \frac{(14c_2+5\sqrt{7}-14)e^{-\sqrt{7}\,t}}{14} + \frac{(14c_1-5\sqrt{7}-14)e^{\sqrt{7}\,t}}{14} + 2\,e^{3t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \frac{((-14c_2+24)\sqrt{7}+28c_2-63)e^{-\sqrt{7}\,t}}{42} + \frac{((14c_1-24)\sqrt{7}+28c_1-63)e^{\sqrt{7}\,t}}{42} + 3\,e^{3t}, x_2(t) = \frac{(14c_2+5\sqrt{7}-14)e^{-\sqrt{7}\,t}}{14}\right.$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 82

```
dsolve([diff(x__1(t),t)=2*x__1(t)+1*x__2(t)+1*exp(3*t),diff(x__2(t),t)=3*x__1(t)-2*x__2(t)+e
```

$$x_1(t) = e^{\sqrt{7}\,t} c_2 + e^{-\sqrt{7}\,t} c_1 + 3\,e^{3t}$$
$$x_2(t) = \sqrt{7}\,e^{\sqrt{7}\,t} c_2 - \sqrt{7}\,e^{-\sqrt{7}\,t} c_1 + 2\,e^{3t} - 2\,e^{\sqrt{7}\,t} c_2 - 2\,e^{-\sqrt{7}\,t} c_1$$

✓ Solution by Mathematica

Time used: 0.411 (sec). Leaf size: 171

```
DSolve[{x1'[t]==2*x1[t]+1*x2[t]+Exp[3*t],x2'[t]==3*x1[t]-2*x2[t]+Exp[3*t]},{x1[t],x2[t]},t,I
```

$$\text{x1}(t) \to \frac{1}{14} e^{-\sqrt{7}t} \left(42 e^{(3+\sqrt{7})t} + \left(\left(7+2\sqrt{7}\right) c_1 + \sqrt{7} c_2\right) e^{2\sqrt{7}t} + \left(7-2\sqrt{7}\right) c_1 - \sqrt{7} c_2\right)$$
$$\text{x2}(t) \to \frac{1}{14} e^{-\sqrt{7}t} \left(28 e^{(3+\sqrt{7})t} + \left(3\sqrt{7} c_1 + \left(7-2\sqrt{7}\right) c_2\right) e^{2\sqrt{7}t} - 3\sqrt{7} c_1 + \left(7+2\sqrt{7}\right) c_2\right)$$

4.10 problem 11

Internal problem ID [1863]
Internal file name [`OUTPUT/1864_Sunday_June_05_2022_02_35_56_AM_70867750/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.12, Systems of differential equations. The nonhomogeneous equation. variation of parameters. Page 366
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = x_1(t) - x_2(t) - t^2$$
$$x_2'(t) = x_1(t) + 3x_2(t) + 2t$$

4.10.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -t^2 \\ 2t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At}=\begin{bmatrix} e^{2t}(1-t) & -e^{2t}t \\ e^{2t}t & e^{2t}(t+1) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{2t}(1-t) & -e^{2t}t \\ e^{2t}t & e^{2t}(t+1) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}(1-t)\,c_1 - e^{2t}tc_2 \\ e^{2t}tc_1 + e^{2t}(t+1)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} -(c_1(t-1)+c_2t)\,e^{2t} \\ e^{2t}(tc_1+c_2t+c_2) \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t)=e^{At}\int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} e^{-2t}(t+1) & t\,e^{-2t} \\ -t\,e^{-2t} & -e^{-2t}(t-1) \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} e^{2t}(1-t) & -e^{2t}t \\ e^{2t}t & e^{2t}(t+1) \end{bmatrix}\int\begin{bmatrix} e^{-2t}(t+1) & t\,e^{-2t} \\ -t\,e^{-2t} & -e^{-2t}(t-1) \end{bmatrix}\begin{bmatrix} -t^2 \\ 2t \end{bmatrix}dt \\
&= \begin{bmatrix} e^{2t}(1-t) & -e^{2t}t \\ e^{2t}t & e^{2t}(t+1) \end{bmatrix}\begin{bmatrix} \frac{e^{-2t}(4t^3+2t^2+2t+1)}{8} \\ -\frac{e^{-2t}(4t^3-2t^2+6t+3)}{8} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2}t+\frac{1}{8}+\frac{3}{4}t^2 \\ -\frac{1}{4}t^2-t-\frac{3}{8} \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
&= \begin{bmatrix} \frac{1}{8} + ((-c_1 - c_2)\, t + c_1)\, e^{2t} + \frac{3t^2}{4} + \frac{t}{2} \\ -\frac{3}{8} + ((c_1 + c_2)\, t + c_2)\, e^{2t} - \frac{t^2}{4} - t \end{bmatrix}
\end{aligned}
$$

4.10.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -t^2 \\ 2t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 4\lambda + 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	2	1	Yes	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 2 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

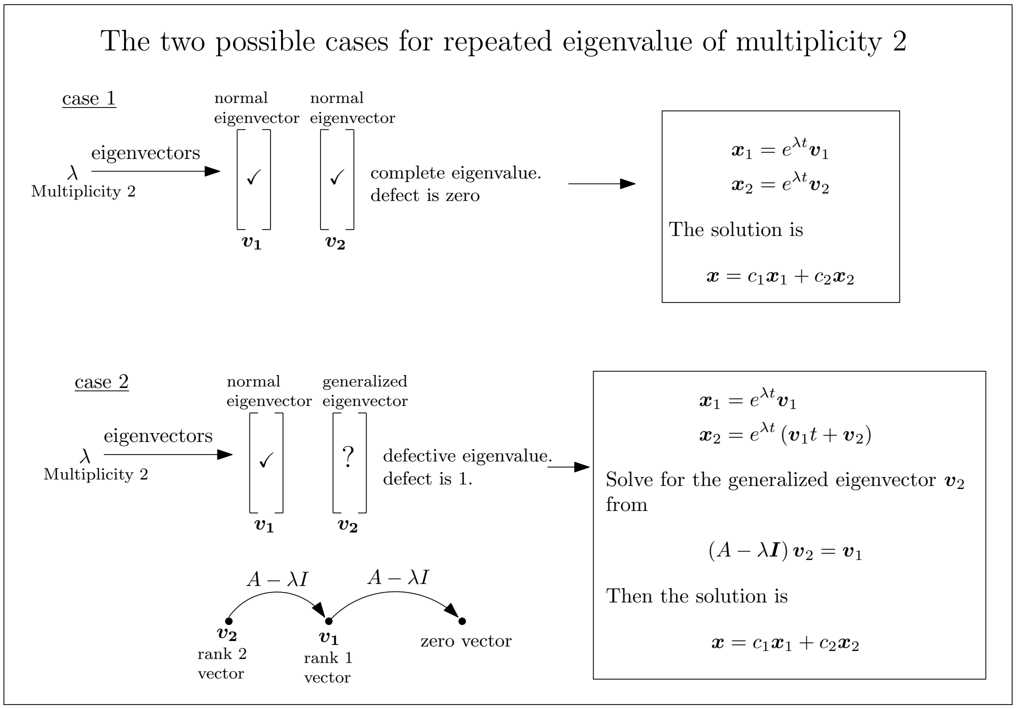


Figure 24: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 2. Therefore the two basis solution associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \mathrm{e}^{2t} \\
&= \begin{bmatrix} -\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \mathrm{e}^{2t} \\
&= \begin{bmatrix} -\mathrm{e}^{2t} t \\ \mathrm{e}^{2t}(t+1) \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{2t} t \\ \mathrm{e}^{2t}(t+1) \end{bmatrix}
$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$
\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}
$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$
\Phi(t) = \begin{bmatrix} -\mathrm{e}^{2t} & -\mathrm{e}^{2t} t \\ \mathrm{e}^{2t} & \mathrm{e}^{2t}(t+1) \end{bmatrix}
$$

The particular solution is then given by

$$
\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt
$$

But

$$
\Phi^{-1} = \begin{bmatrix} -\mathrm{e}^{-2t}(t+1) & -t\,\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} & \mathrm{e}^{-2t} \end{bmatrix}
$$

Hence

$$
\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} -\mathrm{e}^{2t} & -\mathrm{e}^{2t}t \\ \mathrm{e}^{2t} & \mathrm{e}^{2t}(t+1) \end{bmatrix} \int \begin{bmatrix} -\mathrm{e}^{-2t}(t+1) & -t\,\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} & \mathrm{e}^{-2t} \end{bmatrix} \begin{bmatrix} -t^2 \\ 2t \end{bmatrix} dt \\
&= \begin{bmatrix} -\mathrm{e}^{2t} & -\mathrm{e}^{2t}t \\ \mathrm{e}^{2t} & \mathrm{e}^{2t}(t+1) \end{bmatrix} \int \begin{bmatrix} \mathrm{e}^{-2t}t^2(t-1) \\ -\mathrm{e}^{-2t}t(t-2) \end{bmatrix} dt \\
&= \begin{bmatrix} -\mathrm{e}^{2t} & -\mathrm{e}^{2t}t \\ \mathrm{e}^{2t} & \mathrm{e}^{2t}(t+1) \end{bmatrix} \begin{bmatrix} -\frac{\mathrm{e}^{-2t}(4t^3+2t^2+2t+1)}{8} \\ \frac{(2t^2-2t-1)\mathrm{e}^{-2t}}{4} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2}t + \frac{1}{8} + \frac{3}{4}t^2 \\ -\frac{1}{4}t^2 - t - \frac{3}{8} \end{bmatrix}
\end{aligned}
$$

Now that we found particular solution, the final solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} -c_1\mathrm{e}^{2t} \\ c_1\mathrm{e}^{2t} \end{bmatrix} + \begin{bmatrix} -c_2\mathrm{e}^{2t}t \\ c_2\mathrm{e}^{2t}(t+1) \end{bmatrix} + \begin{bmatrix} \frac{1}{2}t + \frac{1}{8} + \frac{3}{4}t^2 \\ -\frac{1}{4}t^2 - t - \frac{3}{8} \end{bmatrix}
\end{aligned}
$$

Which becomes

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{8} + (-tc_2 - c_1)\,\mathrm{e}^{2t} + \frac{3t^2}{4} + \frac{t}{2} \\ -\frac{3}{8} + (tc_2 + c_1 + c_2)\,\mathrm{e}^{2t} - \frac{t^2}{4} - t \end{bmatrix}
$$

4.10.3 Maple step by step solution

Let's solve

$[x_1'(t) = x_1(t) - x_2(t) - t^2, x_2'(t) = x_1(t) + 3x_2(t) + 2t]$

- Define vector

$$\underline{x^{\rightarrow}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t) + \begin{bmatrix} -t^2 \\ 2t \end{bmatrix}$$

- System to solve

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t) + \begin{bmatrix} -t^2 \\ 2t \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} -t^2 \\ 2t \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

- Rewrite the system as

$$\underline{x^{\rightarrow}}'(t) = A \cdot \underline{x^{\rightarrow}}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 2

$$\underline{x^{\rightarrow}}_1(t) = \mathrm{e}^{2t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 2$ is the eigenvalue, an

$\overrightarrow{x}_2(t) = \mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right)$

- Note that the t multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obta
- Substitute $\overrightarrow{x}_2(t)$ into the homogeneous system

  $\lambda\,\mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \left(\mathrm{e}^{\lambda t}A\right) \cdot \left(t\overrightarrow{v} + \overrightarrow{p}\right)$
- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

  $\lambda\,\mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \mathrm{e}^{\lambda t}\left(\lambda t\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$
- Simplify equation

  $\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$
- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$
- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{x}_2(t)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$
- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 2

  $$\left(\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} - 2 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \cdot \overrightarrow{p} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
- Choice of $\overrightarrow{p}$

  $$\overrightarrow{p} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
- Second solution from eigenvalue 2

  $$\overrightarrow{x}_2(t) = \mathrm{e}^{2t} \cdot \left(t \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$
- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{x}$

  $\overrightarrow{x}(t) = c_1\overrightarrow{x}_1(t) + c_2\overrightarrow{x}_2(t) + \overrightarrow{x}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

  $$\phi(t) = \begin{bmatrix} -\mathrm{e}^{2t} & \mathrm{e}^{2t}(1-t) \\ \mathrm{e}^{2t} & \mathrm{e}^{2t}t \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$
- Substitute the value of $\phi(t)$ and $\phi(0)$

  $$\Phi(t) = \begin{bmatrix} -e^{2t} & e^{2t}(1-t) \\ e^{2t} & e^{2t}t \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}}$$
- Evaluate and simplify to get the fundamental matrix

  $$\Phi(t) = \begin{bmatrix} -e^{2t}(t-1) & -e^{2t}t \\ e^{2t}t & e^{2t}(t+1) \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

  $\underline{\vec{x}}_p(t) = \Phi(t) \cdot \vec{v}(t)$
- Take the derivative of the particular solution

  $\underline{\vec{x}}'_p(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$
- Cancel like terms

  $\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$
- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$
- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\,ds$
- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\underline{\vec{x}}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\,ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{x}_{p}(t)=\left[\begin{array}{c}\frac{(-2t-1)\mathrm{e}^{2t}}{8}+\frac{3t^{2}}{4}+\frac{t}{2}+\frac{1}{8}\\ \frac{\mathrm{e}^{2t}(2t+3)}{8}-\frac{t^{2}}{4}-t-\frac{3}{8}\end{array}\right]$$

- Plug particular solution back into general solution

$$\overrightarrow{x}(t)=c_{1}\overrightarrow{x}_{1}(t)+c_{2}\overrightarrow{x}_{2}(t)+\left[\begin{array}{c}\frac{(-2t-1)\mathrm{e}^{2t}}{8}+\frac{3t^{2}}{4}+\frac{t}{2}+\frac{1}{8}\\ \frac{\mathrm{e}^{2t}(2t+3)}{8}-\frac{t^{2}}{4}-t-\frac{3}{8}\end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c}x_{1}(t)\\ x_{2}(t)\end{array}\right]=\left[\begin{array}{c}\frac{((-8c_{2}-2)t-8c_{1}+8c_{2}-1)\mathrm{e}^{2t}}{8}+\frac{3t^{2}}{4}+\frac{t}{2}+\frac{1}{8}\\ \frac{((8c_{2}+2)t+8c_{1}+3)\mathrm{e}^{2t}}{8}-\frac{t^{2}}{4}-t-\frac{3}{8}\end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x_{1}(t)=\frac{((-8c_{2}-2)t-8c_{1}+8c_{2}-1)\mathrm{e}^{2t}}{8}+\frac{3t^{2}}{4}+\frac{t}{2}+\frac{1}{8},x_{2}(t)=\frac{((8c_{2}+2)t+8c_{1}+3)\mathrm{e}^{2t}}{8}-\frac{t^{2}}{4}-t-\frac{3}{8}\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 63

```
dsolve([diff(x__1(t),t)=1*x__1(t)-1*x__2(t)-t^2,diff(x__2(t),t)=1*x__1(t)+3*x__2(t)+2*t],sin
```

$$x_{1}(t)=c_{2}\mathrm{e}^{2t}+\mathrm{e}^{2t}tc_{1}+\frac{3t^{2}}{4}+\frac{t}{2}+\frac{1}{8}$$
$$x_{2}(t)=-\frac{t^{2}}{4}-c_{2}\mathrm{e}^{2t}-\mathrm{e}^{2t}tc_{1}-t-\frac{3}{8}-c_{1}\mathrm{e}^{2t}$$

✓ Solution by Mathematica

Time used: 0.27 (sec). Leaf size: 94

```
DSolve[{x1'[t]==1*x1[t]+3*x2[t]-t^2,x2'[t]==1*x1[t]+3*x2[t]+2*t},{x1[t],x2[t]},t,IncludeSing
```

$$\text{x1}(t)\to\frac{1}{128}\left(-32t^{3}-88t^{2}-44t+32c_{1}\left(e^{4t}+3\right)+96c_{2}e^{4t}-11-96c_{2}\right)$$
$$\text{x2}(t)\to\frac{1}{384}\left(32t^{3}+120t^{2}-132t+96c_{1}\left(e^{4t}-1\right)+288c_{2}e^{4t}-33+96c_{2}\right)$$

4.11 problem 12

Internal problem ID [1864]
Internal file name [OUTPUT/1865_Sunday_June_05_2022_02_35_58_AM_55583981/index.tex]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.12, Systems of differential equations. The nonhomogeneous equation. variation of parameters. Page 366
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned}
x_1'(t) &= x_1(t) + 3x_2(t) + 2x_3(t) + \sin(t) \\
x_2'(t) &= -x_1(t) + 2x_2(t) + x_3(t) \\
x_3'(t) &= 4x_1(t) - x_2(t) - x_3(t)
\end{aligned}$$

4.11.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 2 & 1 \\ 4 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} \sin(t) \\ 0 \\ 0 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(3\,\mathrm{e}^{5t}+\mathrm{e}^{3t}+2)\mathrm{e}^{-2t}}{6} & \frac{(3\,\mathrm{e}^{5t}-2\,\mathrm{e}^{3t}-1)\mathrm{e}^{-2t}}{3} & -\frac{(-3\,\mathrm{e}^{5t}+\mathrm{e}^{3t}+2)\mathrm{e}^{-2t}}{6} \\ -\frac{(\mathrm{e}^{3t}-1)\mathrm{e}^{-2t}}{3} & \frac{(4\,\mathrm{e}^{3t}-1)\mathrm{e}^{-2t}}{3} & \frac{(\mathrm{e}^{3t}-1)\mathrm{e}^{-2t}}{3} \\ \frac{(\mathrm{e}^{5t}+\mathrm{e}^{3t}-2)\mathrm{e}^{-2t}}{2} & (\mathrm{e}^{5t}-2\,\mathrm{e}^{3t}+1)\,\mathrm{e}^{-2t} & \frac{(\mathrm{e}^{5t}-\mathrm{e}^{3t}+2)\mathrm{e}^{-2t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \frac{(3\,\mathrm{e}^{5t}+\mathrm{e}^{3t}+2)\mathrm{e}^{-2t}}{6} & \frac{(3\,\mathrm{e}^{5t}-2\,\mathrm{e}^{3t}-1)\mathrm{e}^{-2t}}{3} & -\frac{(-3\,\mathrm{e}^{5t}+\mathrm{e}^{3t}+2)\mathrm{e}^{-2t}}{6} \\ -\frac{(\mathrm{e}^{3t}-1)\mathrm{e}^{-2t}}{3} & \frac{(4\,\mathrm{e}^{3t}-1)\mathrm{e}^{-2t}}{3} & \frac{(\mathrm{e}^{3t}-1)\mathrm{e}^{-2t}}{3} \\ \frac{(\mathrm{e}^{5t}+\mathrm{e}^{3t}-2)\mathrm{e}^{-2t}}{2} & (\mathrm{e}^{5t}-2\,\mathrm{e}^{3t}+1)\,\mathrm{e}^{-2t} & \frac{(\mathrm{e}^{5t}-\mathrm{e}^{3t}+2)\mathrm{e}^{-2t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(3\,\mathrm{e}^{5t}+\mathrm{e}^{3t}+2)\mathrm{e}^{-2t}c_1}{6} + \frac{(3\,\mathrm{e}^{5t}-2\,\mathrm{e}^{3t}-1)\mathrm{e}^{-2t}c_2}{3} - \frac{(-3\,\mathrm{e}^{5t}+\mathrm{e}^{3t}+2)\mathrm{e}^{-2t}c_3}{6} \\ -\frac{(\mathrm{e}^{3t}-1)\mathrm{e}^{-2t}c_1}{3} + \frac{(4\,\mathrm{e}^{3t}-1)\mathrm{e}^{-2t}c_2}{3} + \frac{(\mathrm{e}^{3t}-1)\mathrm{e}^{-2t}c_3}{3} \\ \frac{(\mathrm{e}^{5t}+\mathrm{e}^{3t}-2)\mathrm{e}^{-2t}c_1}{2} + (\mathrm{e}^{5t}-2\,\mathrm{e}^{3t}+1)\,\mathrm{e}^{-2t}c_2 + \frac{(\mathrm{e}^{5t}-\mathrm{e}^{3t}+2)\mathrm{e}^{-2t}c_3}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{((c_1-4c_2-c_3)\mathrm{e}^{3t}+(3c_1+6c_2+3c_3)\mathrm{e}^{5t}+2c_1-2c_2-2c_3)\mathrm{e}^{-2t}}{6} \\ -\frac{((c_1-4c_2-c_3)\mathrm{e}^{3t}-c_1+c_2+c_3)\mathrm{e}^{-2t}}{3} \\ \frac{((c_1-4c_2-c_3)\mathrm{e}^{3t}+(c_1+2c_2+c_3)\mathrm{e}^{5t}-2c_1+2c_2+2c_3)\mathrm{e}^{-2t}}{2} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} \frac{(2\,\mathrm{e}^{5t}+\mathrm{e}^{2t}+3)\mathrm{e}^{-3t}}{6} & -\frac{(\mathrm{e}^{5t}+2\,\mathrm{e}^{2t}-3)\mathrm{e}^{-3t}}{3} & -\frac{(2\,\mathrm{e}^{5t}+\mathrm{e}^{2t}-3)\mathrm{e}^{-3t}}{6} \\ \frac{\mathrm{e}^{2t}}{3} - \frac{\mathrm{e}^{-t}}{3} & \frac{4\,\mathrm{e}^{-t}}{3} - \frac{\mathrm{e}^{2t}}{3} & -\frac{\mathrm{e}^{2t}}{3} + \frac{\mathrm{e}^{-t}}{3} \\ \frac{(-2\,\mathrm{e}^{5t}+\mathrm{e}^{2t}+1)\mathrm{e}^{-3t}}{2} & (\mathrm{e}^{5t}-2\,\mathrm{e}^{2t}+1)\,\mathrm{e}^{-3t} & -\frac{(-2\,\mathrm{e}^{5t}+\mathrm{e}^{2t}-1)\mathrm{e}^{-3t}}{2} \end{bmatrix} \end{aligned}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} \frac{(3\,e^{5t}+e^{3t}+2)e^{-2t}}{6} & \frac{(3\,e^{5t}-2\,e^{3t}-1)e^{-2t}}{3} & -\frac{(-3\,e^{5t}+e^{3t}+2)e^{-2t}}{6} \\ -\frac{(e^{3t}-1)e^{-2t}}{3} & \frac{(4\,e^{3t}-1)e^{-2t}}{3} & \frac{(e^{3t}-1)e^{-2t}}{3} \\ \frac{(e^{5t}+e^{3t}-2)e^{-2t}}{2} & (e^{5t}-2\,e^{3t}+1)\,e^{-2t} & \frac{(e^{5t}-e^{3t}+2)e^{-2t}}{2} \end{bmatrix} \int \begin{bmatrix} \frac{(2\,e^{5t}+e^{2t}+3)e^{-3t}}{6} & -\frac{(e^{5t}+2\,e^{2t}-\cdots}{3} & \cdots \\ \frac{e^{2t}}{3}-\frac{e^{-t}}{3} & \frac{4\,e^{-t}}{3}-\cdots & \cdots \\ \frac{(-2\,e^{5t}+e^{2t}+1)e^{-3t}}{2} & (e^{5t}-2\,e^{2t}+\cdots & \cdots \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(3\,e^{5t}+e^{3t}+2)e^{-2t}}{6} & \frac{(3\,e^{5t}-2\,e^{3t}-1)e^{-2t}}{3} & -\frac{(-3\,e^{5t}+e^{3t}+2)e^{-2t}}{6} \\ -\frac{(e^{3t}-1)e^{-2t}}{3} & \frac{(4\,e^{3t}-1)e^{-2t}}{3} & \frac{(e^{3t}-1)e^{-2t}}{3} \\ \frac{(e^{5t}+e^{3t}-2)e^{-2t}}{2} & (e^{5t}-2\,e^{3t}+1)\,e^{-2t} & \frac{(e^{5t}-e^{3t}+2)e^{-2t}}{2} \end{bmatrix} \begin{bmatrix} -\frac{\left((\cos(t)+\sin(t))e^{2t}+\frac{4(\cos(t)-2\sin(t))e^{5t}}{5}+\cdots\right.}{12} \\ \frac{e^{-t}(\cos(t)+\sin(t))}{6}-\frac{e^{2t}(\cos(t)\cdots}{15\cdots} \\ -\frac{\left((\cos(t)+\sin(t))e^{2t}+\frac{4(-\cos(t)+2\sin(t))e^{5t}}{5}-\cdots\right.}{4} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{\sin(t)}{10}-\frac{\cos(t)}{5} \\ \frac{3\sin(t)}{10}+\frac{\cos(t)}{10} \\ -\frac{4\sin(t)}{5}-\frac{\cos(t)}{10} \end{bmatrix}$$

Hence the complete solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t)+\vec{x}_p(t) \\ &= \begin{bmatrix} \frac{\left(\frac{(c_1-4c_2-c_3)e^{3t}}{3}+(c_1+2c_2+c_3)e^{5t}+\frac{(-\sin(t)-2\cos(t))e^{2t}}{5}+\frac{2c_1}{3}-\frac{2c_2}{3}-\frac{2c_3}{3}\right)e^{-2t}}{2} \\ -\frac{e^{-2t}\left((c_1-4c_2-c_3)e^{3t}+\frac{3(-\cos(t)-3\sin(t))e^{2t}}{10}-c_1+c_2+c_3\right)}{3} \\ \frac{\left((c_1-4c_2-c_3)e^{3t}+(c_1+2c_2+c_3)e^{5t}+\frac{(-8\sin(t)-\cos(t))e^{2t}}{5}-2c_1+2c_2+2c_3\right)e^{-2t}}{2} \end{bmatrix} \end{aligned}$$

4.11.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)+\vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 2 & 1 \\ 4 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} \sin(t) \\ 0 \\ 0 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t)+\vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 3 & 2 \\ -1 & 2 & 1 \\ 4 & -1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 3 & 2 \\ -1 & 2-\lambda & 1 \\ 4 & -1 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3$$
$$\lambda_2 = -2$$
$$\lambda_3 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
1	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 3 & 2 \\ -1 & 2 & 1 \\ 4 & -1 & -1 \end{bmatrix} - (-2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 & 2 \\ -1 & 4 & 1 \\ 4 & -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 3 & 3 & 2 & 0 \\ -1 & 4 & 1 & 0 \\ 4 & -1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{3} \implies \left[\begin{array}{ccc|c} 3 & 3 & 2 & 0 \\ 0 & 5 & \frac{5}{3} & 0 \\ 4 & -1 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{4R_1}{3} \implies \left[\begin{array}{ccc|c} 3 & 3 & 2 & 0 \\ 0 & 5 & \frac{5}{3} & 0 \\ 0 & -5 & -\frac{5}{3} & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \implies \left[\begin{array}{ccc|c} 3 & 3 & 2 & 0 \\ 0 & 5 & \frac{5}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & 3 & 2 \\ 0 & 5 & \frac{5}{3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{3}, v_2 = -\frac{t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{3} \\ -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{3} \\ -\frac{t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{3} \\ -\frac{t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{3} \\ -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{3} \\ -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 3 & 2 \\ -1 & 2 & 1 \\ 4 & -1 & -1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 & 2 \\ -1 & 1 & 1 \\ 4 & -1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 3 & 2 & 0 \\ -1 & 1 & 1 & 0 \\ 4 & -1 & -2 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 3 & 2 & 0 \\ 4 & -1 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 + 4R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 3 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{3}, v_2 = -\frac{2t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{3} \\ -\frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{3} \\ -\frac{2t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{3} \\ -\frac{2t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{3} \\ -\frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{3} \\ -\frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 3 & 2 \\ -1 & 2 & 1 \\ 4 & -1 & -1 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 3 & 2 \\ -1 & -1 & 1 \\ 4 & -1 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & 3 & 2 & 0 \\ -1 & -1 & 1 & 0 \\ 4 & -1 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \implies \left[\begin{array}{ccc|c} -2 & 3 & 2 & 0 \\ 0 & -\frac{5}{2} & 0 & 0 \\ 4 & -1 & -4 & 0 \end{array}\right]$$

$$
R_3 = R_3 + 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 3 & 2 & 0 \\ 0 & -\frac{5}{2} & 0 & 0 \\ 0 & 5 & 0 & 0 \end{array}\right]
$$

$$
R_3 = R_3 + 2R_2 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 3 & 2 & 0 \\ 0 & -\frac{5}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} -2 & 3 & 2 \\ 0 & -\frac{5}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = 0\}$

Hence the solution is

$$
\begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$
\begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}
$$

Let $t = 1$ the eigenvector becomes

$$
\begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}
$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated

with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
3	1	1	No	$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$
-2	1	1	No	$\begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{3t} \\ &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{3t} \end{aligned}$$

Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_2(t) &= \vec{v}_2 e^{-2t} \\ &= \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix} e^{-2t} \end{aligned}$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_3(t) = \vec{v}_3 e^t$$
$$= \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix} e^t$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{3t} \\ 0 \\ e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{e^{-2t}}{3} \\ -\frac{e^{-2t}}{3} \\ e^{-2t} \end{bmatrix} + c_3 \begin{bmatrix} \frac{e^t}{3} \\ -\frac{2e^t}{3} \\ e^t \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} e^{3t} & -\frac{e^{-2t}}{3} & \frac{e^t}{3} \\ 0 & -\frac{e^{-2t}}{3} & -\frac{2e^t}{3} \\ e^{3t} & e^{-2t} & e^t \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{e^{-3t}}{2} & e^{-3t} & \frac{e^{-3t}}{2} \\ -e^{2t} & e^{2t} & e^{2t} \\ \frac{e^{-t}}{2} & -2e^{-t} & -\frac{e^{-t}}{2} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} e^{3t} & -\frac{e^{-2t}}{3} & \frac{e^{t}}{3} \\ 0 & -\frac{e^{-2t}}{3} & -\frac{2\,e^{t}}{3} \\ e^{3t} & e^{-2t} & e^{t} \end{bmatrix} \int \begin{bmatrix} \frac{e^{-3t}}{2} & e^{-3t} & \frac{e^{-3t}}{2} \\ -e^{2t} & e^{2t} & e^{2t} \\ \frac{e^{-t}}{2} & -2\,e^{-t} & -\frac{e^{-t}}{2} \end{bmatrix} \begin{bmatrix} \sin(t) \\ 0 \\ 0 \end{bmatrix} dt$$

$$= \begin{bmatrix} e^{3t} & -\frac{e^{-2t}}{3} & \frac{e^{t}}{3} \\ 0 & -\frac{e^{-2t}}{3} & -\frac{2\,e^{t}}{3} \\ e^{3t} & e^{-2t} & e^{t} \end{bmatrix} \int \begin{bmatrix} \frac{e^{-3t}\sin(t)}{2} \\ -e^{2t}\sin(t) \\ \frac{e^{-t}\sin(t)}{2} \end{bmatrix} dt$$

$$= \begin{bmatrix} e^{3t} & -\frac{e^{-2t}}{3} & \frac{e^{t}}{3} \\ 0 & -\frac{e^{-2t}}{3} & -\frac{2\,e^{t}}{3} \\ e^{3t} & e^{-2t} & e^{t} \end{bmatrix} \begin{bmatrix} -\frac{e^{-3t}(\cos(t)+3\sin(t))}{20} \\ \frac{e^{2t}(\cos(t)-2\sin(t))}{5} \\ -\frac{e^{-t}(\cos(t)+\sin(t))}{4} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{\sin(t)}{10} - \frac{\cos(t)}{5} \\ \frac{3\sin(t)}{10} + \frac{\cos(t)}{10} \\ -\frac{4\sin(t)}{5} - \frac{\cos(t)}{10} \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{3t} \\ 0 \\ c_1 e^{3t} \end{bmatrix} + \begin{bmatrix} -\frac{c_2 e^{-2t}}{3} \\ -\frac{c_2 e^{-2t}}{3} \\ c_2 e^{-2t} \end{bmatrix} + \begin{bmatrix} \frac{c_3 e^{t}}{3} \\ -\frac{2 c_3 e^{t}}{3} \\ c_3 e^{t} \end{bmatrix} + \begin{bmatrix} -\frac{\sin(t)}{10} - \frac{\cos(t)}{5} \\ \frac{3\sin(t)}{10} + \frac{\cos(t)}{10} \\ -\frac{4\sin(t)}{5} - \frac{\cos(t)}{10} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} e^{-2t}\left(\frac{(-\sin(t)-2\cos(t))e^{2t}}{10} + c_1 e^{5t} + \frac{c_3 e^{3t}}{3} - \frac{c_2}{3}\right) \\ -\frac{\left(\frac{3(-\cos(t)-3\sin(t))e^{2t}}{10} + 2c_3 e^{3t} + c_2\right)e^{-2t}}{3} \\ e^{-2t}\left(\left(-\frac{4\sin(t)}{5} - \frac{\cos(t)}{10}\right)e^{2t} + c_1 e^{5t} + c_3 e^{3t} + c_2\right) \end{bmatrix}$$

4.11.3 Maple step by step solution

Let's solve

$[x_1'(t) = x_1(t) + 3x_2(t) + 2x_3(t) + \sin(t), x_2'(t) = -x_1(t) + 2x_2(t) + x_3(t), x_3'(t) = 4x_1(t) - x_2(t$

- Define vector

$$\underline{x^{\rightarrow}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 2 & 1 \\ 4 & -1 & -1 \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t) + \begin{bmatrix} \sin(t) \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 2 & 1 \\ 4 & -1 & -1 \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t) + \begin{bmatrix} \sin(t) \\ 0 \\ 0 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} \sin(t) \\ 0 \\ 0 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 2 & 1 \\ 4 & -1 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\underline{x^{\rightarrow}}'(t) = A \cdot \underline{x^{\rightarrow}}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_1 = e^{-2t} \cdot \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_2 = e^{t} \cdot \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_3 = e^{3t} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\underline{\vec{x}}$

  $\underline{\vec{x}}(t) = c_1\underline{\vec{x}}_1 + c_2\underline{\vec{x}}_2 + c_3\underline{\vec{x}}_3 + \underline{\vec{x}}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

  $$\phi(t) = \begin{bmatrix} -\frac{e^{-2t}}{3} & \frac{e^{t}}{3} & e^{3t} \\ -\frac{e^{-2t}}{3} & -\frac{2e^{t}}{3} & 0 \\ e^{-2t} & e^{t} & e^{3t} \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

  $$\Phi(t) = \begin{bmatrix} -\frac{e^{-2t}}{3} & \frac{e^{t}}{3} & e^{3t} \\ -\frac{e^{-2t}}{3} & -\frac{2e^{t}}{3} & 0 \\ e^{-2t} & e^{t} & e^{3t} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -\frac{1}{3} & \frac{1}{3} & 1 \\ -\frac{1}{3} & -\frac{2}{3} & 0 \\ 1 & 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

  $$\Phi(t) = \begin{bmatrix} \frac{(3e^{5t}+e^{3t}+2)e^{-2t}}{6} & \frac{(3e^{5t}-2e^{3t}-1)e^{-2t}}{3} & -\frac{(-3e^{5t}+e^{3t}+2)e^{-2t}}{6} \\ -\frac{(e^{3t}-1)e^{-2t}}{3} & \frac{(4e^{3t}-1)e^{-2t}}{3} & \frac{(e^{3t}-1)e^{-2t}}{3} \\ \frac{(e^{5t}+e^{3t}-2)e^{-2t}}{2} & (e^{5t}-2e^{3t}+1)e^{-2t} & \frac{(e^{5t}-e^{3t}+2)e^{-2t}}{2} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

  $\underline{\vec{x}}_p(t) = \Phi(t) \cdot \vec{v}(t)$

- Take the derivative of the particular solution

  $\underline{\vec{x}}_p'(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

  $\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\underline{\vec{x}}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\underline{\vec{x}}_p(t) = \left[\begin{array}{c} -\frac{e^{-2t}\left(6\,e^{2t}\sin(t)+12\,e^{2t}\cos(t)-3\,e^{5t}-5\,e^{3t}-4\right)}{60} \\ \frac{\left(e^{2t}(\cos(t)+3\sin(t))-\frac{5\,e^{3t}}{3}+\frac{2}{3}\right)e^{-2t}}{10} \\ -\frac{\left((8\sin(t)+\cos(t))e^{2t}-\frac{5\,e^{3t}}{2}-\frac{e^{5t}}{2}+2\right)e^{-2t}}{10} \end{array}\right]$$

- Plug particular solution back into general solution

$$\underline{\vec{x}}(t) = c_1\underline{\vec{x}}_1 + c_2\underline{\vec{x}}_2 + c_3\underline{\vec{x}}_3 + \left[\begin{array}{c} -\frac{e^{-2t}\left(6\,e^{2t}\sin(t)+12\,e^{2t}\cos(t)-3\,e^{5t}-5\,e^{3t}-4\right)}{60} \\ \frac{\left(e^{2t}(\cos(t)+3\sin(t))-\frac{5\,e^{3t}}{3}+\frac{2}{3}\right)e^{-2t}}{10} \\ -\frac{\left((8\sin(t)+\cos(t))e^{2t}-\frac{5\,e^{3t}}{2}-\frac{e^{5t}}{2}+2\right)e^{-2t}}{10} \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x_1(t) \\ x_2(t) \\ x_3(t) \end{array}\right] = \left[\begin{array}{c} -\frac{\left(-60c_3e^{5t}-3\,e^{5t}-20c_2e^{3t}+6\,e^{2t}\sin(t)-5\,e^{3t}+12\,e^{2t}\cos(t)+20c_1-4\right)e^{-2t}}{60} \\ \frac{(\cos(t)+3\sin(t))e^{-2t}e^{2t}}{10} - \frac{\left(20c_2e^{3t}+10c_1+5\,e^{3t}-2\right)e^{-2t}}{30} \\ -\frac{\left(-20c_3e^{5t}-e^{5t}-20c_2e^{3t}+16\,e^{2t}\sin(t)-5\,e^{3t}+2\,e^{2t}\cos(t)-20c_1+4\right)e^{-2t}}{20} \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x_1(t) = -\frac{\left(-60c_3e^{5t}-3\,e^{5t}-20c_2e^{3t}+6\,e^{2t}\sin(t)-5\,e^{3t}+12\,e^{2t}\cos(t)+20c_1-4\right)e^{-2t}}{60}, x_2(t) = \frac{(\cos(t)+3\sin(t))e^{-2t}e^{2t}}{10} - \right.$$

✓ Solution by Maple

Time used: 0.14 (sec). Leaf size: 83

```
dsolve([diff(x__1(t),t)=1*x__1(t)+3*x__2(t)+2*x__3(t)+sin(t),diff(x__2(t),t)=-1*x__1(t)+2*x_
```

$$
\begin{aligned}
x_1(t) &= -\frac{\sin(t)}{10} - \frac{\cos(t)}{5} + c_1 e^{3t} - \frac{c_2 e^{t}}{2} + c_3 e^{-2t} \\
x_2(t) &= c_3 e^{-2t} + c_2 e^{t} + \frac{\cos(t)}{10} + \frac{3\sin(t)}{10} \\
x_3(t) &= -\frac{4\sin(t)}{5} - \frac{\cos(t)}{10} + c_1 e^{3t} - \frac{3c_2 e^{t}}{2} - 3c_3 e^{-2t}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.318 (sec). Leaf size: 211

```
DSolve[{x1'[t]==1*x1[t]+3*x2[t]+2*x3[t]+Sin[t],x2'[t]==-1*x1[t]+2*x2[t]+1*x3[t],x3'[t]==4*x1
```

$$
\begin{aligned}
\text{x1}(t) \to \frac{1}{30}\big(&-3\sin(t) - 6\cos(t) \\
&+ 5e^{-2t}\big(c_1\big(e^{3t} + 3e^{5t} + 2\big) + c_2\big(-4e^{3t} + 6e^{5t} - 2\big) + c_3\big(-e^{3t} + 3e^{5t} - 2\big)\big)\big) \\
\text{x2}(t) \to \frac{1}{30}\big(&9\sin(t) + 3\cos(t) - 10e^{-2t}\big(c_1\big(e^{3t} - 1\big) - 4c_2 e^{3t} - c_3 e^{3t} + c_2 + c_3\big)\big) \\
\text{x3}(t) \to \frac{1}{10}\big(&-8\sin(t) - \cos(t) \\
&+ 5e^{-2t}\big(c_1\big(e^{3t} + e^{5t} - 2\big) + 2c_2\big(-2e^{3t} + e^{5t} + 1\big) + c_3\big(-e^{3t} + e^{5t} + 2\big)\big)\big)
\end{aligned}
$$

4.12 problem 13

Internal problem ID [1865]
Internal file name [`OUTPUT/1866_Sunday_June_05_2022_02_36_02_AM_56441829/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.12, Systems of differential equations. The nonhomogeneous equation. variation of parameters. Page 366
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_1(t) + 2x_2(t) - 3x_3(t) + \mathrm{e}^t \\
x_2'(t) &= x_1(t) + x_2(t) + 2x_3(t) \\
x_3'(t) &= x_1(t) - x_2(t) + 4x_3(t) - \mathrm{e}^t
\end{aligned}
$$

4.12.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} \mathrm{e}^t \\ 0 \\ -\mathrm{e}^t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{2t}(1-t) & -\frac{e^{2t}(t-4)t}{2} & \frac{e^{2t}t(t-6)}{2} \\ e^{2t}t & e^{2t}\left(1-t+\frac{1}{2}t^2\right) & -\frac{e^{2t}(t-4)t}{2} \\ e^{2t}t & \frac{e^{2t}t(t-2)}{2} & e^{2t}\left(1-\frac{1}{2}t^2+2t\right) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} e^{2t}(1-t) & -\frac{e^{2t}(t-4)t}{2} & \frac{e^{2t}t(t-6)}{2} \\ e^{2t}t & e^{2t}\left(1-t+\frac{1}{2}t^2\right) & -\frac{e^{2t}(t-4)t}{2} \\ e^{2t}t & \frac{e^{2t}t(t-2)}{2} & e^{2t}\left(1-\frac{1}{2}t^2+2t\right) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} e^{2t}(1-t)\,c_1 - \frac{e^{2t}(t-4)tc_2}{2} + \frac{e^{2t}t(t-6)c_3}{2} \\ e^{2t}tc_1 + e^{2t}\left(1-t+\frac{1}{2}t^2\right)c_2 - \frac{e^{2t}(t-4)tc_3}{2} \\ e^{2t}tc_1 + \frac{e^{2t}t(t-2)c_2}{2} + e^{2t}\left(1-\frac{1}{2}t^2+2t\right)c_3 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{((c_2-c_3)t^2+(2c_1-4c_2+6c_3)t-2c_1)e^{2t}}{2} \\ \frac{((c_2-c_3)t^2+(2c_1-2c_2+4c_3)t+2c_2)e^{2t}}{2} \\ \frac{((c_2-c_3)t^2+(2c_1-2c_2+4c_3)t+2c_3)e^{2t}}{2} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= \left(e^{At}\right)^{-1} \\ &= \begin{bmatrix} e^{-2t}(t+1) & -\frac{e^{-2t}t(t+4)}{2} & \frac{e^{-2t}t(t+6)}{2} \\ -t\,e^{-2t} & \frac{e^{-2t}(t^2+2t+2)}{2} & -\frac{e^{-2t}t(t+4)}{2} \\ -t\,e^{-2t} & \frac{e^{-2t}t(2+t)}{2} & -\frac{(t^2+4t-2)e^{-2t}}{2} \end{bmatrix} \end{aligned}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} e^{2t}(1-t) & -\frac{e^{2t}(t-4)t}{2} & \frac{e^{2t}t(t-6)}{2} \\ e^{2t}t & e^{2t}\left(1-t+\frac{1}{2}t^2\right) & -\frac{e^{2t}(t-4)t}{2} \\ e^{2t}t & \frac{e^{2t}t(t-2)}{2} & e^{2t}\left(1-\frac{1}{2}t^2+2t\right) \end{bmatrix} \int \begin{bmatrix} e^{-2t}(t+1) & -\frac{e^{-2t}t(t+4)}{2} & \frac{e^{-2t}t(t+6)}{2} \\ -t\,e^{-2t} & \frac{e^{-2t}(t^2+2t+2)}{2} & -\frac{e^{-2t}t(t+4)}{2} \\ -t\,e^{-2t} & \frac{e^{-2t}t(2+t)}{2} & -\frac{(t^2+4t-2)e^{-2t}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t}(1-t) & -\frac{e^{2t}(t-4)t}{2} & \frac{e^{2t}t(t-6)}{2} \\ e^{2t}t & e^{2t}\left(1-t+\frac{1}{2}t^2\right) & -\frac{e^{2t}(t-4)t}{2} \\ e^{2t}t & \frac{e^{2t}t(t-2)}{2} & e^{2t}\left(1-\frac{1}{2}t^2+2t\right) \end{bmatrix} \begin{bmatrix} \frac{e^{-t}(t^2+6t+4)}{2} \\ -\frac{e^{-t}(2+t)^2}{2} \\ -\frac{e^{-t}(t^2+4t+2)}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2\,e^{t} \\ -2\,e^{t} \\ -e^{t} \end{bmatrix}$$

Hence the complete solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \begin{bmatrix} \frac{((-c_2+c_3)t^2+(-2c_1+4c_2-6c_3)t+2c_1)e^{2t}}{2} + 2\,e^{t} \\ \frac{((c_2-c_3)t^2+(2c_1-2c_2+4c_3)t+2c_2)e^{2t}}{2} - 2\,e^{t} \\ \frac{((c_2-c_3)t^2+(2c_1-2c_2+4c_3)t+2c_3)e^{2t}}{2} - e^{t} \end{bmatrix} \end{aligned}$$

4.12.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} e^{t} \\ 0 \\ -e^{t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 2 & -3 \\ 1 & 1-\lambda & 2 \\ 1 & -1 & 4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 & -3 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 2 & -3 & 0 \\ 1 & -1 & 2 & 0 \\ 1 & -1 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 2 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 2 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 2 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 2 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	3	1	Yes	$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 2 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

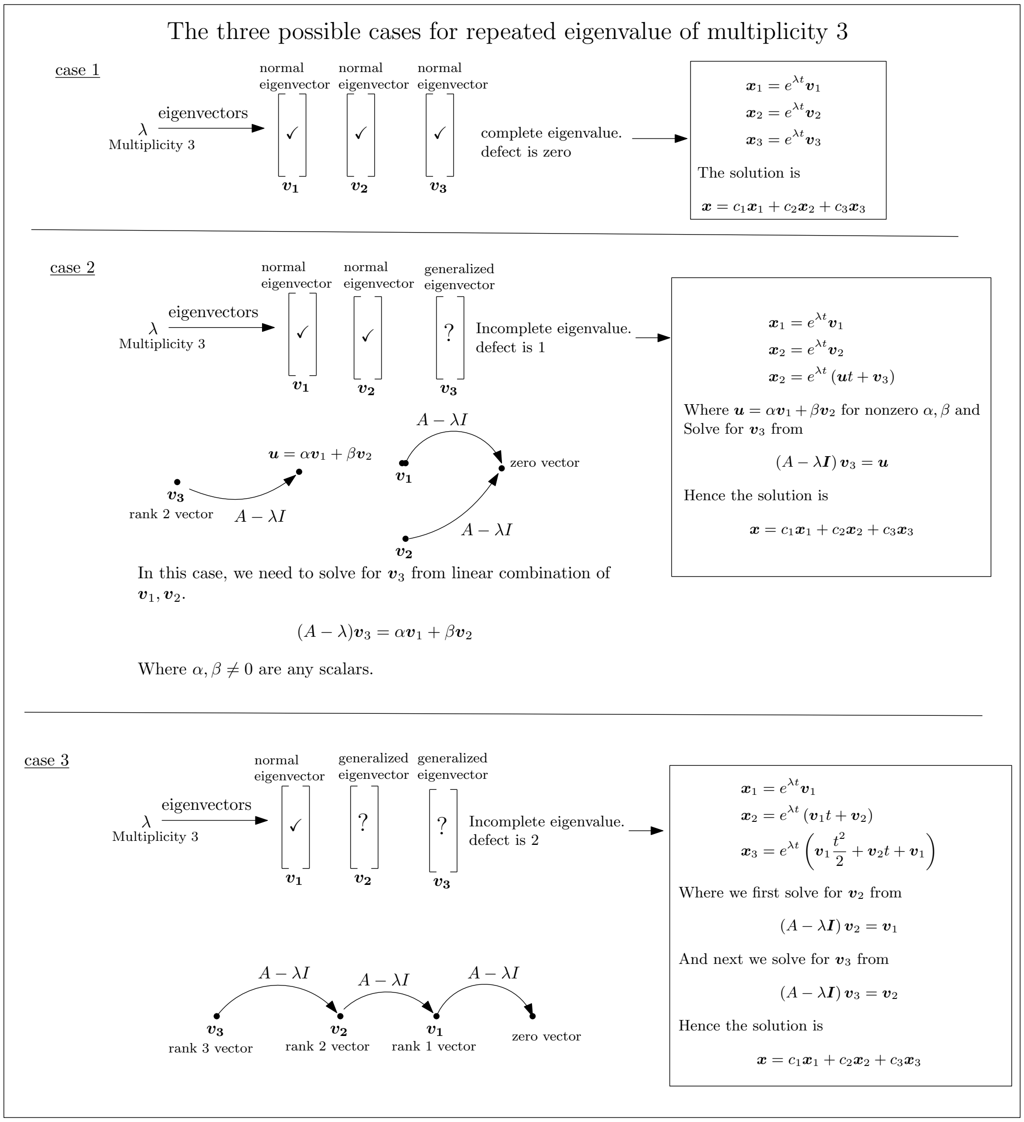


Figure 25: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 2. This falls into case 3 shown above. First we find generalized eigenvector $\vec{v}_2$ of rank 2 and then use this to find generalized eigenvector $\vec{v}_3$ of rank 3.$\vec{v}_2$ is found by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence

$$\left(\begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 & -3 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Now $\vec{v}_3$ is found by solving

$$(A - \lambda I)\,\vec{v}_3 = \vec{v}_2$$

Where $\vec{v}_2$ is the (rank 2) generalized eigenvector found above. Hence

$$\left(\begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 & -3 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_3$ gives

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

We have found three generalized eigenvectors for eigenvalue 2. Therefore the three basis

solutions associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \mathrm{e}^{2t} \\
&= \begin{bmatrix} -\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= e^{\lambda t}(\vec{v}_1 t + \vec{v}_2) \\
&= \mathrm{e}^{2t}\left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) \\
&= \begin{bmatrix} -\mathrm{e}^{2t} t \\ \mathrm{e}^{2t}(t+1) \\ \mathrm{e}^{2t}(t+1) \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_3(t) &= \left(\vec{v}_1 \frac{t^2}{2} + \vec{v}_2 t + \vec{v}_3\right) e^{\lambda t} \\
&= \left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \frac{t^2}{2} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right) \mathrm{e}^{2t} \\
&= \begin{bmatrix} -\frac{\mathrm{e}^{2t}(t^2-2)}{2} \\ \frac{\mathrm{e}^{2t}(t^2+2t+4)}{2} \\ \frac{\mathrm{e}^{2t}(t^2+2t+2)}{2} \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{2t} t \\ \mathrm{e}^{2t}(t+1) \\ \mathrm{e}^{2t}(t+1) \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{2t}\left(1 - \frac{t^2}{2}\right) \\ \mathrm{e}^{2t}\left(\frac{1}{2}t^2 + t + 2\right) \\ \mathrm{e}^{2t}\left(t + \frac{1}{2}t^2 + 1\right) \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -\mathrm{e}^{2t} & -\mathrm{e}^{2t} t & \mathrm{e}^{2t}\left(1 - \frac{t^2}{2}\right) \\ \mathrm{e}^{2t} & \mathrm{e}^{2t}(t+1) & \mathrm{e}^{2t}\left(\frac{1}{2}t^2 + t + 2\right) \\ \mathrm{e}^{2t} & \mathrm{e}^{2t}(t+1) & \mathrm{e}^{2t}\left(t + \frac{1}{2}t^2 + 1\right) \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -\mathrm{e}^{-2t}(t+1) & \frac{\mathrm{e}^{-2t}(t^2+4t+2)}{2} & -\frac{(t^2+6t+2)\mathrm{e}^{-2t}}{2} \\ \mathrm{e}^{-2t} & -\mathrm{e}^{-2t}(2+t) & \mathrm{e}^{-2t}(t+3) \\ 0 & \mathrm{e}^{-2t} & -\mathrm{e}^{-2t} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} -e^{2t} & -e^{2t}t & e^{2t}\left(1-\frac{t^2}{2}\right) \\ e^{2t} & e^{2t}(t+1) & e^{2t}\left(\frac{1}{2}t^2+t+2\right) \\ e^{2t} & e^{2t}(t+1) & e^{2t}\left(t+\frac{1}{2}t^2+1\right) \end{bmatrix} \int \begin{bmatrix} -e^{-2t}(t+1) & \frac{e^{-2t}(t^2+4t+2)}{2} & -\frac{(t^2+6t+2)e^{-2t}}{2} \\ e^{-2t} & -e^{-2t}(2+t) & e^{-2t}(t+3) \\ 0 & e^{-2t} & -e^{-2t} \end{bmatrix} \begin{bmatrix} e^t \\ 0 \\ -e^t \end{bmatrix}$$

$$= \begin{bmatrix} -e^{2t} & -e^{2t}t & e^{2t}\left(1-\frac{t^2}{2}\right) \\ e^{2t} & e^{2t}(t+1) & e^{2t}\left(\frac{1}{2}t^2+t+2\right) \\ e^{2t} & e^{2t}(t+1) & e^{2t}\left(t+\frac{1}{2}t^2+1\right) \end{bmatrix} \int \begin{bmatrix} \frac{e^{-t}t(t+4)}{2} \\ e^{-t}(-t-2) \\ e^{-t} \end{bmatrix} dt$$

$$= \begin{bmatrix} -e^{2t} & -e^{2t}t & e^{2t}\left(1-\frac{t^2}{2}\right) \\ e^{2t} & e^{2t}(t+1) & e^{2t}\left(\frac{1}{2}t^2+t+2\right) \\ e^{2t} & e^{2t}(t+1) & e^{2t}\left(t+\frac{1}{2}t^2+1\right) \end{bmatrix} \begin{bmatrix} -\frac{e^{-t}(t^2+6t+6)}{2} \\ e^{-t}(t+3) \\ -e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} 2e^t \\ -2e^t \\ -e^t \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -c_1e^{2t} \\ c_1e^{2t} \\ c_1e^{2t} \end{bmatrix} + \begin{bmatrix} -c_2e^{2t}t \\ c_2e^{2t}(t+1) \\ c_2e^{2t}(t+1) \end{bmatrix} + \begin{bmatrix} c_3e^{2t}\left(1-\frac{t^2}{2}\right) \\ c_3e^{2t}\left(\frac{1}{2}t^2+t+2\right) \\ c_3e^{2t}\left(t+\frac{1}{2}t^2+1\right) \end{bmatrix} + \begin{bmatrix} 2e^t \\ -2e^t \\ -e^t \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{(-c_3t^2-2tc_2-2c_1+2c_3)e^{2t}}{2} + 2e^t \\ \frac{((t^2+2t+4)c_3+2tc_2+2c_1+2c_2)e^{2t}}{2} - 2e^t \\ \frac{((t^2+2t+2)c_3+2tc_2+2c_1+2c_2)e^{2t}}{2} - e^t \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 123

```
dsolve([diff(x__1(t),t)=1*x__1(t)+2*x__2(t)-3*x__3(t)+exp(t),diff(x__2(t),t)=1*x__1(t)+1*x__
```

$$x_1(t) = 2\,e^t - c_1e^{2t} - c_2e^{2t}t + c_2e^{2t} - e^{2t}c_3t^2 + 2\,e^{2t}c_3t + 4c_3e^{2t}$$
$$x_2(t) = -2\,e^t + c_1e^{2t} + c_2e^{2t}t + e^{2t}c_3t^2$$
$$x_3(t) = -e^t + c_1e^{2t} + c_2e^{2t}t + e^{2t}c_3t^2 - 2c_3e^{2t}$$

✓ Solution by Mathematica

Time used: 0.098 (sec). Leaf size: 133

```
DSolve[{x1'[t]==1*x1[t]+2*x2[t]-3*x3[t]+Exp[t],x2'[t]==1*x1[t]+1*x2[t]+2*x3[t],x3'[t]==1*x1[
```

$$\text{x1}(t) \to \frac{1}{2}e^t\big(4 + e^t(-2c_1(t-1) - c_2(t-4)t + c_3(t-6)t)\big)$$
$$\text{x2}(t) \to \frac{1}{2}e^t\big(-4 + e^t\big((c_2 - c_3)t^2 + 2(c_1 - c_2 + 2c_3)t + 2c_2\big)\big)$$
$$\text{x3}(t) \to \frac{1}{2}e^t\big(-2 + e^t\big((c_2 - c_3)t^2 + 2(c_1 - c_2 + 2c_3)t + 2c_3\big)\big)$$

4.13 problem 14

Internal problem ID [1866]
Internal file name [`OUTPUT/1867_Sunday_June_05_2022_02_36_05_AM_30052516/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.12, Systems of differential equations. The nonhomogeneous equation. variation of parameters. Page 366
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= -x_1(t) - x_2(t) + 1 \\
x_2'(t) &= -4x_2(t) - x_3(t) + t \\
x_3'(t) &= 5x_2(t) + \mathrm{e}^t
\end{aligned}
$$

4.13.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -4 & -1 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1 \\ t \\ \mathrm{e}^t \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-t} & -\frac{e^{-2t}\cos(t)}{2} - \frac{3\,e^{-2t}\sin(t)}{2} + \frac{e^{-t}}{2} & -\frac{e^{-2t}\cos(t)}{2} - \frac{e^{-2t}\sin(t)}{2} + \frac{e^{-t}}{2} \\ 0 & e^{-2t}\cos(t) - 2\,e^{-2t}\sin(t) & -e^{-2t}\sin(t) \\ 0 & 5\,e^{-2t}\sin(t) & e^{-2t}\cos(t) + 2\,e^{-2t}\sin(t) \end{bmatrix}$$
$$= \begin{bmatrix} e^{-t} & \frac{(-\cos(t)-3\sin(t))e^{-2t}}{2} + \frac{e^{-t}}{2} & \frac{(-\cos(t)-\sin(t))e^{-2t}}{2} + \frac{e^{-t}}{2} \\ 0 & e^{-2t}(\cos(t) - 2\sin(t)) & -e^{-2t}\sin(t) \\ 0 & 5\,e^{-2t}\sin(t) & e^{-2t}(\cos(t) + 2\sin(t)) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} e^{-t} & \frac{(-\cos(t)-3\sin(t))e^{-2t}}{2} + \frac{e^{-t}}{2} & \frac{(-\cos(t)-\sin(t))e^{-2t}}{2} + \frac{e^{-t}}{2} \\ 0 & e^{-2t}(\cos(t) - 2\sin(t)) & -e^{-2t}\sin(t) \\ 0 & 5\,e^{-2t}\sin(t) & e^{-2t}(\cos(t) + 2\sin(t)) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} e^{-t}c_1 + \left(\frac{(-\cos(t)-3\sin(t))e^{-2t}}{2} + \frac{e^{-t}}{2}\right)c_2 + \left(\frac{(-\cos(t)-\sin(t))e^{-2t}}{2} + \frac{e^{-t}}{2}\right)c_3 \\ e^{-2t}(\cos(t) - 2\sin(t))\,c_2 - e^{-2t}\sin(t)\,c_3 \\ 5\,e^{-2t}\sin(t)\,c_2 + e^{-2t}(\cos(t) + 2\sin(t))\,c_3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{((-c_2-c_3)\cos(t)-3(c_2+\frac{c_3}{3})\sin(t))e^{-2t}}{2} + e^{-t}\left(c_1 + \frac{c_2}{2} + \frac{c_3}{2}\right) \\ ((-2c_2 - c_3)\sin(t) + c_2\cos(t))\,e^{-2t} \\ ((5c_2 + 2c_3)\sin(t) + c_3\cos(t))\,e^{-2t} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} e^{t} & -\frac{(-1+e^{t}(-3\sin(t)+\cos(t)))e^{t}}{2} & -\frac{(-1+(-\sin(t)+\cos(t))e^{t})e^{t}}{2} \\ 0 & (\cos(t) + 2\sin(t))\,e^{2t} & e^{2t}\sin(t) \\ 0 & -5\,e^{2t}\sin(t) & e^{2t}(\cos(t) - 2\sin(t)) \end{bmatrix} \end{aligned}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} e^{-t} & \frac{(-\cos(t)-3\sin(t))e^{-2t}}{2} + \frac{e^{-t}}{2} & \frac{(-\cos(t)-\sin(t))e^{-2t}}{2} + \frac{e^{-t}}{2} \\ 0 & e^{-2t}(\cos(t) - 2\sin(t)) & -e^{-2t}\sin(t) \\ 0 & 5\,e^{-2t}\sin(t) & e^{-2t}(\cos(t) + 2\sin(t)) \end{bmatrix} \int \begin{bmatrix} e^{t} & -\frac{(-1+e^{t}(-3\sin(t)+\cos(t)))e^{t}}{2} & -\frac{(-\cdots}{} \\ 0 & (\cos(t) + 2\sin(t))\,e^{2t} & \\ 0 & -5\,e^{2t}\sin(t) & e^{2t}(\cdots \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} & \frac{(-\cos(t)-3\sin(t))e^{-2t}}{2} + \frac{e^{-t}}{2} & \frac{(-\cos(t)-\sin(t))e^{-2t}}{2} + \frac{e^{-t}}{2} \\ 0 & e^{-2t}(\cos(t) - 2\sin(t)) & -e^{-2t}\sin(t) \\ 0 & 5\,e^{-2t}\sin(t) & e^{-2t}(\cos(t) + 2\sin(t)) \end{bmatrix} \begin{bmatrix} -\frac{\left(-1+\frac{(2\cos(t)-\sin(t))e^{2t}}{5}+\left(\left(t-\frac{3}{5}\right)\cos(t)-\frac{1}{2}+(\cdots\right.\right.}{2} \\ \frac{((-2+5t)\sin(t)+\cos(t))e^{2t}}{5} - \frac{e^{3t}(-3\sin\cdots}{} \\ \frac{((5t-4)\cos(t)+(3-10t)\sin(t))e^{2t}}{5} + \frac{e^{3t}(\cdots}{} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{e^{t}}{20} + \frac{4}{5} \\ -\frac{e^{t}}{10} + \frac{1}{5} \\ \frac{e^{t}}{2} + t - \frac{4}{5} \end{bmatrix}$$

Hence the complete solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \begin{bmatrix} \frac{e^{-2t}\left(16\,e^{2t}+e^{3t}+10(2c_1+c_2+c_3)e^{t}+10(-c_2-c_3)\cos(t)+10(-3c_2-c_3)\sin(t)\right)}{20} \\ \left(\frac{e^{2t}}{5} - \frac{e^{3t}}{10} + (-2c_2 - c_3)\sin(t) + c_2\cos(t)\right)e^{-2t} \\ \left(\left(-\frac{4}{5} + t\right)e^{2t} + \frac{e^{3t}}{2} + (5c_2 + 2c_3)\sin(t) + c_3\cos(t)\right)e^{-2t} \end{bmatrix} \end{aligned}$$

4.13.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -4 & -1 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1 \\ t \\ e^{t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & -1 & 0 \\ 0 & -4 & -1 \\ 0 & 5 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & -1 & 0 \\ 0 & -4-\lambda & -1 \\ 0 & 5 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + 5\lambda^2 + 9\lambda + 5 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2 + i$$
$$\lambda_2 = -2 - i$$
$$\lambda_3 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
$-2+i$	1	complex eigenvalue
$-2-i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -1 & 0 \\ 0 & -4 & -1 \\ 0 & 5 & 0 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & -3 & -1 \\ 0 & 5 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 5 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - 3R_1 \Longrightarrow \left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 5 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 + 5R_1 \Longrightarrow \left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2, v_3\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -2 - i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -1 & 0 \\ 0 & -4 & -1 \\ 0 & 5 & 0 \end{bmatrix} - (-2 - i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1+i & -1 & 0 \\ 0 & -2+i & -1 \\ 0 & 5 & 2+i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1+i & -1 & 0 & 0 \\ 0 & -2+i & -1 & 0 \\ 0 & 5 & 2+i & 0 \end{array}\right]$$

$$R_3 = R_3 + (2+i)\,R_2 \Longrightarrow \left[\begin{array}{ccc|c} 1+i & -1 & 0 & 0 \\ 0 & -2+i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1+i & -1 & 0 \\ 0 & -2+i & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{3}{10} + \frac{i}{10}\right) t, v_2 = \left(-\frac{2}{5} - \frac{i}{5}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{3}{10} + \frac{I}{10}\right) t \\ \left(-\frac{2}{5} - \frac{I}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{3}{10} + \frac{i}{10}\right) t \\ \left(-\frac{2}{5} - \frac{i}{5}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{3}{10} + \frac{I}{10}\right) t \\ \left(-\frac{2}{5} - \frac{I}{5}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{3}{10} + \frac{i}{10} \\ -\frac{2}{5} - \frac{i}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{3}{10} + \frac{I}{10}\right) t \\ \left(-\frac{2}{5} - \frac{I}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{3}{10} + \frac{i}{10} \\ -\frac{2}{5} - \frac{i}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{3}{10} + \frac{I}{10}\right) t \\ \left(-\frac{2}{5} - \frac{I}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -3+i \\ -4-2i \\ 10 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = -2 + i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -1 & 0 \\ 0 & -4 & -1 \\ 0 & 5 & 0 \end{bmatrix} - (-2+i)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-i & -1 & 0 \\ 0 & -2-i & -1 \\ 0 & 5 & 2-i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1-i & -1 & 0 & 0 \\ 0 & -2-i & -1 & 0 \\ 0 & 5 & 2-i & 0 \end{array}\right]$$

$$R_3 = R_3 + (2-i)\,R_2 \Longrightarrow \left[\begin{array}{ccc|c} 1-i & -1 & 0 & 0 \\ 0 & -2-i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1-i & -1 & 0 \\ 0 & -2-i & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{3}{10} - \frac{i}{10}\right)t, v_2 = \left(-\frac{2}{5} + \frac{i}{5}\right)t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{3}{10} - \frac{I}{10}\right)t \\ \left(-\frac{2}{5} + \frac{I}{5}\right)t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{3}{10} - \frac{i}{10}\right)t \\ \left(-\frac{2}{5} + \frac{i}{5}\right)t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{3}{10} - \frac{I}{10}\right) t \\ \left(-\frac{2}{5} + \frac{I}{5}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{3}{10} - \frac{i}{10} \\ -\frac{2}{5} + \frac{i}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{3}{10} - \frac{I}{10}\right) t \\ \left(-\frac{2}{5} + \frac{I}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{3}{10} - \frac{i}{10} \\ -\frac{2}{5} + \frac{i}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{3}{10} - \frac{I}{10}\right) t \\ \left(-\frac{2}{5} + \frac{I}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -3 - i \\ -4 + 2i \\ 10 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$-2+i$	1	1	No	$\begin{bmatrix} -\frac{3}{10}-\frac{i}{10} \\ -\frac{2}{5}+\frac{i}{5} \\ 1 \end{bmatrix}$
$-2-i$	1	1	No	$\begin{bmatrix} -\frac{3}{10}+\frac{i}{10} \\ -\frac{2}{5}-\frac{i}{5} \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-t}\end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} \left(-\frac{3}{10}-\frac{i}{10}\right) \mathrm{e}^{(-2+i)t} \\ \left(-\frac{2}{5}+\frac{i}{5}\right) \mathrm{e}^{(-2+i)t} \\ \mathrm{e}^{(-2+i)t} \end{bmatrix} + c_2 \begin{bmatrix} \left(-\frac{3}{10}+\frac{i}{10}\right) \mathrm{e}^{(-2-i)t} \\ \left(-\frac{2}{5}-\frac{i}{5}\right) \mathrm{e}^{(-2-i)t} \\ \mathrm{e}^{(-2-i)t} \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{-t} \\ 0 \\ 0 \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \left(-\frac{3}{10} - \frac{i}{10}\right) e^{(-2+i)t} & \left(-\frac{3}{10} + \frac{i}{10}\right) e^{(-2-i)t} & e^{-t} \\ \left(-\frac{2}{5} + \frac{i}{5}\right) e^{(-2+i)t} & \left(-\frac{2}{5} - \frac{i}{5}\right) e^{(-2-i)t} & 0 \\ e^{(-2+i)t} & e^{(-2-i)t} & 0 \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} 0 & -\frac{5i e^{(2-i)t}}{2} & \left(\frac{1}{2} - i\right) e^{(2-i)t} \\ 0 & \frac{5i e^{(2+i)t}}{2} & \left(\frac{1}{2} + i\right) e^{(2+i)t} \\ e^{t} & \frac{e^{t}}{2} & \frac{e^{t}}{2} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \left(-\frac{3}{10} - \frac{i}{10}\right) e^{(-2+i)t} & \left(-\frac{3}{10} + \frac{i}{10}\right) e^{(-2-i)t} & e^{-t} \\ \left(-\frac{2}{5} + \frac{i}{5}\right) e^{(-2+i)t} & \left(-\frac{2}{5} - \frac{i}{5}\right) e^{(-2-i)t} & 0 \\ e^{(-2+i)t} & e^{(-2-i)t} & 0 \end{bmatrix} \int \begin{bmatrix} 0 & -\frac{5i e^{(2-i)t}}{2} & \left(\frac{1}{2} - i\right) e^{(2-i)t} \\ 0 & \frac{5i e^{(2+i)t}}{2} & \left(\frac{1}{2} + i\right) e^{(2+i)t} \\ e^{t} & \frac{e^{t}}{2} & \frac{e^{t}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ t \\ e^{t} \end{bmatrix} dt \\
&= \begin{bmatrix} \left(-\frac{3}{10} - \frac{i}{10}\right) e^{(-2+i)t} & \left(-\frac{3}{10} + \frac{i}{10}\right) e^{(-2-i)t} & e^{-t} \\ \left(-\frac{2}{5} + \frac{i}{5}\right) e^{(-2+i)t} & \left(-\frac{2}{5} - \frac{i}{5}\right) e^{(-2-i)t} & 0 \\ e^{(-2+i)t} & e^{(-2-i)t} & 0 \end{bmatrix} \int \begin{bmatrix} \left(\frac{1}{2} - i\right) e^{(3-i)t} - \frac{5i e^{(2-i)t} t}{2} \\ \left(\frac{1}{2} + i\right) e^{(3+i)t} + \frac{5i e^{(2+i)t} t}{2} \\ e^{t} + \frac{t e^{t}}{2} + \frac{e^{2t}}{2} \end{bmatrix} dt \\
&= \begin{bmatrix} \left(-\frac{3}{10} - \frac{i}{10}\right) e^{(-2+i)t} & \left(-\frac{3}{10} + \frac{i}{10}\right) e^{(-2-i)t} & e^{-t} \\ \left(-\frac{2}{5} + \frac{i}{5}\right) e^{(-2+i)t} & \left(-\frac{2}{5} - \frac{i}{5}\right) e^{(-2-i)t} & 0 \\ e^{(-2+i)t} & e^{(-2-i)t} & 0 \end{bmatrix} \begin{bmatrix} \frac{e^{(2-i)t}(-4+3i+(5-10i)t)}{10} + \left(\frac{1}{4} - \frac{i}{4}\right) e^{(3-i)t} \\ \frac{e^{(2+i)t}(-4-3i+(5+10i)t)}{10} + \left(\frac{1}{4} + \frac{i}{4}\right) e^{(3+i)t} \\ \frac{e^{2t}}{4} + \frac{(2t+2)e^{t}}{4} \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{t}}{20} + \frac{4}{5} \\ -\frac{e^{t}}{10} + \frac{1}{5} \\ \frac{e^{t}}{2} + t - \frac{4}{5} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \left(-\frac{3}{10} - \frac{i}{10}\right) c_1 \mathrm{e}^{(-2+i)t} \\ \left(-\frac{2}{5} + \frac{i}{5}\right) c_1 \mathrm{e}^{(-2+i)t} \\ c_1 \mathrm{e}^{(-2+i)t} \end{bmatrix} + \begin{bmatrix} \left(-\frac{3}{10} + \frac{i}{10}\right) c_2 \mathrm{e}^{(-2-i)t} \\ \left(-\frac{2}{5} - \frac{i}{5}\right) c_2 \mathrm{e}^{(-2-i)t} \\ c_2 \mathrm{e}^{(-2-i)t} \end{bmatrix} + \begin{bmatrix} c_3 \mathrm{e}^{-t} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{\mathrm{e}^t}{20} + \frac{4}{5} \\ -\frac{\mathrm{e}^t}{10} + \frac{1}{5} \\ \frac{\mathrm{e}^t}{2} + t - \frac{4}{5} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \left(-\frac{3}{10} - \frac{i}{10}\right) c_1 \mathrm{e}^{(-2+i)t} + \left(-\frac{3}{10} + \frac{i}{10}\right) c_2 \mathrm{e}^{(-2-i)t} + c_3 \mathrm{e}^{-t} + \frac{\mathrm{e}^t}{20} + \frac{4}{5} \\ \left(-\frac{2}{5} + \frac{i}{5}\right) c_1 \mathrm{e}^{(-2+i)t} + \left(-\frac{2}{5} - \frac{i}{5}\right) c_2 \mathrm{e}^{(-2-i)t} - \frac{\mathrm{e}^t}{10} + \frac{1}{5} \\ c_1 \mathrm{e}^{(-2+i)t} + c_2 \mathrm{e}^{(-2-i)t} + \frac{\mathrm{e}^t}{2} + t - \frac{4}{5} \end{bmatrix}$$

4.13.3 Maple step by step solution

Let's solve

$$[x_1'(t) = -x_1(t) - x_2(t) + 1, x_2'(t) = -4x_2(t) - x_3(t) + t, x_3'(t) = 5x_2(t) + \mathrm{e}^t]$$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -4 & -1 \\ 0 & 5 & 0 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 1 \\ t \\ \mathrm{e}^t \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -4 & -1 \\ 0 & 5 & 0 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 1 \\ t \\ \mathrm{e}^t \end{bmatrix}$$

- Define the forcing function

$$\overrightarrow{f}(t) = \begin{bmatrix} 1 \\ t \\ e^{t} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -4 & -1 \\ 0 & 5 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\underline{x}^{\rightarrow\,\prime}(t) = A \cdot \underline{x}^{\rightarrow}(t) + \overrightarrow{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[-2 - \mathrm{I}, \begin{bmatrix} -\frac{3}{10} + \frac{\mathrm{I}}{10} \\ -\frac{2}{5} - \frac{\mathrm{I}}{5} \\ 1 \end{bmatrix}\right], \left[-2 + \mathrm{I}, \begin{bmatrix} -\frac{3}{10} - \frac{\mathrm{I}}{10} \\ -\frac{2}{5} + \frac{\mathrm{I}}{5} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{x}^{\rightarrow}{}_{1} = e^{-t} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2 - \mathrm{I}, \begin{bmatrix} -\frac{3}{10} + \frac{\mathrm{I}}{10} \\ -\frac{2}{5} - \frac{\mathrm{I}}{5} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{(-2-I)t} \cdot \begin{bmatrix} -\frac{3}{10} + \frac{I}{10} \\ -\frac{2}{5} - \frac{I}{5} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{-2t} \cdot (\cos(t) - I\sin(t)) \cdot \begin{bmatrix} -\frac{3}{10} + \frac{I}{10} \\ -\frac{2}{5} - \frac{I}{5} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{-2t} \cdot \begin{bmatrix} \left(-\frac{3}{10} + \frac{I}{10}\right)(\cos(t) - I\sin(t)) \\ \left(-\frac{2}{5} - \frac{I}{5}\right)(\cos(t) - I\sin(t)) \\ \cos(t) - I\sin(t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_2(t) = e^{-2t} \cdot \begin{bmatrix} -\frac{3\cos(t)}{10} + \frac{\sin(t)}{10} \\ -\frac{2\cos(t)}{5} - \frac{\sin(t)}{5} \\ \cos(t) \end{bmatrix}, \vec{x}_3(t) = e^{-2t} \cdot \begin{bmatrix} \frac{3\sin(t)}{10} + \frac{\cos(t)}{10} \\ \frac{2\sin(t)}{5} - \frac{\cos(t)}{5} \\ -\sin(t) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}$

$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2(t) + c_3\vec{x}_3(t) + \vec{x}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} e^{-t} & e^{-2t}\left(-\frac{3\cos(t)}{10} + \frac{\sin(t)}{10}\right) & e^{-2t}\left(\frac{3\sin(t)}{10} + \frac{\cos(t)}{10}\right) \\ 0 & e^{-2t}\left(-\frac{2\cos(t)}{5} - \frac{\sin(t)}{5}\right) & e^{-2t}\left(\frac{2\sin(t)}{5} - \frac{\cos(t)}{5}\right) \\ 0 & e^{-2t}\cos(t) & -e^{-2t}\sin(t) \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} e^{-t} & e^{-2t}\left(-\frac{3\cos(t)}{10} + \frac{\sin(t)}{10}\right) & e^{-2t}\left(\frac{3\sin(t)}{10} + \frac{\cos(t)}{10}\right) \\ 0 & e^{-2t}\left(-\frac{2\cos(t)}{5} - \frac{\sin(t)}{5}\right) & e^{-2t}\left(\frac{2\sin(t)}{5} - \frac{\cos(t)}{5}\right) \\ 0 & e^{-2t}\cos(t) & -e^{-2t}\sin(t) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & -\frac{3}{10} & \frac{1}{10} \\ 0 & -\frac{2}{5} & -\frac{1}{5} \\ 0 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} e^{-t} & \frac{(-\cos(t)-3\sin(t))e^{-2t}}{2} + \frac{e^{-t}}{2} & \frac{(-\cos(t)-\sin(t))e^{-2t}}{2} + \frac{e^{-t}}{2} \\ 0 & e^{-2t}(\cos(t) - 2\sin(t)) & -e^{-2t}\sin(t) \\ 0 & 5e^{-2t}\sin(t) & e^{-2t}(\cos(t) + 2\sin(t)) \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

  $\underline{\vec{x}}_p(t) = \Phi(t) \cdot \vec{v}(t)$

- Take the derivative of the particular solution

  $\underline{\vec{x}}'_p(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

  $\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\underline{\vec{x}}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{x}_p(t) = \begin{bmatrix} \frac{(e^{3t}+16\,e^{2t}-15\,e^{t}-2\cos(t))e^{-2t}}{20} \\ -\frac{e^{-2t}(-2\,e^{2t}+e^{3t}+\cos(t)+\sin(t))}{10} \\ \frac{e^{-2t}(10\,e^{2t}t-8\,e^{2t}+3\cos(t)+\sin(t)+5\,e^{3t})}{10} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\overrightarrow{x}(t) = c_1\overrightarrow{x}_1 + c_2\overrightarrow{x}_2(t) + c_3\overrightarrow{x}_3(t) + \begin{bmatrix} \frac{(e^{3t}+16\,e^{2t}-15\,e^{t}-2\cos(t))e^{-2t}}{20} \\ -\frac{e^{-2t}(-2\,e^{2t}+e^{3t}+\cos(t)+\sin(t))}{10} \\ \frac{e^{-2t}(10\,e^{2t}t-8\,e^{2t}+3\cos(t)+\sin(t)+5\,e^{3t})}{10} \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{(16\,e^{2t}+e^{3t}+2(-1-3c_2+c_3)\cos(t)+5(-3+4c_1)e^{t}+2(c_2+3c_3)\sin(t))e^{-2t}}{20} \\ -\frac{2\left(-\frac{e^{2t}}{2}+\frac{e^{3t}}{4}+\left(c_2+\frac{c_3}{2}+\frac{1}{4}\right)\cos(t)+\left(\frac{c_2}{2}-c_3+\frac{1}{4}\right)\sin(t)\right)e^{-2t}}{5} \\ \frac{e^{-2t}(10\,e^{2t}t+10c_2\cos(t)-10c_3\sin(t)-8\,e^{2t}+3\cos(t)+\sin(t)+5\,e^{3t})}{10} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \frac{(16\,e^{2t}+e^{3t}+2(-1-3c_2+c_3)\cos(t)+5(-3+4c_1)e^{t}+2(c_2+3c_3)\sin(t))e^{-2t}}{20}, x_2(t) = -\frac{2\left(-\frac{e^{2t}}{2}+\frac{e^{3t}}{4}+\left(c_2+\frac{c_3}{2}+\frac{1}{4}\right)\cos\right.}{5}\right.$$

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 123

```
dsolve([diff(x__1(t),t)=-1*x__1(t)-1*x__2(t)+0*x__3(t)+1,diff(x__2(t),t)=0*x__1(t)-4*x__2(t)
```

$$x_1(t) = -\frac{c_2 e^{-2t}\sin(t)}{2} + \frac{e^{-2t}\sin(t)\,c_3}{2} + \frac{c_2 e^{-2t}\cos(t)}{2} + \frac{e^{-2t}\cos(t)\,c_3}{2} + \frac{e^{t}}{20} + \frac{4}{5} + e^{-t}c_1$$
$$x_2(t) = e^{-2t}\sin(t)\,c_3 + c_2 e^{-2t}\cos(t) + \frac{1}{5} - \frac{e^{t}}{10}$$
$$x_3(t) = -2\,e^{-2t}\sin(t)\,c_3 - e^{-2t}\cos(t)\,c_3 - 2c_2 e^{-2t}\cos(t) + c_2 e^{-2t}\sin(t) + \frac{e^{t}}{2} - \frac{4}{5} + t$$

✓ Solution by Mathematica

Time used: 1.816 (sec). Leaf size: 144

```
DSolve[{x1'[t]==-1*x1[t]-1*x2[t]+0*x3[t]+1,x2'[t]==0*x1[t]-4*x2[t]-1*x3[t]+t,x3'[t]==0*x1[t]
```

$$x1(t) \to \frac{1}{20}e^{-2t}\left(e^t\left(16e^t + e^{2t} + 10(2c_1 + c_2 + c_3)\right) - 10(c_2 + c_3)\cos(t) - 10(3c_2 + c_3)\sin(t)\right)$$
$$x2(t) \to \frac{1}{10}\left(2 - e^t\right) + c_2e^{-2t}\cos(t) - (2c_2 + c_3)e^{-2t}\sin(t)$$
$$x3(t) \to t + \frac{e^t}{2} + c_3e^{-2t}\cos(t) + (5c_2 + 2c_3)e^{-2t}\sin(t) - \frac{4}{5}$$

4.14 problem 16

Internal problem ID [1867]
Internal file name [`OUTPUT/1868_Sunday_June_05_2022_02_36_09_AM_9581674/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.12, Systems of differential equations. The nonhomogeneous equation. variation of parameters. Page 366
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = x_1(t) + x_2(t) - x_3(t) + \mathrm{e}^{2t}$$
$$x_2'(t) = 2x_1(t) + 3x_2(t) - 4x_3(t) + 2\,\mathrm{e}^{2t}$$
$$x_3'(t) = 4x_1(t) + x_2(t) - 4x_3(t) + \mathrm{e}^{2t}$$

4.14.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} \mathrm{e}^{2t} \\ 2\,\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -\frac{(4\,e^{5t}-15\,e^{4t}+1)e^{-3t}}{10} & e^{2t}-e^{t} & -\frac{(6\,e^{5t}-5\,e^{4t}-1)e^{-3t}}{10} \\ -\frac{(8\,e^{5t}-15\,e^{4t}+7)e^{-3t}}{10} & 2\,e^{2t}-e^{t} & -\frac{(12\,e^{5t}-5\,e^{4t}-7)e^{-3t}}{10} \\ -\frac{(4\,e^{5t}-15\,e^{4t}+11)e^{-3t}}{10} & e^{2t}-e^{t} & -\frac{(6\,e^{5t}-5\,e^{4t}-11)e^{-3t}}{10} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} -\frac{(4\,e^{5t}-15\,e^{4t}+1)e^{-3t}}{10} & e^{2t}-e^{t} & -\frac{(6\,e^{5t}-5\,e^{4t}-1)e^{-3t}}{10} \\ -\frac{(8\,e^{5t}-15\,e^{4t}+7)e^{-3t}}{10} & 2\,e^{2t}-e^{t} & -\frac{(12\,e^{5t}-5\,e^{4t}-7)e^{-3t}}{10} \\ -\frac{(4\,e^{5t}-15\,e^{4t}+11)e^{-3t}}{10} & e^{2t}-e^{t} & -\frac{(6\,e^{5t}-5\,e^{4t}-11)e^{-3t}}{10} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{(4\,e^{5t}-15\,e^{4t}+1)e^{-3t}c_1}{10} + (e^{2t}-e^{t})\,c_2 - \frac{(6\,e^{5t}-5\,e^{4t}-1)e^{-3t}c_3}{10} \\ -\frac{(8\,e^{5t}-15\,e^{4t}+7)e^{-3t}c_1}{10} + (2\,e^{2t}-e^{t})\,c_2 - \frac{(12\,e^{5t}-5\,e^{4t}-7)e^{-3t}c_3}{10} \\ -\frac{(4\,e^{5t}-15\,e^{4t}+11)e^{-3t}c_1}{10} + (e^{2t}-e^{t})\,c_2 - \frac{(6\,e^{5t}-5\,e^{4t}-11)e^{-3t}c_3}{10} \end{bmatrix} \\ &= \begin{bmatrix} \frac{3\left(\left(c_1-\frac{2c_2}{3}+\frac{c_3}{3}\right)e^{4t}+\left(-\frac{4c_1}{15}+\frac{2c_2}{3}-\frac{2c_3}{5}\right)e^{5t}-\frac{c_1}{15}+\frac{c_3}{15}\right)e^{-3t}}{2} \\ \frac{3\left(\left(c_1-\frac{2c_2}{3}+\frac{c_3}{3}\right)e^{4t}+\left(-\frac{8c_1}{15}+\frac{4c_2}{3}-\frac{4c_3}{5}\right)e^{5t}-\frac{7c_1}{15}+\frac{7c_3}{15}\right)e^{-3t}}{2} \\ \frac{3\left(\left(c_1-\frac{2c_2}{3}+\frac{c_3}{3}\right)e^{4t}+\left(-\frac{4c_1}{15}+\frac{2c_2}{3}-\frac{2c_3}{5}\right)e^{5t}-\frac{11c_1}{15}+\frac{11c_3}{15}\right)e^{-3t}}{2} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} -\frac{(e^{5t}-15\,e^{t}+4)e^{-2t}}{10} & -e^{-2t}(e^{t}-1) & \frac{(e^{5t}+5\,e^{t}-6)e^{-2t}}{10} \\ -\frac{(7\,e^{5t}-15\,e^{t}+8)e^{-2t}}{10} & -e^{-2t}(e^{t}-2) & \frac{(7\,e^{5t}+5\,e^{t}-12)e^{-2t}}{10} \\ -\frac{(11\,e^{5t}-15\,e^{t}+4)e^{-2t}}{10} & -e^{-2t}(e^{t}-1) & \frac{(11\,e^{5t}+5\,e^{t}-6)e^{-2t}}{10} \end{bmatrix} \end{aligned}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} -\frac{(4\,e^{5t}-15\,e^{4t}+1)e^{-3t}}{10} & e^{2t}-e^{t} & -\frac{(6\,e^{5t}-5\,e^{4t}-1)e^{-3t}}{10} \\ -\frac{(8\,e^{5t}-15\,e^{4t}+7)e^{-3t}}{10} & 2\,e^{2t}-e^{t} & -\frac{(12\,e^{5t}-5\,e^{4t}-7)e^{-3t}}{10} \\ -\frac{(4\,e^{5t}-15\,e^{4t}+11)e^{-3t}}{10} & e^{2t}-e^{t} & -\frac{(6\,e^{5t}-5\,e^{4t}-11)e^{-3t}}{10} \end{bmatrix} \int \begin{bmatrix} -\frac{(e^{5t}-15\,e^{t}+4)e^{-2t}}{10} & -e^{-2t}(e^{t}-1) \\ -\frac{(7\,e^{5t}-15\,e^{t}+8)e^{-2t}}{10} & -e^{-2t}(e^{t}-2) \\ -\frac{(11\,e^{5t}-15\,e^{t}+4)e^{-2t}}{10} & -e^{-2t}(e^{t}-1) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{(4\,e^{5t}-15\,e^{4t}+1)e^{-3t}}{10} & e^{2t}-e^{t} & -\frac{(6\,e^{5t}-5\,e^{4t}-1)e^{-3t}}{10} \\ -\frac{(8\,e^{5t}-15\,e^{4t}+7)e^{-3t}}{10} & 2\,e^{2t}-e^{t} & -\frac{(12\,e^{5t}-5\,e^{4t}-7)e^{-3t}}{10} \\ -\frac{(4\,e^{5t}-15\,e^{4t}+11)e^{-3t}}{10} & e^{2t}-e^{t} & -\frac{(6\,e^{5t}-5\,e^{4t}-11)e^{-3t}}{10} \end{bmatrix} \begin{bmatrix} t \\ 2t \\ t \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t}t \\ 2\,e^{2t}t \\ e^{2t}t \end{bmatrix}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \begin{bmatrix} \left(\left(-\frac{2c_1}{5} + c_2 - \frac{3c_3}{5} + t\right) e^{5t} + \left(\frac{3c_1}{2} - c_2 + \frac{c_3}{2}\right) e^{4t} - \frac{c_1}{10} + \frac{c_3}{10}\right) e^{-3t} \\ 2\left(\left(-\frac{2c_1}{5} + c_2 - \frac{3c_3}{5} + t\right) e^{5t} + \left(\frac{3c_1}{4} - \frac{c_2}{2} + \frac{c_3}{4}\right) e^{4t} - \frac{7c_1}{20} + \frac{7c_3}{20}\right) e^{-3t} \\ \left(\left(-\frac{2c_1}{5} + c_2 - \frac{3c_3}{5} + t\right) e^{5t} + \left(\frac{3c_1}{2} - c_2 + \frac{c_3}{2}\right) e^{4t} - \frac{11c_1}{10} + \frac{11c_3}{10}\right) e^{-3t} \end{bmatrix}$$

4.14.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} e^{2t} \\ 2\,e^{2t} \\ e^{2t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 1 & -1 \\ 2 & 3-\lambda & -4 \\ 4 & 1 & -4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 7\lambda + 6 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$
$$\lambda_2 = -3$$
$$\lambda_3 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
-3	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 & -1 \\ 2 & 6 & -4 \\ 4 & 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 4 & 1 & -1 & 0 \\ 2 & 6 & -4 & 0 \\ 4 & 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 4 & 1 & -1 & 0 \\ 0 & \frac{11}{2} & -\frac{7}{2} & 0 \\ 4 & 1 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 4 & 1 & -1 & 0 \\ 0 & \frac{11}{2} & -\frac{7}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & 1 & -1 \\ 0 & \frac{11}{2} & -\frac{7}{2} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{11}, v_2 = \frac{7t}{11}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{11} \\ \frac{7t}{11} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{11} \\ \frac{7t}{11} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{11} \\ \frac{7t}{11} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{11} \\ \frac{7}{11} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{11} \\ \frac{7t}{11} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{11} \\ \frac{7}{11} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{11} \\ \frac{7t}{11} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 11 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -1 \\ 2 & 2 & -4 \\ 4 & 1 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 2 & 2 & -4 & 0 \\ 4 & 1 & -5 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} 2 & 2 & -4 & 0 \\ 0 & 1 & -1 & 0 \\ 4 & 1 & -5 & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2 & 2 & -4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right]$$

$$R_3 = R_3 + 3R_2 \Longrightarrow \left[\begin{array}{ccc|c} 2 & 2 & -4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 2 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & -1 \\ 2 & 1 & -4 \\ 4 & 1 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 2 & 1 & -4 & 0 \\ 4 & 1 & -6 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 3 & -6 & 0 \\ 4 & 1 & -6 & 0 \end{array}\right]$$

$$R_3 = R_3 + 4R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 3 & -6 & 0 \\ 0 & 5 & -10 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{5R_2}{3} \Longrightarrow \left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 3 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 & -1 \\ 0 & 3 & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = 2t\}$

Hence the solution is

$$\begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity algebraic m	multiplicity geometric k	defective?	eigenvectors
1	1	1	No	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
-3	1	1	No	$\begin{bmatrix} \frac{1}{11} \\ \frac{7}{11} \\ 1 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{t} \\ &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{t} \end{aligned}$$

Since eigenvalue -3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_2(t) &= \vec{v}_2 e^{-3t} \\ &= \begin{bmatrix} \frac{1}{11} \\ \frac{7}{11} \\ 1 \end{bmatrix} e^{-3t} \end{aligned}$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_3(t) &= \vec{v}_3 e^{2t} \\
&= \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{2t}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)
$$

Which is written as

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{t} \\ e^{t} \\ e^{t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{e^{-3t}}{11} \\ \frac{7\,e^{-3t}}{11} \\ e^{-3t} \end{bmatrix} + c_3 \begin{bmatrix} e^{2t} \\ 2\,e^{2t} \\ e^{2t} \end{bmatrix}
$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$
\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}
$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$
\Phi(t) = \begin{bmatrix} e^{t} & \frac{e^{-3t}}{11} & e^{2t} \\ e^{t} & \frac{7\,e^{-3t}}{11} & 2\,e^{2t} \\ e^{t} & e^{-3t} & e^{2t} \end{bmatrix}
$$

The particular solution is then given by

$$
\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt
$$

But

$$
\Phi^{-1} = \begin{bmatrix} \frac{3\,e^{-t}}{2} & -e^{-t} & \frac{e^{-t}}{2} \\ -\frac{11\,e^{3t}}{10} & 0 & \frac{11\,e^{3t}}{10} \\ -\frac{2\,e^{-2t}}{5} & e^{-2t} & -\frac{3\,e^{-2t}}{5} \end{bmatrix}
$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} e^{t} & \frac{e^{-3t}}{11} & e^{2t} \\ e^{t} & \frac{7\,e^{-3t}}{11} & 2\,e^{2t} \\ e^{t} & e^{-3t} & e^{2t} \end{bmatrix} \int \begin{bmatrix} \frac{3\,e^{-t}}{2} & -e^{-t} & \frac{e^{-t}}{2} \\ -\frac{11\,e^{3t}}{10} & 0 & \frac{11\,e^{3t}}{10} \\ -\frac{2\,e^{-2t}}{5} & e^{-2t} & -\frac{3\,e^{-2t}}{5} \end{bmatrix} \begin{bmatrix} e^{2t} \\ 2\,e^{2t} \\ e^{2t} \end{bmatrix} dt$$

$$= \begin{bmatrix} e^{t} & \frac{e^{-3t}}{11} & e^{2t} \\ e^{t} & \frac{7\,e^{-3t}}{11} & 2\,e^{2t} \\ e^{t} & e^{-3t} & e^{2t} \end{bmatrix} \int \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} dt$$

$$= \begin{bmatrix} e^{t} & \frac{e^{-3t}}{11} & e^{2t} \\ e^{t} & \frac{7\,e^{-3t}}{11} & 2\,e^{2t} \\ e^{t} & e^{-3t} & e^{2t} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t}t \\ 2\,e^{2t}t \\ e^{2t}t \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{t} \\ c_1 e^{t} \\ c_1 e^{t} \end{bmatrix} + \begin{bmatrix} \frac{c_2 e^{-3t}}{11} \\ \frac{7c_2 e^{-3t}}{11} \\ c_2 e^{-3t} \end{bmatrix} + \begin{bmatrix} c_3 e^{2t} \\ 2c_3 e^{2t} \\ c_3 e^{2t} \end{bmatrix} + \begin{bmatrix} e^{2t}t \\ 2\,e^{2t}t \\ e^{2t}t \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \left((c_3+t)\,e^{5t} + c_1 e^{4t} + \frac{c_2}{11}\right) e^{-3t} \\ 2\left((c_3+t)\,e^{5t} + \frac{c_1 e^{4t}}{2} + \frac{7c_2}{22}\right) e^{-3t} \\ \left((c_3+t)\,e^{5t} + c_1 e^{4t} + c_2\right) e^{-3t} \end{bmatrix}$$

4.14.3 Maple step by step solution

Let's solve

$$\left[x_1'(t) = x_1(t) + x_2(t) - x_3(t) + \left(e^t\right)^2, x_2'(t) = 2x_1(t) + 3x_2(t) - 4x_3(t) + 2\left(e^t\right)^2, x_3'(t) = 4x_1(t)\right.$$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} \left(e^t\right)^2 \\ 2\left(e^t\right)^2 \\ \left(e^t\right)^2 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} \left(e^t\right)^2 \\ 2\left(e^t\right)^2 \\ \left(e^t\right)^2 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} \left(e^t\right)^2 \\ 2\left(e^t\right)^2 \\ \left(e^t\right)^2 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{bmatrix}$$

- Rewrite the system as

$$\underline{x}^{\rightarrow}{}'(t) = A \cdot \underline{x}^{\rightarrow}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} \frac{1}{11} \\ \frac{7}{11} \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} \frac{1}{11} \\ \frac{7}{11} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{1} = \mathrm{e}^{-3t} \cdot \begin{bmatrix} \frac{1}{11} \\ \frac{7}{11} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{2} = \mathrm{e}^{t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{3} = \mathrm{e}^{2t} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\underline{x}^{\rightarrow}$

  $\underline{x}^{\rightarrow}(t) = c_1 \underline{x}^{\rightarrow}_1 + c_2 \underline{x}^{\rightarrow}_2 + c_3 \underline{x}^{\rightarrow}_3 + \underline{x}^{\rightarrow}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} \frac{e^{-3t}}{11} & e^t & e^{2t} \\ \frac{7\,e^{-3t}}{11} & e^t & 2\,e^{2t} \\ e^{-3t} & e^t & e^{2t} \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} \frac{e^{-3t}}{11} & e^t & e^{2t} \\ \frac{7\,e^{-3t}}{11} & e^t & 2\,e^{2t} \\ e^{-3t} & e^t & e^{2t} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{11} & 1 & 1 \\ \frac{7}{11} & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} -\frac{(4\,e^{5t}-15\,e^{4t}+1)e^{-3t}}{10} & e^{2t} - e^t & -\frac{(6\,e^{5t}-5\,e^{4t}-1)e^{-3t}}{10} \\ -\frac{(8\,e^{5t}-15\,e^{4t}+7)e^{-3t}}{10} & 2\,e^{2t} - e^t & -\frac{(12\,e^{5t}-5\,e^{4t}-7)e^{-3t}}{10} \\ -\frac{(4\,e^{5t}-15\,e^{4t}+11)e^{-3t}}{10} & e^{2t} - e^t & -\frac{(6\,e^{5t}-5\,e^{4t}-11)e^{-3t}}{10} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

  $\underline{x}^{\rightarrow}_p(t) = \Phi(t) \cdot \vec{v}(t)$

- Take the derivative of the particular solution

  $\underline{x}^{\rightarrow}{}'_p(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

  $\Phi(t) \cdot \overrightarrow{v}'(t) = \overrightarrow{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(t) = \frac{1}{\Phi(t)} \cdot \overrightarrow{f}(t)$

- Integrate to solve for $\overrightarrow{v}(t)$

  $\overrightarrow{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$

- Plug $\overrightarrow{v}(t)$ into the equation for the particular solution

  $\underline{x}^{\rightarrow}_{p}(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\underline{x}^{\rightarrow}_{p}(t) = \begin{bmatrix} e^{2t}t \\ 2\,e^{2t}t \\ e^{2t}t \end{bmatrix}$$

- Plug particular solution back into general solution

$$\underline{x}^{\rightarrow}(t) = c_1\underline{x}^{\rightarrow}_{1} + c_2\underline{x}^{\rightarrow}_{2} + c_3\underline{x}^{\rightarrow}_{3} + \begin{bmatrix} e^{2t}t \\ 2\,e^{2t}t \\ e^{2t}t \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} e^{-3t}\left((c_3+t)\,e^{5t} + c_2 e^{4t} + \frac{c_1}{11}\right) \\ 2\left((c_3+t)\,e^{5t} + \frac{c_2 e^{4t}}{2} + \frac{7c_1}{22}\right) e^{-3t} \\ \left((c_3+t)\,e^{5t} + c_2 e^{4t} + c_1\right) e^{-3t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\left\{x_1(t) = e^{-3t}\left((c_3+t)\,e^{5t} + c_2 e^{4t} + \frac{c_1}{11}\right), x_2(t) = 2\left((c_3+t)\,e^{5t} + \frac{c_2 e^{4t}}{2} + \frac{7c_1}{22}\right) e^{-3t}, x_3(t) = ((c_3 + \right.$

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 84

```
dsolve([diff(x__1(t),t)=1*x__1(t)+1*x__2(t)-1*x__3(t)+exp(2*t),diff(x__2(t),t)=2*x__1(t)+3*x
```

$$
\begin{aligned}
x_1(t) &= e^{2t}t + c_1e^{t} + c_2e^{-3t} + c_3e^{2t}\\
x_2(t) &= 2\,e^{2t}t + c_1e^{t} + 7c_2e^{-3t} + 2c_3e^{2t}\\
x_3(t) &= e^{2t}t + c_1e^{t} + 11c_2e^{-3t} + c_3e^{2t}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.128 (sec). Leaf size: 2491

```
DSolve[{x1'[t]==1*x1[t]+1*x2[t]-1*x3[t]+Exp[2*t],x2'[t]==2*x1[t]+3*x2[t]-4*x3[t]+2*Exp[2*t],
```

Too large to display

4.15 problem 17

Internal problem ID [1868]
Internal file name [`OUTPUT/1869_Sunday_June_05_2022_02_36_13_AM_39046103/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.12, Systems of differential equations. The nonhomogeneous equation. variation of parameters. Page 366
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_1(t) - x_2(t) - x_3(t) + e^{3t} \\
x_2'(t) &= x_1(t) + 3x_2(t) + x_3(t) - e^{3t} \\
x_3'(t) &= -3x_1(t) + x_2(t) - x_3(t) - e^{3t}
\end{aligned}
$$

4.15.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} e^{3t} \\ -e^{3t} \\ -e^{3t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -\frac{(e^{5t}-5\,e^{4t}-1)e^{-2t}}{5} & -e^{3t}+e^{2t} & -\frac{(e^{5t}-1)e^{-2t}}{5} \\ \frac{(e^{5t}-1)e^{-2t}}{5} & e^{3t} & \frac{(e^{5t}-1)e^{-2t}}{5} \\ \frac{(e^{5t}-5\,e^{4t}+4)e^{-2t}}{5} & e^{3t}-e^{2t} & \frac{(e^{5t}+4)e^{-2t}}{5} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} -\frac{(e^{5t}-5\,e^{4t}-1)e^{-2t}}{5} & -e^{3t}+e^{2t} & -\frac{(e^{5t}-1)e^{-2t}}{5} \\ \frac{(e^{5t}-1)e^{-2t}}{5} & e^{3t} & \frac{(e^{5t}-1)e^{-2t}}{5} \\ \frac{(e^{5t}-5\,e^{4t}+4)e^{-2t}}{5} & e^{3t}-e^{2t} & \frac{(e^{5t}+4)e^{-2t}}{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{(e^{5t}-5\,e^{4t}-1)e^{-2t}c_1}{5} + \left(-e^{3t}+e^{2t}\right)c_2 - \frac{(e^{5t}-1)e^{-2t}c_3}{5} \\ \frac{(e^{5t}-1)e^{-2t}c_1}{5} + e^{3t}c_2 + \frac{(e^{5t}-1)e^{-2t}c_3}{5} \\ \frac{(e^{5t}-5\,e^{4t}+4)e^{-2t}c_1}{5} + \left(e^{3t}-e^{2t}\right)c_2 + \frac{(e^{5t}+4)e^{-2t}c_3}{5} \end{bmatrix} \\ &= \begin{bmatrix} \left(\left(-\frac{c_1}{5}-c_2-\frac{c_3}{5}\right)e^{5t} + (c_1+c_2)\,e^{4t} + \frac{c_1}{5} + \frac{c_3}{5}\right)e^{-2t} \\ \frac{e^{-2t}\left((c_1+5c_2+c_3)e^{5t}-c_1-c_3\right)}{5} \\ -\left(\left(-\frac{c_1}{5}-c_2-\frac{c_3}{5}\right)e^{5t} + (c_1+c_2)\,e^{4t} - \frac{4c_1}{5} - \frac{4c_3}{5}\right)e^{-2t} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= \left(e^{At}\right)^{-1} \\ &= \begin{bmatrix} \frac{e^{-3t}(e^{5t}+5\,e^{t}-1)}{5} & e^{-3t}(e^{t}-1) & \frac{(e^{5t}-1)e^{-3t}}{5} \\ -\frac{(e^{5t}-1)e^{-3t}}{5} & e^{-3t} & -\frac{(e^{5t}-1)e^{-3t}}{5} \\ \frac{e^{-3t}(4\,e^{5t}-5\,e^{t}+1)}{5} & -e^{-3t}(e^{t}-1) & \frac{(4\,e^{5t}+1)e^{-3t}}{5} \end{bmatrix} \end{aligned}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} -\frac{(e^{5t}-5e^{4t}-1)e^{-2t}}{5} & -e^{3t}+e^{2t} & -\frac{(e^{5t}-1)e^{-2t}}{5} \\ \frac{(e^{5t}-1)e^{-2t}}{5} & e^{3t} & \frac{(e^{5t}-1)e^{-2t}}{5} \\ \frac{(e^{5t}-5e^{4t}+4)e^{-2t}}{5} & e^{3t}-e^{2t} & \frac{(e^{5t}+4)e^{-2t}}{5} \end{bmatrix} \int \begin{bmatrix} \frac{e^{-3t}(e^{5t}+5e^{t}-1)}{5} & e^{-3t}(e^{t}-1) & \frac{(e^{5t}-1)e^{-3t}}{5} \\ -\frac{(e^{5t}-1)e^{-3t}}{5} & e^{-3t} & -\frac{(e^{5t}-1)e^{-3t}}{5} \\ \frac{e^{-3t}(4e^{5t}-5e^{t}+1)}{5} & -e^{-3t}(e^{t}-1) & \frac{(4e^{5t}+1)e^{-3t}}{5} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{(e^{5t}-5e^{4t}-1)e^{-2t}}{5} & -e^{3t}+e^{2t} & -\frac{(e^{5t}-1)e^{-2t}}{5} \\ \frac{(e^{5t}-1)e^{-2t}}{5} & e^{3t} & \frac{(e^{5t}-1)e^{-2t}}{5} \\ \frac{(e^{5t}-5e^{4t}+4)e^{-2t}}{5} & e^{3t}-e^{2t} & \frac{(e^{5t}+4)e^{-2t}}{5} \end{bmatrix} \begin{bmatrix} t \\ -t \\ -t \end{bmatrix}$$

$$= \begin{bmatrix} e^{3t}t \\ -e^{3t}t \\ -e^{3t}t \end{bmatrix}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$= \begin{bmatrix} e^{-2t}\left(\left(-\frac{c_1}{5} - c_2 - \frac{c_3}{5} + t\right)e^{5t} + (c_1+c_2)e^{4t} + \frac{c_1}{5} + \frac{c_3}{5}\right) \\ -\left(\left(-\frac{c_1}{5} - c_2 - \frac{c_3}{5} + t\right)e^{5t} + \frac{c_1}{5} + \frac{c_3}{5}\right)e^{-2t} \\ -\left(\left(-\frac{c_1}{5} - c_2 - \frac{c_3}{5} + t\right)e^{5t} + (c_1+c_2)e^{4t} - \frac{4c_1}{5} - \frac{4c_3}{5}\right)e^{-2t} \end{bmatrix}$$

4.15.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} e^{3t} \\ -e^{3t} \\ -e^{3t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -1 & -1 \\ 1 & 3-\lambda & 1 \\ -3 & 1 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 3\lambda^2 - 4\lambda + 12 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned} \lambda_1 &= 2 \\ \lambda_2 &= 3 \\ \lambda_3 &= -2 \end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
2	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 & -1 \\ 1 & 5 & 1 \\ -3 & 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 3 & -1 & -1 & 0 \\ 1 & 5 & 1 & 0 \\ -3 & 1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} 3 & -1 & -1 & 0 \\ 0 & \frac{16}{3} & \frac{4}{3} & 0 \\ -3 & 1 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} 3 & -1 & -1 & 0 \\ 0 & \frac{16}{3} & \frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -1 & -1 \\ 0 & \frac{16}{3} & \frac{4}{3} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{4}, v_2 = -\frac{t}{4}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{4} \\ -\frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{4} \\ -\frac{t}{4} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{4} \\ -\frac{t}{4} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{4} \\ -\frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{4} \\ -\frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ -3 & 1 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ -3 & 1 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & 1 & -3 & 0 \end{array}\right]$$

$$R_3 = R_3 - 3R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & -1 & -1 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -1 & -1 \\ 1 & 0 & 1 \\ -3 & 1 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & -1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ -3 & 1 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 & -1 & -1 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ -3 & 1 & -4 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{3R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 & -1 & -1 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{5}{2} & -\frac{5}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 + 5R_2 \Longrightarrow \left[\begin{array}{ccc|c} -2 & -1 & -1 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & -1 & -1 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity algebraic m	multiplicity geometric k	defective?	eigenvectors
2	1	1	No	$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$
3	1	1	No	$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$
-2	1	1	No	$\begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{2t} \\ &= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{2t} \end{aligned}$$

Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_2(t) &= \vec{v}_2 e^{3t} \\ &= \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} e^{3t} \end{aligned}$$

Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{-2t} \\ &= \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ 1 \end{bmatrix} e^{-2t}\end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -e^{2t} \\ 0 \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} -e^{3t} \\ e^{3t} \\ e^{3t} \end{bmatrix} + c_3 \begin{bmatrix} \frac{e^{-2t}}{4} \\ -\frac{e^{-2t}}{4} \\ e^{-2t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -e^{2t} & -e^{3t} & \frac{e^{-2t}}{4} \\ 0 & e^{3t} & -\frac{e^{-2t}}{4} \\ e^{2t} & e^{3t} & e^{-2t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -e^{-2t} & -e^{-2t} & 0 \\ \frac{e^{-3t}}{5} & e^{-3t} & \frac{e^{-3t}}{5} \\ \frac{4\,e^{2t}}{5} & 0 & \frac{4\,e^{2t}}{5} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} -e^{2t} & -e^{3t} & \frac{e^{-2t}}{4} \\ 0 & e^{3t} & -\frac{e^{-2t}}{4} \\ e^{2t} & e^{3t} & e^{-2t} \end{bmatrix} \int \begin{bmatrix} -e^{-2t} & -e^{-2t} & 0 \\ \frac{e^{-3t}}{5} & e^{-3t} & \frac{e^{-3t}}{5} \\ \frac{4\,e^{2t}}{5} & 0 & \frac{4\,e^{2t}}{5} \end{bmatrix} \begin{bmatrix} e^{3t} \\ -e^{3t} \\ -e^{3t} \end{bmatrix} dt$$

$$= \begin{bmatrix} -e^{2t} & -e^{3t} & \frac{e^{-2t}}{4} \\ 0 & e^{3t} & -\frac{e^{-2t}}{4} \\ e^{2t} & e^{3t} & e^{-2t} \end{bmatrix} \int \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} dt$$

$$= \begin{bmatrix} -e^{2t} & -e^{3t} & \frac{e^{-2t}}{4} \\ 0 & e^{3t} & -\frac{e^{-2t}}{4} \\ e^{2t} & e^{3t} & e^{-2t} \end{bmatrix} \begin{bmatrix} 0 \\ -t \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} e^{3t}t \\ -e^{3t}t \\ -e^{3t}t \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -c_1 e^{2t} \\ 0 \\ c_1 e^{2t} \end{bmatrix} + \begin{bmatrix} -c_2 e^{3t} \\ c_2 e^{3t} \\ c_2 e^{3t} \end{bmatrix} + \begin{bmatrix} \frac{c_3 e^{-2t}}{4} \\ -\frac{c_3 e^{-2t}}{4} \\ c_3 e^{-2t} \end{bmatrix} + \begin{bmatrix} e^{3t}t \\ -e^{3t}t \\ -e^{3t}t \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \left(\left(-c_2 + t\right) e^{5t} - c_1 e^{4t} + \frac{c_3}{4}\right) e^{-2t} \\ -\left(\left(-c_2 + t\right) e^{5t} + \frac{c_3}{4}\right) e^{-2t} \\ -\left(\left(-c_2 + t\right) e^{5t} - c_1 e^{4t} - c_3\right) e^{-2t} \end{bmatrix}$$

4.15.3 Maple step by step solution

Let's solve

$$\left[x_1'(t) = x_1(t) - x_2(t) - x_3(t) + (e^t)^3, x_2'(t) = x_1(t) + 3x_2(t) + x_3(t) - (e^t)^3, x_3'(t) = -3x_1(t) +\right.$$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} (e^t)^3 \\ -(e^t)^3 \\ -(e^t)^3 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} (e^t)^3 \\ -(e^t)^3 \\ -(e^t)^3 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} (e^t)^3 \\ -(e^t)^3 \\ -(e^t)^3 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\underline{x}^{\rightarrow\prime}(t) = A \cdot \underline{x}^{\rightarrow}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_1 = e^{-2t} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_2 = e^{2t} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_3 = e^{3t} \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\underline{x}_p^{\rightarrow}$

  $\underline{x}^{\rightarrow}(t) = c_1 \underline{x}^{\rightarrow}_1 + c_2 \underline{x}^{\rightarrow}_2 + c_3 \underline{x}^{\rightarrow}_3 + \underline{x}^{\rightarrow}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

  $$\phi(t) = \begin{bmatrix} \frac{e^{-2t}}{4} & -e^{2t} & -e^{3t} \\ -\frac{e^{-2t}}{4} & 0 & e^{3t} \\ e^{-2t} & e^{2t} & e^{3t} \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

  $$\Phi(t) = \begin{bmatrix} \frac{e^{-2t}}{4} & -e^{2t} & -e^{3t} \\ -\frac{e^{-2t}}{4} & 0 & e^{3t} \\ e^{-2t} & e^{2t} & e^{3t} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{4} & -1 & -1 \\ -\frac{1}{4} & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

  $$\Phi(t) = \begin{bmatrix} -\frac{(e^{5t} - 5\,e^{4t} - 1)e^{-2t}}{5} & -e^{3t} + e^{2t} & -\frac{(e^{5t} - 1)e^{-2t}}{5} \\ \frac{(e^{5t} - 1)e^{-2t}}{5} & e^{3t} & \frac{(e^{5t} - 1)e^{-2t}}{5} \\ \frac{(e^{5t} - 5\,e^{4t} + 4)e^{-2t}}{5} & e^{3t} - e^{2t} & \frac{(e^{5t} + 4)e^{-2t}}{5} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

  $\underline{x}^{\rightarrow}_p(t) = \Phi(t) \cdot \vec{v}(t)$

- Take the derivative of the particular solution

  $\underline{x}^{\rightarrow\prime}_p(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

  $\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t) = \begin{bmatrix} e^{3t}t \\ -e^{3t}t \\ -e^{3t}t \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 + \begin{bmatrix} e^{3t}t \\ -e^{3t}t \\ -e^{3t}t \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} e^{-2t}\left((-c_3 + t)\, e^{5t} - c_2 e^{4t} + \frac{c_1}{4}\right) \\ -\left((-c_3 + t)\, e^{5t} + \frac{c_1}{4}\right) e^{-2t} \\ -e^{-2t}\left((-c_3 + t)\, e^{5t} - c_2 e^{4t} - c_1\right) \end{bmatrix}$$

- Solution to the system of ODEs

  $\left\{x_1(t) = e^{-2t}\left((-c_3 + t)\, e^{5t} - c_2 e^{4t} + \frac{c_1}{4}\right), x_2(t) = -\left((-c_3 + t)\, e^{5t} + \frac{c_1}{4}\right) e^{-2t}, x_3(t) = -e^{-2t}((-c_3\right.$

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 87

```
dsolve([diff(x__1(t),t)=1*x__1(t)-1*x__2(t)-1*x__3(t)+exp(3*t),diff(x__2(t),t)=1*x__1(t)+3*x
```

$$
\begin{aligned}
x_1(t) &= \mathrm{e}^{3t}t + c_1\mathrm{e}^{-2t} + c_2\mathrm{e}^{2t} + c_3\mathrm{e}^{3t} \\
x_2(t) &= -\mathrm{e}^{3t}t - c_1\mathrm{e}^{-2t} - c_3\mathrm{e}^{3t} \\
x_3(t) &= -\mathrm{e}^{3t}t + 4c_1\mathrm{e}^{-2t} - c_2\mathrm{e}^{2t} - c_3\mathrm{e}^{3t}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 142

```
DSolve[{x1'[t]==1*x1[t]-1*x2[t]-1*x3[t]+Exp[3*t],x2'[t]==1*x1[t]+3*x2[t]+1*x3[t]-Exp[3*t],x3
```

$$
\begin{aligned}
\mathrm{x1}(t) &\to \frac{1}{5}e^{-2t}\left(5(c_1+c_2)e^{4t} + e^{5t}(5t - c_1 - 5c_2 - c_3) + c_1 + c_3\right) \\
\mathrm{x2}(t) &\to \frac{1}{5}e^{-2t}\left(e^{5t}(-5t + c_1 + 5c_2 + c_3) - c_1 - c_3\right) \\
\mathrm{x3}(t) &\to \frac{1}{5}e^{-2t}\left(-5(c_1+c_2)e^{4t} + e^{5t}(-5t + c_1 + 5c_2 + c_3) + 4(c_1 + c_3)\right)
\end{aligned}
$$

4.16 problem 18

Internal problem ID [1869]
Internal file name [`OUTPUT/1870_Sunday_June_05_2022_02_36_16_AM_64025338/index.tex`]

Book: Differential equations and their applications, 4th ed., M. Braun
Section: Section 3.12, Systems of differential equations. The nonhomogeneous equation. variation of parameters. Page 366
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = 3x_1(t) + 2x_2(t) + 4x_3(t) + 2\,\mathrm{e}^{8t}$$
$$x_2'(t) = 2x_1(t) + 2x_3(t) + \mathrm{e}^{8t}$$
$$x_3'(t) = 4x_1(t) + 2x_2(t) + 3x_3(t) + 2\,\mathrm{e}^{8t}$$

4.16.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 2\,\mathrm{e}^{8t} \\ \mathrm{e}^{8t} \\ 2\,\mathrm{e}^{8t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \left[\begin{array}{ccc} \frac{5\,\mathrm{e}^{-t}}{9}+\frac{4\,\mathrm{e}^{8t}}{9} & -\frac{2\,\mathrm{e}^{-t}}{9}+\frac{2\,\mathrm{e}^{8t}}{9} & -\frac{4\,\mathrm{e}^{-t}}{9}+\frac{4\,\mathrm{e}^{8t}}{9} \\ -\frac{2\,\mathrm{e}^{-t}}{9}+\frac{2\,\mathrm{e}^{8t}}{9} & \frac{8\,\mathrm{e}^{-t}}{9}+\frac{\mathrm{e}^{8t}}{9} & -\frac{2\,\mathrm{e}^{-t}}{9}+\frac{2\,\mathrm{e}^{8t}}{9} \\ -\frac{4\,\mathrm{e}^{-t}}{9}+\frac{4\,\mathrm{e}^{8t}}{9} & -\frac{2\,\mathrm{e}^{-t}}{9}+\frac{2\,\mathrm{e}^{8t}}{9} & \frac{5\,\mathrm{e}^{-t}}{9}+\frac{4\,\mathrm{e}^{8t}}{9} \end{array}\right]$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \left[\begin{array}{ccc} \frac{5\,\mathrm{e}^{-t}}{9}+\frac{4\,\mathrm{e}^{8t}}{9} & -\frac{2\,\mathrm{e}^{-t}}{9}+\frac{2\,\mathrm{e}^{8t}}{9} & -\frac{4\,\mathrm{e}^{-t}}{9}+\frac{4\,\mathrm{e}^{8t}}{9} \\ -\frac{2\,\mathrm{e}^{-t}}{9}+\frac{2\,\mathrm{e}^{8t}}{9} & \frac{8\,\mathrm{e}^{-t}}{9}+\frac{\mathrm{e}^{8t}}{9} & -\frac{2\,\mathrm{e}^{-t}}{9}+\frac{2\,\mathrm{e}^{8t}}{9} \\ -\frac{4\,\mathrm{e}^{-t}}{9}+\frac{4\,\mathrm{e}^{8t}}{9} & -\frac{2\,\mathrm{e}^{-t}}{9}+\frac{2\,\mathrm{e}^{8t}}{9} & \frac{5\,\mathrm{e}^{-t}}{9}+\frac{4\,\mathrm{e}^{8t}}{9} \end{array}\right] \left[\begin{array}{c} c_1 \\ c_2 \\ c_3 \end{array}\right] \\
&= \left[\begin{array}{c} \left(\frac{5\,\mathrm{e}^{-t}}{9}+\frac{4\,\mathrm{e}^{8t}}{9}\right) c_1 + \left(-\frac{2\,\mathrm{e}^{-t}}{9}+\frac{2\,\mathrm{e}^{8t}}{9}\right) c_2 + \left(-\frac{4\,\mathrm{e}^{-t}}{9}+\frac{4\,\mathrm{e}^{8t}}{9}\right) c_3 \\ \left(-\frac{2\,\mathrm{e}^{-t}}{9}+\frac{2\,\mathrm{e}^{8t}}{9}\right) c_1 + \left(\frac{8\,\mathrm{e}^{-t}}{9}+\frac{\mathrm{e}^{8t}}{9}\right) c_2 + \left(-\frac{2\,\mathrm{e}^{-t}}{9}+\frac{2\,\mathrm{e}^{8t}}{9}\right) c_3 \\ \left(-\frac{4\,\mathrm{e}^{-t}}{9}+\frac{4\,\mathrm{e}^{8t}}{9}\right) c_1 + \left(-\frac{2\,\mathrm{e}^{-t}}{9}+\frac{2\,\mathrm{e}^{8t}}{9}\right) c_2 + \left(\frac{5\,\mathrm{e}^{-t}}{9}+\frac{4\,\mathrm{e}^{8t}}{9}\right) c_3 \end{array}\right] \\
&= \left[\begin{array}{c} \frac{(5c_1-2c_2-4c_3)\mathrm{e}^{-t}}{9}+\frac{4\left(c_1+\frac{c_2}{2}+c_3\right)\mathrm{e}^{8t}}{9} \\ \frac{(-2c_1+8c_2-2c_3)\mathrm{e}^{-t}}{9}+\frac{2\left(c_1+\frac{c_2}{2}+c_3\right)\mathrm{e}^{8t}}{9} \\ \frac{(-4c_1-2c_2+5c_3)\mathrm{e}^{-t}}{9}+\frac{4\left(c_1+\frac{c_2}{2}+c_3\right)\mathrm{e}^{8t}}{9} \end{array}\right]
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \left[\begin{array}{ccc} \frac{(5\,\mathrm{e}^{9t}+4)\mathrm{e}^{-8t}}{9} & -\frac{2(\mathrm{e}^{9t}-1)\mathrm{e}^{-8t}}{9} & -\frac{4(\mathrm{e}^{9t}-1)\mathrm{e}^{-8t}}{9} \\ -\frac{2(\mathrm{e}^{9t}-1)\mathrm{e}^{-8t}}{9} & \frac{(8\,\mathrm{e}^{9t}+1)\mathrm{e}^{-8t}}{9} & -\frac{2(\mathrm{e}^{9t}-1)\mathrm{e}^{-8t}}{9} \\ -\frac{4(\mathrm{e}^{9t}-1)\mathrm{e}^{-8t}}{9} & -\frac{2(\mathrm{e}^{9t}-1)\mathrm{e}^{-8t}}{9} & \frac{(5\,\mathrm{e}^{9t}+4)\mathrm{e}^{-8t}}{9} \end{array}\right]
\end{aligned}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} \frac{5\,e^{-t}}{9}+\frac{4\,e^{8t}}{9} & -\frac{2\,e^{-t}}{9}+\frac{2\,e^{8t}}{9} & -\frac{4\,e^{-t}}{9}+\frac{4\,e^{8t}}{9} \\ -\frac{2\,e^{-t}}{9}+\frac{2\,e^{8t}}{9} & \frac{8\,e^{-t}}{9}+\frac{e^{8t}}{9} & -\frac{2\,e^{-t}}{9}+\frac{2\,e^{8t}}{9} \\ -\frac{4\,e^{-t}}{9}+\frac{4\,e^{8t}}{9} & -\frac{2\,e^{-t}}{9}+\frac{2\,e^{8t}}{9} & \frac{5\,e^{-t}}{9}+\frac{4\,e^{8t}}{9} \end{bmatrix} \int \begin{bmatrix} \frac{(5\,e^{9t}+4)e^{-8t}}{9} & -\frac{2(e^{9t}-1)e^{-8t}}{9} & -\frac{4(e^{9t}-1)e^{-8t}}{9} \\ -\frac{2(e^{9t}-1)e^{-8t}}{9} & \frac{(8\,e^{9t}+1)e^{-8t}}{9} & -\frac{2(e^{9t}-1)e^{-8t}}{9} \\ -\frac{4(e^{9t}-1)e^{-8t}}{9} & -\frac{2(e^{9t}-1)e^{-8t}}{9} & \frac{(5\,e^{9t}+4)e^{-8t}}{9} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5\,e^{-t}}{9}+\frac{4\,e^{8t}}{9} & -\frac{2\,e^{-t}}{9}+\frac{2\,e^{8t}}{9} & -\frac{4\,e^{-t}}{9}+\frac{4\,e^{8t}}{9} \\ -\frac{2\,e^{-t}}{9}+\frac{2\,e^{8t}}{9} & \frac{8\,e^{-t}}{9}+\frac{e^{8t}}{9} & -\frac{2\,e^{-t}}{9}+\frac{2\,e^{8t}}{9} \\ -\frac{4\,e^{-t}}{9}+\frac{4\,e^{8t}}{9} & -\frac{2\,e^{-t}}{9}+\frac{2\,e^{8t}}{9} & \frac{5\,e^{-t}}{9}+\frac{4\,e^{8t}}{9} \end{bmatrix} \begin{bmatrix} 2t \\ t \\ 2t \end{bmatrix}$$

$$= \begin{bmatrix} 2t\,e^{8t} \\ t\,e^{8t} \\ 2t\,e^{8t} \end{bmatrix}$$

Hence the complete solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \begin{bmatrix} \frac{2(9t+2c_1+c_2+2c_3)e^{8t}}{9} + \frac{5\left(c_1-\frac{2c_2}{5}-\frac{4c_3}{5}\right)e^{-t}}{9} \\ \frac{(9t+2c_1+c_2+2c_3)e^{8t}}{9} - \frac{2\,e^{-t}(c_1-4c_2+c_3)}{9} \\ \frac{2(9t+2c_1+c_2+2c_3)e^{8t}}{9} - \frac{4\left(c_1+\frac{c_2}{2}-\frac{5c_3}{4}\right)e^{-t}}{9} \end{bmatrix} \end{aligned}$$

4.16.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 2\,e^{8t} \\ e^{8t} \\ 2\,e^{8t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 6\lambda^2 - 15\lambda - 8 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 8$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
8	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 4 & 2 & 4 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 2 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 4 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 2 & 4 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 4 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2} - s\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} - s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} - s \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2} - s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix}$$
$$= t \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} -\frac{t}{2} - s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

Which are normalized to

$$\left(\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

Considering the eigenvalue $\lambda_2 = 8$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} - (8) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 2 & -8 & 2 & 0 \\ 4 & 2 & -5 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{2R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 0 & -\frac{36}{5} & \frac{18}{5} & 0 \\ 4 & 2 & -5 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{4R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 0 & -\frac{36}{5} & \frac{18}{5} & 0 \\ 0 & \frac{18}{5} & -\frac{9}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_2}{2} \Longrightarrow \left[\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 0 & -\frac{36}{5} & \frac{18}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -5 & 2 & 4 \\ 0 & -\frac{36}{5} & \frac{18}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = t, v_2 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} t \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} t \\ \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} t \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
8	1	1	No	$\begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$
-1	2	2	No	$\begin{bmatrix} -\frac{1}{2} & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 8 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{8t}\\ &= \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix} e^{8t}\end{aligned}$$

eigenvalue -1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

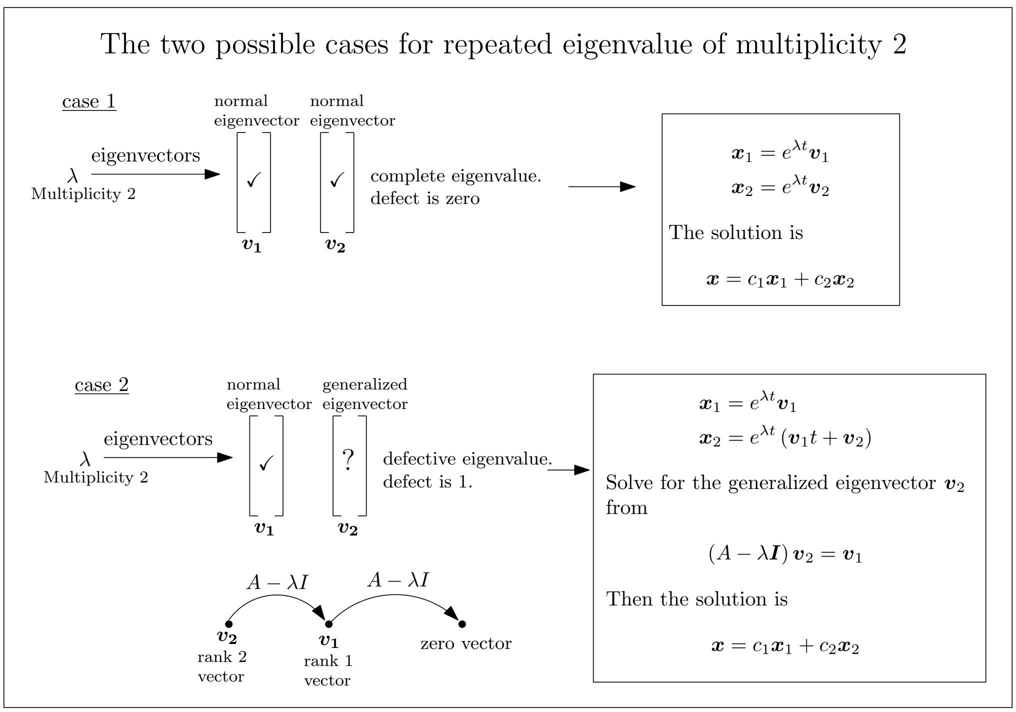


Figure 26: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above.

Hence the corresponding eigenvector basis are

$$\vec{x}_2(t) = \vec{v}_2 e^{-t}$$
$$= \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} e^{-t}$$

$$\vec{x}_3(t) = \vec{v}_3 e^{-t}$$
$$= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-t}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{8t} \\ \frac{e^{8t}}{2} \\ e^{8t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{e^{-t}}{2} \\ e^{-t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} e^{8t} & -\frac{e^{-t}}{2} & -e^{-t} \\ \frac{e^{8t}}{2} & e^{-t} & 0 \\ e^{8t} & 0 & e^{-t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{4\,\mathrm{e}^{-8t}}{9} & \frac{2\,\mathrm{e}^{-8t}}{9} & \frac{4\,\mathrm{e}^{-8t}}{9} \\ -\frac{2\,\mathrm{e}^{t}}{9} & \frac{8\,\mathrm{e}^{t}}{9} & -\frac{2\,\mathrm{e}^{t}}{9} \\ -\frac{4\,\mathrm{e}^{t}}{9} & -\frac{2\,\mathrm{e}^{t}}{9} & \frac{5\,\mathrm{e}^{t}}{9} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \mathrm{e}^{8t} & -\frac{\mathrm{e}^{-t}}{2} & -\mathrm{e}^{-t} \\ \frac{\mathrm{e}^{8t}}{2} & \mathrm{e}^{-t} & 0 \\ \mathrm{e}^{8t} & 0 & \mathrm{e}^{-t} \end{bmatrix} \int \begin{bmatrix} \frac{4\,\mathrm{e}^{-8t}}{9} & \frac{2\,\mathrm{e}^{-8t}}{9} & \frac{4\,\mathrm{e}^{-8t}}{9} \\ -\frac{2\,\mathrm{e}^{t}}{9} & \frac{8\,\mathrm{e}^{t}}{9} & -\frac{2\,\mathrm{e}^{t}}{9} \\ -\frac{4\,\mathrm{e}^{t}}{9} & -\frac{2\,\mathrm{e}^{t}}{9} & \frac{5\,\mathrm{e}^{t}}{9} \end{bmatrix} \begin{bmatrix} 2\,\mathrm{e}^{8t} \\ \mathrm{e}^{8t} \\ 2\,\mathrm{e}^{8t} \end{bmatrix} dt \\
&= \begin{bmatrix} \mathrm{e}^{8t} & -\frac{\mathrm{e}^{-t}}{2} & -\mathrm{e}^{-t} \\ \frac{\mathrm{e}^{8t}}{2} & \mathrm{e}^{-t} & 0 \\ \mathrm{e}^{8t} & 0 & \mathrm{e}^{-t} \end{bmatrix} \int \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} dt \\
&= \begin{bmatrix} \mathrm{e}^{8t} & -\frac{\mathrm{e}^{-t}}{2} & -\mathrm{e}^{-t} \\ \frac{\mathrm{e}^{8t}}{2} & \mathrm{e}^{-t} & 0 \\ \mathrm{e}^{8t} & 0 & \mathrm{e}^{-t} \end{bmatrix} \begin{bmatrix} 2t \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 2t\,\mathrm{e}^{8t} \\ t\,\mathrm{e}^{8t} \\ 2t\,\mathrm{e}^{8t} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} c_1\mathrm{e}^{8t} \\ \frac{c_1\mathrm{e}^{8t}}{2} \\ c_1\mathrm{e}^{8t} \end{bmatrix} + \begin{bmatrix} -\frac{c_2\mathrm{e}^{-t}}{2} \\ c_2\mathrm{e}^{-t} \\ 0 \end{bmatrix} + \begin{bmatrix} -c_3\mathrm{e}^{-t} \\ 0 \\ c_3\mathrm{e}^{-t} \end{bmatrix} + \begin{bmatrix} 2t\,\mathrm{e}^{8t} \\ t\,\mathrm{e}^{8t} \\ 2t\,\mathrm{e}^{8t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{(-c_2-2c_3)\mathrm{e}^{-t}}{2} + 2\left(\frac{c_1}{2} + t\right)\mathrm{e}^{8t} \\ \frac{(c_1+2t)\mathrm{e}^{8t}}{2} + c_2\mathrm{e}^{-t} \\ (c_1 + 2t)\,\mathrm{e}^{8t} + c_3\mathrm{e}^{-t} \end{bmatrix}$$

4.16.3 Maple step by step solution

Let's solve

$\left[x_1'(t) = 3x_1(t) + 2x_2(t) + 4x_3(t) + 2(\mathrm{e}^t)^8, x_2'(t) = 2x_1(t) + 2x_3(t) + (\mathrm{e}^t)^8, x_3'(t) = 4x_1(t) + 2x_2\right.$

- Define vector

$$\underline{\vec{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \cdot \underline{\vec{x}}(t) + \begin{bmatrix} 2(\mathrm{e}^t)^8 \\ (\mathrm{e}^t)^8 \\ 2(\mathrm{e}^t)^8 \end{bmatrix}$$

- System to solve

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \cdot \underline{\vec{x}}(t) + \begin{bmatrix} 2(\mathrm{e}^t)^8 \\ (\mathrm{e}^t)^8 \\ 2(\mathrm{e}^t)^8 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 2(\mathrm{e}^t)^8 \\ (\mathrm{e}^t)^8 \\ 2(\mathrm{e}^t)^8 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

- Rewrite the system as

$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}\right], \left[-1, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right], \left[8, \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-1, \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}\right]$$

- First solution from eigenvalue -1

$$\overrightarrow{x}_1(t) = e^{-t} \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = -1$ is the eigenvalue, a

$$\overrightarrow{x}_2(t) = e^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the t multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obta
- Substitute $\overrightarrow{x}_2(t)$ into the homogeneous system

$$\lambda e^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda t}\overrightarrow{v} = \left(e^{\lambda t}A\right) \cdot \left(t\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda e^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda t}\overrightarrow{v} = e^{\lambda t}\left(\lambda t\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

$$\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{x}_2(t)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue -1

$$\left(\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} - (-1) \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} -\frac{1}{8} \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue -1

$$\underline{\vec{x}}_2(t) = \mathrm{e}^{-t} \cdot \left(t \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{8} \\ 0 \\ 0 \end{bmatrix}\right)$$

- Consider eigenpair

$$\left[8, \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_3 = \mathrm{e}^{8t} \cdot \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\underline{\vec{x}}$

$$\underline{\vec{x}}(t) = c_1 \underline{\vec{x}}_1(t) + c_2 \underline{\vec{x}}_2(t) + c_3 \underline{\vec{x}}_3 + \underline{\vec{x}}_p(t)$$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} -\frac{\mathrm{e}^{-t}}{2} & \mathrm{e}^{-t}\left(-\frac{t}{2} - \frac{1}{8}\right) & \mathrm{e}^{8t} \\ \mathrm{e}^{-t} & t\,\mathrm{e}^{-t} & \frac{\mathrm{e}^{8t}}{2} \\ 0 & 0 & \mathrm{e}^{8t} \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} -\frac{e^{-t}}{2} & e^{-t}\left(-\frac{t}{2} - \frac{1}{8}\right) & e^{8t} \\ e^{-t} & t\,e^{-t} & \frac{e^{8t}}{2} \\ 0 & 0 & e^{8t} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -\frac{1}{2} & -\frac{1}{8} & 1 \\ 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} (4t+1)\,e^{-t} & 2t\,e^{-t} & (-5t-1)\,e^{-t} + e^{8t} \\ -8t\,e^{-t} & e^{-t}(1-4t) & -\frac{e^{-t}}{2} + 10t\,e^{-t} + \frac{e^{8t}}{2} \\ 0 & 0 & e^{8t} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$

- Take the derivative of the particular solution

$\vec{x}_p'(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

$\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

$\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

$\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\,ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

$\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\underline{x}^{\rightarrow}{}_p(t) = \begin{bmatrix} 2t\,e^{8t} \\ t\,e^{8t} \\ 2t\,e^{8t} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\underline{x}^{\rightarrow}(t) = c_1 \underline{x}^{\rightarrow}{}_1(t) + c_2 \underline{x}^{\rightarrow}{}_2(t) + c_3 \underline{x}^{\rightarrow}{}_3 + \begin{bmatrix} 2t\,e^{8t} \\ t\,e^{8t} \\ 2t\,e^{8t} \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{((-4t-1)c_2-4c_1)e^{-t}}{8} + 2\left(\frac{c_3}{2}+t\right)e^{8t} \\ (c_2t+c_1)\,e^{-t} + \left(\frac{c_3}{2}+t\right)e^{8t} \\ e^{8t}(c_3+2t) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \frac{((-4t-1)c_2-4c_1)e^{-t}}{8} + 2\left(\frac{c_3}{2}+t\right)e^{8t}, x_2(t) = (c_2t+c_1)\,e^{-t} + \left(\frac{c_3}{2}+t\right)e^{8t}, x_3(t) = e^{8t}(c_3+2t\right.$$

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 87

```
dsolve([diff(x__1(t),t)=3*x__1(t)+2*x__2(t)+4*x__3(t)+2*exp(8*t),diff(x__2(t),t)=2*x__1(t)+0
```

$$x_1(t) = 2c_3e^{-t} + 2c_2e^{8t} + 2t\,e^{8t} + e^{-t}c_1$$
$$x_2(t) = c_3e^{-t} + c_2e^{8t} + t\,e^{8t}$$
$$x_3(t) = -\frac{5c_3e^{-t}}{2} + 2c_2e^{8t} + 2t\,e^{8t} - e^{-t}c_1$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 139

```
DSolve[{x1'[t]==3*x1[t]+2*x2[t]+4*x3[t]+2*Exp[8*t],x2'[t]==2*x1[t]+0*x2[t]+2*x3[t]+Exp[8*t],
```

$$x1(t) \to \frac{1}{9}e^{-t}\left(2e^{9t}(9t + 2c_1 + c_2 + 2c_3) + 5c_1 - 2(c_2 + 2c_3)\right)$$
$$x2(t) \to \frac{1}{9}e^{-t}\left(e^{9t}(9t + 2c_1 + c_2 + 2c_3) - 2(c_1 - 4c_2 + c_3)\right)$$
$$x3(t) \to \frac{1}{9}e^{-t}\left(2e^{9t}(9t + 2c_1 + c_2 + 2c_3) - 4c_1 - 2c_2 + 5c_3\right)$$