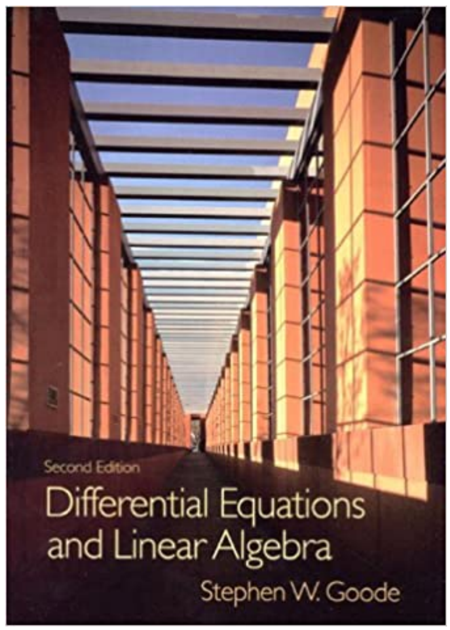
A Solution Manual For

Differential equations and linear algebra, Stephen W. Goode, second edition, 2000



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May 15, 2024

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1.1 problem 1

Internal problem ID [2544]
Internal file name [`OUTPUT/2036_Sunday_June_05_2022_02_45_45_AM_37514662/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.4, page 36
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - 2yx = 0}$$

1.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= 2xy \end{aligned}$$

Where $f(x) = 2x$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y}\,dy &= 2x\,dx \\ \int \frac{1}{y}\,dy &= \int 2x\,dx \\ \ln(y) &= x^2 + c_1 \\ y &= \mathrm{e}^{x^2+c_1} \\ &= c_1 \mathrm{e}^{x^2}\end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{x^2} \tag{1}$$

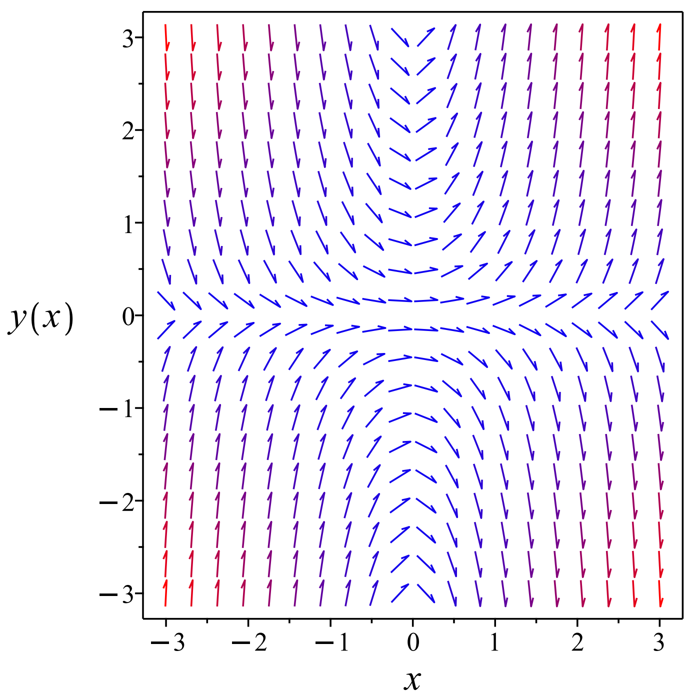


Figure 1: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 \mathrm{e}^{x^2}$$

Verified OK.

1.1.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -2x \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' - 2yx = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -2x dx} \\ &= e^{-x^2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}\mu y &= 0 \\ \frac{d}{dx}\left(y\, e^{-x^2}\right) &= 0 \end{aligned}$$

Integrating gives

$$y\, e^{-x^2} = c_1$$

Dividing both sides by the integrating factor $\mu = e^{-x^2}$ results in

$$y = c_1 e^{x^2}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{x^2} \tag{1}$$

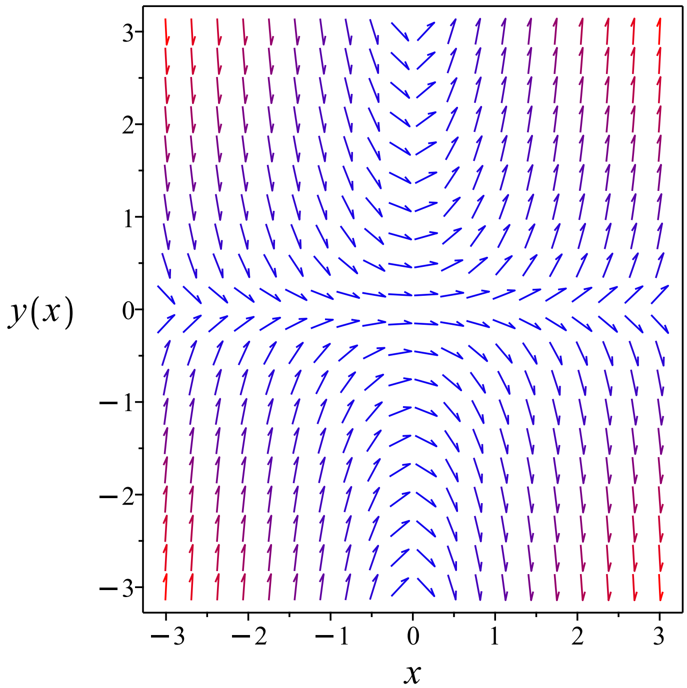


Figure 2: Slope field plot

Verification of solutions

$$y = c_1 e^{x^2}$$

Verified OK.

1.1.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - 2u(x)\,x^2 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u(2x^2-1)}{x} \end{aligned}$$

Where $f(x) = \frac{2x^2-1}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= \frac{2x^2-1}{x}\, dx \\
\int \frac{1}{u}\, du &= \int \frac{2x^2-1}{x}\, dx \\
\ln(u) &= x^2 - \ln(x) + c_2 \\
u &= e^{x^2-\ln(x)+c_2} \\
&= c_2 e^{x^2-\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \frac{c_2 e^{x^2}}{x}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= xu \\
&= c_2 e^{x^2}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_2 e^{x^2} \tag{1}
$$

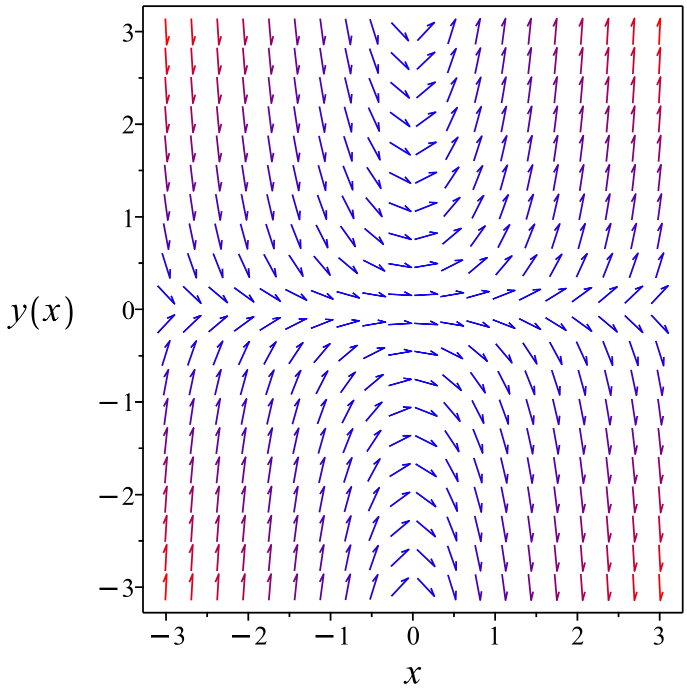


Figure 3: Slope field plot

Verification of solutions

$$y = c_2 e^{x^2}$$

Verified OK.

1.1.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 2xy$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 1: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{x^2}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{x^2}} dy \end{aligned}$$

Which results in

$$S = y\,e^{-x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = 2xy$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -2yx\,e^{-x^2} \\ S_y &= e^{-x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\,\mathrm{e}^{-x^2} = c_1$$

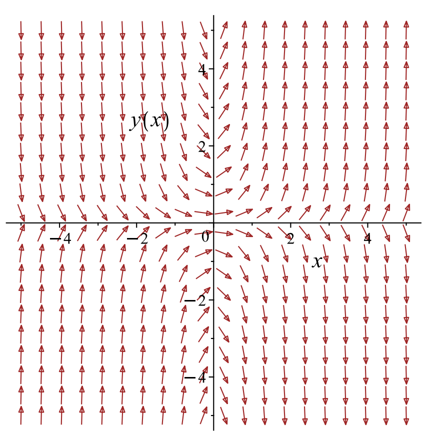
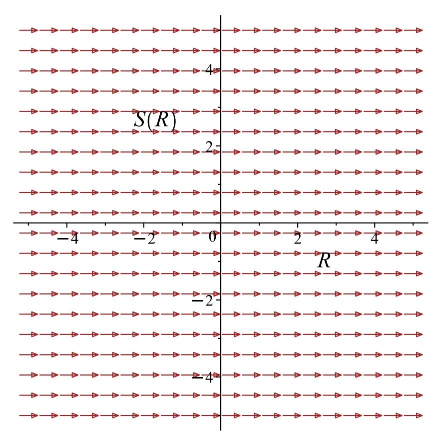
Which simplifies to

$$y\,\mathrm{e}^{-x^2} = c_1$$

Which gives

$$y = c_1 \mathrm{e}^{x^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 2xy$ 	$R = x$ $S = y\,\mathrm{e}^{-x^2}$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{x^2} \tag{1}$$

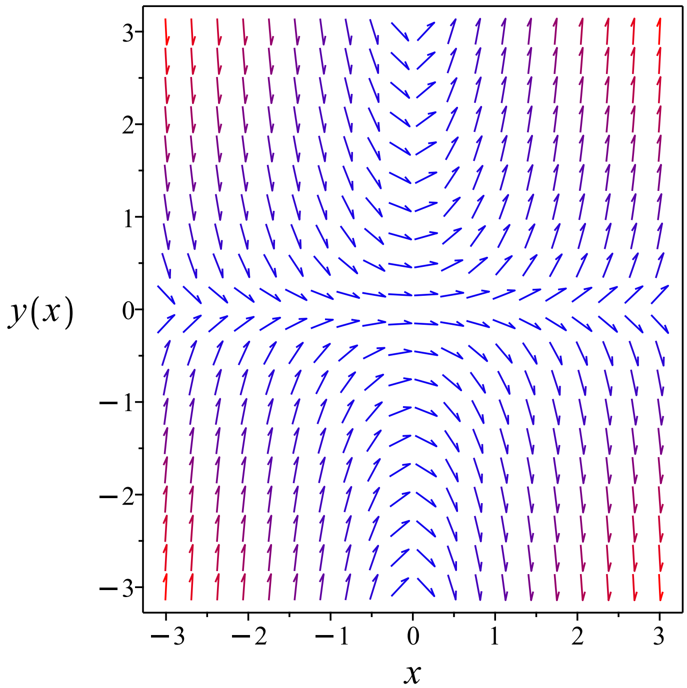


Figure 4: Slope field plot

Verification of solutions

$$y = c_1 e^{x^2}$$

Verified OK.

1.1.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{2y}\right)\mathrm{d}y &= (x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x + \left(\frac{1}{2y}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -x\\ N(x,y) &= \frac{1}{2y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{2y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M\, \mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int -x\, \mathrm{d}x\end{aligned}$$
$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{2y}$. Therefore equation (4) becomes

$$\frac{1}{2y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{2y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{2y}\right) \mathrm{d}y$$
$$f(y) = \frac{\ln(y)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \frac{\ln(y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \frac{\ln(y)}{2}$$

The solution becomes

$$y = \mathrm{e}^{x^2+2c_1}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{x^2+2c_1} \tag{1}$$

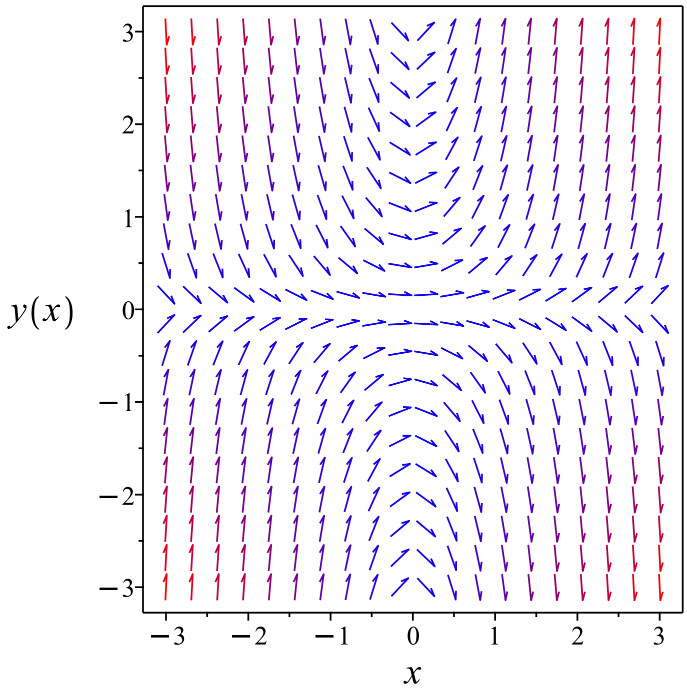


Figure 5: Slope field plot

Verification of solutions

$$y = e^{x^2+2c_1}$$

Verified OK.

1.1.6 Maple step by step solution

Let's solve

$y' - 2yx = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = 2x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int 2x dx + c_1$

- Evaluate integral

$$\ln(y) = x^2 + c_1$$

- Solve for y

$$y = e^{x^2+c_1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(diff(y(x),x)=2*x*y(x),y(x), singsol=all)
```

$$y(x) = e^{x^2} c_1$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 18

```
DSolve[y'[x]==2*x*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{x^2}$$
$$y(x) \to 0$$

1.2 problem 2

Internal problem ID [2545]
Internal file name [`OUTPUT/2037_Sunday_June_05_2022_02_45_47_AM_26852723/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.4, page 36
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - \frac{y^2}{x^2+1} = 0}$$

1.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y^2}{x^2+1} \end{aligned}$$

Where $f(x) = \frac{1}{x^2+1}$ and $g(y) = y^2$. Integrating both sides gives

$$\frac{1}{y^2}\,dy = \frac{1}{x^2+1}\,dx$$

$$\int \frac{1}{y^2}\,dy = \int \frac{1}{x^2+1}\,dx$$

$$-\frac{1}{y} = \arctan(x) + c_1$$

Which results in

$$y = -\frac{1}{\arctan(x) + c_1}$$

<u>Summary</u>

The solution(s) found are the following

$$y = -\frac{1}{\arctan(x) + c_1} \tag{1}$$

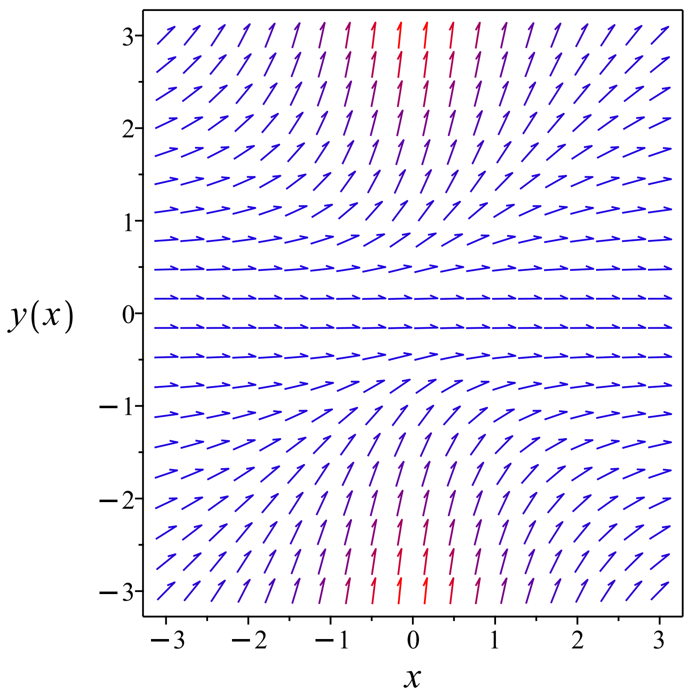


Figure 6: Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{1}{\arctan(x) + c_1}$$

Verified OK.

1.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y^2}{x^2+1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 4: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int b f(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = x^2 + 1$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{x^2+1} dx \end{aligned}$$

Which results in

$$S = \arctan(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y^2}{x^2+1}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{1}{x^2+1} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\arctan(x) = -\frac{1}{y} + c_1$$

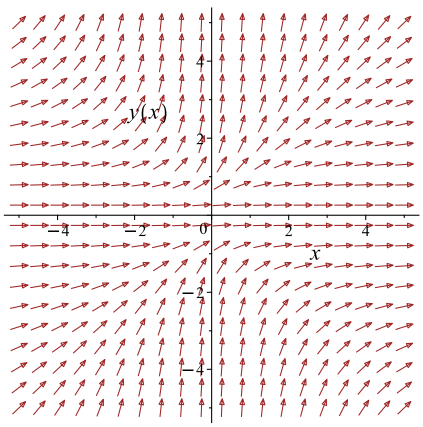
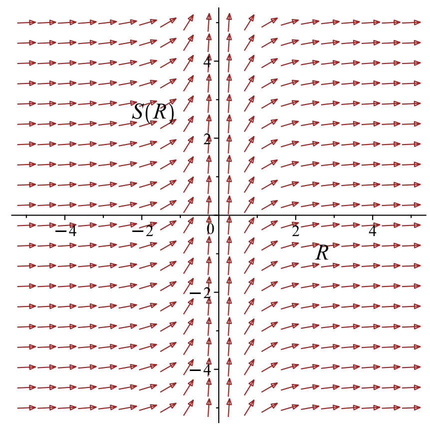
Which simplifies to

$$\arctan(x) = -\frac{1}{y} + c_1$$

Which gives

$$y = -\frac{1}{\arctan(x) - c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2}{x^2+1}$ 	$R = y$ $S = \arctan(x)$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Summary

The solution(s) found are the following

$$y = -\frac{1}{\arctan(x) - c_1} \tag{1}$$

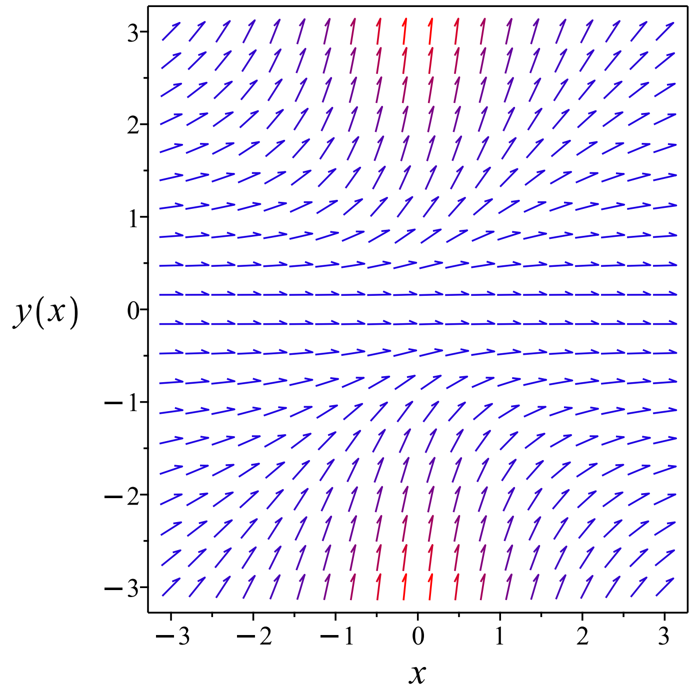


Figure 7: Slope field plot

Verification of solutions

$$y = -\frac{1}{\arctan(x) - c_1}$$

Verified OK.

1.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y^2}\right) \mathrm{d}y = \left(\frac{1}{x^2+1}\right) \mathrm{d}x$$
$$\left(-\frac{1}{x^2+1}\right) \mathrm{d}x + \left(\frac{1}{y^2}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{1}{x^2+1}$$
$$N(x, y) = \frac{1}{y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x^2+1}\right) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^2}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x\\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{1}{x^2+1} \, \mathrm{d}x\end{aligned}$$

$$\phi = -\arctan(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2}$. Therefore equation (4) becomes

$$\frac{1}{y^2} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^2}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y) \, \mathrm{d}y &= \int \left(\frac{1}{y^2}\right) \mathrm{d}y\\ f(y) &= -\frac{1}{y} + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\arctan(x) - \frac{1}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\arctan(x) - \frac{1}{y}$$

Summary

The solution(s) found are the following

$$-\arctan(x) - \frac{1}{y} = c_1 \tag{1}$$

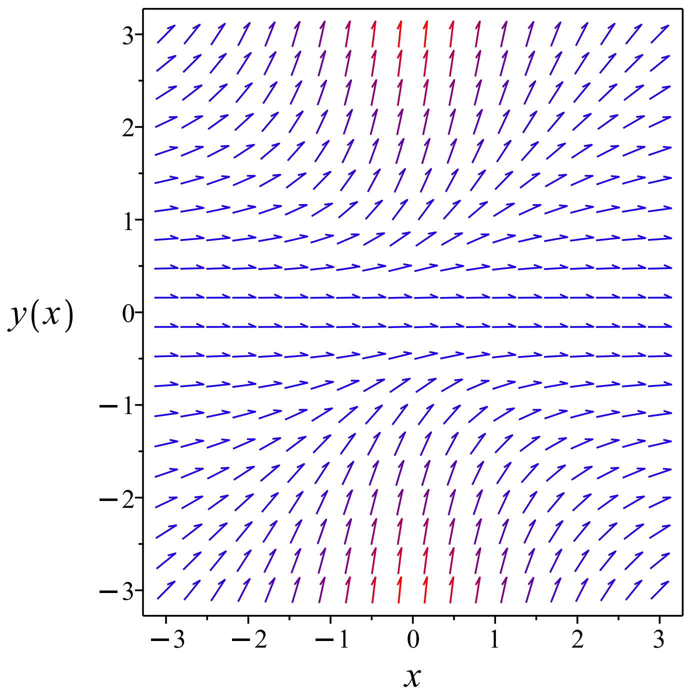


Figure 8: Slope field plot

Verification of solutions

$$-\arctan(x) - \frac{1}{y} = c_1$$

Verified OK.

1.2.4 Solving as riccati ode

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y)\\
&= \frac{y^2}{x^2+1}
\end{aligned}
$$

This is a Riccati ODE. Comparing the ODE to solve

$$
y' = \frac{y^2}{x^2+1}
$$

With Riccati ODE standard form

$$
y' = f_0(x) + f_1(x)y + f_2(x)y^2
$$

Shows that $f_0(x) = 0$, $f_1(x) = 0$ and $f_2(x) = \frac{1}{x^2+1}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u}\\
&= \frac{-u'}{\frac{u}{x^2+1}} \qquad (1)
\end{aligned}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \qquad (2)
$$

But

$$
\begin{aligned}
f_2' &= -\frac{2x}{(x^2+1)^2}\\
f_1 f_2 &= 0\\
f_2^2 f_0 &= 0
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
\frac{u''(x)}{x^2+1} + \frac{2xu'(x)}{(x^2+1)^2} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = c_1 + c_2 \arctan(x)
$$

The above shows that

$$u'(x) = \frac{c_2}{x^2 + 1}$$

Using the above in (1) gives the solution

$$y = -\frac{c_2}{c_1 + c_2 \arctan(x)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{1}{c_3 + \arctan(x)}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{c_3 + \arctan(x)} \tag{1}$$

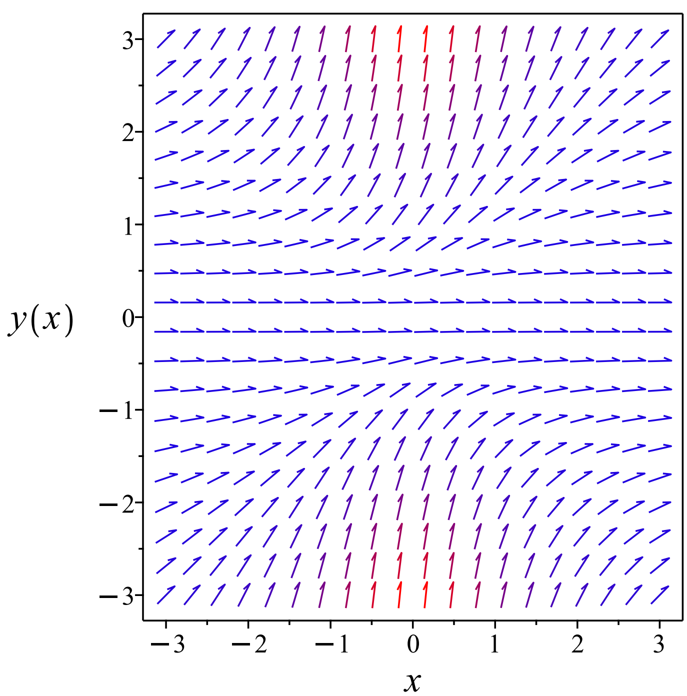


Figure 9: Slope field plot

Verification of solutions

$$y = -\frac{1}{c_3 + \arctan(x)}$$

Verified OK.

1.2.5 Maple step by step solution

Let's solve

$y' - \frac{y^2}{x^2+1} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2} = \frac{1}{x^2+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2}dx = \int \frac{1}{x^2+1}dx + c_1$

- Evaluate integral

  $-\frac{1}{y} = \arctan(x) + c_1$

- Solve for y

  $y = -\frac{1}{\arctan(x)+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x)=y(x)^2/(x^2+1),y(x), singsol=all)
```

$$y(x) = \frac{1}{-\arctan(x) + c_1}$$

✓ Solution by Mathematica

Time used: 0.147 (sec). Leaf size: 19

```
DSolve[y'[x]==y[x]^2/(x^2+1),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{\arctan(x) + c_1}$$
$$y(x) \to 0$$

1.3 problem 3

Internal problem ID [2546]
Internal file name [`OUTPUT/2038_Sunday_June_05_2022_02_45_49_AM_68088448/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.4, page 36
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{e^{y+x}y' = 1}$$

1.3.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= e^{-y}e^{-x} \end{aligned}$$

Where $f(x) = e^{-x}$ and $g(y) = e^{-y}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{e^{-y}}\,dy &= e^{-x}\,dx \\ \int \frac{1}{e^{-y}}\,dy &= \int e^{-x}\,dx \\ e^{y} &= -e^{-x} + c_1 \end{aligned}$$

Which results in

$$y = \ln(-1 + c_1 e^x) - x$$

Summary

The solution(s) found are the following

$$y = \ln(-1 + c_1 e^x) - x \tag{1}$$

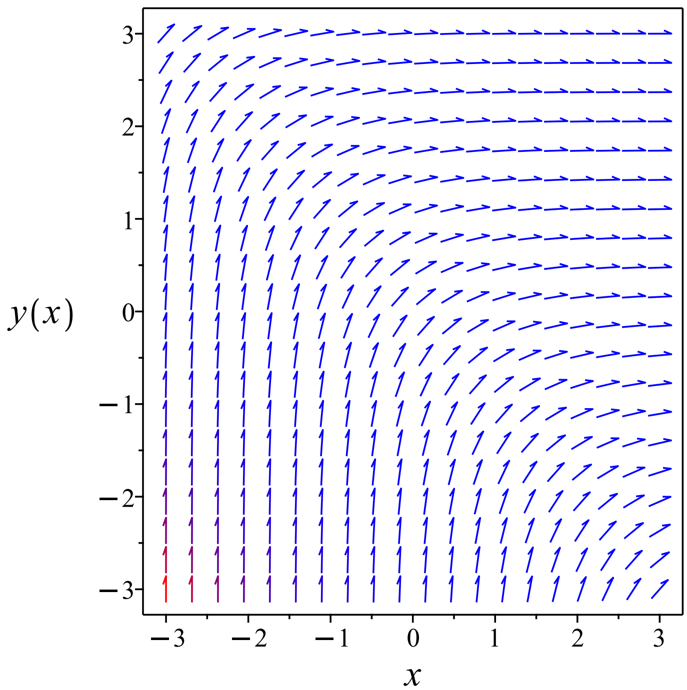


Figure 10: Slope field plot

Verification of solutions

$$y = \ln(-1 + c_1 e^x) - x$$

Verified OK.

1.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = e^{-y-x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 7: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = \mathrm{e}^{x}$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\mathrm{e}^{x}} dx \end{aligned}$$

Which results in

$$S = -\mathrm{e}^{-x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \mathrm{e}^{-y-x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \mathrm{e}^{-x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = e^{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-e^{-x} = e^{y} + c_1$$

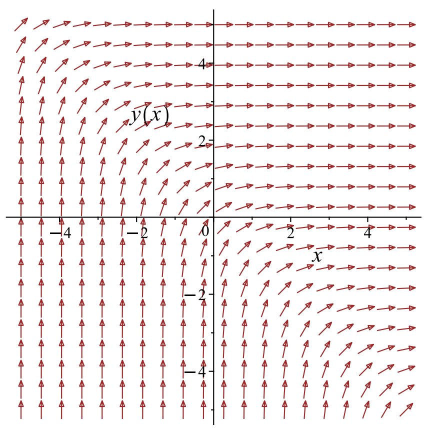
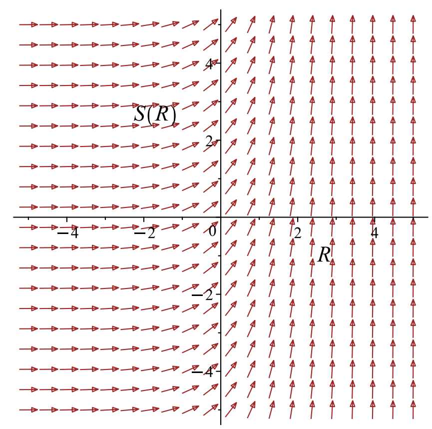
Which simplifies to

$$-e^{-x} = e^{y} + c_1$$

Which gives

$$y = \ln\left(-c_1 e^{x} - 1\right) - x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = \mathrm{e}^{-y-x}$
</td><td>$R = y$
$S = -\mathrm{e}^{-x}$</td><td>$\frac{dS}{dR} = \mathrm{e}^{R}$
</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \ln\left(-c_1 \mathrm{e}^{x} - 1\right) - x \tag{1}$$

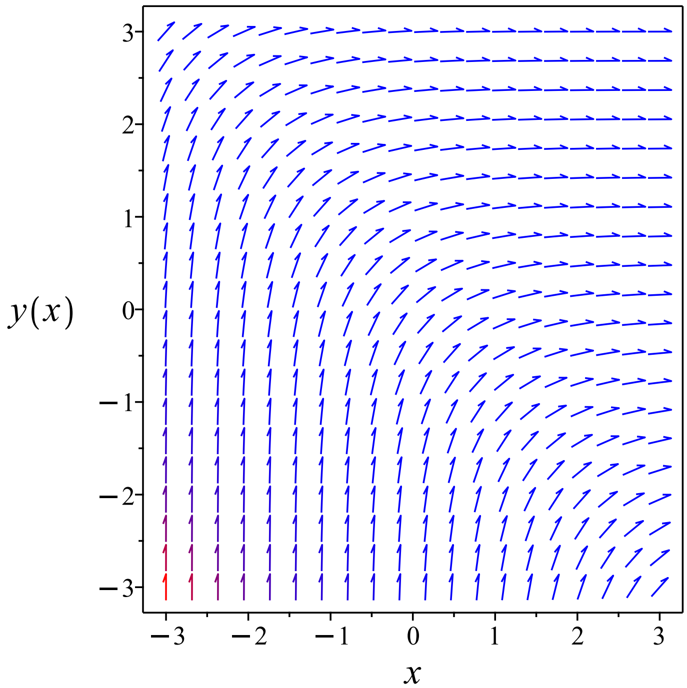


Figure 11: Slope field plot

Verification of solutions

$$y = \ln\left(-c_1 e^x - 1\right) - x$$

Verified OK.

1.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(\mathrm{e}^{y})\,\mathrm{d}y = \left(\mathrm{e}^{-x}\right)\mathrm{d}x$$
$$\left(-\mathrm{e}^{-x}\right)\mathrm{d}x + (\mathrm{e}^{y})\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\mathrm{e}^{-x}$$
$$N(x, y) = \mathrm{e}^{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\mathrm{e}^{-x}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\mathrm{e}^{y})\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\mathrm{e}^{-x} \, \mathrm{d}x$$
$$\phi = \mathrm{e}^{-x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{y}$. Therefore equation (4) becomes

$$\mathrm{e}^{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \mathrm{e}^{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (\mathrm{e}^{y}) \, \mathrm{d}y$$
$$f(y) = \mathrm{e}^{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{-x} + \mathrm{e}^{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = e^{-x} + e^{y}$$

The solution becomes

$$y = \ln(-1 + c_1 e^{x}) - x$$

Summary

The solution(s) found are the following

$$y = \ln(-1 + c_1 e^{x}) - x \tag{1}$$

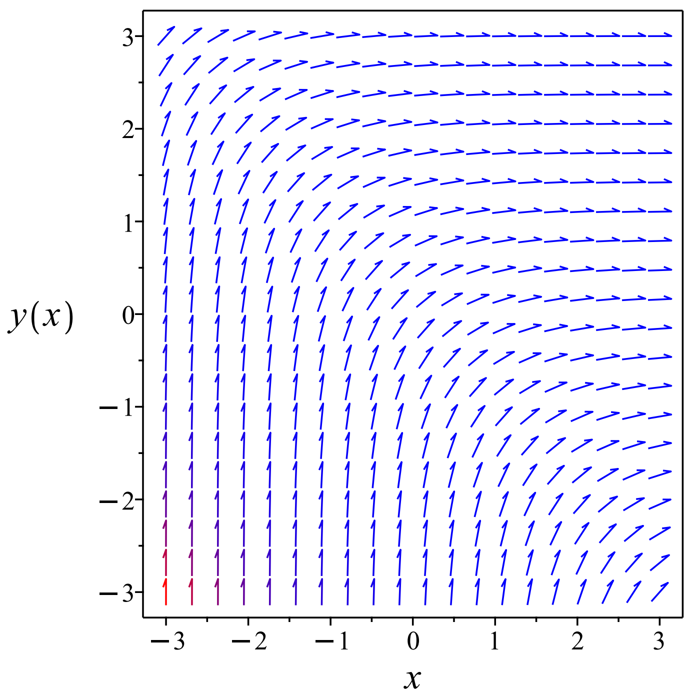


Figure 12: Slope field plot

Verification of solutions

$$y = \ln(-1 + c_1 e^{x}) - x$$

Verified OK.

1.3.4 Maple step by step solution

Let's solve

$e^{y+x}y' = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y'e^y = \frac{1}{e^x}$

- Integrate both sides with respect to x

  $\int y'e^y dx = \int \frac{1}{e^x} dx + c_1$

- Evaluate integral

  $e^y = -\frac{1}{e^x} + c_1$

- Solve for y

  $y = \ln(-1 + c_1 e^x) - x$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(exp(x+y(x))*diff(y(x),x)-1=0,y(x), singsol=all)
```

$$y(x) = \ln(e^x c_1 - 1) - x$$

✓ Solution by Mathematica

Time used: 0.089 (sec). Leaf size: 16

```
DSolve[Exp[x+y[x]]*y'[x]-1==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \log\left(-e^{-x} + c_1\right)$$

1.4 problem 4

Internal problem ID [2547]
Internal file name [`OUTPUT/2039_Sunday_June_05_2022_02_45_52_AM_89899606/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.4, page 36
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{y}{\ln(x)\, x} = 0}$$

1.4.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y}{\ln(x)\, x} \end{aligned}$$

Where $f(x) = \frac{1}{\ln(x)x}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\,dy &= \frac{1}{\ln(x)\,x}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{1}{\ln(x)\,x}\,dx \\
\ln(y) &= \ln(\ln(x)) + c_1 \\
y &= \mathrm{e}^{\ln(\ln(x))+c_1} \\
&= c_1 \ln(x)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 \ln(x) \tag{1}$$

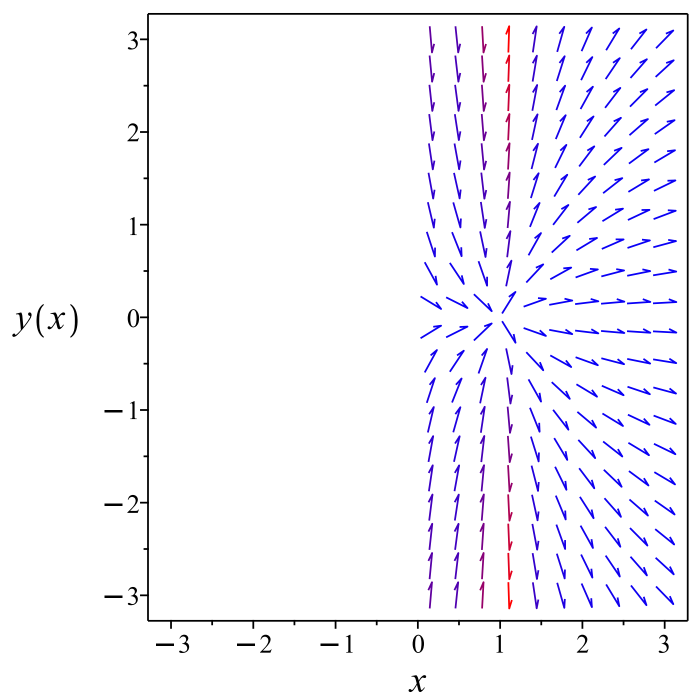


Figure 13: Slope field plot

Verification of solutions

$$y = c_1 \ln(x)$$

Verified OK.

1.4.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{\ln(x)\, x} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' - \frac{y}{\ln(x)\, x} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{1}{\ln(x)x}dx} \\ &= \frac{1}{\ln(x)} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}\mu y &= 0 \\ \frac{d}{dx}\left(\frac{y}{\ln(x)}\right) &= 0 \end{aligned}$$

Integrating gives

$$\frac{y}{\ln(x)} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\ln(x)}$ results in

$$y = c_1 \ln(x)$$

Summary
The solution(s) found are the following

$$y = c_1 \ln(x) \tag{1}$$

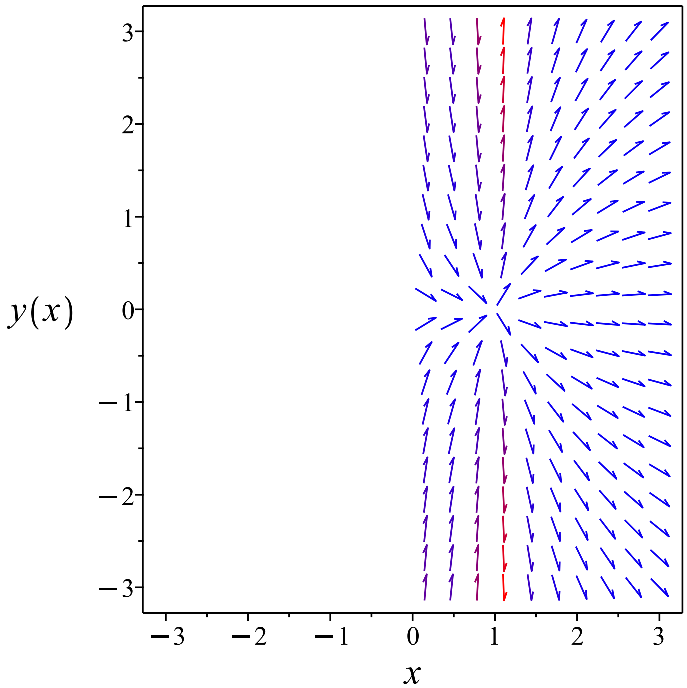


Figure 14: Slope field plot

Verification of solutions

$$y = c_1 \ln(x)$$

Verified OK.

1.4.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{u(x)}{\ln(x)} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u(\ln(x)-1)}{\ln(x)\,x} \end{aligned}$$

Where $f(x) = -\frac{\ln(x)-1}{\ln(x)x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{\ln(x)-1}{\ln(x)\,x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{\ln(x)-1}{\ln(x)\,x}\,dx \\
\ln(u) &= -\ln(x) + \ln(\ln(x)) + c_2 \\
u &= e^{-\ln(x)+\ln(\ln(x))+c_2} \\
&= c_2 e^{-\ln(x)+\ln(\ln(x))}
\end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_2 \ln(x)}{x}$$

Therefore the solution y is

$$\begin{aligned}
y &= xu \\
&= c_2 \ln(x)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2 \ln(x) \tag{1}$$

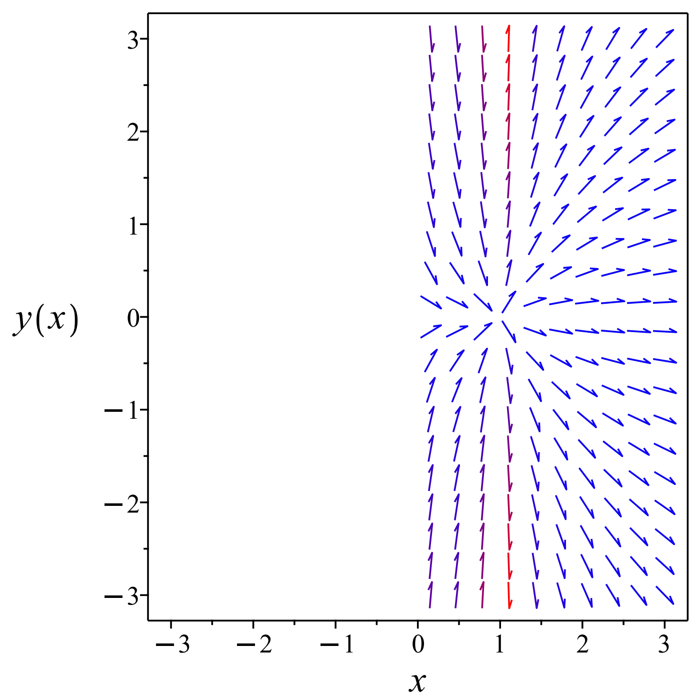


Figure 15: Slope field plot

Verification of solutions

$$y = c_2 \ln(x)$$

Verified OK.

1.4.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y}{\ln(x)\, x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 10: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \ln(x) \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\ln(x)} dy \end{aligned}$$

Which results in

$$S = \frac{y}{\ln(x)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y}{\ln(x) x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{\ln(x)^2 x} \\ S_y &= \frac{1}{\ln(x)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{\ln(x)} = c_1$$

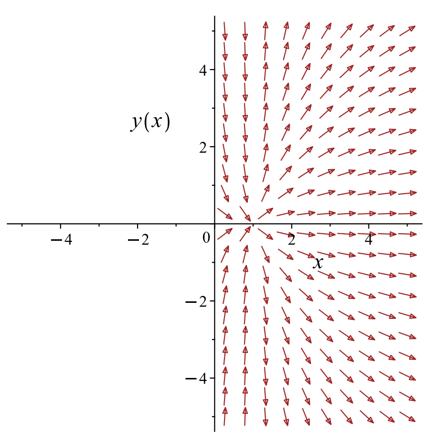
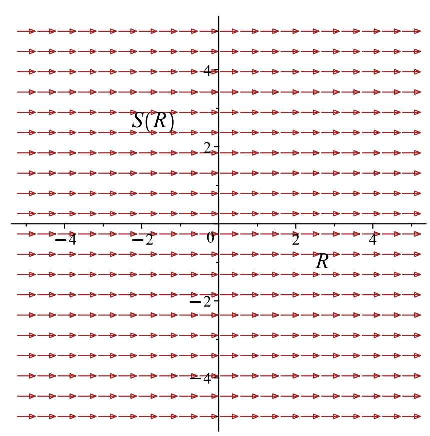
Which simplifies to

$$\frac{y}{\ln(x)} = c_1$$

Which gives

$$y = c_1 \ln(x)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{\ln(x)x}$ 	$R = x$ $S = \frac{y}{\ln(x)}$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$y = c_1 \ln(x) \tag{1}$$

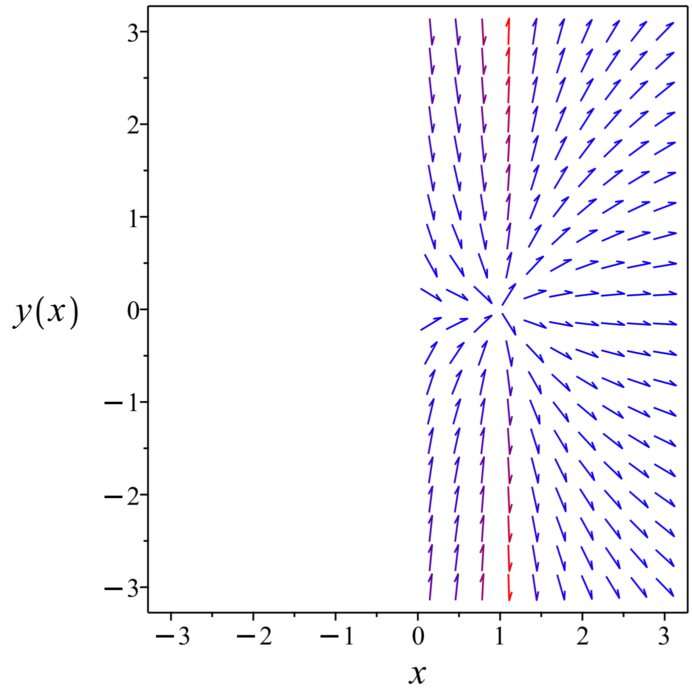


Figure 16: Slope field plot

Verification of solutions

$$y = c_1 \ln(x)$$

Verified OK.

1.4.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y}\right)\mathrm{d}y = \left(\frac{1}{\ln(x)\,x}\right)\mathrm{d}x$$
$$\left(-\frac{1}{\ln(x)\,x}\right)\mathrm{d}x + \left(\frac{1}{y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{1}{\ln(x)\,x}$$
$$N(x,y) = \frac{1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{1}{\ln(x)\,x}\right)$$
$$= 0$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{y}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\frac{1}{\ln(x)\, x} \mathrm{d}x$$
$$\phi = -\ln(\ln(x)) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(\ln(x)) + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(\ln(x)) + \ln(y)$$

The solution becomes

$$y = e^{c_1} \ln(x)$$

Summary

The solution(s) found are the following

$$y = e^{c_1} \ln(x) \tag{1}$$

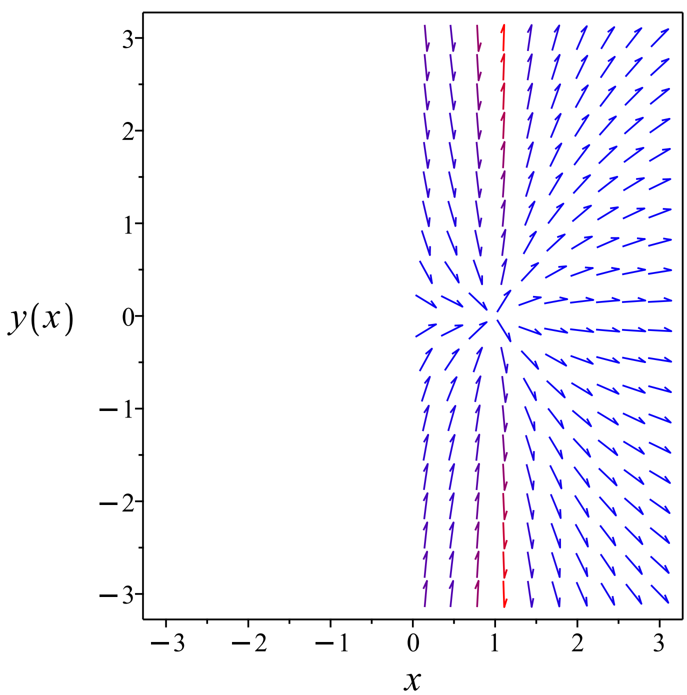


Figure 17: Slope field plot

Verification of solutions

$$y = e^{c_1} \ln(x)$$

Verified OK.

1.4.6 Maple step by step solution

Let's solve

$y' - \frac{y}{\ln(x) x} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{\ln(x) x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \frac{1}{\ln(x) x} dx + c_1$

- Evaluate integral

  $\ln(y) = \ln(\ln(x)) + c_1$

- Solve for y

  $y = e^{c_1} \ln(x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 8

```
dsolve(diff(y(x),x)=y(x)/(x*ln(x)),y(x), singsol=all)
```

$$y(x) = \ln(x) c_1$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 15

```
DSolve[y'[x]==y[x]/(x*Log[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \log(x)$$
$$y(x) \to 0$$

1.5 problem 5

Internal problem ID [2548]
Internal file name [`OUTPUT/2040_Sunday_June_05_2022_02_45_54_AM_15637451/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.4, page 36
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y-(x-2)\,y'=0}$$

1.5.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y)\\
&= f(x)g(y)\\
&= \frac{y}{x-2}
\end{aligned}$$

Where $f(x) = \frac{1}{x-2}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\,dy &= \frac{1}{x-2}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{1}{x-2}\,dx \\
\ln(y) &= \ln(x-2) + c_1 \\
y &= \mathrm{e}^{\ln(x-2)+c_1} \\
&= c_1(x-2)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1(x-2) \tag{1}$$

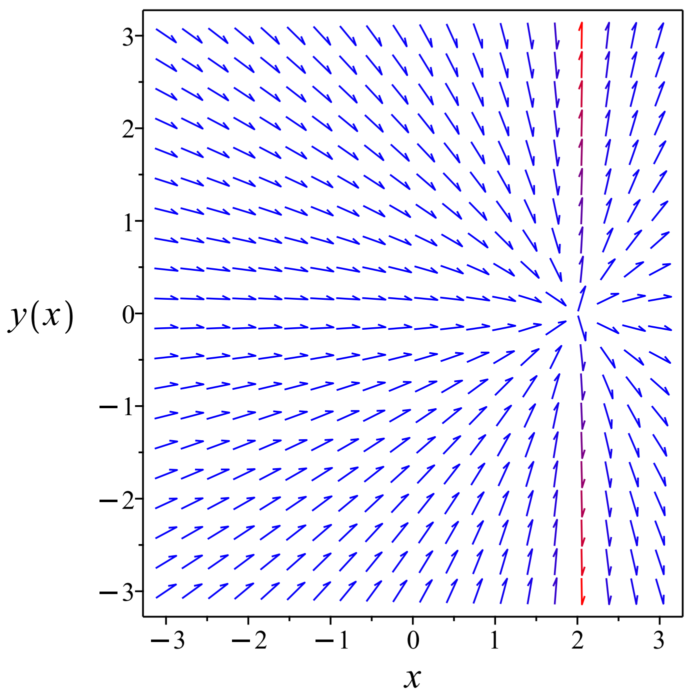


Figure 18: Slope field plot

Verification of solutions

$$y = c_1(x-2)$$

Verified OK.

1.5.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x-2} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' - \frac{y}{x-2} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{1}{x-2}dx} \\ &= \frac{1}{x-2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}\mu y &= 0 \\ \frac{d}{dx}\left(\frac{y}{x-2}\right) &= 0 \end{aligned}$$

Integrating gives

$$\frac{y}{x-2} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x-2}$ results in

$$y = c_1(x-2)$$

Summary
The solution(s) found are the following

$$y = c_1(x-2) \tag{1}$$

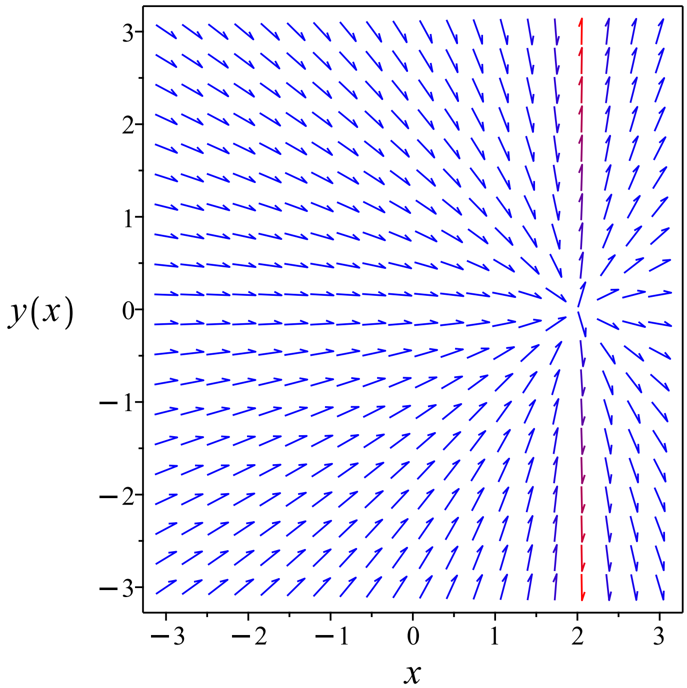


Figure 19: Slope field plot

Verification of solutions

$$y = c_1(x - 2)$$

Verified OK.

1.5.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x - (x - 2)\,(u'(x)\,x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{2u}{x\,(x - 2)} \end{aligned}$$

Where $f(x) = \frac{2}{(x-2)x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= \frac{2}{(x-2)\,x}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{2}{(x-2)\,x}\,dx \\
\ln(u) &= \ln(x-2) - \ln(x) + c_2 \\
u &= \mathrm{e}^{\ln(x-2)-\ln(x)+c_2} \\
&= c_2 \mathrm{e}^{\ln(x-2)-\ln(x)}
\end{aligned}$$

Which simplifies to

$$u(x) = c_2\left(1 - \frac{2}{x}\right)$$

Therefore the solution y is

$$\begin{aligned}
y &= ux \\
&= x c_2\left(1 - \frac{2}{x}\right)
\end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = x c_2\left(1 - \frac{2}{x}\right) \tag{1}$$

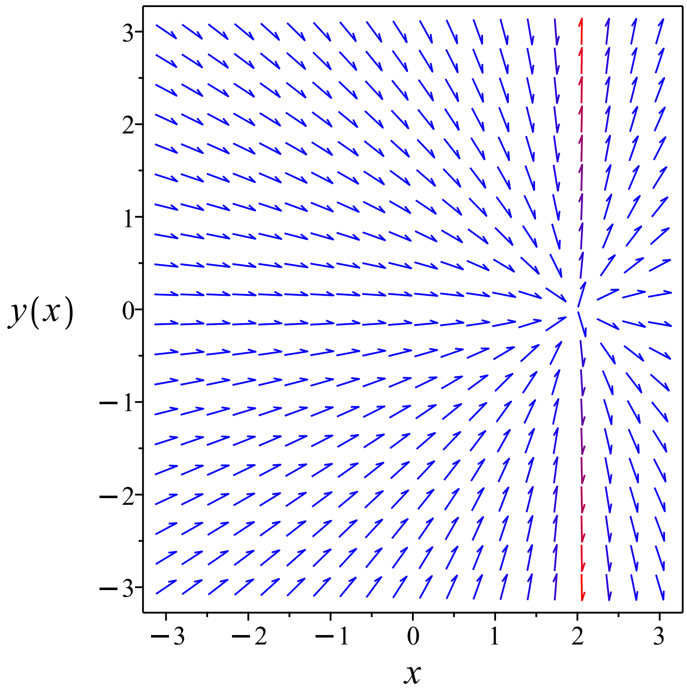


Figure 20: Slope field plot

Verification of solutions

$$y = xc_2\left(1 - \frac{2}{x}\right)$$

Verified OK.

1.5.4 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{Y(X) + y_0}{X + x_0 - 2}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 2$$
$$y_0 = 0$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{Y(X)}{X}$$

In canonical form, the ODE is

$$\begin{aligned}Y' &= F(X, Y)\\ &= \frac{Y}{X}\end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = Y$ and $N = X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= u\\ \frac{\mathrm{d}u}{\mathrm{d}X} &= 0\end{aligned}$$

Or

$$\frac{d}{dX}u(X) = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. Integrating both sides gives

$$\begin{aligned}u(X) &= \int 0 \,\mathrm{d}X\\ &= c_2\end{aligned}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$Y(X) = Xc_2$$

Using the solution for $Y(X)$

$$Y(X) = Xc_2$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y$$
$$X = x + 2$$

Then the solution in y becomes

$$y = c_2(x - 2)$$

Summary
The solution(s) found are the following

$$y = c_2(x - 2) \tag{1}$$

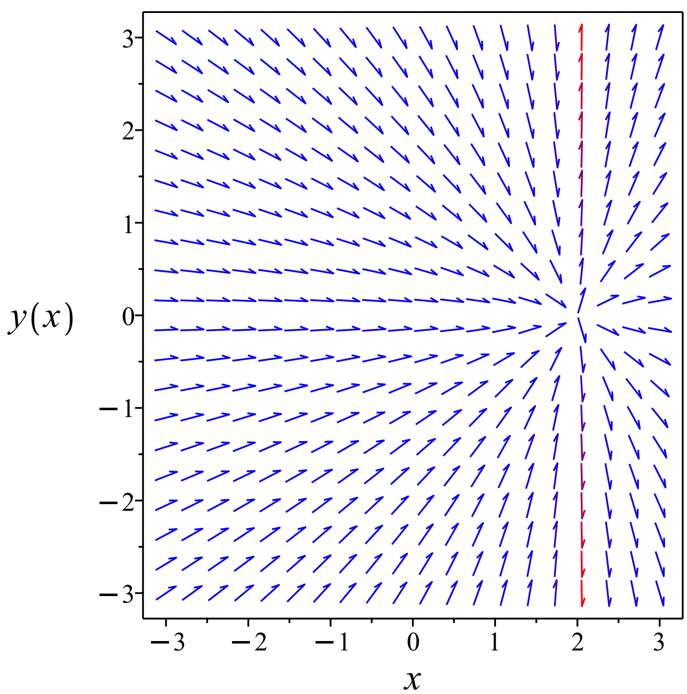


Figure 21: Slope field plot

Verification of solutions

$$y = c_2(x - 2)$$

Verified OK.

1.5.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{y}{x-2} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 13: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= x-2
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{x-2} dy
\end{aligned}
$$

Which results in

$$
S = \frac{y}{x-2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{y}{x-2}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{y}{(x-2)^2} \\
S_y &= \frac{1}{x-2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{y}{x-2} = c_1
$$

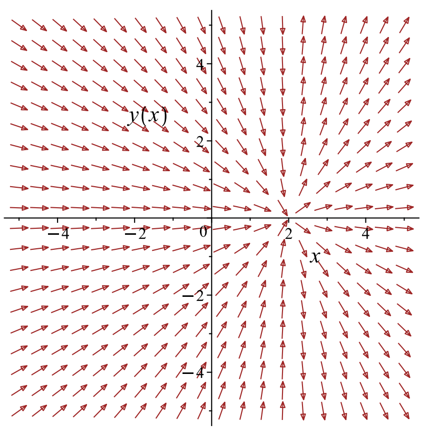
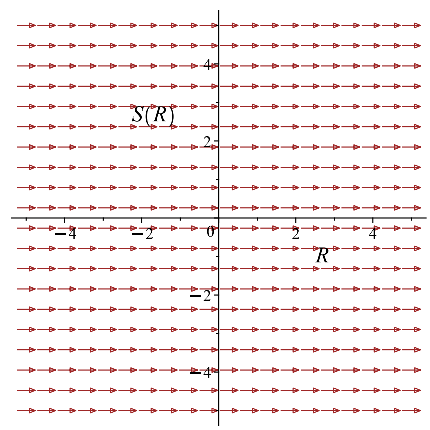
Which simplifies to

$$
\frac{y}{x-2} = c_1
$$

Which gives

$$
y = c_1(x-2)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{x-2}$ 	$R = x$ $S = \frac{y}{x-2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = c_1(x - 2) \tag{1}$$

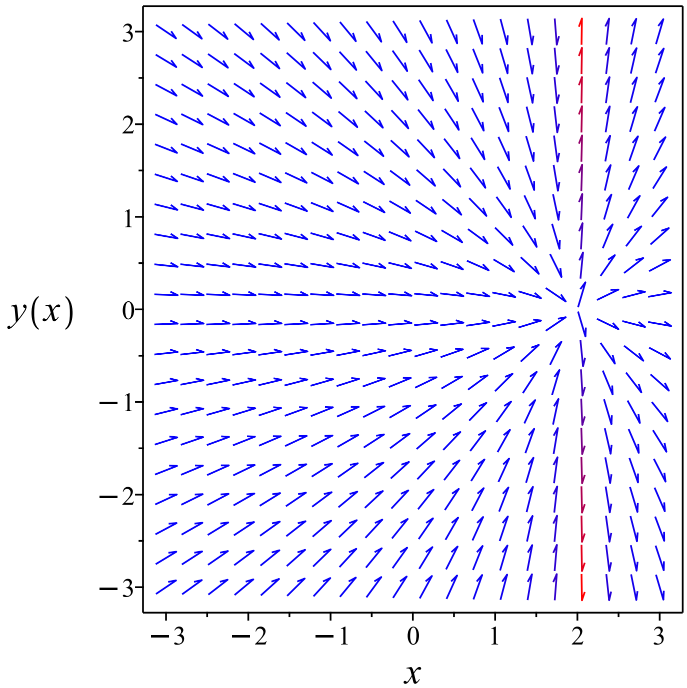


Figure 22: Slope field plot

Verification of solutions

$$
y = c_1(x-2)
$$

Verified OK.

1.5.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$
M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y) = 0
$$

Hence

$$
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y}\right)\mathrm{d}y &= \left(\frac{1}{x-2}\right)\mathrm{d}x\\ \left(-\frac{1}{x-2}\right)\mathrm{d}x + \left(\frac{1}{y}\right)\mathrm{d}y &= 0 \qquad\qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{1}{x-2}\\ N(x, y) &= \frac{1}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x-2}\right)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M\, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int -\frac{1}{x-2}\, \mathrm{d}x \\
\phi &= -\ln(x-2) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$
\frac{1}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\, \mathrm{d}y &= \int \left(\frac{1}{y}\right) \mathrm{d}y \\
f(y) &= \ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x-2) + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x-2) + \ln(y)$$

The solution becomes

$$y = e^{c_1}(x-2)$$

Summary

The solution(s) found are the following

$$y = e^{c_1}(x-2) \tag{1}$$

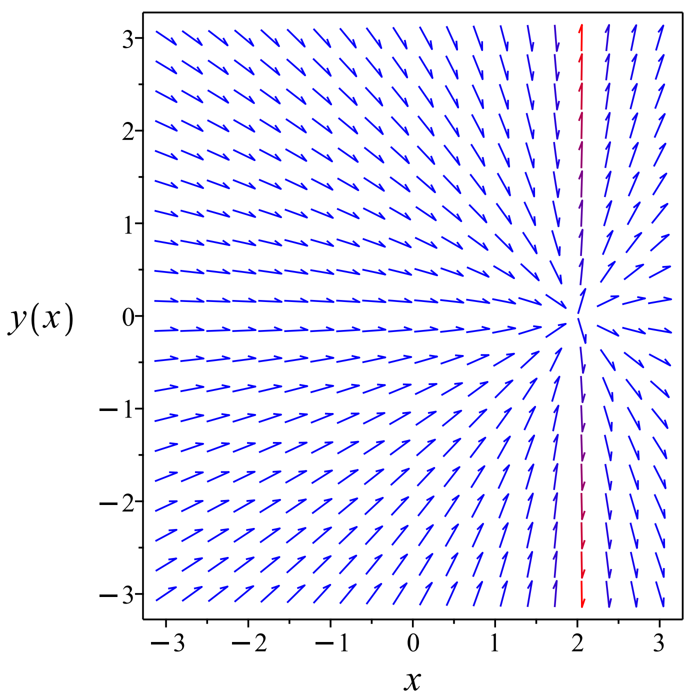


Figure 23: Slope field plot

Verification of solutions

$$y = e^{c_1}(x-2)$$

Verified OK.

1.5.7 Maple step by step solution

Let's solve

$y - (x-2)\, y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{x-2}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \frac{1}{x-2} dx + c_1$

- Evaluate integral

  $\ln(y) = \ln(x-2) + c_1$

- Solve for y

  $y = \mathrm{e}^{c_1}(x-2)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(y(x)-(x-2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1(-2 + x)$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 16

```
DSolve[y[x]-(x-2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1(x-2)$$
$$y(x) \to 0$$

1.6 problem 6

Internal problem ID [2549]
Internal file name [OUTPUT/2041_Sunday_June_05_2022_02_45_55_AM_40885883/index.tex]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.4, page 36
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - \frac{2x(y-1)}{x^2+3} = 0}$$

1.6.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x(2y-2)}{x^2+3} \end{aligned}$$

Where $f(x) = \frac{x}{x^2+3}$ and $g(y) = 2y - 2$. Integrating both sides gives

$$\frac{1}{2y-2}\, dy = \frac{x}{x^2+3}\, dx$$

$$\int \frac{1}{2y-2}\, dy = \int \frac{x}{x^2+3}\, dx$$

$$\frac{\ln(y-1)}{2} = \frac{\ln(x^2+3)}{2} + c_1$$

Raising both side to exponential gives

$$\sqrt{y-1} = e^{\frac{\ln(x^2+3)}{2}+c_1}$$

Which simplifies to

$$\sqrt{y-1} = c_2\sqrt{x^2+3}$$

Which simplifies to

$$y = c_2^2(x^2+3)\,e^{2c_1} + 1$$

Summary

The solution(s) found are the following

$$y = c_2^2(x^2+3)\,e^{2c_1} + 1 \tag{1}$$

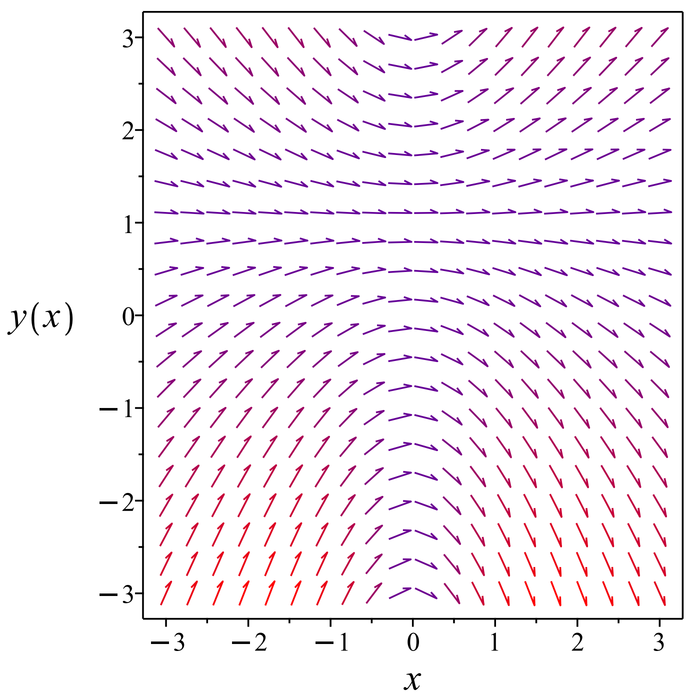


Figure 24: Slope field plot

Verification of solutions

$$y = c_2^2(x^2+3)\,e^{2c_1} + 1$$

Verified OK.

1.6.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{2x}{x^2+3} \\ q(x) &= -\frac{2x}{x^2+3} \end{aligned}$$

Hence the ode is

$$y' - \frac{2xy}{x^2+3} = -\frac{2x}{x^2+3}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{2x}{x^2+3}dx} \\ &= \frac{1}{x^2+3} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(-\frac{2x}{x^2+3}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x^2+3}\right) &= \left(\frac{1}{x^2+3}\right)\left(-\frac{2x}{x^2+3}\right) \\ \mathrm{d}\left(\frac{y}{x^2+3}\right) &= \left(-\frac{2x}{(x^2+3)^2}\right)\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{y}{x^2+3} &= \int -\frac{2x}{(x^2+3)^2}\,\mathrm{d}x \\ \frac{y}{x^2+3} &= \frac{1}{x^2+3} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2+3}$ results in

$$y = 1 + c_1(x^2+3)$$

Summary
The solution(s) found are the following

$$y = 1 + c_1(x^2+3) \tag{1}$$

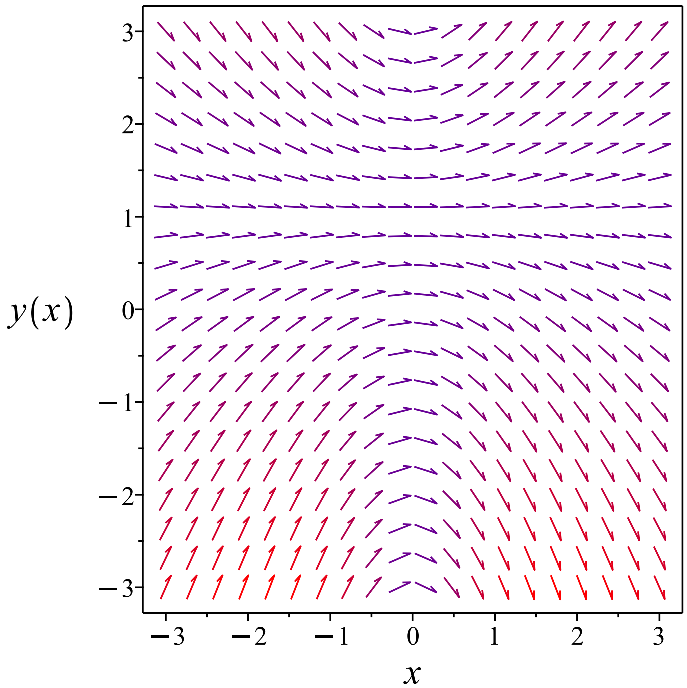


Figure 25: Slope field plot

Verification of solutions

$$y = 1 + c_1(x^2 + 3)$$

Verified OK.

1.6.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{2x(y-1)}{x^2+3}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 16: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x^2 + 3
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^2 + 3} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x^2 + 3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{2x(y-1)}{x^2 + 3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2yx}{(x^2 + 3)^2} \\ S_y &= \frac{1}{x^2 + 3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{2x}{(x^2 + 3)^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{2R}{(R^2 + 3)^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{1}{R^2+3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x^2+3} = \frac{1}{x^2+3} + c_1$$

Which simplifies to

$$\frac{y}{x^2+3} = \frac{1}{x^2+3} + c_1$$

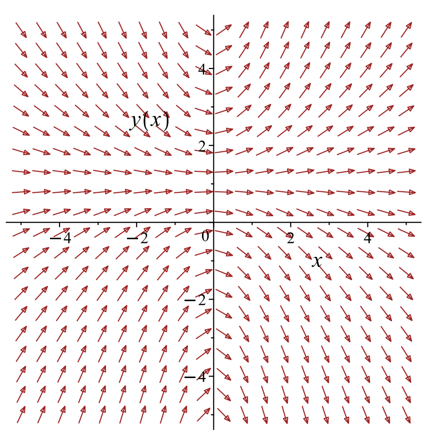
Which gives

$$y = c_1 x^2 + 3c_1 + 1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = \frac{2x(y-1)}{x^2+3}$



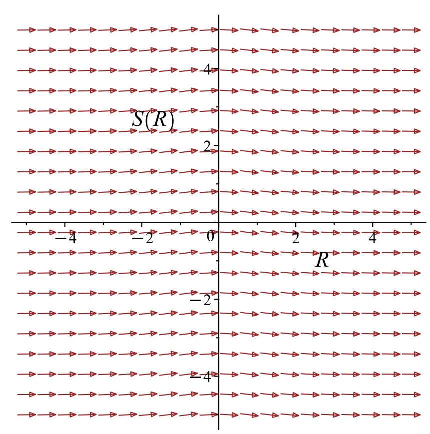
</td><td>

$R = x$

$S = \frac{y}{x^2+3}$

</td><td>

$\frac{dS}{dR} = -\frac{2R}{(R^2+3)^2}$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = c_1 x^2 + 3c_1 + 1 \tag{1}$$

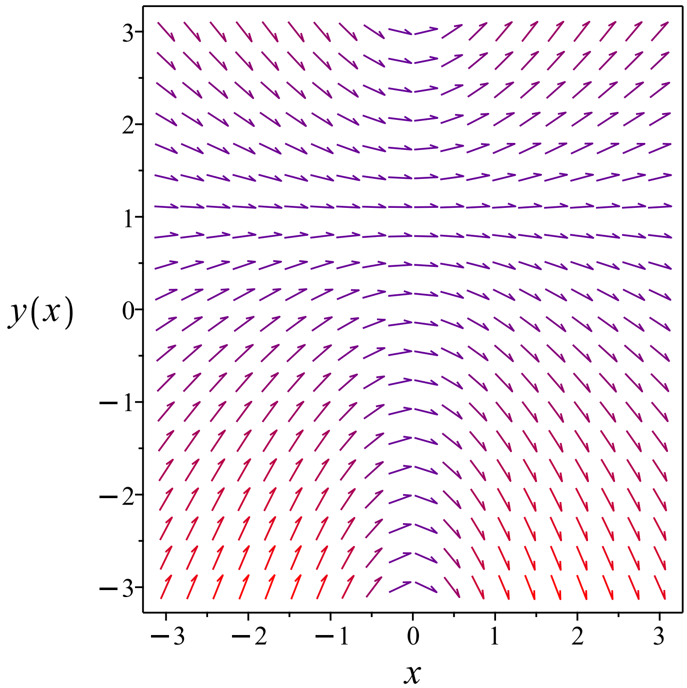


Figure 26: Slope field plot

Verification of solutions

$$y = c_1 x^2 + 3c_1 + 1$$

Verified OK.

1.6.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{2y-2}\right)\mathrm{d}y &= \left(\frac{x}{x^2+3}\right)\mathrm{d}x \\
\left(-\frac{x}{x^2+3}\right)\mathrm{d}x + \left(\frac{1}{2y-2}\right)\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{x}{x^2+3} \\
N(x,y) &= \frac{1}{2y-2}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{x^2+3}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{2y-2}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{x}{x^2+3} \, \mathrm{d}x \\
\phi &= -\frac{\ln(x^2+3)}{2} + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{2y-2}$. Therefore equation (4) becomes

$$
\frac{1}{2y-2} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{2y-2}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{2y-2}\right) \mathrm{d}y$$
$$f(y) = \frac{\ln(y-1)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x^2+3)}{2} + \frac{\ln(y-1)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x^2+3)}{2} + \frac{\ln(y-1)}{2}$$

The solution becomes

$$y = e^{2c_1}x^2 + 3\,e^{2c_1} + 1$$

Summary

The solution(s) found are the following

$$y = e^{2c_1}x^2 + 3\,e^{2c_1} + 1 \tag{1}$$

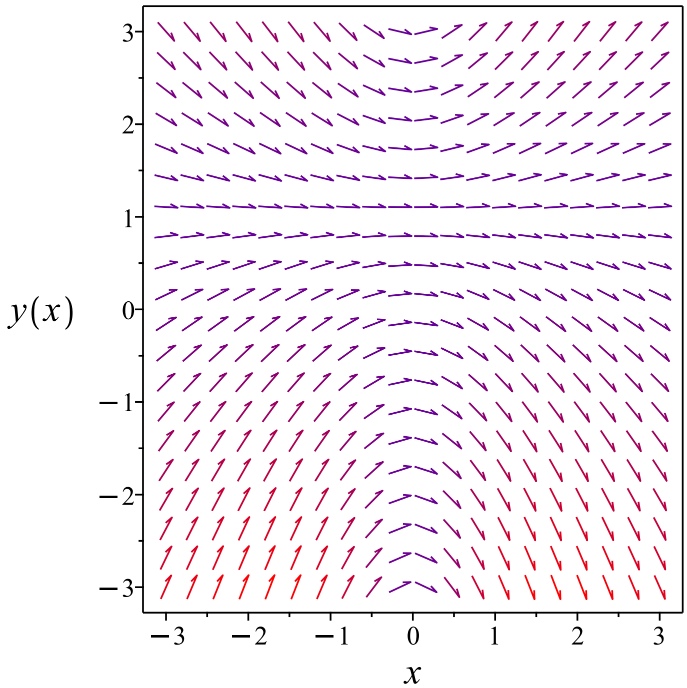


Figure 27: Slope field plot

Verification of solutions

$$y = e^{2c_1}x^2 + 3\,e^{2c_1} + 1$$

Verified OK.

1.6.5 Maple step by step solution

Let's solve

$y' - \frac{2x(y-1)}{x^2+3} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y-1} = \frac{2x}{x^2+3}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y-1}dx = \int \frac{2x}{x^2+3}dx + c_1$

- Evaluate integral

$$\ln(y-1) = \ln(x^2+3) + c_1$$

- Solve for y

$$y = e^{c_1}x^2 + 3\,e^{c_1} + 1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve(diff(y(x),x)=(2*x*(y(x)-1))/(x^2+3),y(x), singsol=all)
```

$$y(x) = c_1x^2 + 3c_1 + 1$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 20

```
DSolve[y'[x]==(2*x*(y[x]-1))/(x^2+3),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 1 + c_1(x^2+3)$$
$$y(x) \to 1$$

1.7 problem 7

Internal problem ID [2550]
Internal file name [`OUTPUT/2042_Sunday_June_05_2022_02_45_57_AM_62513362/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.4, page 36
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y - xy' + 2y'x^2 = 3}$$

1.7.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{-y+3}{x\,(2x-1)} \end{aligned}$$

Where $f(x) = \frac{1}{x(2x-1)}$ and $g(y) = -y + 3$. Integrating both sides gives

$$\begin{aligned} \frac{1}{-y+3}\,dy &= \frac{1}{x\,(2x-1)}\,dx \\ \int \frac{1}{-y+3}\,dy &= \int \frac{1}{x\,(2x-1)}\,dx \\ -\ln(y-3) &= \ln(2x-1) - \ln(x) + c_1 \end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{y-3} = e^{\ln(2x-1)-\ln(x)+c_1}$$

Which simplifies to

$$\frac{1}{y-3} = c_2 e^{\ln(2x-1)-\ln(x)}$$

Which simplifies to

$$y = \frac{3c_2\left(2\,e^{c_1} - \frac{e^{c_1}}{x}\right) + 1}{c_2\left(2\,e^{c_1} - \frac{e^{c_1}}{x}\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{3c_2\left(2\,e^{c_1} - \frac{e^{c_1}}{x}\right) + 1}{c_2\left(2\,e^{c_1} - \frac{e^{c_1}}{x}\right)} \tag{1}$$

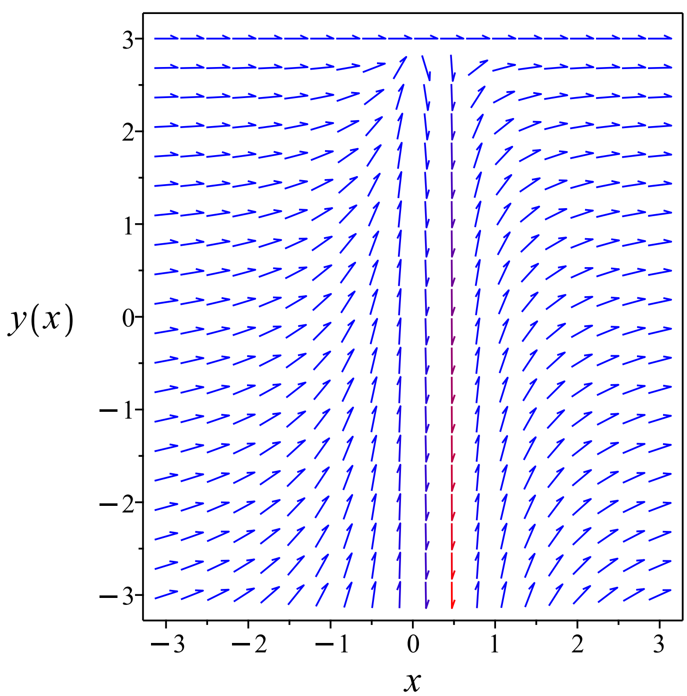


Figure 28: Slope field plot

Verification of solutions

$$y = \frac{3c_2\left(2\,\mathrm{e}^{c_1} - \frac{\mathrm{e}^{c_1}}{x}\right) + 1}{c_2\left(2\,\mathrm{e}^{c_1} - \frac{\mathrm{e}^{c_1}}{x}\right)}$$

Verified OK.

1.7.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x\,(2x-1)} \\ q(x) &= \frac{3}{x\,(2x-1)} \end{aligned}$$

Hence the ode is

$$y' + \frac{y}{x\,(2x-1)} = \frac{3}{x\,(2x-1)}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int \frac{1}{x(2x-1)}dx} \\ &= \mathrm{e}^{\ln(2x-1)-\ln(x)} \end{aligned}$$

Which simplifies to

$$\mu = \frac{2x-1}{x}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{3}{x\,(2x-1)}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(2x-1)\,y}{x}\right) &= \left(\frac{2x-1}{x}\right)\left(\frac{3}{x\,(2x-1)}\right) \\ \mathrm{d}\left(\frac{(2x-1)\,y}{x}\right) &= \left(\frac{3}{x^2}\right)\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{(2x-1)\,y}{x} &= \int \frac{3}{x^2}\,\mathrm{d}x \\ \frac{(2x-1)\,y}{x} &= -\frac{3}{x} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{2x-1}{x}$ results in

$$y = -\frac{3}{2x-1} + \frac{c_1 x}{2x-1}$$

which simplifies to

$$y = \frac{c_1 x - 3}{2x - 1}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 x - 3}{2x - 1} \tag{1}$$

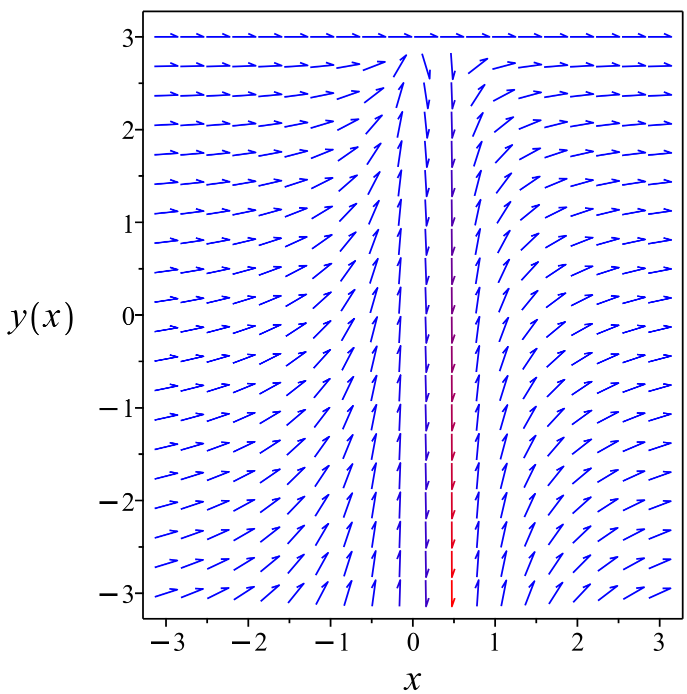


Figure 29: Slope field plot

Verification of solutions

$$y = \frac{c_1 x - 3}{2x - 1}$$

Verified OK.

1.7.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{y-3}{x\left(2x-1\right)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 19: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int b f(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \mathrm{e}^{-\ln(2x-1)+\ln(x)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\mathrm{e}^{-\ln(2x-1)+\ln(x)}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{(2x-1)\,y}{x}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{y-3}{x\,(2x-1)}
$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{y}{x^2} \\
S_y &= \frac{2x-1}{x}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{3}{x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{3}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{3}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{(2x-1)\,y}{x} = -\frac{3}{x} + c_1$$

Which simplifies to

$$\frac{(2x-1)\,y}{x} = -\frac{3}{x} + c_1$$

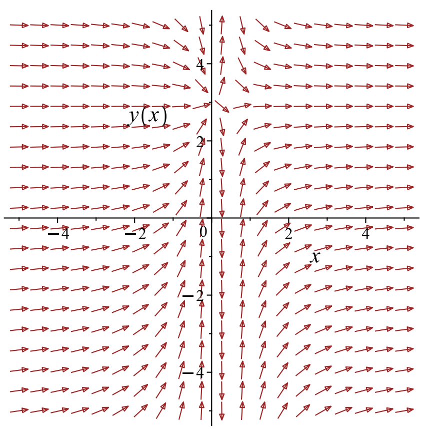
Which gives

$$y = \frac{c_1 x - 3}{2x-1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = -\frac{y-3}{x(2x-1)}$



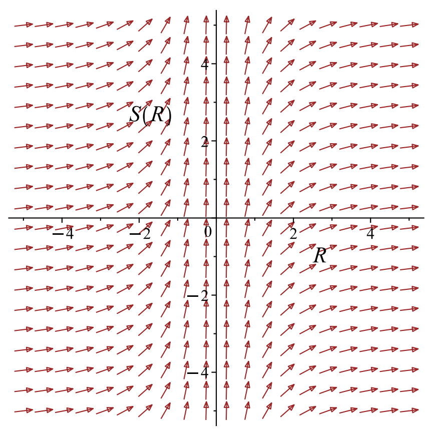
</td><td>

$R = x$

$S = \frac{(2x-1)\,y}{x}$

</td><td>

$\frac{dS}{dR} = \frac{3}{R^2}$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \frac{c_1 x - 3}{2x - 1} \tag{1}$$

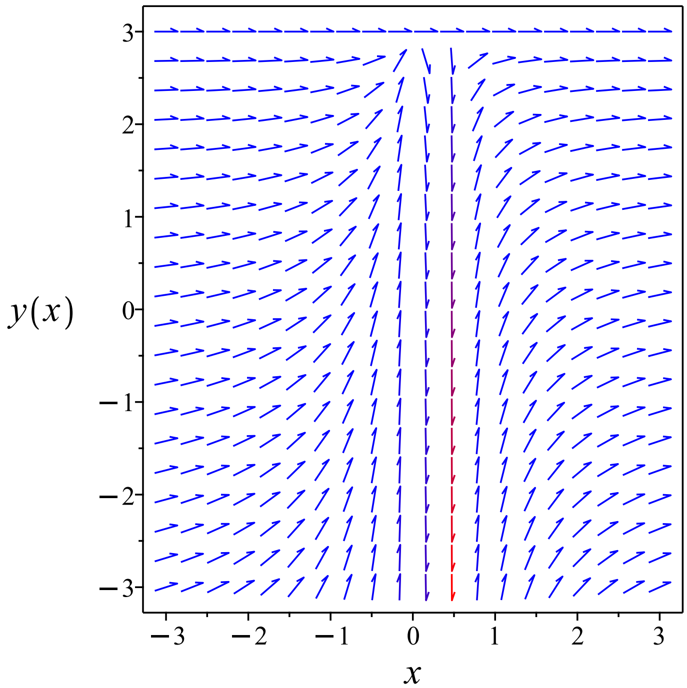


Figure 30: Slope field plot

Verification of solutions

$$y = \frac{c_1 x - 3}{2x - 1}$$

Verified OK.

1.7.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{-y+3}\right)\mathrm{d}y &= \left(\frac{1}{x\,(2x-1)}\right)\mathrm{d}x \\
\left(-\frac{1}{x\,(2x-1)}\right)\mathrm{d}x + \left(\frac{1}{-y+3}\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{1}{x\,(2x-1)} \\
N(x,y) &= \frac{1}{-y+3}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x\,(2x-1)}\right) \\
&= 0
\end{aligned}
$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{-y+3}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int M \, dx$$
$$\int \frac{\partial \phi}{\partial x} \, dx = \int -\frac{1}{x(2x-1)} \, dx$$
$$\phi = -\ln(2x-1) + \ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{-y+3}$. Therefore equation (4) becomes

$$\frac{1}{-y+3} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y-3}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, dy = \int \left(-\frac{1}{y-3}\right) dy$$
$$f(y) = -\ln(y-3) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(2x-1) + \ln(x) - \ln(y-3) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(2x-1) + \ln(x) - \ln(y-3)$$

The solution becomes

$$y = \frac{(6\,\mathrm{e}^{c_1}x - 3\,\mathrm{e}^{c_1} + x)\,\mathrm{e}^{-c_1}}{2x-1}$$

Summary

The solution(s) found are the following

$$y = \frac{(6\,\mathrm{e}^{c_1}x - 3\,\mathrm{e}^{c_1} + x)\,\mathrm{e}^{-c_1}}{2x-1} \tag{1}$$

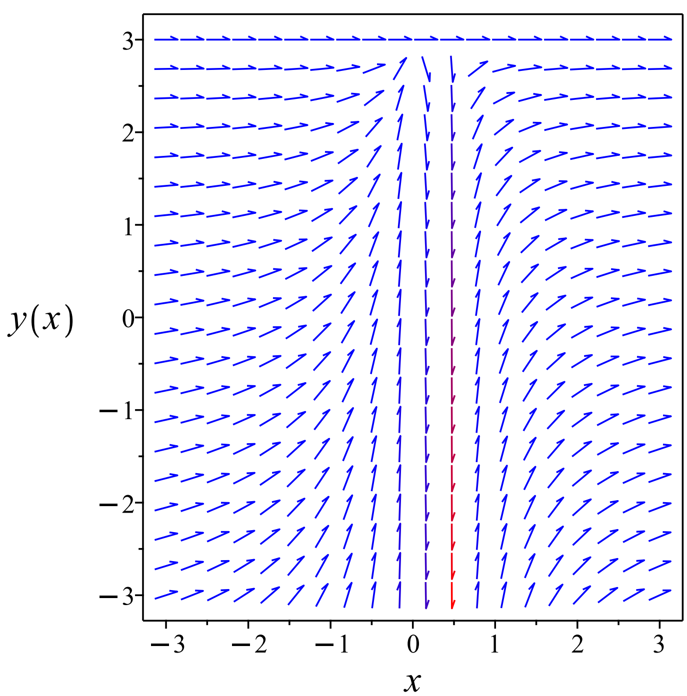


Figure 31: Slope field plot

Verification of solutions

$$y = \frac{(6\,\mathrm{e}^{c_1} x - 3\,\mathrm{e}^{c_1} + x)\,\mathrm{e}^{-c_1}}{2x - 1}$$

Verified OK.

1.7.5 Maple step by step solution

Let's solve

$y - xy' + 2y'x^2 = 3$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-y+3} = \frac{1}{2x^2-x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{-y+3}dx = \int \frac{1}{2x^2-x}dx + c_1$

- Evaluate integral

  $-\ln(-y+3) = \ln(2x-1) - \ln(x) + c_1$

- Solve for y

  $y = \frac{6\,\mathrm{e}^{c_1} x - 3\,\mathrm{e}^{c_1} - x}{\mathrm{e}^{c_1}(2x-1)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(y(x)-x*diff(y(x),x)=3-2*x^2*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \frac{c_1 x - 3}{2x - 1}$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 24

```
DSolve[y[x]-x*y'[x]==3-2*x^2*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to \frac{3+c_1x}{1-2x} \\
y(x) &\to 3
\end{aligned}
$$

1.8 problem 8

Internal problem ID [2551]
Internal file name [`OUTPUT/2043_Sunday_June_05_2022_02_46_00_AM_71161820/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.4, page 36
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - \frac{\cos(-y+x)}{\sin(x)\sin(y)} = -1}$$

1.8.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{\cos(x)\cot(y)}{\sin(x)} \end{aligned}$$

Where $f(x) = \frac{\cos(x)}{\sin(x)}$ and $g(y) = \cot(y)$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\cot(y)}\, dy &= \frac{\cos(x)}{\sin(x)}\, dx \\ \int \frac{1}{\cot(y)}\, dy &= \int \frac{\cos(x)}{\sin(x)}\, dx \\ -\ln(\cos(y)) &= \ln(\sin(x)) + c_1 \end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{\cos(y)} = e^{\ln(\sin(x))+c_1}$$

Which simplifies to

$$\sec(y) = c_2 \sin(x)$$

Summary
The solution(s) found are the following

$$y = \operatorname{arcsec}\left(c_2 e^{c_1} \sin(x)\right) \tag{1}$$

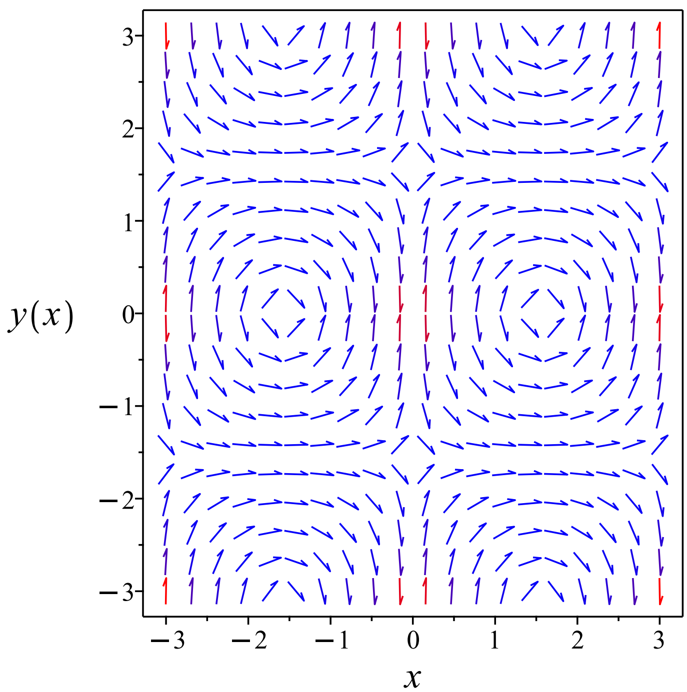


Figure 32: Slope field plot

Verification of solutions

$$y = \operatorname{arcsec}\left(c_2 e^{c_1} \sin(x)\right)$$

Verified OK.

1.8.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{\sin(x)\sin(y)-\cos(-y+x)}{\sin(x)\sin(y)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 22: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y'=f(x)y(x)+g(x)$	0	$e^{\int f dx}$
separable ode	$y'=f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y'=f(x)$	0	1
quadrature ode	$y'=g(y)$	1	0
homogeneous ODEs of Class A	$y'=f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y'=(a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y'=\frac{y}{x}+g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y'=g(x)\,e^{h(x)+by}+f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y'=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y'=f(x)\,y+g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y'=f_1(x)\,y+f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{\sin(x)}{\cos(x)} \\
\eta(x,y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{\sin(x)}{\cos(x)}} dx
\end{aligned}
$$

Which results in

$$
S = \ln(\sin(x))
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{\sin(x)\sin(y) - \cos(-y+x)}{\sin(x)\sin(y)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \cot(x) \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{\cos(x)\sin(y)}{-\sin(x)\sin(y) + \cos(-y + x)} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \tan(R)
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\ln(\cos(R)) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\ln(\sin(x)) = -\ln(\cos(y)) + c_1
$$

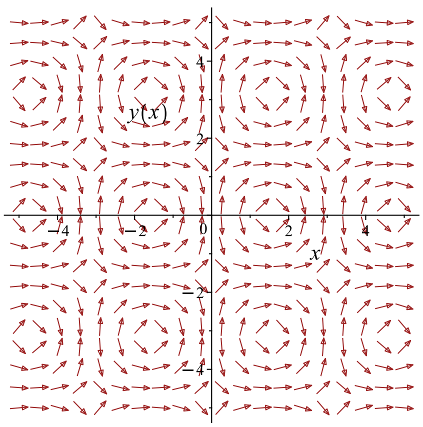
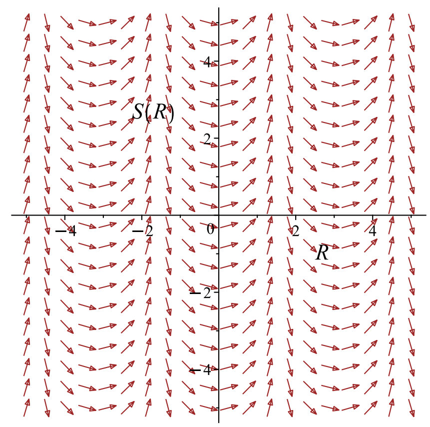
Which simplifies to

$$
\ln(\sin(x)) = -\ln(\cos(y)) + c_1
$$

Which gives

$$
y = \arccos\left(\frac{e^{c_1}}{\sin(x)}\right)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{\sin(x)\sin(y)-\cos(-y+x)}{\sin(x)\sin(y)}$ 	$R = y$ $S = \ln(\sin(x))$	$\frac{dS}{dR} = \tan(R)$ 

Summary

The solution(s) found are the following

$$y = \arccos\left(\frac{e^{c_1}}{\sin(x)}\right) \tag{1}$$

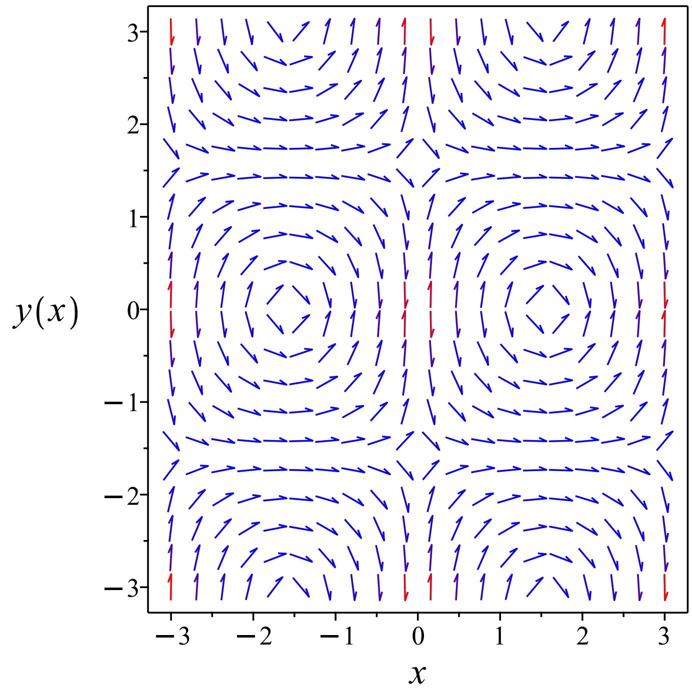


Figure 33: Slope field plot

Verification of solutions

$$y = \arccos\left(\frac{e^{c_1}}{\sin(x)}\right)$$

Verified OK.

1.8.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{\sin(y)}{\cos(y)}\right) \mathrm{d}y = \left(\frac{\cos(x)}{\sin(x)}\right) \mathrm{d}x$$
$$\left(-\frac{\cos(x)}{\sin(x)}\right) \mathrm{d}x + \left(\frac{\sin(y)}{\cos(y)}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{\cos(x)}{\sin(x)}$$
$$N(x, y) = \frac{\sin(y)}{\cos(y)}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{\cos(x)}{\sin(x)}\right)$$
$$= 0$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{\sin(y)}{\cos(y)}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{\cos(x)}{\sin(x)}\,\mathrm{d}x\\ \phi &= -\ln(\sin(x)) + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{\sin(y)}{\cos(y)}$. Therefore equation (4) becomes

$$\frac{\sin(y)}{\cos(y)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned}f'(y) &= \frac{\sin(y)}{\cos(y)}\\ &= \tan(y)\end{aligned}$$

Integrating the above w.r.t y results in

$$\int f'(y)\,\mathrm{d}y = \int (\tan(y))\,\mathrm{d}y$$

$$f(y) = -\ln(\cos(y)) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(\sin(x)) - \ln(\cos(y)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(\sin(x)) - \ln(\cos(y))$$

Summary

The solution(s) found are the following

$$-\ln(\sin(x)) - \ln(\cos(y)) = c_1 \tag{1}$$

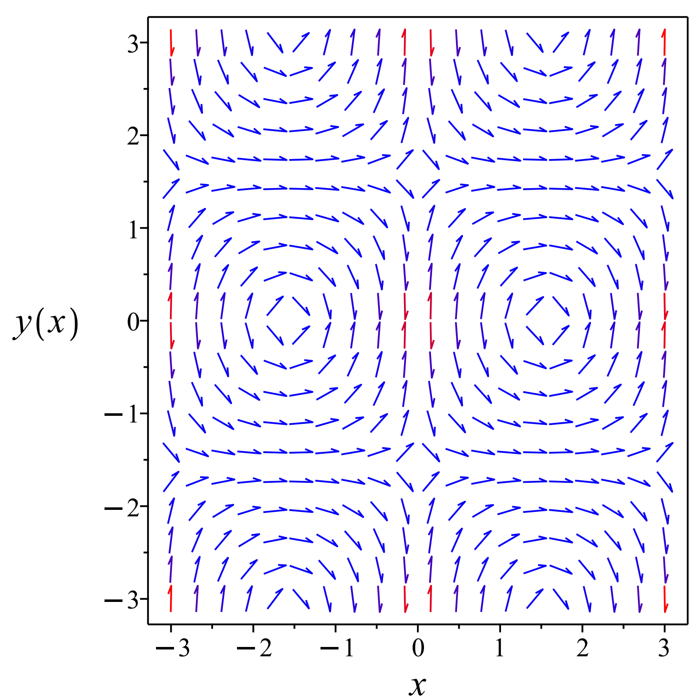


Figure 34: Slope field plot

Verification of solutions

$$-\ln(\sin(x)) - \ln(\cos(y)) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.14 (sec). Leaf size: 11

```
dsolve(diff(y(x),x)=(cos(x-y(x)))/(sin(x)*sin(y(x)))-1,y(x), singsol=all)
```

$$y(x) = \arccos\left(\frac{\csc(x)}{c_1}\right)$$

✓ Solution by Mathematica

Time used: 5.76 (sec). Leaf size: 47

```
DSolve[y'[x]==(Cos[x-y[x]])/(Sin[x]*Sin[y[x]])-1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\arccos\left(-\frac{1}{2}c_1\csc(x)\right)$$
$$y(x) \to \arccos\left(-\frac{1}{2}c_1\csc(x)\right)$$
$$y(x) \to -\frac{\pi}{2}$$
$$y(x) \to \frac{\pi}{2}$$

1.9 problem 9

Internal problem ID [2552]
Internal file name [`OUTPUT/2044_Sunday_June_05_2022_02_46_04_AM_54947365/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.4, page 36
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - \frac{x(y^2-1)}{2(x-2)(x-1)} = 0}$$

1.9.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x\left(\frac{y^2}{2}-\frac{1}{2}\right)}{(x-2)(x-1)} \end{aligned}$$

Where $f(x) = \frac{x}{(x-2)(x-1)}$ and $g(y) = \frac{y^2}{2} - \frac{1}{2}$. Integrating both sides gives

$$\frac{1}{\frac{y^2}{2}-\frac{1}{2}}\, dy = \frac{x}{(x-2)(x-1)}\, dx$$

$$\int \frac{1}{\frac{y^2}{2}-\frac{1}{2}}\,dy = \int \frac{x}{(x-2)(x-1)}\,dx$$

$$-2\,\mathrm{arctanh}(y) = -\ln(x-1) + 2\ln(x-2) + c_1$$

Which results in

$$y = -\tanh\left(-\frac{\ln(x-1)}{2} + \ln(x-2) + \frac{c_1}{2}\right)$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\tanh\left(-\frac{\ln(x-1)}{2} + \ln(x-2) + \frac{c_1}{2}\right) \tag{1}$$

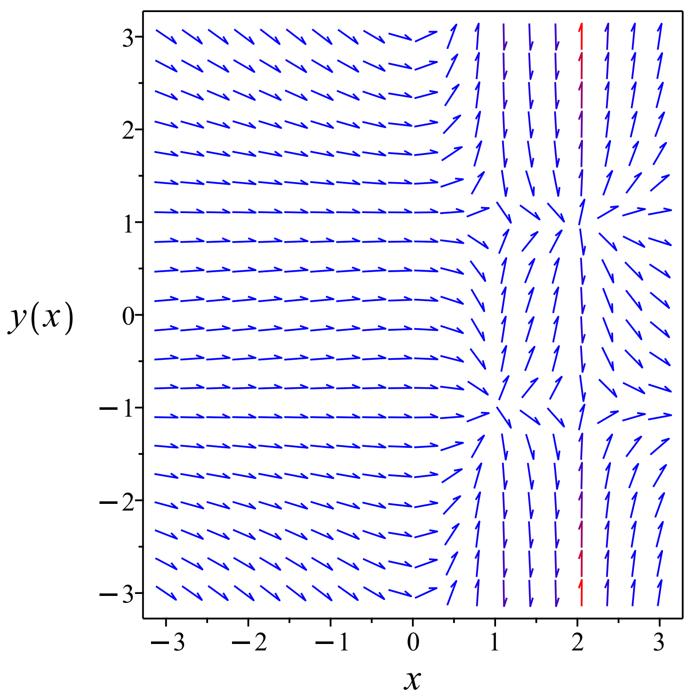


Figure 35: Slope field plot

<u>Verification of solutions</u>

$$y = -\tanh\left(-\frac{\ln(x-1)}{2} + \ln(x-2) + \frac{c_1}{2}\right)$$

Verified OK.

1.9.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{x(y^2-1)}{2(x-2)(x-1)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 24: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f(\frac{y}{x})$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F(\frac{y}{x})$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{(x-2)(x-1)}{x} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{(x-2)(x-1)}{x}} dx
\end{aligned}
$$

Which results in

$$
S = -\ln(x-1) + 2\ln(x-2)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{x(y^2-1)}{2(x-2)(x-1)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{x}{(x-2)(x-1)} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{2}{y^2-1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{2}{R^2-1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -2 \operatorname{arctanh}(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln(x-1) + 2\ln(x-2) = -2 \operatorname{arctanh}(y) + c_1$$

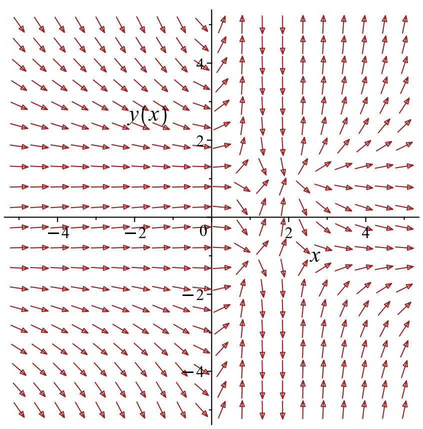
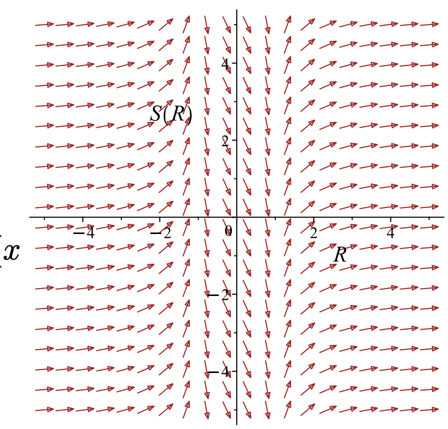
Which simplifies to

$$-\ln(x-1) + 2\ln(x-2) = -2 \operatorname{arctanh}(y) + c_1$$

Which gives

$$y = \tanh\left(\frac{\ln(x-1)}{2} - \ln(x-2) + \frac{c_1}{2}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = \frac{x(y^2-1)}{2(x-2)(x-1)}$
</td><td>$R = y$
$S = -\ln(x-1) + 2\ln(x$</td><td>$\frac{dS}{dR} = \frac{2}{R^2-1}$
</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \tanh\left(\frac{\ln(x-1)}{2} - \ln(x-2) + \frac{c_1}{2}\right) \tag{1}$$

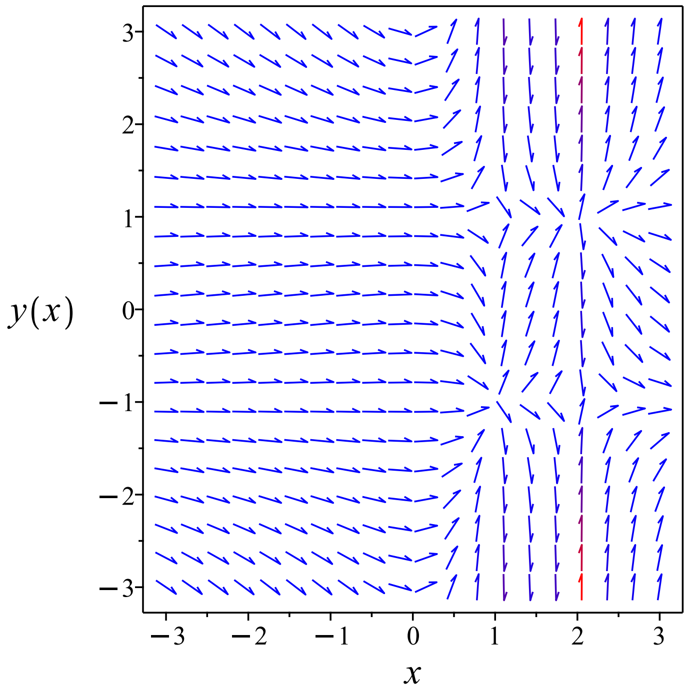


Figure 36: Slope field plot

Verification of solutions

$$y = \tanh\left(\frac{\ln (x-1)}{2} - \ln (x-2) + \frac{c_1}{2}\right)$$

Verified OK.

1.9.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{\frac{y^2}{2} - \frac{1}{2}}\right) \mathrm{d}y &= \left(\frac{x}{(x-2)(x-1)}\right) \mathrm{d}x \\
\left(-\frac{x}{(x-2)(x-1)}\right) \mathrm{d}x + \left(\frac{1}{\frac{y^2}{2} - \frac{1}{2}}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{x}{(x-2)(x-1)} \\
N(x, y) &= \frac{1}{\frac{y^2}{2} - \frac{1}{2}}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{(x-2)(x-1)}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{\frac{y^2}{2}-\frac{1}{2}}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\frac{x}{(x-2)(x-1)} \,\mathrm{d}x \\
\phi &= \ln(x-1) - 2\ln(x-2) + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{\frac{y^2}{2}-\frac{1}{2}}$. Therefore equation (4) becomes

$$
\frac{1}{\frac{y^2}{2}-\frac{1}{2}} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{2}{y^2-1}
$$

Integrating the above w.r.t y gives

$$
\int f'(y)\,\mathrm{d}y = \int \left(\frac{2}{y^2 - 1}\right) \mathrm{d}y
$$
$$
f(y) = -2\ \operatorname{arctanh}(y) + c_1
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = \ln(x-1) - 2\ln(x-2) - 2\ \operatorname{arctanh}(y) + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = \ln(x-1) - 2\ln(x-2) - 2\ \operatorname{arctanh}(y)
$$

The solution becomes

$$
y = -\tanh\left(-\frac{\ln(x-1)}{2} + \ln(x-2) + \frac{c_1}{2}\right)
$$

Summary

The solution(s) found are the following

$$
y = -\tanh\left(-\frac{\ln(x-1)}{2} + \ln(x-2) + \frac{c_1}{2}\right) \tag{1}
$$

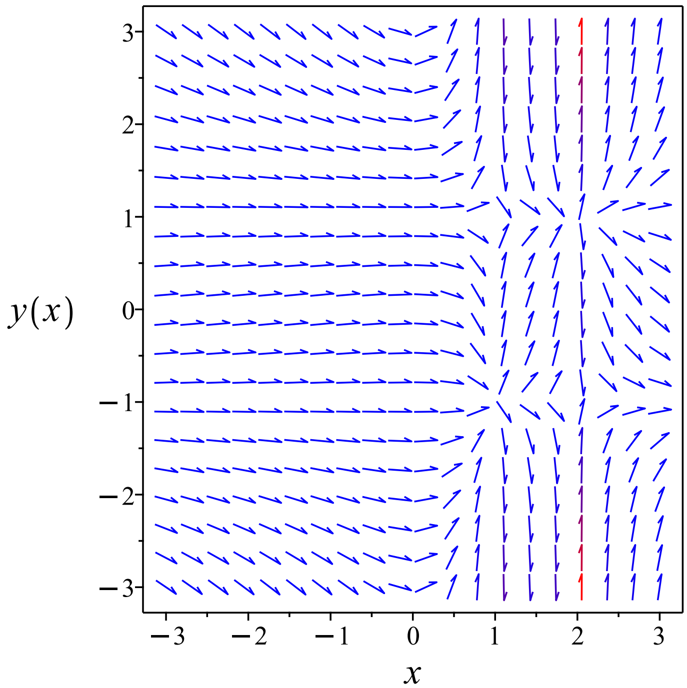


Figure 37: Slope field plot

Verification of solutions

$$y = -\tanh\left(-\frac{\ln(x-1)}{2} + \ln(x-2) + \frac{c_1}{2}\right)$$

Verified OK.

1.9.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{x(y^2-1)}{2(x-2)(x-1)} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{x\,y^2}{2(x-2)(x-1)} - \frac{x}{2(x-2)(x-1)}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\frac{x}{2(x-2)(x-1)}$, $f_1(x) = 0$ and $f_2(x) = \frac{x}{2(x-2)(x-1)}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{\frac{xu}{2(x-2)(x-1)}}
\end{aligned} \tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= \frac{1}{2(x-2)(x-1)} - \frac{x}{2(x-2)^2(x-1)} - \frac{x}{2(x-2)(x-1)^2} \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= -\frac{x^3}{8(x-2)^3(x-1)^3}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
\frac{xu''(x)}{2(x-2)(x-1)} - \left(\frac{1}{2(x-2)(x-1)} - \frac{x}{2(x-2)^2(x-1)} - \frac{x}{2(x-2)(x-1)^2}\right) u'(x) - \frac{x^3 u(x)}{8(x-2)^3(x-}
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = c_1 \sinh\left(-\frac{\ln(x-1)}{2} + \ln(x-2)\right) + c_2 \cosh\left(-\frac{\ln(x-1)}{2} + \ln(x-2)\right)
$$

The above shows that

$$
u'(x) = \frac{x\left(c_1 \cosh\left(-\frac{\ln(x-1)}{2} + \ln(x-2)\right) + c_2 \sinh\left(-\frac{\ln(x-1)}{2} + \ln(x-2)\right)\right)}{2(x-2)(x-1)}
$$

Using the above in (1) gives the solution

$$
y = -\frac{c_1 \cosh\left(-\frac{\ln(x-1)}{2} + \ln(x-2)\right) + c_2 \sinh\left(-\frac{\ln(x-1)}{2} + \ln(x-2)\right)}{c_1 \sinh\left(-\frac{\ln(x-1)}{2} + \ln(x-2)\right) + c_2 \cosh\left(-\frac{\ln(x-1)}{2} + \ln(x-2)\right)}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-c_3 \cosh\left(-\frac{\ln(x-1)}{2} + \ln(x-2)\right) - \sinh\left(-\frac{\ln(x-1)}{2} + \ln(x-2)\right)}{c_3 \sinh\left(-\frac{\ln(x-1)}{2} + \ln(x-2)\right) + \cosh\left(-\frac{\ln(x-1)}{2} + \ln(x-2)\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{-c_3 \cosh\left(-\frac{\ln(x-1)}{2} + \ln(x-2)\right) - \sinh\left(-\frac{\ln(x-1)}{2} + \ln(x-2)\right)}{c_3 \sinh\left(-\frac{\ln(x-1)}{2} + \ln(x-2)\right) + \cosh\left(-\frac{\ln(x-1)}{2} + \ln(x-2)\right)} \tag{1}$$

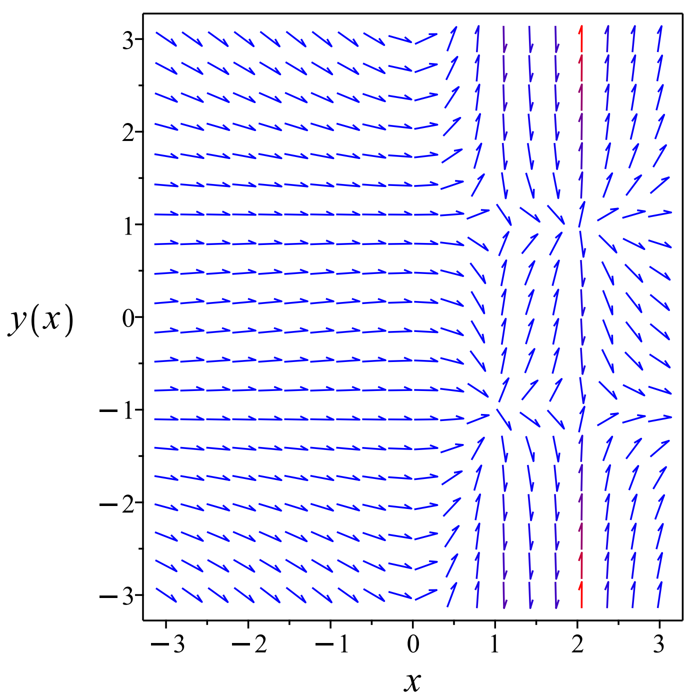


Figure 38: Slope field plot

Verification of solutions

$$y = \frac{-c_3 \cosh\left(-\frac{\ln(x-1)}{2} + \ln(x-2)\right) - \sinh\left(-\frac{\ln(x-1)}{2} + \ln(x-2)\right)}{c_3 \sinh\left(-\frac{\ln(x-1)}{2} + \ln(x-2)\right) + \cosh\left(-\frac{\ln(x-1)}{2} + \ln(x-2)\right)}$$

Verified OK.

1.9.5 Maple step by step solution

Let's solve

$y' - \frac{x(y^2-1)}{2(x-2)(x-1)} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2-1} = \frac{x}{2(x-2)(x-1)}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2-1}dx = \int \frac{x}{2(x-2)(x-1)}dx + c_1$

- Evaluate integral

  $-\mathrm{arctanh}(y) = -\frac{\ln(x-1)}{2} + \ln(x-2) + c_1$

- Solve for y

  $y = -\tanh\left(-\frac{\ln(x-1)}{2} + \ln(x-2) + c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x)=(x*( y(x)^2-1))/(2*(x-2)*(x-1)),y(x), singsol=all)
```

$$y(x) = -\tanh\left(\ln(-2+x) - \frac{\ln(x-1)}{2} + \frac{c_1}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.942 (sec). Leaf size: 51

```
DSolve[y'[x]==(x*( y[x]^2-1))/(2*(x-2)*(x-1)),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x + e^{2c_1}(x-2)^2 - 1}{-x + e^{2c_1}(x-2)^2 + 1}$$
$$y(x) \to -1$$
$$y(x) \to 1$$

1.10 problem 10

Internal problem ID [2553]
Internal file name [`OUTPUT/2045_Sunday_June_05_2022_02_46_06_AM_99634586/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.4, page 36
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - \frac{x^2 y - 32}{-x^2 + 16} = 32}$$

1.10.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{x^2}{x^2 - 16}$$
$$q(x) = \frac{32x^2 - 480}{x^2 - 16}$$

Hence the ode is

$$y' + \frac{x^2 y}{x^2 - 16} = \frac{32x^2 - 480}{x^2 - 16}$$

The integrating factor μ is

$$\mu = e^{\int \frac{x^2}{x^2-16}dx}$$
$$= e^{x-2\ln(x+4)+2\ln(x-4)}$$

Which simplifies to

$$\mu = \frac{(x-4)^2 e^x}{(x+4)^2}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{32x^2-480}{x^2-16}\right)$$
$$\frac{d}{dx}\left(\frac{(x-4)^2 e^x y}{(x+4)^2}\right) = \left(\frac{(x-4)^2 e^x}{(x+4)^2}\right)\left(\frac{32x^2-480}{x^2-16}\right)$$
$$d\left(\frac{(x-4)^2 e^x y}{(x+4)^2}\right) = \left(\frac{32\,e^x(x-4)(x^2-15)}{(x+4)^3}\right) dx$$

Integrating gives

$$\frac{(x-4)^2 e^x y}{(x+4)^2} = \int \frac{32\,e^x(x-4)(x^2-15)}{(x+4)^3}\, dx$$
$$\frac{(x-4)^2 e^x y}{(x+4)^2} = 32\,e^x + \frac{128\,e^x}{(x+4)^2} - \frac{1952\,e^x}{x+4} - 1440\,e^{-4}\,\text{expIntegral}_1\,(-x-4) + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{(x-4)^2 e^x}{(x+4)^2}$ results in

$$y = \frac{(x+4)^2 e^{-x}\left(32\,e^x + \frac{128\,e^x}{(x+4)^2} - \frac{1952\,e^x}{x+4} - 1440\,e^{-4}\,\text{expIntegral}_1\,(-x-4)\right)}{(x-4)^2} + \frac{c_1(x+4)^2 e^{-x}}{(x-4)^2}$$

which simplifies to

$$y = \frac{-1440\,e^{-x-4}(x+4)^2\,\text{expIntegral}_1\,(-x-4) + c_1(x+4)^2 e^{-x} + 32x^2 - 1696x - 7168}{(x-4)^2}$$

Summary
The solution(s) found are the following

$$y = \frac{-1440\,e^{-x-4}(x+4)^2\,\text{expIntegral}_1\,(-x-4) + c_1(x+4)^2 e^{-x} + 32x^2 - 1696x - 7168}{(x-4)^2} \tag{1}$$

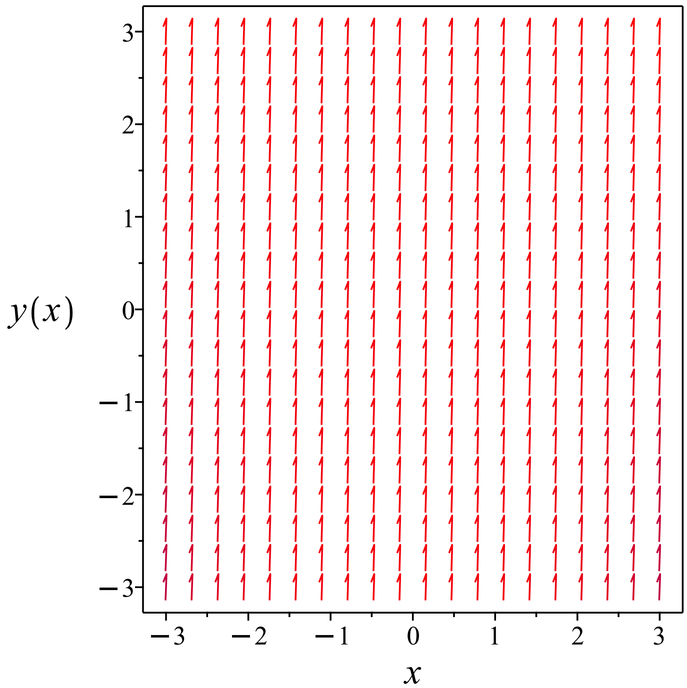


Figure 39: Slope field plot

Verification of solutions

$$y = \frac{-1440\,\mathrm{e}^{-x-4}(x+4)^2\,\mathrm{expIntegral}_1\left(-x-4\right) + c_1(x+4)^2\,\mathrm{e}^{-x} + 32x^2 - 1696x - 7168}{(x-4)^2}$$

Verified OK.

1.10.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x^2y - 32x^2 + 480}{x^2 - 16}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 27: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \mathrm{e}^{-x+2\ln(x+4)-2\ln(x-4)} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-x+2\ln(x+4)-2\ln(x-4)}} dy \end{aligned}$$

Which results in

$$S = \frac{(x-4)^2 e^x y}{(x+4)^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x^2 y - 32x^2 + 480}{x^2 - 16}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{(x-4) e^x y x^2}{(x+4)^3} \\ S_y &= \frac{(x-4)^2 e^x}{(x+4)^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{32 e^x (x-4)(x^2-15)}{(x+4)^3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{32 e^R (R-4)(R^2-15)}{(R+4)^3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 32\,\mathrm{e}^{R} + \frac{128\,\mathrm{e}^{R}}{(R+4)^{2}} - \frac{1952\,\mathrm{e}^{R}}{R+4} - 1440\,\mathrm{e}^{-4}\,\mathrm{expIntegral}_{1}\left(-4-R\right) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{(x-4)^{2}\,\mathrm{e}^{x} y}{(x+4)^{2}} = 32\,\mathrm{e}^{x} + \frac{128\,\mathrm{e}^{x}}{(x+4)^{2}} - \frac{1952\,\mathrm{e}^{x}}{x+4} - 1440\,\mathrm{e}^{-4}\,\mathrm{expIntegral}_{1}\left(-x-4\right) + c_1$$

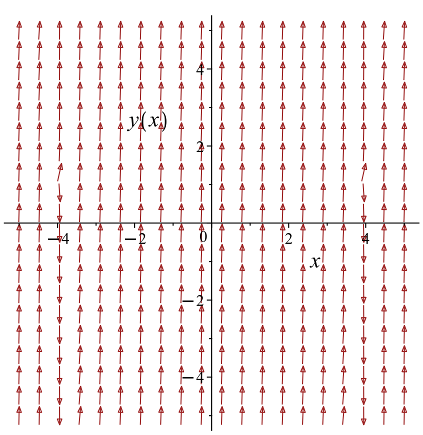
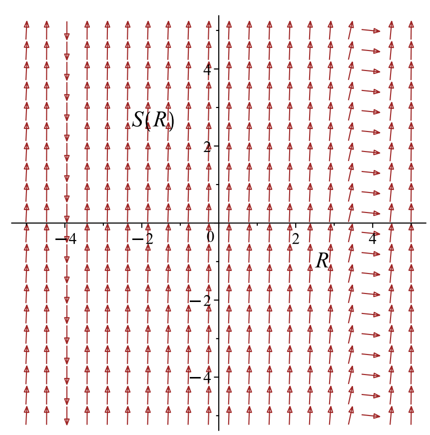
Which simplifies to

$$\frac{(x-4)^{2}\,\mathrm{e}^{x} y}{(x+4)^{2}} = 32\,\mathrm{e}^{x} + \frac{128\,\mathrm{e}^{x}}{(x+4)^{2}} - \frac{1952\,\mathrm{e}^{x}}{x+4} - 1440\,\mathrm{e}^{-4}\,\mathrm{expIntegral}_{1}\left(-x-4\right) + c_1$$

Which gives

$$y = \frac{\left(-1440\,\mathrm{e}^{-4}\,\mathrm{expIntegral}_{1}\left(-x-4\right)x^{2} + 32x^{2}\mathrm{e}^{x} - 11520\,\mathrm{e}^{-4}\,\mathrm{expIntegral}_{1}\left(-x-4\right)x + c_1 x^{2} - 1696x\,\mathrm{e}^{x}\right.}{x^{2}-8x+16}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x^{2}y-32x^{2}+480}{x^{2}-16}$ 	$R = x$ $S = \frac{(x-4)^{2}\,\mathrm{e}^{x} y}{(x+4)^{2}}$	$\frac{dS}{dR} = \frac{32\,\mathrm{e}^{R}(R-4)(R^{2}-15)}{(R+4)^{3}}$ 

Summary

The solution(s) found are the following

$$y = \frac{\left(-1440\,\mathrm{e}^{-4}\,\mathrm{expIntegral}_1\left(-x-4\right)x^2 + 32x^2\mathrm{e}^{x} - 11520\,\mathrm{e}^{-4}\,\mathrm{expIntegral}_1\left(-x-4\right)x + c_1 x^2 - 1696x\,\mathrm{e}^{x} -}{x^2 - 8x + 16} \tag{1}$$

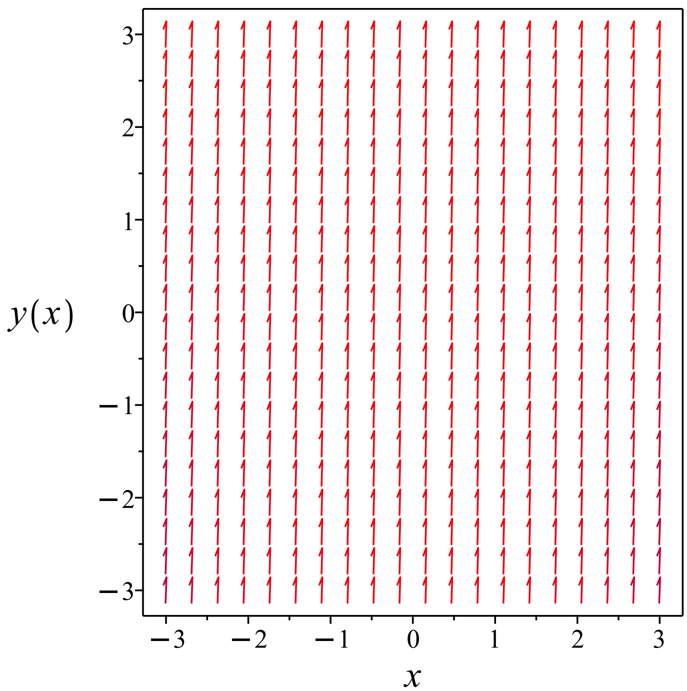


Figure 40: Slope field plot

Verification of solutions

$$y = \frac{\left(-1440\,\mathrm{e}^{-4}\,\mathrm{expIntegral}_1\left(-x-4\right)x^2 + 32x^2\mathrm{e}^{x} - 11520\,\mathrm{e}^{-4}\,\mathrm{expIntegral}_1\left(-x-4\right)x + c_1 x^2 - 1696x\,\mathrm{e}^{x} -}{x^2 - 8x + 16}$$

Verified OK.

1.10.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\mathrm{d}y=\left(\frac{x^2y-32}{-x^2+16}+32\right)\mathrm{d}x$$

$$\left(-\frac{x^2y-32}{-x^2+16}-32\right)\mathrm{d}x+\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{x^2 y - 32}{-x^2 + 16} - 32 \\
N(x,y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x^2 y - 32}{-x^2 + 16} - 32\right) \\
&= \frac{x^2}{x^2 - 16}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(-\frac{x^2}{-x^2 + 16}\right) - (0)\right) \\
&= \frac{x^2}{x^2 - 16}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}x} \\
&= e^{\int \frac{x^2}{x^2 - 16} \,\mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{x - 2\ln(x+4) + 2\ln(x-4)} \\
&= \frac{(x-4)^2 \,\mathrm{e}^x}{(x+4)^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{(x-4)^2 \mathrm{e}^x}{(x+4)^2} \left(-\frac{x^2 y - 32}{-x^2 + 16} - 32 \right) \\
&= \frac{\mathrm{e}^x (x-4) \left(480 + (y-32) x^2\right)}{(x+4)^3}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{(x-4)^2 \mathrm{e}^x}{(x+4)^2} (1) \\
&= \frac{(x-4)^2 \mathrm{e}^x}{(x+4)^2}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N} \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{\mathrm{e}^x (x-4) \left(480 + (y-32) x^2\right)}{(x+4)^3} \right) + \left(\frac{(x-4)^2 \mathrm{e}^x}{(x+4)^2} \right) \frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x
$$

$$
\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{\mathrm{e}^x (x-4) \left(480 + (y-32) x^2\right)}{(x+4)^3} \mathrm{d}x
$$

$$
\begin{aligned}
\phi & \\
&= \frac{1440 \,\mathrm{e}^{-4} (x+4)^2 \,\mathrm{expIntegral}_1 (-x-4) + \left((y-32) x^2 + (-8y + 1696) x + 16y + 7168\right) \mathrm{e}^x}{(x+4)^2} \\
&\quad + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= \frac{e^{x}(x^{2}-8x+16)}{(x+4)^{2}} + f'(y) \qquad (4)\\ &= \frac{(x-4)^{2} e^{x}}{(x+4)^{2}} + f'(y)\end{aligned}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{(x-4)^{2} e^{x}}{(x+4)^{2}}$. Therefore equation (4) becomes

$$\frac{(x-4)^{2} e^{x}}{(x+4)^{2}} = \frac{(x-4)^{2} e^{x}}{(x+4)^{2}} + f'(y) \qquad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1440\,e^{-4}(x+4)^{2}\,\mathrm{expIntegral}_1(-x-4) + ((y-32)\,x^{2} + (-8y+1696)\,x + 16y + 7168)\,e^{x}}{(x+4)^{2}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1440\,e^{-4}(x+4)^{2}\,\mathrm{expIntegral}_1(-x-4) + ((y-32)\,x^{2} + (-8y+1696)\,x + 16y + 7168)\,e^{x}}{(x+4)^{2}}$$

The solution becomes

$$y = \frac{(-1440\,e^{-4}\,\mathrm{expIntegral}_1(-x-4)\,x^{2} + 32x^{2}e^{x} - 11520\,e^{-4}\,\mathrm{expIntegral}_1(-x-4)\,x + c_1x^{2} - 1696x\,e^{x} -}{x^{2}-8x+16}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-1440\,\mathrm{e}^{-4}\,\mathrm{expIntegral}_1\left(-x-4\right)x^2+32x^2\mathrm{e}^x-11520\,\mathrm{e}^{-4}\,\mathrm{expIntegral}_1\left(-x-4\right)x+c_1x^2-1696x\,\mathrm{e}^x-}{x^2-8x+16} \tag{1}$$

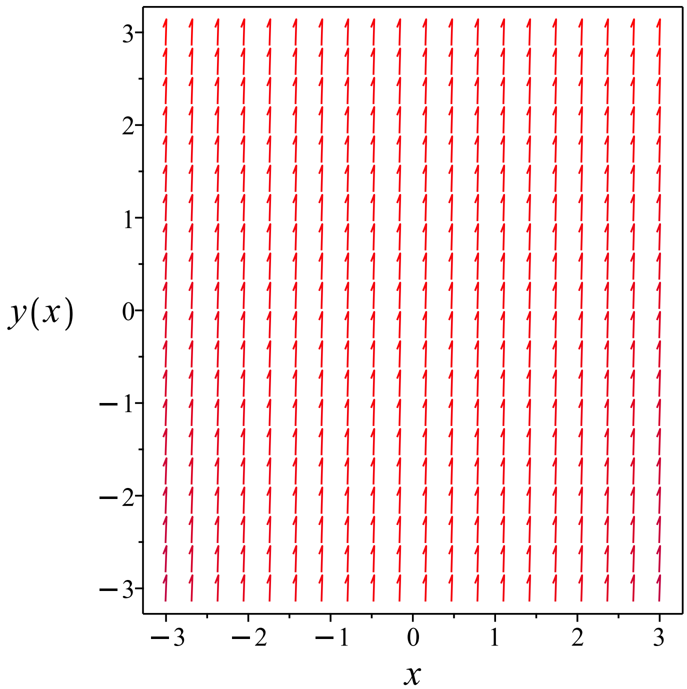


Figure 41: Slope field plot

Verification of solutions

$$y = \frac{\left(-1440\,\mathrm{e}^{-4}\,\mathrm{expIntegral}_1\left(-x-4\right)x^2+32x^2\mathrm{e}^x-11520\,\mathrm{e}^{-4}\,\mathrm{expIntegral}_1\left(-x-4\right)x+c_1x^2-1696x\,\mathrm{e}^x-}{x^2-8x+16}$$

Verified OK.

1.10.4 Maple step by step solution

Let's solve

$y' - \frac{x^2y-32}{-x^2+16} = 32$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{x^2y}{x^2-16} + \frac{32(x^2-15)}{x^2-16}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{x^2y}{x^2-16} = \frac{32(x^2-15)}{x^2-16}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{x^2y}{x^2-16}\right) = \frac{32\mu(x)(x^2-15)}{x^2-16}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{x^2y}{x^2-16}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)x^2}{x^2-16}$

- Solve to find the integrating factor

  $\mu(x) = e^{x-2\ln(x+4)+2\ln(x-4)}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right)dx = \int \frac{32\mu(x)(x^2-15)}{x^2-16}dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \frac{32\mu(x)(x^2-15)}{x^2-16}dx + c_1$

- Solve for y

  $y = \frac{\int \frac{32\mu(x)\left(x^2-15\right)}{x^2-16}dx+c_1}{\mu(x)}$

- Substitute $\mu(x) = e^{x-2\ln(x+4)+2\ln(x-4)}$

  $y = \frac{\int \frac{32\,e^{x-2\ln(x+4)+2\ln(x-4)}\left(x^2-15\right)}{x^2-16}dx+c_1}{e^{x-2\ln(x+4)+2\ln(x-4)}}$

- Evaluate the integrals on the rhs

$$y=\frac{32\,\mathrm{e}^{x}+\frac{128\,\mathrm{e}^{x}}{(x+4)^{2}}-\frac{1952\,\mathrm{e}^{x}}{x+4}-1440\,\mathrm{e}^{-4}\mathrm{Ei}_{1}(-x-4)+c_{1}}{\mathrm{e}^{x-2\ln(x+4)+2\ln(x-4)}}$$

- Simplify

$$y=\frac{-1440\,\mathrm{e}^{-x-4}(x+4)^{2}\mathrm{Ei}_{1}(-x-4)+c_{1}(x+4)^{2}\mathrm{e}^{-x}+32x^{2}-1696x-7168}{(x-4)^{2}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 51

```
dsolve(diff(y(x),x)=(x^2*y(x)-32)/(16-x^2) + 32,y(x), singsol=all)
```

$$y(x)=\frac{-1440\,\mathrm{e}^{-4-x}(x+4)^{2}\,\mathrm{expIntegral}_{1}\left(-4-x\right)+c_{1}(x+4)^{2}\,\mathrm{e}^{-x}+32x^{2}-1696x-7168}{(x-4)^{2}}$$

✓ Solution by Mathematica

Time used: 0.204 (sec). Leaf size: 56

```
DSolve[y'[x]==(x^2*y[x]-32)/(16-x^2) + 32,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{e^{-x-4}(1440(x+4)^{2}\,\mathrm{ExpIntegralEi}(x+4)+e^{4}(32e^{x}(x^{2}-53x-224)+c_{1}(x+4)^{2}))}{(x-4)^{2}}$$

1.11 problem 11

Internal problem ID [2554]
Internal file name [`OUTPUT/2046_Sunday_June_05_2022_02_46_08_AM_33689551/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.4, page 36
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{(x-a)(x-b)\,y'-y=-c}$$

1.11.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y-c}{(-x+a)(-x+b)} \end{aligned}$$

Where $f(x)=\frac{1}{(-x+a)(-x+b)}$ and $g(y)=y-c$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y-c}\,dy &= \frac{1}{(-x+a)(-x+b)}\,dx \\ \int \frac{1}{y-c}\,dy &= \int \frac{1}{(-x+a)(-x+b)}\,dx \end{aligned}$$

$$\ln (y-c) = -\frac{\ln (x-b)}{-b+a} + \frac{\ln (x-a)}{-b+a} + c_1$$

Raising both side to exponential gives

$$y-c = \mathrm{e}^{-\frac{\ln(x-b)}{-b+a}+\frac{\ln(x-a)}{-b+a}+c_1}$$

Which simplifies to

$$y-c = c_2\mathrm{e}^{-\frac{\ln(x-b)}{-b+a}+\frac{\ln(x-a)}{-b+a}}$$

Which simplifies to

$$y = c_2(x-b)^{-\frac{1}{-b+a}} (x-a)^{\frac{1}{-b+a}} \mathrm{e}^{c_1} + c$$

Summary

The solution(s) found are the following

$$y = c_2(x-b)^{-\frac{1}{-b+a}} (x-a)^{\frac{1}{-b+a}} \mathrm{e}^{c_1} + c \tag{1}$$

Verification of solutions

$$y = c_2(x-b)^{-\frac{1}{-b+a}} (x-a)^{\frac{1}{-b+a}} \mathrm{e}^{c_1} + c$$

Verified OK.

1.11.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{(-x+a)(-x+b)}$$
$$q(x) = -\frac{c}{(-x+a)(-x+b)}$$

Hence the ode is

$$y' - \frac{y}{(-x+a)(-x+b)} = -\frac{c}{(-x+a)(-x+b)}$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int -\frac{1}{(-x+a)(-x+b)}dx}$$
$$= \mathrm{e}^{\frac{\ln(x-b)}{-b+a}-\frac{\ln(x-a)}{-b+a}}$$

Which simplifies to

$$\mu = (x-b)^{\frac{1}{-b+a}} (x-a)^{-\frac{1}{-b+a}}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(-\frac{c}{(-x+a)(-x+b)}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left((x-b)^{\frac{1}{-b+a}} (x-a)^{-\frac{1}{-b+a}} y\right) = \left((x-b)^{\frac{1}{-b+a}} (x-a)^{-\frac{1}{-b+a}}\right)\left(-\frac{c}{(-x+a)(-x+b)}\right)$$
$$\mathrm{d}\left((x-b)^{\frac{1}{-b+a}} (x-a)^{-\frac{1}{-b+a}} y\right) = \left(-c(x-b)^{\frac{b-a+1}{-b+a}} (x-a)^{\frac{b-a-1}{-b+a}}\right) \mathrm{d}x$$

Integrating gives

$$(x-b)^{\frac{1}{-b+a}} (x-a)^{-\frac{1}{-b+a}} y = \int -c(x-b)^{\frac{b-a+1}{-b+a}} (x-a)^{\frac{b-a-1}{-b+a}} \mathrm{d}x$$
$$(x-b)^{\frac{1}{-b+a}} (x-a)^{-\frac{1}{-b+a}} y = c(x-a)^{1-\frac{-b+a+1}{-b+a}} (x-b)^{1-\frac{-b+a-1}{-b+a}} + c_1$$

Dividing both sides by the integrating factor $\mu = (x-b)^{\frac{1}{-b+a}} (x-a)^{-\frac{1}{-b+a}}$ results in

$$y = (x-b)^{-\frac{1}{-b+a}} (x-a)^{\frac{1}{-b+a}} c(x-a)^{1-\frac{-b+a+1}{-b+a}} (x-b)^{1-\frac{-b+a-1}{-b+a}} + c_1(x-b)^{-\frac{1}{-b+a}} (x-a)^{\frac{1}{-b+a}}$$

which simplifies to

$$y = c + c_1(x-b)^{-\frac{1}{-b+a}} (x-a)^{\frac{1}{-b+a}}$$

Summary

The solution(s) found are the following

$$y = c + c_1(x-b)^{-\frac{1}{-b+a}} (x-a)^{\frac{1}{-b+a}} \tag{1}$$

Verification of solutions

$$y = c + c_1(x-b)^{-\frac{1}{-b+a}} (x-a)^{\frac{1}{-b+a}}$$

Verified OK.

1.11.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y-c}{(-x+a)(-x+b)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 30: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \mathrm{e}^{-\frac{\ln(x-b)}{-b+a} + \frac{\ln(x-a)}{-b+a}} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-\frac{\ln(x-b)}{-b+a} + \frac{\ln(x-a)}{-b+a}}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{\frac{\ln(x-b)-\ln(x-a)}{-b+a}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y - c}{(-x + a)(-x + b)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y\left((x-b)^{\frac{b-a+1}{-b+a}} (x-a)^{-\frac{1}{-b+a}} - (x-b)^{\frac{1}{-b+a}} (x-a)^{\frac{b-a-1}{-b+a}}\right)}{-b+a} \\ S_y &= (x-b)^{\frac{1}{-b+a}} (x-a)^{-\frac{1}{-b+a}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\left(-(-y+c)(-b+a)(x-b)^{\frac{b-a+1}{-b+a}} - y(x-b)^{\frac{1}{-b+a}}\right)(x-a)^{\frac{b-a-1}{-b+a}} + (x-a)^{-\frac{1}{-b+a}} (x-b)^{\frac{b-a+1}{-b+a}} y}{-b+a} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -(R-a)^{\frac{b-a-1}{-b+a}} (R-b)^{\frac{b-a+1}{-b+a}} c$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c(R-a)^{-\frac{1}{-b+a}} (R-b)^{\frac{1}{-b+a}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$(x-b)^{\frac{1}{-b+a}} (x-a)^{-\frac{1}{-b+a}} y = c(x-b)^{\frac{1}{-b+a}} (x-a)^{-\frac{1}{-b+a}} + c_1$$

Which simplifies to

$$-(x-b)^{\frac{1}{-b+a}} (-y+c) (x-a)^{-\frac{1}{-b+a}} - c_1 = 0$$

Which gives

$$y = \left(c(x-b)^{\frac{1}{-b+a}} (x-a)^{-\frac{1}{-b+a}} + c_1\right) (x-b)^{-\frac{1}{-b+a}} (x-a)^{\frac{1}{-b+a}}$$

Summary

The solution(s) found are the following

$$y = \left(c(x-b)^{\frac{1}{-b+a}} (x-a)^{-\frac{1}{-b+a}} + c_1\right) (x-b)^{-\frac{1}{-b+a}} (x-a)^{\frac{1}{-b+a}} \tag{1}$$

Verification of solutions

$$y = \left(c(x-b)^{\frac{1}{-b+a}} (x-a)^{-\frac{1}{-b+a}} + c_1\right) (x-b)^{-\frac{1}{-b+a}} (x-a)^{\frac{1}{-b+a}}$$

Verified OK.

1.11.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y-c}\right)\mathrm{d}y&=\left(\frac{1}{(-x+a)(-x+b)}\right)\mathrm{d}x\\ \left(-\frac{1}{(-x+a)(-x+b)}\right)\mathrm{d}x+\left(\frac{1}{y-c}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\frac{1}{(-x+a)(-x+b)}\\ N(x,y)&=\frac{1}{y-c}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{(-x+a)(-x+b)}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y-c}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{(-x+a)(-x+b)}\,\mathrm{d}x\\ \phi &= \frac{\ln(x-b)-\ln(x-a)}{-b+a}+f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=\frac{1}{y-c}$. Therefore equation (4) becomes

$$\frac{1}{y-c} = 0+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{-y+c}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{-y+c}\right)\mathrm{d}y$$
$$f(y) = \ln(-y+c) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\ln(x-b) - \ln(x-a)}{-b+a} + \ln(-y+c) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\ln(x-b) - \ln(x-a)}{-b+a} + \ln(-y+c)$$

The solution becomes

$$y = -\mathrm{e}^{-\frac{-c_1a+c_1b+\ln(x-b)-\ln(x-a)}{-b+a}} + c$$

Summary

The solution(s) found are the following

$$y = -\mathrm{e}^{-\frac{-c_1a+c_1b+\ln(x-b)-\ln(x-a)}{-b+a}} + c \tag{1}$$

Verification of solutions

$$y = -\mathrm{e}^{-\frac{-c_1a+c_1b+\ln(x-b)-\ln(x-a)}{-b+a}} + c$$

Verified OK.

1.11.5 Maple step by step solution

Let's solve

$(x-a)(x-b)y' - y = -c$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

$$\frac{y'}{y-c} = \frac{1}{(x-a)(x-b)}$$

- Integrate both sides with respect to x

$$\int \frac{y'}{y-c}dx = \int \frac{1}{(x-a)(x-b)}dx + c_1$$

- Evaluate integral

$$\ln(y-c) = -\frac{\ln(x-b)}{-b+a} + \frac{\ln(x-a)}{-b+a} + c_1$$

- Solve for y

$$y = e^{-\frac{-c_1a+c_1b+\ln\left(\frac{-x+b}{-x+a}\right)}{-b+a}} + c$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 36

```
dsolve((x-a)*(x-b)*diff(y(x),x)-(y(x)-c)=0,y(x), singsol=all)
```

$$y(x) = c + (x-b)^{-\frac{1}{a-b}}(x-a)^{\frac{1}{a-b}}c_1$$

✓ Solution by Mathematica

Time used: 0.287 (sec). Leaf size: 41

```
DSolve[(x-a)*(x-b)*y'[x]-(y[x]-c)==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c + c_1(x-b)^{\frac{1}{b-a}}(x-a)^{\frac{1}{a-b}}$$
$$y(x) \to c$$

1.12 problem 12

Internal problem ID [2555]
Internal file name [`OUTPUT/2047_Sunday_June_05_2022_02_46_11_AM_44380788/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.4, page 36
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(x^2+1)\,y'+y^2=-1}$$

With initial conditions

$$[y(0)=1]$$

1.12.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{y^2+1}{x^2+1} \end{aligned}$$

The x domain of $f(x,y)$ when $y=1$ is

$$\{-\infty<x<\infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y^2+1}{x^2+1}\right)\\ &= -\frac{2y}{x^2+1}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

1.12.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= f(x)g(y)\\ &= \frac{-y^2-1}{x^2+1}\end{aligned}$$

Where $f(x) = \frac{1}{x^2+1}$ and $g(y) = -y^2 - 1$. Integrating both sides gives

$$\begin{aligned}\frac{1}{-y^2-1}\,dy &= \frac{1}{x^2+1}\,dx\\ \int \frac{1}{-y^2-1}\,dy &= \int \frac{1}{x^2+1}\,dx\\ -\arctan(y) &= \arctan(x) + c_1\end{aligned}$$

Which results in

$$y = -\tan(\arctan(x) + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\tan(c_1)$$

$$c_1 = -\frac{\pi}{4}$$

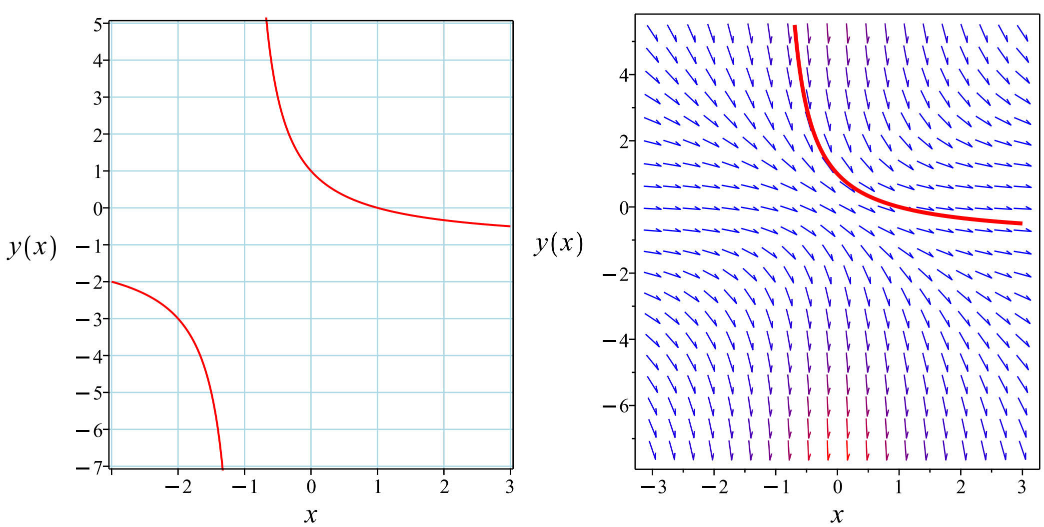
Substituting c_1 found above in the general solution gives

$$y = \frac{1-x}{x+1}$$

Summary

The solution(s) found are the following

$$y = \frac{1-x}{x+1} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{1-x}{x+1}$$

Verified OK.

1.12.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y^2+1}{x^2+1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 33: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= x^2+1 \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{x^2+1} dx
\end{aligned}
$$

Which results in

$$
S = \arctan(x)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{y^2+1}{x^2+1}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{1}{x^2+1} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{y^2+1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R^2+1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\arctan(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\arctan(x) = -\arctan(y) + c_1$$

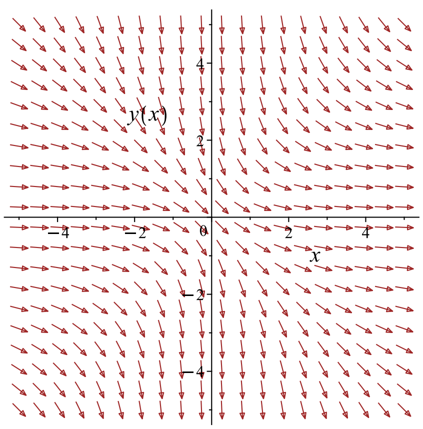
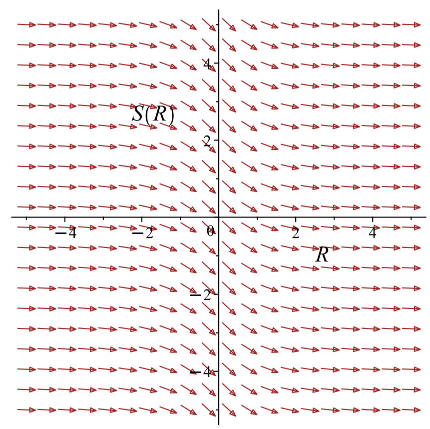
Which simplifies to

$$\arctan(x) = -\arctan(y) + c_1$$

Which gives

$$y = \tan(-\arctan(x) + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y^2+1}{x^2+1}$ 	$R = y$ $S = \arctan(x)$	$\frac{dS}{dR} = -\frac{1}{R^2+1}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \tan(c_1)$$

$$c_1 = \frac{\pi}{4}$$

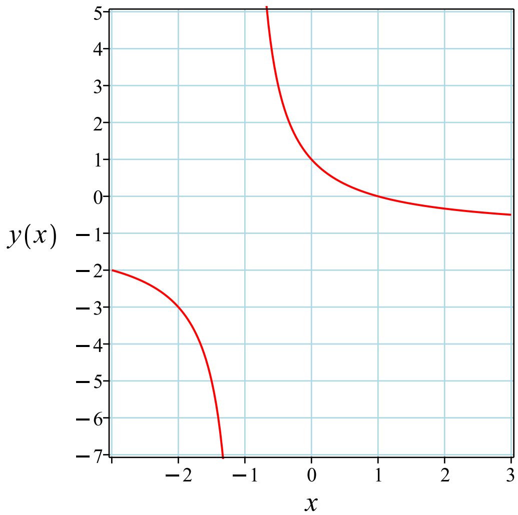
Substituting c_1 found above in the general solution gives

$$y = \frac{1 - x}{x + 1}$$

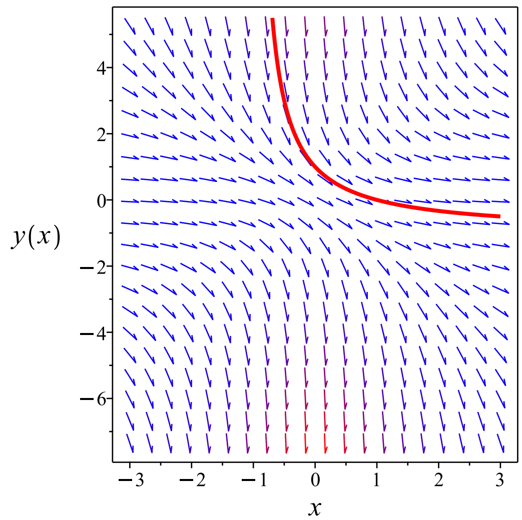
Summary

The solution(s) found are the following

$$y = \frac{1 - x}{x + 1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{1 - x}{x + 1}$$

Verified OK.

1.12.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{-y^2-1}\right)\mathrm{d}y &= \left(\frac{1}{x^2+1}\right)\mathrm{d}x\\ \left(-\frac{1}{x^2+1}\right)\mathrm{d}x + \left(\frac{1}{-y^2-1}\right)\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{1}{x^2+1}\\ N(x, y) &= \frac{1}{-y^2-1}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x^2+1}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{-y^2-1}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\frac{1}{x^2+1} \mathrm{d}x$$
$$\phi = -\arctan(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{-y^2-1}$. Therefore equation (4) becomes

$$\frac{1}{-y^2-1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y^2+1}$$

Integrating the above w.r.t y gives

$$\int f'(y) \mathrm{d}y = \int \left(-\frac{1}{y^2+1}\right) \mathrm{d}y$$
$$f(y) = -\arctan(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\arctan(x) - \arctan(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\arctan(x) - \arctan(y)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{\pi}{4} = c_1$$

$$c_1 = -\frac{\pi}{4}$$

Substituting c_1 found above in the general solution gives

$$-\arctan(x) - \arctan(y) = -\frac{\pi}{4}$$

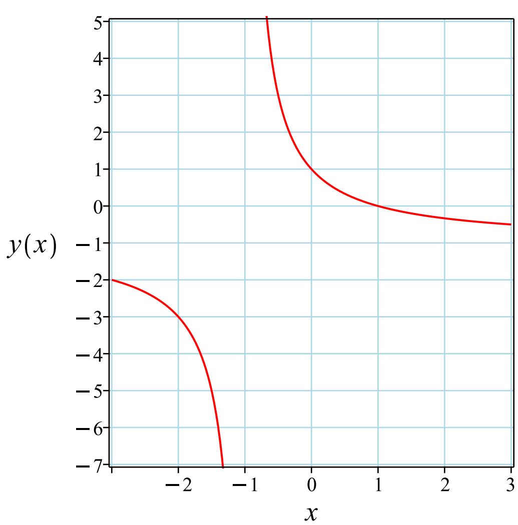
Solving for y from the above gives

$$y = \cot\left(\arctan(x) + \frac{\pi}{4}\right)$$

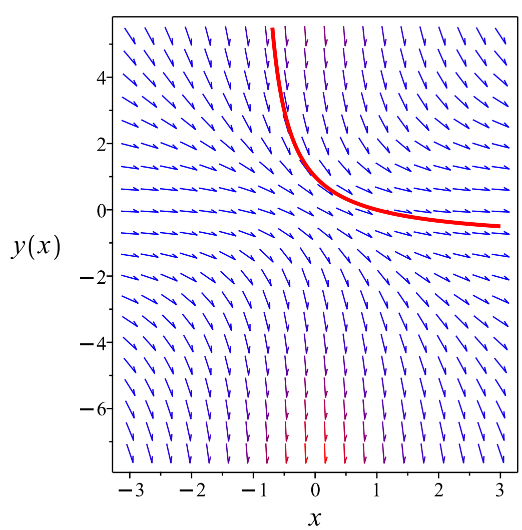
Summary

The solution(s) found are the following

$$y = \cot\left(\arctan(x) + \frac{\pi}{4}\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \cot\left(\arctan(x) + \frac{\pi}{4}\right)$$

Verified OK.

1.12.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y^2 + 1}{x^2 + 1} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{y^2}{x^2 + 1} - \frac{1}{x^2 + 1}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\frac{1}{x^2+1}$, $f_1(x) = 0$ and $f_2(x) = -\frac{1}{x^2+1}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{u}{x^2+1}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{2x}{(x^2 + 1)^2} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= -\frac{1}{(x^2 + 1)^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{u''(x)}{x^2 + 1} - \frac{2x u'(x)}{(x^2 + 1)^2} - \frac{u(x)}{(x^2 + 1)^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{c_1 x + c_2}{\sqrt{x^2 + 1}}$$

The above shows that

$$u'(x) = \frac{-c_2 x + c_1}{(x^2 + 1)^{\frac{3}{2}}}$$

Using the above in (1) gives the solution

$$y = \frac{-c_2 x + c_1}{c_1 x + c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{c_3 - x}{c_3 x + 1}$$

Initial conditions are used to solve for c_3. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_3$$

$$c_3 = 1$$

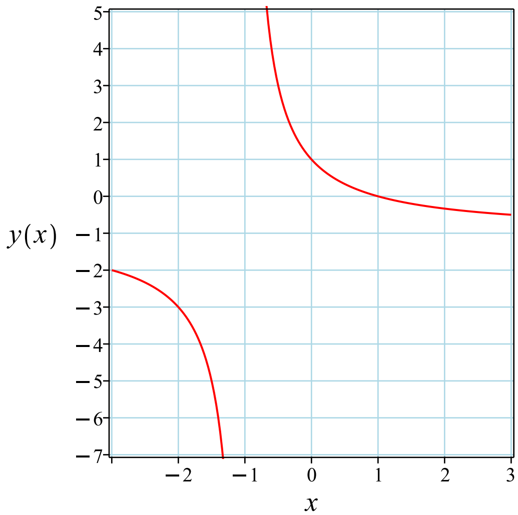
Substituting c_3 found above in the general solution gives

$$y = -\frac{x - 1}{x + 1}$$

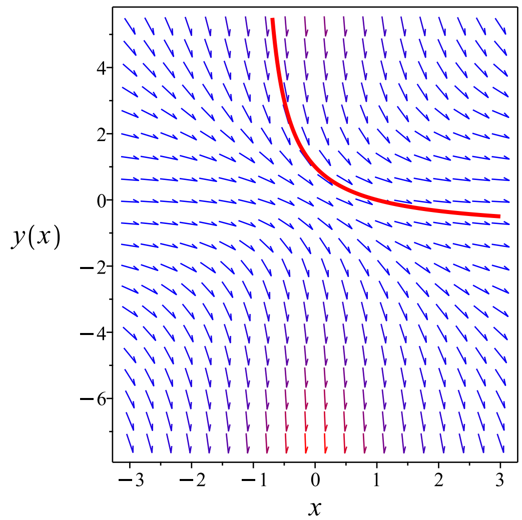
Summary

The solution(s) found are the following

$$y = -\frac{x - 1}{x + 1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{x-1}{x+1}$$

Verified OK.

1.12.6 Maple step by step solution

Let's solve

$[(x^2+1)\, y' + y^2 = -1, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-y^2-1} = \frac{1}{x^2+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{-y^2-1} dx = \int \frac{1}{x^2+1} dx + c_1$

- Evaluate integral

  $-\arctan(y) = \arctan(x) + c_1$

- Solve for y

  $y = -\tan(\arctan(x) + c_1)$

- Use initial condition $y(0) = 1$

  $1 = -\tan(c_1)$
- Solve for c_1

  $c_1 = -\frac{\pi}{4}$
- Substitute $c_1 = -\frac{\pi}{4}$ into general solution and simplify

  $y = \cot\left(\arctan(x) + \frac{\pi}{4}\right)$
- Solution to the IVP

  $y = \cot\left(\arctan(x) + \frac{\pi}{4}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple
Time used: 0.047 (sec). Leaf size: 11

```
dsolve([(x^2+1)*diff(y(x),x)+y(x)^2=-1,y(0) = 1],y(x), singsol=all)
```

$$y(x) = \cot\left(\arctan(x) + \frac{\pi}{4}\right)$$

✓ Solution by Mathematica
Time used: 0.242 (sec). Leaf size: 14

```
DSolve[{(x^2+1)*y'[x]+y[x]^2==-1,y[0]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \cot\left(\arctan(x) + \frac{\pi}{4}\right)$$

1.13 problem 13

Internal problem ID [2556]
Internal file name [`OUTPUT/2048_Sunday_June_05_2022_02_46_13_AM_4917224/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.4, page 36
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(-x^2+1)\,y'+yx=ax}$$

With initial conditions

$$[y(0)=2a]$$

1.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'+p(x)y=q(x)$$

Where here

$$\begin{aligned}p(x)&=-\frac{x}{x^2-1}\\q(x)&=-\frac{ax}{x^2-1}\end{aligned}$$

Hence the ode is

$$y' - \frac{xy}{x^2 - 1} = -\frac{ax}{x^2 - 1}$$

The domain of $p(x) = -\frac{x}{x^2-1}$ is

$$\{-\infty \le x < -1, -1 < x < 1, 1 < x \le \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -\frac{ax}{x^2-1}$ is

$$\{-\infty \le x < -1, -1 < x < 1, 1 < x \le \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

1.13.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{x(-a + y)}{x^2 - 1} \end{aligned}$$

Where $f(x) = \frac{x}{x^2-1}$ and $g(y) = -a + y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{-a + y}\, dy &= \frac{x}{x^2 - 1}\, dx \\ \int \frac{1}{-a + y}\, dy &= \int \frac{x}{x^2 - 1}\, dx \\ \ln(-a + y) &= \frac{\ln(x - 1)}{2} + \frac{\ln(x + 1)}{2} + c_1 \end{aligned}$$

Raising both side to exponential gives

$$-a + y = \mathrm{e}^{\frac{\ln(x-1)}{2} + \frac{\ln(x+1)}{2} + c_1}$$

Which simplifies to

$$-a + y = c_2 \mathrm{e}^{\frac{\ln(x-1)}{2} + \frac{\ln(x+1)}{2}}$$

Which can be simplified to become

$$y = c_2\sqrt{x-1}\,\sqrt{x+1}\,\mathrm{e}^{c_1} + a$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2a$ in the above solution gives an equation to solve for the constant of integration.

$$2a = i\mathrm{e}^{c_1}c_2 + a$$

$$c_1 = \ln\left(-\frac{ia}{c_2}\right)$$

Substituting c_1 found above in the general solution gives

$$y = -i\sqrt{x-1}\,\sqrt{x+1}\,a + a$$

Summary

The solution(s) found are the following

$$y = -i\sqrt{x-1}\,\sqrt{x+1}\,a + a \tag{1}$$

Verification of solutions

$$y = -i\sqrt{x-1}\,\sqrt{x+1}\,a + a$$

Verified OK.

1.13.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{x}{x^2-1}dx} \\ &= \mathrm{e}^{-\frac{\ln(x-1)}{2}-\frac{\ln(x+1)}{2}}\end{aligned}$$

Which simplifies to

$$\mu = \frac{1}{\sqrt{x-1}\,\sqrt{x+1}}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(-\frac{ax}{x^2-1}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{\sqrt{x-1}\,\sqrt{x+1}}\right) &= \left(\frac{1}{\sqrt{x-1}\,\sqrt{x+1}}\right)\left(-\frac{ax}{x^2-1}\right) \\ \mathrm{d}\left(\frac{y}{\sqrt{x-1}\,\sqrt{x+1}}\right) &= \left(-\frac{ax}{(x^2-1)\sqrt{x-1}\,\sqrt{x+1}}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\frac{y}{\sqrt{x-1}\,\sqrt{x+1}} = \int -\frac{ax}{(x^2-1)\sqrt{x-1}\,\sqrt{x+1}}\,\mathrm{d}x$$
$$\frac{y}{\sqrt{x-1}\,\sqrt{x+1}} = \frac{\sqrt{x-1}\,\sqrt{x+1}\,a}{x^2-1} + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{x-1}\,\sqrt{x+1}}$ results in

$$y = \frac{(x-1)(x+1)\,a}{x^2-1} + c_1\sqrt{x-1}\,\sqrt{x+1}$$

which simplifies to

$$y = a + c_1\sqrt{x-1}\,\sqrt{x+1}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2a$ in the above solution gives an equation to solve for the constant of integration.

$$2a = c_1 i + a$$

$$c_1 = -ia$$

Substituting c_1 found above in the general solution gives

$$y = -i\sqrt{x-1}\,\sqrt{x+1}\,a + a$$

Summary

The solution(s) found are the following

$$y = -i\sqrt{x-1}\,\sqrt{x+1}\,a + a \tag{1}$$

Verification of solutions

$$y = -i\sqrt{x-1}\,\sqrt{x+1}\,a + a$$

Verified OK.

1.13.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x(-a+y)}{x^2-1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 36: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= \mathrm{e}^{\frac{\ln(x-1)}{2} + \frac{\ln(x+1)}{2}} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{\ln(x-1)}{2} + \frac{\ln(x+1)}{2}}} dy \end{aligned}$$

Which results in

$$S = e^{\ln\left(\frac{1}{\sqrt{x-1}}\right) + \ln\left(\frac{1}{\sqrt{x+1}}\right)} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x(-a + y)}{x^2 - 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{yx}{(x-1)^{\frac{3}{2}} (x+1)^{\frac{3}{2}}} \\ S_y &= \frac{1}{\sqrt{x-1}\,\sqrt{x+1}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{ax}{(x-1)^{\frac{3}{2}} (x+1)^{\frac{3}{2}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{aR}{(R-1)^{\frac{3}{2}} (R+1)^{\frac{3}{2}}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{a}{\sqrt{R-1}\,\sqrt{R+1}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{\sqrt{x-1}\,\sqrt{x+1}} = \frac{a}{\sqrt{x-1}\,\sqrt{x+1}} + c_1$$

Which simplifies to

$$\frac{y}{\sqrt{x-1}\,\sqrt{x+1}} = \frac{a}{\sqrt{x-1}\,\sqrt{x+1}} + c_1$$

Which gives

$$y = a + c_1\sqrt{x-1}\,\sqrt{x+1}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2a$ in the above solution gives an equation to solve for the constant of integration.

$$2a = c_1 i + a$$

$$c_1 = -ia$$

Substituting c_1 found above in the general solution gives

$$y = -i\sqrt{x-1}\,\sqrt{x+1}\,a + a$$

Summary
The solution(s) found are the following

$$y = -i\sqrt{x-1}\,\sqrt{x+1}\,a + a \tag{1}$$

Verification of solutions

$$y = -i\sqrt{x-1}\,\sqrt{x+1}\,a + a$$

Verified OK.

1.13.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{-a+y}\right)\mathrm{d}y&=\left(\frac{x}{x^2-1}\right)\mathrm{d}x\\ \left(-\frac{x}{x^2-1}\right)\mathrm{d}x+\left(\frac{1}{-a+y}\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{x}{x^2-1} \\
N(x,y) &= \frac{1}{-a+y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{x^2-1}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{-a+y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -\frac{x}{x^2-1} \,\mathrm{d}x \\
\phi &= -\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{-a+y}$. Therefore equation (4) becomes

$$\frac{1}{-a+y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{a-y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{a-y}\right) \mathrm{d}y$$

$$f(y) = \ln(a-y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + \ln(a-y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + \ln(a-y)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2a$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{i\pi}{2} + \ln(-a) = c_1$$

$$c_1 = -\frac{i\pi}{2} + \ln(-a)$$

Substituting c_1 found above in the general solution gives

$$-\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + \ln(a-y) = -\frac{i\pi}{2} + \ln(-a)$$

Solving for y from the above gives

$$y = \left(-i\sqrt{x-1}\,\sqrt{x+1}+1\right) a$$

Summary

The solution(s) found are the following

$$y = \left(-i\sqrt{x-1}\,\sqrt{x+1}+1\right) a \tag{1}$$

Verification of solutions

$$y = \left(-i\sqrt{x-1}\,\sqrt{x+1}+1\right) a$$

Verified OK. `{positive}`

1.13.6 Maple step by step solution

Let's solve

$[(-x^2+1)\, y' + yx = ax, y(0) = 2a]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{a-y} = -\frac{x}{(x-1)(x+1)}$

- Integrate both sides with respect to x

  $\int \frac{y'}{a-y}dx = \int -\frac{x}{(x-1)(x+1)}dx + c_1$

- Evaluate integral

  $-\ln(a-y) = -\frac{\ln((x-1)(x+1))}{2} + c_1$

- Solve for y

  $y = -\mathrm{e}^{\frac{\ln((x-1)(x+1))}{2} - c_1} + a$

- Use initial condition $y(0) = 2a$

  $2a = -\mathrm{e}^{\frac{\mathrm{I}\pi}{2} - c_1} + a$

- Solve for c_1

  $c_1 = \frac{\mathrm{I}\pi}{2} - \ln(-a)$

- Substitute $c_1 = \frac{\mathrm{I}\pi}{2} - \ln(-a)$ into general solution and simplify

$$y = a\left(1 - \mathrm{I}\sqrt{x^2 - 1}\right)$$

- Solution to the IVP

$$y = a\left(1 - \mathrm{I}\sqrt{x^2 - 1}\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 20

```
dsolve([(1-x^2)*diff(y(x),x)+x*y(x)=a*x,y(0) = 2*a],y(x), singsol=all)
```

$$y(x) = a\left(1 - i\sqrt{x - 1}\,\sqrt{x + 1}\right)$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 21

```
DSolve[{(1-x^2)*y'[x]+x*y[x]==a*x,y[0]==2*a},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to a - ia\sqrt{x^2 - 1}$$

1.14 problem 14

Internal problem ID [2557]
Internal file name [`OUTPUT/2049_Sunday_June_05_2022_02_46_15_AM_48430184/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.4, page 36
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + \frac{\sin(y+x)}{\sin(y)\cos(x)} = 1}$$

With initial conditions

$$\left[y\left(\frac{\pi}{4}\right) = \frac{\pi}{4}\right]$$

1.14.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{\cos(x)\sin(y) - \sin(y+x)}{\sin(y)\cos(x)} \end{aligned}$$

The x domain of $f(x,y)$ when $y = \frac{\pi}{4}$ is

$$\left\{x < \frac{1}{2}\pi + \pi\_Z140 \vee \frac{1}{2}\pi + \pi\_Z140 < x\right\}$$

And the point $x_0 = \frac{\pi}{4}$ is inside this domain. The y domain of $f(x,y)$ when $x = \frac{\pi}{4}$ is

$$\{y < \pi\_Z141 \lor \pi\_Z141 < y\}$$

And the point $y_0 = \frac{\pi}{4}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{\cos(x)\sin(y) - \sin(y+x)}{\sin(y)\cos(x)}\right) \\ &= \frac{\cos(x)\cos(y) - \cos(y+x)}{\sin(y)\cos(x)} - \frac{(\cos(x)\sin(y) - \sin(y+x))\cos(y)}{\sin(y)^2\cos(x)}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = \frac{\pi}{4}$ is

$$\left\{x < \frac{1}{2}\pi + \pi\_Z140 \lor \frac{1}{2}\pi + \pi\_Z140 < x\right\}$$

And the point $x_0 = \frac{\pi}{4}$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = \frac{\pi}{4}$ is

$$\{y < \pi\_Z141 \lor \pi\_Z141 < y\}$$

And the point $y_0 = \frac{\pi}{4}$ is inside this domain. Therefore solution exists and is unique.

1.14.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{\sin(x)\cot(y)}{\cos(x)}\end{aligned}$$

Where $f(x) = -\frac{\sin(x)}{\cos(x)}$ and $g(y) = \cot(y)$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\cot(y)}\,dy &= -\frac{\sin(x)}{\cos(x)}\,dx \\ \int \frac{1}{\cot(y)}\,dy &= \int -\frac{\sin(x)}{\cos(x)}\,dx \\ -\ln(\cos(y)) &= \ln(\cos(x)) + c_1\end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{\cos(y)} = \mathrm{e}^{\ln(\cos(x))+c_1}$$

Which simplifies to

$$\sec(y) = c_2 \cos(x)$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{4}$ and $y = \frac{\pi}{4}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\pi}{4} = \frac{\pi}{2} - \arcsin\left(\frac{\sqrt{2}\,\mathrm{e}^{-c_1}}{c_2}\right)$$

$$c_1 = -\ln\left(\frac{c_2}{2}\right)$$

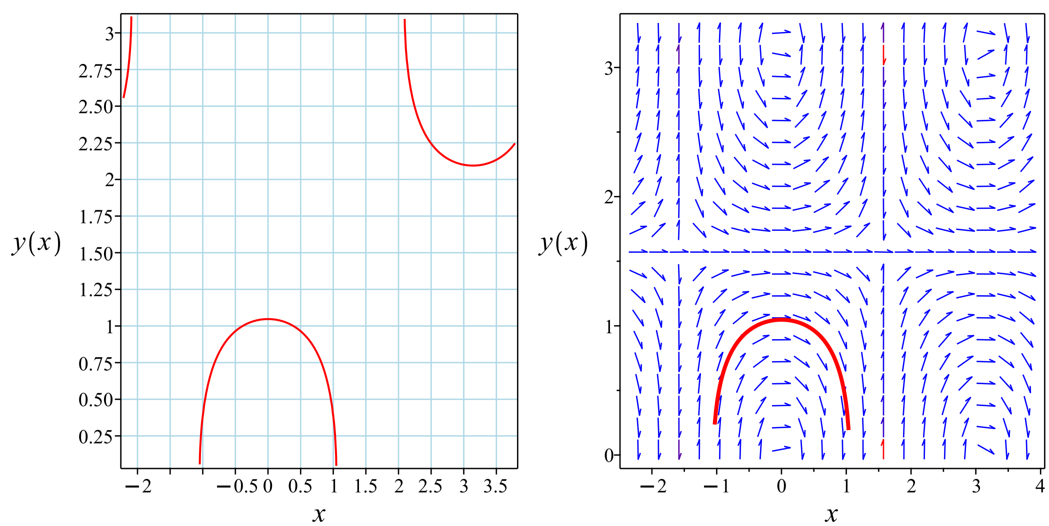
Substituting c_1 found above in the general solution gives

$$y = \frac{\pi}{2} - \arcsin\left(\frac{1}{2\cos(x)}\right)$$

Summary

The solution(s) found are the following

$$y = \frac{\pi}{2} - \arcsin\left(\frac{1}{2\cos(x)}\right) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{\pi}{2} - \arcsin\left(\frac{1}{2\cos(x)}\right)$$

Verified OK.

1.14.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}
y' &= \frac{\cos(x)\sin(y) - \sin(y+x)}{\sin(y)\cos(x)} \\
y' &= \omega(x,y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 39: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -\frac{\cos(x)}{\sin(x)} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{\cos(x)}{\sin(x)}} dx \end{aligned}$$

Which results in

$$S = \ln(\cos(x))$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{\cos(x)\sin(y) - \sin(y + x)}{\sin(y)\cos(x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -\tan(x) \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \tan(y) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \tan(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\ln(\cos(R)) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(\cos(x)) = -\ln(\cos(y)) + c_1$$

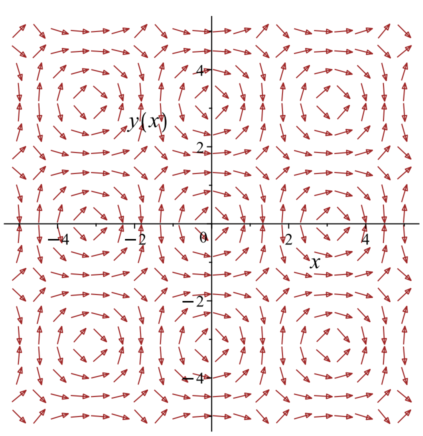
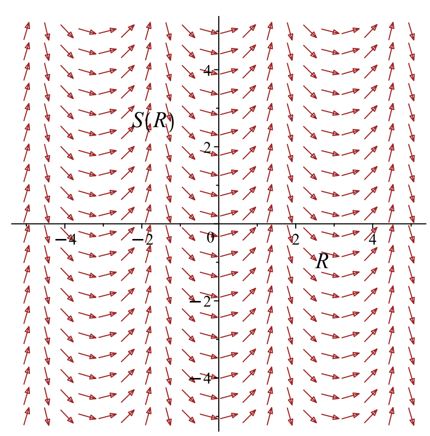
Which simplifies to

$$\ln(\cos(x)) = -\ln(\cos(y)) + c_1$$

Which gives

$$y = \arccos\left(\frac{e^{c_1}}{\cos(x)}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\cos(x)\sin(y)-\sin(y+x)}{\sin(y)\cos(x)}$ 	$R = y$ $S = \ln(\cos(x))$	$\frac{dS}{dR} = \tan(R)$ 

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{4}$ and $y = \frac{\pi}{4}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\pi}{4} = \frac{\pi}{2} - \arcsin\left(\sqrt{2}\, e^{c_1}\right)$$

$$c_1 = -\ln(2)$$

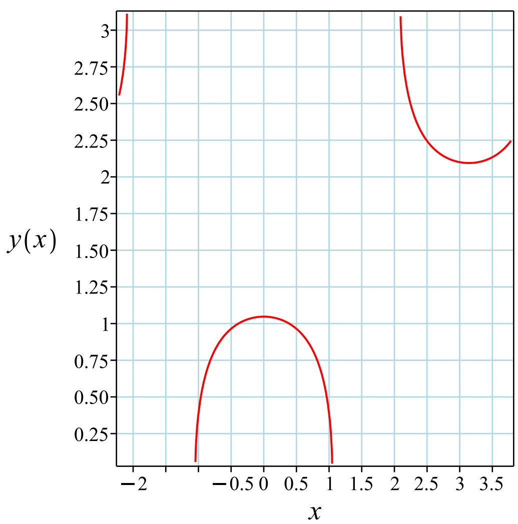
Substituting c_1 found above in the general solution gives

$$y = \frac{\pi}{2} - \arcsin\left(\frac{1}{2\cos(x)}\right)$$

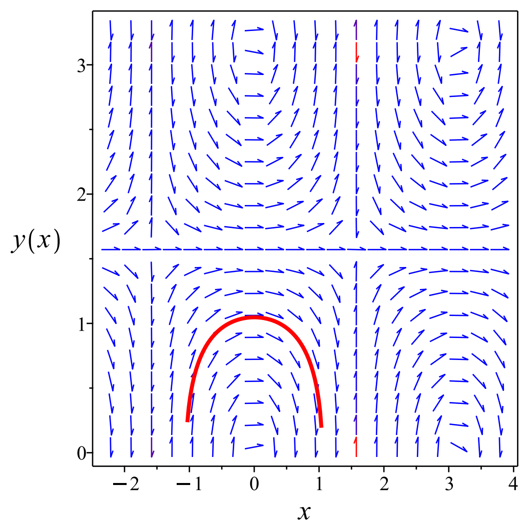
Summary

The solution(s) found are the following

$$y = \frac{\pi}{2} - \arcsin\left(\frac{1}{2\cos(x)}\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\pi}{2} - \arcsin\left(\frac{1}{2\cos(x)}\right)$$

Verified OK.

1.14.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{\sin(y)}{\cos(y)}\right)\mathrm{d}y&=\left(\frac{\sin(x)}{\cos(x)}\right)\mathrm{d}x\\ \left(-\frac{\sin(x)}{\cos(x)}\right)\mathrm{d}x+\left(-\frac{\sin(y)}{\cos(y)}\right)\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{\sin(x)}{\cos(x)}$$
$$N(x,y) = -\frac{\sin(y)}{\cos(y)}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\sin(x)}{\cos(x)}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{\sin(y)}{\cos(y)}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{\sin(x)}{\cos(x)}\,\mathrm{d}x\\ \phi &= \ln(\cos(x)) + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{\sin(y)}{\cos(y)}$. Therefore equation (4) becomes

$$-\frac{\sin(y)}{\cos(y)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned} f'(y) &= -\frac{\sin(y)}{\cos(y)} \\ &= -\tan(y) \end{aligned}$$

Integrating the above w.r.t y results in

$$\begin{aligned} \int f'(y)\,\mathrm{d}y &= \int (-\tan(y))\,\mathrm{d}y \\ f(y) &= \ln(\cos(y)) + c_1 \end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(\cos(x)) + \ln(\cos(y)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(\cos(x)) + \ln(\cos(y))$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{4}$ and $y = \frac{\pi}{4}$ in the above solution gives an equation to solve for the constant of integration.

$$-\ln(2) = c_1$$

$$c_1 = -\ln(2)$$

Substituting c_1 found above in the general solution gives

$$\ln(\cos(x)) + \ln(\cos(y)) = -\ln(2)$$

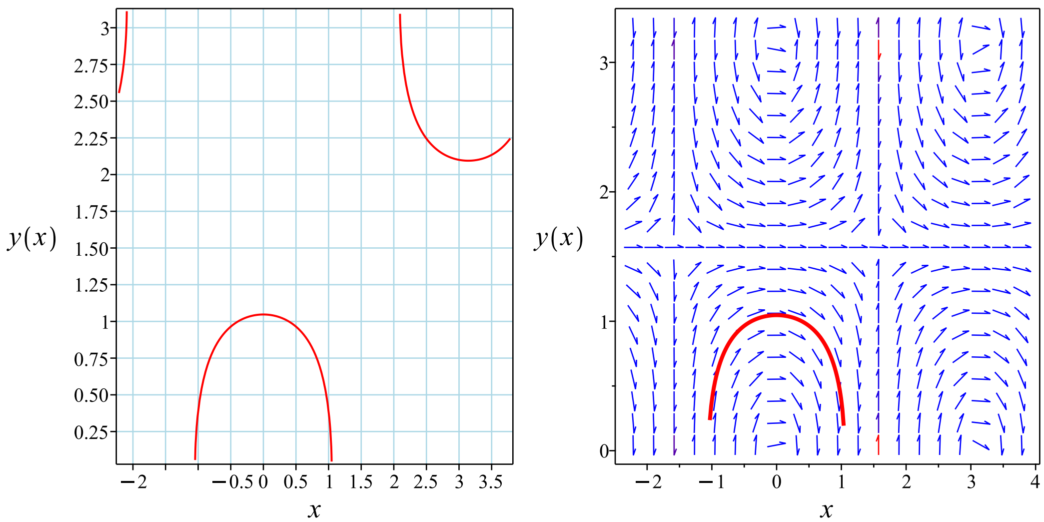
Solving for y from the above gives

$$y = \arccos\left(\frac{\sec(x)}{2}\right)$$

Summary

The solution(s) found are the following

$$y = \arccos\left(\frac{\sec(x)}{2}\right) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \arccos\left(\frac{\sec(x)}{2}\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.312 (sec). Leaf size: 9

```
dsolve([diff(y(x),x)=1- (sin(x+y(x)))/(sin(y(x))*cos(x)),y(1/4*Pi) = 1/4*Pi],y(x), singsol=a
```

$$y(x) = \frac{\pi}{2} - \arcsin\left(\frac{\sec(x)}{2}\right)$$

✓ Solution by Mathematica

Time used: 6.234 (sec). Leaf size: 12

```
DSolve[{y'[x]==1- Sin[x+y[x]]/(Sin[y[x]]*Cos[x]),y[Pi/4]==Pi/4},y[x],x,IncludeSingularSoluti
```

$$y(x) \to \arccos\left(\frac{\sec(x)}{2}\right)$$

1.15 problem 15

Internal problem ID [2558]
Internal file name [`OUTPUT/2050_Sunday_June_05_2022_02_46_22_AM_38738506/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.4, page 36
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - y^3 \sin(x) = 0}$$

1.15.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= y^3 \sin(x) \end{aligned}$$

Where $f(x) = \sin(x)$ and $g(y) = y^3$. Integrating both sides gives

$$\frac{1}{y^3}\,dy = \sin(x)\,dx$$

$$\int \frac{1}{y^3}\,dy = \int \sin(x)\,dx$$

$$-\frac{1}{2y^2} = -\cos(x) + c_1$$

Which results in

$$y = -\frac{1}{\sqrt{-2c_1 + 2\cos(x)}}$$

$$y = \frac{1}{\sqrt{-2c_1 + 2\cos(x)}}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{\sqrt{-2c_1 + 2\cos(x)}} \tag{1}$$

$$y = \frac{1}{\sqrt{-2c_1 + 2\cos(x)}} \tag{2}$$

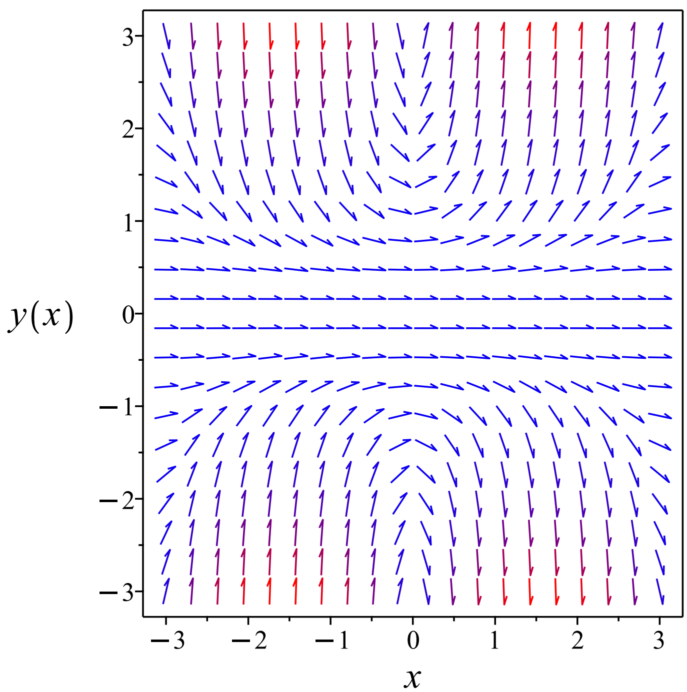


Figure 49: Slope field plot

Verification of solutions

$$y = -\frac{1}{\sqrt{-2c_1 + 2\cos(x)}}$$

Verified OK.

$$y = \frac{1}{\sqrt{-2c_1 + 2\cos(x)}}$$

Verified OK.

1.15.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = y^3 \sin(x)$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 41: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{1}{\sin(x)} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{\sin(x)}} dx \end{aligned}$$

Which results in

$$S = -\cos(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = y^3 \sin(x)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \sin(x) \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{2R^2} + c_1 \tag{4}$$

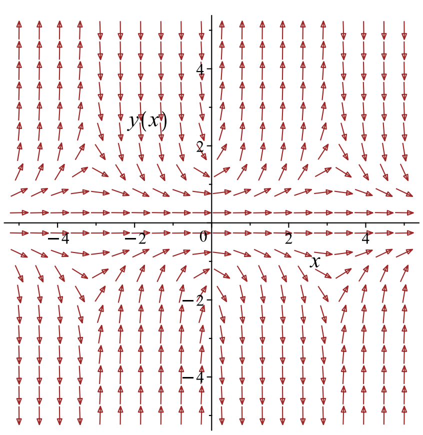
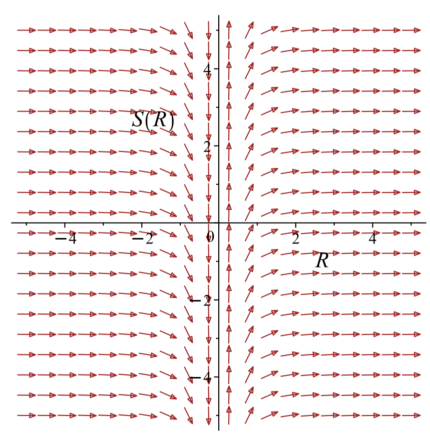
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\cos(x) = -\frac{1}{2y^2} + c_1$$

Which simplifies to

$$-\cos(x) = -\frac{1}{2y^2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = y^3 \sin(x)$ 	$R = y$ $S = -\cos(x)$	$\frac{dS}{dR} = \frac{1}{R^3}$ 

Summary

The solution(s) found are the following

$$-\cos(x) = -\frac{1}{2y^2} + c_1 \tag{1}$$

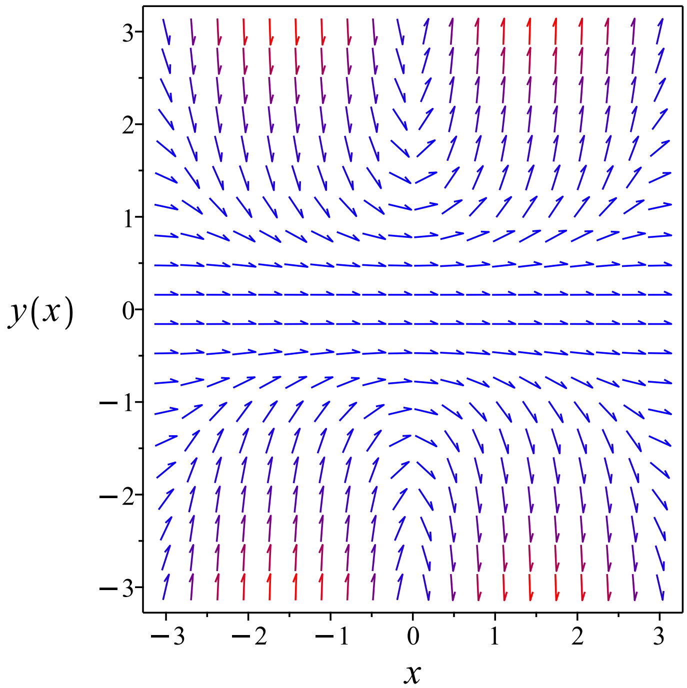


Figure 50: Slope field plot

Verification of solutions

$$-\cos(x) = -\frac{1}{2y^2} + c_1$$

Verified OK.

1.15.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y^3}\right)\mathrm{d}y = (\sin(x))\,\mathrm{d}x$$
$$(-\sin(x))\,\mathrm{d}x + \left(\frac{1}{y^3}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\sin(x)$$
$$N(x, y) = \frac{1}{y^3}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(-\sin(x))$$
$$= 0$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^3}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\sin(x) \, \mathrm{d}x \\
\phi &= \cos(x) + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^3}$. Therefore equation (4) becomes

$$
\frac{1}{y^3} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y^3}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(\frac{1}{y^3}\right) \mathrm{d}y \\
f(y) &= -\frac{1}{2y^2} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \cos(x) - \frac{1}{2y^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \cos(x) - \frac{1}{2y^2}$$

Summary

The solution(s) found are the following

$$\cos(x) - \frac{1}{2y^2} = c_1 \tag{1}$$

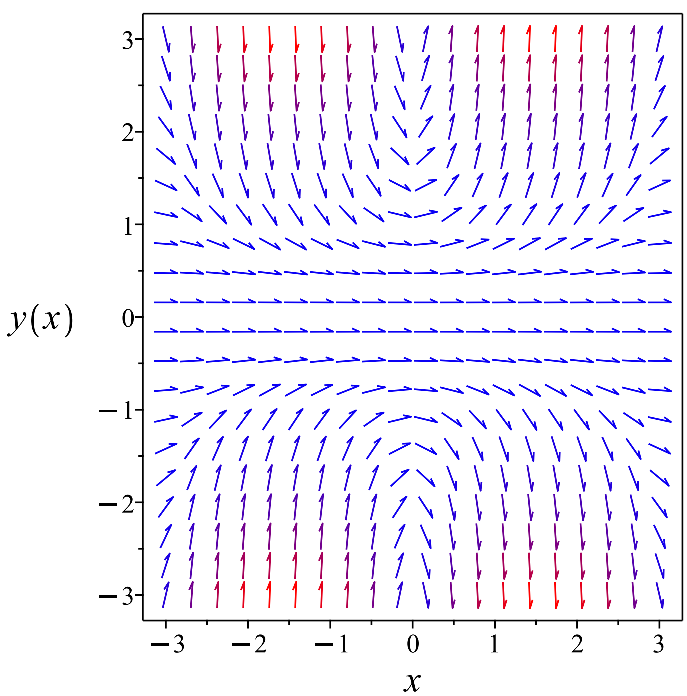


Figure 51: Slope field plot

Verification of solutions

$$\cos(x) - \frac{1}{2y^2} = c_1$$

Verified OK.

1.15.4 Maple step by step solution

Let's solve

$y' - y^3 \sin(x) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^3} = \sin(x)$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^3} dx = \int \sin(x)\, dx + c_1$

- Evaluate integral

  $-\frac{1}{2y^2} = -\cos(x) + c_1$

- Solve for y

  $\left\{y = \frac{1}{\sqrt{-2c_1 + 2\cos(x)}}, y = -\frac{1}{\sqrt{-2c_1 + 2\cos(x)}}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(diff(y(x),x)=y(x)^3*sin(x),y(x), singsol=all)
```

$$y(x) = \frac{1}{\sqrt{c_1 + 2\cos(x)}}$$
$$y(x) = -\frac{1}{\sqrt{c_1 + 2\cos(x)}}$$

✓ Solution by Mathematica

Time used: 0.207 (sec). Leaf size: 49

```
DSolve[y'[x]==y[x]^3*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{\sqrt{2}\sqrt{\cos(x) - c_1}}$$
$$y(x) \to \frac{1}{\sqrt{2}\sqrt{\cos(x) - c_1}}$$
$$y(x) \to 0$$

2 1.6, page 50

2.1 problem 1

Internal problem ID [2559]
Internal file name [`OUTPUT/2051_Sunday_June_05_2022_02_46_25_AM_77727377/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.6, page 50
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y = e^{2x}}$$

2.1.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -1$$
$$q(x) = e^{2x}$$

Hence the ode is

$$y' - y = e^{2x}$$

The integrating factor μ is

$$\mu = e^{\int (-1)dx}$$
$$= e^{-x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(e^{2x}\right)$$
$$\frac{d}{dx}\left(e^{-x}y\right) = \left(e^{-x}\right)\left(e^{2x}\right)$$
$$d\left(e^{-x}y\right) = e^{x}\, dx$$

Integrating gives

$$e^{-x}y = \int e^{x}\, dx$$
$$e^{-x}y = e^{x} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{-x}$ results in

$$y = e^{2x} + c_1 e^{x}$$

Summary
The solution(s) found are the following

$$y = e^{2x} + c_1 e^{x} \tag{1}$$

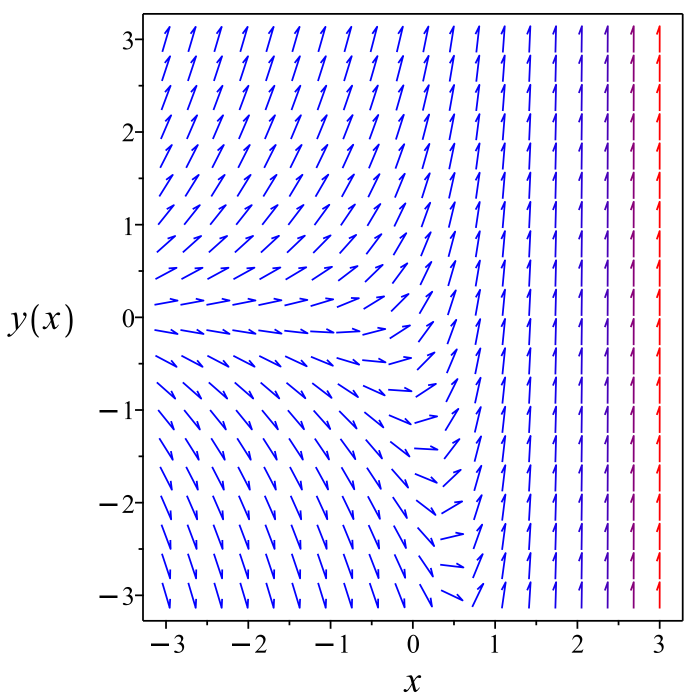


Figure 52: Slope field plot

Verification of solutions

$$y = e^{2x} + c_1 e^{x}$$

Verified OK.

2.1.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= y + e^{2x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 44: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^x} dy \end{aligned}$$

Which results in

$$S = e^{-x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = y + e^{2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -e^{-x} y \\ S_y &= e^{-x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \mathrm{e}^{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{-x} y = \mathrm{e}^{x} + c_1$$

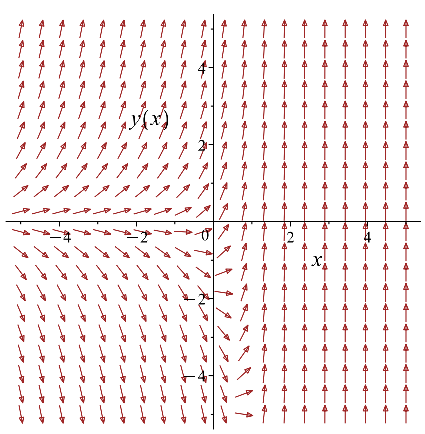
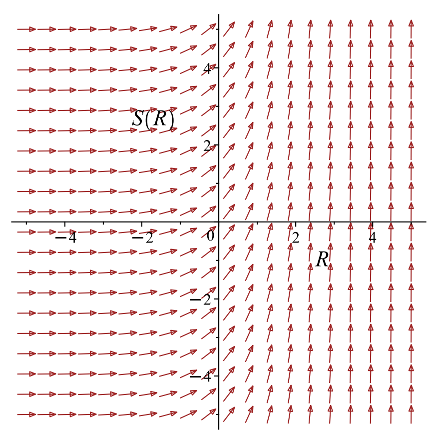
Which simplifies to

$$\mathrm{e}^{-x} y = \mathrm{e}^{x} + c_1$$

Which gives

$$y = \mathrm{e}^{x}\left(\mathrm{e}^{x} + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = y + \mathrm{e}^{2x}$

</td><td>$R = x$
$S = \mathrm{e}^{-x} y$</td><td>$\frac{dS}{dR} = \mathrm{e}^{R}$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{x}\left(\mathrm{e}^{x} + c_1\right) \tag{1}$$

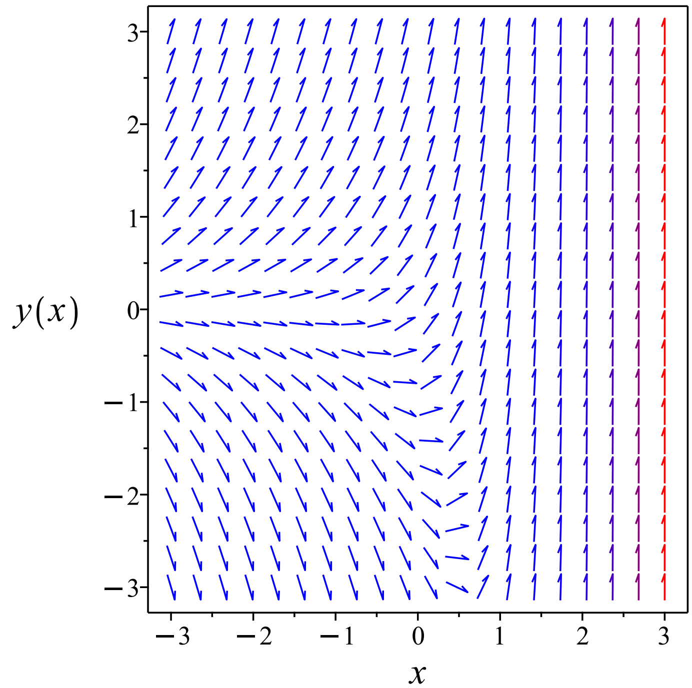


Figure 53: Slope field plot

Verification of solutions

$$y = e^{x}(e^{x} + c_1)$$

Verified OK.

2.1.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(y + \mathrm{e}^{2x}\right)\mathrm{d}x\\ \left(-y - \mathrm{e}^{2x}\right)\mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -y - \mathrm{e}^{2x}\\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y - \mathrm{e}^{2x}\right)\\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((-1) - (0)) \\ &= -1 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -1 \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-x} \\ &= \mathrm{e}^{-x} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{-x}\left(-y - \mathrm{e}^{2x}\right) \\ &= -\mathrm{e}^{-x} y - \mathrm{e}^{x} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{-x}(1) \\ &= \mathrm{e}^{-x} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-\mathrm{e}^{-x} y - \mathrm{e}^{x}\right) + \left(\mathrm{e}^{-x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int \overline{M} \, dx$$

$$\int \frac{\partial \phi}{\partial x} \, dx = \int -e^{-x} y - e^{x} \, dx$$

$$\phi = e^{-x} y - e^{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = e^{-x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = e^{-x}$. Therefore equation (4) becomes

$$e^{-x} = e^{-x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = e^{-x} y - e^{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = e^{-x} y - e^{x}$$

The solution becomes

$$y = e^{x}(e^{x} + c_1)$$

Summary

The solution(s) found are the following

$$y = e^{x}(e^{x} + c_1) \tag{1}$$

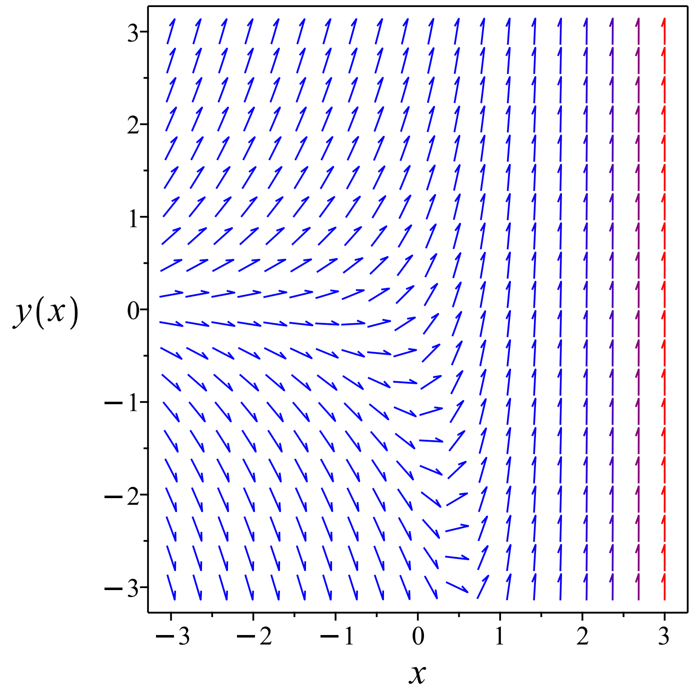


Figure 54: Slope field plot

<u>Verification of solutions</u>

$$y = e^{x}(e^{x} + c_1)$$

Verified OK.

2.1.4 Maple step by step solution

Let's solve

$y' - y = e^{2x}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = y + e^{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y = e^{2x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)(y' - y) = \mu(x)\, e^{2x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\,(y' - y) = \mu'(x)\, y + \mu(x)\, y'$
- Isolate $\mu'(x)$

  $\mu'(x) = -\mu(x)$
- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{-x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, \mathrm{e}^{2x} dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, \mathrm{e}^{2x} dx + c_1$
- Solve for y

  $y = \frac{\int \mu(x)\mathrm{e}^{2x} dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{-x}$

  $y = \frac{\int \mathrm{e}^{2x}\mathrm{e}^{-x} dx + c_1}{\mathrm{e}^{-x}}$
- Evaluate the integrals on the rhs

  $y = \frac{\mathrm{e}^{x} + c_1}{\mathrm{e}^{-x}}$
- Simplify

  $y = \mathrm{e}^{x}(\mathrm{e}^{x} + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(diff(y(x),x)-y(x)=exp(2*x),y(x), singsol=all)
```

$$y(x) = (e^x + c_1) e^x$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 15

```
DSolve[y'[x]-y[x]==Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow e^x(e^x + c_1)$$

2.2 problem 2

Internal problem ID [2560]
Internal file name [`OUTPUT/2052_Sunday_June_05_2022_02_46_27_AM_62150205/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.6, page 50
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y'x^2 - 4yx = x^7 \sin(x)}$$

2.2.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{4}{x}$$
$$q(x) = x^5 \sin(x)$$

Hence the ode is

$$y' - \frac{4y}{x} = x^5 \sin(x)$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{4}{x}dx} \\
&= \frac{1}{x^4}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(x^5 \sin(x)\right) \\
\frac{d}{dx}\left(\frac{y}{x^4}\right) &= \left(\frac{1}{x^4}\right)\left(x^5 \sin(x)\right) \\
d\left(\frac{y}{x^4}\right) &= (x \sin(x))\ dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{x^4} &= \int x \sin(x)\ dx \\
\frac{y}{x^4} &= \sin(x) - \cos(x)\, x + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^4}$ results in

$$
y = x^4(\sin(x) - \cos(x)\, x) + c_1 x^4
$$

which simplifies to

$$
y = x^4(\sin(x) - \cos(x)\, x + c_1)
$$

Summary

The solution(s) found are the following

$$
y = x^4(\sin(x) - \cos(x)\, x + c_1) \tag{1}
$$

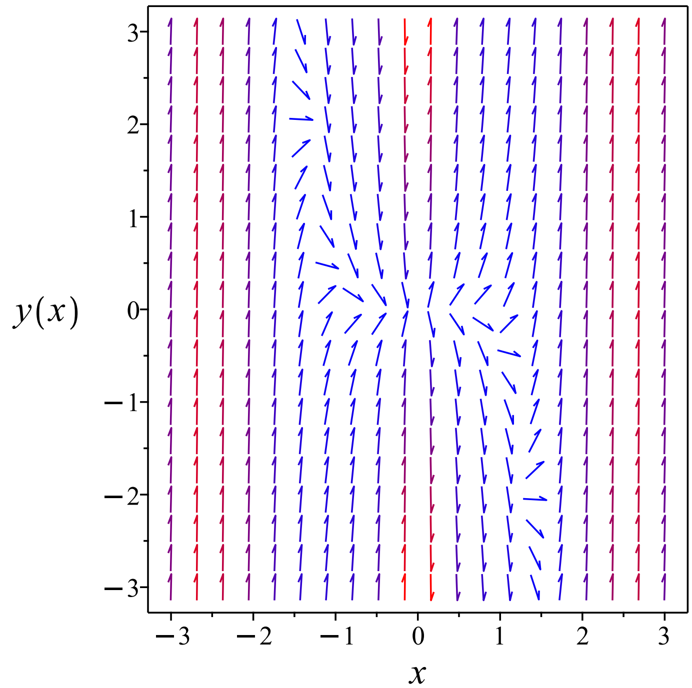


Figure 55: Slope field plot

Verification of solutions

$$y = x^4(\sin(x) - \cos(x)\, x + c_1)$$

Verified OK.

2.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{\sin(x)\, x^6 + 4y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 47: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x^4
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^4} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x^4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{\sin(x) x^6 + 4y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{4y}{x^5} \\ S_y &= \frac{1}{x^4} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x \sin(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R \sin(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \sin(R) - R\cos(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x^4} = \sin(x) - \cos(x)\, x + c_1$$

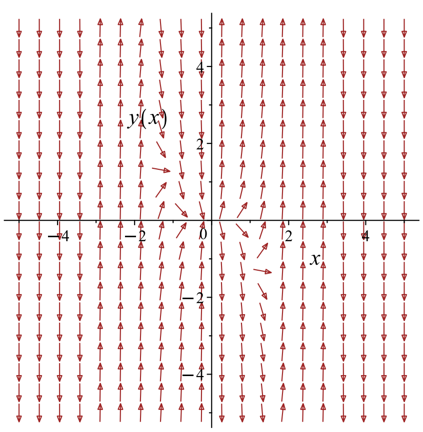
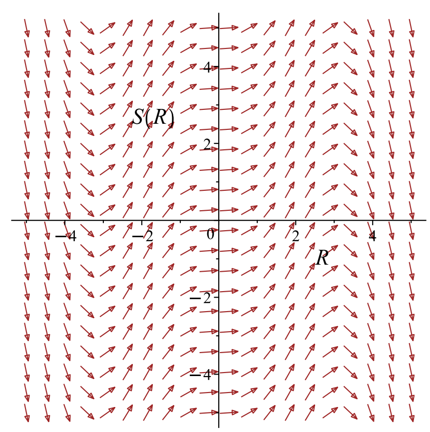
Which simplifies to

$$\frac{y}{x^4} = \sin(x) - \cos(x)\, x + c_1$$

Which gives

$$y = x^4(\sin(x) - \cos(x)\, x + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\sin(x)x^6+4y}{x}$ 	$R = x$ $S = \frac{y}{x^4}$	$\frac{dS}{dR} = R\sin(R)$ 

Summary
The solution(s) found are the following

$$y = x^4(\sin(x) - \cos(x)\, x + c_1) \tag{1}$$

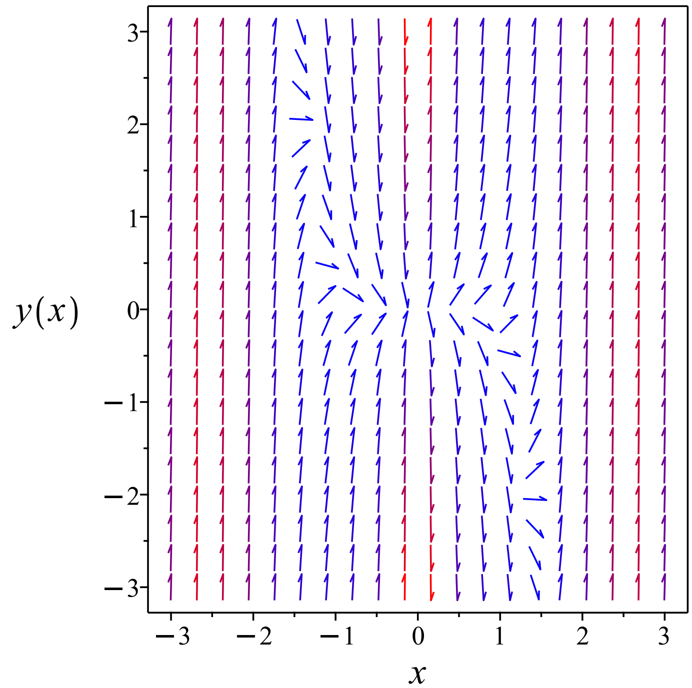


Figure 56: Slope field plot

Verification of solutions

$$y = x^4(\sin(x) - \cos(x)\, x + c_1)$$

Verified OK.

2.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(x^2\right)\mathrm{d}y = \left(4xy + x^7 \sin(x)\right)\mathrm{d}x$$
$$\left(-x^7 \sin(x) - 4xy\right)\mathrm{d}x + \left(x^2\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x^7 \sin(x) - 4xy$$
$$N(x, y) = x^2$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^7 \sin(x) - 4xy\right) \\ &= -4x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^2\right) \\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x^2}((-4x) - (2x)) \\ &= -\frac{6}{x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -\frac{6}{x} \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-6\ln(x)} \\ &= \frac{1}{x^6} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{x^6}\left(-x^7 \sin(x) - 4xy\right) \\ &= \frac{-\sin(x)\, x^6 - 4y}{x^5} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{x^6}\left(x^2\right) \\ &= \frac{1}{x^4} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{-\sin(x)\, x^6 - 4y}{x^5}\right) + \left(\frac{1}{x^4}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{-\sin(x) x^6 - 4y}{x^5} \, \mathrm{d}x$$

$$\phi = -\sin(x) + \cos(x) x + \frac{y}{x^4} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x^4} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x^4}$. Therefore equation (4) becomes

$$\frac{1}{x^4} = \frac{1}{x^4} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\sin(x) + \cos(x) x + \frac{y}{x^4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\sin(x) + \cos(x) x + \frac{y}{x^4}$$

The solution becomes

$$y = x^4(\sin(x) - \cos(x)\,x + c_1)$$

Summary

The solution(s) found are the following

$$y = x^4(\sin(x) - \cos(x)\,x + c_1) \tag{1}$$

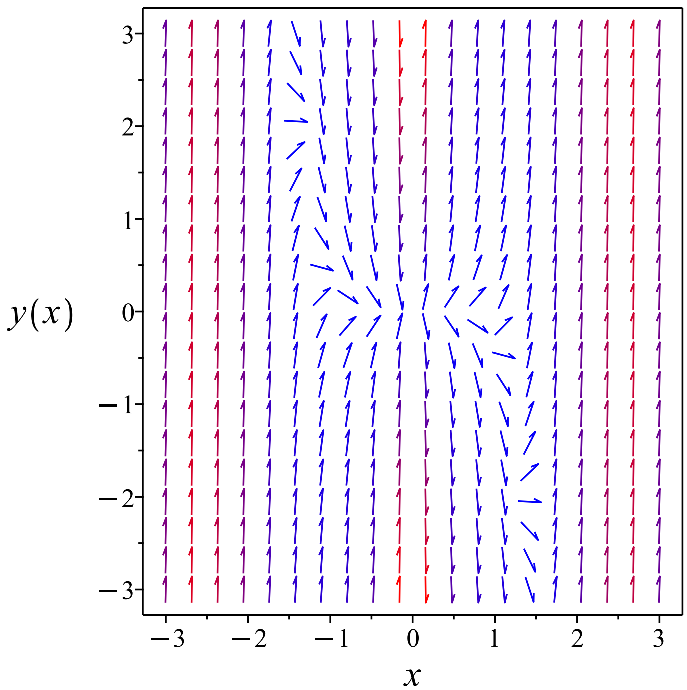


Figure 57: Slope field plot

Verification of solutions

$$y = x^4(\sin(x) - \cos(x)\,x + c_1)$$

Verified OK.

2.2.4 Maple step by step solution

Let's solve

$y'x^2 - 4yx = x^7 \sin(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{4y}{x} + x^5 \sin(x)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{4y}{x} = x^5 \sin(x)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x) \left(y' - \frac{4y}{x}\right) = \mu(x) x^5 \sin(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x) y)$

  $\mu(x) \left(y' - \frac{4y}{x}\right) = \mu'(x) y + \mu(x) y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{4\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x^4}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x) y)\right) dx = \int \mu(x) x^5 \sin(x) dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x) y = \int \mu(x) x^5 \sin(x) dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) x^5 \sin(x) dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x^4}$

  $y = x^4 \left(\int x \sin(x) dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x^4(\sin(x) - \cos(x) x + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(x^2*diff(y(x),x)-4*x*y(x)=x^7*sin(x),y(x), singsol=all)
```

$$y(x) = (-x\cos(x) + \sin(x) + c_1)\, x^4$$

✓ Solution by Mathematica

Time used: 0.065 (sec). Leaf size: 19

```
DSolve[x^2*y'[x]-4*x*y[x]==x^7*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^4(\sin(x) - x\cos(x) + c_1)$$

2.3 problem 3

Internal problem ID [2561]
Internal file name [`OUTPUT/2053_Sunday_June_05_2022_02_46_29_AM_73525458/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.6, page 50
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + 2yx = 2x^3}$$

2.3.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 2x \\ q(x) &= 2x^3 \end{aligned}$$

Hence the ode is

$$y' + 2yx = 2x^3$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 2x dx} \\ &= e^{x^2} \end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(2x^3\right)$$
$$\frac{d}{dx}\left(e^{x^2} y\right) = \left(e^{x^2}\right)\left(2x^3\right)$$
$$d\left(e^{x^2} y\right) = \left(2x^3 e^{x^2}\right)\, dx$$

Integrating gives

$$e^{x^2} y = \int 2x^3 e^{x^2}\, dx$$
$$e^{x^2} y = \left(x^2-1\right) e^{x^2} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{x^2}$ results in

$$y = e^{-x^2}\left(x^2-1\right) e^{x^2} + c_1 e^{-x^2}$$

which simplifies to

$$y = x^2 - 1 + c_1 e^{-x^2}$$

Summary

The solution(s) found are the following

$$y = x^2 - 1 + c_1 e^{-x^2} \tag{1}$$

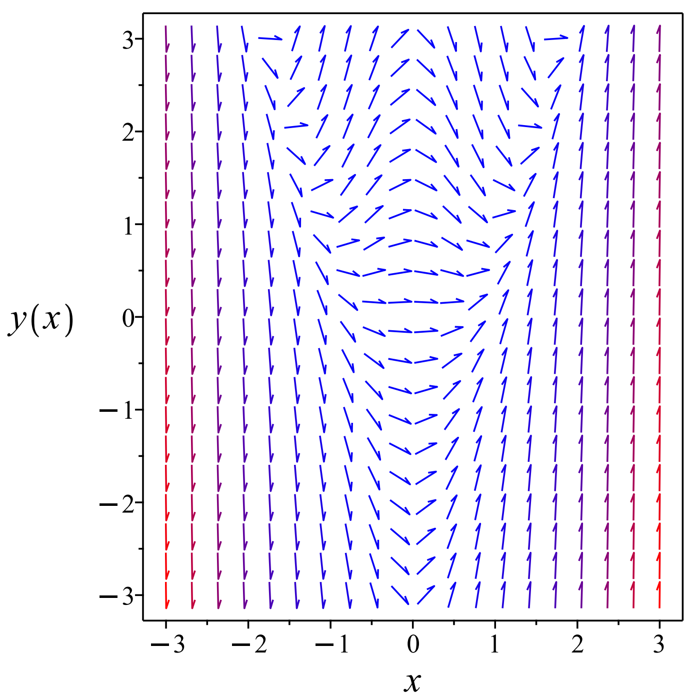


Figure 58: Slope field plot

Verification of solutions

$$y = x^2 - 1 + c_1 \mathrm{e}^{-x^2}$$

Verified OK.

2.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= 2x^3 - 2xy \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 50: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-x^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-x^2}} dy \end{aligned}$$

Which results in

$$S = e^{x^2} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = 2x^3 - 2xy$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2x\, e^{x^2} y \\ S_y &= e^{x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2x^3 e^{x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2R^3 e^{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = (R^2 - 1)\,\mathrm{e}^{R^2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{x^2} y = (x^2 - 1)\,\mathrm{e}^{x^2} + c_1$$

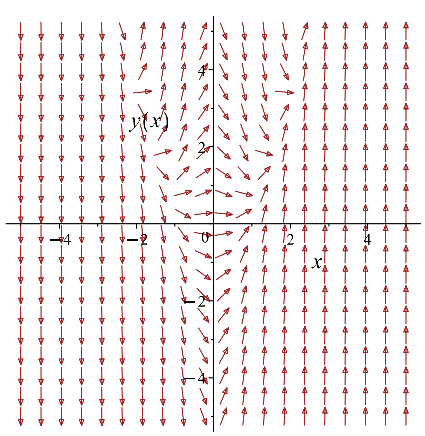
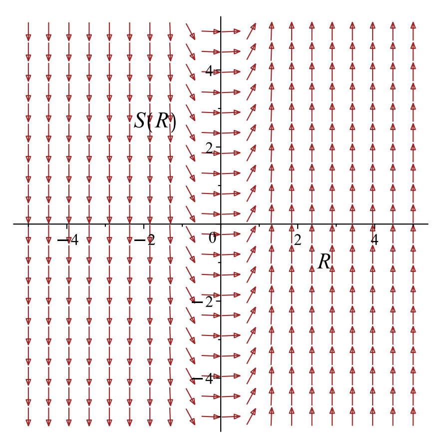
Which simplifies to

$$\mathrm{e}^{x^2} y = (x^2 - 1)\,\mathrm{e}^{x^2} + c_1$$

Which gives

$$y = \left(x^2 \mathrm{e}^{x^2} - \mathrm{e}^{x^2} + c_1\right) \mathrm{e}^{-x^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 2x^3 - 2xy$ 	$R = x$ $S = \mathrm{e}^{x^2} y$	$\frac{dS}{dR} = 2R^3 \mathrm{e}^{R^2}$ 

<u>Summary</u>
The solution(s) found are the following

$$y = \left(x^2 \mathrm{e}^{x^2} - \mathrm{e}^{x^2} + c_1\right) \mathrm{e}^{-x^2} \tag{1}$$

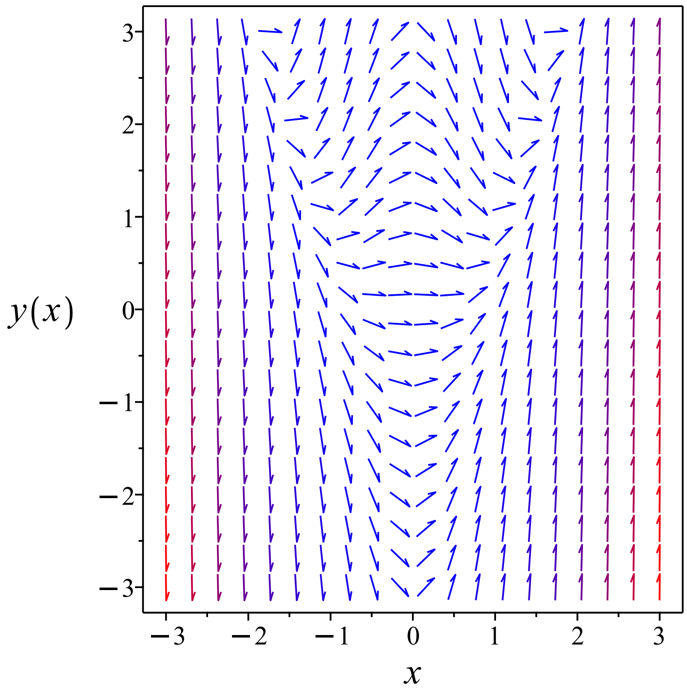


Figure 59: Slope field plot

Verification of solutions

$$y = \left(x^2 e^{x^2} - e^{x^2} + c_1\right) e^{-x^2}$$

Verified OK.

2.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= \left(2x^3 - 2xy\right) \mathrm{d}x \\
\left(-2x^3 + 2xy\right) \mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -2x^3 + 2xy \\
N(x, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-2x^3 + 2xy\right) \\
&= 2x
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((2x) - (0)) \\ &= 2x \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, dx} \\ &= e^{\int 2x \, dx} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{x^2} \\ &= \mathrm{e}^{x^2} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{x^2}(-2x^3 + 2xy) \\ &= -2x(x^2 - y)\,\mathrm{e}^{x^2} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{x^2}(1) \\ &= \mathrm{e}^{x^2} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-2x(x^2 - y)\,\mathrm{e}^{x^2}\right) + \left(\mathrm{e}^{x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -2x\left(x^2 - y\right)\mathrm{e}^{x^2}\,\mathrm{d}x$$
$$\phi = -\left(x^2 - y - 1\right)\mathrm{e}^{x^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{x^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{x^2}$. Therefore equation (4) becomes

$$\mathrm{e}^{x^2} = \mathrm{e}^{x^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\left(x^2 - y - 1\right)\mathrm{e}^{x^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\left(x^2 - y - 1\right)\mathrm{e}^{x^2}$$

The solution becomes

$$y = \left(x^2\mathrm{e}^{x^2} - \mathrm{e}^{x^2} + c_1\right)\mathrm{e}^{-x^2}$$

Summary

The solution(s) found are the following

$$y = \left(x^2 e^{x^2} - e^{x^2} + c_1\right) e^{-x^2} \tag{1}$$

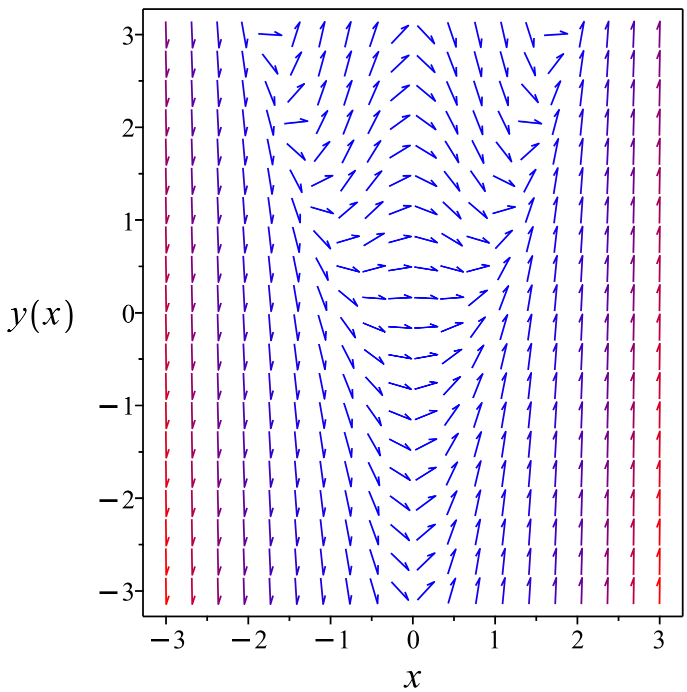


Figure 60: Slope field plot

Verification of solutions

$$y = \left(x^2 e^{x^2} - e^{x^2} + c_1\right) e^{-x^2}$$

Verified OK.

2.3.4 Maple step by step solution

Let's solve

$y' + 2yx = 2x^3$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -2yx + 2x^3$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE
  $y' + 2yx = 2x^3$
- The ODE is linear; multiply by an integrating factor $\mu(x)$
  $\mu(x)\,(y' + 2yx) = 2\mu(x)\,x^3$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$
  $\mu(x)\,(y' + 2yx) = \mu'(x)\,y + \mu(x)\,y'$
- Isolate $\mu'(x)$
  $\mu'(x) = 2\mu(x)\,x$
- Solve to find the integrating factor
  $\mu(x) = \mathrm{e}^{x^2}$
- Integrate both sides with respect to x
  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int 2\mu(x)\,x^3 dx + c_1$
- Evaluate the integral on the lhs
  $\mu(x)\,y = \int 2\mu(x)\,x^3 dx + c_1$
- Solve for y
  $y = \frac{\int 2\mu(x)x^3 dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{x^2}$
  $y = \frac{\int 2x^3\mathrm{e}^{x^2} dx + c_1}{\mathrm{e}^{x^2}}$
- Evaluate the integrals on the rhs
  $y = \frac{(x^2-1)\mathrm{e}^{x^2} + c_1}{\mathrm{e}^{x^2}}$
- Simplify
  $y = x^2 - 1 + c_1\mathrm{e}^{-x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve(diff(y(x),x)+2*x*y(x)=2*x^3,y(x), singsol=all)
```

$$y(x) = x^2 - 1 + c_1 \mathrm{e}^{-x^2}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 20

```
DSolve[y'[x]+2*x*y[x]==2*x^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2 + c_1 e^{-x^2} - 1$$

2.4 problem 4

Internal problem ID [2562]
Internal file name [`OUTPUT/2054_Sunday_June_05_2022_02_46_32_AM_34693024/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.6, page 50
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y' + \frac{2xy}{x^2 + 1} = 4x}$$

2.4.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{2x}{x^2 + 1} \\ q(x) &= 4x \end{aligned}$$

Hence the ode is

$$y' + \frac{2xy}{x^2 + 1} = 4x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2x}{x^2+1}dx} \\ &= x^2 + 1\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(4x) \\ \frac{d}{dx}\left(\left(x^2+1\right) y\right) &= \left(x^2+1\right)(4x) \\ d\left(\left(x^2+1\right) y\right) &= \left(4x\left(x^2+1\right)\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\left(x^2+1\right) y &= \int 4x\left(x^2+1\right) dx \\ \left(x^2+1\right) y &= \left(x^2+1\right)^2 + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2 + 1$ results in

$$y = x^2 + 1 + \frac{c_1}{x^2+1}$$

Summary
The solution(s) found are the following

$$y = x^2 + 1 + \frac{c_1}{x^2+1} \tag{1}$$

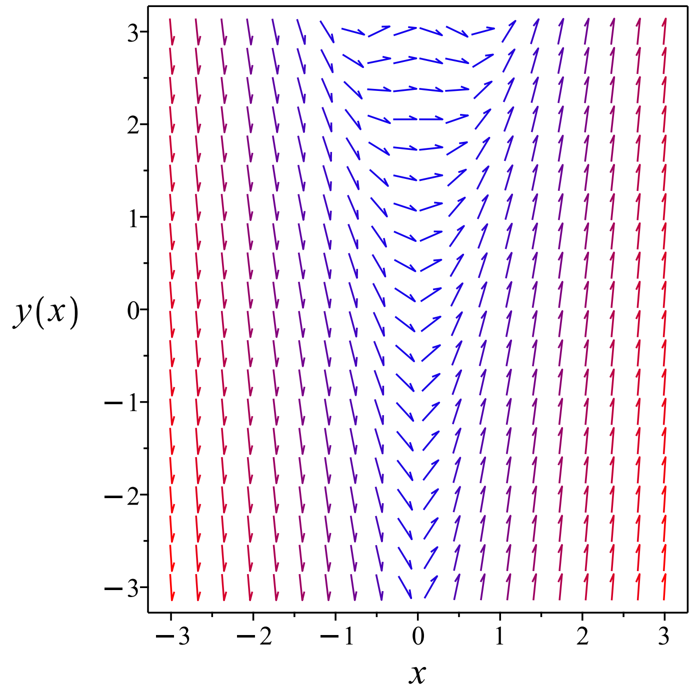


Figure 61: Slope field plot

Verification of solutions

$$y = x^2 + 1 + \frac{c_1}{x^2 + 1}$$

Verified OK.

2.4.2 Solving as differentialType ode

Writing the ode as

$$y' = -\frac{2xy}{x^2 + 1} + 4x \tag{1}$$

Which becomes

$$0 = (-x^2 - 1)\, dy + (2x(2x^2 - y + 2))\, dx \tag{2}$$

But the RHS is complete differential because

$$(-x^2 - 1)\, dy + (2x(2x^2 - y + 2))\, dx = d\left(\frac{(-2x^2 + y - 2)^2}{4} - \frac{y^2}{4}\right)$$

Hence (2) becomes

$$0 = d\left(\frac{(-2x^2+y-2)^2}{4} - \frac{y^2}{4}\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{x^4+2x^2+c_1+1}{x^2+1} + c_1$$

Summary
The solution(s) found are the following

$$y = \frac{x^4+2x^2+c_1+1}{x^2+1} + c_1 \tag{1}$$

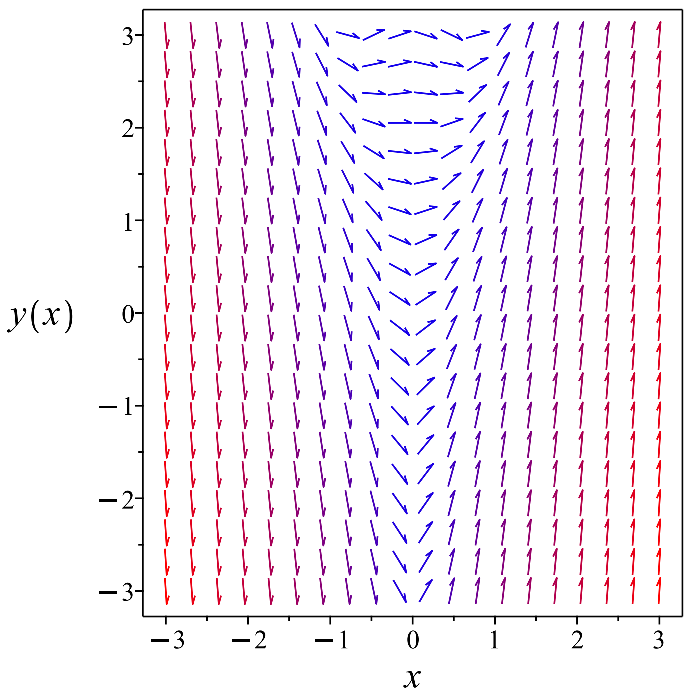


Figure 62: Slope field plot

Verification of solutions

$$y = \frac{x^4+2x^2+c_1+1}{x^2+1} + c_1$$

Verified OK.

2.4.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{2x(-2x^2+y-2)}{x^2+1} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 53: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{x^2+1}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{1}{x^2+1}} dy
\end{aligned}
$$

Which results in

$$
S = \left(x^2+1\right) y
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{2x(-2x^2+y-2)}{x^2+1}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= 2xy \\
S_y &= x^2 + 1
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 4x^3 + 4x \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 4R^3 + 4R
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \left(R^2 + 1\right)^2 + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\left(x^2 + 1\right) y = \left(x^2 + 1\right)^2 + c_1
$$

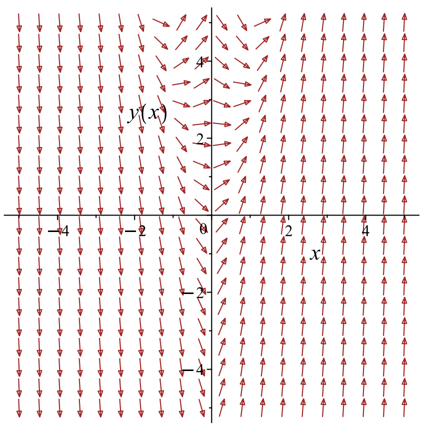
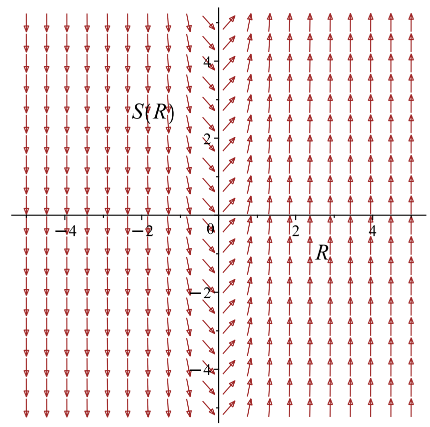
Which simplifies to

$$
\left(x^2 + 1\right) y = \left(x^2 + 1\right)^2 + c_1
$$

Which gives

$$
y = \frac{x^4 + 2x^2 + c_1 + 1}{x^2 + 1}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2x(-2x^2+y-2)}{x^2+1}$ 	$R = x$ $S = (x^2+1)\,y$	$\frac{dS}{dR} = 4R^3 + 4R$ 

Summary

The solution(s) found are the following

$$y = \frac{x^4 + 2x^2 + c_1 + 1}{x^2 + 1} \tag{1}$$

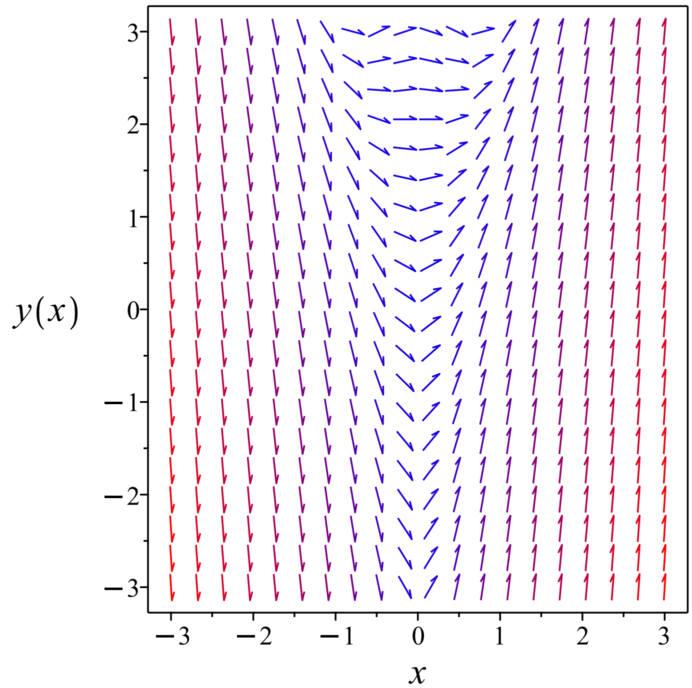


Figure 63: Slope field plot

Verification of solutions

$$y = \frac{x^4 + 2x^2 + c_1 + 1}{x^2 + 1}$$

Verified OK.

2.4.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^2+1)\,\mathrm{d}y &= \left(-2x\left(-2x^2+y-2\right)\right)\mathrm{d}x \\ \left(2x\left(-2x^2+y-2\right)\right)\mathrm{d}x + (x^2+1)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 2x\left(-2x^2+y-2\right) \\ N(x,y) &= x^2+1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2x\left(-2x^2+y-2\right)\right) \\ &= 2x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^2+1\right) \\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 2x(-2x^2 + y - 2) \, \mathrm{d}x$$
$$\phi = -\frac{(2x^2 - y + 2)^2}{4} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 - \frac{y}{2} + 1 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2 + 1$. Therefore equation (4) becomes

$$x^2 + 1 = x^2 - \frac{y}{2} + 1 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{y}{2}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{y}{2}\right) \mathrm{d}y$$
$$f(y) = \frac{y^2}{4} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{(2x^2 - y + 2)^2}{4} + \frac{y^2}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{(2x^2 - y + 2)^2}{4} + \frac{y^2}{4}$$

The solution becomes

$$y = \frac{x^4 + 2x^2 + c_1 + 1}{x^2 + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{x^4 + 2x^2 + c_1 + 1}{x^2 + 1} \tag{1}$$

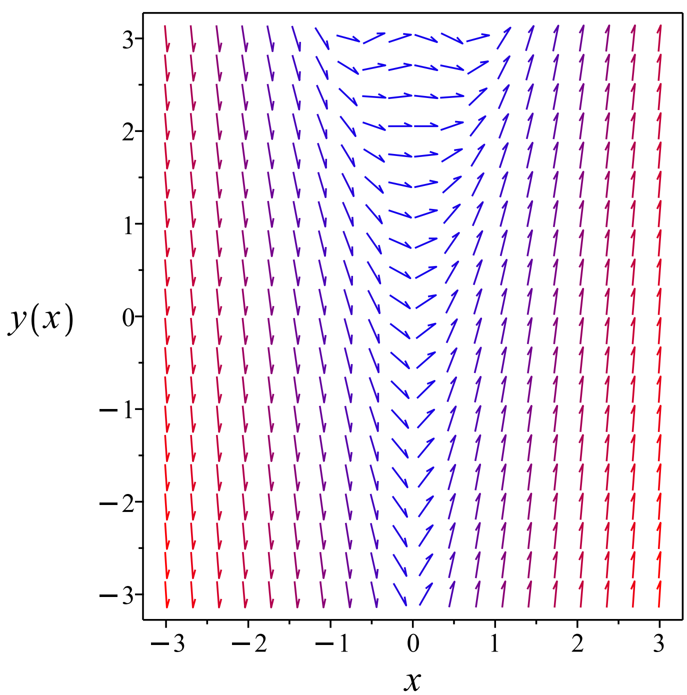


Figure 64: Slope field plot

Verification of solutions

$$y = \frac{x^4 + 2x^2 + c_1 + 1}{x^2 + 1}$$

Verified OK.

2.4.5 Maple step by step solution

Let's solve

$y' + \frac{2xy}{x^2+1} = 4x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{2xy}{x^2+1} + 4x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2xy}{x^2+1} = 4x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{2xy}{x^2+1}\right) = 4\mu(x)\, x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{2xy}{x^2+1}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{2\mu(x)x}{x^2+1}$

- Solve to find the integrating factor

  $\mu(x) = x^2 + 1$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int 4\mu(x)\, x dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int 4\mu(x)\, x dx + c_1$

- Solve for y

  $y = \frac{\int 4\mu(x)x dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x^2 + 1$

$y = \frac{\int 4x(x^2+1)dx + c_1}{x^2+1}$

- Evaluate the integrals on the rhs

  $y = \frac{(x^2+1)^2 + c_1}{x^2+1}$

- Simplify

  $y = \frac{x^4+2x^2+c_1+1}{x^2+1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(x),x)+2*x/(1+x^2)*y(x)=4*x,y(x), singsol=all)
```

$$y(x) = x^2 + 1 + \frac{c_1}{x^2+1}$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 24

```
DSolve[y'[x]+2*x/(1+x^2)*y[x]==4*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^4+2x^2+c_1}{x^2+1}$$

2.5 problem 5

Internal problem ID [2563]
Internal file name [`OUTPUT/2055_Sunday_June_05_2022_02_46_34_AM_26089267/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.6, page 50
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \frac{2xy}{x^2+1} = \frac{4}{(x^2+1)^2}}$$

2.5.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2x}{x^2+1}$$
$$q(x) = \frac{4}{(x^2+1)^2}$$

Hence the ode is

$$y' + \frac{2xy}{x^2+1} = \frac{4}{(x^2+1)^2}$$

The integrating factor μ is

$$\mu = e^{\int \frac{2x}{x^2+1}dx}$$
$$= x^2 + 1$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{4}{(x^2+1)^2}\right)$$
$$\frac{d}{dx}\left((x^2+1)\, y\right) = (x^2+1)\left(\frac{4}{(x^2+1)^2}\right)$$
$$d\left((x^2+1)\, y\right) = \left(\frac{4}{x^2+1}\right) dx$$

Integrating gives

$$(x^2+1)\, y = \int \frac{4}{x^2+1}\, dx$$
$$(x^2+1)\, y = 4\arctan(x) + c_1$$

Dividing both sides by the integrating factor $\mu = x^2 + 1$ results in

$$y = \frac{4\arctan(x)}{x^2+1} + \frac{c_1}{x^2+1}$$

which simplifies to

$$y = \frac{4\arctan(x) + c_1}{x^2+1}$$

Summary

The solution(s) found are the following

$$y = \frac{4\arctan(x) + c_1}{x^2+1} \tag{1}$$

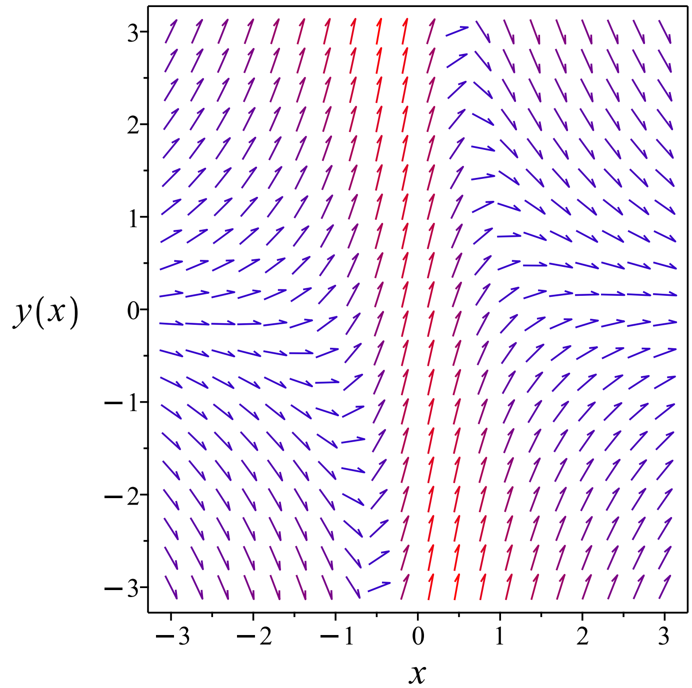


Figure 65: Slope field plot

Verification of solutions

$$y = \frac{4\arctan(x) + c_1}{x^2 + 1}$$

Verified OK.

2.5.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{2(x^3y + xy - 2)}{(x^2 + 1)^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 56: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x^2 + 1}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^2+1}} dy \end{aligned}$$

Which results in

$$S = (x^2 + 1) y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2(x^3 y + xy - 2)}{(x^2+1)^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2xy \\ S_y &= x^2 + 1 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{4}{x^2 + 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{4}{R^2 + 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = 4 \arctan(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$(x^2 + 1)\, y = 4 \arctan(x) + c_1$$

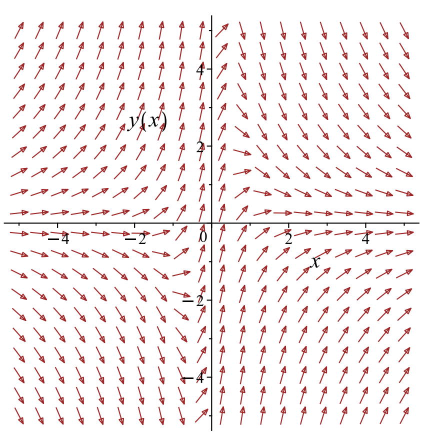
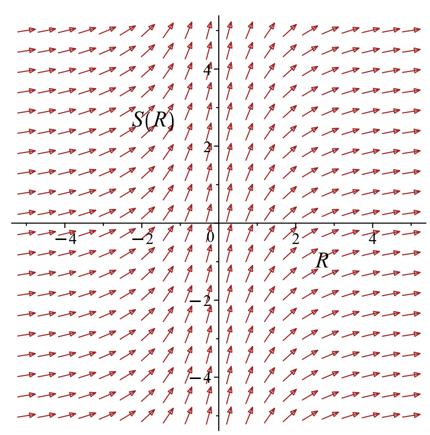
Which simplifies to

$$(x^2 + 1)\, y = 4 \arctan(x) + c_1$$

Which gives

$$y = \frac{4 \arctan(x) + c_1}{x^2 + 1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2(x^3y+xy-2)}{(x^2+1)^2}$ 	$R = x$ $S = (x^2 + 1)\, y$	$\frac{dS}{dR} = \frac{4}{R^2+1}$ 

Summary

The solution(s) found are the following

$$y = \frac{4 \arctan(x) + c_1}{x^2 + 1} \tag{1}$$

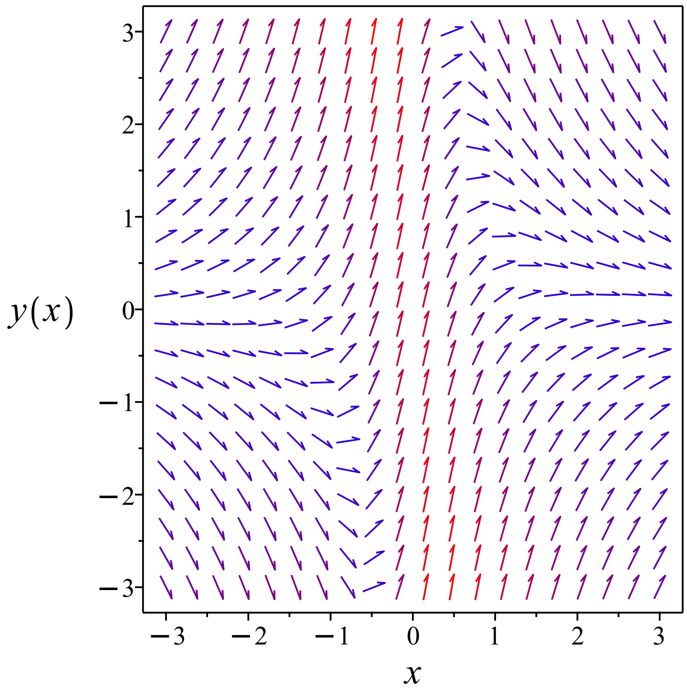


Figure 66: Slope field plot

Verification of solutions

$$y = \frac{4\arctan(x) + c_1}{x^2 + 1}$$

Verified OK.

2.5.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-\frac{2xy}{x^2+1} + \frac{4}{(x^2+1)^2}\right)\mathrm{d}x$$

$$\left(\frac{2xy}{x^2+1} - \frac{4}{(x^2+1)^2}\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = \frac{2xy}{x^2+1} - \frac{4}{(x^2+1)^2}$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{2xy}{x^2+1} - \frac{4}{(x^2+1)^2}\right)\\ &= \frac{2x}{x^2+1}\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(\frac{2x}{x^2+1}\right) - (0)\right) \\
&= \frac{2x}{x^2+1}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{2x}{x^2+1} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(x^2+1)} \\
&= x^2+1
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= x^2+1\left(\frac{2xy}{x^2+1} - \frac{4}{(x^2+1)^2}\right) \\
&= \frac{2x^3y + 2xy - 4}{x^2+1}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= x^2+1(1) \\
&= x^2+1
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{2x^3y + 2xy - 4}{x^2 + 1}\right) + \left(x^2 + 1\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \frac{2x^3y + 2xy - 4}{x^2 + 1}\,\mathrm{d}x$$
$$\phi = x^2y - 4\arctan(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = x^2 + 1$. Therefore equation (4) becomes

$$x^2 + 1 = x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 1$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (1)\,\mathrm{d}y$$
$$f(y) = y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x^2 y - 4 \arctan(x) + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^2 y - 4 \arctan(x) + y$$

Summary

The solution(s) found are the following

$$x^2 y - 4 \arctan(x) + y = c_1 \tag{1}$$

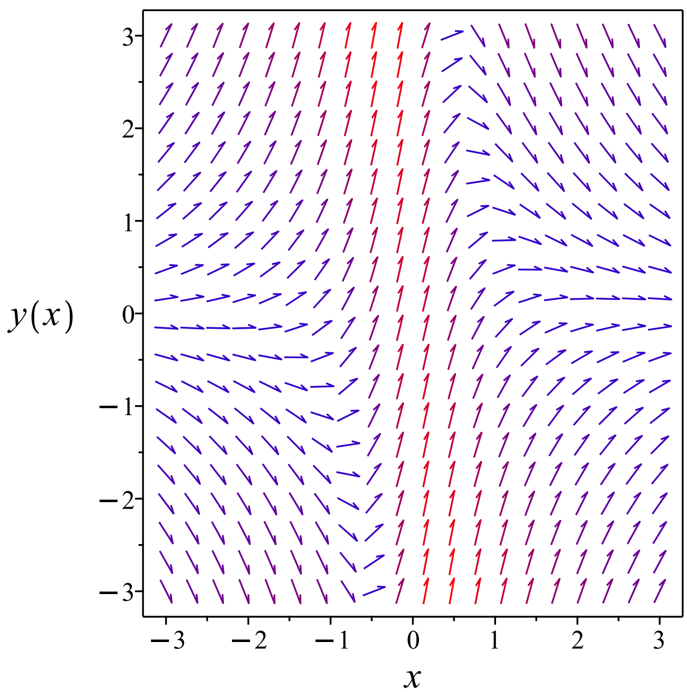


Figure 67: Slope field plot

Verification of solutions

$$x^2 y - 4 \arctan(x) + y = c_1$$

Verified OK.

2.5.4 Maple step by step solution

Let's solve

$y' + \frac{2xy}{x^2+1} = \frac{4}{(x^2+1)^2}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{2xy}{x^2+1} + \frac{4}{(x^2+1)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2xy}{x^2+1} = \frac{4}{(x^2+1)^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{2xy}{x^2+1}\right) = \frac{4\mu(x)}{(x^2+1)^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{2xy}{x^2+1}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{2\mu(x)x}{x^2+1}$

- Solve to find the integrating factor

  $\mu(x) = x^2 + 1$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{4\mu(x)}{(x^2+1)^2} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{4\mu(x)}{(x^2+1)^2} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{4\mu(x)}{\left(x^2+1\right)^2} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x^2 + 1$

  $y = \frac{\int \frac{4}{x^2+1} dx + c_1}{x^2+1}$

- Evaluate the integrals on the rhs

  $y = \frac{4 \arctan(x) + c_1}{x^2+1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 18

```
dsolve(diff(y(x),x)+2*x/(1+x^2)*y(x)=4/(1+x^2)^2,y(x), singsol=all)
```

$$y(x) = \frac{4 \arctan(x) + c_1}{x^2 + 1}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 20

```
DSolve[y'[x]+2*x/(1+x^2)*y[x]==4/(1+x^2)^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{4 \arctan(x) + c_1}{x^2 + 1}$$

2.6 problem 6

Internal problem ID [2564]
Internal file name [`OUTPUT/2056_Sunday_June_05_2022_02_46_36_AM_61261930/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.6, page 50
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{2\cos(x)^2 y' + y\sin(2x) = 4\cos(x)^4}$$

2.6.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \tan(x) \\ q(x) &= 2\cos(x)^2 \end{aligned}$$

Hence the ode is

$$y' + y\tan(x) = 2\cos(x)^2$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \tan(x)dx} \\ &= \frac{1}{\cos(x)} \end{aligned}$$

Which simplifies to

$$\mu = \sec(x)$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(2\cos(x)^2\right) \\
\frac{d}{dx}(\sec(x)\, y) &= (\sec(x))\left(2\cos(x)^2\right) \\
d(\sec(x)\, y) &= (2\cos(x))\, dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\sec(x)\, y &= \int 2\cos(x)\, dx \\
\sec(x)\, y &= 2\sin(x) + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sec(x)$ results in

$$y = 2\sin(x)\cos(x) + c_1\cos(x)$$

which simplifies to

$$y = \cos(x)\,(2\sin(x) + c_1)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \cos(x)\,(2\sin(x) + c_1) \tag{1}$$

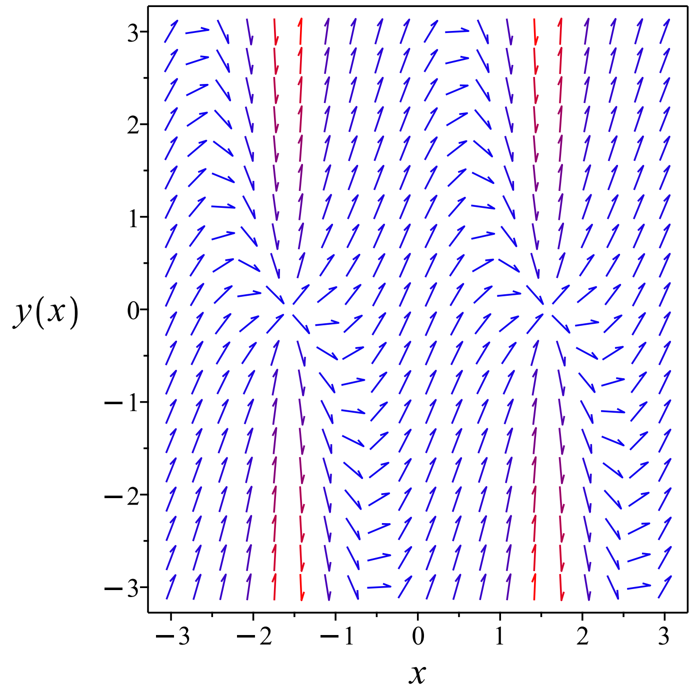


Figure 68: Slope field plot

Verification of solutions

$$y = \cos(x)(2\sin(x) + c_1)$$

Verified OK.

2.6.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-y\sin(2x) + 4\cos(x)^4}{2\cos(x)^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 59: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= \cos(x) \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\cos(x)} dy \end{aligned}$$

Which results in

$$S = \frac{y}{\cos(x)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-y \sin(2x) + 4 \cos(x)^4}{2 \cos(x)^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \sec(x) \tan(x) y \\ S_y &= \sec(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2 \cos(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2 \cos(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = 2\sin(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\sec(x)\, y = 2\sin(x) + c_1$$

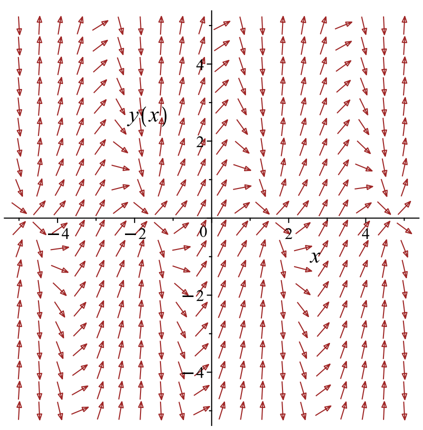
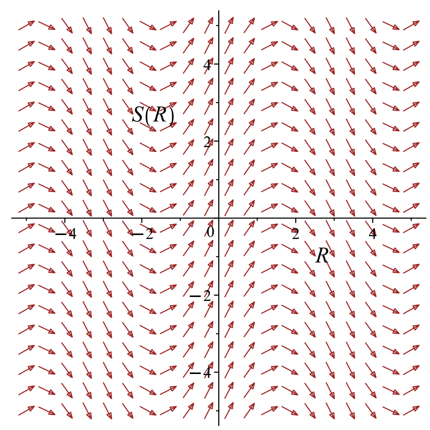
Which simplifies to

$$\sec(x)\, y = 2\sin(x) + c_1$$

Which gives

$$y = \frac{2\sin(x) + c_1}{\sec(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-y\sin(2x)+4\cos(x)^4}{2\cos(x)^2}$ 	$R = x$ $S = \sec(x)\, y$	$\frac{dS}{dR} = 2\cos(R)$ 

Summary

The solution(s) found are the following

$$y = \frac{2\sin(x) + c_1}{\sec(x)} \tag{1}$$

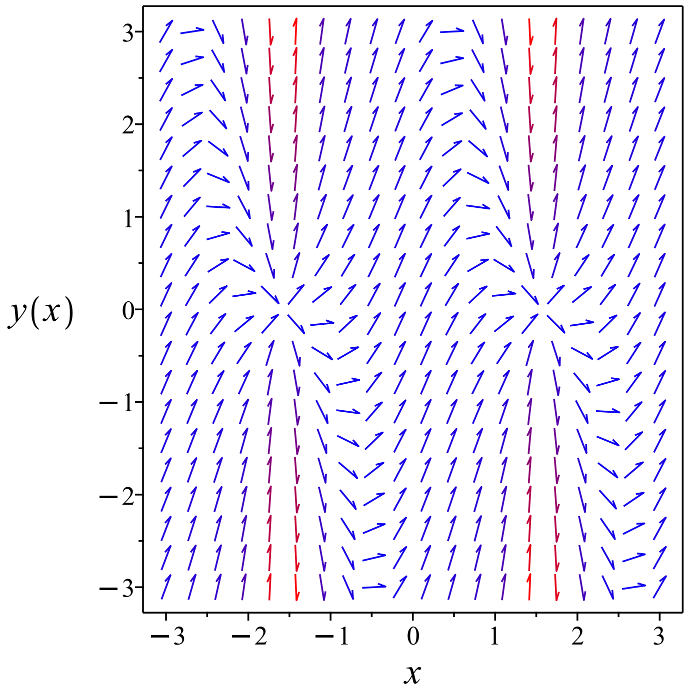


Figure 69: Slope field plot

Verification of solutions

$$y = \frac{2\sin(x) + c_1}{\sec(x)}$$

Verified OK.

2.6.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(2\cos(x)^2\right)\mathrm{d}y = \left(-y\sin(2x) + 4\cos(x)^4\right)\mathrm{d}x$$
$$\left(y\sin(2x) - 4\cos(x)^4\right)\mathrm{d}x + \left(2\cos(x)^2\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = y\sin(2x) - 4\cos(x)^4$$
$$N(x,y) = 2\cos(x)^2$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y\sin(2x) - 4\cos(x)^4\right)\\ &= \sin(2x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(2\cos(x)^2\right)\\ &= -2\sin(2x)\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{\sec(x)^2}{2}((\sin(2x)) - (-4\sin(x)\cos(x))) \\
&= 3\tan(x)
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}x} \\
&= e^{\int 3\tan(x) \,\mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-3\ln(\cos(x))} \\
&= \sec(x)^3
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \sec(x)^3 \left(y\sin(2x) - 4\cos(x)^4\right) \\
&= -4\cos(x) + 2\sec(x)\tan(x)\,y
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \sec(x)^3 \left(2\cos(x)^2\right) \\
&= 2\sec(x)
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
(-4\cos(x) + 2\sec(x)\tan(x)\,y) + (2\sec(x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -4\cos(x) + 2\sec(x)\tan(x)\, y \, \mathrm{d}x$$
$$\phi = 2\sec(x)\, y - 4\sin(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2\sec(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2\sec(x)$. Therefore equation (4) becomes

$$2\sec(x) = 2\sec(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = 2\sec(x)\, y - 4\sin(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = 2\sec(x)\, y - 4\sin(x)$$

The solution becomes

$$y = \frac{4\sin(x) + c_1}{2\sec(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{4\sin(x) + c_1}{2\sec(x)} \tag{1}$$

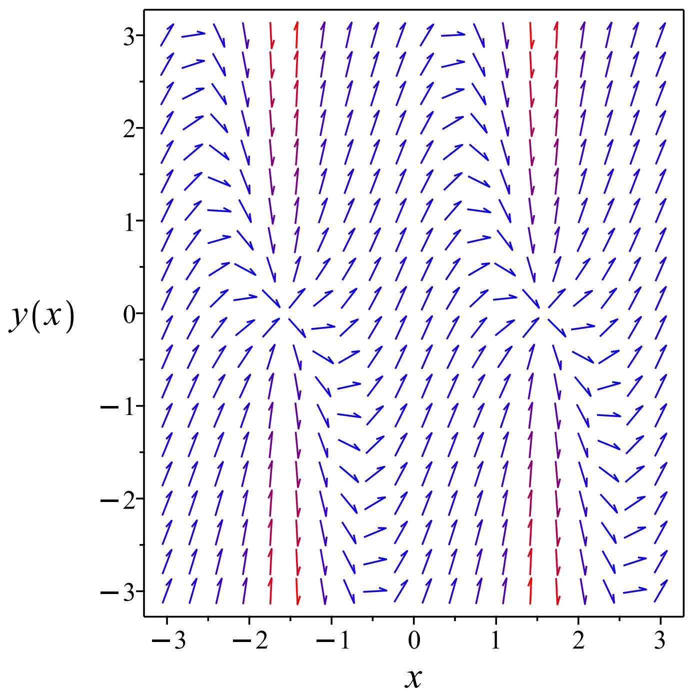


Figure 70: Slope field plot

Verification of solutions

$$y = \frac{4\sin(x) + c_1}{2\sec(x)}$$

Verified OK.

2.6.4 Maple step by step solution

Let's solve

$2\cos(x)^2 y' + y\sin(2x) = 4\cos(x)^4$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{\sin(2x)y}{2\cos(x)^2} + 2\cos(x)^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{\sin(2x)y}{2\cos(x)^2} = 2\cos(x)^2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{\sin(2x)y}{2\cos(x)^2}\right) = 2\mu(x)\cos(x)^2$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{\sin(2x)y}{2\cos(x)^2}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)\sin(2x)}{2\cos(x)^2}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{\cos(x)}$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx = \int 2\mu(x)\cos(x)^2\,dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int 2\mu(x)\cos(x)^2\,dx + c_1$

- Solve for y

  $y = \frac{\int 2\mu(x)\cos(x)^2dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{\cos(x)}$

  $y = \cos(x)\left(\int 2\cos(x)\,dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = \cos(x)\,(2\sin(x) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(2*cos(x)^2*diff(y(x),x)+y(x)*sin(2*x)=4*cos(x)^4,y(x), singsol=all)
```

$$y(x) = (2\sin(x) + c_1)\cos(x)$$

✓ Solution by Mathematica

Time used: 0.058 (sec). Leaf size: 15

```
DSolve[2*Cos[x]^2*y'[x]+y[x]*Sin[2*x]==4*Cos[x]^4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \cos(x)(2\sin(x) + c_1)$$

2.7 problem 7

Internal problem ID [2565]
Internal file name [`OUTPUT/2057_Sunday_June_05_2022_02_46_38_AM_20398170/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.6, page 50
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \frac{y}{\ln(x)\, x} = 9x^2}$$

2.7.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{\ln(x)\, x} \\ q(x) &= 9x^2 \end{aligned}$$

Hence the ode is

$$y' + \frac{y}{\ln(x)\, x} = 9x^2$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{1}{\ln(x)x}dx} \\
&= \ln(x)
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(9x^2\right) \\
\frac{d}{dx}(\ln(x)\, y) &= (\ln(x))\left(9x^2\right) \\
d(\ln(x)\, y) &= \left(9\ln(x)\, x^2\right)\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\ln(x)\, y &= \int 9\ln(x)\, x^2\, dx \\
\ln(x)\, y &= 3x^3\ln(x) - x^3 + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \ln(x)$ results in

$$
y = \frac{3x^3\ln(x) - x^3}{\ln(x)} + \frac{c_1}{\ln(x)}
$$

which simplifies to

$$
y = \frac{3x^3\ln(x) - x^3 + c_1}{\ln(x)}
$$

Summary

The solution(s) found are the following

$$
y = \frac{3x^3\ln(x) - x^3 + c_1}{\ln(x)} \tag{1}
$$

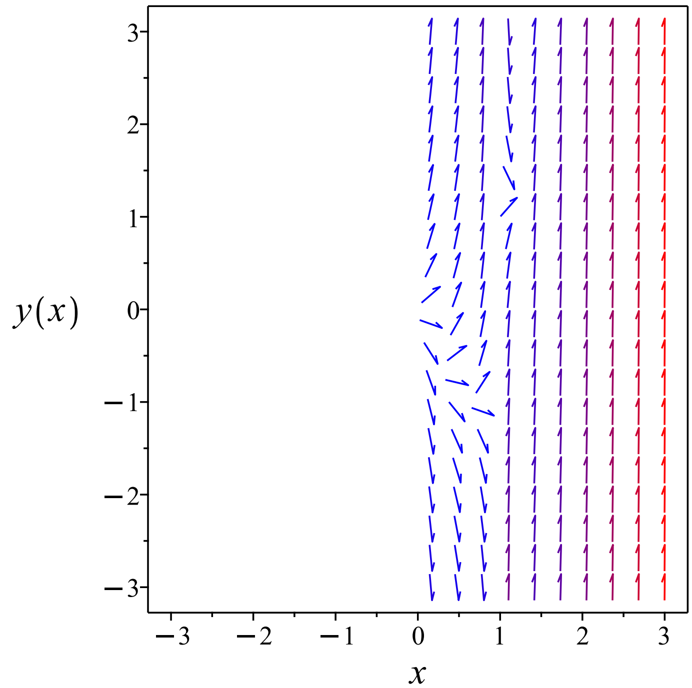


Figure 71: Slope field plot

Verification of solutions

$$y = \frac{3x^3 \ln(x) - x^3 + c_1}{\ln(x)}$$

Verified OK.

2.7.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{-9x^3 \ln(x) + y}{\ln(x)\, x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 62: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= \frac{1}{\ln(x)} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\ln(x)}} dy \end{aligned}$$

Which results in

$$S = \ln(x)\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-9x^3 \ln(x) + y}{\ln(x)\, x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y}{x} \\ S_y &= \ln(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 9 \ln(x)\, x^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 9 \ln(R)\, R^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 3R^3 \ln(R) - R^3 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y \ln(x) = 3x^3 \ln(x) - x^3 + c_1$$

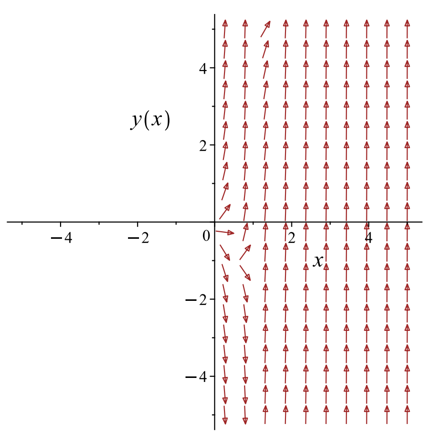
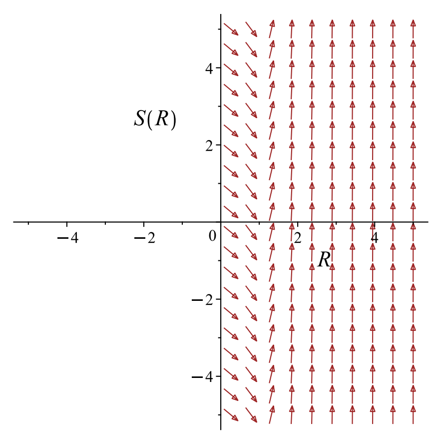
Which simplifies to

$$y \ln(x) = 3x^3 \ln(x) - x^3 + c_1$$

Which gives

$$y = \frac{3x^3 \ln(x) - x^3 + c_1}{\ln(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-9x^3 \ln(x)+y}{\ln(x)x}$ 	$R = x$ $S = \ln(x)\, y$	$\frac{dS}{dR} = 9\ln(R)\, R^2$ 

Summary

The solution(s) found are the following

$$y = \frac{3x^3 \ln(x) - x^3 + c_1}{\ln(x)} \tag{1}$$

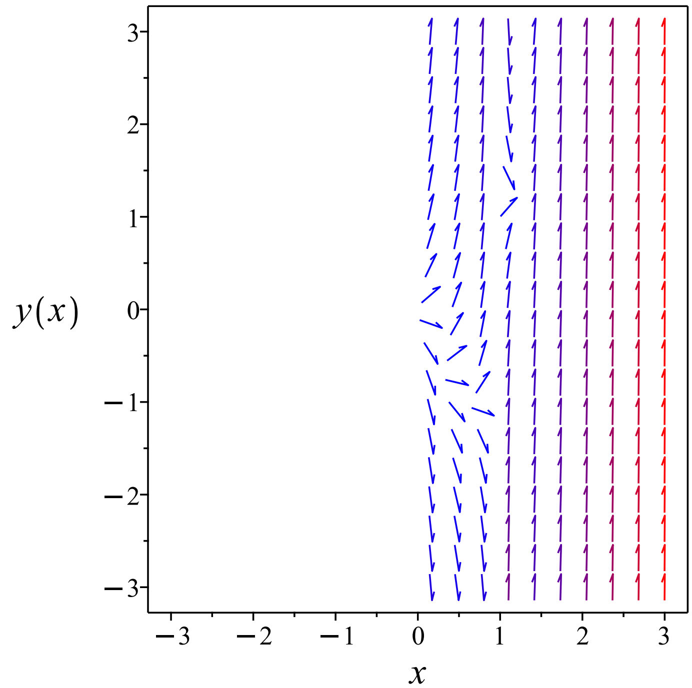


Figure 72: Slope field plot

Verification of solutions

$$y = \frac{3x^3 \ln(x) - x^3 + c_1}{\ln(x)}$$

Verified OK.

2.7.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-\frac{y}{\ln(x)\,x} + 9x^2\right)\mathrm{d}x$$

$$\left(-9x^2 + \frac{y}{\ln(x)\,x}\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -9x^2 + \frac{y}{\ln(x)\,x}$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-9x^2 + \frac{y}{\ln(x)\,x}\right) \\ &= \frac{1}{\ln(x)\,x}\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(\frac{1}{\ln(x)\, x}\right) - (0)\right) \\
&= \frac{1}{\ln(x)\, x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{1}{\ln(x)x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(\ln(x))} \\
&= \ln(x)
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \ln(x)\left(-9x^2 + \frac{y}{\ln(x)\, x}\right) \\
&= \frac{-9x^3 \ln(x) + y}{x}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \ln(x)(1) \\
&= \ln(x)
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{-9x^3 \ln(x) + y}{x}\right) + (\ln(x))\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{-9x^3 \ln(x) + y}{x} \mathrm{d}x$$
$$\phi = (-3x^3 + y) \ln(x) + x^3 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \ln(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \ln(x)$. Therefore equation (4) becomes

$$\ln(x) = \ln(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (-3x^3 + y) \ln(x) + x^3 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (-3x^3 + y)\ln(x) + x^3$$

The solution becomes

$$y = \frac{3x^3 \ln(x) - x^3 + c_1}{\ln(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{3x^3 \ln(x) - x^3 + c_1}{\ln(x)} \tag{1}$$

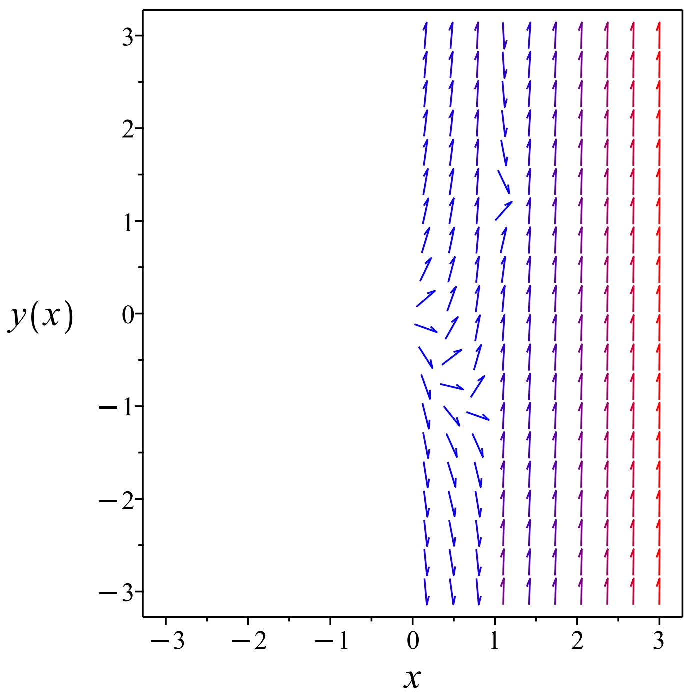


Figure 73: Slope field plot

Verification of solutions

$$y = \frac{3x^3 \ln(x) - x^3 + c_1}{\ln(x)}$$

Verified OK.

2.7.4 Maple step by step solution

Let's solve

$y' + \frac{y}{\ln(x)x} = 9x^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{\ln(x)x} + 9x^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{\ln(x)x} = 9x^2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{\ln(x)x}\right) = 9\mu(x)\,x^2$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{y}{\ln(x)x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{\ln(x)x}$

- Solve to find the integrating factor

  $\mu(x) = \ln(x)$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right)dx = \int 9\mu(x)\,x^2dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int 9\mu(x)\,x^2dx + c_1$

- Solve for y

  $y = \frac{\int 9\mu(x)x^2dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \ln(x)$

  $y = \frac{\int 9\ln(x)x^2dx + c_1}{\ln(x)}$

- Evaluate the integrals on the rhs

  $y = \frac{3x^3\ln(x) - x^3 + c_1}{\ln(x)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x)+1/(x*ln(x))*y(x)=9*x^2,y(x), singsol=all)
```

$$y(x) = \frac{3x^3 \ln(x) - x^3 + c_1}{\ln(x)}$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 25

```
DSolve[y'[x]+1/(x*Log[x])*y[x]==9*x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-x^3 + 3x^3 \log(x) + c_1}{\log(x)}$$

2.8 problem 8

Internal problem ID [2566]
Internal file name [`OUTPUT/2058_Sunday_June_05_2022_02_46_40_AM_61184153/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.6, page 50
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - y\tan(x) = 8\sin(x)^3}$$

2.8.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\tan(x) \\ q(x) &= 8\sin(x)^3 \end{aligned}$$

Hence the ode is

$$y' - y\tan(x) = 8\sin(x)^3$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\tan(x)dx} \\ &= \cos(x) \end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(8\sin(x)^3\right)$$
$$\frac{d}{dx}(\cos(x)\,y) = (\cos(x))\left(8\sin(x)^3\right)$$
$$d(\cos(x)\,y) = \left(8\sin(x)^3\cos(x)\right)\,dx$$

Integrating gives

$$\cos(x)\,y = \int 8\sin(x)^3\cos(x)\,dx$$
$$\cos(x)\,y = 2\sin(x)^4 + c_1$$

Dividing both sides by the integrating factor $\mu = \cos(x)$ results in

$$y = 2\sec(x)\sin(x)^4 + c_1\sec(x)$$

which simplifies to

$$y = \sec(x)\left(2\sin(x)^4 + c_1\right)$$

Summary

The solution(s) found are the following

$$y = \sec(x)\left(2\sin(x)^4 + c_1\right) \tag{1}$$

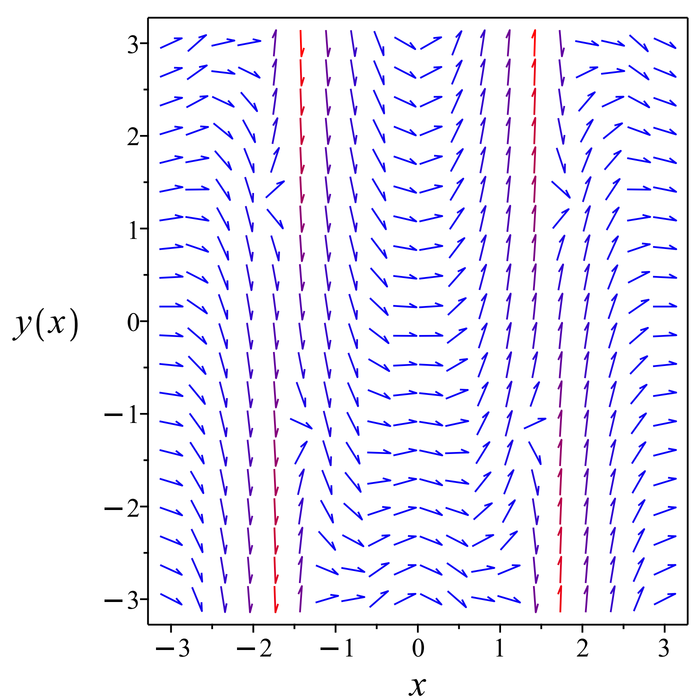


Figure 74: Slope field plot

Verification of solutions

$$y = \sec(x)\left(2\sin(x)^4 + c_1\right)$$

Verified OK.

2.8.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \tan(x)\,y + 8\sin(x)^3$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 65: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{\cos(x)}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\cos(x)}} dy \end{aligned}$$

Which results in

$$S = \cos(x) y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \tan(x) y + 8 \sin(x)^3$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\sin(x) y \\ S_y &= \cos(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 8 \sin(x)^3 \cos(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 8 \sin(R)^3 \cos(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 2\sin(R)^4 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\cos(x)\, y = 2\sin(x)^4 + c_1$$

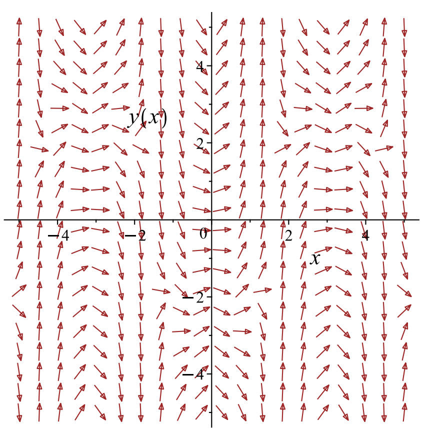
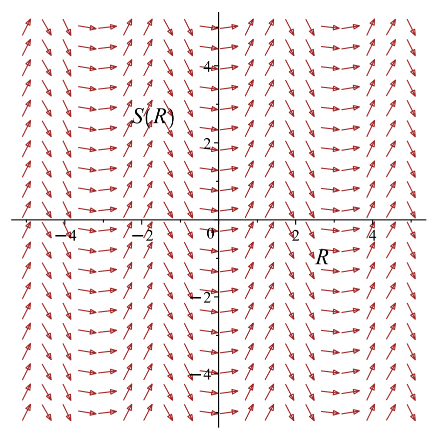
Which simplifies to

$$\cos(x)\, y = 2\sin(x)^4 + c_1$$

Which gives

$$y = \frac{2\sin(x)^4 + c_1}{\cos(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \tan(x)\, y + 8\sin(x)^3$ 	$R = x$ $S = \cos(x)\, y$	$\frac{dS}{dR} = 8\sin(R)^3\cos(R)$ 

Summary
The solution(s) found are the following

$$y = \frac{2\sin(x)^4 + c_1}{\cos(x)} \tag{1}$$

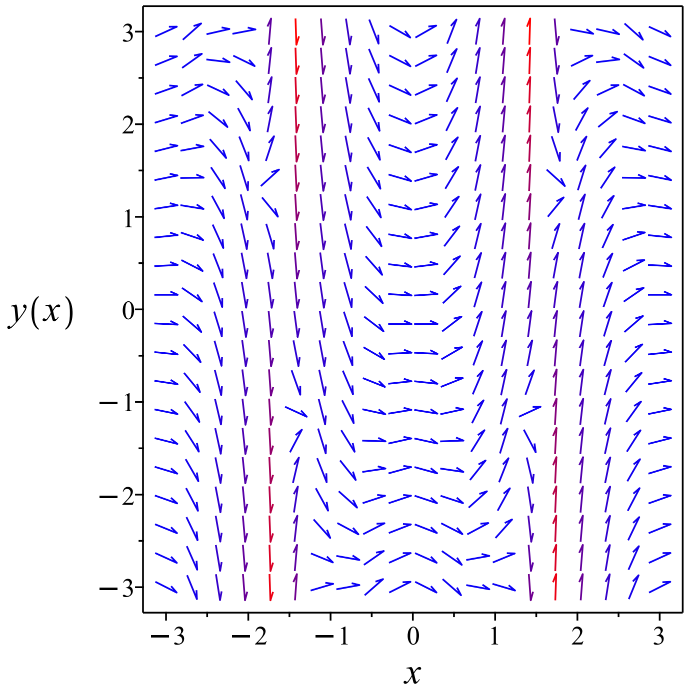


Figure 75: Slope field plot

Verification of solutions

$$y = \frac{2 \sin(x)^4 + c_1}{\cos(x)}$$

Verified OK.

2.8.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(\tan(x)\, y + 8 \sin(x)^3\right) \mathrm{d}x$$
$$\left(-\tan(x)\, y - 8 \sin(x)^3\right) \mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\tan(x)\, y - 8 \sin(x)^3$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\tan(x)\, y - 8 \sin(x)^3\right) \\ &= -\tan(x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((-\tan(x)) - (0)) \\ &= -\tan(x) \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -\tan(x) \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\ln(\cos(x))} \\ &= \cos(x) \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \cos(x)\left(-\tan(x)\, y - 8\sin(x)^3\right) \\ &= \sin(x)\left(-8\sin(x)^2\cos(x) - y\right) \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \cos(x)\,(1) \\ &= \cos(x) \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\sin(x)\left(-8\sin(x)^2\cos(x) - y\right)\right) + (\cos(x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \sin(x) \left(-8 \sin(x)^2 \cos(x) - y\right) \mathrm{d}x$$

$$\phi = \cos(x) \left(-2 \cos(x)^3 + 4 \cos(x) + y\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \cos(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \cos(x)$. Therefore equation (4) becomes

$$\cos(x) = \cos(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \cos(x) \left(-2 \cos(x)^3 + 4 \cos(x) + y\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \cos(x) \left(-2 \cos(x)^3 + 4 \cos(x) + y\right)$$

The solution becomes

$$y = \frac{2 \cos(x)^4 - 4 \cos(x)^2 + c_1}{\cos(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{2\cos(x)^4 - 4\cos(x)^2 + c_1}{\cos(x)} \tag{1}$$

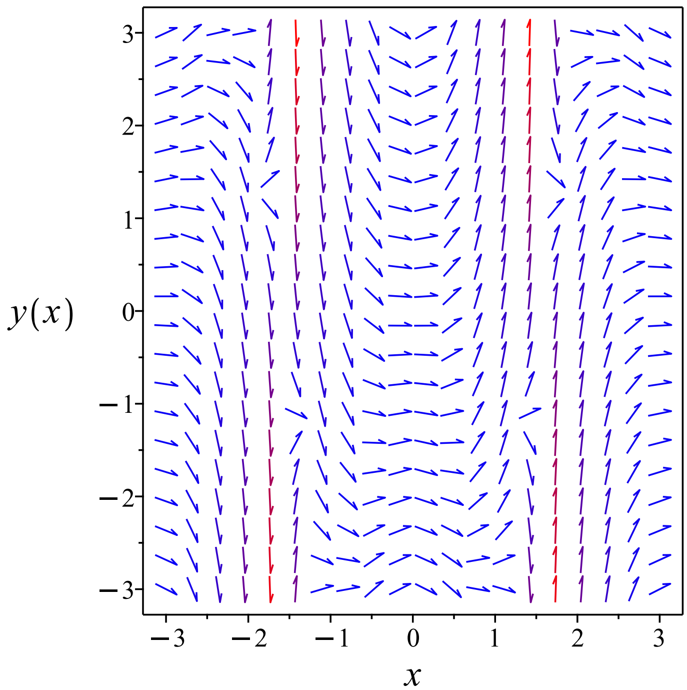


Figure 76: Slope field plot

Verification of solutions

$$y = \frac{2\cos(x)^4 - 4\cos(x)^2 + c_1}{\cos(x)}$$

Verified OK.

2.8.4 Maple step by step solution

Let's solve

$y' - y\tan(x) = 8\sin(x)^3$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$$y' = y \tan(x) + 8 \sin(x)^3$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' - y \tan(x) = 8 \sin(x)^3$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x) \left(y' - y \tan(x)\right) = 8\mu(x) \sin(x)^3$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

$$\mu(x) \left(y' - y \tan(x)\right) = \mu'(x)\, y + \mu(x)\, y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = -\mu(x) \tan(x)$$

- Solve to find the integrating factor

$$\mu(x) = \cos(x)$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int 8\mu(x) \sin(x)^3\, dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\, y = \int 8\mu(x) \sin(x)^3\, dx + c_1$$

- Solve for y

$$y = \frac{\int 8\mu(x) \sin(x)^3 dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \cos(x)$

$$y = \frac{\int 8 \sin(x)^3 \cos(x) dx + c_1}{\cos(x)}$$

- Evaluate the integrals on the rhs

$$y = \frac{2 \sin(x)^4 + c_1}{\cos(x)}$$

- Simplify

$$y = \sec(x) \left(2 \sin(x)^4 + c_1\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x)-y(x)*tan(x)=8*sin(x)^3,y(x), singsol=all)
```

$$y(x) = 2\cos(x)^3 - 4\cos(x) + \frac{\sec(x)(4c_1 + 5)}{4}$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 19

```
DSolve[y'[x]-y[x]*Tan[x]==8*Sin[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2\sin^3(x)\tan(x) + c_1\sec(x)$$

2.9 problem 9

Internal problem ID [2567]
Internal file name [`OUTPUT/2059_Sunday_June_05_2022_02_46_43_AM_51064587/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.6, page 50
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{x't + 2x = 4\,\mathrm{e}^{t}}$$

2.9.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$p(t) = \frac{2}{t}$$
$$q(t) = \frac{4\,\mathrm{e}^{t}}{t}$$

Hence the ode is

$$x' + \frac{2x}{t} = \frac{4\,\mathrm{e}^{t}}{t}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{2}{t}dt} \\
&= t^2
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dt}(\mu x) &= (\mu)\left(\frac{4\,e^t}{t}\right) \\
\frac{d}{dt}\left(t^2 x\right) &= \left(t^2\right)\left(\frac{4\,e^t}{t}\right) \\
d\left(t^2 x\right) &= \left(4\,e^t t\right)\, dt
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
t^2 x &= \int 4\,e^t t\, dt \\
t^2 x &= 4(t-1)\,e^t + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = t^2$ results in

$$
x = \frac{4(t-1)\,e^t}{t^2} + \frac{c_1}{t^2}
$$

which simplifies to

$$
x = \frac{(4t-4)\,e^t + c_1}{t^2}
$$

Summary
The solution(s) found are the following

$$
x = \frac{(4t-4)\,e^t + c_1}{t^2} \tag{1}
$$

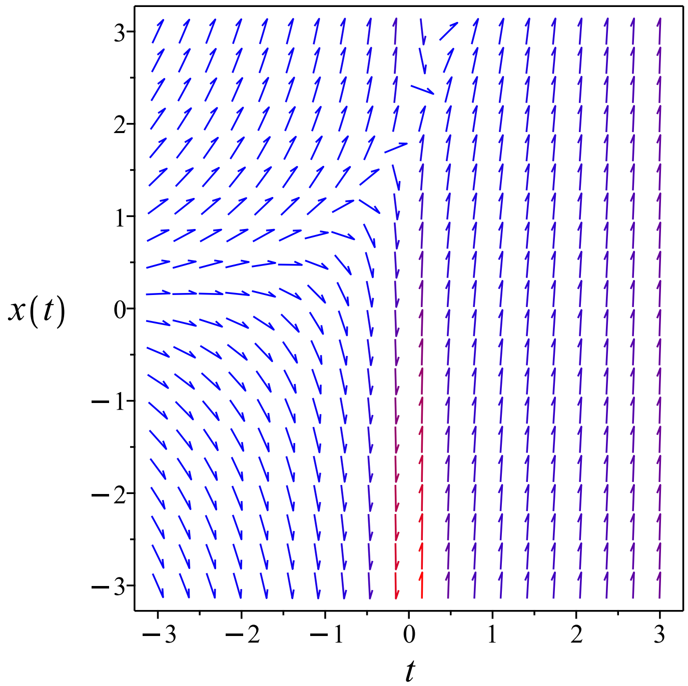


Figure 77: Slope field plot

Verification of solutions

$$x = \frac{(4t - 4)\,\mathrm{e}^{t} + c_1}{t^2}$$

Verified OK.

2.9.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$x' = \frac{-2x + 4\,\mathrm{e}^{t}}{t}$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 68: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, x) &= 0 \\
\eta(t, x) &= \frac{1}{t^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{t^2}} dy \end{aligned}$$

Which results in

$$S = t^2 x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x) S_x}{R_t + \omega(t,x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = \frac{-2x + 4\,\mathrm{e}^t}{t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= 2tx \\ S_x &= t^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 4\,\mathrm{e}^t t \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 4\,\mathrm{e}^R R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 4(R-1)\,\mathrm{e}^{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$t^2 x = 4(t-1)\,\mathrm{e}^{t} + c_1$$

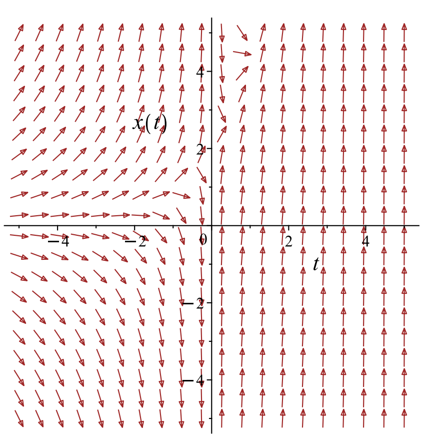
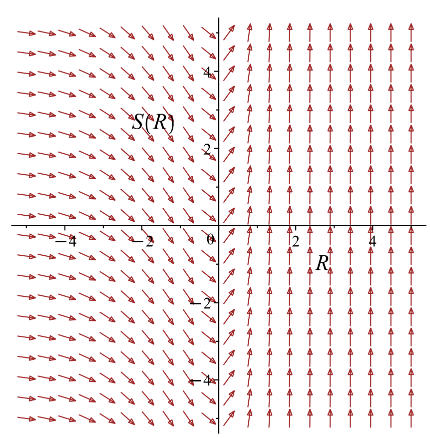
Which simplifies to

$$t^2 x = 4(t-1)\,\mathrm{e}^{t} + c_1$$

Which gives

$$x = \frac{4\,\mathrm{e}^{t} t - 4\,\mathrm{e}^{t} + c_1}{t^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = \frac{-2x+4\,\mathrm{e}^{t}}{t}$ 	$R = t$ $S = t^2 x$	$\frac{dS}{dR} = 4\,\mathrm{e}^{R} R$ 

Summary
The solution(s) found are the following

$$x = \frac{4\,\mathrm{e}^{t} t - 4\,\mathrm{e}^{t} + c_1}{t^2} \tag{1}$$

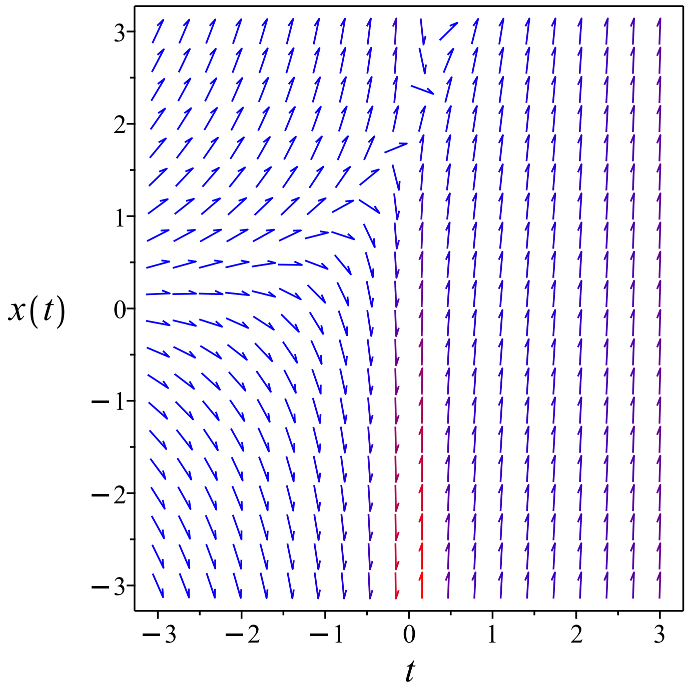


Figure 78: Slope field plot

Verification of solutions

$$x = \frac{4\,\mathrm{e}^{t} t - 4\,\mathrm{e}^{t} + c_1}{t^2}$$

Verified OK.

2.9.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t, x)\, \mathrm{d}t + N(t, x)\, \mathrm{d}x = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(t)\, \mathrm{d}x &= \left(-2x + 4\,\mathrm{e}^{t}\right) \mathrm{d}t \\
\left(2x - 4\,\mathrm{e}^{t}\right) \mathrm{d}t + (t)\, \mathrm{d}x &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, x) &= 2x - 4\,\mathrm{e}^{t} \\
N(t, x) &= t
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(2x - 4\,\mathrm{e}^{t}\right) \\
&= 2
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(t) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right) \\
&= \frac{1}{t}((2) - (1)) \\
&= \frac{1}{t}
\end{aligned}
$$

Since A does not depend on x, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}t} \\
&= e^{\int \frac{1}{t} \,\mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(t)} \\
&= t
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= t(2x - 4\,\mathrm{e}^t) \\
&= 2t(x - 2\,\mathrm{e}^t)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= t(t) \\
&= t^2
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \\
(2t(x - 2\,\mathrm{e}^t)) + (t^2)\frac{\mathrm{d}x}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t,x)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial t} &= \overline{M} \qquad (1)\\
\frac{\partial \phi}{\partial x} &= \overline{N} \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t &= \int \overline{M} \,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t &= \int 2t(x - 2\,\mathrm{e}^{t}) \,\mathrm{d}t \\
\phi &= (-4t+4)\,\mathrm{e}^{t} + t^{2}x + f(x) \qquad (3)
\end{aligned}
$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$
\frac{\partial \phi}{\partial x} = t^{2} + f'(x) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = t^{2}$. Therefore equation (4) becomes

$$
t^{2} = t^{2} + f'(x) \qquad (5)
$$

Solving equation (5) for $f'(x)$ gives

$$
f'(x) = 0
$$

Therefore

$$
f(x) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$
\phi = (-4t+4)\,\mathrm{e}^{t} + t^{2}x + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = (-4t+4)\,\mathrm{e}^{t} + t^{2}x
$$

The solution becomes

$$x = \frac{4\,\mathrm{e}^{t} t - 4\,\mathrm{e}^{t} + c_1}{t^2}$$

<u>Summary</u>

The solution(s) found are the following

$$x = \frac{4\,\mathrm{e}^{t} t - 4\,\mathrm{e}^{t} + c_1}{t^2} \tag{1}$$

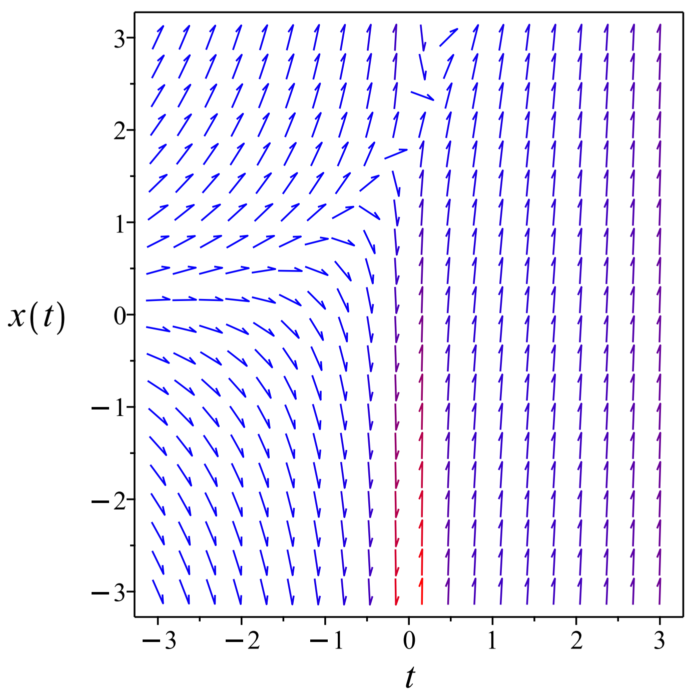


Figure 79: Slope field plot

<u>Verification of solutions</u>

$$x = \frac{4\,\mathrm{e}^{t} t - 4\,\mathrm{e}^{t} + c_1}{t^2}$$

Verified OK.

2.9.4 Maple step by step solution

Let's solve

$x't + 2x = 4\,e^{t}$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

  $x' = -\frac{2x}{t} + \frac{4\,e^{t}}{t}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $x' + \frac{2x}{t} = \frac{4\,e^{t}}{t}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(x' + \frac{2x}{t}\right) = \frac{4\mu(t)e^{t}}{t}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,x)$

  $\mu(t)\left(x' + \frac{2x}{t}\right) = \mu'(t)\,x + \mu(t)\,x'$

- Isolate $\mu'(t)$

  $\mu'(t) = \frac{2\mu(t)}{t}$

- Solve to find the integrating factor

  $\mu(t) = t^{2}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,x)\right) dt = \int \frac{4\mu(t)e^{t}}{t} dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,x = \int \frac{4\mu(t)e^{t}}{t} dt + c_1$

- Solve for x

  $x = \frac{\int \frac{4\mu(t)e^{t}}{t} dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = t^{2}$

  $x = \frac{\int 4\,e^{t} t dt + c_1}{t^{2}}$

- Evaluate the integrals on the rhs

  $x = \frac{4(t-1)e^{t} + c_1}{t^{2}}$

- Simplify

$$x = \frac{(4t-4)e^t + c_1}{t^2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(t*diff(x(t),t)+2*x(t)=4*exp(t),x(t), singsol=all)
```

$$x(t) = \frac{(4t-4)e^t + c_1}{t^2}$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 20

```
DSolve[t*x'[t]+2*x[t]==4*Exp[t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{4e^t(t-1) + c_1}{t^2}$$

2.10 problem 10

Internal problem ID [2568]
Internal file name [`OUTPUT/2060_Sunday_June_05_2022_02_46_45_AM_32706070/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.6, page 50
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - \sin(x)(y\sec(x) - 2) = 0}$$

2.10.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\tan(x) \\ q(x) &= -2\sin(x) \end{aligned}$$

Hence the ode is

$$y' - y\tan(x) = -2\sin(x)$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\tan(x)dx} \\ &= \cos(x) \end{aligned}$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(-2\sin(x)\right) \\
\frac{d}{dx}(\cos(x)\, y) &= (\cos(x))\left(-2\sin(x)\right) \\
d(\cos(x)\, y) &= \left(-\sin(2x)\right)\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\cos(x)\, y &= \int -\sin(2x)\, dx \\
\cos(x)\, y &= \frac{\cos(2x)}{2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \cos(x)$ results in

$$
y = \frac{\sec(x)\cos(2x)}{2} + c_1 \sec(x)
$$

which simplifies to

$$
y = \cos(x) - \frac{\sec(x)}{2} + c_1 \sec(x)
$$

Summary

The solution(s) found are the following

$$
y = \cos(x) - \frac{\sec(x)}{2} + c_1 \sec(x) \tag{1}
$$

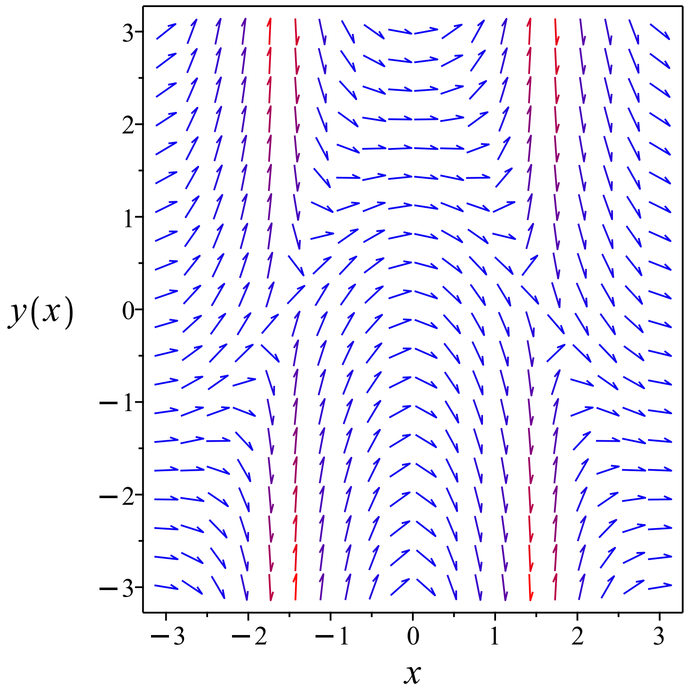


Figure 80: Slope field plot

Verification of solutions

$$y = \cos(x) - \frac{\sec(x)}{2} + c_1 \sec(x)$$

Verified OK.

2.10.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \sin(x)(\sec(x)\, y - 2)$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 71: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{\cos(x)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\cos(x)}} dy \end{aligned}$$

Which results in

$$S = \cos(x)\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \sin(x)(\sec(x)\, y - 2)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\sin(x)\, y \\ S_y &= \cos(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\sin(2x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\sin(2R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\cos(2R)}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\cos(x)\,y = \frac{\cos(2x)}{2} + c_1$$

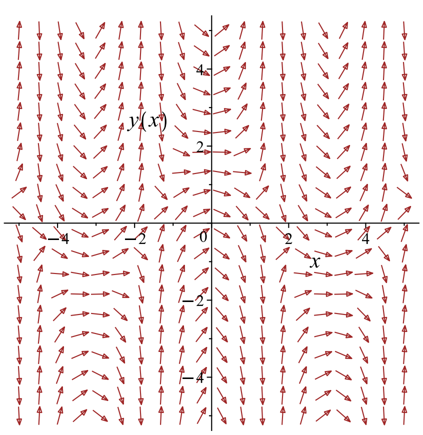
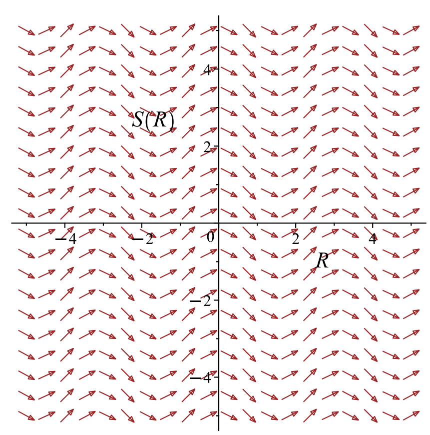
Which simplifies to

$$\cos(x)\,y = \frac{\cos(2x)}{2} + c_1$$

Which gives

$$y = \frac{\cos(2x) + 2c_1}{2\cos(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \sin(x)(\sec(x)\,y - 2)$ 	$R = x$ $S = \cos(x)\,y$	$\frac{dS}{dR} = -\sin(2R)$ 

Summary
The solution(s) found are the following

$$y = \frac{\cos(2x) + 2c_1}{2\cos(x)} \tag{1}$$

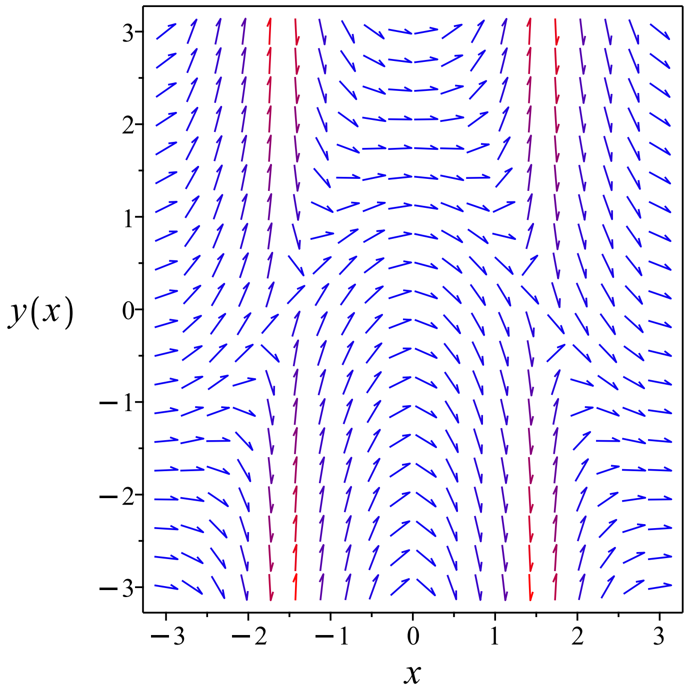


Figure 81: Slope field plot

Verification of solutions

$$y = \frac{\cos(2x) + 2c_1}{2\cos(x)}$$

Verified OK.

2.10.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} \mathrm{d}y &= (\sin(x)(\sec(x)\,y - 2))\,\mathrm{d}x \\ (-\sin(x)(\sec(x)\,y - 2))\,\mathrm{d}x + \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -\sin(x)(\sec(x)\,y - 2) \\ N(x, y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\sin(x)(\sec(x)\,y - 2)) \\ &= -\tan(x) \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((-\sec(x)\sin(x)) - (0)) \\ &= -\tan(x) \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -\tan(x) \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\ln(\cos(x))} \\ &= \cos(x) \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \cos(x)\left(-\sin(x)\left(\sec(x)\, y - 2\right)\right) \\ &= -\sin(x)\left(y - 2\cos(x)\right) \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \cos(x)\,(1) \\ &= \cos(x) \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-\sin(x)\left(y - 2\cos(x)\right)\right) + \left(\cos(x)\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\sin(x)(y - 2\cos(x)) \, \mathrm{d}x$$
$$\phi = \cos(x)(-\cos(x) + y) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \cos(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \cos(x)$. Therefore equation (4) becomes

$$\cos(x) = \cos(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \cos(x)(-\cos(x) + y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \cos(x)(-\cos(x) + y)$$

The solution becomes

$$y = \frac{\cos(x)^2 + c_1}{\cos(x)}$$

Summary
The solution(s) found are the following

$$y = \frac{\cos(x)^2 + c_1}{\cos(x)} \tag{1}$$

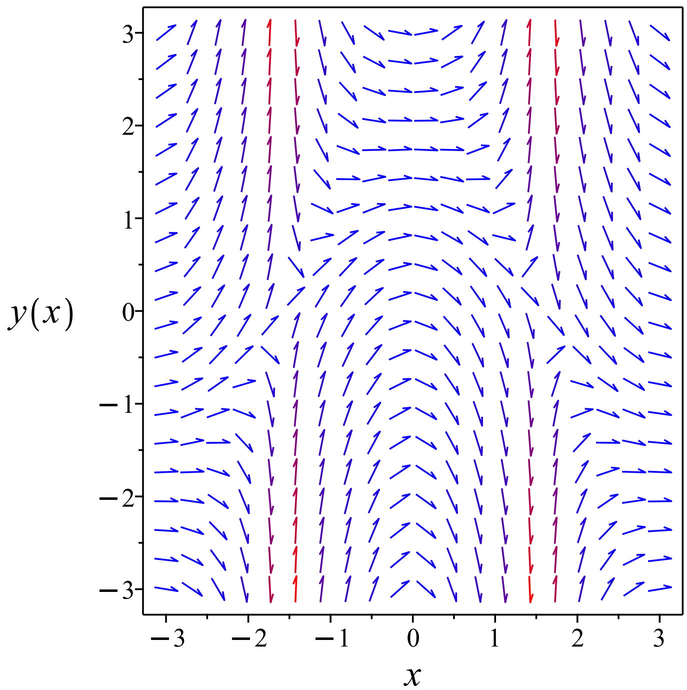


Figure 82: Slope field plot

Verification of solutions

$$y = \frac{\cos(x)^2 + c_1}{\cos(x)}$$

Verified OK.

2.10.4 Maple step by step solution

Let's solve

$y' - \sin(x)(y \sec(x) - 2) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$$y' = \sin(x) y \sec(x) - 2\sin(x)$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$-\sin(x) y \sec(x) + y' = -2\sin(x)$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x) (-\sin(x) y \sec(x) + y') = -2\mu(x) \sin(x)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x) y)$

$$\mu(x) (-\sin(x) y \sec(x) + y') = \mu'(x) y + \mu(x) y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = -\mu(x) \sec(x) \sin(x)$$

- Solve to find the integrating factor

$$\mu(x) = \frac{1}{\sec(x)}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x) y)\right) dx = \int -2\mu(x) \sin(x) dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x) y = \int -2\mu(x) \sin(x) dx + c_1$$

- Solve for y

$$y = \frac{\int -2\mu(x) \sin(x) dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \frac{1}{\sec(x)}$

$$y = \sec(x) \left(\int -\frac{2\sin(x)}{\sec(x)} dx + c_1\right)$$

- Evaluate the integrals on the rhs

$$y = \sec(x) \left(-\sin(x)^2 + c_1\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x)=sin(x)*(y(x)*sec(x)-2),y(x), singsol=all)
```

$$y(x) = \cos(x) - \frac{\sec(x)}{2} + \sec(x) c_1$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 20

```
DSolve[y'[x]==Sin[x]*(y[x]*Sec[x]-2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} \sec(x)(\cos(2x) + 2c_1)$$

2.11 problem 11

Internal problem ID [2569]
Internal file name [`OUTPUT/2061_Sunday_June_05_2022_02_46_48_AM_66628410/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.6, page 50
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{-y\sin(x) - y'\cos(x) = -1}$$

2.11.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \tan(x)$$
$$q(x) = \sec(x)$$

Hence the ode is

$$y' + y\tan(x) = \sec(x)$$

The integrating factor μ is

$$\mu = e^{\int \tan(x)dx}$$
$$= \frac{1}{\cos(x)}$$

Which simplifies to

$$\mu = \sec(x)$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)(\sec(x)) \\
\frac{d}{dx}(\sec(x)\, y) &= (\sec(x))(\sec(x)) \\
d(\sec(x)\, y) &= \sec(x)^2\, dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\sec(x)\, y &= \int \sec(x)^2\, dx \\
\sec(x)\, y &= \tan(x) + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sec(x)$ results in

$$y = \tan(x)\cos(x) + c_1 \cos(x)$$

which simplifies to

$$y = c_1 \cos(x) + \sin(x)$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 \cos(x) + \sin(x) \tag{1}$$

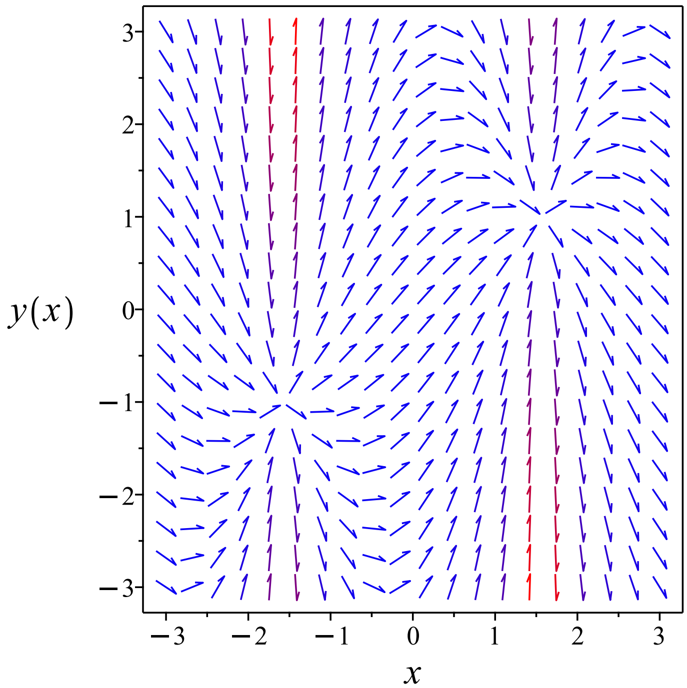


Figure 83: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + \sin(x)$$

Verified OK.

2.11.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{-1 + \sin(x) y}{\cos(x)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 74: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \cos(x) \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\cos(x)} dy\end{aligned}$$

Which results in

$$S = \frac{y}{\cos(x)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-1 + \sin(x)\, y}{\cos(x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1 \\ R_y &= 0 \\ S_x &= \sec(x)\tan(x)\, y \\ S_y &= \sec(x)\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \sec(x)^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \sec(R)^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \tan(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y \sec(x) = \tan(x) + c_1$$

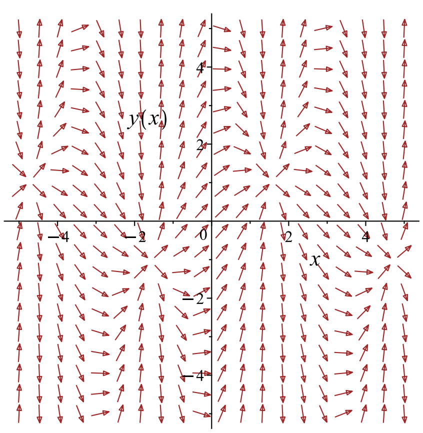
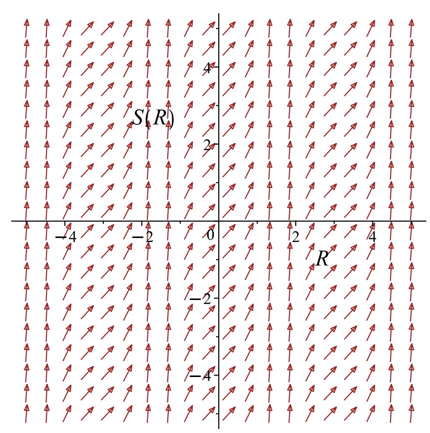
Which simplifies to

$$y \sec(x) = \tan(x) + c_1$$

Which gives

$$y = \frac{\tan(x) + c_1}{\sec(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-1+\sin(x)y}{\cos(x)}$ 	$R = x$ $S = \sec(x)\, y$	$\frac{dS}{dR} = \sec(R)^2$ 

Summary

The solution(s) found are the following

$$y = \frac{\tan(x) + c_1}{\sec(x)} \tag{1}$$

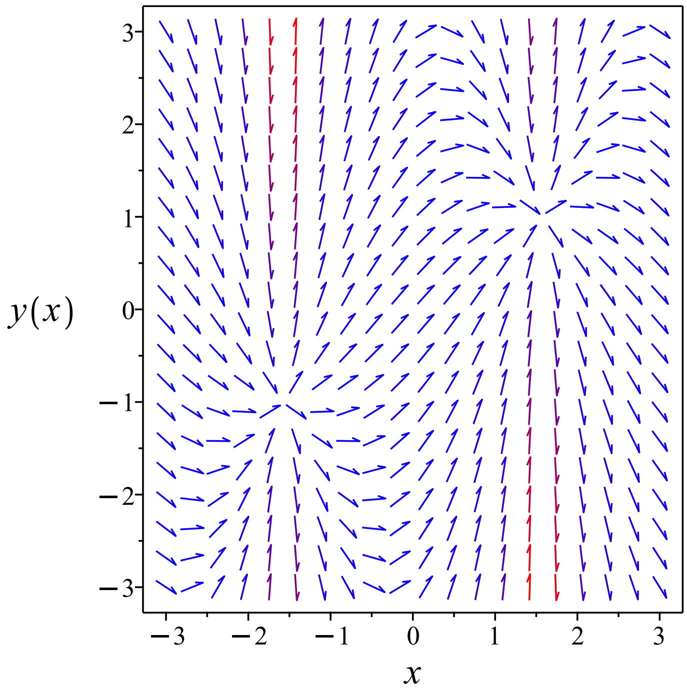


Figure 84: Slope field plot

Verification of solutions

$$y = \frac{\tan(x) + c_1}{\sec(x)}$$

Verified OK.

2.11.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-\cos(x))\,\mathrm{d}y &= (-1 + \sin(x)\,y)\,\mathrm{d}x \\ (-\sin(x)\,y + 1)\,\mathrm{d}x + (-\cos(x))\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\sin(x)\,y + 1 \\ N(x, y) &= -\cos(x)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\sin(x)\,y + 1) \\ &= -\sin(x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-\cos(x)) \\ &= \sin(x)\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= -\sec(x)\left((-\sin(x)) - (\sin(x))\right) \\
&= 2\tan(x)
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}x} \\
&= e^{\int 2\tan(x)\,\mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-2\ln(\cos(x))} \\
&= \sec(x)^2
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \sec(x)^2\left(-\sin(x)\,y + 1\right) \\
&= \left(-\sin(x)\,y + 1\right)\sec(x)^2
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \sec(x)^2\left(-\cos(x)\right) \\
&= -\sec(x)
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\left(-\sin(x)\,y + 1\right)\sec(x)^2\right) + \left(-\sec(x)\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int (-\sin(x) y + 1) \sec(x)^2 \, \mathrm{d}x$$
$$\phi = -\sec(x) y + \tan(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\sec(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\sec(x)$. Therefore equation (4) becomes

$$-\sec(x) = -\sec(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\sec(x) y + \tan(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\sec(x) y + \tan(x)$$

The solution becomes

$$y = \frac{\tan(x) - c_1}{\sec(x)}$$

Summary
The solution(s) found are the following

$$y = \frac{\tan(x) - c_1}{\sec(x)} \tag{1}$$

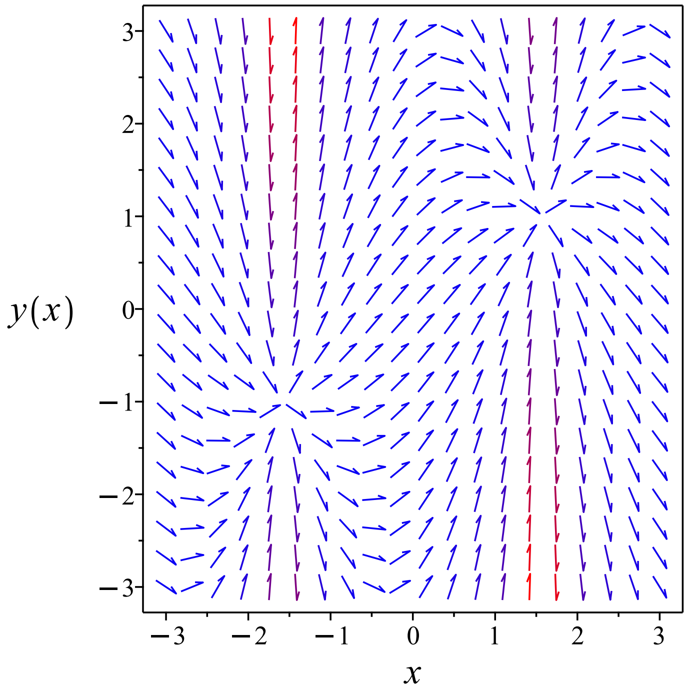


Figure 85: Slope field plot

Verification of solutions

$$y = \frac{\tan(x) - c_1}{\sec(x)}$$

Verified OK.

2.11.4 Maple step by step solution

Let's solve

$-y\sin(x) - y'\cos(x) = -1$

- Highest derivative means the order of the ODE is 1

y'

- Isolate the derivative

$y' = -\frac{\sin(x) y}{\cos(x)} + \frac{1}{\cos(x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{\sin(x) y}{\cos(x)} = \frac{1}{\cos(x)}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x) \left(y' + \frac{\sin(x) y}{\cos(x)}\right) = \frac{\mu(x)}{\cos(x)}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x) \left(y' + \frac{\sin(x) y}{\cos(x)}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x) \sin(x)}{\cos(x)}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{\cos(x)}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)}{\cos(x)} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)}{\cos(x)} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)}{\cos(x)} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{\cos(x)}$

  $y = \cos(x) \left(\int \frac{1}{\cos(x)^2} dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = \cos(x) (\tan(x) + c_1)$

- Simplify

  $y = c_1 \cos(x) + \sin(x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve((1-y(x)*sin(x))-cos(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \cos(x) c_1 + \sin(x)$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 13

```
DSolve[(1-y[x]*Sin[x])-Cos[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sin(x) + c_1 \cos(x)$$

2.12 problem 12

Internal problem ID [2570]
Internal file name [`OUTPUT/2062_Sunday_June_05_2022_02_46_50_AM_96056443/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.6, page 50
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - \frac{y}{x} = 2\ln(x)\, x^2}$$

2.12.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = 2\ln(x)\, x^2$$

Hence the ode is

$$y' - \frac{y}{x} = 2\ln(x)\, x^2$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(2\ln(x)\,x^2\right) \\ \frac{d}{dx}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)\left(2\ln(x)\,x^2\right) \\ d\left(\frac{y}{x}\right) &= (2\ln(x)\,x)\ dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x} &= \int 2\ln(x)\,x\,dx \\ \frac{y}{x} &= \ln(x)\,x^2 - \frac{x^2}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = x\left(\ln(x)\,x^2 - \frac{x^2}{2}\right) + c_1 x$$

which simplifies to

$$y = x^3\ln(x) - \frac{x^3}{2} + c_1 x$$

Summary

The solution(s) found are the following

$$y = x^3\ln(x) - \frac{x^3}{2} + c_1 x \tag{1}$$

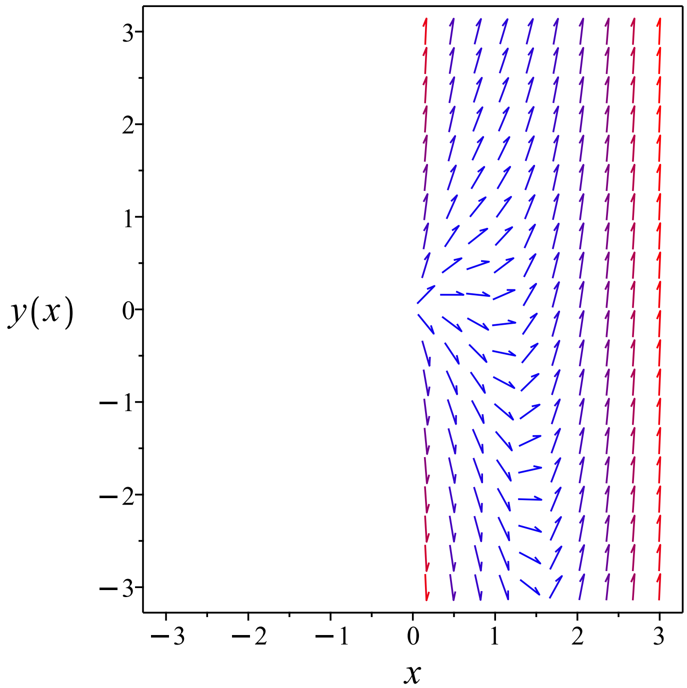


Figure 86: Slope field plot

Verification of solutions

$$y = x^3 \ln(x) - \frac{x^3}{2} + c_1 x$$

Verified OK.

2.12.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u'(x)\, x = 2 \ln(x)\, x^2$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int 2 \ln(x)\, x \, \mathrm{d}x \\ &= \ln(x)\, x^2 - \frac{x^2}{2} + c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= x\left(\ln(x)\, x^2 - \frac{x^2}{2} + c_2\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x\left(\ln(x)\,x^2 - \frac{x^2}{2} + c_2\right) \tag{1}$$

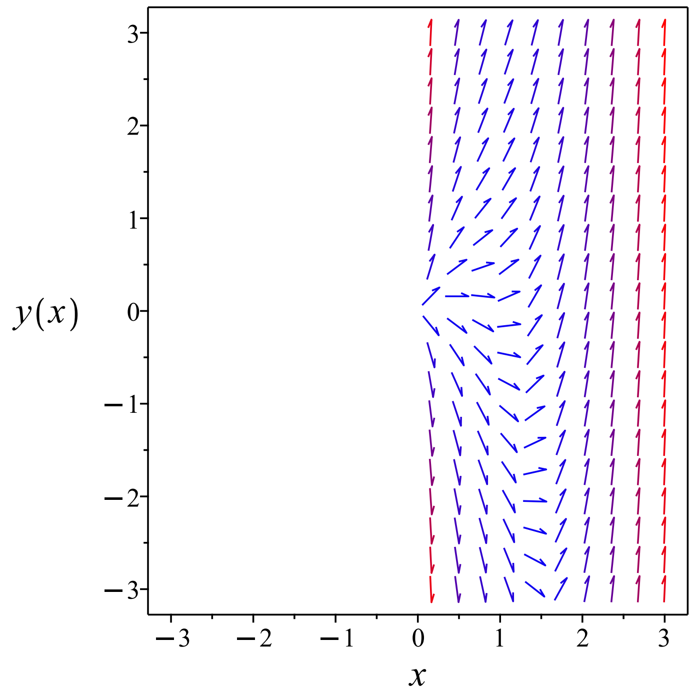


Figure 87: Slope field plot

Verification of solutions

$$y = x\left(\ln(x)\,x^2 - \frac{x^2}{2} + c_2\right)$$

Verified OK.

2.12.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{2x^3\ln(x) + y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 77: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{2x^3 \ln(x) + y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2 \ln(x) x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2 \ln(R) R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^2 \ln(R) - \frac{R^2}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = \ln(x)\, x^2 - \frac{x^2}{2} + c_1$$

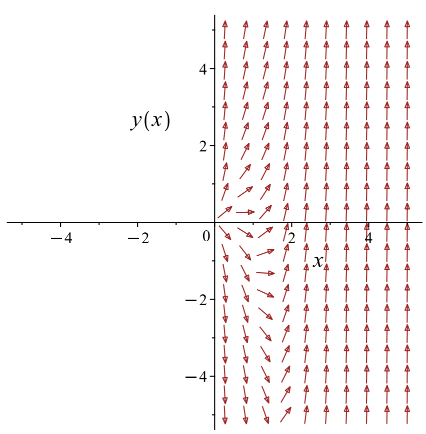
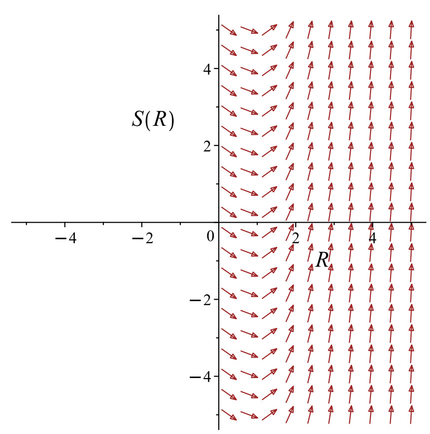
Which simplifies to

$$\frac{y}{x} = \ln(x)\, x^2 - \frac{x^2}{2} + c_1$$

Which gives

$$y = \frac{x(2\ln(x)\, x^2 - x^2 + 2c_1)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2x^3\ln(x)+y}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = 2\ln(R)\, R$ 

Summary

The solution(s) found are the following

$$y = \frac{x(2\ln(x)\,x^2 - x^2 + 2c_1)}{2} \tag{1}$$

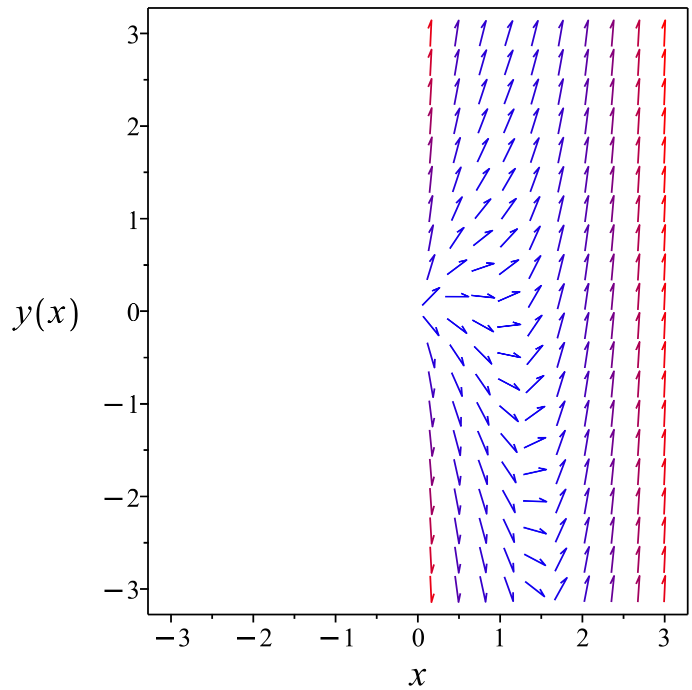


Figure 88: Slope field plot

Verification of solutions

$$y = \frac{x(2\ln(x)\,x^2 - x^2 + 2c_1)}{2}$$

Verified OK.

2.12.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(\frac{y}{x} + 2\ln(x)\, x^2\right)\mathrm{d}x\\ \left(-\frac{y}{x} - 2\ln(x)\, x^2\right)\mathrm{d}x + \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{y}{x} - 2\ln(x)\, x^2\\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y}{x} - 2\ln(x)\, x^2\right)\\ &= -\frac{1}{x}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1\left(\left(-\frac{1}{x}\right) - (0)\right) \\ &= -\frac{1}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -\frac{1}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(x)} \\ &= \frac{1}{x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{x}\left(-\frac{y}{x} - 2\ln(x)\, x^2\right) \\ &= \frac{-2x^3 \ln(x) - y}{x^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{x}(1) \\ &= \frac{1}{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{-2x^3 \ln(x) - y}{x^2}\right) + \left(\frac{1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{-2x^3 \ln(x) - y}{x^2}\,\mathrm{d}x$$
$$\phi = -\ln(x)\,x^2 + \frac{x^2}{2} + \frac{y}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x}$. Therefore equation (4) becomes

$$\frac{1}{x} = \frac{1}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x)\, x^2 + \frac{x^2}{2} + \frac{y}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x)\, x^2 + \frac{x^2}{2} + \frac{y}{x}$$

The solution becomes

$$y = \frac{x(2\ln(x)\, x^2 - x^2 + 2c_1)}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{x(2\ln(x)\, x^2 - x^2 + 2c_1)}{2} \tag{1}$$

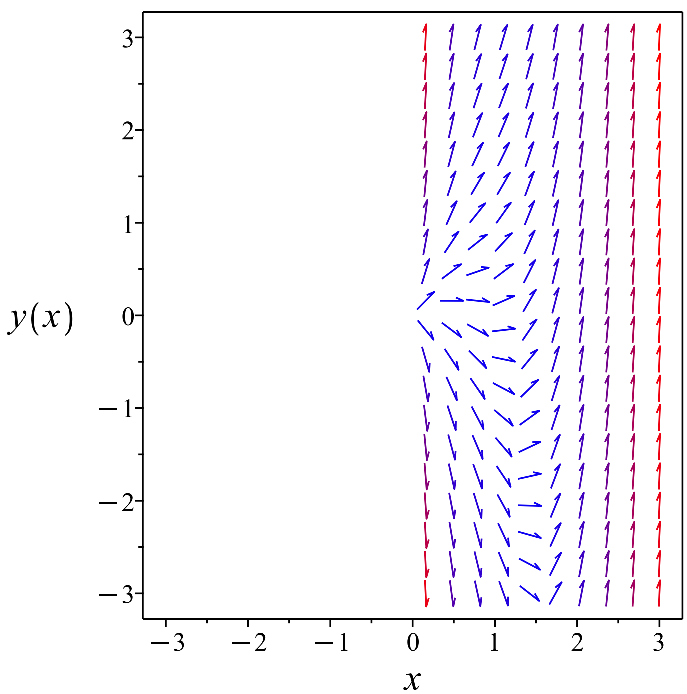


Figure 89: Slope field plot

Verification of solutions

$$y = \frac{x(2\ln(x)\,x^2 - x^2 + 2c_1)}{2}$$

Verified OK.

2.12.5 Maple step by step solution

Let's solve

$y' - \frac{y}{x} = 2\ln(x)\,x^2$

- Highest derivative means the order of the ODE is 1

  y'
- Isolate the derivative

  $y' = \frac{y}{x} + 2\ln(x)\,x^2$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x} = 2\ln(x)\,x^2$
- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = 2\mu(x)\ln(x)\,x^2$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x}$
- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$
- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx = \int 2\mu(x)\ln(x)\,x^2dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y = \int 2\mu(x)\ln(x)\,x^2dx + c_1$
- Solve for y

  $y = \frac{\int 2\mu(x)\ln(x)x^2dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \frac{1}{x}$

$$y = x\left(\int 2\ln(x)\, x dx + c_1\right)$$

- Evaluate the integrals on the rhs

$$y = \left(\ln(x)\, x^2 - \frac{x^2}{2} + c_1\right) x$$

- Simplify

$$y = x^3 \ln(x) - \frac{x^3}{2} + c_1 x$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x)-y(x)/x=2*x^2*ln(x),y(x), singsol=all)
```

$$y(x) = x^3 \ln(x) - \frac{x^3}{2} + c_1 x$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 23

```
DSolve[y'[x]-y[x]/x==2*x^2*Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x^3}{2} + x^3 \log(x) + c_1 x$$

2.13 problem 13

Internal problem ID [2571]
Internal file name [`OUTPUT/2063_Sunday_June_05_2022_02_46_52_AM_32115168/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.6, page 50
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + \alpha y = e^{\beta x}}$$

2.13.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \alpha$$
$$q(x) = e^{\beta x}$$

Hence the ode is

$$y' + \alpha y = e^{\beta x}$$

The integrating factor μ is

$$\mu = e^{\int \alpha dx}$$
$$= e^{\alpha x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(e^{\beta x}\right)$$
$$\frac{d}{dx}(e^{\alpha x}y) = (e^{\alpha x})\left(e^{\beta x}\right)$$
$$d(e^{\alpha x}y) = e^{x(\alpha+\beta)}\,dx$$

Integrating gives

$$e^{\alpha x}y = \int e^{x(\alpha+\beta)}\,dx$$
$$e^{\alpha x}y = \frac{e^{x(\alpha+\beta)}}{\alpha+\beta} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{\alpha x}$ results in

$$y = \frac{e^{-\alpha x}e^{x(\alpha+\beta)}}{\alpha+\beta} + c_1 e^{-\alpha x}$$

which simplifies to

$$y = \frac{c_1(\alpha+\beta)\,e^{-\alpha x} + e^{\beta x}}{\alpha+\beta}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1(\alpha+\beta)\,e^{-\alpha x} + e^{\beta x}}{\alpha+\beta} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(\alpha+\beta)\,e^{-\alpha x} + e^{\beta x}}{\alpha+\beta}$$

Verified OK.

2.13.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\alpha y + e^{\beta x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 80: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-\alpha x}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-\alpha x}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{\alpha x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\alpha y + \mathrm{e}^{\beta x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \alpha\, \mathrm{e}^{\alpha x} y \\ S_y &= \mathrm{e}^{\alpha x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \mathrm{e}^{x(\alpha+\beta)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \mathrm{e}^{R(\alpha+\beta)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{e^{R(\alpha+\beta)}}{\alpha + \beta} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{\alpha x} y = \frac{e^{x(\alpha+\beta)}}{\alpha + \beta} + c_1$$

Which simplifies to

$$e^{\alpha x} y = \frac{e^{x(\alpha+\beta)}}{\alpha + \beta} + c_1$$

Which gives

$$y = \frac{\left(\alpha c_1 + \beta c_1 + e^{x(\alpha+\beta)}\right) e^{-\alpha x}}{\alpha + \beta}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(\alpha c_1 + \beta c_1 + e^{x(\alpha+\beta)}\right) e^{-\alpha x}}{\alpha + \beta} \tag{1}$$

Verification of solutions

$$y = \frac{\left(\alpha c_1 + \beta c_1 + e^{x(\alpha+\beta)}\right) e^{-\alpha x}}{\alpha + \beta}$$

Verified OK.

2.13.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y&=\left(-\alpha y+\mathrm{e}^{\beta x}\right)\mathrm{d}x\\ \left(\alpha y-\mathrm{e}^{\beta x}\right)\mathrm{d}x+\mathrm{d}y&=0 \qquad (2\text{A})\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=\alpha y-\mathrm{e}^{\beta x}\\ N(x,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(\alpha y-\mathrm{e}^{\beta x}\right)\\ &=\alpha\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1((\alpha) - (0))\\ &= \alpha\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \,\mathrm{d}x}\\ &= e^{\int \alpha \,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\alpha x}\\ &= \mathrm{e}^{\alpha x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{\alpha x}\left(\alpha y - \mathrm{e}^{\beta x}\right)\\ &= \left(\alpha y - \mathrm{e}^{\beta x}\right)\mathrm{e}^{\alpha x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{\alpha x}(1)\\ &= \mathrm{e}^{\alpha x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\left(\alpha y - \mathrm{e}^{\beta x}\right)\mathrm{e}^{\alpha x}\right) + \left(\mathrm{e}^{\alpha x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \left(\alpha y - \mathrm{e}^{\beta x}\right)\mathrm{e}^{\alpha x}\,\mathrm{d}x$$

$$\phi = \frac{-\mathrm{e}^{x(\alpha+\beta)} + y(\alpha+\beta)\,\mathrm{e}^{\alpha x}}{\alpha+\beta} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = \mathrm{e}^{\alpha x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \mathrm{e}^{\alpha x}$. Therefore equation (4) becomes

$$\mathrm{e}^{\alpha x} = \mathrm{e}^{\alpha x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-\mathrm{e}^{x(\alpha+\beta)} + y(\alpha+\beta)\,\mathrm{e}^{\alpha x}}{\alpha+\beta} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-\mathrm{e}^{x(\alpha+\beta)} + y(\alpha+\beta)\,\mathrm{e}^{\alpha x}}{\alpha+\beta}$$

The solution becomes

$$y = \frac{\left(\alpha c_1 + \beta c_1 + e^{x(\alpha+\beta)}\right) e^{-\alpha x}}{\alpha + \beta}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(\alpha c_1 + \beta c_1 + e^{x(\alpha+\beta)}\right) e^{-\alpha x}}{\alpha + \beta} \tag{1}$$

Verification of solutions

$$y = \frac{\left(\alpha c_1 + \beta c_1 + e^{x(\alpha+\beta)}\right) e^{-\alpha x}}{\alpha + \beta}$$

Verified OK.

2.13.4 Maple step by step solution

Let's solve

$y' + \alpha y = e^{\beta x}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\alpha y + e^{\beta x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \alpha y = e^{\beta x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \alpha y\right) = \mu(x) e^{\beta x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \alpha y\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)\, \alpha$

- Solve to find the integrating factor

  $\mu(x) = e^{\alpha x}$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, \mathrm{e}^{\beta x} dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\, y = \int \mu(x)\, \mathrm{e}^{\beta x} dx + c_1$$

- Solve for y

$$y = \frac{\int \mu(x) \mathrm{e}^{\beta x} dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \mathrm{e}^{\alpha x}$

$$y = \frac{\int \mathrm{e}^{\beta x} \mathrm{e}^{\alpha x} dx + c_1}{\mathrm{e}^{\alpha x}}$$

- Evaluate the integrals on the rhs

$$y = \frac{\frac{\mathrm{e}^{\alpha x + \beta x}}{\alpha + \beta} + c_1}{\mathrm{e}^{\alpha x}}$$

- Simplify

$$y = \frac{(\mathrm{e}^{x(\alpha+\beta)} + (\alpha+\beta) c_1) \mathrm{e}^{-\alpha x}}{\alpha + \beta}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x)+alpha*y(x)=exp(beta*x),y(x), singsol=all)
```

$$y(x) = \frac{\mathrm{e}^{-\alpha x} \left(\mathrm{e}^{x(\alpha+\beta)} + c_1(\alpha + \beta)\right)}{\alpha + \beta}$$

✓ Solution by Mathematica

Time used: 0.065 (sec). Leaf size: 31

```
DSolve[y'[x]+\[Alpha]*y[x]==Exp[\[Beta]*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{\alpha(-x)}\left(e^{x(\alpha+\beta)} + c_1(\alpha + \beta)\right)}{\alpha + \beta}$$

2.14 problem 14

Internal problem ID [2572]
Internal file name [`OUTPUT/2064_Sunday_June_05_2022_02_46_55_AM_66233644/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.6, page 50
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = -\frac{m}{x} + \ln(x)}$$

2.14.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{\ln(x)\, x - m}{x}\, \mathrm{d}x \\ &= \ln(x)\, x - x - m\ln(x) + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \ln(x)\, x - x - m\ln(x) + c_1 \tag{1}$$

Verification of solutions

$$y = \ln(x)\, x - x - m\ln(x) + c_1$$

Verified OK.

2.14.2 Maple step by step solution

Let's solve

$y' = -\frac{m}{x} + \ln(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int \left(-\frac{m}{x} + \ln(x)\right) dx + c_1$

- Evaluate integral

  $y = \ln(x)\, x - x - m \ln(x) + c_1$

- Solve for y

  $y = \ln(x)\, x - x - m \ln(x) + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x)+m/x=ln(x),y(x), singsol=all)
```

$$y(x) = (-m + x)\ln(x) + c_1 - x$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 19

```
DSolve[y'[x]+m/x==Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (x - m)\log(x) - x + c_1$$

3 1.8, page 68

3.1 problem 9

Internal problem ID [2573]
Internal file name [`OUTPUT/2065_Sunday_June_05_2022_02_46_57_AM_25630428/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.8, page 68
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{(3x - y)\, y' - 3y = 0}$$

3.1.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$(3x - u(x)\, x)\, (u'(x)\, x + u(x)) - 3u(x)\, x = 0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x) g(u) \\
&= -\frac{u^2}{(u - 3)\, x}
\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2}{u-3}$. Integrating both sides gives

$$\frac{1}{\frac{u^2}{u-3}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{u^2}{u-3}}\,du = \int -\frac{1}{x}\,dx$$

$$\frac{3}{u} + \ln(u) = -\ln(x) + c_2$$

The solution is

$$\frac{3}{u(x)} + \ln(u(x)) + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{3x}{y} + \ln\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

$$\frac{3x}{y} + \ln\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

<u>Summary</u>

The solution(s) found are the following

$$\frac{3x}{y} + \ln\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0 \tag{1}$$

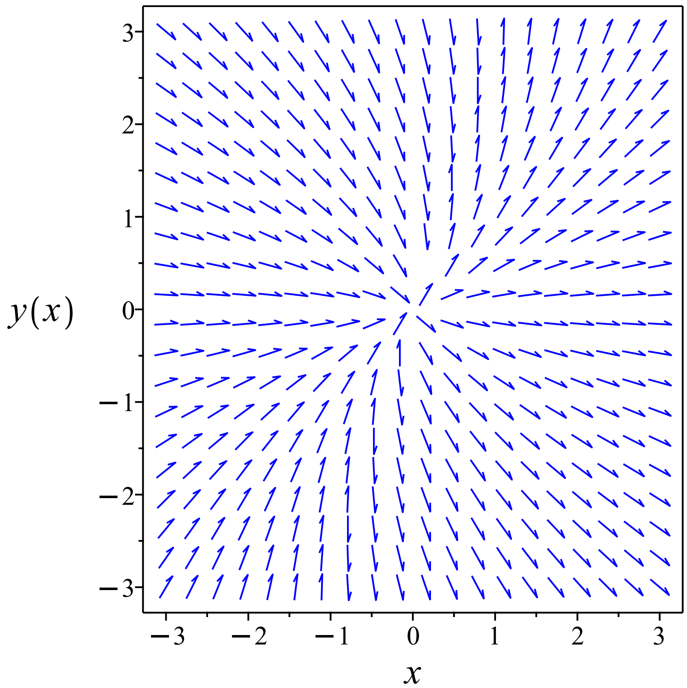


Figure 90: Slope field plot

Verification of solutions

$$\frac{3x}{y} + \ln\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

Verified OK.

3.1.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{3y}{-3x + y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{3y(b_3 - a_2)}{-3x + y} - \frac{9y^2 a_3}{(-3x + y)^2} + \frac{9y(x a_2 + y a_3 + a_1)}{(-3x + y)^2} - \left(-\frac{3}{-3x + y} + \frac{3y}{(-3x + y)^2}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{6xyb_2 - 3y^2 a_2 - y^2 b_2 + 3y^2 b_3 + 9x b_1 - 9y a_1}{(3x - y)^2} = 0$$

Setting the numerator to zero gives

$$-6xyb_2 + 3y^2 a_2 + y^2 b_2 - 3y^2 b_3 - 9x b_1 + 9y a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$3a_2 v_2^2 - 6b_2 v_1 v_2 + b_2 v_2^2 - 3b_3 v_2^2 + 9a_1 v_2 - 9b_1 v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-6b_2 v_1 v_2 - 9b_1 v_1 + (3a_2 + b_2 - 3b_3)\, v_2^2 + 9a_1 v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
9a_1 &= 0 \\
-9b_1 &= 0 \\
-6b_2 &= 0 \\
3a_2 + b_2 - 3b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= a_3 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(-\frac{3y}{-3x + y}\right)(x) \\
&= -\frac{y^2}{3x - y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\frac{y^2}{3x-y}} dy \end{aligned}$$

Which results in

$$S = \frac{3x}{y} + \ln(y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{3y}{-3x+y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{3}{y} \\ S_y &= \frac{-3x+y}{y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y)\,y + 3x}{y} = c_1$$

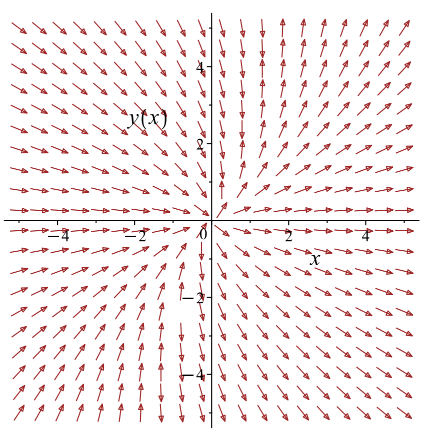
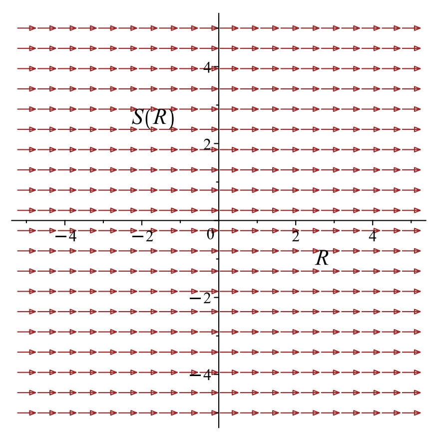
Which simplifies to

$$\frac{\ln(y)\,y + 3x}{y} = c_1$$

Which gives

$$y = \mathrm{e}^{\mathrm{LambertW}\left(-3x\,\mathrm{e}^{-c_1}\right)+c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3y}{-3x+y}$ 	$R = x$ $S = \frac{\ln(y)\,y + 3x}{y}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\mathrm{LambertW}\left(-3x\,\mathrm{e}^{-c_1}\right)+c_1} \tag{1}$$

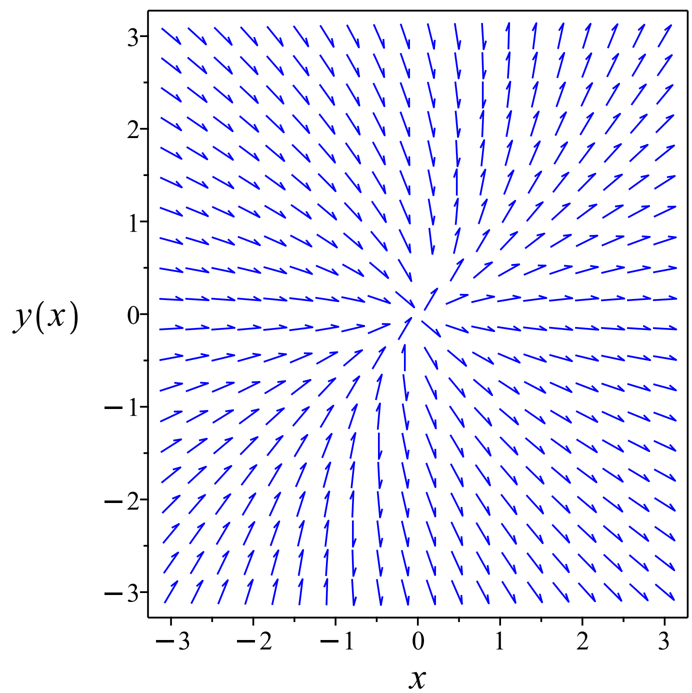


Figure 91: Slope field plot

Verification of solutions

$$y = e^{\text{LambertW}\left(-3x\,e^{-c_1}\right)+c_1}$$

Verified OK.

3.1.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \,\mathrm{d}x + N(x, y) \,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(3x - y) \,\mathrm{d}y &= (3y) \,\mathrm{d}x\\ (-3y) \,\mathrm{d}x + (3x - y) \,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -3y\\ N(x, y) &= 3x - y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-3y)\\ &= -3\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(3x - y)\\ &= 3\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{3x - y}((-3) - (3)) \\
&= -\frac{6}{3x - y}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= -\frac{1}{3y}((3) - (-3)) \\
&= -\frac{2}{y}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int -\frac{2}{y} \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(y)} \\
&= \frac{1}{y^2}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{y^2}(-3y) \\
&= -\frac{3}{y}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^2}(3x - y) \\
&= \frac{3x - y}{y^2}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\frac{3}{y}\right) + \left(\frac{3x - y}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{3}{y}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{3x}{y} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{3x}{y^2} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{3x-y}{y^2}$. Therefore equation (4) becomes

$$
\frac{3x - y}{y^2} = \frac{3x}{y^2} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{y}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{y}\right)\mathrm{d}y$$
$$f(y) = -\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{3x}{y} - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{3x}{y} - \ln(y)$$

The solution becomes

$$y = \mathrm{e}^{\mathrm{LambertW}\left(-3\,\mathrm{e}^{c_1}x\right)-c_1}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\mathrm{LambertW}\left(-3\,\mathrm{e}^{c_1}x\right)-c_1} \tag{1}$$

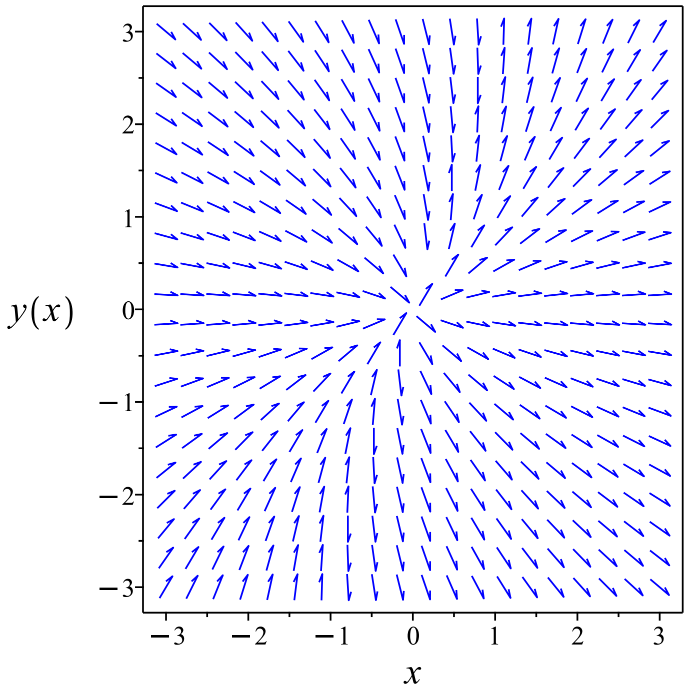


Figure 92: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{\mathrm{LambertW}\left(-3\,\mathrm{e}^{c_1}x\right)-c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve((3*x-y(x))*diff(y(x),x)=3*y(x),y(x), singsol=all)
```

$$y(x) = -\frac{3x}{\text{LambertW}\left(-3x\,\mathrm{e}^{-3c_1}\right)}$$

✓ Solution by Mathematica

Time used: 6.016 (sec). Leaf size: 25

```
DSolve[(3*x-y[x])*y'[x]==3*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{3x}{W\left(-3e^{-c_1}x\right)}$$
$$y(x) \to 0$$

3.2 problem 10

Internal problem ID [2574]
Internal file name [`OUTPUT/2066_Sunday_June_05_2022_02_47_00_AM_18701538/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.8, page 68
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Riccati]
```

$$\boxed{y' - \frac{(y + x)^2}{2x^2} = 0}$$

3.2.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{(u(x)\,x + x)^2}{2x^2} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{\frac{u^2}{2} + \frac{1}{2}}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \frac{u^2}{2} + \frac{1}{2}$. Integrating both sides gives

$$\frac{1}{\frac{u^2}{2} + \frac{1}{2}} du = \frac{1}{x} dx$$

$$\int \frac{1}{\frac{u^2}{2} + \frac{1}{2}} du = \int \frac{1}{x} dx$$

$$2 \arctan(u) = \ln(x) + c_2$$

The solution is

$$2 \arctan(u(x)) - \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$2 \arctan\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0$$

$$2 \arctan\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0$$

Summary
The solution(s) found are the following

$$2 \arctan\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0 \tag{1}$$

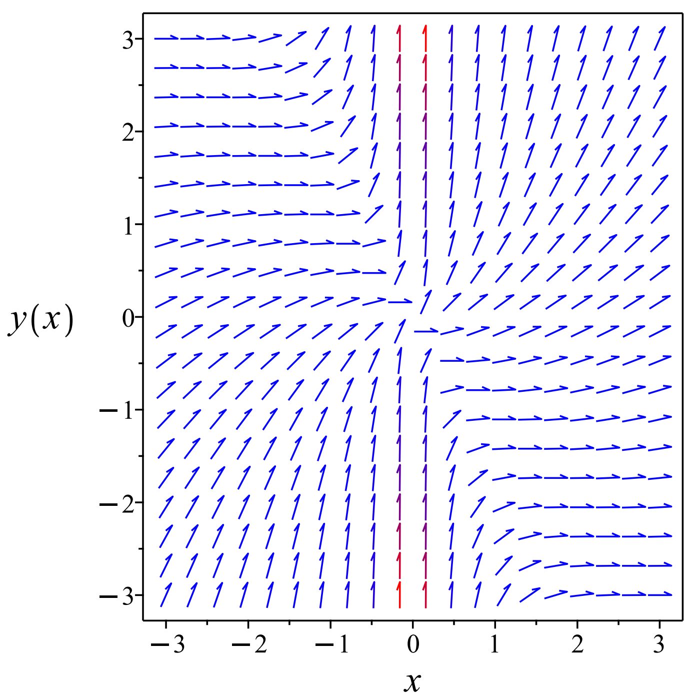


Figure 93: Slope field plot

Verification of solutions

$$2\arctan\left(\frac{y}{x}\right)-\ln(x)-c_2=0$$

Verified OK.

3.2.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{(y+x)^2}{2x^2} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1,a_2,a_3,b_1,b_2,b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2+\frac{(y+x)^2(b_3-a_2)}{2x^2}-\frac{(y+x)^4 a_3}{4x^4} - \left(\frac{y+x}{x^2}-\frac{(y+x)^2}{x^3}\right)(xa_2+ya_3+a_1)-\frac{(y+x)(xb_2+yb_3+b_1)}{x^2}=0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2x^4a_2+x^4a_3-2x^4b_3+4x^3ya_3+4x^3yb_2-2x^2y^2a_2+2x^2y^2a_3+2x^2y^2b_3+y^4a_3+4x^3b_1-4x^2ya_1+4x}{4x^4} = 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-2x^4a_2 - x^4a_3 + 2x^4b_3 - 4x^3ya_3 - 4x^3yb_2 + 2x^2y^2a_2 - 2x^2y^2a_3 \\
&- 2x^2y^2b_3 - y^4a_3 - 4x^3b_1 + 4x^2ya_1 - 4x^2yb_1 + 4x\,y^2a_1 = 0
\end{aligned}
\tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\{x = v_1, y = v_2\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-2a_2v_1^4 + 2a_2v_1^2v_2^2 - a_3v_1^4 - 4a_3v_1^3v_2 - 2a_3v_1^2v_2^2 - a_3v_2^4 - 4b_2v_1^3v_2 \\
&+ 2b_3v_1^4 - 2b_3v_1^2v_2^2 + 4a_1v_1^2v_2 + 4a_1v_1v_2^2 - 4b_1v_1^3 - 4b_1v_1^2v_2 = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&(-2a_2 - a_3 + 2b_3)\,v_1^4 + (-4a_3 - 4b_2)\,v_1^3v_2 - 4b_1v_1^3 \\
&+ (2a_2 - 2a_3 - 2b_3)\,v_1^2v_2^2 + (4a_1 - 4b_1)\,v_1^2v_2 + 4a_1v_1v_2^2 - a_3v_2^4 = 0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
4a_1 &= 0 \\
-a_3 &= 0 \\
-4b_1 &= 0 \\
4a_1 - 4b_1 &= 0 \\
-4a_3 - 4b_2 &= 0 \\
-2a_2 - a_3 + 2b_3 &= 0 \\
2a_2 - 2a_3 - 2b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{(y+x)^2}{2x^2}\right)(x) \\
&= \frac{-x^2 - y^2}{2x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-x^2 - y^2}{2x}} dy
\end{aligned}
$$

Which results in

$$S = -2\arctan\left(\frac{y}{x}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{(y+x)^2}{2x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2y}{x^2+y^2} \\ S_y &= -\frac{2x}{x^2+y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-2\arctan\left(\frac{y}{x}\right) = -\ln(x) + c_1$$

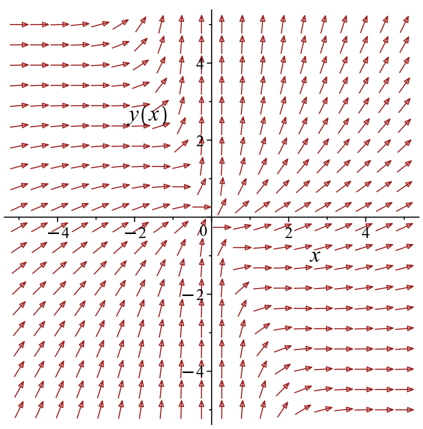
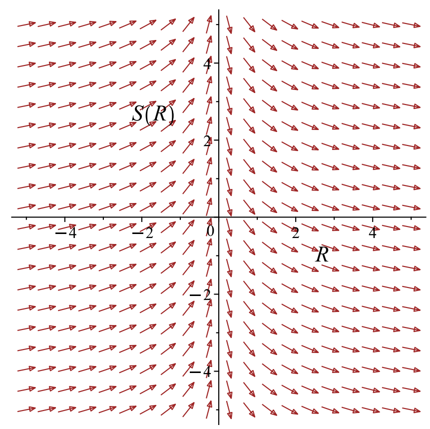
Which simplifies to

$$-2\arctan\left(\frac{y}{x}\right) = -\ln(x) + c_1$$

Which gives

$$y = -\tan\left(-\frac{\ln(x)}{2} + \frac{c_1}{2}\right)x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{(y+x)^2}{2x^2}$ 	$R = x$ $S = -2\arctan\left(\frac{y}{x}\right)$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = -\tan\left(-\frac{\ln(x)}{2} + \frac{c_1}{2}\right)x \tag{1}$$

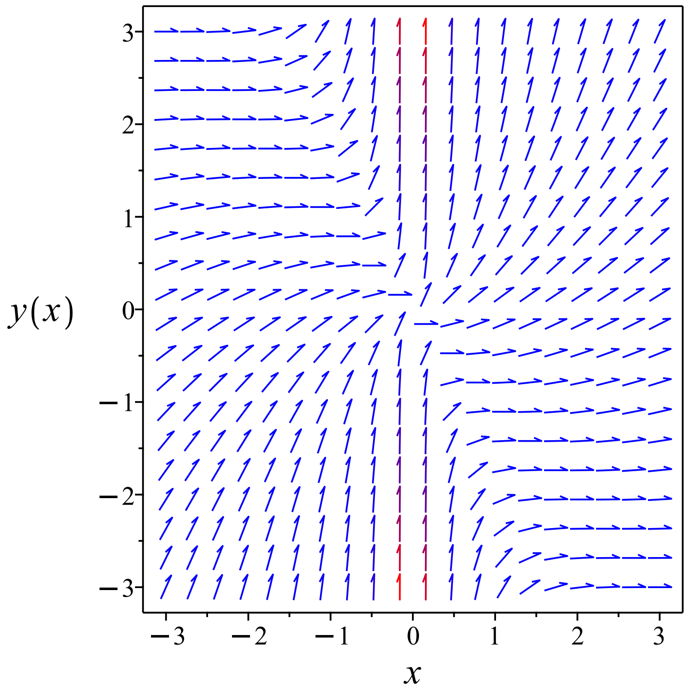


Figure 94: Slope field plot

Verification of solutions

$$y = -\tan\left(-\frac{\ln(x)}{2} + \frac{c_1}{2}\right) x$$

Verified OK.

3.2.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{(y + x)^2}{2x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{1}{2} + \frac{y}{x} + \frac{y^2}{2x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{1}{2}$, $f_1(x) = \frac{1}{x}$ and $f_2(x) = \frac{1}{2x^2}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{\frac{u}{2x^2}}
\end{aligned}
\tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= -\frac{1}{x^3} \\
f_1 f_2 &= \frac{1}{2x^3} \\
f_2^2 f_0 &= \frac{1}{8x^4}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
\frac{u''(x)}{2x^2} + \frac{u'(x)}{2x^3} + \frac{u(x)}{8x^4} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = c_1 \sin\left(\frac{\ln(x)}{2}\right) + c_2 \cos\left(\frac{\ln(x)}{2}\right)
$$

The above shows that

$$
u'(x) = \frac{c_1 \cos\left(\frac{\ln(x)}{2}\right) - c_2 \sin\left(\frac{\ln(x)}{2}\right)}{2x}
$$

Using the above in (1) gives the solution

$$
y = -\frac{x\left(c_1 \cos\left(\frac{\ln(x)}{2}\right) - c_2 \sin\left(\frac{\ln(x)}{2}\right)\right)}{c_1 \sin\left(\frac{\ln(x)}{2}\right) + c_2 \cos\left(\frac{\ln(x)}{2}\right)}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(-c_3 \cos\left(\frac{\ln(x)}{2}\right) + \sin\left(\frac{\ln(x)}{2}\right)\right) x}{c_3 \sin\left(\frac{\ln(x)}{2}\right) + \cos\left(\frac{\ln(x)}{2}\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-c_3 \cos\left(\frac{\ln(x)}{2}\right) + \sin\left(\frac{\ln(x)}{2}\right)\right) x}{c_3 \sin\left(\frac{\ln(x)}{2}\right) + \cos\left(\frac{\ln(x)}{2}\right)} \tag{1}$$

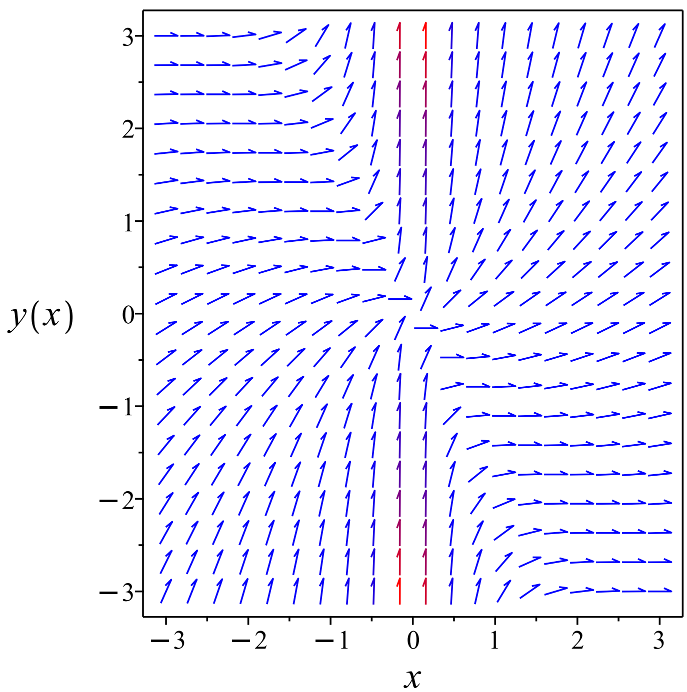


Figure 95: Slope field plot

Verification of solutions

$$y = \frac{\left(-c_3 \cos\left(\frac{\ln(x)}{2}\right) + \sin\left(\frac{\ln(x)}{2}\right)\right) x}{c_3 \sin\left(\frac{\ln(x)}{2}\right) + \cos\left(\frac{\ln(x)}{2}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve(diff(y(x),x)=(x+y(x))^2/(2*x^2),y(x), singsol=all)
```

$$y(x) = \tan\left(\frac{\ln(x)}{2} + \frac{c_1}{2}\right) x$$

✓ Solution by Mathematica

Time used: 0.234 (sec). Leaf size: 17

```
DSolve[y'[x]==(x+y[x])^2/(2*x^2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x \tan\left(\frac{\log(x)}{2} + c_1\right)$$

3.3 problem 11

Internal problem ID [2575]
Internal file name [`OUTPUT/2067_Sunday_June_05_2022_02_47_02_AM_4957607/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.8, page 68
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{\sin\left(\frac{y}{x}\right)(xy'-y)-x\cos\left(\frac{y}{x}\right)=0}$$

3.3.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$\sin\left(u(x)\right)\left(x(u'(x)\,x+u(x))-u(x)\,x\right)-x\cos\left(u(x)\right)=0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{\cot\left(u\right)}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \cot(u)$. Integrating both sides gives

$$\frac{1}{\cot(u)}\,du = \frac{1}{x}\,dx$$

$$\int \frac{1}{\cot(u)}\,du = \int \frac{1}{x}\,dx$$

$$-\ln(\cos(u)) = \ln(x) + c_2$$

Raising both side to exponential gives

$$\frac{1}{\cos(u)} = e^{\ln(x)+c_2}$$

Which simplifies to

$$\sec(u) = c_3 x$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= x\,\operatorname{arcsec}\left(c_3 e^{c_2} x\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = x\,\operatorname{arcsec}\left(c_3 e^{c_2} x\right) \tag{1}$$

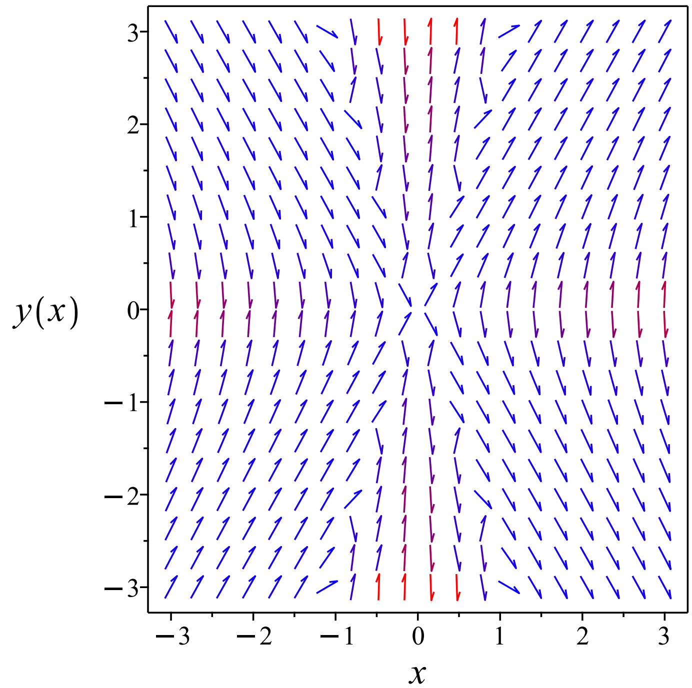


Figure 96: Slope field plot

Verification of solutions

$$y = x \operatorname{arcsec}\left(c_3 e^{c_2} x\right)$$

Verified OK.

3.3.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{x \cos\left(\frac{y}{x}\right) + \sin\left(\frac{y}{x}\right) y}{\sin\left(\frac{y}{x}\right) x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2+\frac{\left(x \cos\left(\frac{y}{x}\right)+\sin\left(\frac{y}{x}\right) y\right)(b_3-a_2)}{\sin\left(\frac{y}{x}\right) x}-\frac{\left(x \cos\left(\frac{y}{x}\right)+\sin\left(\frac{y}{x}\right) y\right)^2 a_3}{\sin\left(\frac{y}{x}\right)^2 x^2} \\ & -\left(\frac{\cos\left(\frac{y}{x}\right)+\frac{y \sin\left(\frac{y}{x}\right)}{x}-\frac{y^2 \cos\left(\frac{y}{x}\right)}{x^2}}{\sin\left(\frac{y}{x}\right) x}+\frac{\left(x \cos\left(\frac{y}{x}\right)+\sin\left(\frac{y}{x}\right) y\right) y \cos\left(\frac{y}{x}\right)}{\sin\left(\frac{y}{x}\right)^2 x^3}\right. \\ & \left.-\frac{x \cos\left(\frac{y}{x}\right)+\sin\left(\frac{y}{x}\right) y}{\sin\left(\frac{y}{x}\right) x^2}\right)(x a_2+y a_3+a_1) \\ & -\left(\frac{\cos\left(\frac{y}{x}\right) y}{x^2 \sin\left(\frac{y}{x}\right)}-\frac{\left(x \cos\left(\frac{y}{x}\right)+\sin\left(\frac{y}{x}\right) y\right) \cos\left(\frac{y}{x}\right)}{\sin\left(\frac{y}{x}\right)^2 x^2}\right)(x b_2+y b_3+b_1)=0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{\cos\left(\frac{y}{x}\right)^2 x^2 a_3-\cos\left(\frac{y}{x}\right)^2 x^2 b_2+\cos\left(\frac{y}{x}\right)^2 x y a_2-\cos\left(\frac{y}{x}\right)^2 x y b_3+\cos\left(\frac{y}{x}\right)^2 y^2 a_3+\cos\left(\frac{y}{x}\right) \sin\left(\frac{y}{x}\right) x^2 a_2-\mathrm{co}}{}$$

$$=0$$

Setting the numerator to zero gives

$$\begin{aligned} & -\cos\left(\frac{y}{x}\right)^2 x^2 a_3+\cos\left(\frac{y}{x}\right)^2 x^2 b_2-\cos\left(\frac{y}{x}\right)^2 x y a_2+\cos\left(\frac{y}{x}\right)^2 x y b_3 \\ & -\cos\left(\frac{y}{x}\right)^2 y^2 a_3-\cos\left(\frac{y}{x}\right) \sin\left(\frac{y}{x}\right) x^2 a_2+\cos\left(\frac{y}{x}\right) \sin\left(\frac{y}{x}\right) x^2 b_3 \\ & -2 \cos\left(\frac{y}{x}\right) \sin\left(\frac{y}{x}\right) x y a_3+b_2 \sin\left(\frac{y}{x}\right)^2 x^2-\sin\left(\frac{y}{x}\right)^2 x y a_2 \\ & +\sin\left(\frac{y}{x}\right)^2 x y b_3-\sin\left(\frac{y}{x}\right)^2 y^2 a_3+\cos\left(\frac{y}{x}\right)^2 x b_1-\cos\left(\frac{y}{x}\right)^2 y a_1=0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$-\frac{x\left(x^2 a_2 \sin\left(\frac{2 y}{x}\right)-x^2 b_3 \sin\left(\frac{2 y}{x}\right)+2 x y a_3 \sin\left(\frac{2 y}{x}\right)+x^2 a_3 \cos\left(\frac{2 y}{x}\right)-x b_1 \cos\left(\frac{2 y}{x}\right)+y a_1 \cos\left(\frac{2 y}{x}\right)+x^2 a_3-2\right.}{2} \tag{6E}$$

$$=0$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \cos\left(\frac{2y}{x}\right), \sin\left(\frac{2y}{x}\right)\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \cos\left(\frac{2y}{x}\right) = v_3, \sin\left(\frac{2y}{x}\right) = v_4\right\}$$

The above PDE (6E) now becomes

$$-\frac{v_1(v_1^2 a_2 v_4 + v_1^2 a_3 v_3 + 2v_1 v_2 a_3 v_4 - v_1^2 b_3 v_4 + v_2 a_1 v_3 + 2v_1 v_2 a_2 + v_1^2 a_3 + 2v_2^2 a_3 - v_1 b_1 v_3 - 2v_1^2 b_2 - 2v_1 v_2 b_3}{2} = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$\left(-\frac{a_3}{2} + b_2\right) v_1^3 - \frac{a_3 v_3 v_1^3}{2} + \left(-\frac{a_2}{2} + \frac{b_3}{2}\right) v_4 v_1^3 + \frac{b_1 v_1^2}{2} + (b_3 - a_2) v_2 v_1^2 + \frac{b_1 v_3 v_1^2}{2} - a_3 v_4 v_2 v_1^2 - a_3 v_2^2 v_1 - \frac{a_1 v_2 v_1}{2} - \frac{a_1 v_2 v_3 v_1}{2} = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-\frac{a_1}{2} &= 0 \\
-a_3 &= 0 \\
-\frac{a_3}{2} &= 0 \\
\frac{b_1}{2} &= 0 \\
-\frac{a_2}{2} + \frac{b_3}{2} &= 0 \\
-\frac{a_3}{2} + b_2 &= 0 \\
b_3 - a_2 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{x\cos\left(\frac{y}{x}\right) + \sin\left(\frac{y}{x}\right) y}{\sin\left(\frac{y}{x}\right) x}\right)(x) \\
&= -\frac{\cos\left(\frac{y}{x}\right) x}{\sin\left(\frac{y}{x}\right)} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\frac{\cos\left(\frac{y}{x}\right)x}{\sin\left(\frac{y}{x}\right)}} dy
\end{aligned}
$$

Which results in

$$
S = \ln\left(\cos\left(\frac{y}{x}\right)\right)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{x\cos\left(\frac{y}{x}\right) + \sin\left(\frac{y}{x}\right) y}{\sin\left(\frac{y}{x}\right) x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{\tan\left(\frac{y}{x}\right) y}{x^2} \\
S_y &= -\frac{\tan\left(\frac{y}{x}\right)}{x}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln\left(\cos\left(\frac{y}{x}\right)\right) = -\ln(x) + c_1$$

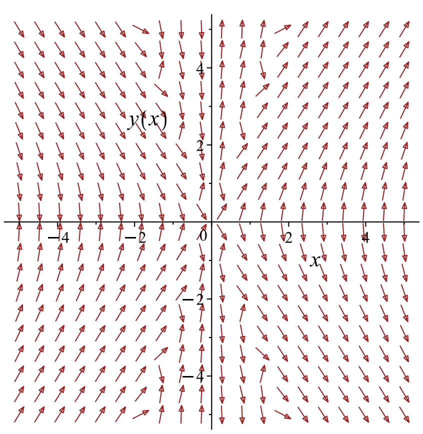
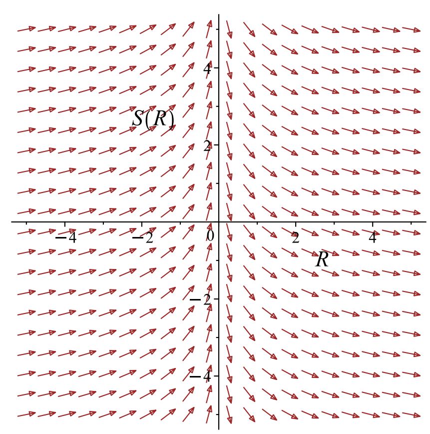
Which simplifies to

$$\ln\left(\cos\left(\frac{y}{x}\right)\right) = -\ln(x) + c_1$$

Which gives

$$y = x \arccos\left(\frac{e^{c_1}}{x}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x\cos\left(\frac{y}{x}\right)+\sin\left(\frac{y}{x}\right)y}{\sin\left(\frac{y}{x}\right)x}$ 	$R = x$ $S = \ln\left(\cos\left(\frac{y}{x}\right)\right)$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary
The solution(s) found are the following

$$y = x \arccos\left(\frac{e^{c_1}}{x}\right) \tag{1}$$

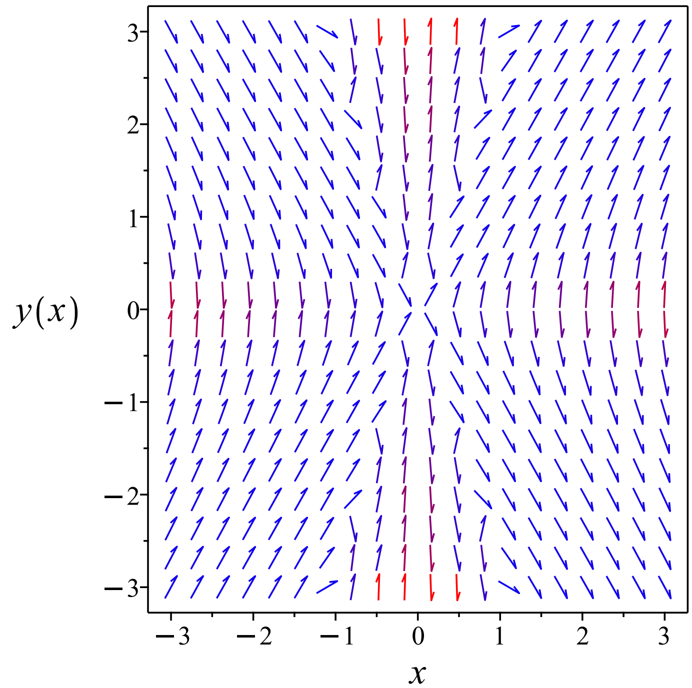


Figure 97: Slope field plot

Verification of solutions

$$y = x \arccos\left(\frac{e^{c_1}}{x}\right)$$

Verified OK.

3.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\sin\left(\frac{y}{x}\right)x\right)\mathrm{d}y = \left(x\cos\left(\frac{y}{x}\right) + \sin\left(\frac{y}{x}\right)y\right)\mathrm{d}x$$
$$\left(-x\cos\left(\frac{y}{x}\right) - \sin\left(\frac{y}{x}\right)y\right)\mathrm{d}x + \left(\sin\left(\frac{y}{x}\right)x\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -x\cos\left(\frac{y}{x}\right) - \sin\left(\frac{y}{x}\right)y$$
$$N(x,y) = \sin\left(\frac{y}{x}\right)x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x\cos\left(\frac{y}{x}\right) - \sin\left(\frac{y}{x}\right)y\right)\\ &= -\frac{\cos\left(\frac{y}{x}\right)y}{x}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\sin\left(\frac{y}{x}\right)x\right)\\ &= -\frac{\cos\left(\frac{y}{x}\right)y}{x} + \sin\left(\frac{y}{x}\right)\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{\csc\left(\frac{y}{x}\right)}{x}\left(\left(-\frac{\cos\left(\frac{y}{x}\right)y}{x}\right) - \left(-\frac{\cos\left(\frac{y}{x}\right)y}{x} + \sin\left(\frac{y}{x}\right)\right)\right)\\ &= -\frac{1}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\frac{1}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(x)}\\ &= \frac{1}{x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x}\left(-x\cos\left(\frac{y}{x}\right) - \sin\left(\frac{y}{x}\right)y\right)\\ &= \frac{-x\cos\left(\frac{y}{x}\right) - \sin\left(\frac{y}{x}\right)y}{x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x}\left(\sin\left(\frac{y}{x}\right)x\right)\\ &= \sin\left(\frac{y}{x}\right)\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\left(\frac{-x\cos\left(\frac{y}{x}\right) - \sin\left(\frac{y}{x}\right)y}{x}\right) + \left(\sin\left(\frac{y}{x}\right)\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \quad (1)$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \quad (2)$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{-x\cos\left(\frac{y}{x}\right) - \sin\left(\frac{y}{x}\right)y}{x}\,\mathrm{d}x$$

$$\phi = -x\cos\left(\frac{y}{x}\right) + f(y) \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \sin\left(\frac{y}{x}\right) + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sin\left(\frac{y}{x}\right)$. Therefore equation (4) becomes

$$\sin\left(\frac{y}{x}\right) = \sin\left(\frac{y}{x}\right) + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x\cos\left(\frac{y}{x}\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x\cos\left(\frac{y}{x}\right)$$

Summary

The solution(s) found are the following

$$-x\cos\left(\frac{y}{x}\right) = c_1 \tag{1}$$

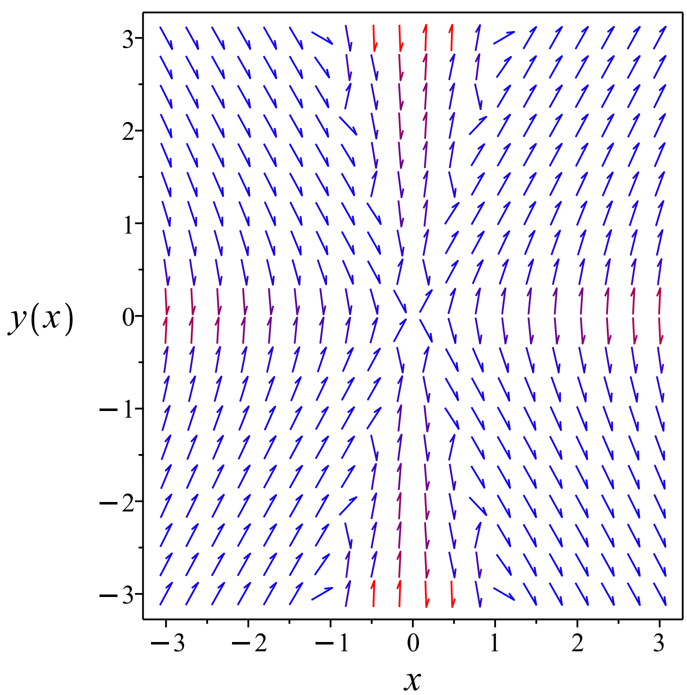


Figure 98: Slope field plot

Verification of solutions

$$-x\cos\left(\frac{y}{x}\right) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve(sin(y(x)/x)*(x*diff(y(x),x)-y(x))=x*cos(y(x)/x),y(x), singsol=all)
```

$$y(x) = x \arccos\left(\frac{1}{c_1 x}\right)$$

✓ Solution by Mathematica

Time used: 25.589 (sec). Leaf size: 56

```
DSolve[Sin[y[x]/x]*(x*y'[x]-y[x])==x*Cos[y[x]/x],y[x],x,IncludeSingularSolutions -> True]
```

$$\begin{aligned} y(x) &\to -x \arccos\left(\frac{e^{-c_1}}{x}\right) \\ y(x) &\to x \arccos\left(\frac{e^{-c_1}}{x}\right) \\ y(x) &\to -\frac{\pi x}{2} \\ y(x) &\to \frac{\pi x}{2} \end{aligned}$$

3.4 problem 12

Internal problem ID [2576]
Internal file name [`OUTPUT/2068_Sunday_June_05_2022_02_47_07_AM_35449455/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.8, page 68
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{xy' - \sqrt{16x^2 - y^2} - y = 0}$$

3.4.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{\sqrt{16x^2 - y^2} + y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 + \frac{\left(\sqrt{16x^2-y^2}+y\right)(b_3-a_2)}{x} - \frac{\left(\sqrt{16x^2-y^2}+y\right)^2 a_3}{x^2} \\
&- \left(\frac{16}{\sqrt{16x^2-y^2}} - \frac{\sqrt{16x^2-y^2}+y}{x^2}\right)(xa_2+ya_3+a_1) \\
&- \frac{\left(-\frac{y}{\sqrt{16x^2-y^2}}+1\right)(xb_2+yb_3+b_1)}{x} = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{\left(16x^2-y^2\right)^{\frac{3}{2}} a_3 + 16x^3a_2 - 16x^3b_3 + 32x^2ya_3 - x^2yb_2 - y^3a_3 + \sqrt{16x^2-y^2}\,xb_1 - \sqrt{16x^2-y^2}\,ya_1 - xy}{\sqrt{16x^2-y^2}\,x^2}
$$
$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-\left(16x^2-y^2\right)^{\frac{3}{2}} a_3 - 16x^3a_2 + 16x^3b_3 - 32x^2ya_3 + x^2yb_2 + y^3a_3 \\
&- \sqrt{16x^2-y^2}\,xb_1 + \sqrt{16x^2-y^2}\,ya_1 + xyb_1 - y^2a_1 = 0
\end{aligned} \tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
&-\left(16x^2-y^2\right)^{\frac{3}{2}} a_3 + \left(16x^2-y^2\right)xb_3 - \left(16x^2-y^2\right)ya_3 \\
&- 16x^3a_2 - 16x^2ya_3 + x^2yb_2 + x\,y^2b_3 + \left(16x^2-y^2\right)a_1 \\
&- \sqrt{16x^2-y^2}\,xb_1 + \sqrt{16x^2-y^2}\,ya_1 - 16x^2a_1 + xyb_1 = 0
\end{aligned} \tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&-16x^3a_2 + 16x^3b_3 - 16x^2\sqrt{16x^2-y^2}\,a_3 - 32x^2ya_3 + x^2yb_2 + \sqrt{16x^2-y^2}\,y^2a_3 \\
&+ y^3a_3 - \sqrt{16x^2-y^2}\,xb_1 + xyb_1 + \sqrt{16x^2-y^2}\,ya_1 - y^2a_1 = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{16x^2-y^2}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{16x^2 - y^2} = v_3\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-16v_1^3a_2 - 32v_1^2v_2a_3 - 16v_1^2v_3a_3 + v_2^3a_3 + v_3v_2^2a_3 + v_1^2v_2b_2 \\ &+ 16v_1^3b_3 - v_2^2a_1 + v_3v_2a_1 + v_1v_2b_1 - v_3v_1b_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned} &(-16a_2 + 16b_3)\, v_1^3 + (-32a_3 + b_2)\, v_1^2v_2 - 16v_1^2v_3a_3 \\ &+ v_1v_2b_1 - v_3v_1b_1 + v_2^3a_3 + v_3v_2^2a_3 - v_2^2a_1 + v_3v_2a_1 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ a_3 &= 0 \\ b_1 &= 0 \\ -a_1 &= 0 \\ -16a_3 &= 0 \\ -b_1 &= 0 \\ -16a_2 + 16b_3 &= 0 \\ -32a_3 + b_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{\sqrt{16x^2 - y^2} + y}{x}\right)(x) \\
&= -\sqrt{16x^2 - y^2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\sqrt{16x^2 - y^2}} dy
\end{aligned}$$

Which results in

$$S = -\arctan\left(\frac{y}{\sqrt{16x^2 - y^2}}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{\sqrt{16x^2 - y^2} + y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y}{\sqrt{16x^2 - y^2}\, x} \\ S_y &= -\frac{1}{\sqrt{16x^2 - y^2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\arctan\left(\frac{y}{\sqrt{16x^2 - y^2}}\right) = -\ln(x) + c_1$$

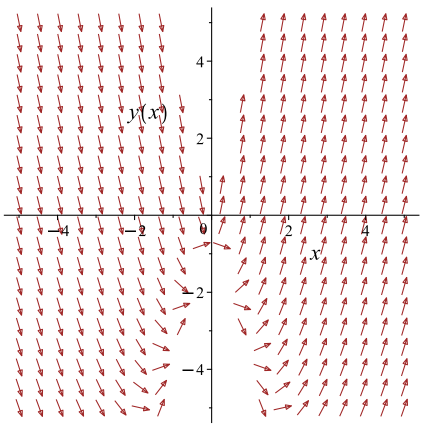
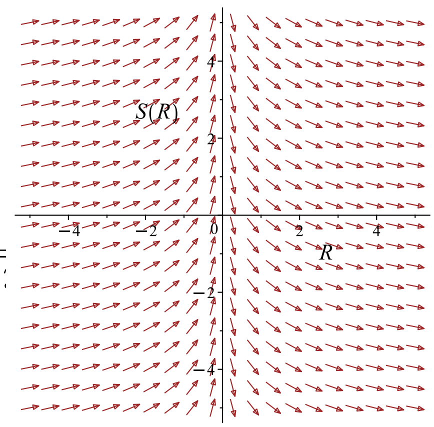
Which simplifies to

$$-\arctan\left(\frac{y}{\sqrt{16x^2 - y^2}}\right) = -\ln(x) + c_1$$

Which gives

$$y = -4\tan(-\ln(x) + c_1)\sqrt{\frac{x^2}{\tan(-\ln(x) + c_1)^2 + 1}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\sqrt{16x^2-y^2}+y}{x}$ 	$R = x$ $S = -\arctan\left(\frac{y}{\sqrt{16x^2 -}}\right.$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = -4\tan(-\ln(x) + c_1)\sqrt{\frac{x^2}{\tan(-\ln(x) + c_1)^2 + 1}} \tag{1}$$

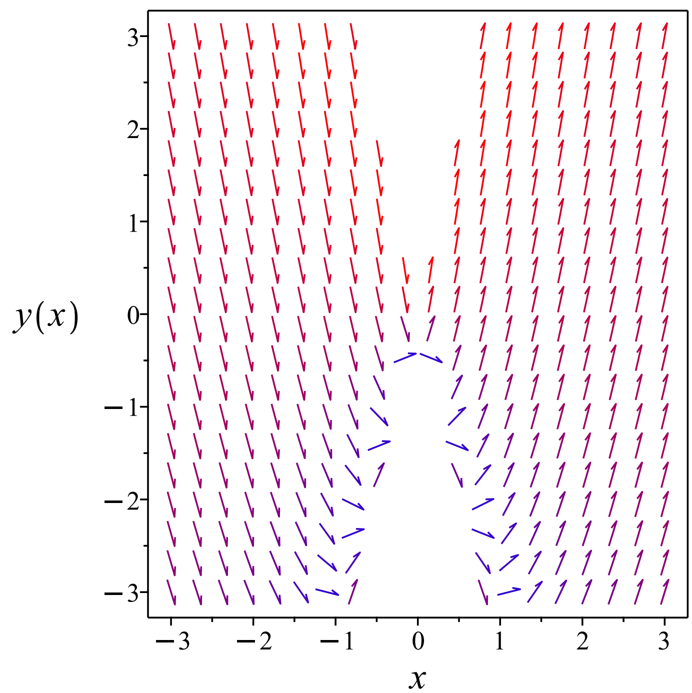


Figure 99: Slope field plot

Verification of solutions

$$y = -4\tan(-\ln(x) + c_1)\sqrt{\frac{x^2}{\tan(-\ln(x) + c_1)^2 + 1}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(x*diff(y(x),x)=sqrt(16*x^2-y(x)^2)+y(x),y(x), singsol=all)
```

$$-\arctan\left(\frac{y(x)}{\sqrt{16x^2-y(x)^2}}\right)+\ln(x)-c_1=0$$

✓ Solution by Mathematica

Time used: 0.43 (sec). Leaf size: 18

```
DSolve[x*y'[x]==Sqrt[16*x^2-y[x]^2]+y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -4x\cosh(i\log(x)+c_1)$$

3.5 problem 13

Internal problem ID [2577]
Internal file name [`OUTPUT/2069_Sunday_June_05_2022_02_47_11_AM_32996207/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.8, page 68
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

`[[_homogeneous, `class A`], _rational, _dAlembert]`

$$\boxed{xy' - y - \sqrt{9x^2 + y^2} = 0}$$

3.5.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y + \sqrt{9x^2 + y^2}}{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 + \frac{\left(y+\sqrt{9x^2+y^2}\right)(b_3-a_2)}{x} - \frac{\left(y+\sqrt{9x^2+y^2}\right)^2 a_3}{x^2} \\
&- \left(\frac{9}{\sqrt{9x^2+y^2}} - \frac{y+\sqrt{9x^2+y^2}}{x^2}\right)(xa_2+ya_3+a_1) \\
&- \frac{\left(1+\frac{y}{\sqrt{9x^2+y^2}}\right)(xb_2+yb_3+b_1)}{x} = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{\left(9x^2+y^2\right)^{\frac{3}{2}} a_3 + 9x^3a_2 - 9x^3b_3 + 18x^2ya_3 + x^2yb_2 + y^3a_3 + \sqrt{9x^2+y^2}\,xb_1 - \sqrt{9x^2+y^2}\,ya_1 + xyb_1 - y}{\sqrt{9x^2+y^2}\,x^2}
$$
$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-\left(9x^2+y^2\right)^{\frac{3}{2}} a_3 - 9x^3a_2 + 9x^3b_3 - 18x^2ya_3 - x^2yb_2 - y^3a_3 \\
&- \sqrt{9x^2+y^2}\,xb_1 + \sqrt{9x^2+y^2}\,ya_1 - xyb_1 + y^2a_1 = 0
\end{aligned} \tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
&-\left(9x^2+y^2\right)^{\frac{3}{2}} a_3 + \left(9x^2+y^2\right)xb_3 - \left(9x^2+y^2\right)ya_3 - 9x^3a_2 - 9x^2ya_3 - x^2yb_2 \\
&- x\,y^2b_3 + \left(9x^2+y^2\right)a_1 - \sqrt{9x^2+y^2}\,xb_1 + \sqrt{9x^2+y^2}\,ya_1 - 9x^2a_1 - xyb_1 = 0
\end{aligned} \tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&-9x^3a_2 + 9x^3b_3 - 9x^2\sqrt{9x^2+y^2}\,a_3 - 18x^2ya_3 - x^2yb_2 - \sqrt{9x^2+y^2}\,y^2a_3 \\
&- y^3a_3 - \sqrt{9x^2+y^2}\,xb_1 - xyb_1 + \sqrt{9x^2+y^2}\,ya_1 + y^2a_1 = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{9x^2+y^2}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{9x^2 + y^2} = v_3\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}&-9v_1^3a_2 - 18v_1^2v_2a_3 - 9v_1^2v_3a_3 - v_2^3a_3 - v_3v_2^2a_3 - v_1^2v_2b_2 \\ &+ 9v_1^3b_3 + v_2^2a_1 + v_3v_2a_1 - v_1v_2b_1 - v_3v_1b_1 = 0\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned}&(-9a_2 + 9b_3)\,v_1^3 + (-18a_3 - b_2)\,v_1^2v_2 - 9v_1^2v_3a_3 - v_1v_2b_1 \\ &- v_3v_1b_1 - v_2^3a_3 - v_3v_2^2a_3 + v_2^2a_1 + v_3v_2a_1 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}a_1 &= 0\\ -9a_3 &= 0\\ -a_3 &= 0\\ -b_1 &= 0\\ -9a_2 + 9b_3 &= 0\\ -18a_3 - b_2 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0\\ a_2 &= b_3\\ a_3 &= 0\\ b_1 &= 0\\ b_2 &= 0\\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{y + \sqrt{9x^2 + y^2}}{x}\right)(x) \\
&= -\sqrt{9x^2 + y^2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\sqrt{9x^2 + y^2}} dy
\end{aligned}$$

Which results in

$$S = -\ln\left(y + \sqrt{9x^2 + y^2}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y + \sqrt{9x^2 + y^2}}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{9x}{\sqrt{9x^2 + y^2}\,\left(y + \sqrt{9x^2 + y^2}\right)} \\ S_y &= -\frac{1}{\sqrt{9x^2 + y^2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{2\left(\sqrt{9x^2 + y^2}\, y + 9x^2 + y^2\right)}{x\sqrt{9x^2 + y^2}\,\left(y + \sqrt{9x^2 + y^2}\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{2}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -2\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln\left(y + \sqrt{9x^2 + y^2}\right) = -2\ln(x) + c_1$$

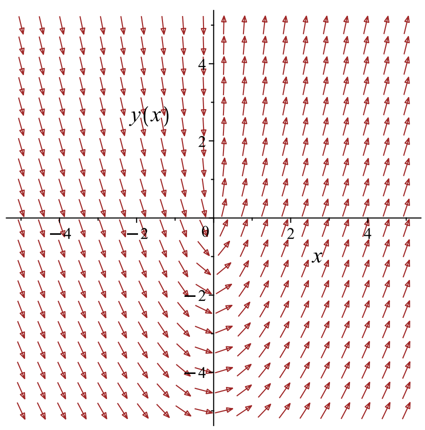
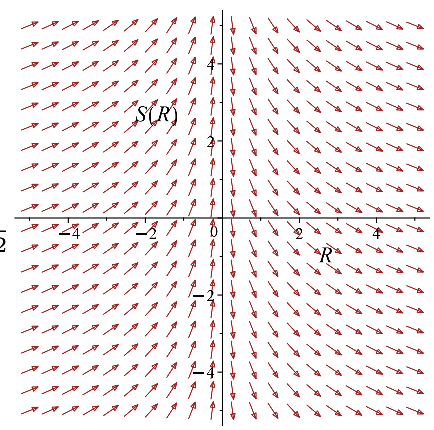
Which simplifies to

$$-\ln\left(y + \sqrt{9x^2 + y^2}\right) = -2\ln(x) + c_1$$

Which gives

$$y = -\frac{\mathrm{e}^{-c_1}\left(9\,\mathrm{e}^{2c_1} - x^2\right)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+\sqrt{9x^2+y^2}}{x}$ 	$R = x$ $S = -\ln\left(y + \sqrt{9x^2 + y^2}\right)$	$\frac{dS}{dR} = -\frac{2}{R}$ 

Summary

The solution(s) found are the following

$$y = -\frac{e^{-c_1}\left(9\, e^{2c_1} - x^2\right)}{2} \tag{1}$$

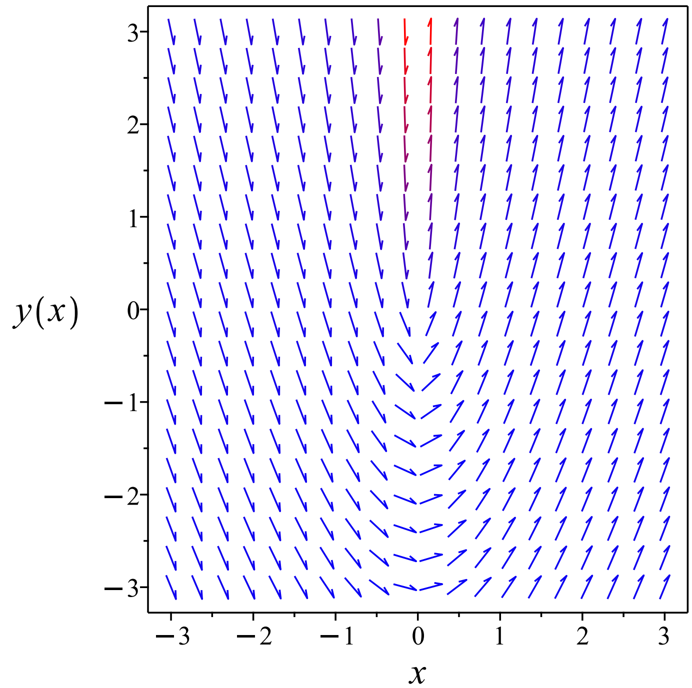


Figure 100: Slope field plot

Verification of solutions

$$y = -\frac{e^{-c_1}(9\,e^{2c_1} - x^2)}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(x*diff(y(x),x)-y(x)=sqrt(9*x^2+y(x)^2),y(x), singsol=all)
```

$$\frac{-c_1x^2+\sqrt{9x^2+y\left(x\right)^2}+y(x)}{x^2}=0$$

✓ Solution by Mathematica

Time used: 0.376 (sec). Leaf size: 27

```
DSolve[x*y'[x]-y[x]==Sqrt[9*x^2+y[x]^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{9e^{c_1}x^2}{2}-\frac{e^{-c_1}}{2}$$

3.6 problem 14

Internal problem ID [2578]
Internal file name [`OUTPUT/2070_Sunday_June_05_2022_02_47_16_AM_42539443/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.8, page 68
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{x(x^2-y^2)-x(y^2+x^2)\,y'=0}$$

3.6.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x(x^2-u(x)^2\,x^2)-x(u(x)^2\,x^2+x^2)\,(u'(x)\,x+u(x))=0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^3+u^2+u-1}{(u^2+1)\,x} \end{aligned}$$

Where $f(x)=-\frac{1}{x}$ and $g(u)=\frac{u^3+u^2+u-1}{u^2+1}$. Integrating both sides gives

$$\frac{1}{\frac{u^3+u^2+u-1}{u^2+1}}\,du=-\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{u^3+u^2+u-1}{u^2+1}}\, du = \int -\frac{1}{x}\, dx$$

$$\int^{u} \frac{\_a^2+1}{\_a^3+\_a^2+\_a-1} d\_a = -\ln(x) + c_2$$

Which results in

$$\int^{u} \frac{\_a^2+1}{\_a^3+\_a^2+\_a-1} d\_a = -\ln(x) + c_2$$

The solution is

$$\int^{u(x)} \frac{\_a^2+1}{\_a^3+\_a^2+\_a-1} d\_a + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\int^{\frac{y}{x}} \frac{\_a^2+1}{\_a^3+\_a^2+\_a-1} d\_a + \ln(x) - c_2 = 0$$

$$\int^{\frac{y}{x}} \frac{\_a^2+1}{\_a^3+\_a^2+\_a-1} d\_a + \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\int^{\frac{y}{x}} \frac{\_a^2+1}{\_a^3+\_a^2+\_a-1} d\_a + \ln(x) - c_2 = 0 \tag{1}$$

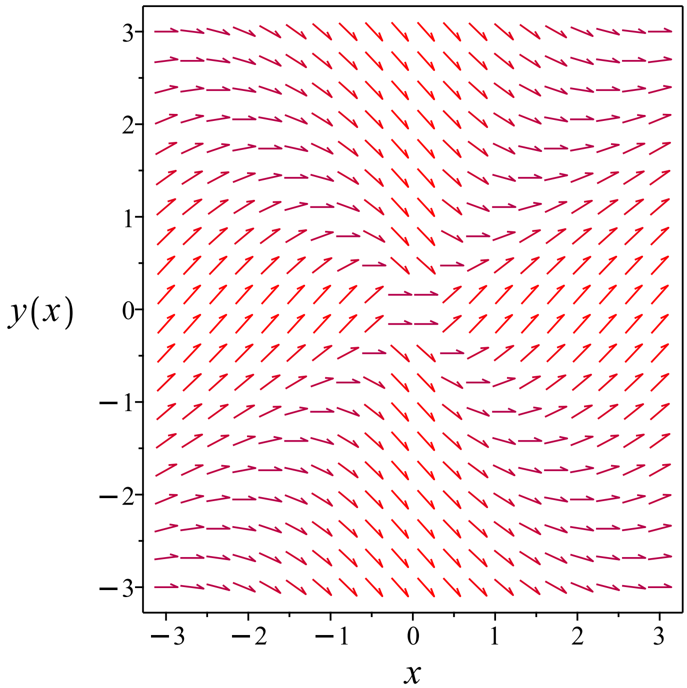


Figure 101: Slope field plot

Verification of solutions

$$\int^{\frac{y}{x}} \frac{\_a^2 + 1}{\_a^3 + \_a^2 + \_a - 1} d\_a + \ln(x) - c_2 = 0$$

Verified OK.

3.6.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{-x^2 + y^2}{x^2 + y^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{(-x^2+y^2)(b_3-a_2)}{x^2+y^2} - \frac{(-x^2+y^2)^2 a_3}{(x^2+y^2)^2} \\ & - \left(\frac{2x}{x^2+y^2} + \frac{2(-x^2+y^2)x}{(x^2+y^2)^2}\right)(xa_2+ya_3+a_1) \\ & - \left(-\frac{2y}{x^2+y^2} + \frac{2(-x^2+y^2)y}{(x^2+y^2)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^4a_2 + x^4a_3 - x^4b_2 - x^4b_3 - 4x^3yb_2 + 4x^2y^2a_2 - 2x^2y^2a_3 - 2x^2y^2b_2 - 4x^2y^2b_3 + 4x\,y^3a_3 - y^4a_2 + y^4a_3}{(x^2+y^2)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} & -x^4a_2 - x^4a_3 + x^4b_2 + x^4b_3 + 4x^3yb_2 - 4x^2y^2a_2 + 2x^2y^2a_3 + 2x^2y^2b_2 \\ & + 4x^2y^2b_3 - 4x\,y^3a_3 + y^4a_2 - y^4a_3 + y^4b_2 - y^4b_3 + 4x^2yb_1 - 4x\,y^2a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & -a_2v_1^4 - 4a_2v_1^2v_2^2 + a_2v_2^4 - a_3v_1^4 + 2a_3v_1^2v_2^2 - 4a_3v_1v_2^3 - a_3v_2^4 + b_2v_1^4 + 4b_2v_1^3v_2 \\ & + 2b_2v_1^2v_2^2 + b_2v_2^4 + b_3v_1^4 + 4b_3v_1^2v_2^2 - b_3v_2^4 - 4a_1v_1v_2^2 + 4b_1v_1^2v_2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(-a_2 - a_3 + b_2 + b_3) v_1^4 + 4b_2 v_1^3 v_2 + (-4a_2 + 2a_3 + 2b_2 + 4b_3) v_1^2 v_2^2 + 4b_1 v_1^2 v_2 - 4a_3 v_1 v_2^3 - 4a_1 v_1 v_2^2 + (a_2 - a_3 + b_2 - b_3) v_2^4 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-4a_1 &= 0 \\
-4a_3 &= 0 \\
4b_1 &= 0 \\
4b_2 &= 0 \\
-4a_2 + 2a_3 + 2b_2 + 4b_3 &= 0 \\
-a_2 - a_3 + b_2 + b_3 &= 0 \\
a_2 - a_3 + b_2 - b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned} \frac{dy}{dx} &= \frac{\eta}{\xi} \\ &= \frac{y}{x} \\ &= \frac{y}{x} \end{aligned}$$

This is easily solved to give

$$y = c_1 x$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{y}{x}$$

And S is found from

$$\begin{aligned} dS &= \frac{dx}{\xi} \\ &= \frac{dx}{x} \end{aligned}$$

Integrating gives

$$\begin{aligned} S &= \int \frac{dx}{T} \\ &= \ln(x) \end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-x^2 + y^2}{x^2 + y^2}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= -\frac{y}{x^2} \\
R_y &= \frac{1}{x} \\
S_x &= \frac{1}{x} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{x(x^2+y^2)}{x^3 - x^2 y - x\, y^2 - y^3} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{-R^2 - 1}{R^3 + R^2 + R - 1}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \int -\frac{R^2+1}{R^3 + R^2 + R - 1} dR + c_1 \tag{4}
$$

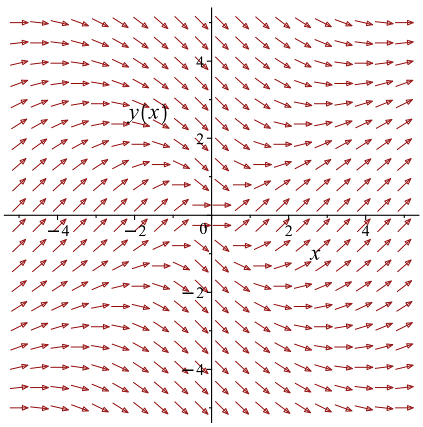
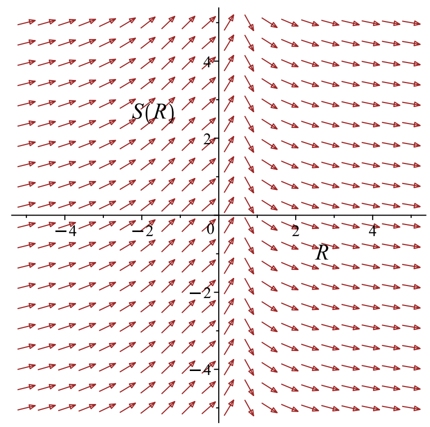
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\ln(x) = \int^{\frac{y}{x}} -\frac{\_a^2+1}{\_a^3 + \_a^2 + \_a - 1} d\_a + c_1
$$

Which simplifies to

$$
\ln(x) = \int^{\frac{y}{x}} -\frac{\_a^2+1}{\_a^3 + \_a^2 + \_a - 1} d\_a + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-x^2+y^2}{x^2+y^2}$ 	$R = \frac{y}{x}$ $S = \ln(x)$	$\frac{dS}{dR} = \frac{-R^2-1}{R^3+R^2+R-1}$ 

Summary

The solution(s) found are the following

$$\ln(x) = \int^{\frac{y}{x}} -\frac{\_a^2+1}{\_a^3+\_a^2+\_a-1} d\_a + c_1 \tag{1}$$

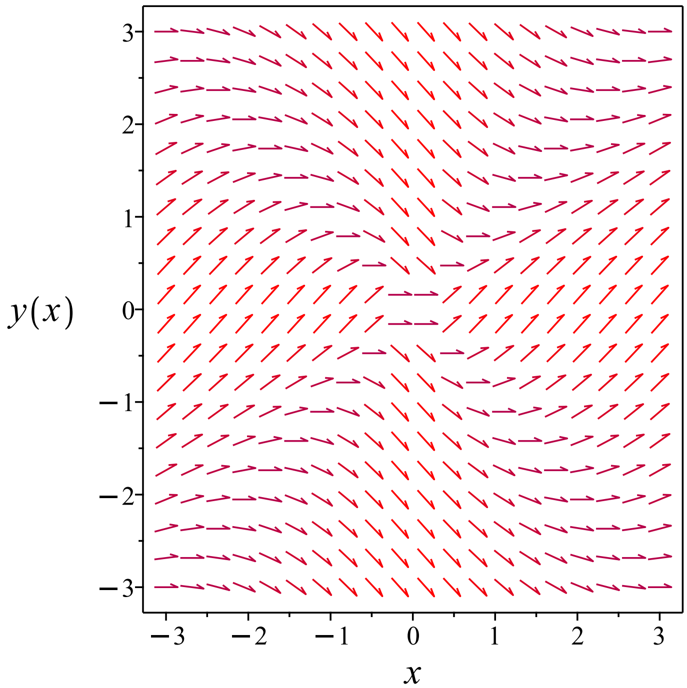


Figure 102: Slope field plot

Verification of solutions

$$\ln(x) = \int^{\frac{y}{x}} -\frac{\_a^2+1}{\_a^3+\_a^2+\_a-1} d\_a + c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve(x*(x^2-y(x)^2)-x*(x^2+y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{RootOf}\left(\int^{\_Z} \frac{\_a^2+1}{\_a^3+\_a^2+\_a-1}d\_a+\ln(x)+c_1\right)x$$

✓ Solution by Mathematica

Time used: 0.133 (sec). Leaf size: 71

```
DSolve[x*(x^2-y[x]^2)-x*(x^2+y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[\mathrm{RootSum}\left[\#1^3+\#1^2+\#1-1\&,\frac{\#1^2\log\left(\frac{y(x)}{x}-\#1\right)+\log\left(\frac{y(x)}{x}-\#1\right)}{3\#1^2+2\#1+1}\&\right]=-\log(x)+c_1,y(x)\right]$$

3.7 problem 15

Internal problem ID [2579]
Internal file name [`OUTPUT/2071_Sunday_June_05_2022_02_47_18_AM_35225455/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.8, page 68
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{xy' + y\ln(x) - y\ln(y) = 0}$$

3.7.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y(\ln(x) - \ln(y))}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{y(\ln(x) - \ln(y))(b_3 - a_2)}{x} - \frac{y^2(\ln(x) - \ln(y))^2 a_3}{x^2} - \left(-\frac{y}{x^2} + \frac{y(\ln(x) - \ln(y))}{x^2}\right)(xa_2 + ya_3 + a_1) - \left(-\frac{\ln(x) - \ln(y)}{x} + \frac{1}{x}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{\ln(x)^2 y^2 a_3 - 2\ln(x)\ln(y) y^2 a_3 + \ln(y)^2 y^2 a_3 - \ln(x) x^2 b_2 + \ln(x) y^2 a_3 + \ln(y) x^2 b_2 - \ln(y) y^2 a_3 - \ln}{x^2} = 0$$

Setting the numerator to zero gives

$$-\ln(x)^2 y^2 a_3 + 2\ln(x)\ln(y) y^2 a_3 - \ln(y)^2 y^2 a_3 + \ln(x) x^2 b_2 - \ln(x) y^2 a_3 - \ln(y) x^2 b_2 + \ln(y) y^2 a_3 + \ln(x) x b_1 - \ln(x) y a_1 - \ln(y) x b_1 + \ln(y) y a_1 + x y a_2 - x y b_3 + y^2 a_3 - x b_1 + y a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \ln(x), \ln(y)\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \ln(x) = v_3, \ln(y) = v_4\}$$

The above PDE (6E) now becomes

$$-v_3^2 v_2^2 a_3 + 2v_3 v_4 v_2^2 a_3 - v_4^2 v_2^2 a_3 - v_3 v_2^2 a_3 + v_4 v_2^2 a_3 + v_3 v_1^2 b_2 - v_4 v_1^2 b_2 - v_3 v_2 a_1 + v_4 v_2 a_1 + v_1 v_2 a_2 + v_2^2 a_3 + v_3 v_1 b_1 - v_4 v_1 b_1 - v_1 v_2 b_3 + v_2 a_1 - v_1 b_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$\begin{aligned}
&v_3 v_1^2 b_2 - v_4 v_1^2 b_2 + (-b_3 + a_2) v_1 v_2 + v_3 v_1 b_1 - v_4 v_1 b_1 - v_1 b_1 - v_3^2 v_2^2 a_3 \\
&+ 2 v_3 v_4 v_2^2 a_3 - v_3 v_2^2 a_3 - v_4^2 v_2^2 a_3 + v_4 v_2^2 a_3 + v_2^2 a_3 - v_3 v_2 a_1 + v_4 v_2 a_1 + v_2 a_1 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
a_1 &= 0 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
-a_1 &= 0 \\
-a_3 &= 0 \\
2a_3 &= 0 \\
-b_1 &= 0 \\
-b_2 &= 0 \\
-b_3 + a_2 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{y(\ln(x) - \ln(y))}{x}\right)(x) \\
&= \ln(x)\,y - \ln(y)\,y + y \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\ln(x)\,y - \ln(y)\,y + y} dy
\end{aligned}$$

Which results in

$$S = -\ln(\ln(x) - \ln(y) + 1)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(\ln(x) - \ln(y))}{x}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{1}{x(\ln(x) - \ln(y) + 1)} \\
S_y &= \frac{1}{y(\ln(x) - \ln(y) + 1)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln(\ln(x) - \ln(y) + 1) = -\ln(x) + c_1$$

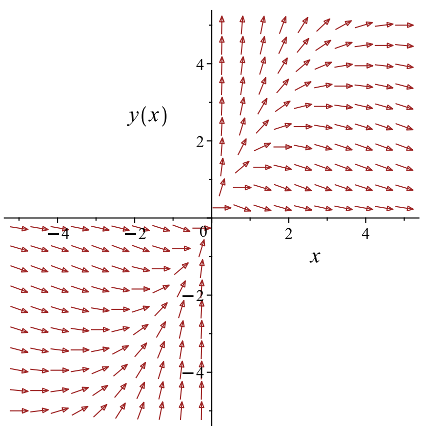
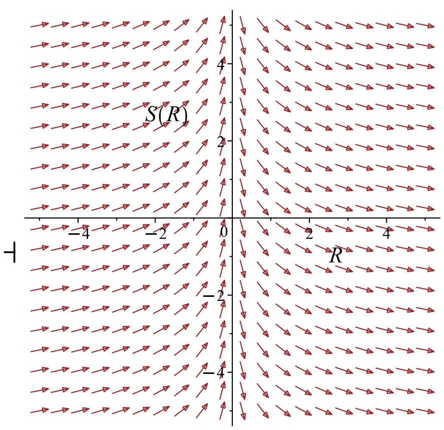
Which simplifies to

$$-\ln(\ln(x) - \ln(y) + 1) = -\ln(x) + c_1$$

Which gives

$$y = \mathrm{e}^{(\mathrm{e}^{c_1}\ln(x) + \mathrm{e}^{c_1} - x)\mathrm{e}^{-c_1}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(\ln(x)-\ln(y))}{x}$ 	$R = x$ $S = -\ln(\ln(x) - \ln(y) +$	$\frac{dS}{dR} = -\frac{1}{R}$ 

<u>Summary</u>

The solution(s) found are the following

$$y = e^{(e^{c_1} \ln(x) + e^{c_1} - x) e^{-c_1}} \tag{1}$$

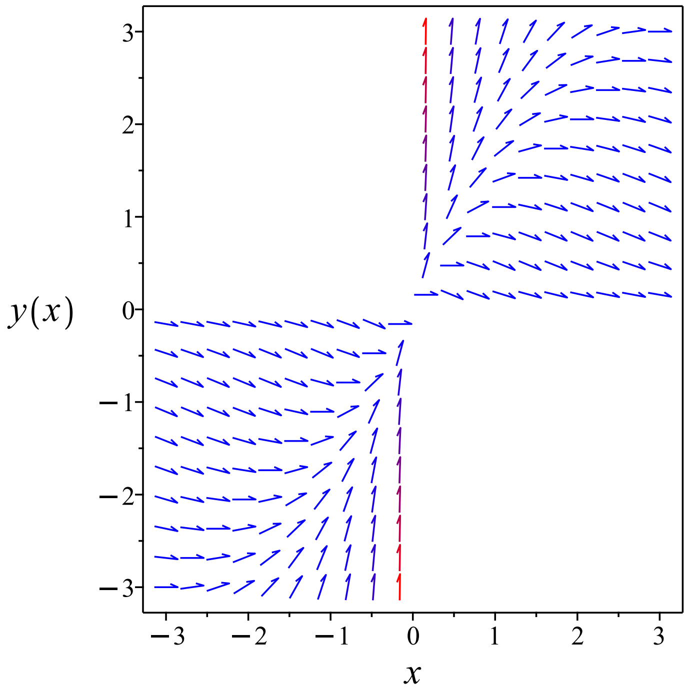


Figure 103: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{(\mathrm{e}^{c_1} \ln(x) + \mathrm{e}^{c_1} - x)\mathrm{e}^{-c_1}}$$

Verified OK.

3.7.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(x)\,\mathrm{d}y = (-\ln(x)\,y + \ln(y)\,y)\,\mathrm{d}x$$
$$(\ln(x)\,y - \ln(y)\,y)\,\mathrm{d}x + (x)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = \ln(x)\,y - \ln(y)\,y$$
$$N(x, y) = x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\ln(x)\,y - \ln(y)\,y) \\ &= -1 + \ln(x) - \ln(y)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{x^2 y}$ is an integrating factor. Therefore by multiplying $M = y \ln(x) - y \ln(y)$ and $N = x$ by this integrating factor the ode becomes exact. The new M, N are

$$M = \frac{y \ln(x) - y \ln(y)}{x^2 y}$$

$$N = \frac{1}{xy}$$

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$

$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{xy}\right) dy = \left(-\frac{\ln(x)\, y - \ln(y)\, y}{x^2 y}\right) dx$$

$$\left(\frac{\ln(x)\, y - \ln(y)\, y}{x^2 y}\right) dx + \left(\frac{1}{xy}\right) dy = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= \frac{\ln(x)\, y - \ln(y)\, y}{x^2 y} \\ N(x,y) &= \frac{1}{xy} \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{\ln(x)\, y - \ln(y)\, y}{x^2 y}\right) \\ &= -\frac{1}{x^2 y} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{xy}\right) \\ &= -\frac{1}{x^2 y} \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial x}\, dx &= \int M\, dx \\ \int \frac{\partial \phi}{\partial x}\, dx &= \int \frac{\ln(x)\, y - \ln(y)\, y}{x^2 y}\, dx \end{aligned}$$

$$\phi = \frac{-\ln(x) + \ln(y) - 1}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{xy} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{xy}$. Therefore equation (4) becomes

$$\frac{1}{xy} = \frac{1}{xy} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-\ln(x) + \ln(y) - 1}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-\ln(x) + \ln(y) - 1}{x}$$

The solution becomes

$$y = e^{c_1 x + 1} x$$

Summary

The solution(s) found are the following

$$y = e^{c_1 x + 1} x \tag{1}$$

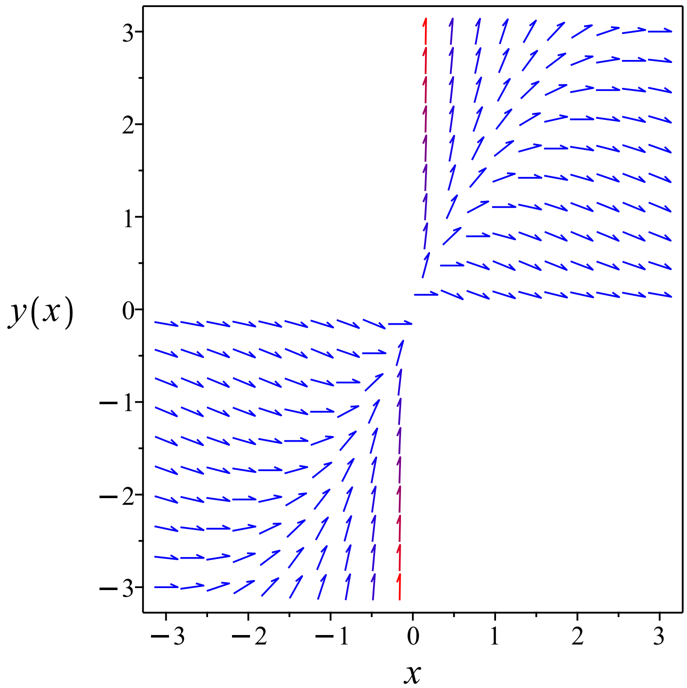


Figure 104: Slope field plot

Verification of solutions

$$y = e^{c_1 x + 1} x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 12

```
dsolve(x*diff(y(x),x)+y(x)*ln(x)=y(x)*ln(y(x)),y(x), singsol=all)
```

$$y(x) = x\,\mathrm{e}^{c_1 x+1}$$

✓ Solution by Mathematica

Time used: 0.264 (sec). Leaf size: 24

```
DSolve[x*y'[x]+y[x]*Log[x]==y[x]*Log[y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to xe^{1+e^{c_1}x}$$
$$y(x) \to ex$$

3.8 problem 16

Internal problem ID [2580]
Internal file name [`OUTPUT/2072_Sunday_June_05_2022_02_47_23_AM_6298508/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.8, page 68
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{y' - \frac{y^2 + 2yx - 2x^2}{x^2 - yx + y^2} = 0}$$

3.8.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{u(x)^2\,x^2 + 2u(x)\,x^2 - 2x^2}{x^2 - u\,(x)\,x^2 + u\,(x)^2\,x^2} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^3 - 2u^2 - u + 2}{x\,(u^2 - u + 1)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^3-2u^2-u+2}{u^2-u+1}$. Integrating both sides gives

$$\frac{1}{\frac{u^3-2u^2-u+2}{u^2-u+1}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{u^3-2u^2-u+2}{u^2-u+1}}\,du = \int -\frac{1}{x}\,dx$$

$$\frac{\ln(u+1)}{2} - \frac{\ln(u-1)}{2} + \ln(u-2) = -\ln(x) + c_2$$

Raising both side to exponential gives

$$e^{\frac{\ln(u+1)}{2} - \frac{\ln(u-1)}{2} + \ln(u-2)} = e^{-\ln(x)+c_2}$$

Which simplifies to

$$\frac{\sqrt{u+1}\,(u-2)}{\sqrt{u-1}} = \frac{c_3}{x}$$

The solution is

$$\frac{\sqrt{u(x)+1}\,(u(x)-2)}{\sqrt{u(x)-1}} = \frac{c_3}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{\sqrt{\frac{y}{x}+1}\,\left(\frac{y}{x}-2\right)}{\sqrt{\frac{y}{x}-1}} = \frac{c_3}{x}$$

$$\frac{\sqrt{\frac{y+x}{x}}\,(-2x+y)}{\sqrt{\frac{y-x}{x}}\,x} = \frac{c_3}{x}$$

Which simplifies to

$$\frac{(-2x+y)\sqrt{\frac{y+x}{x}}}{\sqrt{\frac{y-x}{x}}} = c_3$$

Summary

The solution(s) found are the following

$$\frac{(-2x+y)\sqrt{\frac{y+x}{x}}}{\sqrt{\frac{y-x}{x}}} = c_3 \tag{1}$$

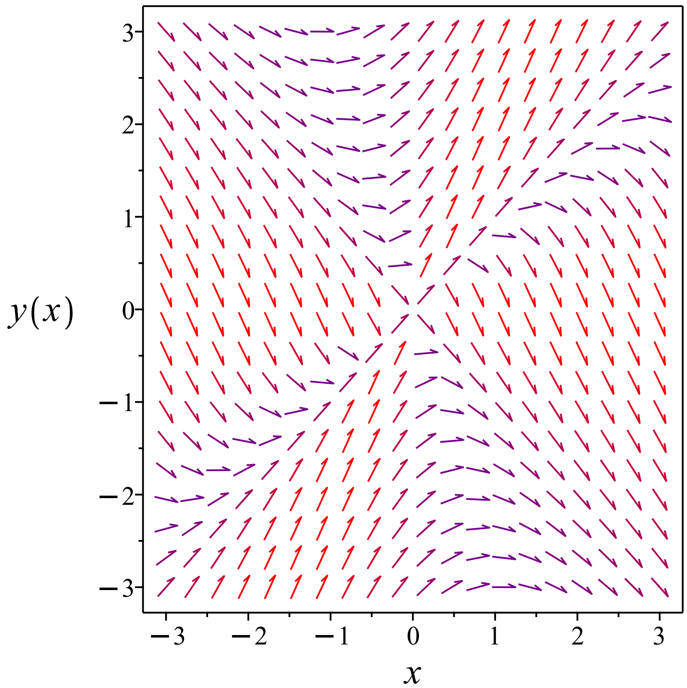


Figure 105: Slope field plot

Verification of solutions

$$\frac{(-2x+y)\sqrt{\frac{y+x}{x}}}{\sqrt{\frac{y-x}{x}}} = c_3$$

Verified OK.

3.8.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{-2x^2+2xy+y^2}{x^2-xy+y^2}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{(-2x^2 + 2xy + y^2)(b_3 - a_2)}{x^2 - xy + y^2} - \frac{(-2x^2 + 2xy + y^2)^2 a_3}{(x^2 - xy + y^2)^2} \\ &- \left(\frac{-4x + 2y}{x^2 - xy + y^2} - \frac{(-2x^2 + 2xy + y^2)(2x - y)}{(x^2 - xy + y^2)^2}\right)(x a_2 + y a_3 + a_1) \\ &- \left(\frac{2x + 2y}{x^2 - xy + y^2} - \frac{(-2x^2 + 2xy + y^2)(-x + 2y)}{(x^2 - xy + y^2)^2}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^4a_2 - 4x^4a_3 + x^4b_2 - 2x^4b_3 - 4x^3ya_2 + 8x^3ya_3 - 8x^3yb_2 + 4x^3yb_3 + 9x^2y^2a_2 + 6x^2y^2b_2 - 9x^2y^2b_3 - 4x}{(x^2 - xy} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} &2x^4a_2 - 4x^4a_3 + x^4b_2 - 2x^4b_3 - 4x^3ya_2 + 8x^3ya_3 - 8x^3yb_2 + 4x^3yb_3 \\ &+ 9x^2y^2a_2 + 6x^2y^2b_2 - 9x^2y^2b_3 - 4x\,y^3a_2 + 2x\,y^3a_3 - 2x\,y^3b_2 + 4x\,y^3b_3 \\ &- y^4a_2 - 4y^4a_3 + y^4b_2 + y^4b_3 - 6x^2yb_1 + 6x\,y^2a_1 + 3x\,y^2b_1 - 3y^3a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$2a_2v_1^4 - 4a_2v_1^3v_2 + 9a_2v_1^2v_2^2 - 4a_2v_1v_2^3 - a_2v_2^4 - 4a_3v_1^4 + 8a_3v_1^3v_2 + 2a_3v_1v_2^3 - 4a_3v_2^4 + b_2v_1^4 - 8b_2v_1^3v_2 + 6b_2v_1^2v_2^2 - 2b_2v_1v_2^3 + b_2v_2^4 - 2b_3v_1^4 + 4b_3v_1^3v_2 - 9b_3v_1^2v_2^2 + 4b_3v_1v_2^3 + b_3v_2^4 + 6a_1v_1v_2^2 - 3a_1v_2^3 - 6b_1v_1^2v_2 + 3b_1v_1v_2^2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(2a_2 - 4a_3 + b_2 - 2b_3)\,v_1^4 + (-4a_2 + 8a_3 - 8b_2 + 4b_3)\,v_1^3v_2 + (9a_2 + 6b_2 - 9b_3)\,v_1^2v_2^2 - 6b_1v_1^2v_2 + (-4a_2 + 2a_3 - 2b_2 + 4b_3)\,v_1v_2^3 + (6a_1 + 3b_1)\,v_1v_2^2 + (-a_2 - 4a_3 + b_2 + b_3)\,v_2^4 - 3a_1v_2^3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-3a_1 &= 0\\
-6b_1 &= 0\\
6a_1 + 3b_1 &= 0\\
9a_2 + 6b_2 - 9b_3 &= 0\\
-4a_2 + 2a_3 - 2b_2 + 4b_3 &= 0\\
-4a_2 + 8a_3 - 8b_2 + 4b_3 &= 0\\
-a_2 - 4a_3 + b_2 + b_3 &= 0\\
2a_2 - 4a_3 + b_2 - 2b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0\\
a_2 &= b_3\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\,\xi \\ &= y - \left(\frac{-2x^2 + 2xy + y^2}{x^2 - xy + y^2}\right)(x) \\ &= \frac{2x^3 - x^2y - 2x\,y^2 + y^3}{x^2 - xy + y^2} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{2x^3 - x^2y - 2x\,y^2 + y^3}{x^2 - xy + y^2}} dy\end{aligned}$$

Which results in

$$S = \frac{\ln(y + x)}{2} - \frac{\ln(y - x)}{2} + \ln(-2x + y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-2x^2 + 2xy + y^2}{x^2 - xy + y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{1}{2x+2y} - \frac{1}{2x-2y} - \frac{2}{-2x+y} \\
S_y &= \frac{1}{2x+2y} + \frac{1}{2x-2y} + \frac{1}{-2x+y}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

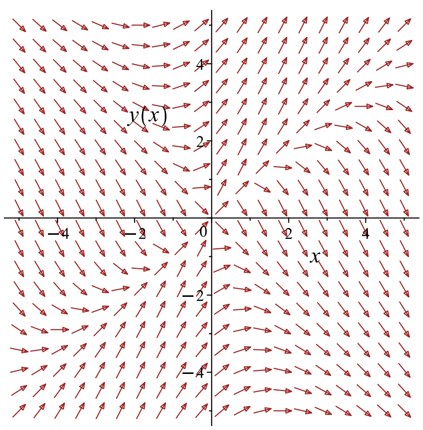
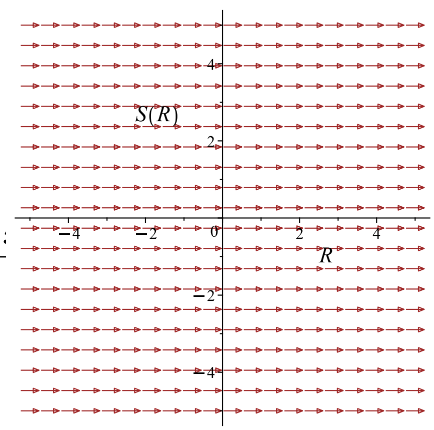
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y+x)}{2} - \frac{\ln(y-x)}{2} + \ln(-2x+y) = c_1$$

Which simplifies to

$$\frac{\ln(y+x)}{2} - \frac{\ln(y-x)}{2} + \ln(-2x+y) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = \frac{-2x^2+2xy+y^2}{x^2-xy+y^2}$
</td><td>$R = x$
$S = \frac{\ln(y+x)}{2} - \frac{\ln(y-x)}{2}$</td><td>$\frac{dS}{dR} = 0$
</td></tr>
</table>

Summary

The solution(s) found are the following

$$\frac{\ln(y+x)}{2} - \frac{\ln(y-x)}{2} + \ln(-2x+y) = c_1 \tag{1}$$

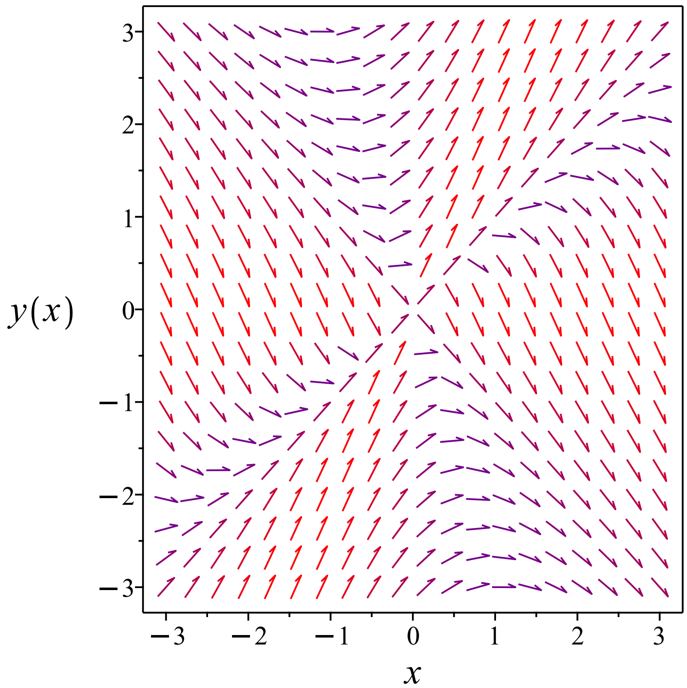


Figure 106: Slope field plot

Verification of solutions

$$\frac{\ln(y+x)}{2} - \frac{\ln(y-x)}{2} + \ln(-2x+y) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.86 (sec). Leaf size: 80

```
dsolve(diff(y(x),x)= (y(x)^2+2*x*y(x)-2*x^2)/(x^2-x*y(x)+y(x)^2),y(x), singsol=all)
```

$$y(x) = \frac{x\left(-\operatorname{RootOf}\left(2\_Z^6 + (9c_1x^2 - 1)\_Z^4 - 6x^2c_1\_Z^2 + c_1x^2\right)^2 + 1\right)}{\operatorname{RootOf}\left(2\_Z^6 + (9c_1x^2 - 1)\_Z^4 - 6x^2c_1\_Z^2 + c_1x^2\right)^2}$$

✓ Solution by Mathematica

Time used: 60.179 (sec). Leaf size: 373

```
DSolve[y'[x]== (y[x]^2+2*x*y[x]-2*x^2)/(x^2-x*y[x]+y[x]^2),y[x],x,IncludeSingularSolutions -
```

$$y(x) \to \frac{\sqrt[3]{-54x^3 + 2\sqrt{729x^6 + (-9x^2 + 3e^{2c_1})^3}}}{3\sqrt[3]{2}} - \frac{\sqrt[3]{2}(-3x^2 + e^{2c_1})}{\sqrt[3]{-54x^3 + 2\sqrt{729x^6 + (-9x^2 + 3e^{2c_1})^3}}} + x$$

$$y(x) \to \frac{\left(-1 + i\sqrt{3}\right)\sqrt[3]{-54x^3 + 2\sqrt{729x^6 + (-9x^2 + 3e^{2c_1})^3}}}{6\sqrt[3]{2}} + \frac{\left(1 + i\sqrt{3}\right)(-3x^2 + e^{2c_1})}{2^{2/3}\sqrt[3]{-54x^3 + 2\sqrt{729x^6 + (-9x^2 + 3e^{2c_1})^3}}} + x$$

$$y(x) \to -\frac{\left(1 + i\sqrt{3}\right)\sqrt[3]{-54x^3 + 2\sqrt{729x^6 + (-9x^2 + 3e^{2c_1})^3}}}{6\sqrt[3]{2}} + \frac{\left(1 - i\sqrt{3}\right)(-3x^2 + e^{2c_1})}{2^{2/3}\sqrt[3]{-54x^3 + 2\sqrt{729x^6 + (-9x^2 + 3e^{2c_1})^3}}} + x$$

3.9 problem 17

Internal problem ID [2581]
Internal file name [`OUTPUT/2073_Sunday_June_05_2022_02_47_27_AM_56589216/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.8, page 68
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`]]
```

$$\boxed{2xyy' - x^2 e^{-\frac{y^2}{x^2}} - 2y^2 = 0}$$

3.9.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$2x^2 u(x)\,(u'(x)\,x + u(x)) - x^2 e^{-u(x)^2} - 2u(x)^2\,x^2 = 0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{e^{-u^2}}{2ux}
\end{aligned}$$

Where $f(x) = \frac{1}{2x}$ and $g(u) = \frac{e^{-u^2}}{u}$. Integrating both sides gives

$$\frac{1}{\frac{e^{-u^2}}{u}}\,du = \frac{1}{2x}\,dx$$

$$\int \frac{1}{\frac{e^{-u^2}}{u}}\,du = \int \frac{1}{2x}\,dx$$

$$\frac{e^{u^2}}{2} = \frac{\ln(x)}{2} + c_2$$

The solution is

$$\frac{e^{u(x)^2}}{2} - \frac{\ln(x)}{2} - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{e^{\frac{y^2}{x^2}}}{2} - \frac{\ln(x)}{2} - c_2 = 0$$

$$\frac{e^{\frac{y^2}{x^2}}}{2} - \frac{\ln(x)}{2} - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{e^{\frac{y^2}{x^2}}}{2} - \frac{\ln(x)}{2} - c_2 = 0 \tag{1}$$

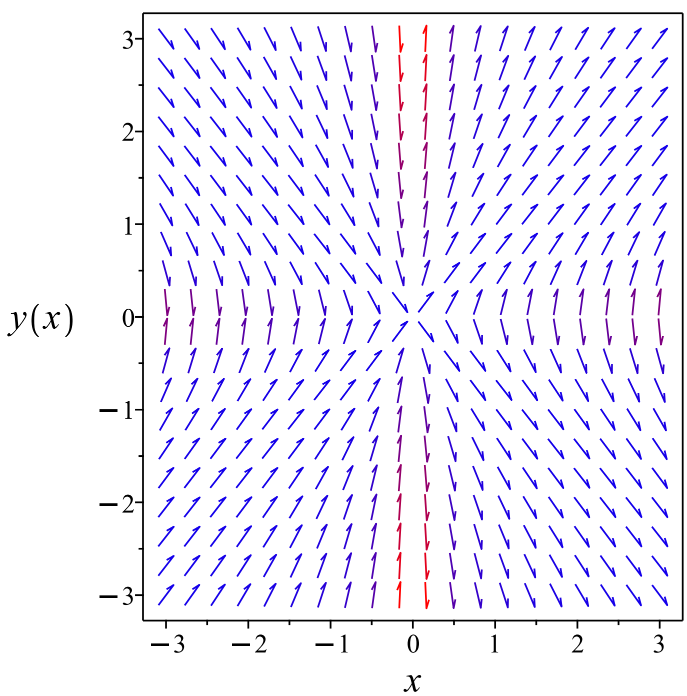


Figure 107: Slope field plot

Verification of solutions

$$\frac{\mathrm{e}^{\frac{y^2}{x^2}}}{2} - \frac{\ln(x)}{2} - c_2 = 0$$

Verified OK.

3.9.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{x^2 \mathrm{e}^{-\frac{y^2}{x^2}} + 2y^2}{2xy}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 &+ \frac{\left(x^2 \mathrm{e}^{-\frac{y^2}{x^2}} + 2y^2\right)(b_3 - a_2)}{2xy} - \frac{\left(x^2 \mathrm{e}^{-\frac{y^2}{x^2}} + 2y^2\right)^2 a_3}{4x^2 y^2} \\ &- \left(\frac{2x\,\mathrm{e}^{-\frac{y^2}{x^2}} + \frac{2y^2 \mathrm{e}^{-\frac{y^2}{x^2}}}{x}}{2xy} - \frac{x^2 \mathrm{e}^{-\frac{y^2}{x^2}} + 2y^2}{2x^2 y}\right)(x a_2 + y a_3 + a_1) \\ &- \left(\frac{-2y\,\mathrm{e}^{-\frac{y^2}{x^2}} + 4y}{2xy} - \frac{x^2 \mathrm{e}^{-\frac{y^2}{x^2}} + 2y^2}{2x\,y^2}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$
-\frac{\mathrm{e}^{-\frac{2y^2}{x^2}} x^4 a_3 - 2\,\mathrm{e}^{-\frac{y^2}{x^2}} x^4 b_2 + 4\,\mathrm{e}^{-\frac{y^2}{x^2}} x^3 y a_2 - 4\,\mathrm{e}^{-\frac{y^2}{x^2}} x^3 y b_3 + 6\,\mathrm{e}^{-\frac{y^2}{x^2}} x^2 y^2 a_3 - 4\,\mathrm{e}^{-\frac{y^2}{x^2}} x^2 y^2 b_2 + 4\,\mathrm{e}^{-\frac{y^2}{x^2}} x\, y^3 a_2 - 4}{4y^2 x}
$$
$$= 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-\mathrm{e}^{-\frac{2y^2}{x^2}} x^4 a_3 + 2\,\mathrm{e}^{-\frac{y^2}{x^2}} x^4 b_2 - 4\,\mathrm{e}^{-\frac{y^2}{x^2}} x^3 y a_2 + 4\,\mathrm{e}^{-\frac{y^2}{x^2}} x^3 y b_3 - 6\,\mathrm{e}^{-\frac{y^2}{x^2}} x^2 y^2 a_3 \\
&+ 4\,\mathrm{e}^{-\frac{y^2}{x^2}} x^2 y^2 b_2 - 4\,\mathrm{e}^{-\frac{y^2}{x^2}} x\, y^3 a_2 + 4\,\mathrm{e}^{-\frac{y^2}{x^2}} x\, y^3 b_3 - 4\,\mathrm{e}^{-\frac{y^2}{x^2}} y^4 a_3 + 2\,\mathrm{e}^{-\frac{y^2}{x^2}} x^3 b_1 \\
&- 2\,\mathrm{e}^{-\frac{y^2}{x^2}} x^2 y a_1 + 4\,\mathrm{e}^{-\frac{y^2}{x^2}} x\, y^2 b_1 - 4\,\mathrm{e}^{-\frac{y^2}{x^2}} y^3 a_1 - 4x\, y^2 b_1 + 4y^3 a_1 = 0
\end{aligned}
\tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
&-\mathrm{e}^{-\frac{2y^2}{x^2}} x^4 a_3 + 2\,\mathrm{e}^{-\frac{y^2}{x^2}} x^4 b_2 - 4\,\mathrm{e}^{-\frac{y^2}{x^2}} x^3 y a_2 + 4\,\mathrm{e}^{-\frac{y^2}{x^2}} x^3 y b_3 - 6\,\mathrm{e}^{-\frac{y^2}{x^2}} x^2 y^2 a_3 \\
&+ 4\,\mathrm{e}^{-\frac{y^2}{x^2}} x^2 y^2 b_2 - 4\,\mathrm{e}^{-\frac{y^2}{x^2}} x\, y^3 a_2 + 4\,\mathrm{e}^{-\frac{y^2}{x^2}} x\, y^3 b_3 - 4\,\mathrm{e}^{-\frac{y^2}{x^2}} y^4 a_3 + 2\,\mathrm{e}^{-\frac{y^2}{x^2}} x^3 b_1 \\
&- 2\,\mathrm{e}^{-\frac{y^2}{x^2}} x^2 y a_1 + 4\,\mathrm{e}^{-\frac{y^2}{x^2}} x\, y^2 b_1 - 4\,\mathrm{e}^{-\frac{y^2}{x^2}} y^3 a_1 - 4x\, y^2 b_1 + 4y^3 a_1 = 0
\end{aligned}
\tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \mathrm{e}^{-\frac{2y^2}{x^2}}, \mathrm{e}^{-\frac{y^2}{x^2}}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x = v_1, y = v_2, \mathrm{e}^{-\frac{2y^2}{x^2}} = v_3, \mathrm{e}^{-\frac{y^2}{x^2}} = v_4\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-4v_4 v_1^3 v_2 a_2 - 4v_4 v_1 v_2^3 a_2 - v_3 v_1^4 a_3 - 6v_4 v_1^2 v_2^2 a_3 - 4v_4 v_2^4 a_3 \\
&+ 2v_4 v_1^4 b_2 + 4v_4 v_1^2 v_2^2 b_2 + 4v_4 v_1^3 v_2 b_3 + 4v_4 v_1 v_2^3 b_3 - 2v_4 v_1^2 v_2 a_1 \\
&- 4v_4 v_2^3 a_1 + 2v_4 v_1^3 b_1 + 4v_4 v_1 v_2^2 b_1 + 4v_2^3 a_1 - 4v_1 v_2^2 b_1 = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3, v_4\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&-v_3 v_1^4 a_3 + 2v_4 v_1^4 b_2 + (-4a_2 + 4b_3) v_1^3 v_2 v_4 + 2v_4 v_1^3 b_1 \\
&+ (-6a_3 + 4b_2) v_1^2 v_2^2 v_4 - 2v_4 v_1^2 v_2 a_1 + (-4a_2 + 4b_3) v_1 v_2^3 v_4 \\
&+ 4v_4 v_1 v_2^2 b_1 - 4v_1 v_2^2 b_1 - 4v_4 v_2^4 a_3 - 4v_4 v_2^3 a_1 + 4v_2^3 a_1 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-4a_1 &= 0 \\
-2a_1 &= 0 \\
4a_1 &= 0 \\
-4a_3 &= 0 \\
-a_3 &= 0 \\
-4b_1 &= 0 \\
2b_1 &= 0 \\
4b_1 &= 0 \\
2b_2 &= 0 \\
-4a_2 + 4b_3 &= 0 \\
-6a_3 + 4b_2 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{x^2 e^{-\frac{y^2}{x^2}} + 2y^2}{2xy}\right)(x) \\
&= -\frac{x^2 e^{-\frac{y^2}{x^2}}}{2y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\frac{x^2 e^{-\frac{y^2}{x^2}}}{2y}} dy
\end{aligned}
$$

Which results in

$$
S = -e^{\frac{y^2}{x^2}}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x^2 e^{-\frac{y^2}{x^2}} + 2y^2}{2xy}$$

Evaluating all the partial derivatives gives

$$R_x = 1$$
$$R_y = 0$$
$$S_x = \frac{2y^2 e^{\frac{y^2}{x^2}}}{x^3}$$
$$S_y = -\frac{2 e^{\frac{y^2}{x^2}} y}{x^2}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

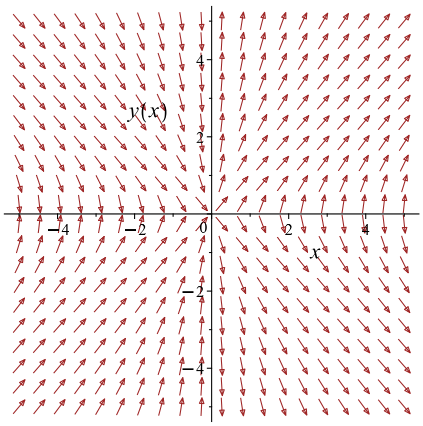
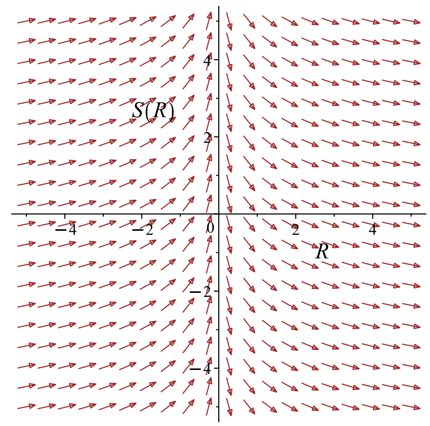
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-e^{\frac{y^2}{x^2}} = -\ln(x) + c_1$$

Which simplifies to

$$-e^{\frac{y^2}{x^2}} = -\ln(x) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2 e^{-\frac{y^2}{x^2}} + 2y^2}{2xy}$ 	$R = x$ $S = -e^{\frac{y^2}{x^2}}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$-e^{\frac{y^2}{x^2}} = -\ln(x) + c_1 \tag{1}$$

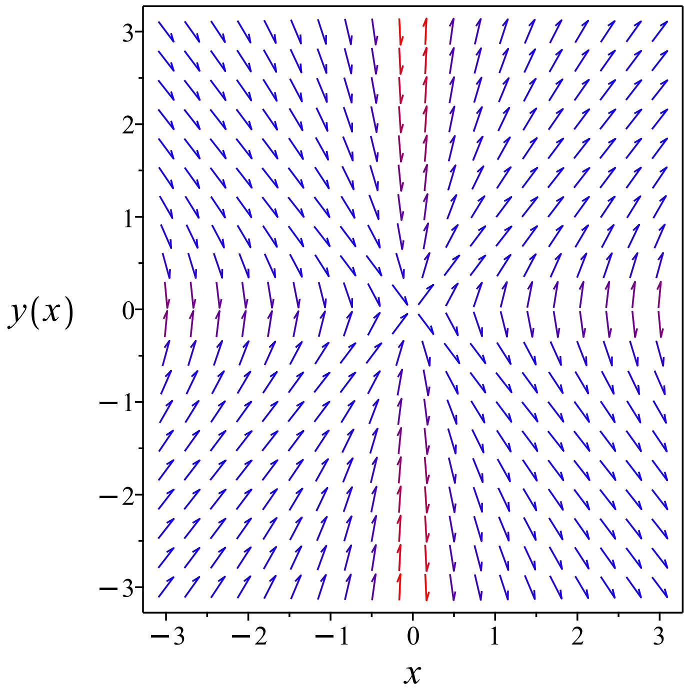


Figure 108: Slope field plot

Verification of solutions

$$-e^{\frac{y^2}{x^2}} = -\ln(x) + c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 26

```
dsolve(2*x*y(x)*diff(y(x),x)-(x^2*exp(-y(x)^2/x^2)+2*y(x)^2)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{\ln(\ln(x) + c_1)}\, x$$
$$y(x) = -\sqrt{\ln(\ln(x) + c_1)}\, x$$

✓ Solution by Mathematica

Time used: 2.155 (sec). Leaf size: 38

```
DSolve[2*x*y[x]*y'[x]-(x^2*Exp[-y[x]^2/x^2]+2*y[x]^2)==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to -x\sqrt{\log(\log(x) + 2c_1)}$$
$$y(x) \to x\sqrt{\log(\log(x) + 2c_1)}$$

3.10 problem 18

Internal problem ID [2582]
Internal file name [`OUTPUT/2074_Sunday_June_05_2022_02_47_31_AM_12116909/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.8, page 68
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Riccati]
```

$$\boxed{y'x^2 - y^2 - 3yx = x^2}$$

3.10.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\,x^2 - u(x)^2\,x^2 - 3u(x)\,x^2 = x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u^2 + 2u + 1}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u^2 + 2u + 1$. Integrating both sides gives

$$\frac{1}{u^2 + 2u + 1}\,du = \frac{1}{x}\,dx$$

$$\int \frac{1}{u^2+2u+1}du = \int \frac{1}{x}dx$$
$$-\frac{1}{u+1} = \ln(x) + c_2$$

The solution is

$$-\frac{1}{u(x)+1} - \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$-\frac{1}{\frac{y}{x}+1} - \ln(x) - c_2 = 0$$
$$\frac{(-c_2 - \ln(x))\,y - x(c_2 + \ln(x) + 1)}{y+x} = 0$$

Which simplifies to

$$-\frac{y\ln(x) + c_2 y + \ln(x)\,x + c_2 x + x}{y+x} = 0$$

Summary

The solution(s) found are the following

$$-\frac{y\ln(x) + c_2 y + \ln(x)\,x + c_2 x + x}{y+x} = 0 \tag{1}$$

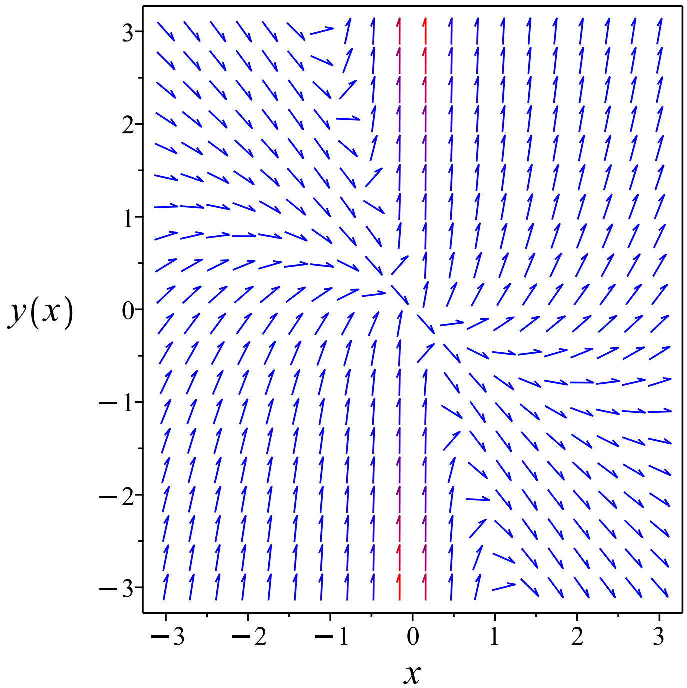


Figure 109: Slope field plot

Verification of solutions

$$-\frac{y \ln (x) + c_2 y + \ln (x) x + c_2 x + x}{y + x} = 0$$

Verified OK.

3.10.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{x^2 + 3xy + y^2}{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{(x^2 + 3xy + y^2)(b_3 - a_2)}{x^2} - \frac{(x^2 + 3xy + y^2)^2 a_3}{x^4} \\ & - \left(\frac{2x + 3y}{x^2} - \frac{2(x^2 + 3xy + y^2)}{x^3}\right)(xa_2 + ya_3 + a_1) \\ & - \frac{(3x + 2y)(xb_2 + yb_3 + b_1)}{x^2} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^4a_2 + x^4a_3 + 2b_2x^4 - x^4b_3 + 6x^3ya_3 + 2x^3yb_2 - x^2y^2a_2 + 8x^2y^2a_3 + x^2y^2b_3 + 4x\,y^3a_3 + y^4a_3 + 3x^3b_1 -}{x^4}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & -x^4a_2 - x^4a_3 - 2b_2x^4 + x^4b_3 - 6x^3ya_3 - 2x^3yb_2 + x^2y^2a_2 - 8x^2y^2a_3 \\ & - x^2y^2b_3 - 4x\,y^3a_3 - y^4a_3 - 3x^3b_1 + 3x^2ya_1 - 2x^2yb_1 + 2x\,y^2a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & -a_2v_1^4 + a_2v_1^2v_2^2 - a_3v_1^4 - 6a_3v_1^3v_2 - 8a_3v_1^2v_2^2 - 4a_3v_1v_2^3 - a_3v_2^4 - 2b_2v_1^4 \\ & - 2b_2v_1^3v_2 + b_3v_1^4 - b_3v_1^2v_2^2 + 3a_1v_1^2v_2 + 2a_1v_1v_2^2 - 3b_1v_1^3 - 2b_1v_1^2v_2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}(-a_2 - a_3 - 2b_2 + b_3)\, v_1^4 + (-6a_3 - 2b_2)\, v_1^3 v_2 - 3b_1 v_1^3 + (a_2 - 8a_3 - b_3)\, v_1^2 v_2^2 \\ + (3a_1 - 2b_1)\, v_1^2 v_2 - 4a_3 v_1 v_2^3 + 2a_1 v_1 v_2^2 - a_3 v_2^4 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 2a_1 &= 0 \\ -4a_3 &= 0 \\ -a_3 &= 0 \\ -3b_1 &= 0 \\ 3a_1 - 2b_1 &= 0 \\ -6a_3 - 2b_2 &= 0 \\ a_2 - 8a_3 - b_3 &= 0 \\ -a_2 - a_3 - 2b_2 + b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= y - \left(\frac{x^2+3xy+y^2}{x^2}\right)(x) \\ &= \frac{-x^2-2xy-y^2}{x} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-x^2-2xy-y^2}{x}} dy\end{aligned}$$

Which results in

$$S = \frac{x}{y+x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^2+3xy+y^2}{x^2}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{y}{(y+x)^2} \\
S_y &= -\frac{x}{(y+x)^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\ln(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{x}{y+x} = -\ln(x) + c_1
$$

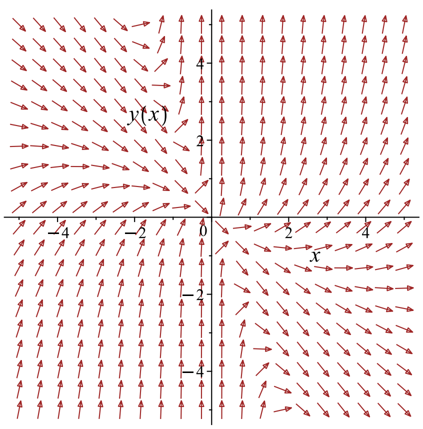
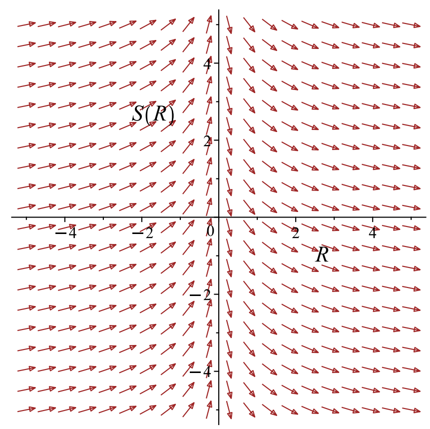
Which simplifies to

$$
\frac{x}{y+x} = -\ln(x) + c_1
$$

Which gives

$$
y = -\frac{x(\ln(x) - c_1 + 1)}{\ln(x) - c_1}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2+3xy+y^2}{x^2}$ 	$R = x$ $S = \frac{x}{y+x}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = -\frac{x(\ln(x) - c_1 + 1)}{\ln(x) - c_1} \tag{1}$$

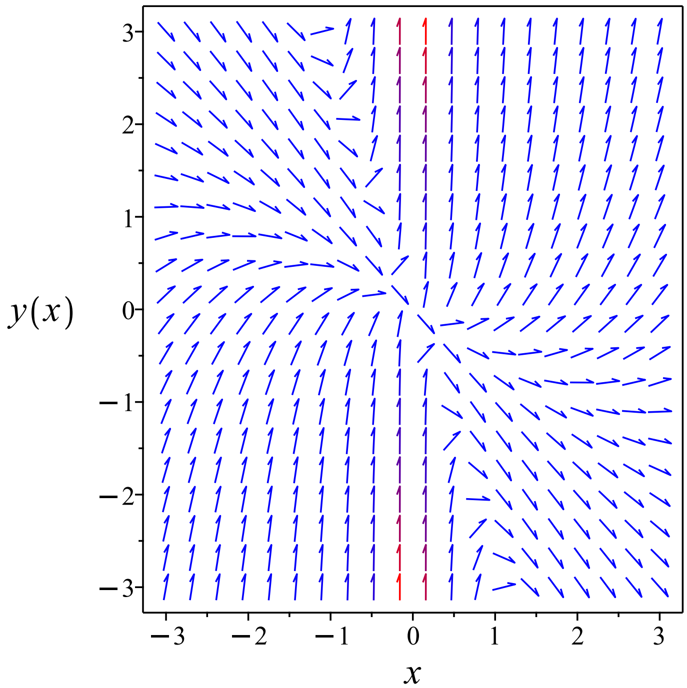


Figure 110: Slope field plot

Verification of solutions

$$y = -\frac{x(\ln(x) - c_1 + 1)}{\ln(x) - c_1}$$

Verified OK.

3.10.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{x^2 + 3xy + y^2}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = 1 + \frac{3y}{x} + \frac{y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 1$, $f_1(x) = \frac{3}{x}$ and $f_2(x) = \frac{1}{x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{2}{x^3} \\ f_1 f_2 &= \frac{3}{x^3} \\ f_2^2 f_0 &= \frac{1}{x^4} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x^2} - \frac{u'(x)}{x^3} + \frac{u(x)}{x^4} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x(\ln(x)\, c_2 + c_1)$$

The above shows that

$$u'(x) = \ln(x)\, c_2 + c_1 + c_2$$

Using the above in (1) gives the solution

$$y = -\frac{(\ln(x)\, c_2 + c_1 + c_2)\, x}{\ln(x)\, c_2 + c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{(\ln(x) + c_3 + 1)\, x}{\ln(x) + c_3}$$

Summary

The solution(s) found are the following

$$y = -\frac{(\ln(x) + c_3 + 1)\, x}{\ln(x) + c_3} \tag{1}$$

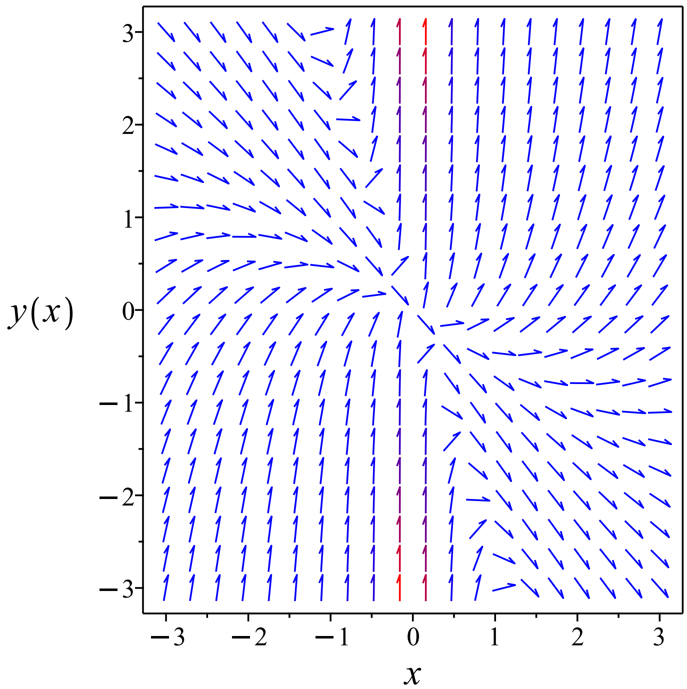


Figure 111: Slope field plot

Verification of solutions

$$y = -\frac{(\ln(x) + c_3 + 1)\, x}{\ln(x) + c_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(x^2*diff(y(x),x)=y(x)^2+3*x*y(x)+x^2,y(x), singsol=all)
```

$$y(x) = -\frac{x(\ln(x) + c_1 + 1)}{\ln(x) + c_1}$$

✓ Solution by Mathematica

Time used: 0.145 (sec). Leaf size: 28

```
DSolve[x^2*y'[x]==y[x]^2+3*x*y[x]+x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x(\log(x) + 1 + c_1)}{\log(x) + c_1}$$
$$y(x) \to -x$$

3.11 problem 19

Internal problem ID [2583]
Internal file name [`OUTPUT/2075_Sunday_June_05_2022_02_47_34_AM_41897557/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.8, page 68
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{yy' - \sqrt{y^2 + x^2} = -x}$$

3.11.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{\sqrt{x^2 + y^2} - x}{y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{\left(\sqrt{x^2+y^2}-x\right)(b_3-a_2)}{y} - \frac{\left(\sqrt{x^2+y^2}-x\right)^2 a_3}{y^2} - \frac{\left(\frac{x}{\sqrt{x^2+y^2}}-1\right)(xa_2+ya_3+a_1)}{y} - \left(\frac{1}{\sqrt{x^2+y^2}} - \frac{\sqrt{x^2+y^2}-x}{y^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{(x^2+y^2)^{\frac{3}{2}} a_3 + \sqrt{x^2+y^2}\, x^2a_3 + \sqrt{x^2+y^2}\, x^2b_2 - 2\sqrt{x^2+y^2}\, xya_2 + 2\sqrt{x^2+y^2}\, xyb_3 - \sqrt{x^2+y^2}\, y^2a_3 -}{}$$

$$= 0$$

Setting the numerator to zero gives

$$-(x^2+y^2)^{\frac{3}{2}} a_3 - \sqrt{x^2+y^2}\, x^2a_3 - \sqrt{x^2+y^2}\, x^2b_2 + 2\sqrt{x^2+y^2}\, xya_2 - 2\sqrt{x^2+y^2}\, xyb_3 + \sqrt{x^2+y^2}\, y^2a_3 + b_2\sqrt{x^2+y^2}\, y^2 + 2x^3a_3 + x^3b_2 - 2x^2ya_2 + 2x^2yb_3 + x\,y^2a_3 - y^3a_2 + y^3b_3 - \sqrt{x^2+y^2}\, xb_1 + \sqrt{x^2+y^2}\, ya_1 + x^2b_1 - xya_1 = 0 \tag{6E}$$

Simplifying the above gives

$$-(x^2+y^2)^{\frac{3}{2}} a_3 + 2(x^2+y^2)\, xa_3 + (x^2+y^2)\, xb_2 - (x^2+y^2)\, ya_2 + 2(x^2+y^2)\, yb_3 - \sqrt{x^2+y^2}\, x^2a_3 - \sqrt{x^2+y^2}\, x^2b_2 + 2\sqrt{x^2+y^2}\, xya_2 - 2\sqrt{x^2+y^2}\, xyb_3 + \sqrt{x^2+y^2}\, y^2a_3 + b_2\sqrt{x^2+y^2}\, y^2 - x^2ya_2 - x\,y^2a_3 - x\,y^2b_2 - y^3b_3 + (x^2+y^2)\, b_1 - \sqrt{x^2+y^2}\, xb_1 + \sqrt{x^2+y^2}\, ya_1 - xya_1 - y^2b_1 = 0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$2x^3a_3 + x^3b_2 - 2\sqrt{x^2+y^2}\, x^2a_3 - \sqrt{x^2+y^2}\, x^2b_2 - 2x^2ya_2 + 2x^2yb_3 + 2\sqrt{x^2+y^2}\, xya_2 - 2\sqrt{x^2+y^2}\, xyb_3 + x\,y^2a_3 + b_2\sqrt{x^2+y^2}\, y^2 - y^3a_2 + y^3b_3 + x^2b_1 - \sqrt{x^2+y^2}\, xb_1 - xya_1 + \sqrt{x^2+y^2}\, ya_1 = 0$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{x^2+y^2}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{x^2+y^2} = v_3\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}&-2v_1^2v_2a_2 + 2v_3v_1v_2a_2 - v_2^3a_2 + 2v_1^3a_3 - 2v_3v_1^2a_3 + v_1v_2^2a_3 + v_1^3b_2 - v_3v_1^2b_2 \\ &+ b_2v_3v_2^2 + 2v_1^2v_2b_3 - 2v_3v_1v_2b_3 + v_2^3b_3 - v_1v_2a_1 + v_3v_2a_1 + v_1^2b_1 - v_3v_1b_1 = 0\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned}&(2a_3 + b_2)\, v_1^3 + (-2a_2 + 2b_3)\, v_1^2v_2 + (-2a_3 - b_2)\, v_1^2v_3 + v_1^2b_1 + v_1v_2^2a_3 \\ &+ (2a_2 - 2b_3)\, v_1v_2v_3 - v_1v_2a_1 - v_3v_1b_1 + (b_3 - a_2)\, v_2^3 + b_2v_3v_2^2 + v_3v_2a_1 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}a_1 &= 0\\ a_3 &= 0\\ b_1 &= 0\\ b_2 &= 0\\ -a_1 &= 0\\ -b_1 &= 0\\ -2a_2 + 2b_3 &= 0\\ 2a_2 - 2b_3 &= 0\\ -2a_3 - b_2 &= 0\\ 2a_3 + b_2 &= 0\\ b_3 - a_2 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0\\
a_2 &= b_3\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x\\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi\\
&= y - \left(\frac{\sqrt{x^2+y^2}-x}{y}\right)(x)\\
&= \frac{x^2 - x\sqrt{x^2+y^2}+y^2}{y}\\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy\\
&= \int \frac{1}{\frac{x^2 - x\sqrt{x^2+y^2}+y^2}{y}} dy
\end{aligned}
$$

Which results in

$$S = \ln(y) - \frac{x \ln\left(\frac{2x^2 + 2\sqrt{x^2}\,\sqrt{x^2+y^2}}{y}\right)}{\sqrt{x^2}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\sqrt{x^2+y^2} - x}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{\sqrt{x^2+y^2} + x}{x\sqrt{x^2+y^2}} \\
S_y &= \frac{2x^2 + y^2 + 2x\sqrt{x^2+y^2}}{y\sqrt{x^2+y^2}\,\left(\sqrt{x^2+y^2} + x\right)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{x\sqrt{x^2+y^2} + x^2 + y^2}{x\sqrt{x^2+y^2}\,\left(\sqrt{x^2+y^2} + x\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

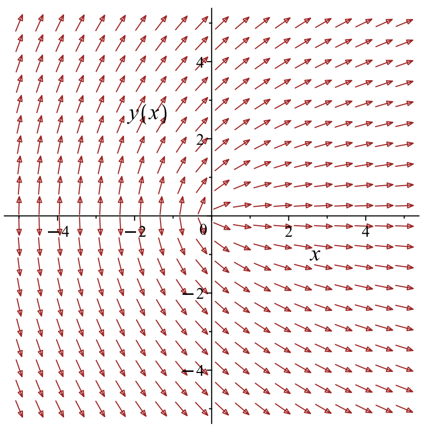
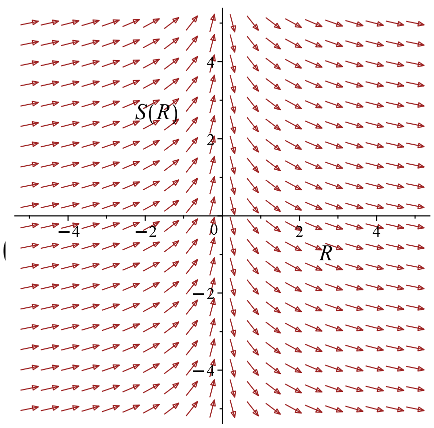
The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

Which gives

$$y = \mathrm{e}^{\frac{\ln(2)}{2} + \frac{\ln(2\,\mathrm{e}^{c_1} + 2x)}{2} + \frac{c_1}{2}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\sqrt{x^2+y^2}-x}{y}$ 	$R = x$ $S = 2\ln(y) - \ln(2) - \ln($	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\frac{\ln(2)}{2} + \frac{\ln(2\,\mathrm{e}^{c_1} + 2x)}{2} + \frac{c_1}{2}} \tag{1}$$

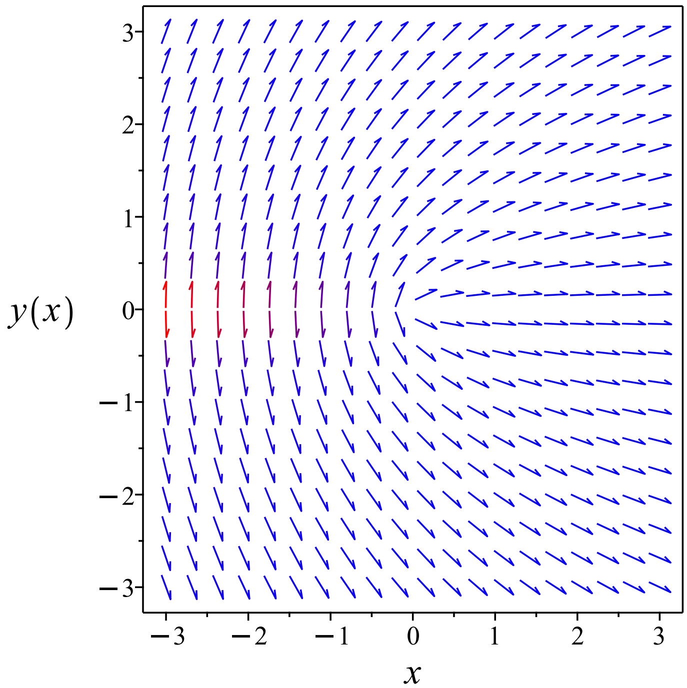


Figure 112: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{\frac{\ln(2)}{2}+\frac{\ln\left(2\,\mathrm{e}^{c_1}+2x\right)}{2}+\frac{c_1}{2}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 27

```
dsolve(y(x)*diff(y(x),x)=sqrt(x^2+y(x)^2)-x,y(x), singsol=all)
```

$$\frac{-c_1 y(x)^2+\sqrt{x^2+y(x)^2}+x}{y(x)^2}=0$$

✓ Solution by Mathematica

Time used: 0.432 (sec). Leaf size: 57

```
DSolve[y[x]*y'[x]==Sqrt[x^2+y[x]^2]-x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -e^{\frac{c_1}{2}}\sqrt{2x+e^{c_1}}$$
$$y(x)\to e^{\frac{c_1}{2}}\sqrt{2x+e^{c_1}}$$
$$y(x)\to 0$$

3.12 problem 20

Internal problem ID [2584]
Internal file name [`OUTPUT/2076_Sunday_June_05_2022_02_47_39_AM_18342098/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.8, page 68
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{2x(y+2x)\,y'-y(4x-y)=0}$$

3.12.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$2x(u(x)\,x+2x)\,(u'(x)\,x+u(x))-u(x)\,x(4x-u(x)\,x)=0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= -\frac{3u^2}{2x\,(u+2)}\end{aligned}$$

Where $f(x) = -\frac{3}{2x}$ and $g(u) = \frac{u^2}{u+2}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{u^2}{u+2}} \, du &= -\frac{3}{2x} \, dx \\
\int \frac{1}{\frac{u^2}{u+2}} \, du &= \int -\frac{3}{2x} \, dx \\
-\frac{2}{u} + \ln(u) &= -\frac{3 \ln(x)}{2} + c_2
\end{aligned}
$$

The solution is

$$
-\frac{2}{u(x)} + \ln(u(x)) + \frac{3 \ln(x)}{2} - c_2 = 0
$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$
\begin{aligned}
-\frac{2x}{y} + \ln\left(\frac{y}{x}\right) + \frac{3 \ln(x)}{2} - c_2 = 0 \\
-\frac{2x}{y} + \ln\left(\frac{y}{x}\right) + \frac{3 \ln(x)}{2} - c_2 = 0
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
-\frac{2x}{y} + \ln\left(\frac{y}{x}\right) + \frac{3 \ln(x)}{2} - c_2 = 0 \tag{1}
$$

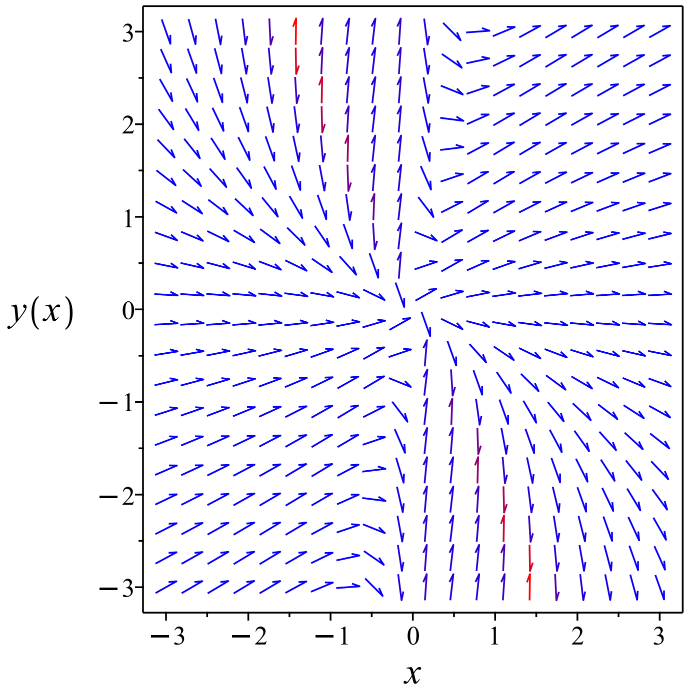


Figure 113: Slope field plot

Verification of solutions

$$-\frac{2x}{y}+\ln\left(\frac{y}{x}\right)+\frac{3\ln(x)}{2}-c_2=0$$

Verified OK.

3.12.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{y(-4x+y)}{2x\,(2x+y)}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 - \frac{y(-4x+y)\,(b_3-a_2)}{2x\,(2x+y)} - \frac{y^2(-4x+y)^2\,a_3}{4x^2\,(2x+y)^2} \\
&- \left(\frac{2y}{x\,(2x+y)} + \frac{y(-4x+y)}{2x^2\,(2x+y)} + \frac{y(-4x+y)}{x\,(2x+y)^2}\right)(xa_2+ya_3+a_1) \\
&- \left(-\frac{-4x+y}{2x\,(2x+y)} - \frac{y}{2x\,(2x+y)} + \frac{y(-4x+y)}{2x\,(2x+y)^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{24x^3yb_2 - 12x^2y^2a_2 + 6x^2y^2b_2 + 12x^2y^2b_3 - 3y^4a_3 - 16x^3b_1 + 16x^2ya_1 + 8x^2yb_1 - 8x\,y^2a_1 + 2x\,y^2b_1 - 2y}{4x^2\,(2x+y)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&24x^3yb_2 - 12x^2y^2a_2 + 6x^2y^2b_2 + 12x^2y^2b_3 - 3y^4a_3 - 16x^3b_1 \\
&+ 16x^2ya_1 + 8x^2yb_1 - 8x\,y^2a_1 + 2x\,y^2b_1 - 2y^3a_1 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-12a_2v_1^2v_2^2 - 3a_3v_2^4 + 24b_2v_1^3v_2 + 6b_2v_1^2v_2^2 + 12b_3v_1^2v_2^2 + 16a_1v_1^2v_2 \\
&- 8a_1v_1v_2^2 - 2a_1v_2^3 - 16b_1v_1^3 + 8b_1v_1^2v_2 + 2b_1v_1v_2^2 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &24b_2v_1^3v_2 - 16b_1v_1^3 + (-12a_2 + 6b_2 + 12b_3)\, v_1^2v_2^2 \\ &+ (16a_1 + 8b_1)\, v_1^2v_2 + (-8a_1 + 2b_1)\, v_1v_2^2 - 3a_3v_2^4 - 2a_1v_2^3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -2a_1 &= 0 \\ -3a_3 &= 0 \\ -16b_1 &= 0 \\ 24b_2 &= 0 \\ -8a_1 + 2b_1 &= 0 \\ 16a_1 + 8b_1 &= 0 \\ -12a_2 + 6b_2 + 12b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\, \xi \\ &= y - \left(-\frac{y(-4x + y)}{2x\,(2x + y)}\right)(x) \\ &= \frac{3y^2}{2y + 4x} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{3y^2}{2y+4x}} dy \end{aligned}$$

Which results in

$$S = -\frac{4x}{3y} + \frac{2\ln(y)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y(-4x + y)}{2x\,(2x + y)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{4}{3y} \\ S_y &= \frac{2y + 4x}{3y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{3x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{3R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\ln(R)}{3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{2y\ln(y) - 4x}{3y} = -\frac{\ln(x)}{3} + c_1$$

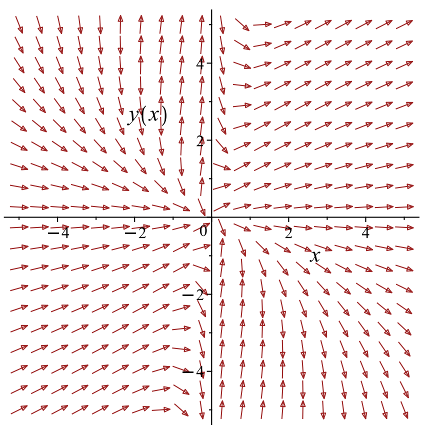
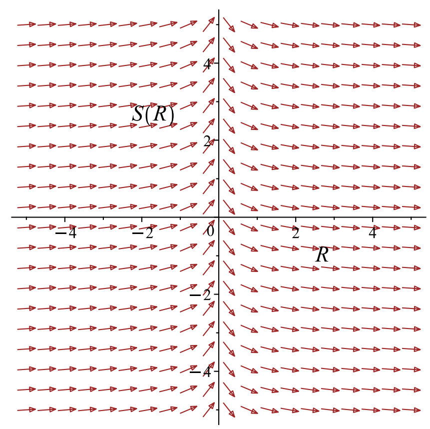
Which simplifies to

$$\frac{2y\ln(y) - 4x}{3y} = -\frac{\ln(x)}{3} + c_1$$

Which gives

$$y = \mathrm{e}^{\mathrm{LambertW}\left(2x\,\mathrm{e}^{\frac{\ln(x)}{2}-\frac{3c_1}{2}}\right)-\frac{\ln(x)}{2}+\frac{3c_1}{2}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(-4x+y)}{2x(2x+y)}$ 	$R = x$ $S = \frac{2\ln(y)\,y - 4x}{3y}$	$\frac{dS}{dR} = -\frac{1}{3R}$ 

Summary
The solution(s) found are the following

$$y = e^{\operatorname{LambertW}\left(2x\,e^{\frac{\ln(x)}{2}-\frac{3c_1}{2}}\right)-\frac{\ln(x)}{2}+\frac{3c_1}{2}} \tag{1}$$

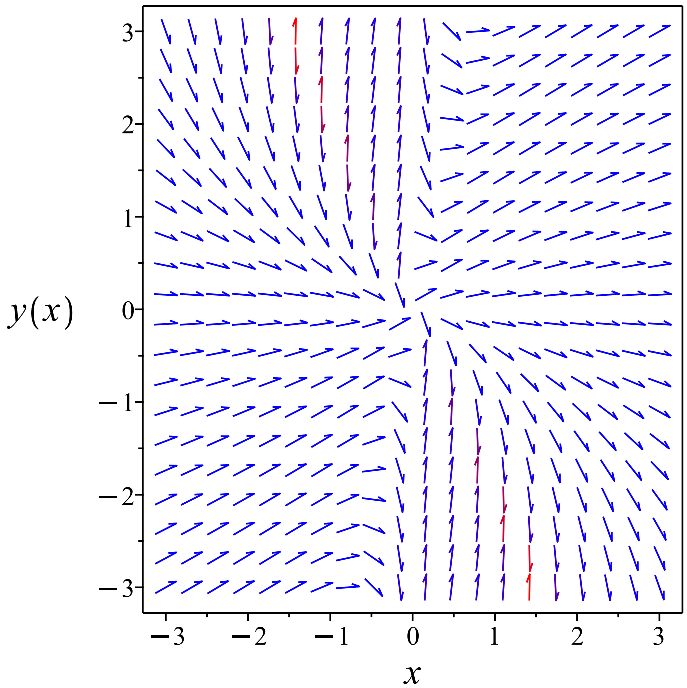


Figure 114: Slope field plot

Verification of solutions

$$
y = e^{\mathrm{LambertW}\left(2x\,e^{\frac{\ln(x)}{2}-\frac{3c_1}{2}}\right)-\frac{\ln(x)}{2}+\frac{3c_1}{2}}
$$

Verified OK.

3.12.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$
M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y) = 0
$$

Hence

$$
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2x(2x + y)) \, \mathrm{d}y &= (y(4x - y)) \, \mathrm{d}x \\ (-y(4x - y)) \, \mathrm{d}x + (2x(2x + y)) \, \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -y(4x - y) \\ N(x, y) &= 2x(2x + y)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y(4x - y)) \\ &= -4x + 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2x(2x + y)) \\ &= 8x + 2y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{xy^2}$ is an integrating factor. Therefore by multiplying $M = -y(4x - y)$ and $N = 2x(y + 2x)$ by this integrating factor the ode becomes exact. The new M, N are

$$
\begin{aligned}
M &= -\frac{4x - y}{xy} \\
N &= \frac{2y + 4x}{y^2}
\end{aligned}
$$

To solve an ode of the form

$$
M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x, y) = 0
$$

Hence

$$
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$\left(\frac{2y+4x}{y^2}\right)\mathrm{d}y = \left(\frac{4x-y}{xy}\right)\mathrm{d}x$$

$$\left(-\frac{4x-y}{xy}\right)\mathrm{d}x + \left(\frac{2y+4x}{y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{4x-y}{xy}$$
$$N(x,y) = \frac{2y+4x}{y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{4x-y}{xy}\right)\\ &= \frac{4}{y^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{2y+4x}{y^2}\right)\\ &= \frac{4}{y^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\frac{4x-y}{xy}\,\mathrm{d}x$$
$$\phi = \ln(x) - \frac{4x}{y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{4x}{y^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2y+4x}{y^2}$. Therefore equation (4) becomes

$$\frac{2y+4x}{y^2} = \frac{4x}{y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{2}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{2}{y}\right) \mathrm{d}y$$
$$f(y) = 2\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(x) - \frac{4x}{y} + 2\ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(x) - \frac{4x}{y} + 2\ln(y)$$

The solution becomes

$$y = \mathrm{e}^{\mathrm{LambertW}\left(2x\,\mathrm{e}^{\frac{\ln(x)}{2}-\frac{c_1}{2}}\right)-\frac{\ln(x)}{2}+\frac{c_1}{2}}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \mathrm{e}^{\mathrm{LambertW}\left(2x\,\mathrm{e}^{\frac{\ln(x)}{2}-\frac{c_1}{2}}\right)-\frac{\ln(x)}{2}+\frac{c_1}{2}} \tag{1}$$

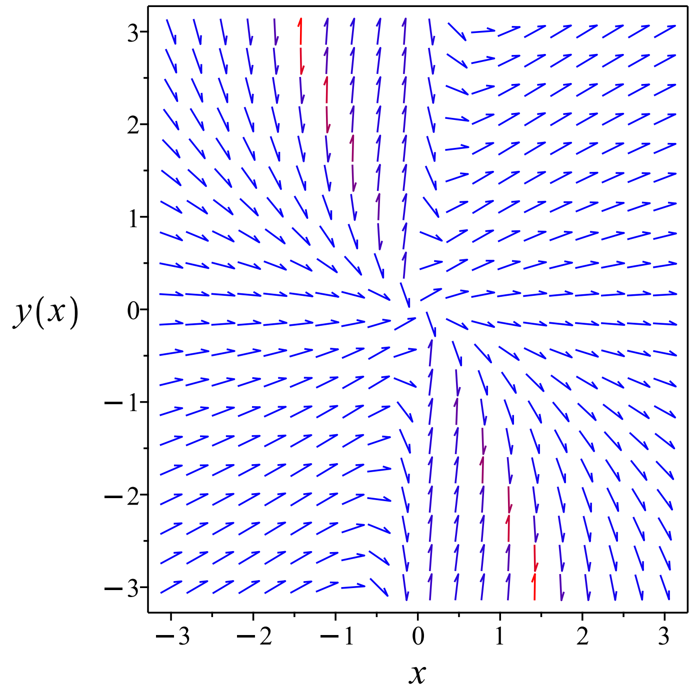


Figure 115: Slope field plot

Verification of solutions

$$y = e^{\text{LambertW}\left(2x\,e^{\frac{\ln(x)}{2}-\frac{c_1}{2}}\right)-\frac{\ln(x)}{2}+\frac{c_1}{2}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 19

```
dsolve(2*x*(y(x)+2*x)*diff(y(x),x)=y(x)*(4*x-y(x)),y(x), singsol=all)
```

$$y(x) = \frac{2x}{\text{LambertW}\left(2\,\mathrm{e}^{\frac{3c_1}{2}} x^{\frac{3}{2}}\right)}$$

✓ Solution by Mathematica

Time used: 5.384 (sec). Leaf size: 29

```
DSolve[2*x*(y[x]+2*x)*y'[x]==y[x]*(4*x-y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2x}{W\left(2e^{-c_1}x^{3/2}\right)}$$
$$y(x) \to 0$$

3.13 problem 21

Internal problem ID [2585]
Internal file name [`OUTPUT/2077_Sunday_June_05_2022_02_47_42_AM_65586110/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.8, page 68
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{xy' - x\tan\left(\frac{y}{x}\right) - y = 0}$$

3.13.1 Solving as homogeneousTypeD ode

Writing the ode as

$$y' = \tan\left(\frac{y}{x}\right) + \frac{y}{x} \tag{A}$$

The given ode has the form

$$y' = \frac{y}{x} + g(x)\, f\left(b\frac{y}{x}\right)^{\frac{n}{m}} \tag{1}$$

Where b is scalar and $g(x)$ is function of x and n, m are integers. The solution is given in Kamke page 20. Using the substitution $y(x) = u(x)\, x$ then

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Hence the given ode becomes

$$\frac{du}{dx}x + u = u + g(x)\, f(bu)^{\frac{n}{m}}$$

$$u' = \frac{1}{x}g(x)\, f(bu)^{\frac{n}{m}} \tag{2}$$

The above ode is always separable. This is easily solved for u assuming the integration can be resolved, and then the solution to the original ode becomes $y = ux$. Comapring the given ode (A) with the form (1) shows that

$$
\begin{aligned}
g(x) &= 1 \\
b &= 1 \\
f\left(\frac{bx}{y}\right) &= \tan\left(\frac{y}{x}\right)
\end{aligned}
$$

Substituting the above in (2) results in the $u(x)$ ode as

$$
u'(x) = \frac{\tan(u(x))}{x}
$$

Which is now solved as separable In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= \frac{\tan(u)}{x}
\end{aligned}
$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \tan(u)$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\tan(u)}\, du &= \frac{1}{x}\, dx \\
\int \frac{1}{\tan(u)}\, du &= \int \frac{1}{x}\, dx \\
\ln(\sin(u)) &= \ln(x) + c_1
\end{aligned}
$$

Raising both side to exponential gives

$$
\sin(u) = \mathrm{e}^{\ln(x)+c_1}
$$

Which simplifies to

$$
\sin(u) = c_2 x
$$

Therefore the solution is

$$
\begin{aligned}
y &= ux \\
&= x \arcsin\left(c_2 \mathrm{e}^{c_1} x\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = x \arcsin\left(c_2 \mathrm{e}^{c_1} x\right) \tag{1}$$

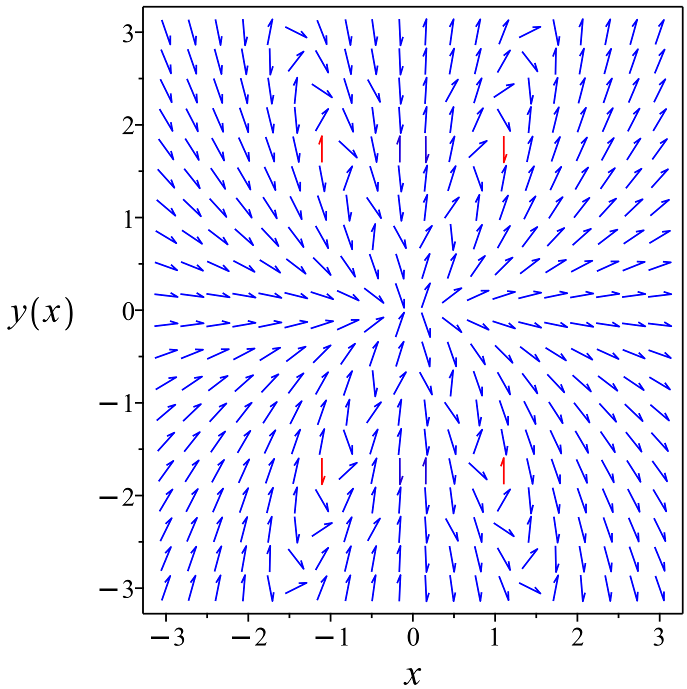


Figure 116: Slope field plot

Verification of solutions

$$y = x \arcsin\left(c_2 \mathrm{e}^{c_1} x\right)$$

Verified OK.

3.13.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$x(u'(x)\, x + u(x)) - x \tan\left(u(x)\right) - u(x)\, x = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x) g(u) \\ &= \frac{\tan(u)}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \tan(u)$. Integrating both sides gives

$$\frac{1}{\tan(u)}\,du = \frac{1}{x}\,dx$$

$$\int \frac{1}{\tan(u)}\,du = \int \frac{1}{x}\,dx$$

$$\ln(\sin(u)) = \ln(x) + c_2$$

Raising both side to exponential gives

$$\sin(u) = \mathrm{e}^{\ln(x)+c_2}$$

Which simplifies to

$$\sin(u) = c_3 x$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= x \arcsin\left(c_3 \mathrm{e}^{c_2} x\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x \arcsin\left(c_3 \mathrm{e}^{c_2} x\right) \tag{1}$$

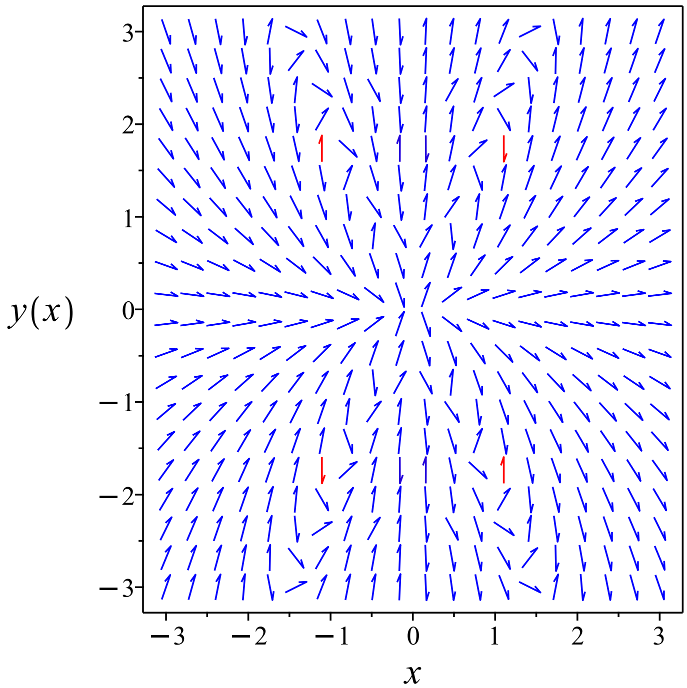


Figure 117: Slope field plot

Verification of solutions

$$y = x \arcsin\left(c_3 \mathrm{e}^{c_2} x\right)$$

Verified OK.

3.13.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x \tan\left(\frac{y}{x}\right) + y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `homogeneous Type D`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 84: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= x^2 \\
\eta(x, y) &= xy
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi} \\ &= \frac{xy}{x^2} \\ &= \frac{y}{x}\end{aligned}$$

This is easily solved to give

$$y = c_1 x$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{y}{x}$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi} \\ &= \frac{dx}{x^2}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T} \\ &= -\frac{1}{x}\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x \tan\left(\frac{y}{x}\right) + y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= -\frac{y}{x^2} \\ R_y &= \frac{1}{x} \\ S_x &= \frac{1}{x^2} \\ S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\cot\left(\frac{y}{x}\right)}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\cot(R)\, S(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{c_1}{\sin(R)} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{x} = \frac{c_1}{\sin\left(\frac{y}{x}\right)}$$

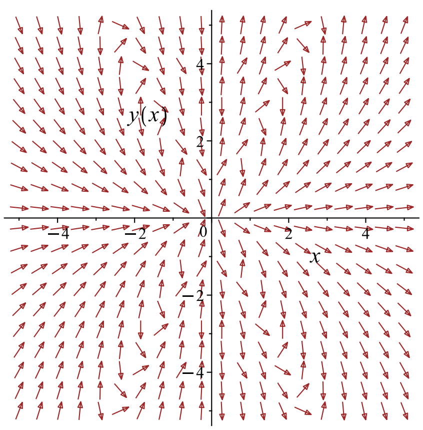
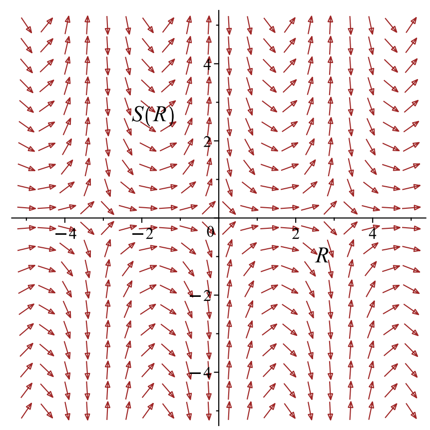
Which simplifies to

$$-\frac{1}{x} = \frac{c_1}{\sin\left(\frac{y}{x}\right)}$$

Which gives

$$y = -\arcsin(c_1 x)\, x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x \tan\left(\frac{y}{x}\right)+y}{x}$ 	$R = \frac{y}{x}$ $S = -\frac{1}{x}$	$\frac{dS}{dR} = -\cot(R)\, S(R)$ 

Summary

The solution(s) found are the following

$$y = -\arcsin(c_1 x)\, x \tag{1}$$

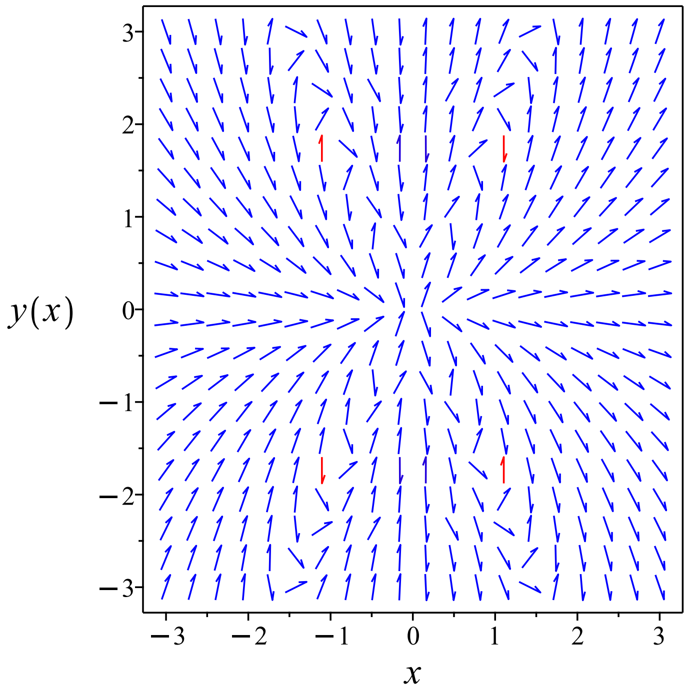


Figure 118: Slope field plot

Verification of solutions

$$y = -\arcsin(c_1 x)\, x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 10

```
dsolve(x*diff(y(x),x)=x*tan(y(x)/x)+y(x),y(x), singsol=all)
```

$$y(x) = \arcsin(c_1 x)\, x$$

✓ Solution by Mathematica

Time used: 4.369 (sec). Leaf size: 19

```
DSolve[x*y'[x]==x*Tan[y[x]/x]+y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x \arcsin(e^{c_1} x)$$
$$y(x) \to 0$$

3.14 problem 22

Internal problem ID [2586]
Internal file name [`OUTPUT/2078_Sunday_June_05_2022_02_47_45_AM_59997785/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode, second edition, 2000
Section: 1.8, page 68
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{y' - \frac{x\sqrt{y^2+x^2}+y^2}{yx} = 0}$$

3.14.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{x\sqrt{x^2+y^2}+y^2}{yx} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
b_2 + \frac{\left(x\sqrt{x^2+y^2}+y^2\right)(b_3-a_2)}{yx} - \frac{\left(x\sqrt{x^2+y^2}+y^2\right)^2 a_3}{y^2x^2} - \left(\frac{\sqrt{x^2+y^2}+\frac{x^2}{\sqrt{x^2+y^2}}}{yx} - \frac{x\sqrt{x^2+y^2}+y^2}{y\,x^2}\right)(xa_2+ya_3+a_1) - \left(\frac{\frac{xy}{\sqrt{x^2+y^2}}+2y}{yx} - \frac{x\sqrt{x^2+y^2}+y^2}{y^2x}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{\left(x^2+y^2\right)^{\frac{3}{2}} x^2 a_3 - x^5 b_2 + 2x^4 y a_2 - 2x^4 y b_3 + 3x^3 y^2 a_3 + x^2 y^3 a_2 - x^2 y^3 b_3 + 2x\, y^4 a_3 + \sqrt{x^2+y^2}\, x\, y^2 b_1 -}{\sqrt{x^2+y^2}\, y^2 x^2} = 0
$$

Setting the numerator to zero gives

$$
-\left(x^2+y^2\right)^{\frac{3}{2}} x^2 a_3 + x^5 b_2 - 2x^4 y a_2 + 2x^4 y b_3 - 3x^3 y^2 a_3 - x^2 y^3 a_2 + x^2 y^3 b_3 - 2x\, y^4 a_3 - \sqrt{x^2+y^2}\, x\, y^2 b_1 + \sqrt{x^2+y^2}\, y^3 a_1 + x^4 b_1 - x^3 y a_1 = 0 \tag{6E}
$$

Simplifying the above gives

$$
-\left(x^2+y^2\right)^{\frac{3}{2}} x^2 a_3 + \left(x^2+y^2\right) x^3 b_2 - \left(x^2+y^2\right) x^2 y a_2 + 2\left(x^2+y^2\right) x^2 y b_3 - 2\left(x^2+y^2\right) x\, y^2 a_3 - x^4 y a_2 - x^3 y^2 a_3 - x^3 y^2 b_2 - x^2 y^3 b_3 + \left(x^2+y^2\right) x^2 b_1 - \sqrt{x^2+y^2}\, x\, y^2 b_1 + \sqrt{x^2+y^2}\, y^3 a_1 - x^3 y a_1 - x^2 y^2 b_1 = 0 \tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
x^5 b_2 - x^4 \sqrt{x^2+y^2}\, a_3 - 2x^4 y a_2 + 2x^4 y b_3 - 3x^3 y^2 a_3 - x^2 \sqrt{x^2+y^2}\, y^2 a_3 - x^2 y^3 a_2 + x^2 y^3 b_3 - 2x\, y^4 a_3 + x^4 b_1 - x^3 y a_1 - \sqrt{x^2+y^2}\, x\, y^2 b_1 + \sqrt{x^2+y^2}\, y^3 a_1 = 0
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{x^2+y^2}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{x^2+y^2} = v_3\right\}$$

The above PDE (6E) now becomes

$$-2v_1^4 v_2 a_2 - v_1^2 v_2^3 a_2 - v_1^4 v_3 a_3 - 3v_1^3 v_2^2 a_3 - v_1^2 v_3 v_2^2 a_3 - 2v_1 v_2^4 a_3 + v_1^5 b_2 + 2v_1^4 v_2 b_3 + v_1^2 v_2^3 b_3 - v_1^3 v_2 a_1 + v_3 v_2^3 a_1 + v_1^4 b_1 - v_3 v_1 v_2^2 b_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$v_1^5 b_2 + (-2a_2 + 2b_3) v_1^4 v_2 - v_1^4 v_3 a_3 + v_1^4 b_1 - 3v_1^3 v_2^2 a_3 - v_1^3 v_2 a_1 + (b_3 - a_2) v_1^2 v_2^3 - v_1^2 v_3 v_2^2 a_3 - 2v_1 v_2^4 a_3 - v_3 v_1 v_2^2 b_1 + v_3 v_2^3 a_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
a_1 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
-a_1 &= 0 \\
-3a_3 &= 0 \\
-2a_3 &= 0 \\
-a_3 &= 0 \\
-b_1 &= 0 \\
-2a_2 + 2b_3 &= 0 \\
b_3 - a_2 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{x\sqrt{x^2+y^2}+y^2}{yx}\right)(x) \\
&= -\frac{x\sqrt{x^2+y^2}}{y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\frac{x\sqrt{x^2+y^2}}{y}} dy
\end{aligned}
$$

Which results in

$$S = -\frac{\sqrt{x^2+y^2}}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x\sqrt{x^2+y^2}+y^2}{yx}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1\\
R_y &= 0\\
S_x &= \frac{y^2}{x^2\sqrt{x^2+y^2}}\\
S_y &= -\frac{y}{x\sqrt{x^2+y^2}}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

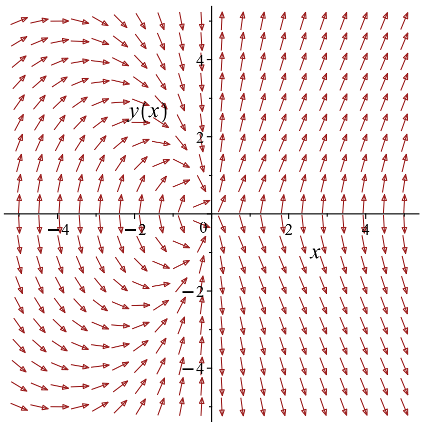
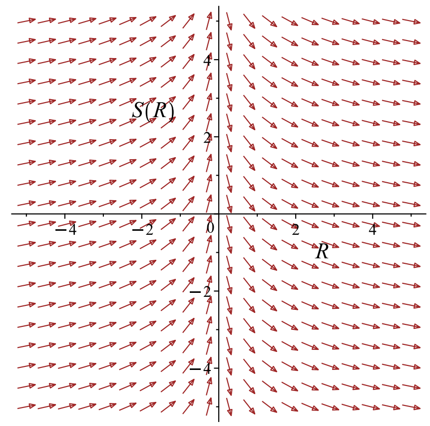
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\sqrt{y^2+x^2}}{x} = -\ln(x) + c_1$$

Which simplifies to

$$-\frac{\sqrt{y^2+x^2}}{x} = -\ln(x) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x\sqrt{x^2+y^2}+y^2}{yx}$ 	$R = x$ $S = -\frac{\sqrt{x^2+y^2}}{x}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$-\frac{\sqrt{y^2+x^2}}{x} = -\ln(x) + c_1 \tag{1}$$

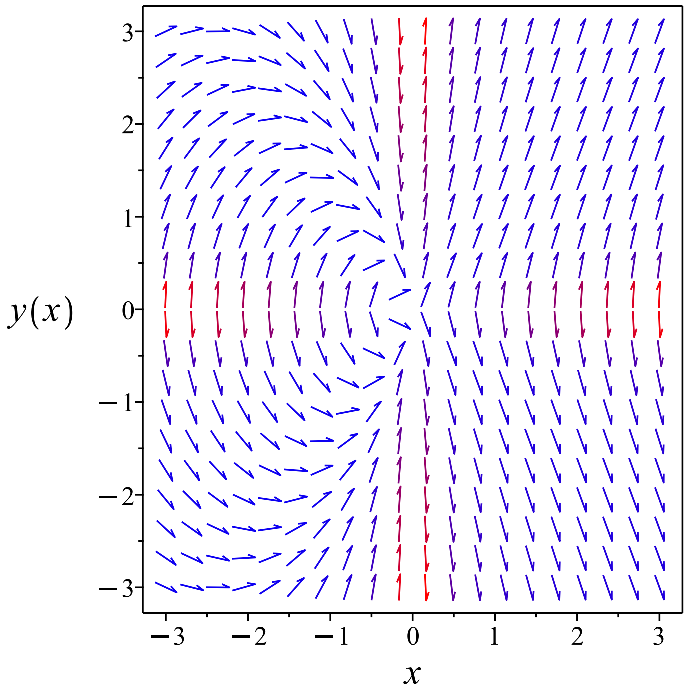


Figure 119: Slope field plot

Verification of solutions

$$-\frac{\sqrt{y^2+x^2}}{x} = -\ln(x) + c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 28

```
dsolve(diff(y(x),x)=(x*sqrt(x^2+y(x)^2)+y(x)^2)/(x*y(x)),y(x), singsol=all)
```

$$\frac{x \ln \left(x\right)-c_{1} x-\sqrt{x^{2}+y \left(x\right)^{2}}}{x}=0$$

✓ Solution by Mathematica

Time used: 0.283 (sec). Leaf size: 54

```
DSolve[y'[x]==(x*Sqrt[x^2+y[x]^2]+y[x]^2)/(x*y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -x\sqrt{\log^2(x)+2c_1\log(x)-1+{c_1}^2}$$
$$y(x)\to x\sqrt{\log^2(x)+2c_1\log(x)-1+{c_1}^2}$$